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Generalized Classes of Distributions with Applications to Income and Lifetime Data

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GENERALIZED CLASSES OF DISTRIBUTIONS WITH APPLICATIONS TO INCOME AND LIFETIME DATA

by

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(Under the Direction of Broderick O. Oluyede)

ABSTRACT

In this thesis, new classes of distributions namely: exponentiated Kumaraswamy-Dagum (EKD), Log-exponentiated Kumaraswamy-Dagum (Log-EKD), McDonald Log-logistic (McLLog) and Gamma-Dagum (GD) distributions are presented. A thorough and comprehensive investigation of these classes of distributions is conducted. Mathematical properties of these classes of distributions including series expansion, hazard and reverse hazard functions, moments, generating functions, mean and median deviations, Bonferroni and Lorenz curves, distribution of order statistics, moments of order statistics and entropies are presented. Estimation of parameters of these distributions via maximum likelihood technique, Fisher information and asymptotic confidence intervals are given. Maximum likelihood estimation of the parameters of the exponentiated Kumaraswamy-Dagum distribution for censored data is constructed. Real data examples are presented to illustrate the usefulness and applicability of these proposed classes of distributions.

Key Words: Kumaraswamy Distribution; Dagum Distribution; McDonald Distribution; Log-logistic Distribution; Gamma Distribution

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GENERALIZED CLASSES OF DISTRIBUTIONS WITH APPLICATIONS TO INCOME AND LIFETIME DATA

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INTRODUCTION

1.1 Introduction

For complex systems including electronic and mechanical systems, the hazard rate function often exhibits non-monotonic (upside down bathtub or bathtub shaped) hazard rates. In this thesis, several new classes of distributions are presented. These distributions can be used for modeling data in several areas including engineering, medical sciences, biological studies and economics.

The purpose here is to define new families of probability distributions that extend well-known families of distributions and at the same time provide greater flexibility in modeling data in practice. For a cumulative distribution function (cdf) $F(x)$ of any random variable, one such generalized class of distributions is referred to as Kumaraswamy generalized distribution [19] with cdf $G_{KG}(x)$ is given by

$$G_{KG}(x) = 1 - (1 - F^{\psi}(x))^{\phi},$$

for $\psi > 0$ and $\phi > 0$. Another generalized class of distributions is the McDonald generalized distribution:

$$G_{McG}(x) = \frac{1}{B(ac^{-1}, b)} \int_0^{F^c(x)} w^{ac^{-1}-1}(1-w)^{b-1} dw,$$

for a, b and $c > 0$. The McDonald generalization is a generalization of the beta-F distributions. See [10], [14], [18] and references therein. Generalizations via weighted distributions are also of tremendous practical importance. See [26], [27], [29], [32] for details. Additional generalizations can be obtained via the gamma distribution, and such generalized class of distributions are referred to as gamma generalized distribution:

$$G_{GG}(x) = \frac{\gamma(-\theta^{-1}\log(\bar{F}(x)), \alpha)}{\Gamma(\alpha)},$$

[36], for $\alpha, \theta > 0$, where $\bar{F}(x) = 1 - F(x)$, and $\gamma(x, \alpha) = \int_0^x t^{\alpha-1} e^{-t} dt$ is the lower incomplete gamma function. Ristić and Balakrishnan [33] also proposed an alternative gamma-generator defined by the cdf:

$$G(x) = 1 - \frac{1}{\Gamma(\alpha)} \int_0^{-\log F(x)} t^{\alpha-1} e^{-t} dt,$$

for $\alpha > 0$, where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the gamma function.

1.2 Outline of Thesis

The outline of this thesis is as follows: In Chapter 2, the exponentiated Kumaraswamy-Dagum distribution, sub-models, hazard and reverse hazard functions, moments, moment generating function, Bonferroni and Lorenz curves, mean and median deviations, reliability, simulation and applications are given. Chapter 3 presents maximum likelihood estimation in the exponentiated Kumaraswamy-Dagum distribution with type I right censored and type II double censored data. Applications to real data are also presented. Log-exponentiated Kumaraswamy-Dagum distribution and its statistical properties are discussed in Chapter 4. Chapter 5 contains McLLog distribution, its statistical properties, simulation and applications. The Gamma-Dagum and related distributions are presented in Chapter 6, followed by areas of further or additional research.

CHAPTER 2

THE EXPONENTIATED KUMARASWAMY-DAGUM DISTRIBUTION

2.1 Introduction

Camilo Dagum proposed the distribution which is referred to as Dagum distribution in 1977. This proposal enable the development of statistical distributions used to fit empirical income and wealth data, that could accommodate heavy tails in income and wealth distributions. Dagum's proposed distribution has both Type-I and Type-II specification, where Type-I is the three parameter specification and Type-II deals with four parameter specification. This distribution is a special case of generalized beta distribution of the second kind (GB2), McDonald [22], McDonald and Xu [23], when the parameter $q = 1$, where the probability density function (pdf) of the GB2 distribution is given by:

$$f_{GB2}(y; a, b, p, q) = \frac{ay^{ap-1}}{b^{ap}B(p,q)[1 + (\frac{y}{b})^a]^{p+q}}, \quad \text{for } y > 0.$$

Note that $a > 0$, $p > 0$, $q > 0$ are the shape parameters, b is the scale parameter and $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is the beta function. Kleiber [17] traced the genesis of Dagum distribution and summarized several statistical properties of this distribution. Domma et al. [7] obtained the maximum likelihood estimates of the parameters of Dagum distribution for censored data. Domma and Condino [8] presented the beta-Dagum distribution.

The pdf and cdf of Dagum distribution are given by:

$$g_D(x; \lambda, \beta, \delta) = \beta\lambda\delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1} \tag{2.1}$$

and

$$G_D(x; \lambda, \beta, \delta) = (1 + \lambda x^{-\delta})^{-\beta}, \tag{2.2}$$

for $x > 0$, where λ is a scale parameter, δ and β are shape parameters. Dagum [6] refers to his model as the generalized logistic-Burr distribution. The k^{th} raw or non central moments are given by

$$E\left(X^k\right) = \beta\lambda^{\frac{k}{\delta}} B\left(\beta + \frac{k}{\delta}, 1 - \frac{k}{\delta}\right),$$

for $k < \delta$, and $\lambda, \beta > 0$, where $B(\cdot, \cdot)$ is the beta function. The q^{th} percentile is

$$x_q = \lambda^{\frac{1}{\delta}} \left(q^{-\frac{1}{\beta}} - 1\right)^{-\frac{1}{\delta}}.$$

In this chapter, we present generalizations of the Dagum distribution via Kumaraswamy distribution and its exponentiated version. This leads to the exponentiated Kumaraswamy-Dagum distribution.

The motivation for the development of this distribution is the modeling of size distribution of personal income and lifetime data with a diverse model that takes into consideration not only shape and scale, but also skewness, kurtosis and tail variation.

This chapter is organized as follows. In section 2.2, we present the exponentiated Kumaraswamy-Dagum distribution and its sub models, as well as series expansion, hazard and reverse hazard functions. Moments, moment generating function, Lorenz and Bonferroni curves, mean and median deviations, and reliability are obtained in section 2.3. Section 2.4 contains results on the distribution of the order statistics and Rényi entropy. Estimation of model parameters via the method of maximum likelihood is presented in section 2.5. In section 2.6, various simulations are conducted for different sample sizes. Section 2.7 contains examples and applications of the EKD distribution and its sub-models, followed by concluding remarks.

2.2 The Exponentiated Kumaraswamy-Dagum Distribution

In this section, we present the proposed distribution and its sub-models. Series expansion, hazard and reverse hazard functions are also studied in this section.

2.2.1 The Kumaraswamy-Dagum Distribution

Kumaraswamy [19] introduced a two-parameter distribution on $(0,1)$. Its cdf is given by

$$G(x) = 1 - (1 - x^{\psi})^{\phi}, x \in (0,1),$$

for $\psi > 0$ and $\phi > 0$.

For an arbitrary cdf $F(x)$ with pdf $f(x) = \frac{dF(x)}{dx}$, the family of Kumaraswamy-G distributions with cdf $G_K(x)$ is given by

$$G_K(x) = 1 - (1 - F^{\psi}(x))^{\phi},$$

for $\psi > 0$ and $\phi > 0$. By letting $F(x) = G_D(x)$, we obtain the Kumaraswamy-Dagum (KD) distribution, with cdf

$$G_{KD}(x) = 1 - (1 - G_D^{\psi}(x))^{\phi},$$

where $G_D(x)$ is the cdf of Dagum distribution.

2.2.2 The Exponentiated Kumaraswamy-Dagum Distribution

Replacing the dependent parameter $\beta\psi$ by α, the cdf and pdf of the EKD distribution are given by

$$G_{_{EKD}}(x; \alpha, \lambda, \delta, \phi, \theta) = \{1 - [1 - (1 + \lambda x^{-\delta})^{-\alpha}]^{\phi}\}^{\theta}, \quad (2.3)$$

and

$$\begin{aligned} g_{_{EKD}}(x; \alpha, \lambda, \delta, \phi, \theta) &= \alpha\lambda\delta\phi\theta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\alpha-1}[1 - (1 + \lambda x^{-\delta})^{-\alpha}]^{\phi-1} \\ &\times \{1 - [1 - (1 + \lambda x^{-\delta})^{-\alpha}]^{\phi}\}^{\theta-1}, \quad (2.4) \end{aligned}$$

for $\alpha, \lambda, \delta, \phi, \theta > 0$, and $x > 0$, respectively. The quantile function of the EKD distribution is in closed form,

$$G^{-1}_{_{EKD}}(q) = x_q = \lambda^{\frac{1}{\delta}}\left\{[1 - (1 - q^{\frac{1}{\theta}})^{\frac{1}{\phi}}]^{-\frac{1}{\alpha}} - 1\right\}^{-\frac{1}{\delta}}. \quad (2.5)$$

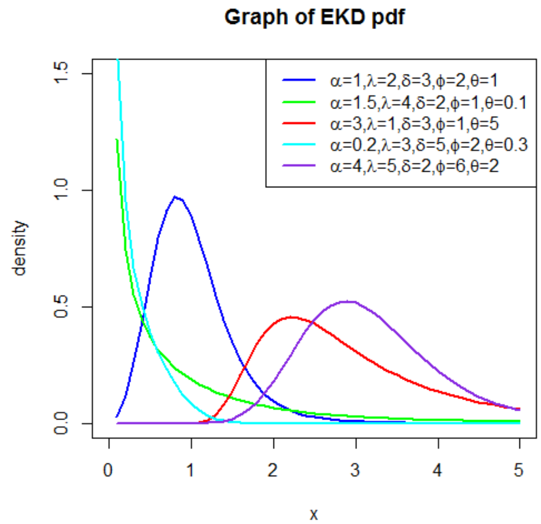

Figure 2.1: EKD Density Functions

Plots of the pdf for selected values of the model parameters are given in Figure 2.1. The plots indicate that the EKD pdf can be decreasing or right skewed.

2.2.3 Sub-models

Sub-models of the EKD distribution for selected values of the parameters are presented in this section.

① When $\theta = 1$, we obtain Kumaraswamy-Dagum distribution with cdf:

$$G(x; \alpha, \lambda, \delta, \phi) = 1 - [1 - (1 + \lambda x^{-\delta})^{-\alpha}]^{\phi},$$

for $\alpha, \lambda, \delta, \phi > 0$ and $x > 0$.

② When $\phi = 1$, $\phi = \theta = 1$ and $\alpha = \phi = 1$, we obtain Dagum distribution with cdfs:

$$G(x; \alpha, \lambda, \delta, \theta) = (1 + \lambda x^{-\delta})^{-\alpha\theta},$$

$$G(x;\alpha,\lambda,\delta) = (1+\lambda x^{-\delta})^{-\alpha},$$

and

$$G(x;\lambda,\delta,\theta) = (1+\lambda x^{-\delta})^{-\theta},$$

respectively.

③ When $\lambda = 1$, we obtain exponentiated Kumaraswamy-Burr III distribution with cdf:

$$G(x;\alpha,\delta,\phi,\theta) = \{1-[1-(1+x^{-\delta})^{-\alpha}]^{\phi}\}^{\theta},$$

for $\alpha,\delta,\phi,\theta > 0$ and $x > 0$.

④ When $\lambda = \theta = 1$, we obtain Kumaraswamy-Burr III distribution with cdf:

$$G(x;\alpha,\delta,\phi) = 1-[1-(1+x^{-\delta})^{-\alpha}]^{\phi},$$

for $\alpha,\delta,\phi > 0$ and $x > 0$.

⑤ When $\lambda = \phi = \theta = 1$, we obtain Burr III distribution with cdf:

$$G(x;\alpha,\delta) = (1+x^{-\delta})^{-\alpha},$$

for $\alpha,\delta > 0$ and $x > 0$.

⑥ When $\alpha = 1$, we obtain exponentiated Kumaraswamy-Fisk or Kumaraswamy Log-logistic distribution with cdf:

$$G(x;\lambda,\delta,\phi,\theta) = \{1-[1-(1+\lambda x^{-\delta})^{-1}]^{\phi}\}^{\theta},$$

for $\lambda,\delta,\phi,\theta > 0$ and $x > 0$.

⑦ When $\alpha = \theta = 1$, we obtain Kumaraswamy-Fisk or Kumaraswamy Log-logistic distribution with cdf:

$$G(x;\lambda,\delta,\phi) = 1-[1-(1+\lambda x^{-\delta})^{-1}]^{\phi},$$

for $\lambda,\delta,\phi > 0$ and $x > 0$.

⑧ When $\alpha = \phi = \theta = 1$, we obtain Fisk or Log-logistic distribution with cdf:

$$G(x; \lambda, \delta) = (1 + \lambda x^{-\delta})^{-1},$$

for $\lambda, \delta > 0$ and $x > 0$.

2.2.4 Series Expansion

We apply the series expansion

$$(1 - z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b - j) j!} z^j, \tag{2.6}$$

for $b > 0$ and $|z| < 1$, to obtain the series expansion of the EKD distribution.

By using equation (2.6),

$$g_{_{EKD}}(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega(i, j) x^{-\delta-1} (1 + \lambda x^{-\delta})^{-\alpha(j+1)-1}, \tag{2.7}$$

where $\omega(i, j) = \alpha\lambda\delta\phi\theta \frac{(-1)^{i+j} \Gamma(\theta)\Gamma(\phi i+\phi)}{\Gamma(\theta-i)\Gamma(\phi i+\phi-j) i! j!}$.

Note that in the Dagum$(\alpha, \delta, \lambda)$ distribution, α and δ are shape parameters, and λ is a scale parameter. In the Exponentiated-Kum(ψ, ϕ, θ) distribution, ψ is a skewness parameter, ϕ is a tail variation parameter, and the parameter θ characterizes the skewness, kurtosis, and tail of the distribution.

Consequently, for the EKD$(\alpha, \lambda, \delta, \phi, \theta)$ distribution, α is a shape and skewness parameter, δ is a shape parameter, λ is a scale parameter, ϕ is a tail variation parameter, and the parameter θ characterizes the skewness, kurtosis, and tail of the distribution.

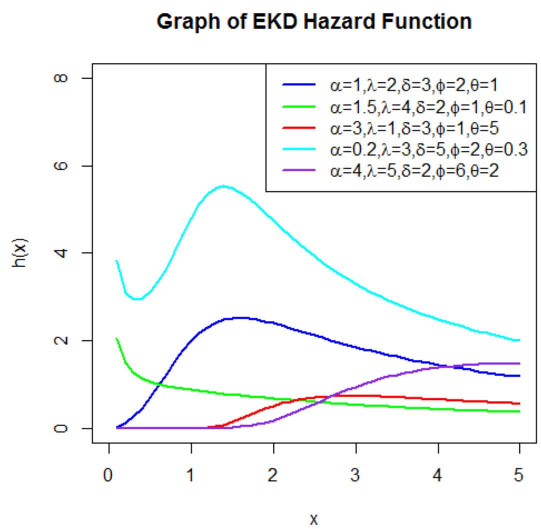

Figure 2.2: EKD Hazard Functions

2.2.5 Hazard and Reverse Hazard Functions

The hazard function of the EKD distribution is

$$\begin{aligned} h_{EKD}(x) &= \frac{g_{EKD}(x)}{1 - G_{EKD}(x)} \\ &= \alpha\lambda\delta\phi\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\alpha-1}[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi-1} \\ &\times \{1-[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi}\}^{\theta-1} \\ &\times \left(1-\{1-[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi}\}^{\theta}\right)^{-1}. \end{aligned} \tag{2.8}$$

Plots of the hazard function are presented in Figure 2.2. The plots show various shapes including monotonically decreasing, unimodal, bathtub, and upside down bathtub shapes for the selected values of the parameters.

The reverse hazard function of the EKD distribution is

$$\begin{aligned} \tau_{EKD}(x) &= \frac{g_{EKD}(x)}{G_{EKD}(x)} \\ &= \alpha\lambda\delta\phi\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\alpha-1}[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi-1} \\ &\times \{1-[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi}\}^{-1}. \end{aligned} \tag{2.9}$$

2.3 Moments, Moment Generating Function, Bonferroni and Lorenz Curves, Mean and Median Deviations, and Reliability

In this section, we present the moments, moment generating function, Bonferroni and Lorenz curves, mean and median deviations as well as the reliability of the EKD distribution. The moments of the sub-models can be readily obtained from the general results.

2.3.1 Moments and Moment Generating Function

Let $t = (1 + \lambda x^{-\delta})^{-1}$ in equation (2.7), then the s^{th} raw moment of the EKD distribution is given by

$$\begin{aligned} E(X^s) &= \int_0^\infty x^s \cdot g_{_{EKD}}(x)dx \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty \omega(i,j)\lambda^{\frac{s}{\delta}-1} \cdot \frac{1}{\delta} \cdot B\left(\alpha(j+1) + \frac{s}{\delta}, 1 - \frac{s}{\delta}\right) \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty \omega(i,j,s)B\left(\alpha(j+1) + \frac{s}{\delta}, 1 - \frac{s}{\delta}\right), \end{aligned} \tag{2.10}$$

where $\omega(i,j,s) = \alpha\phi\theta\lambda^{\frac{s}{\delta}} \frac{(-1)^{i+j}\Gamma(\theta)\Gamma(\phi i+\phi)}{\Gamma(\theta-i)\Gamma(\phi i+\phi-j)i!j!}$, and $s < \delta$.

The moment generating function of the EKD distribution is given by

$$M(t) = \sum_{r=0}^\infty \sum_{i=0}^\infty \sum_{j=0}^\infty \omega(i,j,r)\frac{t^r}{r!}B\left(\alpha(j+1) + \frac{r}{\delta}, 1 - \frac{r}{\delta}\right),$$

for $r < \delta$.

2.3.2 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves are widely used tool for analyzing and visualizing income inequality. Lorenz curve, $L(p)$ can be regarded as the proportion of total income volume accumulated by those units with income lower than or equal to the volume

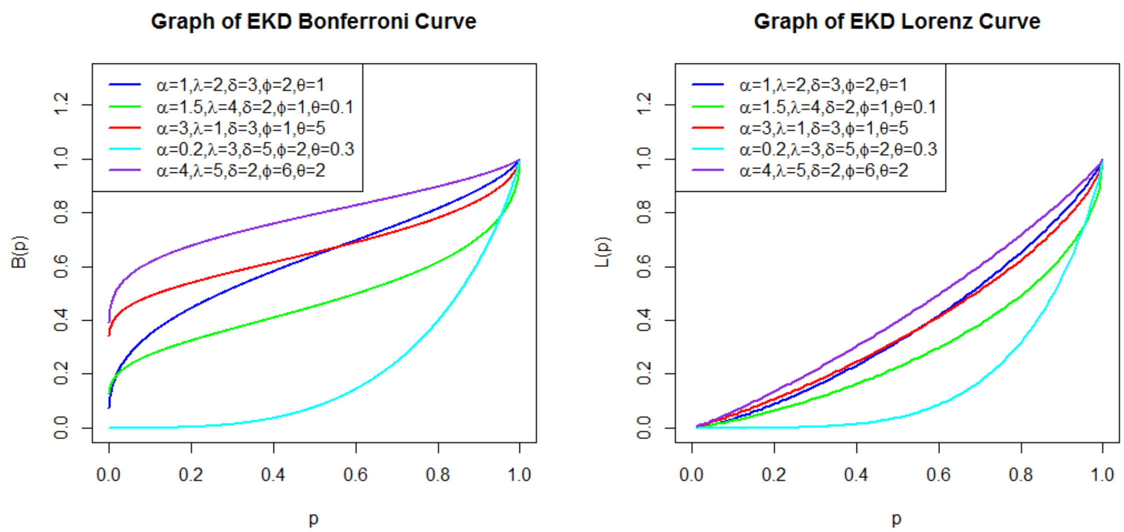

Figure 2.3: EKD Bonferroni and Lorenz Curves

x, and Bonferroni curve, $B(p)$ is the scaled conditional mean curve, that is, ratio of group mean income of the population. Plots of Bonferroni and Lorenz curves are given in Figure 2.3.

Let $I(a) = \int_0^a x \cdot g_{EKD}(x)dx$ and $\mu = E(X)$, then Bonferroni and Lorenz curves are given by

$$B(p) = \frac{I(q)}{p\mu} \quad \text{and} \quad L(p) = \frac{I(q)}{\mu},$$

respectively, for $0 \leq p \leq 1$, and $q = G_{EKD}^{-1}(p)$. The mean of the EKD distribution is obtained from equation (2.10) with $s = 1$, and the quantile function is given in equation (2.5). Consequently,

$$I(a) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega(i,j,1) B_{t(a)}\left(\alpha(j+1) + \frac{1}{\delta}, 1 - \frac{1}{\delta}\right), \tag{2.11}$$

for $\delta > 1$, where $t(a) = (1 + \lambda a^{-\delta})^{-1}$, and $B_{G(x)}(c,d) = \int_0^{G(x)} t^{c-1}(1-t)^{d-1}dt$ for $|G(x)| < 1$ is incomplete beta function.

2.3.3 Mean and Median Deviations

If X has the EKD distribution, we can derive the mean deviation about the mean $\mu = E(X)$ and the median deviation about the median M from

$$\delta_1 = \int_0^\infty |x-\mu|\, g_{EKD}(x)dx \quad \text{and} \quad \delta_2 = \int_0^\infty |x-M|\, g_{EKD}(x)dx,$$

respectively. The mean μ is obtained from equation (2.10) with $s = 1$, and the median M is given by equation (2.5) when $q = \frac{1}{2}$.

The measure δ_1 and δ_2 can be calculated by the following relationships:

$$\delta_1 = 2\mu\, G_{EKD}(\mu) - 2\mu + 2T(\mu) \quad \text{and} \quad \delta_2 = 2T(M) - \mu,$$

where $T(a) = \int_a^\infty x \cdot g_{EKD}(x)dx$ follows from equation (2.11), that is

$$T(a) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega(i,j,1)\left[B\left(\alpha(j+1)+\frac{1}{\delta}, 1-\frac{1}{\delta}\right) - B_{t(a)}\left(\alpha(j+1)+\frac{1}{\delta}, 1-\frac{1}{\delta}\right)\right].$$

2.3.4 Reliability

The reliability $R = P(X_1 > X_2)$ when X_1 and X_2 have independent $EKD(\alpha_1, \lambda_1, \delta_1, \phi_1, \theta_1)$ and $EKD(\alpha_2, \lambda_2, \delta_2, \phi_2, \theta_2)$ distributions is given by

$$\begin{aligned} R &= \int_0^\infty g_1(x)G_2(x)dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\zeta(i,j,k,l)\int_0^\infty x^{-\delta_1-1}(1+\lambda_1 x^{-\delta_1})^{-\alpha_1(j+1)-1}(1+\lambda_2 x^{-\delta_2})^{-\alpha_2 l}dx, \end{aligned}$$

where $\zeta(i,j,k,l) = \alpha_1\lambda_1\delta_1\phi_1\theta_1 \frac{(-1)^{i+j+k+l}\Gamma(\theta_1)\Gamma(\phi_1 i+\phi_1)\Gamma(\theta_2+1)\Gamma(\phi_2 k+1)}{\Gamma(\theta_1-i)\Gamma(\phi_1 i+\phi_1-j)\Gamma(\theta_2+1-k)\Gamma(\phi_2 k+1-l)i!j!k!l!}$.

If $\lambda = \lambda_1 = \lambda_2$ and $\delta = \delta_1 = \delta_2$, then reliability can be reduced to

$$R = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\zeta(i,j,k,l)}{\lambda\delta[\alpha_1(j+1)+\alpha_2 l]}.$$

2.4 Order Statistics and Entropy

In this section, the distribution of the k^{th} order statistic and Rényi entropy (Rényi [31]) for the EKD distribution are presented. The entropy of a random variable is a measure of variation of the uncertainty.

2.4.1 Order Statistics

The pdf of the k^{th} order statistics from a cdf $F(x)$ and associated pdf $f(x)$ is given by

$$\begin{aligned} f_{k:n}(x) &= \frac{f(x)}{B(k, n-k+1)} F^{k-1}(x)[1-F(x)]^{n-k} \\ &= k\binom{n}{k} f(x) F^{k-1}(x)[1-F(x)]^{n-k}. \end{aligned} \tag{2.12}$$

Using equation (2.6), the pdf of the k^{th} order statistic from EKD distribution is given by

$$g_{k:n}(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty} K(i,j,p,k) \cdot x^{-\delta-1}(1+\lambda x^{-\delta})^{-\alpha-\alpha p-1},$$

where $K(i,j,p,k) = \frac{(-1)^{i+j+p}\Gamma(n-k+1)\Gamma(\theta k+\theta i)\Gamma(\phi j+\phi)}{\Gamma(n-k+1-i)\Gamma(\theta k+\theta i-j)\Gamma(\phi j+\phi-p)i!j!p!} k\binom{n}{k}\alpha\lambda\delta\phi\theta.$

2.4.2 Entropy

Rényi entropy of a distribution with pdf $f(x)$ is defined as

$$I_R(\tau) = (1-\tau)^{-1}\log\left\{\int_{\mathbb{R}} f^{\tau}(x)dx\right\}, \tau > 0, \tau \neq 1.$$

Using equation (2.6), Rényi entropy of the EKD distribution is given by

$$\begin{aligned} I_R(\tau) &= (1-\tau)^{-1}\log\Bigg[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\Gamma(\theta\tau-\tau+1)\Gamma(\phi\tau-\tau+\phi i+1)}{\Gamma(\theta\tau-\tau+1-i)\Gamma(\phi\tau-\tau+\phi i+1-j)i!j!} \\ &\times \alpha^{\tau}\lambda^{-\frac{\tau}{\delta}+\frac{1}{\delta}}\delta^{\tau-1}\phi^{\tau}\theta^{\tau}B(\alpha\tau+\alpha j+\frac{1-\tau}{\delta}, \tau+\frac{\tau-1}{\delta})\Bigg]. \end{aligned}$$

for $\alpha\tau + \alpha j + \frac{1-\tau}{\delta} > 0$ and $\tau + \frac{\tau-1}{\delta} > 0$. Rényi entropy for the sub-models can be readily obtained.

2.5 Estimation of Model Parameters

In this section, we present estimation of the parameters of the EKD distribution via method of maximum likelihood estimation. The elements of the score function are presented. There are no closed form solutions to the nonlinear equations obtained by setting the elements of the score function to zero.

2.5.1 Maximum Likelihood Estimation

Let $\mathbf{x} = (x_1, \cdots, x_n)^T$ be a random sample of the EKD distribution with unknown parameter vector $\mathbf{\Theta} = (\alpha, \lambda, \delta, \phi, \theta)^T$.

The log-likelihood function for $\mathbf{\Theta}$ is

$$\begin{aligned} l(\mathbf{\Theta}) &= n(\ln\alpha + \ln\lambda + \ln\delta + \ln\phi + \ln\theta) - (\delta+1)\sum_{i=1}^{n}\ln x_i \\ &- (\alpha+1)\sum_{i=1}^{n}\ln(1+\lambda x_i^{-\delta}) + (\phi-1)\sum_{i=1}^{n}\ln[1-(1+\lambda x_i^{-\delta})^{-\alpha}] \\ &+ (\theta-1)\sum_{i=1}^{n}\ln\{1-[1-(1+\lambda x_i^{-\delta})^{-\alpha}]^{\phi}\}. \end{aligned} \tag{2.13}$$

The partial derivatives of $l(\mathbf{\Theta})$ with respect to the parameters are

$$\begin{aligned} \frac{\partial l}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=1}^{n}\ln(1+\lambda x_i^{-\delta}) + (\phi-1)\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-\alpha}\ln(1+\lambda x_i^{-\delta})}{1-(1+\lambda x_i^{-\delta})^{-\alpha}} \\ &- (\theta-1)\phi\sum_{i=1}^{n}\frac{[1-(1+\lambda x_i^{-\delta})^{-\alpha}]^{\phi-1}(1+\lambda x_i^{-\delta})^{-\alpha}\ln(1+\lambda x_i^{-\delta})}{1-[1-(1+\lambda x_i^{-\delta})^{-\alpha}]^{\phi}}, \end{aligned}$$

$$\begin{aligned}\frac{\partial l}{\partial \lambda} &= \frac{n}{\lambda} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i^{-\delta}}{1+\lambda x_i^{-\delta}} + (\phi-1)\alpha\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-\alpha-1}x_i^{-\delta}}{1-(1+\lambda x_i^{-\delta})^{-\alpha}}\\ &- (\theta-1)\phi\alpha\sum_{i=1}^{n}\frac{[1-(1+\lambda x_i^{-\delta})^{-\alpha}]^{\phi-1}(1+\lambda x_i^{-\delta})^{-\alpha-1}x_i^{-\delta}}{1-[1-(1+\lambda x_i^{-\delta})^{-\alpha}]^{\phi}},\end{aligned}$$

$$\begin{aligned}\frac{\partial l}{\partial \delta} &= \frac{n}{\delta} - \sum_{i=1}^{n}\ln x_i + (\alpha+1)\lambda\sum_{i=1}^{n}\frac{x_i^{-\delta}\ln x_i}{1+\lambda x_i^{-\delta}}\\ &- (\phi-1)\alpha\lambda\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-\alpha-1}x_i^{-\delta}\ln x_i}{1-(1+\lambda x_i^{-\delta})^{-\alpha}}\\ &+ (\theta-1)\phi\alpha\lambda\sum_{i=1}^{n}\frac{[1-(1+\lambda x_i^{-\delta})^{-\alpha}]^{\phi-1}(1+\lambda x_i^{-\delta})^{-\alpha-1}x_i^{-\delta}\ln x_i}{1-[1-(1+\lambda x_i^{-\delta})^{-\alpha}]^{\phi}},\end{aligned}$$

$$\frac{\partial l}{\partial \phi} = \frac{n}{\phi} + \sum_{i=1}^{n}\ln[1-(1+\lambda x_i^{-\delta})^{-\alpha}] - (\theta-1)\sum_{i=1}^{n}\frac{[1-(1+\lambda x_i^{-\delta})^{-\alpha}]^{\phi}\ln[1-(1+\lambda x_i^{-\delta})^{-\alpha}]}{1-[1-(1+\lambda x_i^{-\delta})^{-\alpha}]^{\phi}},$$

and

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln\{1-[1-(1+\lambda x_i^{-\delta})^{-\alpha}]^{\phi}\},$$

respectively. The MLE of the parameters $\alpha, \lambda, \delta, \phi$, and θ, say $\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\phi}$, and $\hat{\theta}$, must be obtained by numerical methods.

2.5.2 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the EKD distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\mathbf{\Theta}} = (\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\phi}, \hat{\theta})$ be the maximum likelihood estimate of $\mathbf{\Theta} = (\alpha, \lambda, \delta, \phi, \theta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Theta}}-\mathbf{\Theta}) \xrightarrow{d} N_5(\underline{0}, I^{-1}(\mathbf{\Theta}))$, where $I(\mathbf{\Theta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Theta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$, that is $J(\hat{\mathbf{\Theta}})$. The multivariate normal distribution $N_5(\underline{0}, J(\hat{\mathbf{\Theta}})^{-1})$, where the mean vector $\underline{0} = (0,0,0,0,0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

The approximate individual $100(1-\eta)\%$ two-sided confidence intervals for α, λ, δ, ϕ and θ are given by:

$$\widehat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\alpha\alpha}(\widehat{\mathbf{\Theta}})}, \quad \widehat{\lambda} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\lambda\lambda}(\widehat{\mathbf{\Theta}})}, \quad \widehat{\delta} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\delta\delta}(\widehat{\mathbf{\Theta}})}$$

$$\widehat{\phi} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\phi\phi}(\widehat{\mathbf{\Theta}})}, \quad \widehat{\theta} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\theta\theta}(\widehat{\mathbf{\Theta}})}$$

respectively, where $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the EKD distribution with its sub-models for a given data set. For example, to test $\theta = 1$, the LR statistic is

$$\omega = 2[\ln(L(\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\phi}, \hat{\theta})) - \ln(L(\tilde{\alpha}, \tilde{\lambda}, \tilde{\delta}, \tilde{\phi}, 1))],$$

where $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\delta}$, $\hat{\phi}$ and $\hat{\theta}$ are the unrestricted estimates, and $\tilde{\alpha}$, $\tilde{\lambda}$, $\tilde{\delta}$ and $\tilde{\phi}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_d$, where χ^2_d denote the upper $100d\%$ point of the χ^2 distribution with 1 degree of freedom.

2.6 Simulation Study

In this section, we examine the performance of the EKD distribution by conducting various simulations for different sizes (n=200, 400, 800, 1200) via the subroutine NLP in SAS. We simulate 2000 samples for the true parameter values $I : \alpha = 2, \lambda = 1, \delta = 3, \phi = 2, \theta = 2$ and $II : \alpha = 1, \lambda = 1, \delta = 1, \phi = 1, \theta = 1$. Table 2.1 lists the means

MLEs of the five model parameters along with the respective root mean squared errors (RMSE). From the results, we can verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

2.7 Applications

In this section, applications based on real data, as well as comparison of the EKD distribution with its sub-models are presented. We provide examples to illustrate the flexibility of the EKD distribution in contrast to other models, including the exponentiated Kumaraswamy-Weibull (EKW), and beta-Kumaraswamy-Weibull (BKW) distributions for data modeling. The pdfs of EKW and BKW distributions are

$$\begin{aligned} f_{EKW}(x) &= \theta abc\lambda^c x^{c-1} e^{-(\lambda x)^c}\left[1-e^{-(\lambda x)^c}\right]^{a-1}\left\{1-\left[1-e^{-(\lambda x)^c}\right]^{a}\right\}^{b-1} \\ &\times \left[1-\left\{1-\left[1-e^{-(\lambda x)^c}\right]^{a}\right\}^{b}\right]^{\theta-1}, \end{aligned}$$

and

$$\begin{aligned} f_{BKW}(x) &= \frac{1}{B(a,b)}\alpha\beta c\lambda^c x^{c-1} e^{-(\lambda x)^c}\left[1-e^{-(\lambda x)^c}\right]^{\alpha-1} \\ &\times \left\{1-\left[1-e^{-(\lambda x)^c}\right]^{\alpha}\right\}^{\beta b-1}\left[1-\left\{1-\left[1-e^{-(\lambda x)^c}\right]^{\alpha}\right\}^{\beta}\right]^{a-1}, \end{aligned}$$

respectively.

The first data set consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes (Proschan [28]). The data is presented in Table 2.3. The second data set consists of the salaries of 818 professional baseball players for the year 2009 (USA TODAY). The third data set represents the poverty rate of 533 districts with more than 15,000 students in 2009 (Digest of Education Statistics "`http://nces.ed.gov/programs/digest/d11/`

Table 2.1: EKD Monte Carlo Simulation Results

		I		II	
n	Parameter	Mean	RMSE	Mean	RMSE
200	α	4.41621	3.979304324	1.7899006	1.992043574
	λ	1.3580866	2.642335804	1.4287071	1.528578784
	δ	3.1167852	2.601663026	1.0337146	0.5898521
	ϕ	5.7270324	6.535452517	2.4702434	3.712081559
	θ	4.5560563	4.306946865	2.8884959	3.689669972
400	α	3.5972873	3.071770841	1.5456974	1.513782811
	λ	1.1196079	0.900800533	1.1382897	0.732002869
	δ	2.9333424	1.821450521	1.0064105	0.377302664
	ϕ	4.6989703	5.277876069	1.5488732	1.872088246
	θ	4.1188983	3.616692978	2.4213684	2.969367761
800	α	3.1040595	2.417025941	1.4359333	1.278449373
	λ	1.0626388	0.609066006	1.0432761	0.346996974
	δ	2.8960167	1.36814261	1.0017278	0.250650155
	ϕ	3.7437056	3.919777583	1.176675	0.766203302
	θ	3.4890255	2.748229594	1.9733522	2.197844717
1200	α	2.8399564	2.058703427	1.3884174	1.169251427
	λ	1.0429655	0.501712467	1.021836	0.258884917
	δ	2.9152476	1.133666485	1.0014919	0.193825437
	ϕ	3.1751818	3.043071803	1.083574	0.392293513
	θ	3.164176	2.346236284	1.731924	1.788360814

Table 2.2: Descriptive Statistics of Application Data Sets

Data	Mean	Median	Mode	SD	Variance	Skewness	Kurtosis	Min.	Max.
I	92.07	54.00	14.00	107.92	11646	2.16	5.19	1.0	603.0
II	3.26	1.15	0.40	4.36	19.05	2.10	5.13	0.4	33.0
III	17.71	16.80	9.30	8.80	77.38	0.80	0.73	2.7	53.6

`tables/dt11_096.asp`"). These data sets are modeled by the EKD distribution and compared with the corresponding sub-models, the Kumaraswamy-Dagum and Dagum distributions, and as well as EKW, BKW distributions. Table 2.2 gives a descriptive summary of each sample. The air conditioning system sample has far more variability and the baseball player salary sample has the smallest variability.

The maximum likelihood estimates (MLEs) of the parameters are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2 Log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\mathbf{\Theta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters for the EKD distribution and its sub-distributions are tabulated.

Fitted densities plots, probability plots (Chambers et al [1]) and plots of the empirical and estimated survival functions are presented in Figure 2.4, Figure 2.5 and Figure 2.6. For the EKD distribution, we plotted for example,

$$G(x_{(j)}) = \{1 - [1 - (1 + \hat{\lambda} x_{(j)}^{-\hat{\delta}})^{-\hat{\alpha}}]^{\hat{\phi}}\}^{\hat{\theta}}$$

against $\frac{j-0.375}{n+0.25}$, $j = 1, 2, ..., n$, where $x_{(j)}$ are the ordered values of the observed data.

Table 2.3: Air Conditioning System Data

194	413	90	74	55	23	97	50	359	50	130	487	57
102	15	14	10	57	320	261	51	44	9	254	493	33
18	209	41	58	60	48	56	87	11	102	12	5	14
14	29	37	186	29	104	7	4	72	270	283	7	61
100	61	502	220	120	141	22	603	35	98	54	100	11
181	65	49	12	239	14	18	39	3	12	5	32	9
438	43	134	184	20	386	182	71	80	188	230	152	5
36	79	59	33	246	1	79	3	27	201	84	27	156
21	16	88	130	14	118	44	15	42	106	46	230	26
59	153	104	20	206	5	66	34	29	26	35	5	82
31	118	326	12	54	36	34	18	25	120	31	22	18
216	139	67	310	3	46	210	57	76	14	111	97	62
39	30	7	44	11	63	23	22	23	14	18	13	34
16	18	130	90	163	208	1	24	70	16	101	52	208
95	62	11	191	14	71							

A measure of closeness of the plot to the diagonal line given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[G(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2$$

was calculated for each plot. The plot with the smallest SS corresponds to the model with points that are closer to the diagonal line.

For the air conditioning system data, initial values $\alpha = 1, \lambda = 2, \delta = 0.6, \phi = 3, \theta = 1$ are used in SAS code for EKD model. The LR statistics for the test of the hypothesis $H_0 : KD$ against $H_a : EKD$ and $H_0 : D$ against $H_a : EKD$ are 1.9 (p-value= 0.17) and 13.4 (p-value= 0.0012). There is no significant difference

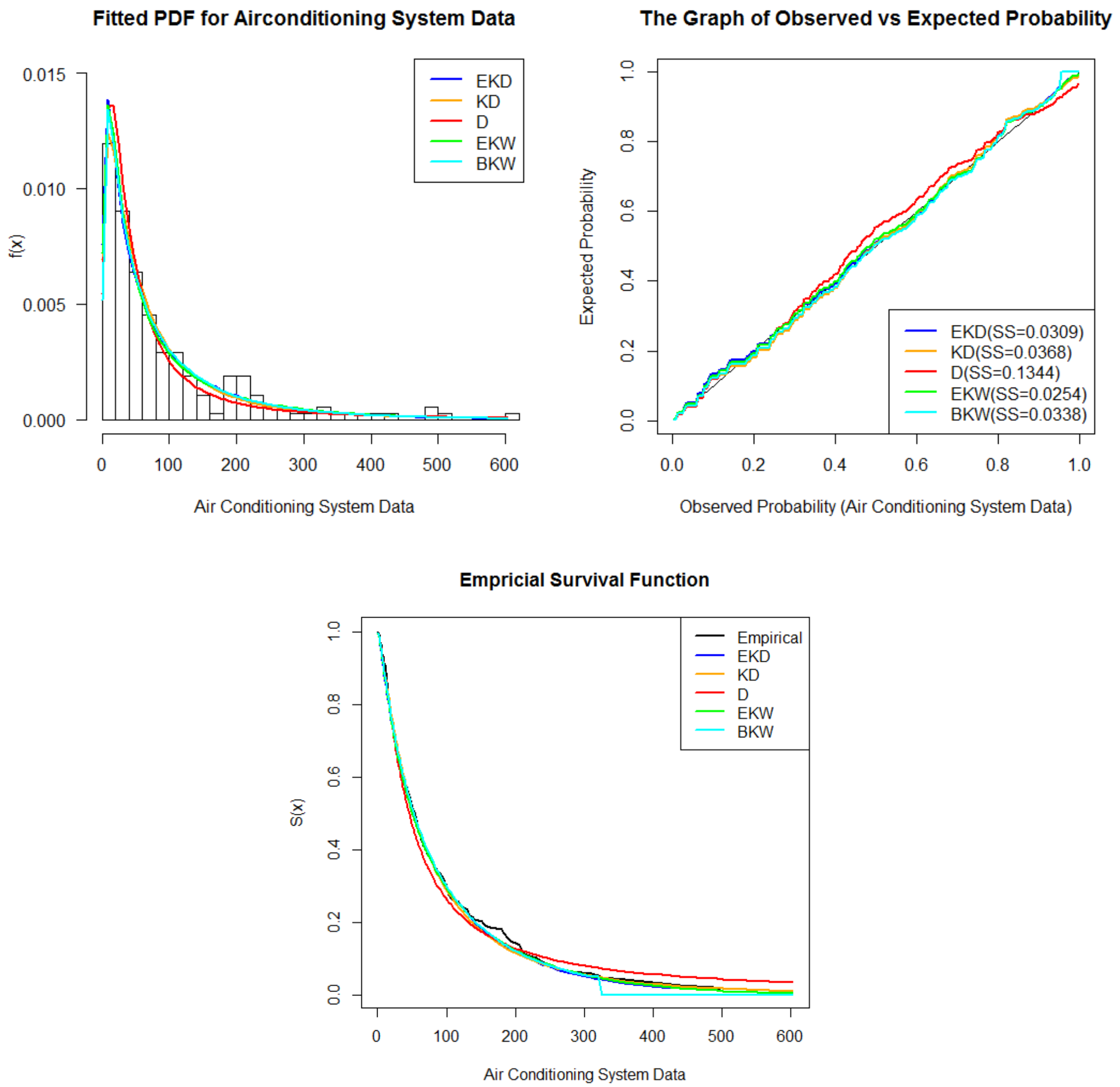

Figure 2.4: EKD Fitted Densities, Observed Probabilities and Empirical Survival Curves for Air Conditioning System Data

Table 2.4: EKD Estimation for Air Conditioning System Data

	Estimates						Statistics				
Distribution	α	λ	δ	ϕ	θ		-2 Log Likelihood	AIC	AICC	BIC	SS
EKD	20.6164	4.7323	0.6192	18.1616	0.1657		2065.0	2075.0	2075.3	2091.2	0.0309
	(1.2347)	(0.4174)	(0.0459)	(5.8028)	(0.0089)						
KD	5.0354	4.3846	0.3762	21.7047	1		2066.9	2074.9	2075.2	2087.9	0.0368
	(2.1177)	(3.0727)	(0.1253)	(27.9167)	-						
D	1.2390	94.1526	1.2626	1	1		2078.4	2084.4	2084.5	2094.1	0.1344
	(0.1749)	(33.7549)	(0.0663)	-	-						
	a	b	c	λ	θ						
EKW	3.7234	0.1219	1.0595	0.0495	0.3784		2063.7	2073.7	2074.0	2089.8	0.0254
	(0.8783)	(0.0183)	(0.1448)	(0.0224)	(0.1136)						
	a	b	α	β	c	λ					
BKW	1.4342	0.0830	2.0054	1.9100	0.7412	0.1809	2064.6	2076.6	2077.1	2096.1	0.0338
	(1.2507)	(0.0875)	(1.6573)	(1.9807)	(0.0343)	(0.0388)					

between EKD and KD. The KD distribution gives smaller SS value than Dagum distribution and slightly bigger than EKD. For the non-nested models, the values of AIC and AICC for KD and EKD models are very close, however the BIC value for KD distribution is slightly smaller than the corresponding value for the EKD distribution. We conclude that KD model compares favorably with the EKD distribution and thus provides a good fit for the air conditioning system data.

For the baseball player salary data set, initial values for EKD model in SAS code are $\alpha = 70, \lambda = 0.01, \delta = 1.026, \phi = 0.1, \theta = 1$. The EKD distribution is a better fit than KD and Dagum distributions for this data, as well as the other distributions. The values of the statistics AIC, AICC and BIC for KD distribution are smaller compared to the non-nested distributions. The LR statistics for the test of the hypotheses $H_0 : KD$ against $H_a : EKD$ and $H_0 : D$ against $H_a : EKD$ are 93.1 (p-value< 0.0001) and 361.5 (p-value< 0.0001). Consequently, we reject the null hypothesis in favor of the EKD distribution and conclude that the EKD distribution is significantly better than the KD and Dagum distributions based on the LR statistic. The value of AIC,

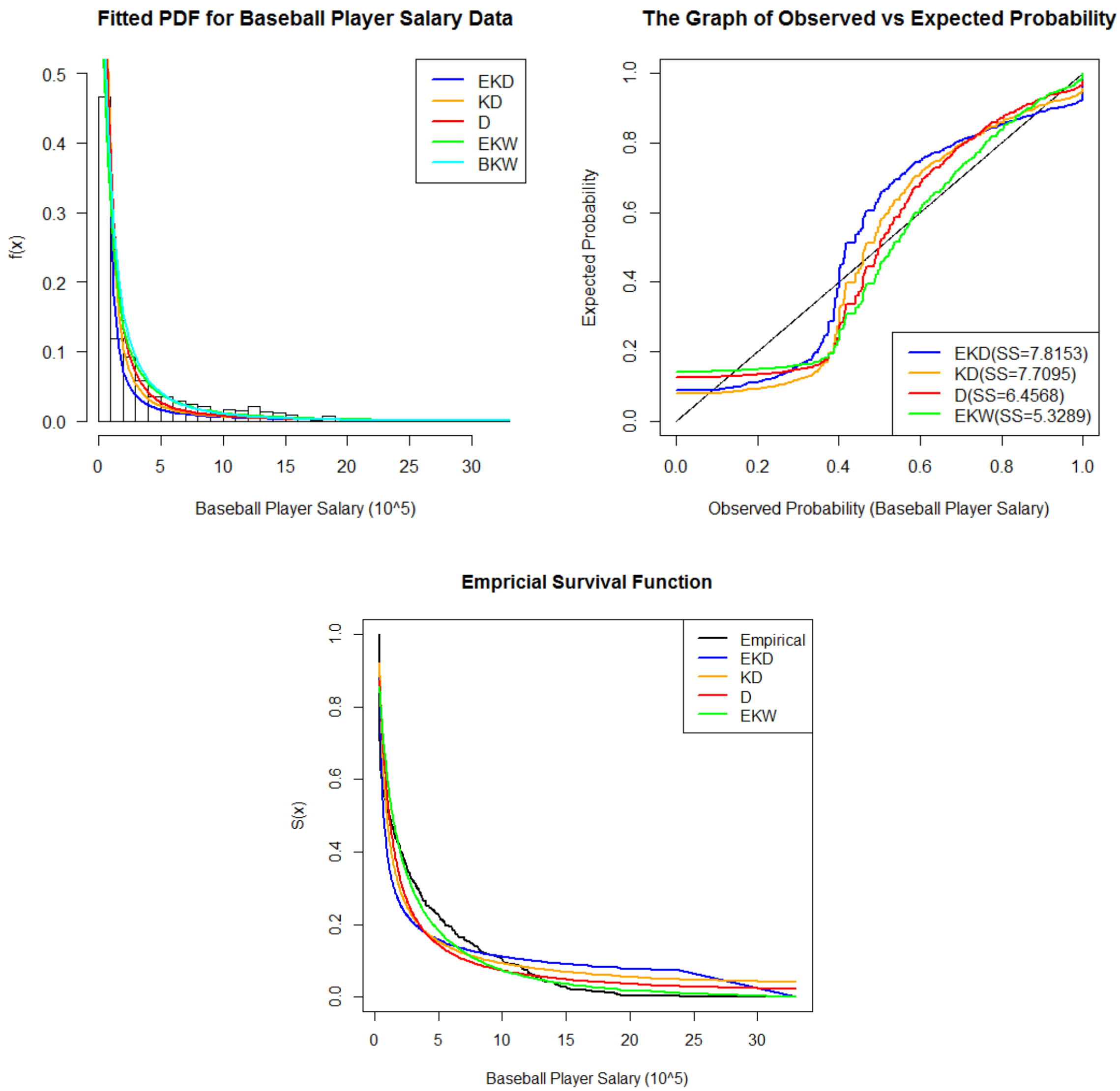

Figure 2.5: EKD Fitted Densities, Observed Probabilities and Empirical Survival Curves for Baseball Player Salary Data

Table 2.5: EKD Estimation for Baseball Player Salary Data

	Estimates						Statistics				
Distribution	α	λ	δ	ϕ	θ		-2 Log Likelihood	AIC	AICC	BIC	SS
EKD	69.1586	0.000043	7.6321	0.0591	0.4075		2864.1	2874.1	2874.2	2897.7	7.8153
	(0.000036)	(0.0000058)	(0.0557)	(0.0044)	(0.0327)						
KD	69.0839	0.000011	7.2375	0.0996	1		2957.2	2965.2	2965.2	2984.0	7.7095
	(0.000061)	(0.00000133)	(0.037)	(0.0036)	-						
D	70.0780	0.0116	1.0312	1	1		3225.6	3231.6	3231.6	3245.7	6.4568
	(34.4988)	(0.0058)	(0.0301)	-	-						
	a	b	c	λ	θ						
EKW	15.0514	0.1368	0.6376	8.8903	0.5419		3209.8	3219.8	3219.9	3243.3	5.3289
	(2.0692)	(0.0266)	(0.0756)	(4.9198)	(0.2098)						
	a	b	α	β	c	λ					
BKW	24.0047	0.03783	14.4799	4.6029	0.5168	32.1184	3088.4	3100.4	3100.5	3128.7	18.0516
	(0.6879)	(0.0039)	(0.2069)	(0.4549)	(0.006)	(2.4559)					

AICC and BIC statistics are lower for the EKD distribution when compared to those for the EKW and BKW distributions.

For poverty rate data, initial values for EKD model are $\alpha = 73, \lambda = 0.1, \delta = 0.15, \phi = 60, \theta = 0.33$. The LR statistic for the test of the hypotheses $H_0 : KD$ against $H_a : EKD$ and $H_0 : D$ against $H_a : EKD$ are 8.2 (p-value= 0.0042) and 81.1 (p-value < 0.0001), respectively. The values of AIC, AICC and BIC statistics shows EKD distributions is a better model and the SS value of the EKD model is comparatively smaller than the corresponding values for the KD and D distributions. Consequently, we conclude that EKD distribution is the best fit for the poverty rate data.

2.8 Concluding Remarks

We have proposed and presented results on a new class of distributions called the EKD distribution. This class of distributions has applications in income and lifetime data analysis. Properties of this class of distributions including the series expansion

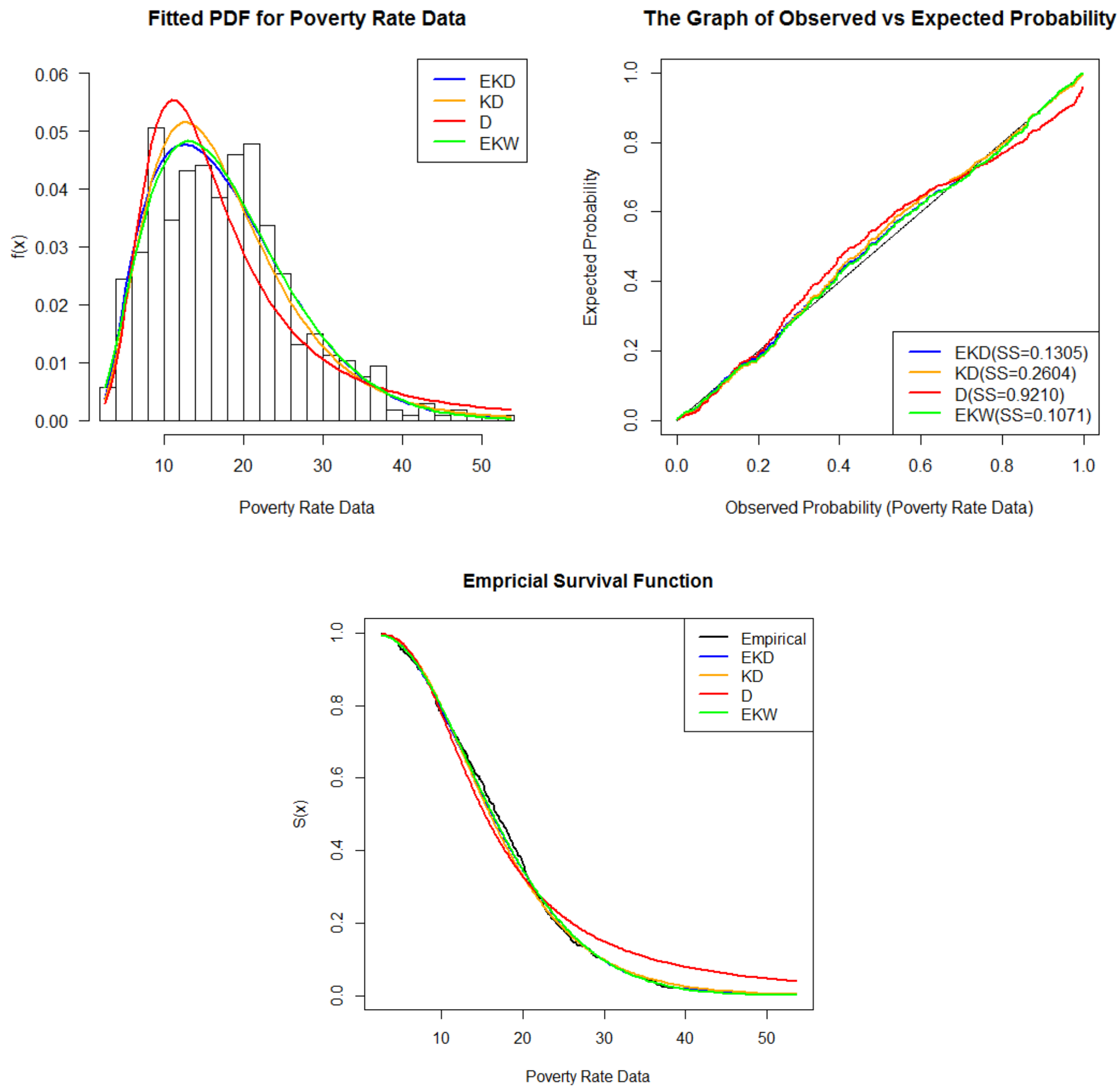

Figure 2.6: EKD Fitted Densities, Observed Probabilities and Empirical Survival Curves for Poverty Rate Data

Table 2.6: EKD Estimation for Poverty Rate Data

	Estimates						Statistics				
Distribution	α	λ	δ	ϕ	θ		-2 Log Likelihood	AIC	AICC	BIC	SS
EKD	75.5803	0.851500	0.8183	60.9069	0.3091		3750.7	3760.7	3760.8	3782.1	0.1305
	(11.1276)	(0.32)	(0.0714)	(29.1324)	(0.02229)						
KD	60.8898	0.304000	0.4666	90.2889	1		3758.9	3766.9	3767.0	3784.0	0.2604
	(17.5714)	(0.0963)	(0.0555)	(54.8283)	-						
D	1.7954	350.0100	2.4175	1	1		3831.8	3837.8	3837.9	3850.7	0.9210
	(0.2034)	(105.94)	(0.0784)	-	-						
	a	b	c	λ	θ						
EKW	0.1013	2.2289	2.741	0.02545	20.0336		3752.8	3762.8	3762.9	3784.2	0.1071
	(0.0944)	(1.8026)	(2.2276)	(0.0199)	(30.0233)						
	a	b	α	β	c	λ					
BKW	0.9985	1.0006	1.9999	0.03989	2.0006	0.1141	4727.5	4739.5	4739.7	4765.2	80.9942
	(0.0069)	(0.0431)	(0.0584)	(0.0017)	(0.2564)	(0.0075)					

of pdf, cdf, moments, hazard function, reverse hazard function, income inequality measures such as Lorenz and Bonferroni curves are derived. Rényi entropy, order statistics, reliability, mean and median deviations are presented. Estimation of the parameters of the models and applications are also given.

CHAPTER 3

MAXIMUM LIKELIHOOD ESTIMATION IN THE EXPONENTIATED KUMARASWAMY-DAGUM DISTRIBUTION WITH CENSORED SAMPLES

3.1 Introduction

The main motivation for the development of this distribution is the modeling of size distribution of personal income and lifetime data for censored data with a diverse model that takes into consideration not only shape and scale but also skewness, kurtosis and tail variation. Also, motivated by the various applications of Dagum distributions in several areas including exponential tilting (weighting) in finance and actuarial sciences, as well as economics, where Dagum distribution plays an important role in size distribution of personal income, we construct, develop and show that this new class of generalized Dagum-type distribution called the exponentiated-Kumaraswamy-Dagum distribution is applicable to real lifetime censored data in order to demonstrate the competitiveness, as well as usefulness of the proposed distribution in reliability and survival analysis problems.

In this chapter, we present maximum likelihood estimation as well as comparisons with other parametric models in the exponentiated Kumaraswamy-Dagum distribution under Type I right censored and Type II doubly censored schemes. This chapter is organized as follows. Maximum likelihood estimates of the model parameters under Type I right censored and Type II doubly censored plans are presented in section 3.2. Applications, case studies and comparisons with the exponentiated Kumaraswamy-Weibull distribution are given in section 3.3, followed by concluding remarks in section 3.4.

3.2 Maximum Likelihood Estimation

Different censoring mechanisms lead to different likelihood functions. In the following sections, we construct log-likelihood functions of the EKD distribution to deal with type I right and type II doubly censored observations.

Although the maximum likelihood estimates are not available in closed form, they can be evaluated with the help of numerical techniques. The difficulties in dealing with the EKD distribution due to its complicated mathematical tractability are easily overcome by using iterative methods which do not require high-computational efforts even in the presence of censoring.

3.2.1 Type I Right Censoring

This is the most common form of incomplete data often encountered in survival analysis. Type I right censored data arises when the study is conducted over a specified time period that can terminate before all the units have failed. Each individual has a fixed censoring time C_i, which would be the time between the date of entry and the end of study, so that the complete failure time of an individual will be known only if it is less than or equal to the censoring time $T_i \leq C_i$; otherwise, only a lower bound of the individual lifetime is available $T_i > C_i$. The data for this design are conveniently indicated by pairs of random variables (T_i, ϵ_i), $i = 1, \ldots, n$. Consider a sample size n of independent positive random variables $T_1, \ldots, T_n$ such that T_i is associated with an indicator variable $\epsilon_i = 0$ if T_i is a censoring time. Let $\mathbf{\Theta} = (\alpha, \lambda, \delta, \phi, \theta)^T$, then the likelihood function, $L(\mathbf{\Theta})$, of a type I right censored sample $(t_1, \epsilon_1), ..., (t_n, \epsilon_n)$ from EKD distribution with pdf $g_{_{EKD}}(\cdot)$ and survival function $S_{_{EKD}}(\cdot)$ can be written as

$$L(\mathbf{\Theta}) = \prod_{i=1}^{n} g_{_{EKD}}(t_i)^{\epsilon_i} S_{_{EKD}}(t_i)^{1-\epsilon_i},$$

where $S_{_{EKD}}(t_i) = 1 - G_{_{EKD}}(t_i)$. The log-likelihood function, $l(\mathbf{\Theta})$, based on data, from Equation (2.3) and (2.4) is

$$\begin{aligned} l(\mathbf{\Theta}) &= \sum_{i=1}^{n} \epsilon_i \Big\{ \ln\alpha + \ln\lambda + \ln\delta + \ln\phi + \ln\theta - (\delta+1)\ln t_i \\ &- (\alpha+1)\ln(1+\lambda t_i^{-\delta}) + (\phi-1)\ln[1-(1+\lambda t_i^{-\delta})^{-\alpha}] \\ &+ (\theta-1)\ln\{1-[1-(1+\lambda t_i^{-\delta})^{-\alpha}]^{\phi}\} \Big\} \\ &+ \sum_{i=1}^{n} (1-\epsilon_i)\ln\Big\{1-\{1-[1-(1+\lambda t_i^{-\delta})^{-\alpha}]^{\phi}\}^{\theta}\Big\}. \end{aligned} \tag{3.1}$$

The MLEs $\hat{\mathbf{\Theta}} = (\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\phi}, \hat{\theta})$ are obtained from the numerical maximization of Equation (3.1). Let $\mathbf{\Theta} = (\alpha, \lambda, \delta, \phi, \theta)^T$ be the parameter vector and $\hat{\mathbf{\Theta}} = (\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\phi}, \hat{\theta})$ be the maximum likelihood estimate of $\mathbf{\Theta} = (\alpha, \beta, \theta, \lambda, \delta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary [11], we have: $\sqrt{n}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) \xrightarrow{d} N_5(\underline{0}, I^{-1}(\mathbf{\Theta}))$, where $I(\mathbf{\Theta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Theta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$, that is $J(\hat{\mathbf{\Theta}})$. The multivariate normal distribution $N_5(\underline{0}, J(\hat{\mathbf{\Theta}})^{-1})$, where the mean vector $\underline{0} = (0,0,0,0,0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

3.2.2 Type II Doubly Censoring

Type II doubly censored data is used to indicate that, in an ordered sample of size n, a known number of observations is missing at both ends, while in type I censoring, the number of censored observations is a random variable and the time of study is fixed. The data consists of the remaining ordered observations $t_{r+1}, \ldots, t_m$ when the r smallest observations and the $n-m$ largest observations are out of a sample of size n from the EKD distribution. The likelihood function, $L(\mathbf{\Theta})$, of the type II doubly censored sample $t_{(r+1)}, ..., t_{(m)}$ from EKD distribution with pdf $g_{_{EKD}}(\cdot)$, cdf $G_{_{EKD}}(\cdot)$ and survival function $S_{_{EKD}}(\cdot)$ is given by

$$L(\mathbf{\Theta}) = \frac{n!}{r!(n-m)!}\{G_{_{EKD}}(t_{(r+1)})\}^r\{S_{_{EKD}}(t_{(m)})\}^{n-m}\prod_{i=r+1}^{m} g_{_{EKD}}(t_{(i)}).$$

The log-likelihood function, $l(\mathbf{\Theta})$, for a type II doubly censored sample $t_{(r+1)}, ..., t_{(m)}$ from EKD distribution is given by

$$\begin{aligned} l(\mathbf{\Theta}) &= \ln\left(\frac{n!}{r!(n-m)!}\right) + r\theta\ln\{1-[1-(1+\lambda t_{(r+1)}^{-\delta})^{-\alpha}]^{\phi}\} \\ &+ (n-m)\ln\left(1-\{1-[1-(1+\lambda t_{(m)}^{-\delta})^{-\alpha}]^{\phi}\}^{\theta}\right) \\ &+ \sum_{i=r+1}^{m}\Bigg(\ln\alpha+\ln\lambda+\ln\delta+\ln\phi+\ln\theta-(\delta+1)\ln t_{(i)} \\ &\quad -(\alpha+1)\ln(1+\lambda t_{(i)}^{-\delta})+(\phi-1)\ln[1-(1+\lambda t_{(i)}^{-\delta})^{-\alpha}] \\ &\quad +(\theta-1)\ln\{1-[1-(1+\lambda t_{(i)}^{-\delta})^{-\alpha}]^{\phi}\}\Bigg). \end{aligned} \tag{3.2}$$

As in the type I right censoring scheme, the MLEs $\hat{\mathbf{\Theta}} = (\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\phi}, \hat{\theta})$ are only obtained by numerical methods.

3.3 Applications

In this section, we give some applications to real life data. Maximum likelihood estimates of the model parameters under type I right and type II doubly censored data are obtained and comparisons with the exponentiated Kumaraswamy-Weibull distribution and its sub-models, which are widely used in reliability and survival data analysis are presented. We compared EKD and its sub-models, as well as exponentiated Kumaraswamy-Weibull (EKW) distribution, with the aid of the statistics: -2 Log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\mathbf{\Theta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters in the model. The EKW pdf is given by

$$\begin{aligned} f_{EKW}(x) &= \theta abc\lambda^c x^{c-1} e^{-(\lambda x)^c}\left[1-e^{-(\lambda x)^c}\right]^{a-1}\left\{1-\left[1-e^{-(\lambda x)^c}\right]^a\right\}^{b-1} \\ &\times \left[1-\left\{1-\left[1-e^{-(\lambda x)^c}\right]^a\right\}^b\right]^{\theta-1}, \end{aligned}$$

Probability plots (Chambers et al. [1]) are also presented. For the probability plot, we plotted $G_{EKD}(x_{(j)};\hat{\alpha},\hat{\lambda},\hat{\delta},\hat{\phi},\hat{\theta})$ against $\dfrac{j-0.375}{n+0.25}$, $j=1,\,2,\,\cdots,\,n$, where $x_{(j)}$ are the ordered values of the observed data. We also computed a measure of closeness of each plot to the diagonal line. This measure of closeness of the plot to the diagonal line is given by the sum of squares

$$SS=\sum_{j=1}^{n}\left[G(x_{(j)})-\left(\frac{j-0.375}{n+0.25}\right)\right]^2,$$

where the censored data will be deleted for calculating in type I right censored data.

3.3.1 Case I: Remission Times of Cancer Patients

For the first example, we consider the data set on remission times (in months) for 137 cancer patients [21]. Estimates of the parameters for type I right censored models, AIC, AICC, BIC, and SS are given in Table 3.1. In the plot comparing the survival functions of the EKD, KD, D and EKW distributions with the Kaplan-Meier curve, we see that the EKD distribution is preferred, while the other models tend to over or underestimate, mostly overestimate the empirical curve.

Plots of the fitted densities and the histogram, hazard functions, empirical survival functions, and observed probability vs predicted probability for the remission times data are given in Figure 3.1.

The LR test statistic of the hypothesis $H_0 : KD$ against $H_a : EKD$ and $H_0 : D$ against $H_a : EKD$ are 107.9 (p-value <0.0001) and 20.4 (p-value <0.0001). Also,

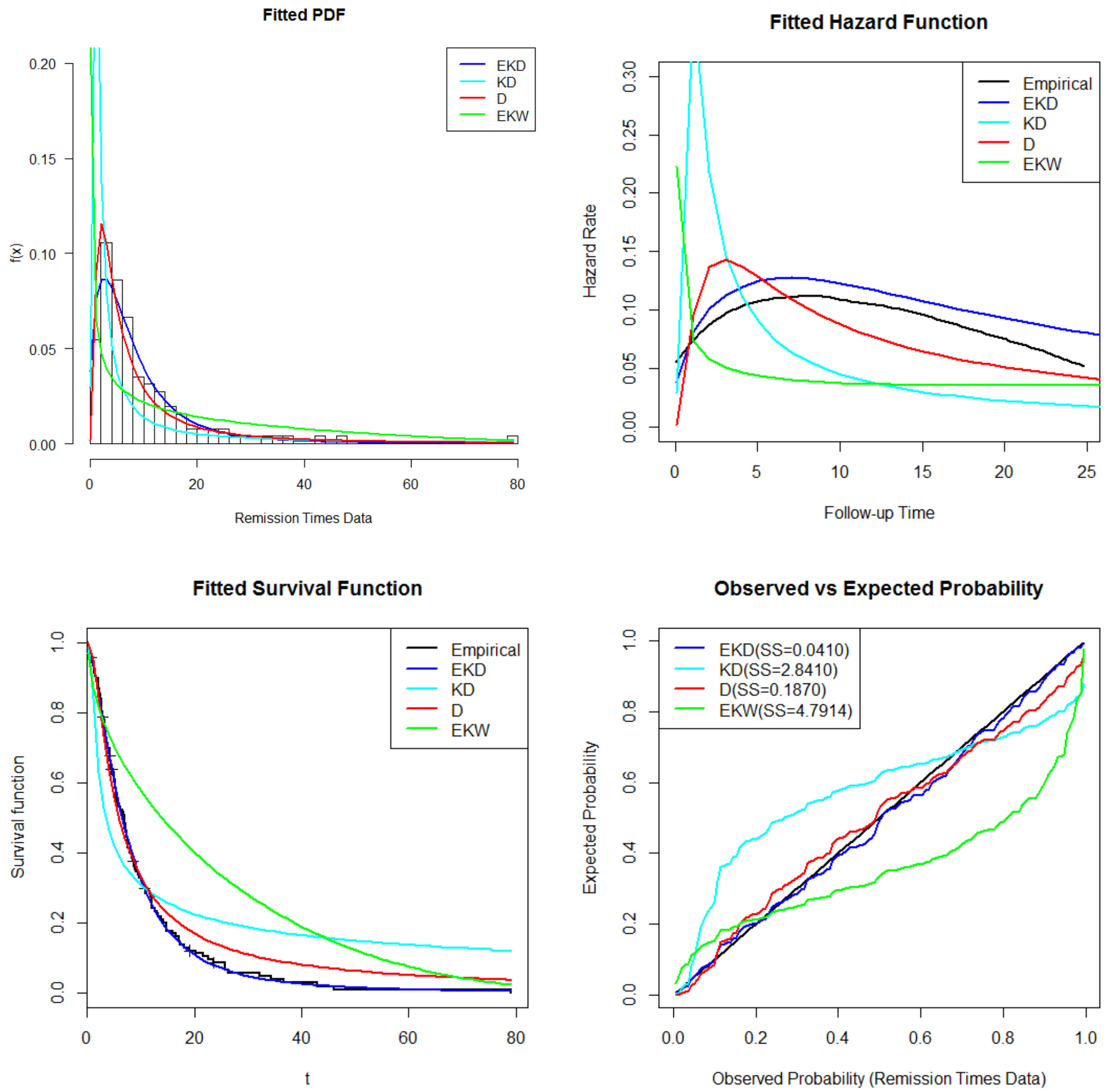

Figure 3.1: Type I EKD Fitted Densities, Hazard Functions, Empirical Survival Curves and Observed Probabilities for Remission Times Data

Table 3.1: Type I EKD Estimation for Remission Times Data

	Estimates					Statistics				
Distribution	α	λ	δ	ϕ	θ	-2 Log Likelihood	AIC	AICC	BIC	SS
EKD	4.0674	52.0264	1.7826	1.425	0.1792	836.7	846.7	847.2	861.3	0.0410
	(0.6287)	(32.0067)	(0.2623)	(0.6799)	(0.02053)					
KD	0.1862	0.5639	7.8559	0.0577	1	944.6	952.6	952.9	964.3	2.8410
	(0.05143)	(0.2804)	(0.1391)	(0.005263)	-					
D	2.5445	2.5344	1.1709	1	1	857.1	863.1	863.2	871.8	0.1870
	(0.8068)	(1.3466)	(0.1174)	-	-					
	a	b	c	λ	θ					
EKW	1.1633	0.05785	2.4018	0.05674	0.1918	927.2	937.2	937.7	951.8	4.7914
	(0.2468)	(0.01698)	(0.1602)	(0.005661)	(0.04479)					

EKD distribution gives the smallest SS value. We can conclude that EKD is the best fit for remission times data.

3.3.2 Case II: 2004 New Car and Truck Data

The second example consists of price of 428 new vehicles for the 2004 year (Kiplinger's Personal Finance, Dec 2003). After sorting the data, we note the 4 largest numbers are far away from others. So we drop the last 4 number. As a result, 424 numbers are analyzed, where $n = 428, r = 0, m = 424$. Estimates of the parameters, $-2\log(L)$, AIC, AICC, BIC, and SS are given in Table 3.2.

Plots of the fitted density and the histogram, and observed probability versus predicted probability are given in Figure 3.2. The sub-model, Dagum distribution seem to be the "superior" fit for this data, based on the plots and the statistics given in Table 3.2.

The LR test statistic of the hypothesis H_0 : KD against H_a : EKD is 489.2 (p-value < 0.0001). Dagum gives the smallest AIC, AICC, BIC and SS values.

Table 3.2: Type II EKD Estimation for Price of Cars Data ($\times 10^4$)

	Estimates					Statistics				
Distribution	α	λ	δ	ϕ	θ	-2 Log Likelihood	AIC	AICC	BIC	SS
EKD	0.006567	0.3440	13.7987	0.1609	16.6667	1489.9	1499.9	1500.0	1520.1	0.2816
	(0.000265)	(0.03825)	(0.1624)	(0.00789)	(2.5266)					
KD	9.4739	0.0027	13.5703	0.0537	1	1979.1	1987.1	1987.2	2003.3	14.8571
	(1.2372)	(0.000734)	(0.1214)	(0.00266)	-					
D	2.453	6.3435	2.9237	1	1	1472.4	1478.4	1478.4	1490.5	0.0478
	(0.7916)	(3.6931)	(0.2089)	-	-					
	a	b	c	λ	θ					
EKW	6.0407	0.0273	4.1272	0.2520	0.0705	1582.5	1592.5	1592.7	1612.8	3.5570
	(0.1765)	(0.00367)	(0.07324)	(0.003682)	(0.003311)					

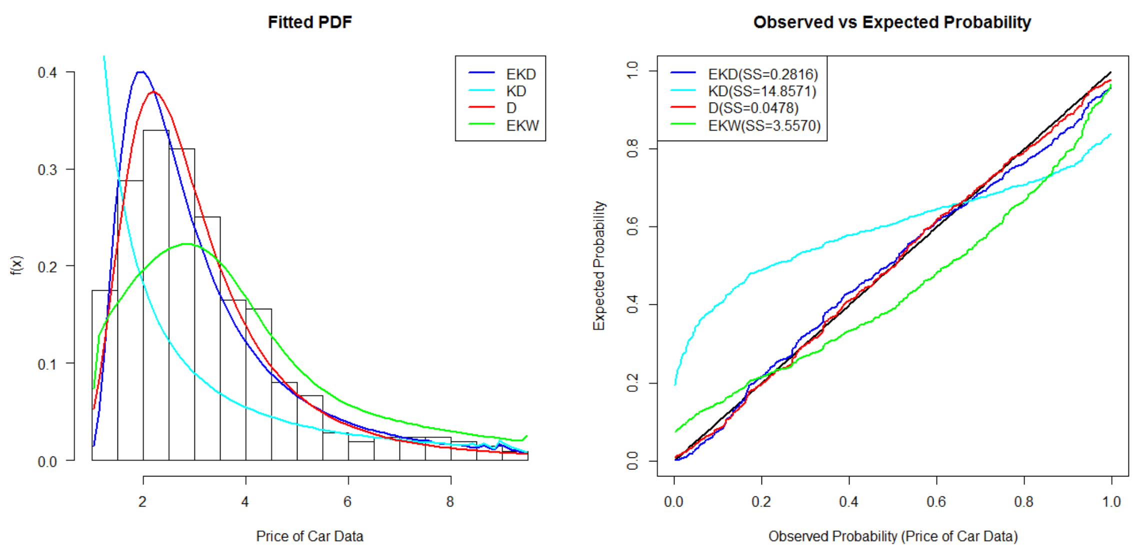

Figure 3.2: Type II EKD Fitted Densities, Observed Probabilities for Price of Cars Data

3.4 Concluding Remarks

We employed a new class of distributions called the Exponentiated Kumaraswamy-Dagum (EKD) distribution to fit censored data. We showed that this class of distributions is competitive class of models as far as censored observations in lifetime and reliability analysis are concerned. Estimation of the parameters of the models under Type I right and Type II doubly censoring plans and applications are also given.

CHAPTER 4

THE LOG-EXPONENTIATED KUMARASWAMY-DAGUM DISTRIBUTION

4.1 Introduction

The log transformation can be used to make highly skewed distribution less skewed. This can be valuable both for making patterns in the data more interpretable and for helping to meet the assumptions of inferential statistics. If X is any random variable, and F is the cdf of X, then $Y = \log(X)$ is called Log-F distribution.

In this chapter, we introduce the log-exponentiated Kumaraswamy-Dagum distribution which can be useful to model lifetime data. This chapter is organized as follows. In section 4.2, we present the log-exponentiated Kumaraswamy-Dagum distribution and its sub models, as well as series expansion, hazard and reverse hazard functions. Moments, moment generating function, Bonferroni and Lorenz curves, mean and median deviations, and reliability are given in section 4.3. Section 4.4 contains results on the distribution of the order statistics and measure of uncertainty. Estimation of model parameters via the method of maximum likelihood is presented in section 4.5. In section 4.6, various simulations are conducted for different sample sizes followed by concluding remarks.

4.2 The Log-Exponentiated Kumaraswamy-Dagum Distribution

In this section, we present the proposed distribution and its sub-models. Series expansion, hazard and reverse hazard functions and reliability are also studied in this section.

4.2.1 The Log-Exponentiated Kumaraswamy-Dagum Distribution

If $X \sim EKD(\alpha, \lambda, \delta, \phi, \theta)$, we define the Log-EKD distribution by the distribution of $Y = \log X$. Its density function is

$$\begin{aligned} g_{_{Log-EKD}}(y) &= \alpha\lambda\delta\phi\theta e^{-\delta y}(1+\lambda e^{-\delta y})^{-\alpha-1}[1-(1+\lambda e^{-\delta y})^{-\alpha}]^{\phi-1} \\ &\times \{1-[1-(1+\lambda e^{-\delta y})^{-\alpha}]^{\phi}\}^{\theta-1}, \end{aligned} \tag{4.1}$$

for $\alpha, \lambda, \delta, \phi, \theta > 0$ and $-\infty < y < \infty$.

The cdf of the Log-EKD distribution is

$$G_{_{Log-EKD}}(y) = \{1-[1-(1+\lambda e^{-\delta y})^{-\alpha}]^{\phi}\}^{\theta}, \tag{4.2}$$

for $\alpha, \lambda, \delta, \phi, \theta > 0$ and $-\infty < y < \infty$.

The quantile function is given by

$$y_q = \delta^{-1}\ln\lambda - \delta^{-1}\ln\{[1-(1-q^{\frac{1}{\theta}})^{\frac{1}{\phi}}]^{-\frac{1}{\alpha}} - 1\}. \tag{4.3}$$

Plots of the pdf for selected values of the model parameters are given in Figure 4.1.

4.2.2 Sub-models

Sub-models of the Log-EKD distribution for selected values of the parameters are presented in this section.

① When $\theta = 1$, we obtain Log-Kumaraswamy-Dagum distribution with cdf:

$$G_{_{Log-KD}}(y) = 1-[1-(1+\lambda e^{-\delta y})^{-\alpha}]^{\phi},$$

for $\alpha, \lambda, \delta, \phi > 0$ and $-\infty < y < \infty$.

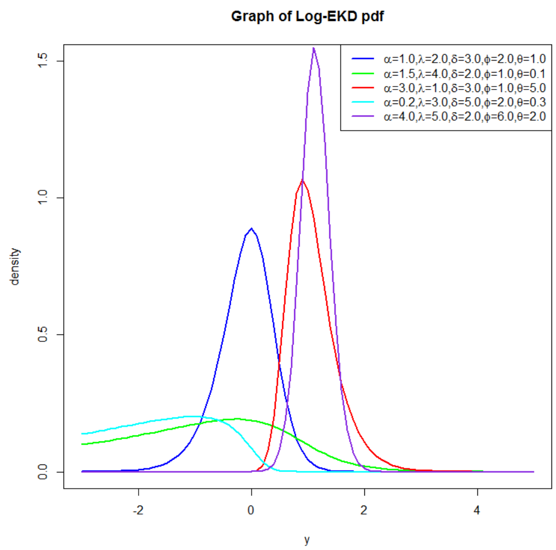

Figure 4.1: Log-EKD Density Functions

② When $\phi = \theta = 1$, we obtain Log-Dagum distribution with cdf:

$$G_{_{Log-D}}(y) \quad = \quad (1 + \lambda e^{-\delta y})^{-\alpha},$$

for $\alpha, \lambda, \delta > 0$ and $-\infty < y < \infty$.

③ When $\lambda = 1$, we obtain Log-exponentiated Kumarawamy-Burr III distribution with cdf:

$$G_{_{Log-EKB}}(y) \quad = \quad \{1 - [1 - (1 + e^{-\delta y})^{-\alpha}]^{\phi}\}^{\theta},$$

for $\alpha, \delta, \phi, \theta > 0$ and $-\infty < y < \infty$.

④ When $\lambda = \theta = 1$, we obtain Log-Kumarawamy-Burr III distribution with cdf:

$$G_{_{Log-KB}}(y) \quad = \quad 1 - [1 - (1 + e^{-\delta y})^{-\alpha}]^{\phi},$$

for $\alpha, \delta, \phi > 0$ and $-\infty < y < \infty$.

⑤ When $\lambda = \phi = \theta = 1$, we obtain Log-Burr III distribution with cdf:

$$G_{_{Log-B}}(y) \quad = \quad (1 + e^{-\delta y})^{-\alpha},$$

for $\alpha, \delta > 0$ and $-\infty < y < \infty$.

⑥ When $\alpha = 1$, we obtain Log-exponentiated Kumarawamy-Fisk or Log-exponentiated Kumarawamy Log-logistic distribution with cdf:

$$G_{_{Log-EKF}}(y) \;=\; \{1 - [1 - (1 + \lambda e^{-\delta y})^{-1}]^{\phi}\}^{\theta},$$

for $\lambda, \delta, \phi, \theta > 0$ and $-\infty < y < \infty$.

⑦ When $\alpha = \theta = 1$, we obtain Log-Kumarawamy-Fisk or Log-Kumaraswamy Log-logistic distribution with cdf:

$$G_{_{Log-KF}}(y) \;=\; 1 - [1 - (1 + \lambda e^{-\delta y})^{-1}]^{\phi},$$

for $\lambda, \delta, \phi > 0$ and $-\infty < y < \infty$.

⑧ When $\alpha = \phi = \theta = 1$, we obtain Log-Fisk or Log Log-logistic distribution with cdf:

$$G_{_{Log-F}}(y) \;=\; (1 + \lambda e^{-\delta y})^{-1},$$

for $\lambda, \delta > 0$ and $-\infty < y < \infty$.

4.2.3 Series Expansion

By using equation (2.6), we obtain the series expansion of the Log-EKD distribution:

$$g_{_{Log-EKD}}(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega(i,j) e^{-\delta y}(1 + \lambda e^{-\delta y})^{-\alpha-\alpha j-1}, \tag{4.4}$$

where $\omega(i,j) = \alpha\lambda\delta\phi\theta \frac{(-1)^{i+j}\Gamma(\theta)\Gamma(\phi i+\phi)}{\Gamma(\theta-i)\Gamma(\phi i+\phi-j)i!j!}$.

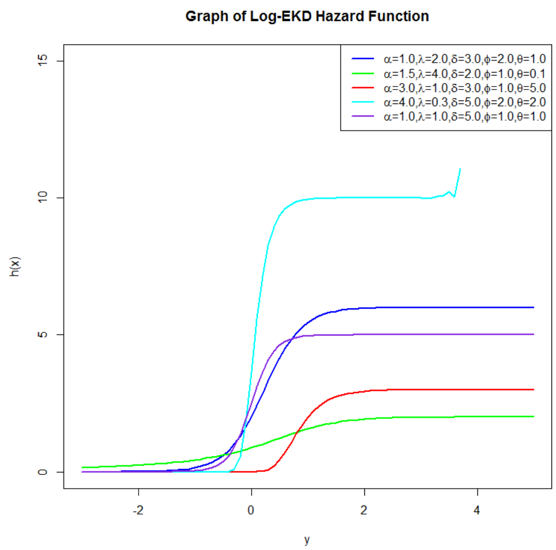

Figure 4.2: Log-EKD Hazard Functions

4.2.4 Hazard and Reverse Hazard Functions

The hazard function of the Log-EKD distribution is

$$\begin{aligned} h_{Log-EKD}(x) &= \frac{g_{Log-EKD}(x)}{1-G_{Log-EKD}(x)} \\ &= \alpha\lambda\delta\phi\theta e^{-\delta y}(1+\lambda e^{-\delta y})^{-\alpha-1}[1-(1+\lambda e^{-\delta y})^{-\alpha}]^{\phi-1} \\ &\times \{1-[1-(1+\lambda e^{-\delta y})^{-\alpha}]^{\phi}\}^{\theta-1} \\ &\times \left(1-\{1-[1-(1+\lambda e^{-\delta y})^{-\alpha}]^{\phi}\}^{\theta}\right)^{-1}. \end{aligned} \tag{4.5}$$

Plots of the hazard function are presented in Figure 4.2 for five different combinations of the parameter values. The graphs include increasing hazard rate and 'sigmoidal' shapes.

The reverse hazard function of the Log-EKD distribution is

$$\begin{aligned} \tau_{Log-EKD}(x) &= \frac{g_{Log-EKD}(x)}{G_{Log-EKD}(x)} \\ &= \alpha\lambda\delta\phi\theta e^{-\delta y}(1+\lambda e^{-\delta y})^{-\alpha-1}[1-(1+\lambda e^{-\delta y})^{-\alpha}]^{\phi-1} \\ &\times \{1-[1-(1+\lambda e^{-\delta y})^{-\alpha}]^{\phi}\}^{-1}. \end{aligned} \tag{4.6}$$

4.3 Moments, Moment Generating Function, Bonferroni and Lorenz Curves, Mean and Median Deviations, and Reliability

4.3.1 Moments and Moment Generating Function

The s^{th} moment of the Log-EKD distribution is given by

$$\begin{aligned} E(Y^s) &= \int_{-\infty}^{\infty} y^s g_{_{Log-EKD}}(y)dy \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega(i,j)\int_{-\infty}^{\infty} y^s e^{-\delta y}(1+\lambda e^{-\delta y})^{-\alpha-\alpha j-1}dy. \end{aligned}$$

Let $t=(1+\lambda e^{-\delta y})^{-1}$, then $y=\delta^{-1}\ln\frac{\lambda t}{1-t}$ and the s^{th} moment of the Log-EKD distribution is given by

$$\begin{aligned} E(Y^s) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega(i,j)\delta^{-s-1}\lambda^{-1}\int_0^1\left(\ln\frac{\lambda t}{1-t}\right)^s t^{\alpha+\alpha j-1}dt, \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega(i,j,s)\Sigma(\alpha,\lambda,j,s), \end{aligned} \tag{4.7}$$

where $\omega(i,j,s)=\omega(i,j)\delta^{-s-1}\lambda^{-1}$ and

$$\Sigma(\alpha,\lambda,j,s)=\int_0^1\left(\ln\frac{\lambda t}{1-t}\right)^s t^{\alpha+\alpha j-1}dt.$$

The moment generating function of the Log-EKD distribution is

$$M(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\omega(i,j,r)\frac{t^r}{r!}\Sigma(\alpha,\lambda,j,r).$$

4.3.2 Bonferroni and Lorenz Curves

The definition of Bonferroni and Lorenz curves are given by

$$B(p)=\frac{I(q)}{p\mu} \text{ and } L(p)=\frac{I(q)}{\mu},$$

where $I(a) = \int_{-\infty}^{a} y \cdot g_{_{Log-EKD}}(y)dy, \mu = E(Y)$ and $q = G_{_{Log-EKD}}^{-1}(p)$. The mean of the Log-EKD distribution is obtained from equation (4.7) with $s = 1$ and the quantile function is given in equation (4.3).

Let $t(y) = (1 + \lambda e^{-\delta y})^{-1}$, then

$$I(a) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega(i, j, 1) \mathit{\Sigma}_{t(a)}(\alpha, \lambda, j, 1),$$

where

$$\mathit{\Sigma}_{t(a)}(\alpha, \lambda, j, s) = \int_{0}^{t(a)} \left(\ln \frac{\lambda t}{1-t} \right)^{s} t^{\alpha+\alpha j-1} dt.$$

4.3.3 Mean and Median Deviations

If Y has the Log-EKD distribution, the mean and median deviations about the mean μ and the median M are given by

$$\delta_1 = \int_{-\infty}^{\infty} |x - \mu| g_{_{Log-EKD}}(y)dy \text{ and } \delta_2 = \int_{-\infty}^{\infty} |x - M| g_{_{Log-EKD}}(y)dy,$$

respectively. The measure δ_1 and δ_2 can be calculated by the following relationships:

$$\delta_1 = 2\mu G_{_{Log-EKD}}(\mu) - 2\mu + 2T(\mu) \text{ and } \delta_2 = 2T(M) - \mu,$$

where $T(a) = \int_{a}^{\infty} y \cdot g_{_{Log-EKD}}(y)dy$ follows as

$$T(a) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega(i, j, 1) \Big[\mathit{\Sigma}(\alpha, \lambda, j, 1) - \mathit{\Sigma}_{t(a)}(\alpha, \lambda, j, 1) \Big].$$

4.3.4 Reliability

The reliability $R = P(X_1 > X_2)$ when X_1 and X_2 have independent $Log-EKD(\alpha_1, \lambda_1, \delta_1, \phi_1, \theta_1)$ and $Log-EKD(\alpha_2, \lambda_2, \delta_2, \phi_2, \theta_2)$ distributions is given by

$$\begin{aligned} R &= \int_{-\infty}^{\infty} g_1(y)G_2(y)dy \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\zeta(i,j,k,l)\int_{-\infty}^{\infty} e^{-\delta_1 y}(1+\lambda_1 e^{-\delta_1 y})^{-\alpha_1-\alpha_1 j-1}(1+\lambda_2 e^{-\delta_2 y})^{-\alpha_2 l}dy, \end{aligned}$$

where $\zeta(i,j,k,l) = \alpha_1\lambda_1\delta_1\phi_1\theta_1 \frac{(-1)^{i+j+k+l}\Gamma(\theta_1)\Gamma(\phi_1 i+\phi_1)\Gamma(\theta_2+1)\Gamma(\phi_2 k+1)}{\Gamma(\theta_1-i)\Gamma(\phi_1 i+\phi_1-j)\Gamma(\theta_2+1-k)\Gamma(\phi_2 k+1-l)i!j!k!l!}$.

If $\lambda = \lambda_1 = \lambda_2$ and $\delta = \delta_1 = \delta_2$, then reliability can be reduced to

$$R = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\zeta(i,j,k,l)}{\delta\lambda(\alpha_1+\alpha_1 j+\alpha_2 l)}.$$

4.4 Order Statistics and Entropy

In this section, the distribution of the k^{th} order statistic and Rényi entropy (Rényi [31]) for the Log-EKD distribution are presented. The entropy of a random variable is a measure of variation of the uncertainty.

4.4.1 Order Statistics

The pdf of the k^{th} order statistic from a pdf $f(x)$ is

$$\begin{aligned} f_{k:n}(x) &= \frac{f(x)}{B(k,n-k+1)}F^{k-1}(x)[1-F(x)]^{n-k} \\ &= k\binom{n}{k}f(x)F^{k-1}(x)[1-F(x)]^{n-k}. \end{aligned} \tag{4.8}$$

Using equation (2.6), the pdf of the k^{th} order statistic from Log-EKD distribution is given by

$$g_{k:n}(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}K(i,j,p,k)\cdot e^{-\delta y}(1+\lambda e^{-\delta y})^{-\alpha-\alpha p-1},$$

where $K(i,j,p,k) = \frac{(-1)^{i+j+p}\Gamma(n-k+1)\Gamma(\theta k+\theta i)\Gamma(\phi j+\phi)}{\Gamma(n-k+1-i)\Gamma(\theta k+\theta i-j)\Gamma(\phi j+\phi-p)i!j!p!}k\binom{n}{k}\alpha\lambda\delta\phi\theta$.

4.4.2 Entropy

Rényi entropy of a distribution with pdf $f(x)$ is defined as

$$I_R(\tau) = (1-\tau)^{-1}\log\left\{\int_{\mathbb{R}} f^{\tau}(x)dx\right\}, \tau > 0, \tau \neq 1.$$

Using equation (2.6), Rényi entropy of Log-EKD distribution is given by

$$\begin{aligned} I_R(\tau) &= (1-\tau)^{-1}\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\Gamma(\theta\tau-\tau+1)\Gamma(\phi\tau-\tau+\phi i+1)}{\Gamma(\theta\tau-\tau+1-i)\Gamma(\phi\tau-\tau+\phi i+1-j)i!j!}\right. \\ &\times \left.(\alpha\phi\theta)^{\tau}\delta^{\tau-1}B(\alpha\tau+\alpha j,\tau)\right]. \end{aligned}$$

for $\alpha\tau + \alpha j > 0$. Rényi entropy for the sub-models can be readily obtained.

4.5 Estimation of Model Parameters

In this section, we present estimates of the parameters of the Log-EKD distribution via method of maximum likelihood estimation. The elements of the score function are presented. There are no closed form solutions to the nonlinear equations obtained by setting the elements of the score function to zero. Thus, the estimates of the model parameters must be obtained via numerical methods.

4.5.1 Maximum Likelihood Estimation

Let $\mathbf{x} = (x_1, \cdots, x_n)^T$ be a random sample of the Log-EKD distribution with unknown parameter vector $\mathbf{\Theta} = (\alpha, \lambda, \delta, \phi, \theta)^T$. The log-likelihood function for $\mathbf{\Theta}$ is

$$\begin{aligned} l(\mathbf{\Theta}) &= n(\ln\alpha+\ln\lambda+\ln\delta+\ln\phi+\ln\theta) - \delta\sum_{i=1}^{n} y_i \\ &- (\alpha+1)\sum_{i=1}^{n}\ln(1+\lambda e^{-\delta y_i}) + (\phi-1)\sum_{i=1}^{n}\ln[1-(1+\lambda e^{-\delta y_i})^{-\alpha}] \\ &+ (\theta-1)\sum_{i=1}^{n}\ln\{1-[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]^{\phi}\}. \end{aligned} \quad (4.9)$$

The partial derivatives of $l(\boldsymbol{\Theta})$ with respect to the parameters are

$$\begin{aligned}\frac{\partial l}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=1}^{n} \ln(1+\lambda e^{-\delta y_i}) + (\phi - 1)\sum_{i=1}^{n} \frac{(1+\lambda e^{-\delta y_i})^{-\alpha}\ln(1+\lambda e^{-\delta y_i})}{1-(1+\lambda e^{-\delta y_i})^{-\alpha}} \\ &- (\theta-1)\phi\sum_{i=1}^{n}\frac{[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]^{\phi-1}(1+\lambda e^{-\delta y_i})^{-\alpha}\ln(1+\lambda e^{-\delta y_i})}{1-[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]^{\phi}},\end{aligned}$$

$$\begin{aligned}\frac{\partial l}{\partial \lambda} &= \frac{n}{\lambda} - (\alpha+1)\sum_{i=1}^{n}\frac{e^{-\delta y_i}}{1+\lambda e^{-\delta y_i}} \\ &+ (\phi-1)\alpha\sum_{i=1}^{n}\frac{(1+\lambda e^{-\delta y_i})^{-\alpha-1}e^{-\delta y_i}}{1-(1+\lambda e^{-\delta y_i})^{-\alpha}} \\ &- (\theta-1)\phi\alpha\sum_{i=1}^{n}\frac{[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]^{\phi-1}(1+\lambda e^{-\delta y_i})^{-\alpha-1}e^{-\delta y_i}}{1-[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]^{\phi}},\end{aligned}$$

$$\begin{aligned}\frac{\partial l}{\partial \delta} &= \frac{n}{\delta} - \sum_{i=1}^{n} y_i + (\alpha+1)\lambda\sum_{i=1}^{n}\frac{e^{-\delta y_i}y_i}{1+\lambda e^{-\delta y_i}} \\ &- (\phi-1)\alpha\lambda\sum_{i=1}^{n}\frac{(1+\lambda e^{-\delta y_i})^{-\alpha-1}e^{-\delta y_i}y_i}{1-(1+\lambda e^{-\delta y_i})^{-\alpha}} \\ &+ (\theta-1)\phi\alpha\lambda\sum_{i=1}^{n}\frac{[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]^{\phi-1}(1+\lambda e^{-\delta y_i})^{-\alpha-1}e^{-\delta y_i}y_i}{1-[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]^{\phi}},\end{aligned}$$

$$\begin{aligned}\frac{\partial l}{\partial \phi} &= \frac{n}{\phi} + \sum_{i=1}^{n}\ln[1-(1+\lambda e^{-\delta y_i})^{-\alpha}] \\ &- (\theta-1)\sum_{i=1}^{n}\frac{[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]^{\phi}\ln[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]}{1-[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]^{\phi}},\end{aligned}$$

and

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln\{1-[1-(1+\lambda e^{-\delta y_i})^{-\alpha}]^{\phi}\},$$

respectively. The MLEs of the parameters $\alpha, \lambda, \delta, \phi$, and θ, say $\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\phi}$, and $\hat{\theta}$, must be obtained by numerical methods.

4.5.2 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the Log-EKD distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\mathbf{\Theta}} = (\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\phi}, \hat{\theta})$ be the maximum likelihood estimate of $\mathbf{\Theta} = (\alpha, \lambda, \delta, \phi, \theta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) \xrightarrow{d} N_5(\underline{0}, I^{-1}(\mathbf{\Theta}))$, where $I(\mathbf{\Theta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Theta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$, that is $J(\hat{\mathbf{\Theta}})$. The multivariate normal distribution $N_5(\underline{0}, J(\hat{\mathbf{\Theta}})^{-1})$, where the mean vector $\underline{0} = (0, 0, 0, 0, 0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

The approximate $100(1-\eta)\%$ two-sided confidence intervals for α, λ, δ, ϕ and θ are given by:

$$\widehat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Theta}})}, \quad \widehat{\lambda} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\widehat{\mathbf{\Theta}})}, \quad \widehat{\delta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\delta\delta}^{-1}(\widehat{\mathbf{\Theta}})}$$

$$\widehat{\phi} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\phi\phi}^{-1}(\widehat{\mathbf{\Theta}})}, \text{ and } \quad \widehat{\theta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\theta\theta}^{-1}(\widehat{\mathbf{\Theta}})}$$

respectively, where $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the Log-EKD distribution with its sub-models for a given data set. For example, to test $\theta = 1$, the LR statistic is

$$\omega = 2[\ln(L(\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\phi}, \hat{\theta})) - \ln(L(\tilde{\alpha}, \tilde{\lambda}, \tilde{\delta}, \tilde{\phi}, 1))],$$

where $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\delta}$, $\hat{\phi}$ and $\hat{\theta}$ are the unrestricted estimates, and $\tilde{\alpha}$, $\tilde{\lambda}$, $\tilde{\delta}$ and $\tilde{\phi}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi_d^2$, where χ_d^2 denote the upper $100d\%$ point of the χ^2 distribution with 1 degrees of freedom.

4.6 Simulation Study

In this section, we examine the performance of the Log-EKD distribution by conducting various simulations for different sizes (n=200, 400, 800, 1200) via the subroutine NLP in SAS. We simulate 2000 samples for the true parameters values $I : \alpha = 2, \lambda = 1, \delta = 3, \phi = 2, \theta = 2$ and $II : \alpha = 1, \lambda = 1, \delta = 1, \phi = 1, \theta = 1$. Table 4.1 lists the means MLEs of the five model parameters along with the respective root mean squared errors (RMSE). From the results, we can verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

4.7 Applications

In this section, we present examples to illustrate the flexibility of the Log-EKD distribution and its sub-models.

The maximum likelihood estimates (MLEs) of the Log-EKD parameters α, λ, δ, ϕ, and θ are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\boldsymbol{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Table 4.2, 4.3 and 4.5. Also, values of the Kolmogorov-Smirnov statistic, $KS = max_{1\leq i\leq n}\{G_{_{Log-EKD}}(x_{(i)}) - \frac{i-1}{n}, \frac{i}{n} - G_{_{Log-EKD}}(x_{(i)})\}$, and the sum of squares $SS = \sum_{j=1}^{n}\left[G_{_{Log-EKD}}(x_{(j)}) - \left(\frac{j-0.375}{n+0.25}\right)\right]^2$ are presented. These statistics are used to compare the distributions presented in these tables. Plots of the fitted densities and

Table 4.1: Log-EKD Monte Carlo Simulation Results

		I		II	
n	Parameter	Mean	RMSE	Mean	RMSE
200	α	4.413567	3.975001597	1.7936395	1.999236829
	λ	1.3590802	2.660520645	1.4262762	1.506448705
	δ	3.1165707	2.601430068	1.0336256	0.58947307
	ϕ	5.7306381	6.543780841	2.463487	3.692871552
	θ	4.5556804	4.304073408	2.8852635	3.689011832
400	α	3.6016435	3.077083246	1.5472599	1.516016227
	λ	1.1189229	0.900938233	1.1378167	0.730888227
	δ	2.9330833	1.821683864	1.0064811	0.377391441
	ϕ	4.7024933	5.282620694	1.5487748	1.874007577
	θ	4.1166788	3.615296336	2.4197854	2.969633816
800	α	3.1044653	2.416169282	1.4362126	1.279387041
	λ	1.0625975	0.60917083	1.043214	0.346919155
	δ	2.8962218	1.367997003	1.0017939	0.250660128
	ϕ	3.7409284	3.911557669	1.1763134	0.763548623
	θ	3.4876455	2.746251136	1.9721072	2.195744771
1200	α	2.8414335	2.059748577	1.3876676	1.167100296
	λ	1.0429831	0.501683566	1.0217614	0.258797025
	δ	2.9150681	1.133929716	1.0014528	0.193822599
	ϕ	3.1775145	3.051107995	1.0836516	0.3923371
	θ	3.1649001	2.349326286	1.7327449	1.789785015

Table 4.2: Log-EKD Estimation for Traffic Data

	Estimates					Statistics					
Model	α	λ	δ	ϕ	θ	-2 Log L	AIC	AICC	BIC	SS	KS
Log-EKD	23.7377	0.8084	0.3006	0.0944	0.2837	674.8	684.8	685.6	697.0	0.0750	0.0581
	(19.6741)	(0.8897)	(0.01182)	(0.02677)	(0.08761)						
Log-KD	47.9714	0.04924	0.2754	0.1906	1	689.4	697.4	697.9	707.1	0.5419	0.1392
	(38.6781)	(0.04179)	(0.01133)	(0.02293)	-						
Log-D	4.2567	0.7007	0.07852	1	1	731.3	737.3	737.6	744.6	0.6941	0.1648
	(2.5951)	(0.5251)	(0.008062)	-	-						
Log-Fisk	1	4.3565	0.08595	1	1	756.4	760.4	760.5	765.2	0.7132	0.2215
	-	(0.9326)	(0.008169)	-	-						

the histogram of the data are given in Figures 4.3, 4.4 and 4.5. Probability plots (Chambers et al. [1]) are also presented in above Figures. For the probability plot, we plotted $G_{_{Log-EKD}}(x_{(j)};\hat{\alpha},\hat{\lambda},\hat{\delta},\hat{\phi},\hat{\theta})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1,\ 2,\ \cdots,\ n$, where $x_{(j)}$ are the ordered values of the observed data.

The first example consists of the length of intervals between the times at which vehicles pass a point on a road [16]. Initial values for Log-EKD model in SAS code are $\alpha = 0.15, \lambda = 0.0001, \delta = 0.001, \phi = 0.1, \theta = 1.5$. Estimates of the parameters of Log-EKD distribution and its related sub-models (standard error in parentheses), AIC, AICC, BIC, KS and SS for traffic data are given in Table 4.2. The estimated covariance matrix for Log-EKD distribution is given by:

$$\begin{pmatrix} 387.07 & -15.7106 & -0.1793 & 0.06776 & 0.1128 \\ -15.7106 & 0.7916 & 0.009276 & -0.00938 & -0.0361 \\ -0.1793 & 0.009276 & 0.00014 & -0.00012 & -0.00044 \\ 0.06776 & -0.00938 & -0.00012 & 0.000716 & 0.001674 \\ 0.1128 & -0.0361 & -0.00044 & 0.001674 & 0.007675 \end{pmatrix}$$

The 95% asymptotic confidence intervals are: $\alpha \in 23.7377 \pm 1.96(19.6741), \lambda \in$

$0.8084 \pm 1.96(0.8897), \delta \in 0.3006 \pm 1.96(0.01182), \phi \in 0.0944 \pm 1.96(0.02677)$ and $\theta \in 0.02837 \pm 1.96(0.08761)$.

Plots of the fitted densities and the histogram, observed probability vs predicted probability, and empirical survival function for the traffic data are given in Figure 4.3.

The LR test statistic of the hypothesis H_0: Log-KD against H_a: Log-EKD and H_0: Log-D against H_a: Log-EKD are 14.6 (p-value < 0.0002) and 56.5 (p-value < 0.0001). We can conclude that there is a significant difference between Log-EKD and Log-KD, Log-D distributions.

Table 4.3: Log-EKD Estimation for Active Repair Times (hours) Data

	Estimates					Statistics					
Model	α	λ	δ	ϕ	θ	-2 Log L	AIC	AICC	BIC	SS	KS
Log-EKD	1.621	1.1789	1.4754	0.1845	0.9213	197.5	207.5	209.2	215.9	0.1657	0.1364
	(0.7701)	(0.7553)	(0.04401)	(0.04319)	(0.3424)						
Log-KD	1.702	1.3545	1.4934	0.1964	1	197.4	205.4	206.6	212.2	0.2395	0.1350
	(0.8004)	(0.8303)	(0.04107)	(0.03373)	-						
Log-D	5.2301	0.542	0.4223	1	1	214.4	220.4	221.0	225.4	0.3268	0.1624
	(4.5503)	(0.5578)	(0.06166)	-	-						
Log-Fisk	1	4.11	0.4658	1	1	228.1	232.1	232.5	235.5	0.3354	0.2350
	-	(1.2575)	(0.06454)	-	-						

The second example consists of the active repair times (hours) for an airborne communication transceiver [16]. Initial value for Log-EKD model in SAS code are $\alpha = 1, \lambda = 0.1, \delta = 0.1, \phi = 0.1, \theta = 1$. Estimates of the parameters of Log-EKD distribution and its related sub-models (standard error in parentheses), AIC, AICC, BIC, KS and SS are given in Table 4.3. The estimated covariance matrix for Log-EKD model is given by:

$$\begin{pmatrix} 0.5931 & 0.05057 & 0.00937 & -0.00866 & -0.1506 \\ 0.05057 & 0.5705 & 0.02152 & -0.00977 & -0.1677 \\ 0.00937 & 0.02152 & 0.001937 & -0.00059 & -0.00803 \\ -0.00866 & -0.00977 & -0.00059 & 0.001865 & 0.009607 \\ -0.1506 & -0.1677 & -0.00803 & 0.009607 & 0.1173 \end{pmatrix}$$

Plots of the fitted densities and the histogram, observed probability vs predicted probability, and empirical survival function are given in Figure 4.4.

The LR test statistic of the hypothesis H_0: Log-D against H_a: Log-EKD is 16.9 (p-value = 0.0002). We can conclude that there is a significant difference between Log-EKD and Log-D. There is no significant difference between Log-EKD and Log-KD distributions. Log-KD gives the smallest AIC, AICC, BIC values and comparatively small SS, KS value, except for the Log-EKD model. We conclude that the Log-KD distribution provides the "best" fit for the active repair times data.

The third example represents INPC data and is given in Table 4.4. The INPC is a national index of consumer prices in Brazil, produced since 1979. Collection period extends from day 01 to 30 of the reference month. INPC measures the cost of living of households with heads employees. Initial value for Log-EKD model in SAS code are $\alpha = 0.1, \lambda = 0.1, \delta = 0.1, \phi = 0.1, \theta = 1$. Estimates of the parameters of Log-EKD distribution and its related sub-models (standard error in parentheses), AIC, AICC, BIC SS and KS are given in Table 4.5. The estimated covariance matrix for Log-EKD

Table 4.4: INPC Data

0.69	0.97	0.43	0.30	0.25	0.59	0.32	0.31	0.26	0.26
0.44	0.42	0.49	0.62	0.42	0.43	0.16	-0.02	0.11	-0.07
0.13	0.12	0.27	0.23	0.38	0.40	0.54	0.58	0.15	0.00
0.03	-0.11	0.70	0.91	0.73	0.44	0.57	0.86	0.44	0.17
0.17	0.50	0.73	0.50	0.40	0.41	0.57	0.39	0.83	0.54
0.37	0.39	0.82	0.18	0.04	-0.06	0.99	1.38	1.37	1.46
2.47	2.70	3.39	1.57	0.83	0.86	1.15	0.61	0.09	0.68
0.62	0.31	1.07	0.74	1.29	0.94	0.44	0.79	1.11	0.60
0.57	0.84	0.48	0.49	0.77	0.55	0.29	0.16	0.43	1.21
1.39	0.30	-0.05	0.09	0.13	0.05	0.61	0.74	0.94	0.96
0.39	0.55	0.74	0.07	0.05	0.47	1.28	1.29	0.65	0.42
-0.18	0.11	-0.31	-0.49	-0.28	0.15	0.72	0.45	0.49	0.54
0.85	0.57	0.15	0.29	0.10	-0.03	0.18	0.35	0.11	0.60
0.68	0.45	0.81	0.33	0.34	0.38	0.02	0.50	1.20	1.33
1.28	0.93	0.29	0.71	1.46	1.65	1.51	1.40	1.17	1.02
2.46	2.18	2.10	2.49	1.62	1.01	1.44			

Table 4.5: Log-EKD Estimation for INPC Data

	Estimates					Statistics					
Model	α	λ	δ	ϕ	θ	-2 Log L	AIC	AICC	BIC	SS	KS
Log-EKD	3.6042	3.503	5.283	0.3042	0.4004	232.1	242.1	242.5	257.4	0.0274	0.0388
	(9.6258)	(6.0352)	(4.3398)	(0.2153)	(0.6422)						
Log-KD	1.0054	3.0204	7.2134	0.2569	1	232.5	240.5	240.8	252.8	0.0343	0.0405
	(0.4941)	(2.4859)	(1.5043)	(0.05679)							
Log-D	4.5406	0.7089	2.5971	1	1	239.9	245.9	246.0	255.0	0.1239	0.0699
	(3.0103)	(0.5982)	(0.2457)								
Log-Fisk	1	6.4846	3.2693	1	1	263.8	267.8	267.9	273.9	0.2485	0.0785
		(1.1577)	(0.2217)								

model is given by:

$$\begin{pmatrix} 92.6562 & -46.5209 & -40.5145 & 1.7494 & -5.9026 \\ -46.5209 & 36.4231 & 23.296 & -1.1953 & 2.2874 \\ -40.5145 & 23.296 & 18.8336 & -0.8782 & 2.4294 \\ 1.7494 & -1.1953 & -0.8782 & 0.04635 & -0.0944 \\ -5.9026 & 2.2874 & 2.4294 & -0.0944 & 0.4124 \end{pmatrix}$$

Plots of the fitted densities and the histogram, observed probability vs predicted probability, and empirical survival function are given in Figure 4.5.

There is no significant difference between the Log-EKD and Log-KD distributions, however Log-KD gives smaller AIC, AICC, BIC values, and second smallest SS, KS values. So Log-KD distribution provides the "best" fit for this data.

4.8 Concluding Remarks

We have proposed and presented results on a new class of distributions called the Log-EKD distribution. This class of distributions have applications in cost and lifetime data analysis. Properties of this class of distributions including the series expansion of pdf, cdf, moments, hazard function, reverse hazard function, income inequality measures such as Lorenz and Bonferroni curves are derived. Rényi entropy, order statistics, reliability, mean and median deviations are presented. Estimation of the parameters of the models and applications are also given.

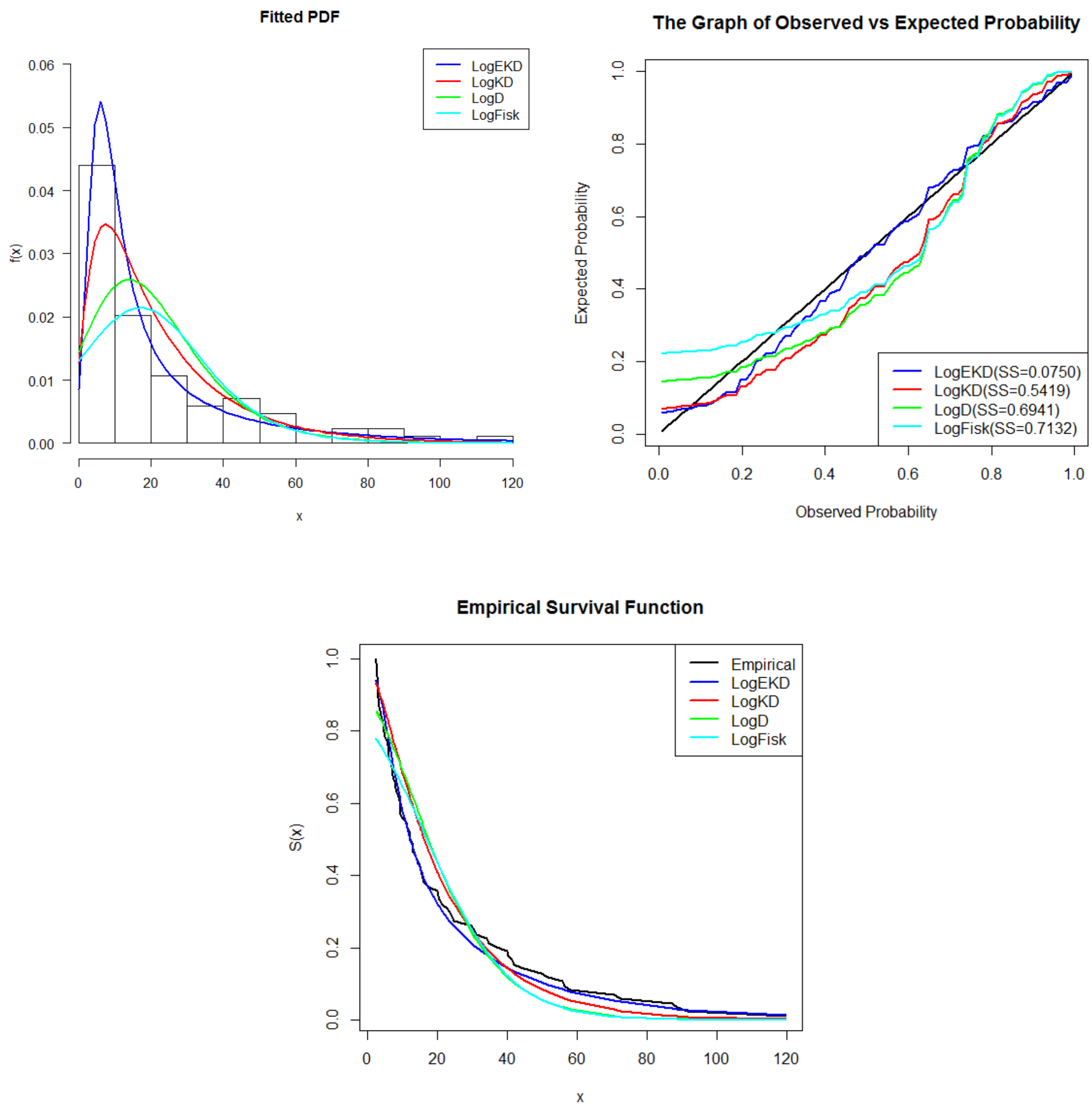

Figure 4.3: Log-EKD Fitted Densities, Observed Probabilities and Empirical Survival Curves for Traffic Data

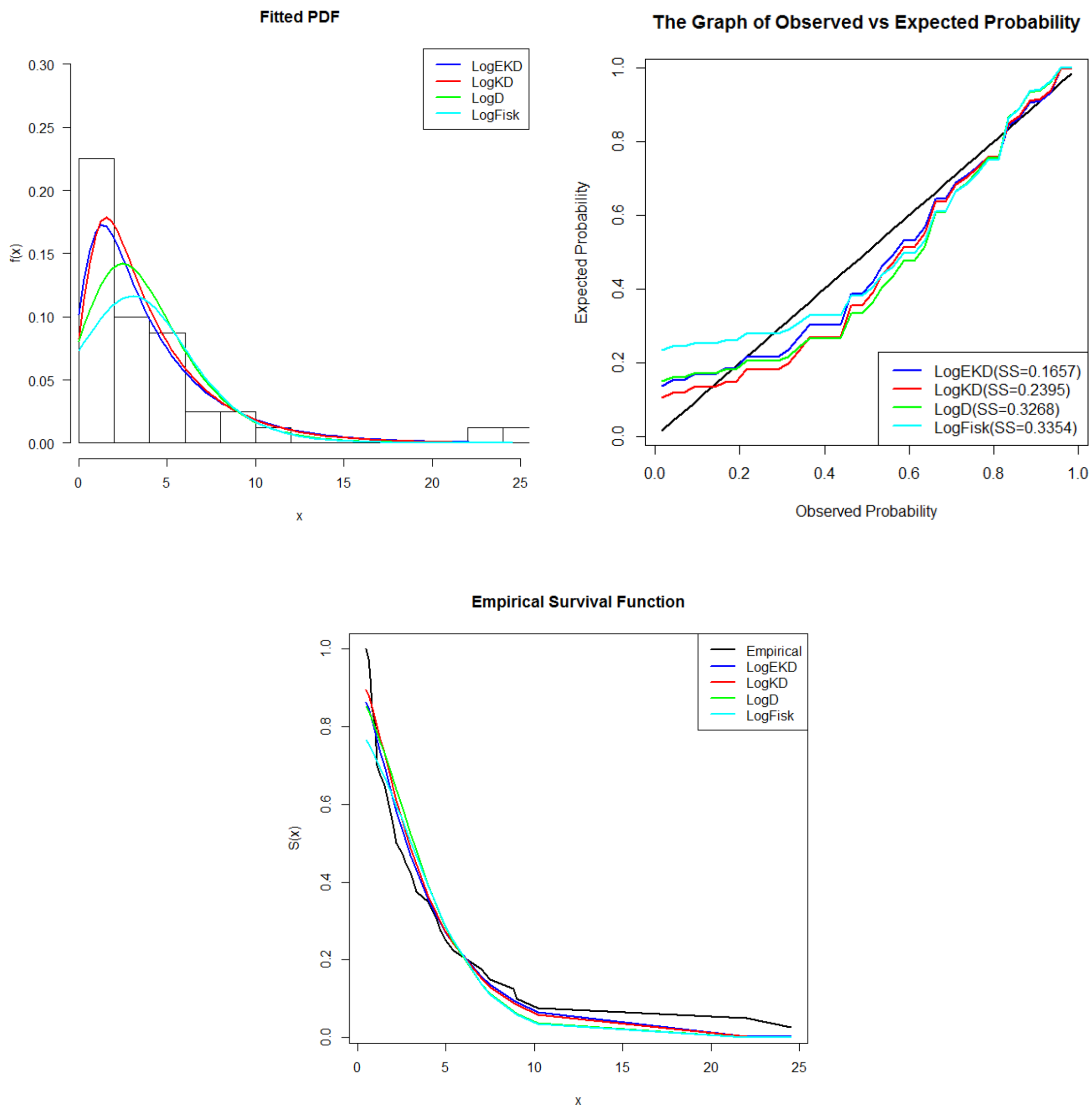

Figure 4.4: Log-EKD Fitted Densities, Observed Probabilities and Empirical Survival Curves for Active Repair Times

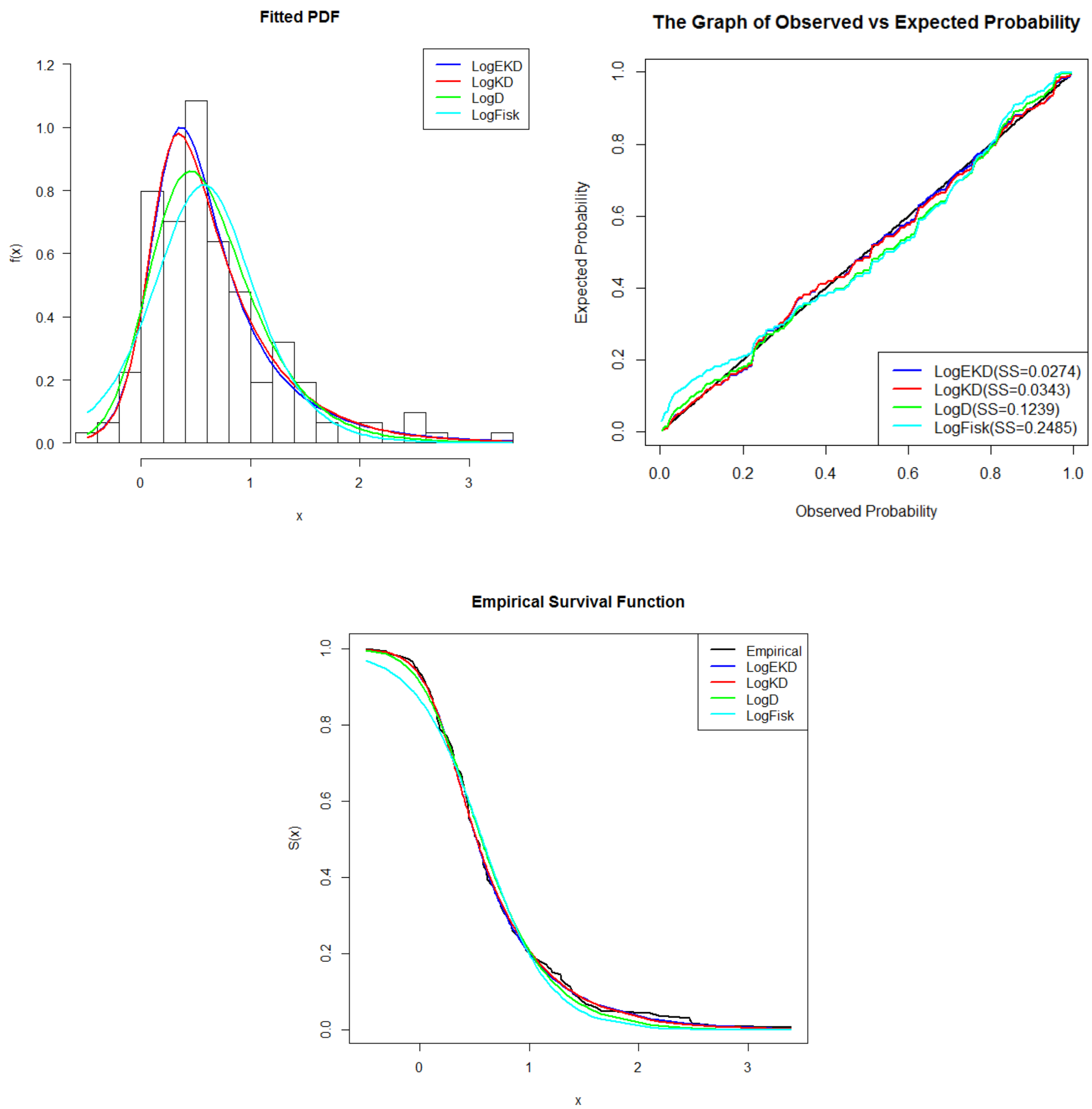

Figure 4.5: Log-EKD Fitted Densities, Observed Probabilities and Empirical Survival Curves for INPC Data

CHAPTER 5

THE MCDONALD LOG-LOGISTIC DISTRIBUTION

5.1 Introduction

The log-logistic distribution is a very useful distribution with applications in several areas including survival analysis, hydrology and economics. There are several generalizations of this distribution including the beta log-logistic, presented by Lamonte [20] following the generator approach introduced by Eugene et al. [9]. In this note, we present the McDonald log-logistic distribution and its statistical properties with applications to lifetime data. Some McDonald generalized distributions in the literature include work by Cordeiro et al. [4] on the McDonald extended distributions generalizing the exponential, generalized exponential, Kumaraswamy exponential and beta exponential distributions, and the McDonald Normal distribution by Cordeiro et al. [2]. Recently, Merovci and Elbatal [24] generalized the Sarhan and Zaindin [34] three parameters modified Weibull distribution via the McDonald distribution. See references therein for additional results. More recently, Cordeiro et al. [3] developed the five parameters McDonald Weibull distribution and applied the log-McDonald Weibull regression model to censored data.

In this chapter, we present the McDonald log-logistic distribution, which is more flexible distribution than the log-logistic and beta log-logistic distributions, and show that it is an appealing alternative to several lifetime models including the gamma, McDonald Weibull and its sub-models including the beta Weibull distribution.

The log-logistic distribution, also called Fisk distribution, has the pdf and cdf:

$$G_{_{LLog}}(x) = (1 + \lambda x^{-\delta})^{-1}, \tag{5.1}$$

and

$$g_{LLog}(x) = \frac{\lambda\delta x^{-\delta-1}}{(1+\lambda x^{-\delta})^2}, \tag{5.2}$$

respectively. The r^{th} moments for $r < \delta$ is given by

$$E\left(X^r\right) = \lambda^{\frac{r}{\delta}} B\left(1+\frac{r}{\delta}, 1-\frac{r}{\delta}\right), \tag{5.3}$$

where $B(\cdot,\cdot)$ is the beta function.

This chapter is organized as follows. In section 5.2, we obtain the McLLog distribution and discuss some of its statistical and mathematical properties. In section 5.3, moments and moment generating function of the McLLog distribution are presented. Bonferroni and Lorenz curves are obtained in section 5.4. The mean and median deviations are given in section 5.5. Section 5.6 contains results on the distribution of the order statistics and measures of uncertainty. Estimation of model parameters via the method of maximum likelihood is given in section 5.7. In Section 5.8, various simulations are conducted for different sample sizes. Section 5.9 contains examples and applications of the McLLog distribution and its sub-models, followed by concluding remarks.

5.2 The McDonald Log-logistic Distribution

In this section, we consider a generalization of the Log-logistic distribution via McDonald distributions.

5.2.1 The McDonald Log-logistic Distribution

Let G denotes a baseline cdf of a random variable X, then the class of Mc-G distribution is defined as

$$F_{McG}(x) = I_{G^c(x)}(ac^{-1}, b) = \frac{1}{B(ac^{-1}, b)} \int_0^{G^c(x)} w^{ac^{-1}-1}(1-w)^{b-1} dw, \tag{5.4}$$

for $a, b, c > 0$, where $I_y(a,b) = \frac{B_y(a,b)}{B(a,b)}$ is the incomplete beta function ratio, (Gradshteyn and Ryzhik [13]). $B_y(a,b) = \int_0^y w^{a-1}(1-w)^{b-1}dw$ is the incomplete beta function, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function, and $\Gamma(.)$ is the gamma function. The pdf corresponding to equation (5.4) is of the form

$$f_{McG}(x) = \frac{cg(x)}{B(ac^{-1},b)} G^{a-1}(x)[1-G^c(x)]^{b-1}. \tag{5.5}$$

Note that $g(x)$ is tractable when the cdf $G(x)$ and its pdf $g(x) = dG(x)/dx$ have simple analytic forms. The additional shape parameters allows for the introduction of skewness and variation of tail weight. When $c = 1$, we obtain the beta log-logistic distribution and when $a = c$, the resulting distribution is the Kumaraswamy generalization. We define the McLLog distribution by taking $G(x) = G_{LLog}(x)$ to be the LLog distribution in equation (5.1). The McLLog cdf and pdf are therefore given by

$$\begin{aligned} F_{McLLog}(x;a,b,c,\lambda,\delta) &= \frac{1}{B(ac^{-1},b)} \int_0^{(1+\lambda x^{-\delta})^{-c}} \omega^{ac^{-1}-1}(1-\omega)^{b-1}d\omega \\ &= I_{(1+\lambda x^{-\delta})^{-c}}(ac^{-1},b), \end{aligned} \tag{5.6}$$

and

$$f_{McLLog}(x;a,b,c,\lambda,\delta) = \frac{c\lambda\delta}{B(ac^{-1},b)} x^{-\delta-1}(1+\lambda x^{-\delta})^{a-1}[1-(1+\lambda x^{-\delta})^{-c}]^{b-1}, \tag{5.7}$$

for $a, b, c, \lambda, \delta > 0$ and $x > 0$, respectively. The quantile function is

$$x_q = \left\{ \left[\left(I_q^{-1}(ac^{-1},b) \right)^{-\frac{1}{c}} - 1 \right] \frac{1}{\lambda} \right\}^{-\frac{1}{\delta}}. \tag{5.8}$$

Graphs of the pdf of McLLog distribution are given in Figure 5.1 for selected values of the parameters. The plots show that the McLLog pdf can be decreasing or right skewed. The distribution has positive asymmetry.

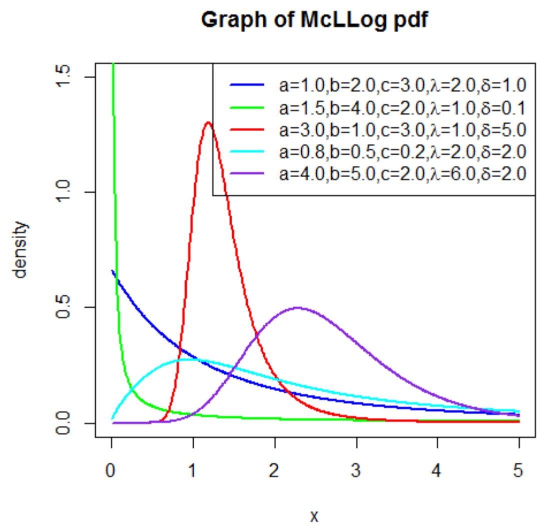

Figure 5.1: McLLog Density Functions

5.2.2 Sub-models

In this section, some sub-models of the McLLog distribution are presented.

① When $c = 1$ we obtain the beta log-logistic (BLLog) distribution with cdf

$$F_{BLLog}(x) = I_{(1+\lambda x^{-\delta})^{-1}}(a, b),$$

for $a, b, \lambda, \delta > 0$ and $x > 0$.

② When $b = c = 1$, we get exponentiated log-logistic (ELLog) or Dagum distribution with cdf

$$F_{ELLog}(x) = (1 + \lambda x^{-\delta})^{-a},$$

for $a, \lambda, \delta > 0$ and $x > 0$.

③ When $a = b = c = 1$, the log-logistic (LLog) distribution is obtained with the cdf

$$F_{LLog}(x) = (1 + \lambda x^{-\delta})^{-1},$$

for $\lambda, \delta > 0$ and $x > 0$.

5.2.3 Series Expansion

In this section , we apply the series expansion

$$(1 - z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j) j!} z^j, \tag{5.9}$$

for $b > 0$ and $|z| < 1$, to obtain a series representation of the McLLog pdf:

$$f_{McLLog}(x; a, b, c, \lambda, \delta) = \sum_{i=0}^{\infty} \omega(i) x^{-\delta-1} (1 + \lambda x^{-\delta})^{-a-ci-1}, \tag{5.10}$$

where $\omega(i) = \frac{(-1)^i \Gamma(b)}{\Gamma(b-i) i!} \cdot \frac{c\lambda\delta}{B(ac^{-1}, b)}$.

5.2.4 Hazard and Reverse Hazard Functions

The hazard and reverse hazard functions of the McLLog distribution are given by

$$\begin{aligned} h_{McLLog}(x; a, b, c, \lambda, \delta) &= \frac{f_{McLLog}(x; a, b, c, \lambda, \delta)}{1 - F_{McLLog}(x; a, b, c, \lambda, \delta)} \\ &= \frac{c\lambda\delta}{B(ac^{-1}, b)} x^{-\delta-1} (1 + \lambda x^{-\delta})^{-a-1} [1 - (1 + \lambda x^{-\delta})^{-c}]^{b-1} \\ &\times \left[1 - I_{(1+\lambda x^{-\delta})^{-c}}(ac^{-1}, b)\right]^{-1} \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} \tau_{McLLog}(x; a, b, c, \lambda, \delta) &= \frac{f_{McLLog}(x; a, b, c, \lambda, \delta)}{F_{McLLog}(x; a, b, c, \lambda, \delta)} \\ &= \frac{c\lambda\delta}{B(ac^{-1}, b)} x^{-\delta-1} (1 + \lambda x^{-\delta})^{-a-1} [1 - (1 + \lambda x^{-\delta})^{-c}]^{b-1} \\ &\times \left[I_{(1+\lambda x^{-\delta})^{-c}}(ac^{-1}, b)\right]^{-1}, \end{aligned} \tag{5.12}$$

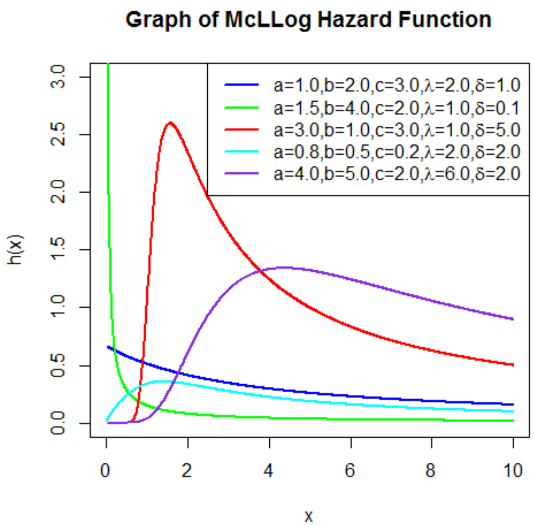

Figure 5.2: McLLog Hazard Functions

respectively. Plots of the hazard function for selected values of the parameters are given in Figure 5.2. The plot shows various shapes including monotonically decreasing, unimodal, and upside down bathtub shapes for five combinations of the values of the parameters. This flexibility makes the McLLog hazard rate function suitable for both monotonic and non-monotonic empirical hazard behaviors that are likely to be encountered in real life situations.

5.2.5 Mean Residual Life Function

The mean residual life function (MRLF) of a distribution F is given by $\mu(t) = E_F(X - t|X \geq t)$. Setting $u(x) = (1 + \lambda x^{-\delta})^{-1}$ in the series expansion of the McLLog pdf, the

MRLF of the McLLog distribution is given by

$$\begin{aligned}\mu(t) &= \frac{\int_t^{\infty}(x-t)f_{_{McLLog}}(x)dx}{\int_t^{\infty}f_{_{McLLog}}(x)dx}\\ &= \frac{\sum_{i=0}^{\infty}\omega(i)\int_t^{\infty}(x-t)x^{-\delta-1}(1+\lambda x^{-\delta})^{-a-ci-1}dx}{1-G_{_{McLLog}}(t;a,b,c,\lambda,\delta)}\\ &= \frac{\sum_{i=0}^{\infty}\omega(i)\mathit{\Delta}(t,i)}{1-G_{_{McLLog}}(t;a,b,c,\lambda,\delta)}, \end{aligned} \tag{5.13}$$

where

$$\begin{aligned}\mathit{\Delta}(t,i) &= \lambda^{\frac{1}{\delta}-1}\delta^{-1}\left[B\left(a+ci+\frac{1}{\delta},1-\frac{1}{\delta}\right)-B_{u(t)}\left(a+ci+\frac{1}{\delta},1-\frac{1}{\delta}\right)\right]\\ &- \frac{t}{\lambda\delta(a+ci)}\left[1-u(t)^{a+ci}\right].\end{aligned}$$

5.3 Moments and Moment Generating Function

In this section, we present the moments and moment generating function of the McLLog distribution. The s^{th} non-central moment of the McLLog distribution is given by

$$\begin{aligned}E(X^s) &= \int_0^{\infty}x^s\cdot\sum_{i=0}^{\infty}\omega(i)x^{-\delta-1}(1+\lambda x^{-\delta})^{-a-ci-1}dx\\ &= \sum_{i=0}^{\infty}\omega(i)\lambda^{\frac{s}{\delta}-1}\delta^{-1}B\left(a+ci+\frac{s}{\delta},1-\frac{s}{\delta}\right)\\ &= \sum_{i=0}^{\infty}\omega(s,i)B\left(a+ci+\frac{s}{\delta},1-\frac{s}{\delta}\right),\end{aligned} \tag{5.14}$$

for $s<\delta$, where $\omega(s,i)=\frac{(-1)^i\Gamma(b)}{\Gamma(b-i)i!}\cdot\frac{c\lambda^{\frac{s}{\delta}}}{B(ac^{-1},b)}$.

The moment generating function of the McLLog distribution is given by

$$M(t)=\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\frac{t^r}{r!}\omega(r,i)B\left(a+ci+\frac{r}{\delta},1-\frac{r}{\delta}\right), \tag{5.15}$$

for $r<\delta$.

5.4 Bonferroni and Lorenz Curves

In this section, inequality measures, namely Bonferroni and Lorenz curves, are presented. Bonferroni and Lorenz curves have played basic roles, for example, in the analysis of income concentration and earning inequality.

Let $I(q) = \int_0^q x \cdot f_{_{McLLog}}(x)dx$, $\nu = E(X)$ and $q = F^{-1}(p)$, then Bonferroni and Lorenz curves are given by

$$B(p) = \frac{I(q)}{p\nu} \quad \text{and} \quad L(p) = \frac{I(q)}{\nu},$$

respectively. Letting $u(x) = (1 + \lambda x^{-\delta})^{-1}$, then

$$I(q) \;=\; \sum_{i=0}^{\infty} \omega(1, i) B_{u(q)}\left(a + ci + \frac{1}{\delta}, 1 - \frac{1}{\delta}\right), \tag{5.16}$$

for $\delta > 1$. Also, the mean ν of the McLLog distribution is

$$\nu = \int_0^{\infty} x \cdot g(x)dx = \sum_{i=0}^{\infty} \omega(1, i) B\left(a + ci + \frac{1}{\delta}, 1 - \frac{1}{\delta}\right), \tag{5.17}$$

for $\delta > 1$.

Graphs of Bonferroni and Lorenz curves for selected values of the model parameters are given in Figure 5.3.

5.5 Mean and Median Deviations

If X has the McLLog distribution, we can derive the mean deviation about the mean $\nu = E(X)$ and the median deviation about the median M from

$$\delta_1 = \int_0^{\infty} |x - \nu| \, f_{_{McLLog}}(x)dx \quad \text{and} \quad \delta_2 = \int_0^{\infty} |x - M| \, f_{_{McLLog}}(x)dx,$$

respectively. The mean ν is obtained in equation (5.17) and the median M can be given by $x_{_{0.5}}$ from equation (5.8). Also, these measures can be calculated by the following relationships

$$\delta_1 = 2\nu \, F_{_{McLLog}}(\nu) - 2\nu + 2T(\nu) \quad \text{and} \quad \delta_2 = 2T(M) - \nu,$$

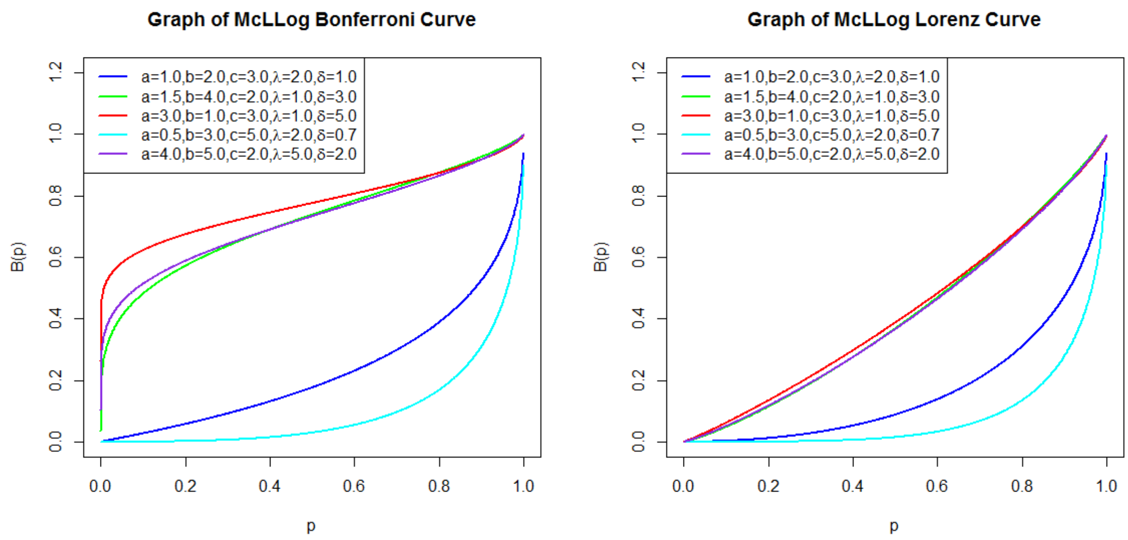

Figure 5.3: McLLog Bonferroni and Lorenz Curves

where $T(a) = \int_a^\infty x \cdot f_{_{McLLog}}(x)dx$ follows from equation (5.16) as

$$T(a) = \sum_{i=0}^{\infty} \omega(1,i)\left[B\left(a+ci+\frac{1}{\delta}, 1-\frac{1}{\delta}\right) - B_{u(q)}\left(a+ci+\frac{1}{\delta}, 1-\frac{1}{\delta}\right)\right].$$

5.6 Order Statistics and Entropy

In this section, the distribution of the k^{th} order statistic and measures of uncertainty including Rényi entropy are presented.

5.6.1 Order Statistics

The pdf of the k^{th} order statistic from a pdf $f(x)$ is given by

$$\begin{aligned} f_{k:n}(x) &= \frac{f(x)}{B(k,n-k+1)}F^{k-1}(x)[1-F(x)]^{n-k} \\ &= k\binom{n}{k}f(x)F^{k-1}(x)[1-F(x)]^{n-k}. \end{aligned}$$

Note that $\left[1-F(x)\right]^{n-k} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(n-k+1)}{\Gamma(n-k+1-j)j!}F^j(x)$, where $n-k+1>0$ and $0 \leqslant F(x) \leqslant 1$. The pdf of the k^{th} order statistic from the McLLog distribution is given by

$$\begin{aligned} f_{k:n}(x) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega(i)\frac{(-1)^j\Gamma(n-k+1)}{\Gamma(n-k+1-j)j!}k\binom{n}{k}x^{-\delta}(1+\lambda x^{-\delta})^{-a-ci-1} \\ &\times \left(I_{(1+\lambda x^{-\delta})^{-c}}(ac^{-1},b)\right)^{k+i-1}. \end{aligned}$$

5.6.2 Entropy

Rényi entropy is given by

$$I_R(\tau) = (1-\tau)^{-1}\log\left[\int_R f^{\tau}(x)dx\right],$$

for $\tau > 0$ and $\tau \neq 1$. Note that the τ^{th} power of the McLLog density has the series representation

$$\begin{aligned} f^{\tau}_{McLLog}(x) &= \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(b\tau-\tau+1)}{\Gamma(b\tau-\tau+1-i)i!}\left(\frac{c\lambda\delta}{B(ac^{-1},b)}\right)^{\tau} \\ &\times x^{-(\delta+1)\tau}(1+\lambda x^{-\delta})^{-a\tau-\tau-ci}. \end{aligned}$$

Let $u(x) = (1+\lambda x^{-\delta})^{-1}$, then Rényi entropy for the McLLog distribution is given by

$$\begin{aligned} I_R(\tau) &= (1-\tau)^{-1}\log\left\{\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(b\tau-\tau+1)}{\Gamma(b\tau-\tau+1-i)i!}\left(\frac{c}{B(ac^{-1},b)}\right)^{\tau}\lambda^{-\frac{\tau}{\delta}+\frac{1}{\delta}}\delta^{\tau-1}\right. \\ &\times \left. B\left(a\tau+ci-\frac{\tau}{\delta}+\frac{1}{\delta},\tau+\frac{\tau}{\delta}-\frac{1}{\delta}\right)\right\}. \qquad (5.18) \end{aligned}$$

5.7 Estimation of Model Parameters

In this section, we present estimates of the parameters of the McLLog distribution via the method of maximum likelihood estimation.

5.7.1 Maximum Likelihood Estimation

Let $\mathbf{x} = (x_1, \cdots, x_n)^T$ be a random sample of the McLLog distribution with unknown parameter vector $\boldsymbol{\Theta} = (a, b, c, \lambda, \delta)^T$. The log-likelihood function is

$$\begin{aligned} l(\boldsymbol{\Theta}) &= n(\ln c + \ln \lambda + \ln \delta - \ln B(ac^{-1}, b)) \\ &- (\delta+1)\sum_{i=1}^{n} \ln x_i - (a+1)\sum_{i=1}^{n} \ln(1+\lambda x_i^{-\delta}) \\ &+ (b-1)\sum_{i=1}^{n} \ln[1-(1+\lambda x_i^{-\delta})^{-c}]. \end{aligned} \tag{5.19}$$

The partial derivative of $l(\boldsymbol{\Theta})$ with respect to the parameters are:

$$\frac{\partial l}{\partial a} = -\frac{n}{c}\Big(\psi(ac^{-1}) - \psi(ac^{-1}+b)\Big) - \sum_{i=1}^{n} \ln(1+\lambda x_i^{-\delta}),$$

$$\frac{\partial l}{\partial b} = -n\Big(\psi(b) - \psi(ac^{-1}+b)\Big) + \sum_{i=1}^{n} \ln\left[1-(1+\lambda x_i^{-\delta})^{-c}\right],$$

$$\begin{aligned} \frac{\partial l}{\partial c} &= \frac{n}{c} + \frac{na}{c^2}\Big(\psi(ac^{-1}) - \psi(ac^{-1}+b)\Big) \\ &+ (b-1)\sum_{i=1}^{n}(1+\lambda x_i^{-\delta})^{-c}\ln(1+\lambda x_i^{-\delta})\left[1-(1+\lambda x_i^{-\delta})^{-c}\right]^{-1}, \end{aligned}$$

$$\begin{aligned} \frac{\partial l}{\partial \lambda} &= \frac{n}{\lambda} - (a+1)\sum_{i=1}^{n} x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-1} \\ &+ (b-1)c\sum_{i=1}^{n} x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-c-1}[1-(1+\lambda x_i^{-\delta})^{-c}]^{-1}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial l}{\partial \delta} &= \frac{n}{\delta} - \sum_{i=1}^{n} \ln x_i + (a+1)\lambda\sum_{i=1}^{n} x_i^{-\delta}\ln x_i(1+\lambda x_i^{-\delta})^{-1} \\ &- (b-1)c\lambda\sum_{i=1}^{n} x_i^{-\delta}\ln x_i(1+\lambda x_i^{-\delta})^{-c-1}[1-(1+\lambda x_i^{-\delta})^{-c}]^{-1}. \end{aligned}$$

The maximum likelihood estimates (MLEs) of the parameters a, b, c, λ and δ, says $\hat{a}, \hat{b}, \hat{c}, \hat{\lambda}$ and $\hat{\delta}$ are obtained by solving the following equations $\frac{\partial l}{\partial a} = \frac{\partial l}{\partial b} = \frac{\partial l}{\partial c} = \frac{\partial l}{\partial \lambda} = \frac{\partial l}{\partial \delta} = 0$. A numerical technique must be applied, since there is no closed form solution to these equations.

5.7.2 Asymptotic Confidence Intervals

In this section, asymptotic confidence intervals for the McLLog distribution are presented. Let $\hat{\mathbf{\Theta}} = (\hat{a}, \hat{b}, \hat{c}, \hat{\lambda}, \hat{\delta})$ be the MLE of $\mathbf{\Theta}$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) \xrightarrow{d} N_5(\underline{0}, I^{-1}(\mathbf{\Theta}))$, where $I(\mathbf{\Theta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Theta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$, that is $J(\hat{\mathbf{\Theta}})$. Elements of the observed information matrix are available from the authors upon request. The multivariate normal distribution $N_5(\underline{0}, J(\hat{\mathbf{\Theta}})^{-1})$, where the mean vector $\underline{0} = (0, 0, 0, 0, 0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. The approximate $100(1-\eta)\%$ two-sided confidence intervals for a, b, c, λ, and δ are given by:

$$\widehat{a} \pm Z_{\frac{\eta}{2}} \left[\widehat{Var(\hat{a})} \right]^{\frac{1}{2}}, \quad \widehat{b} \pm Z_{\frac{\eta}{2}} \left[\widehat{Var(\hat{b})} \right]^{\frac{1}{2}}, \quad \widehat{c} \pm Z_{\frac{\eta}{2}} \left[\widehat{Var(\hat{c})} \right]^{\frac{1}{2}},$$

$$\widehat{\lambda} \pm Z_{\frac{\eta}{2}} \left[\widehat{Var(\hat{\lambda})} \right]^{\frac{1}{2}}, \text{ and } \quad \widehat{\delta} \pm Z_{\frac{\eta}{2}} \left[\widehat{Var(\hat{\delta})} \right]^{\frac{1}{2}}$$

respectively, where $\widehat{Var(\cdot)}$ is the diagonal element of $J(\hat{\Theta})^{-1}$ for each parameter and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the McLLog distribution with its sub-models for a given data set. For example, to test $b = c = 1$, the LR statistic is

$$\omega = 2[\ln(L(\hat{a}, \hat{b}, \hat{c}, \hat{\lambda}, \hat{\delta})) - \ln(L(\tilde{a}, 1, 1, \tilde{\lambda}, \tilde{\delta}))],$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{\lambda}$ and $\hat{\delta}$ are the unrestricted estimates, and $\tilde{a}$, $\tilde{\lambda}$ and $\tilde{\delta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_d$, where χ^2_d denote the upper 100d% point of the χ^2 distribution with 2 degrees of freedom.

5.8 Simulation Study

In this section, we examine the performance of the McLLog distribution by conducting various simulations for different sample sizes. We simulate 1000 samples for the true parameters values $I : a = 2, b = 1, c = 3, \lambda = 4, \delta = 2$ and $II : a = 1, b = 1, c = 1, \lambda = 1, \delta = 1$. Table 5.1 lists the mean MLEs of the five parameters along with their respective root mean squared errors (RMSE) for sample sizes $n = 50, n = 100, n = 200$ and $n = 400$. For a parameter θ and its estimate $\hat{\theta}$, RMSE is given by

$$RMSE(\theta) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\hat{\theta} - \theta)^2}.$$

In our case, $n = 1000$ is sample number. From the results Table 5.1, we can verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

5.9 Applications

In this section, we present applications of McLLog distribution to real data, as well as comparison of the distribution with its sub-models, Weibull, McDonald Weibull (McW), beta Weibull (BW) and gamma distributions.The cdfs of these distributions are given by

$$F_{Weibull}(x; k, \lambda) = 1 - e^{-(x/\lambda)^k},$$

Table 5.1: McLLog Monte Carlo Simulation Results

		I		II	
n	Parameter	Mean	RMSE	Mean	RMSE
50	a	2.5366677	0.953764594	1.2147011	0.591153618
	b	1.7247068	1.63386407	1.2435729	0.742338198
	c	2.541612	0.971367129	0.8496807	0.463244428
	λ	3.8753448	0.813458604	1.1902982	0.723070467
	δ	1.960098	0.717810351	1.1232576	0.517693346
100	a	2.273306	0.612963213	1.1549044	0.495121601
	b	1.3439854	1.060377008	1.1414762	0.566895581
	c	2.7627228	0.663192506	0.8665471	0.384943243
	λ	3.8767538	0.621510257	1.0991431	0.534468615
	δ	1.9754614	0.488438123	1.0789402	0.387445867
200	a	2.1230394	0.369084679	1.1051537	0.411659932
	b	1.1425864	0.562945823	1.0876159	0.419099869
	c	2.8907186	0.416792274	0.8956349	0.321894237
	λ	3.9543896	0.518317277	1.0643899	0.383613999
	δ	1.990635	0.320518642	1.0536146	0.304602528
400	a	2.0636928	0.222489326	1.0800026	0.338342282
	b	1.0607148	0.28944084	1.0701165	0.341335758
	c	2.9485626	0.230661657	0.9186915	0.275012545
	λ	3.9463409	0.397750047	1.0426499	0.269199554
	δ	1.9905656	0.212761134	1.0278817	0.236813429

$$F_{McW}(x;k,\lambda,a,b,c) = \frac{1}{B(ac^{-1},b)} \int_0^{F^c_{Weibull}(x;k,\lambda)} w^{ac^{-1}-1}(1-w)^{b-1}dw,$$

$$F_{BW}(x;k,\lambda,a,b) = \frac{1}{B(a,b)} \int_0^{F_{Weibull}(x;k,\lambda)} w^{a-1}(1-w)^{b-1}dw,$$

and

$$F_{Gamma}(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)}\gamma(\alpha,\beta x),$$

respectively.

The maximum likelihood estimates (MLEs) of the parameters are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2 log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\boldsymbol{\Theta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters for the McLLog distribution and its sub-distributions. A measure of closeness of the plot to the diagonal line given by the sum of squares

$$SS = \sum_{j=1}^{n}\left[F(x_{(j)}) - \left(\frac{j-0.375}{n+0.25}\right)\right]^2$$

was calculated for each plot. The plot with the smallest SS corresponds to the model with points that are closer to the diagonal line. We also obtained Kolmogorov-Smirnov (KS) statistic (smaller is better) for each model, where

$$KS = \max_{1\le i\le n}\left\{F(x_i) - \frac{i-1}{n}, \frac{i}{n} - F(x_i)\right\}.$$

The first example represents the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang [21], where 9 censored observations have been removed. Initial values for McLLog model in SAS code are $a = 1, b = 1, c = 2, \lambda = 0.023$ and $\delta = 1$. Estimates of the parameters, AIC, AICC,

Table 5.2: McLLog Estimation for Remission Times Data

	Estimates					Statistics					
Distribution	a	b	c	λ	δ	-2 Log Likelihood	AIC	AICC	BIC	SS	KS
McLLog	3.7812	51.8809	1.2523	25.3143	0.505	821.4	831.4	831.9	845.6	0.0374	0.0453
	(0.2105)	(0.09516)	(0.04213)	(3.3147)	(0.03542)						
BLLog	63.2938	342.68	1	6.8431	0.1299	827.3	835.3	835.6	846.7	0.0843	0.0549
	(0.1879)	(0.1148)	-	(0.1365)	(0.008203)						
ELLog	1.1892	1	1	12.1923	1.5563	825.8	831.8	832.0	840.3	0.0703	0.0565
	(0.2385)	-	-	(5.6362)	(0.1376)						
LLog	1	1	1	11.2011	1.5048	830.2	834.2	834.3	839.9	0.5669	0.1141
	-	-	-	(2.0498)	(0.0953)						
	k	λ									
Weibull	1.0478	9.5607				828.2	832.2	832.3	837.9	0.1499	0.0700
	(0.06758)	(0.8529)									
	k	λ	a	b	c						
McW	0.4755	3.823	3.9874	2.424	4.9933	821.1	831.1	831.6	845.4	0.0358	0.0448
	(0.05287)	(1.7312)	(1.1422)	(0.149)	(1.4181)						
	k	λ	a	b							
BW	0.6662	3.1098	2.7346	0.9082		821.4	829.4	829.7	840.8	0.0382	0.0449
	(0.06853)	(1.4446)	(0.5328)	(0.2885)							
	α	β									
Gamma	1.1725	0.1252				826.7	830.7	830.8	836.4	0.1326	0.0733
	(0.1308)	(0.01731)									

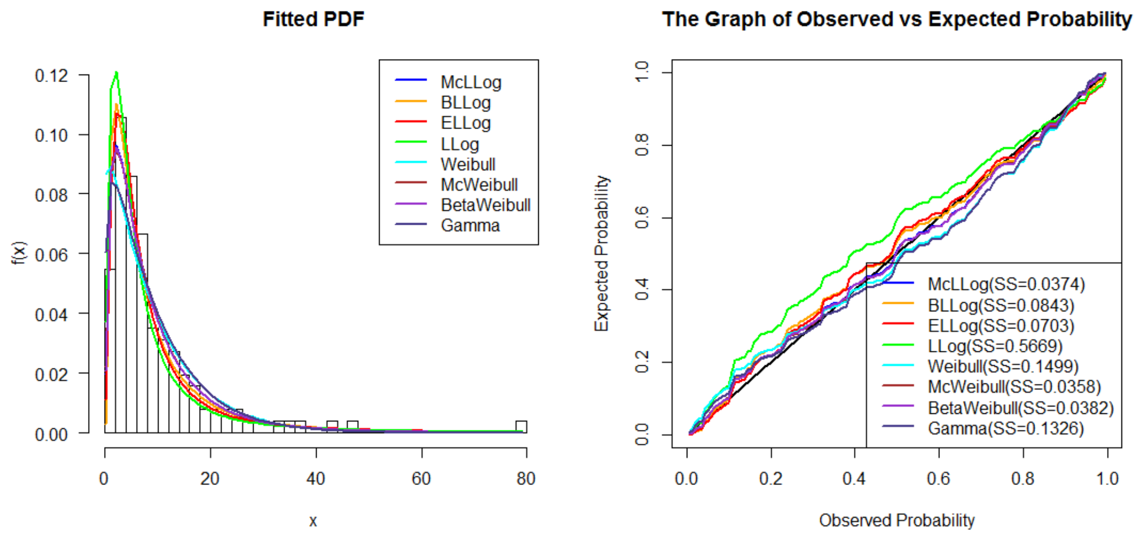

Figure 5.4: McLLog Fitted Densities, Observed Probabilities for Remission Times Data

BIC, SS and KS are given in Table 5.2. Plots of the fitted density and observed vs expected probability are given in Figure 5.4. The estimated covariance matrix is given by:

$$\begin{pmatrix} 0.04432 & -0.0323 & -0.00155 & -0.3684 & -0.00267 \\ -0.0323 & 0.009056 & -0.01946 & 1.0685 & 0.004104 \\ -0.00155 & -0.01946 & 0.001775 & -0.06236 & -0.00007 \\ -0.3684 & 1.0685 & -0.06236 & 10.9872 & 0.09017 \\ -0.00267 & 0.004104 & -0.00007 & 0.09017 & 0.001254 \end{pmatrix},$$

and the 95% confidence intervals for the model parameters are given by $a \in (3.7812 \pm 1.96 \times 0.2105)$, $b \in (51.889 \pm 1.96 \times 0.09516)$, $c \in (1.2523 \pm 1.96 \times 0.04213)$, $\lambda \in (25.3143 \pm 1.96 \times 3.3147)$, and $\delta \in (0.505 \pm 1.96 \times 0.03542)$, respectively.

The LR test statistics of the hypothesis H_0 : BLLog against H_a : McLLog, H_0 : LLog against H_a : McLLog are 5.9 (p-value=0.015) and 8.8 (p-value=0.032). The

Table 5.3: McLLog Estimation for Price of Cars Data

	Estimates					Statistics					
Distribution	a	b	c	λ	δ	-2 Log Likelihood	AIC	AICC	BIC	SS	KS
McLLog	27.1372	3.3695	1.1617	0.4129	1.2703	1489.0	1499.0	1499.1	1519.3	0.0270	0.0196
	(0.739)	(0.1111)	(0.02272)	(0.02745)	(0.04871)						
BLLog	20.143	3.0866	1	0.5537	1.3504	1489.1	1497.1	1497.2	1513.4	0.0271	0.0199
	(0.5018)	(0.1533)	-	(0.03197)	(0.05834)						
ELLog	2.466	1	1	6.2724	2.917	1493.9	1499.9	1500.0	1512.1	0.0503	0.0279
	(0.4929)	-	-	(2.333)	(0.1564)						
LLog	1	1	1	44.1835	3.6393	1505.6	1509.6	1509.7	1517.8	0.0598	0.0254
	-	-	-	(7.615)	(0.1461)						
	k	λ									
Weibull	1.839	3.712				1638.4	1642.4	1642.4	1650.5	1.7156	0.0989
	(0.05965)	(0.1037)									
	k	λ	a	b	c						
McW	0.4667	0.8049	26.8973	5.096	0.001932	1511.8	1521.8	1522	1542.1	0.1630	0.0382
	(0.02209)	(0.1235)	(2.3193)	(0.04712)	(0.000164)						
	k	λ	a	b							
BW	0.3188	0.007203	722.63	1.2263		1488.7	1496.7	1496.8	1513	0.0302	0.0196
	(0.01198)	(0.001562)	(5.4931)	(0.02912)							
	α	β									
Gamma	4.0703	1.2419				1555.4	1559.4	1559.5	1567.6	0.6973	0.0688
	(0.2676)	(0.08691)									

McLLog distribution is significantly better for this data set than the BLLog and LLog models. The McLLog distribution is comparable to the McW distribution with the values of the statistics AIC, AICC, BIC, SS and KS just about the same for both McLLog and McW distributions. The McLLog distribution provides a very good fit to the histogram. The probability plot also shows the adequacy of the model for the data set.

The second example consists of price of 428 new vehicles for the 2004 year (Kiplinger's Personal Finance, Dec 2003). Initial values for McLLog model in SAS code are $a = 18, b = 2.9, c = 0.7, \lambda = 0.01, \delta = 1.38$. The MLE, AIC, AICC, BIC, SS and KS statistics are given in Table 5.3. Plots of the fitted density and observed vs

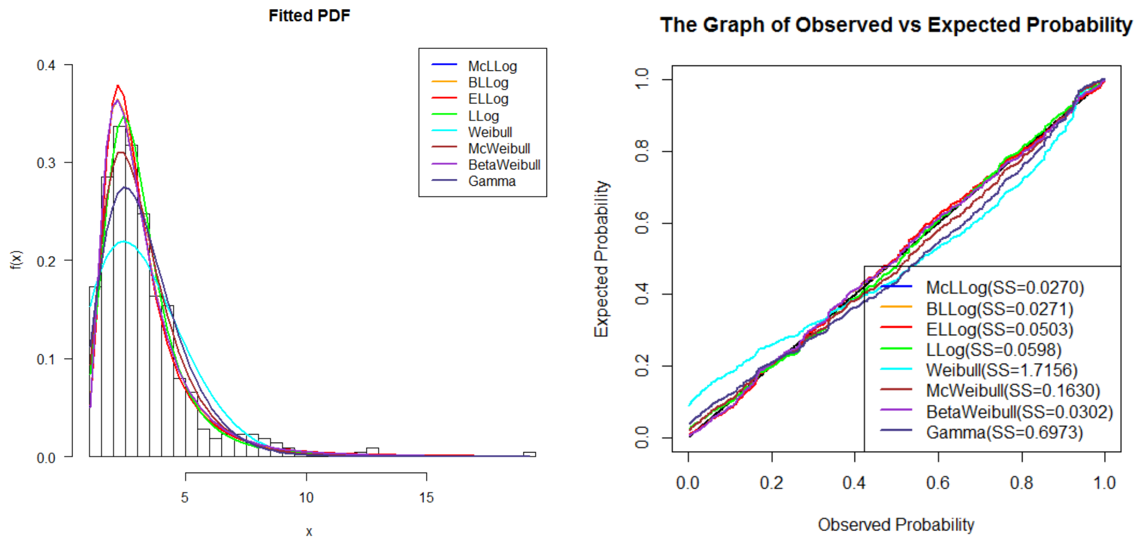

Figure 5.5: McLLog Fitted Densities, Observed Probabilities for Price of Cars Data

expected probability are given in Figure 5.5.

McLLog distribution gives a little smaller -2 log likelihood value than BLLog distribution, but the LR test statistics of the hypothesis H_0 : BLLog against H_a : McLLog is 0.1 (p-value=0.7518). Consequently, the McLLog distribution is not significantly better than BLLog distribution. BLLog distribution gives comparatively smaller AIC, AICC and BIC values than McLLog and ELLog distributions, while McLLog distribution gives smallest SS and KS statistics. We conclude that BLLog distribution is a good model for this data set.

5.10 Concluding Remarks

A new multi-parameter distribution called the McDonald Log-logistic (McLLog) distribution, is proposed and studied in detail. This class of distributions contains a number of distributions with potential applications to a wide area of economics, finance, reliability and medicine. Properties of the class of McLLog distributions including the pdfs, cdfs, moments, hazard function, reverse hazard function, mean residual life function, Bonferroni and Lorenz curves, Rényi entropy are derived. A simulation study is carried out to examine the performance of the McLLog distribution, via the root mean square error of the maximum likelihood estimators of the model parameters. Maximum likelihood estimates of the parameters and applications are presented to illustrate the usefulness and applicability of the proposed model and its sub-models.

CHAPTER 6

THE GAMMA-DAGUM DISTRIBUTION

6.1 Introduction

Kleiber [17] traced the genesis of Dagum distribution and summarized several statistical properties of this distribution. Domma et al. [7] obtained the maximum likelihood estimates of the parameters of Dagum distribution from censored samples. Dagum [6] distribution is a special case of generalized beta distribution of the second kind (GB2), McDonald [22], McDonald and Xu [23], when the parameter $q = 1$, where the probability density function (pdf) of the GB2 distribution is given by:

$$f_{GB2}(y; a, b, p, q) = \frac{ay^{ap-1}}{b^{ap}B(p,q)[1 + (\frac{y}{b})^a]^{p+q}} \quad \text{for } y > 0. \tag{6.1}$$

Note that $a > 0, p > 0, q > 0$, are the shape parameters and $b > 0$ is the scale parameter and $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is the beta function. Domma and Condino [8] obtained statistical properties of the beta-Dagum distribution. The pdf and cumulative distribution function (cdf) of Dagum distribution are given by:

$$f_D(y; \lambda, \beta, \delta) = \beta\lambda\delta y^{-\delta-1}(1 + \lambda y^{-\delta})^{-\beta-1}, \tag{6.2}$$

and

$$F_D(y; \lambda, \beta, \delta) = (1 + \lambda y^{-\delta})^{-\beta}, \quad y > 0, \lambda, \beta, \delta > 0, \tag{6.3}$$

respectively. The hazard and the reverse hazard functions are given by:

$$h_D(y; \lambda, \beta, \delta) = \frac{f_D(y; \lambda, \beta, \delta)}{\overline{F}_D(y; \lambda, \beta, \delta)} = \frac{\beta\lambda\delta y^{-\delta-1}(1 + \lambda y^{-\delta})^{-\beta-1}}{1 - (1 + \lambda y^{-\delta})^{-\beta}}, \tag{6.4}$$

and

$$\tau_D(y; \lambda, \beta, \delta) = \beta\lambda\delta y^{-\delta-1}(\lambda y^{-\delta} + 1)^{-1}, \tag{6.5}$$

respectively. The k^{th} raw or non-central moments are:

$$E(Y^k) = E(Y^k|\beta, \delta, \lambda) = \beta\lambda^{\frac{k}{\delta}} B\left(\beta + \frac{k}{\delta}, 1 - \frac{k}{\delta}\right), \quad \text{for } \delta > k.$$

Motivated by the various applications of Dagum distribution in finance and actuarial sciences, as well as in economics, where Dagum distribution plays an important role in size distribution of personal income, we construct a new class of Dagum-type distribution called the gamma-Dagum (GD) distribution and apply the model to real lifetime data.

For any baseline cdf $F(x)$, and $x \in \mathbf{R}$, Zografos and Balakrishnan [36] defined the distribution (when $\theta = 1$) with pdf $g(x)$ and cdf $G(x)$ (for $\alpha > 0$) as follows

$$g(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}[-\log(\overline{F}(x))]^{\alpha-1}(1-F(x))^{(1/\theta)-1}f(x), \tag{6.6}$$

and

$$G(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^{-\log(\overline{F}(x))} t^{\alpha-1}e^{-t/\theta}dt = \frac{\gamma(-\theta^{-1}\log(\overline{F}(x)),\alpha)}{\Gamma(\alpha)}, \tag{6.7}$$

respectively, where $g(x) = dG(x)/dx$, $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt$ denotes the gamma function, and $\gamma(z,\alpha) = \int_0^{z} t^{\alpha-1}e^{-t}dt$ denotes the incomplete gamma function. The corresponding hazard rate function (hrf) is

$$h_G(x) = \frac{[-\log(1-F(x))]^{\alpha-1}f(x)(1-F(x))^{(1/\theta)-1}}{\theta^{\alpha}(\Gamma(\alpha)-\gamma(-\theta^{-1}\log(1-F(x)),\alpha))}. \tag{6.8}$$

The class of distributions for the special case of $\theta = 1$, is referred to as the ZB-G family of distributions. Also, (when $\theta = 1$), Ristić and Balakrishnan [33] proposed an alternative gamma-generator defined by the cdf and pdf

$$G_2(x) = 1 - \frac{1}{\Gamma(\alpha)\theta^{\alpha}}\int_0^{-\log F(x)} t^{\alpha-1}e^{-t/\theta}dt, \quad \alpha > 0, \tag{6.9}$$

and

$$g_2(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}[-\log(F(x))]^{\alpha-1}(F(x))^{(1/\theta)-1}f(x), \tag{6.10}$$

respectively.

In this chapter, we consider the generalized family of distributions given in equation (6.6) via Dagum distribution. Zografos and Balakrishnan [36] motivated the

ZB-G model as follows. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be lower record values from a sequence of independent and identically distributed (i.i.d.) random variables from a population with pdf $f(x)$. Then, the pdf of the n^{th} upper record value is given by equation (6.6), when $\theta = 1$. A logarithmic transformation of the parent distribution F transforms the random variable X with density (6.6) to a gamma distribution. That is, if X has the density (6.6), then the random variable $Y = -\log[1 - F(X)]$ has a gamma distribution $GAM(\alpha; 1)$ with density $k(y; \alpha) = \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}$, $y > 0$. The opposite is also true, if Y has a gamma $GAM(\alpha; 1)$ distribution, then the random variable $X = G^{-1}(1 - e^{-Y})$ has a ZB-G distribution (Zografos and Balakrishnan [36]). In addition to the motivations provided by Zografos and Balakrishnan [36], we are interested in the generalization of the Dagum distribution via the gamma-generator and establishing the relationship between the distributions in equations (6.6) and (6.10), and weighted distributions in general.

Weighted distribution provides an approach to deal with model specification and data interpretation problems. It adjusts the probabilities of actual occurrence of events to arrive at a specification of the probabilities when those events are recorded. Fisher [12] first introduced the concept of weighted distribution, in order to study the effect of ascertainment upon estimation of frequencies. Rao [32] unified concept of weighted distribution and use it to identify various sampling situations. Cox [5] and Zelen [37] introduced weighted distribution to present length biased sampling. Patil [29] used weighted distribution as stochastic models in the study of harvesting and predation. The usefulness and applications of weighted distribution to biased samples in various areas including medicine, ecology, reliability, and branching processes can also be seen in Nanda and Jain [25], Gupta and Keating [15], Oluyede [26] and in references therein.

Suppose Y is a non-negative random variable with its natural pdf $f(y; \underline{\theta})$, where

$\underline{\theta}$ is a vector of parameters, then the pdf of the weighted random variable Y^w is given by:

$$f^w(y;\underline{\theta},\underline{\beta}) = \frac{w(y,\underline{\beta})f(y;\underline{\theta})}{\omega}, \tag{6.11}$$

where the weight function $w(y,\underline{\beta})$ is a non-negative function, that may depend on the vector of parameters $\underline{\beta}$, and $0 < \omega = E(w(Y,\underline{\beta})) < \infty$ is a normalizing constant. In general, consider the weight function $w(y)$ defined as follows:

$$w(y) = y^k e^{ly} F^i(y) \overline{F}^j(y). \tag{6.12}$$

Setting $k = 0$; $k = j = i = 0$; $l = i = j = 0$; $k = l = 0$; $i \to i-1$; $j = n-i$; $k = l = i = 0$ and $k = l = j = 0$ in this weight function, one at a time, implies probability weighted moments, moment-generating functions, moments, order statistics, proportional hazards and proportional reversed hazards, respectively, where $F(y) = P(Y \leq y)$ and $\overline{F}(y) = 1 - F(y)$. If $w(y) = y$, then $Y^* = Y^w$ is called the size-biased version of Y.

Ristić and Balakrishnan [33], provided motivations for the new family of distributions given in equation (6.9) when $\theta = 1$, that is for $n \in N$, equation (6.9) above is the pdf of the n^{th} lower record value of a sequence of i.i.d. variables from a population with density $f(x)$. Ristić and Balakrishnan [33] used the exponentiated exponential (EE) distribution with cdf $F(x) = (1 - e^{-\beta x})^\alpha$, where $\alpha > 0$ and $\beta > 0$, in equation (6.10) to obtained and study the gamma-exponentiated exponential (GEE) model. See references therein for additional results on the GEE model. Pinho et al. [30] presented the statistical properties of the gamma-exponentiated Weibull distribution. In this note, we obtain a natural extension for Dagum distribution, which we call the gamma-Dagum (GD) distribution.

Note that if $\theta = 1$ and $\alpha = n + 1$, in equation (6.6), we obtain the cdf and pdf of the upper record values U given by

$$F_U(u) = \frac{1}{n!}\int_0^{-\log(1-G(u))} y^n e^{-y} dy, \tag{6.13}$$

and

$$f_U(u) = g(u)[-\log(1 - G(u))]^n/n!. \tag{6.14}$$

Similarly, from equation (6.10), the pdf of the lower record values is given by

$$f_L(t) = g(t)[-\log(G(t))]^n/n!. \tag{6.15}$$

This chapter is organized as follows. In section 6.2, some basic results, the gamma-Dagum (GD) distribution, series expansion and its sub-models, hazard and reverse hazard functions and the quantile function are presented. The moments and moment generating function, mean and median deviations are given in section 6.3. Section 6.4 contains some additional useful results on the distribution of order statistics and Rényi entropy. In section 6.5, results on the estimation of the parameters of the GD distribution via the method of maximum likelihood are presented. Applications are given in section 6.6, followed by concluding remarks.

6.2 The Gamma-Dagum Distribution

In this section, the GD distribution, series expansion of its pdf and some sub-models are presented.

6.2.1 The Gamma-Dagum Distribution

By inserting Dagum distribution in equation (6.7), the cdf $G_{GD}(x) = G(x)$ of the GD distribution is obtained as follows:

$$\begin{aligned} G_{GD}(x) &= \frac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{-\log[1-(1+\lambda x^{-\delta})^{-\beta}]} t^{\alpha-1} e^{-t/\theta} dt \\ &= \frac{\gamma(-\theta^{-1}\log[1-(1+\lambda x^{-\delta})^{-\beta}], \alpha)}{\Gamma(\alpha)}, \end{aligned} \tag{6.16}$$

where $x > 0$, $\lambda > 0$, $\beta > 0$, $\delta > 0$, $\alpha > 0$, $\theta > 0$, and $\gamma(x, \alpha) = \int_0^x t^{\alpha-1}e^{-t}dt$ is the lower incomplete gamma function. The corresponding GD pdf is given by

$$\begin{aligned} g_{GD}(x) &= \frac{\lambda\beta\delta x^{-\delta-1}}{\Gamma(\alpha)\theta^{\alpha}}(1+\lambda x^{-\delta})^{-\beta-1}\Big(-\log[1-(1+\lambda x^{-\delta})^{-\beta}]\Big)^{\alpha-1} \\ &\times \ [1-(1+\lambda x^{-\delta})^{-\beta}]^{(1/\theta)-1}. \end{aligned} \tag{6.17}$$

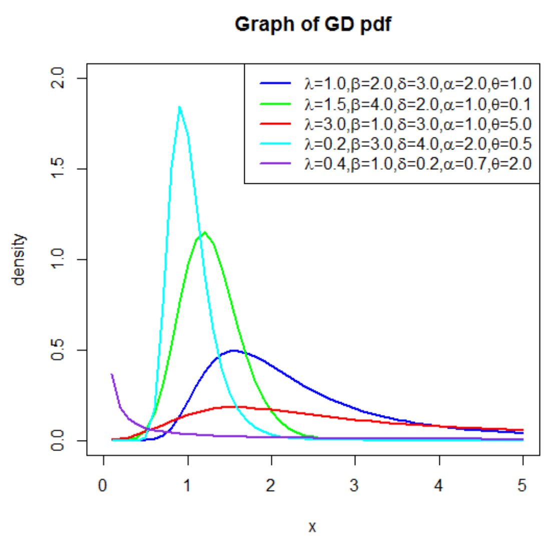

Figure 6.1: GD Density Functions

The graph of the GD pdf can be seen in Figure 6.1. The plots indicate that the GD distribution can be decreasing (L-shaped) or right skewed.

If a random variable X has the GD density, we write $X \sim GD(\lambda, \beta, \delta, \alpha, \theta)$. Let $y = [1 + \lambda x^{-\delta}]^{-\beta}$, and $\psi = 1/\theta$, then using the series representation $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, we have

$$\Big[-\log(1-y)\Big]^{\alpha-1} = y^{\alpha-1}\Bigg[\sum_{m=0}^{\infty}\binom{\alpha-1}{m} y^m \Bigg(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\Bigg)^m\Bigg],$$

and applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\Bigg(\sum_{s=0}^{\infty} a_s y^s\Bigg)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \tag{6.18}$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1) - s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, (Gradshteyn and Ryzhik [13]), the GD pdf can be written as

$$\begin{aligned}
g_{GD}(x) &= \frac{\lambda\beta\delta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}}{\Gamma(\alpha)\theta^{\alpha}} y^{\alpha-1} \\
&\times \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\alpha-1}{m} b_{s,m} y^{m+s}\sum_{k=0}^{\infty}\binom{\psi-1}{k}(-1)^k y^k \\
&= \frac{\lambda\beta\delta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}}{\Gamma(\alpha)\theta^{\alpha}} \\
&\times \sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\alpha-1}{m}\binom{\psi-1}{k}(-1)^k b_{s,m} y^{\alpha+m+s+k-1} \\
&= \sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\alpha-1}{m}\binom{\psi-1}{k}(-1)^k b_{s,m} \\
&\times \frac{\delta\lambda\beta}{\theta^{\alpha}\Gamma(\alpha)} x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta(m+s+k+\alpha)-1} \\
&= \sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\alpha-1}{m}\binom{\psi-1}{k}(-1)^k \frac{b_{s,m}}{(m+s+k+\alpha)\theta^{\alpha}\Gamma(\alpha)} \\
&\times \lambda\beta(m+s+k+\alpha)\delta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta(m+s+k+\alpha)-1},
\end{aligned}$$

where $f(x; \lambda, \beta(m+s+k+\alpha), \delta)$ is the Dagum pdf with parameters λ, $\beta(m+s+k+\alpha)$, and δ. Let $C = \{(m, s, k) \in \mathbf{Z}_+^3\}$, and $\psi = 1/\theta$, then the weights in the GD pdf are

$$w_{\nu} = (-1)^k\binom{\alpha-1}{m}\binom{\psi-1}{k}\frac{b_{m,s}}{(m+s+k+\alpha)\theta^{\alpha}\Gamma(\alpha)},$$

and

$$g_{GD}(x) = \sum_{\nu \in C} w_v f_D(x; \lambda, \beta(m+s+k+\alpha), \delta). \tag{6.19}$$

It follows therefore that the GD density is a linear combination of the Dagum pdfs. The statistical and mathematical properties can be readily obtained from those of the Dagum distribution. Note that $g_{GD}(x)$ is a weighted pdf with the weight function

$$w(x) = [-\log(1-F(x))]^{\alpha-1}[1-F(x)]^{\frac{1}{\theta}-1}, \tag{6.20}$$

that is,

$$\begin{aligned} g_{GD}(x) &= \frac{[-\log(1-F(x))]^{\alpha-1}[1-F(x)]^{\frac{1}{\theta}-1}}{\theta^{\alpha}\Gamma(\alpha)} f(x) \\ &= \frac{w(x)f(x)}{E_F(w(X))}, \end{aligned} \tag{6.21}$$

where $0 < E_F[[-\log(1-F(x))]^{\alpha-1}[1-F(x)]^{\frac{1}{\theta}-1}] = \theta^{\alpha}\Gamma(\alpha) < \infty$, is the normalizing constant. Similarly,

$$g_2(x) = \frac{[-\log(F(X))]^{\alpha-1}[F(X)]^{\frac{1}{\theta}-1}}{\theta^{\alpha}\Gamma(\alpha)} f(x) = \frac{w(x)f(x)}{E_F(w(X))}, \tag{6.22}$$

where $0 < E_F(w(X)) = E_F([-\log(F(X))]^{\alpha-1}[F(X)]^{\frac{1}{\theta}-1}) = \theta^{\alpha}\Gamma(\alpha) < \infty$.

6.2.2 Sub-models

Some of the sub-models of the GD distribution are listed below:

- If $\theta = 1$, we obtain the gamma-Dagum distribution via the ZB-Dagum (ZB-D) distribution.

- When $\lambda = \theta = 1$, we have the ZB-Burr III (ZB-B III) distribution.

- When $\beta = \theta = 1$, we obtain the ZB-Fisk or ZB-Log logistic (ZB-F or ZB-LLog) distribution.

- If $\alpha = 1$, we get the exponentiated Dagum (ED) distribution, which is also a Dagum distribution.
- When $\beta = 1$, we have the gamma-Fisk or gamma-Log logistic (GF or GLLog) distribution.
- If $\lambda = 1$, we obtain the gamma-Burr III (GB III) distribution.
- If $\theta = 1$ and $\alpha = 1$, we have Dagum (D) distribution.
- When $\lambda = \alpha = \theta = 1$, we have Burr III (B III) distribution.
- When $\lambda = \alpha = 1$, we have exponentiated Burr III (EB III) distribution.
- When $\beta = \alpha = 1$, we obtain Fisk or Log-logistic (F or LLog) distribution.

6.2.3 Hazard and Reverse Hazard Functions

In general, if X is a continuous random variable with distribution function F, and probability density function (pdf) f, then the hazard function, reverse hazard function and mean residual life functions are given by $h_F(x) = f(x)/\overline{F}(x)$, $\tau_F(x) = f(x)/F(x)$, and $\delta_F(x) = \int_x^\infty \overline{F}(u)du/\overline{F}(x)$ respectively. The functions $h_F(x)$, $\delta_F(x)$, and $\overline{F}(x)$ are equivalent (Shaked and Shanthikumar [35]). The hazard and reverse hazard functions of the GD distribution are

$$h_G(x) = \frac{\lambda\beta\delta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}(-\log(1-[1+\lambda x^{-\delta}]^{-\beta}))^{\alpha-1}[1-(1+\lambda x^{-\delta})^{-\beta}]^{(1/\theta)-1}}{\theta^{\alpha}(\Gamma(\alpha)-\gamma(-\theta^{-1}\log(1-(1+\lambda x^{-\delta})^{-\beta}),\alpha))}, \tag{6.23}$$

and

$$\tau_G(x) = \frac{\lambda\beta\delta x^{-\delta-1}[1+\lambda x^{-\delta}]^{-\beta-1}(-\log(1-[1+\lambda x^{-\delta}]^{-\beta}))^{\alpha-1}[1-(1+\lambda x^{-\delta})^{-\beta}]^{(1/\theta)-1}}{\theta^{\alpha}(\gamma(-\theta^{-1}\log(1-(1+\lambda x^{-\delta})^{-\beta}),\alpha))}, \tag{6.24}$$

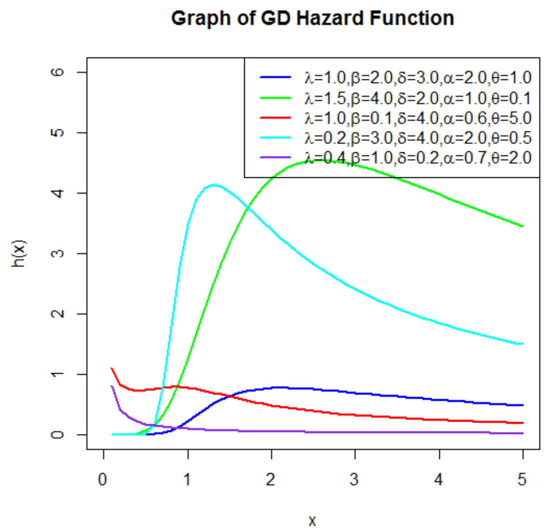

Figure 6.2: GD Hazard Functions

for $x \geq 0$, $\lambda > 0$, $\beta > 0$, $\delta > 0$, $\alpha > 0$, and $\theta > 0$, respectively.

The graph of hazard function for selected parameters can be found in Figure 6.2. The plots shows various shapes including monotonically decreasing, unimodal and upside down bathtub shapes for five different combinations of the parameter values.

6.2.4 Quantile Function

The quantile function of GD distribution is obtained by solving the equation

$$G(Q(y)) = y, \quad 0 < y < 1. \tag{6.25}$$

Note that the inverse or quantile function of Dagum distribution, $F_D(x) = [1+\lambda x^{-\delta}]^{-\beta}$ is given by $Q_D(.)$, that is

$$Q_D(y) = \lambda^{\frac{1}{\delta}} \left(y^{\frac{-1}{\beta}} - 1\right)^{\frac{-1}{\delta}}. \tag{6.26}$$

The quantile function of the GD distribution is obtained by inverting equation (6.16) to obtain

$$Q_{GD}(y) = \lambda^{\frac{1}{\delta}}\left[\left(1 - e^{-\theta u}\right)^{\frac{-1}{\beta}} - 1\right]^{\frac{-1}{\delta}}, \tag{6.27}$$

where $u = \gamma^{-1}(y\Gamma(\alpha), \alpha)$.

6.3 Moments, Moment Generating Function, Mean and Median Deviations

In this section, moments, moment generating function, mean and median deviations of the GD distribution are presented.

6.3.1 Moments and Moment Generating Function

Let $\beta^* = \beta(m+s+k+\alpha)$, and $Y \sim D(\lambda, \beta^*, \delta)$. Note that from $Y \sim D(\alpha, \beta^*, \delta)$, the r^{th} moment of the random variable Y is

$$E(Y^r) = \beta^* \lambda^{r/\delta} B\left(\beta^* + \frac{r}{\delta}, 1 - \frac{r}{\delta}\right), \tag{6.28}$$

$r < \delta$, so that the r^{th} raw moment of GD distribution is given by:

$$E(X^r) = \sum_{\nu \in C} w_\nu E(Y^r) = \sum_{\nu \in C} w_\nu \beta^* \lambda^{r/\delta} B\left(\beta^* + \frac{r}{\delta}, 1 - \frac{r}{\delta}\right), \tag{6.29}$$

$r < \delta$. The moment generating function (MGF) , for $|t| < 1$, is given by:

$$\begin{aligned} M_X(t) &= \sum_{\nu \in C} w_\nu M_Y(t) \\ &= \sum_{\nu \in C}\sum_{i=0}^{\infty} w_\nu \frac{t^i}{i!} \beta^* \lambda^{r/\delta} B\left(\beta^* + \frac{r}{\delta}, 1 - \frac{r}{\delta}\right), \end{aligned} \tag{6.30}$$

for $r < \delta$.

Theorem 6.1.

$$E\{[-\log(1 - F(X))]^r[(1 - F(X))^s]\} = \frac{\theta^{r+\alpha}\Gamma(r+\alpha)}{(s\theta+1)^\alpha \theta^\alpha \Gamma(\alpha)}. \tag{6.31}$$

Proof:

$$\begin{aligned}
& E\{[-\log(1-F(X))]^r[(1-F(X))^s]\} \\
= & \int_0^\infty \frac{f(x)}{\theta^\alpha \Gamma(\alpha)}[-\log(1-F(x))]^{r+\alpha-1}[1-F(x)]^{s+(1/\theta)-1}dx \\
= & \frac{\theta^{r+\alpha}\Gamma(r+\alpha)}{(s\theta+1)^\alpha \theta^\alpha \Gamma(\alpha)}. \qquad (6.32)
\end{aligned}$$

Corollary 6.2. *If $s=0$, we have $E[-\log(1-F(X))^r] = \frac{\theta^{r+\alpha}\Gamma(r+\alpha)}{\theta^\alpha\Gamma(\alpha)}$, and if $r=0$, $E[(1-F(X))^s] = [s\theta+1]^{-\alpha}$.*

Proof: Let $s=0$ in equation (6.31), then

$$E[-\log(1-F(X))^r] = \frac{\theta^{r+\alpha}\Gamma(r+\alpha)}{\theta^\alpha\Gamma(\alpha)}. \qquad (6.33)$$

Let $\theta^* = s + \frac{1}{\theta}$, then with $r=0$ in equation (6.31), we obtain

$$E[(1-F(X))^s] = [s\theta+1]^{-\alpha}. \qquad (6.34)$$

6.3.2 Mean and Median Deviations

If X has the GD distribution, we can derive the mean deviation about the mean μ by

$$\delta_1 = \int_0^\infty |x-\mu| g_{GD}(x)dx = 2\mu G_{GD}(\mu) - 2\mu + 2T(\mu), \qquad (6.35)$$

and the median deviation about the median M by

$$\delta_2 = \int_0^\infty |x-M| g_{GD}(x)dx = 2T(M) - \mu, \qquad (6.36)$$

where $\mu = E(X)$ is given in equation (6.29), $M = Q_{GD}(0.5)$ in equation (6.27) and $T(a) = \int_a^\infty x \cdot g_{GD}(x)dx$. Let $\beta^* = \beta(m+s+k+\alpha)$, then

$$\begin{aligned}
T(a) &= \sum_{\nu\in C} w_\nu T_{D(\lambda,\beta^*,\delta)}(a) \\
&= \sum_{\nu\in C} w_\nu \beta^* \lambda^{\frac{1}{\delta}}\left[B\left(\beta^*+\frac{1}{\delta}, 1-\frac{1}{\delta}\right) - B\left(t(a);\beta^*+\frac{1}{\delta}, 1-\frac{1}{\delta}\right)\right], \qquad (6.37)
\end{aligned}$$

where $t(a) = (1+\lambda a^{-\delta})^{-1}$, and $B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt$.

6.4 Order Statistics and Rényi Entropy

Order statistics plays an important role in probability and statistics. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. In this section, we present the distribution of the order statistics, and Rényi entropy for the GD distribution.

6.4.1 Order Statistics

The pdf of the i^{th} order statistic from the GD pdf $g(x)$ is given by

$$\begin{aligned} g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-i} \\ &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[G(x)]^{i+j-1} \\ &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left[\frac{\gamma(-\theta^{-1}\log(1-\overline{F}(x),\alpha))}{\Gamma(\alpha)}\right]^{i+j-1}. \end{aligned}$$

Using the fact that

$$\gamma(x,\alpha)=\sum_{m=0}^{\infty}\frac{(-1)^m x^{m+\alpha}}{(m+\alpha)m!}, \tag{6.38}$$

and setting $c_m = (-1)^m/((m+\alpha)m!)$, we have

$$\begin{aligned} g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\frac{(-1)^j}{[\Gamma(\alpha)]^{i+j-1}}[-\theta^{-1}\log(\overline{F}(x))]^{\alpha(i+j-1)} \\ &\times \left[\sum_{m=0}^{\infty}\frac{(-1)^m(-\theta^{-1}\log(\overline{F}(x)))^m}{(m+\alpha)m!}\right]^{i+j-1} \\ &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{[\Gamma(\alpha)]^{i+j-1}}[-\theta^{-1}\log(\overline{F}(x))]^{\alpha(i+j-1)} \\ &\times \sum_{m=0}^{\infty}d_{m,i+j-1}(-\theta^{-1}\log(\overline{F}(x)))^m, \end{aligned}$$

where $d_0 = c_0^{(i+j-1)}$, $d_{m,i+j-1} = (mc_0)^{-1}\sum_{l=1}^{m}[(i+j-1)l-m+l]c_l d_{m-l,i+j-1}$. Now,

$$\begin{aligned} g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,n-i+j}}{[\Gamma(\alpha)]^{i+j-1}}[-\theta^{-1}\log(\overline{F}(x))]^{\alpha(i+j-1)+m} \\ &= \frac{n![-\log(\overline{F}(x))]^{\alpha-1}[\overline{F}(x)]^{\psi-1}f(x)}{(i-1)!(n-i)!\Gamma(\alpha)\theta^{\alpha}}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\alpha)]^{i+j-1}} \\ &\times [-\theta^{-1}\log(\overline{F}(x))]^{\alpha(i+j-1)+m} \\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\alpha)]^{i+j}} \\ &\times \frac{[\log(\overline{F}(x))]^{\alpha(i+j)+m-1}}{\theta^{\alpha(i+j)+m}}[\overline{F}(x)]^{\psi-1}f(x) \\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\alpha)]^{i+j}} \\ &\times \frac{\Gamma(\alpha(i+j)+m)}{\theta^{\alpha(i+j)+m}}\frac{[-\log(\overline{F}(x))]^{\alpha(i+j)+m-1}}{\Gamma(\alpha(i+j)+m)}[\overline{F}(x)]^{\psi-1}f(x). \end{aligned}$$

That is,

$$\begin{aligned} g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}\Gamma(\alpha(i+j)+m)}{[\Gamma(\alpha)]^{i+j}} \\ &\times g(x;\alpha(i+j)+m,\beta,\lambda,\delta,\theta), \end{aligned}$$

where $g(x;\alpha(i+j)+m,\beta,\lambda,\delta,\theta)$ is the GD pdf with parameters $\lambda,\beta,\delta,\theta$, and shape parameter $\alpha^*=\alpha(i+j)+m$. It follows therefore that

$$\begin{aligned} E(X_{i:n}^j) &= \frac{n!}{(i-1)!(n-i)!}\sum_{\nu\in C}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j w_\nu d_{m,i+j-1}}{[\Gamma(\alpha)]^{i+j}} \\ &\times \Gamma(\alpha(i+j)+m)\beta^*\lambda^{j/\delta}B\left(\beta^*+\frac{j}{\delta},1-\frac{j}{\delta}\right), \end{aligned}$$

for $j<\delta$, where $B(.,.)$ is the beta function. These moments are often used in several areas including reliability, insurance and quality control for the prediction of future failures times from a set of past or previous failures.

6.4.2 Rényi Entropy

Rényi entropy is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^\infty[g(x;\lambda,\beta,\delta,\alpha,\theta)]^v dx\right), v\neq 1, v>0. \tag{6.39}$$

Rényi entropy tends to Shannon entropy as $v\to 1$. Let $y=[1+\lambda x^{-\delta}]^{-\beta}$. Note that for $\alpha>1$ and ν/θ a natural number,

$$\begin{aligned} g_{GD}^v(x) &= \frac{(\lambda\beta\delta)^v}{(\theta\Gamma(\alpha))^v}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v(\alpha-1)}{m}\binom{v/\alpha-1}{k}x^{-v\delta-v} \\ &\times [1+\lambda x^{-\delta}]^{-v\beta-v}y^{m+s+v\alpha-v+k} \\ &= \frac{(\lambda\beta\delta)^v}{(\theta\Gamma(\alpha))^v}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v(\alpha-1)}{m}\binom{v/\alpha-1}{k} \\ &\times x^{-v\delta-v}[1+\lambda x^{-\delta}]^{-\beta(m+s+k+v\alpha)-v}. \end{aligned}$$

Now,

$$\begin{aligned} \int_0^\infty g_{GD}^v(x)dx &= \frac{(\lambda\beta\delta)^v}{(\theta\Gamma(\alpha))^v}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v(\alpha-1)}{m}\binom{v/\alpha-1}{k} \\ &\times \int_0^\infty x^{-v\delta-v}[1+\lambda x^{-\delta}]^{-\beta(m+s+k+v\alpha)-v}dx. \end{aligned}$$

Let $t = [1 + \lambda x^{-\delta}]^{-1}$, then

$$\begin{aligned} & \int_0^\infty x^{-v\delta - v}[1 + \lambda x^{-\delta}]^{-\beta(m+s+k+v\alpha)-v}dx \\ = & \frac{\lambda^{-v-\frac{v}{\delta}+\delta}}{\delta}\int_0^1 t^{\beta(m+s+k+v\alpha)-\frac{v}{\delta}+\delta-1}(1-t)^{v+\frac{v}{\delta}-\delta-1}dt \\ = & \frac{\lambda^{-v-\frac{v}{\delta}+\delta}}{\delta}B\bigg(\beta(m+s+k+v\alpha)+\delta-\frac{v}{\delta}, v+\frac{v}{\delta}-\delta\bigg), \end{aligned}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the beta function. Consequently, Rényi entropy for GD distribution is given by

$$\begin{aligned} I_R(v) &= \frac{1}{1-v}\log\left[\frac{\lambda^{v-\frac{v}{\delta}}\beta^v\delta^{v-1}}{\theta^{v\alpha}(\Gamma(\alpha))^v}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v(\alpha-1)}{m}\binom{(v/\theta)-1}{k}b_{s,m}\right. \\ &\times \left. B\bigg(\beta(m+s+k+v\alpha)+\delta-\frac{v}{\delta}, v+\frac{v}{\delta}-\delta\bigg)\right], \end{aligned}$$

for $v > 0$, $v \neq 1$.

6.5 Estimation of Model Parameters

In this section, we present estimates of the parameters of the GD distribution.

6.5.1 Maximum Likelihood Estimation

Consider a random sample $x_1, x_2, ..., x_n$ from the gamma-Dagum distribution. The likelihood function is given by

$$\begin{aligned} L(\lambda,\beta,\delta,\theta,\alpha) &= \frac{(\lambda\beta\delta)^n}{[\theta^\alpha\Gamma(\alpha)]^n}\prod_{i=1}^{n}\Bigg\{x_i^{-\delta-1}[1+\lambda x_i^{-\delta}]^{-\beta-1} \\ &\times \left[-\log\bigg(1-(1+\lambda x_i^{-\delta})^{-\beta}\bigg)\right]^{\alpha-1}\left[1-(1+\lambda x_i^{-\delta})^{-\beta}\right]^{(1/\theta)-1}\Bigg\}. \end{aligned}$$

Now, the log-likelihood function denoted by ℓ is

$$\begin{aligned}
\ell &= \log[L(\lambda,\beta,\delta,\theta,\alpha)] \\
&= n\log(\lambda)+n\log(\beta)+n\log(\delta)+(-\delta-1)\sum_{i=1}^{n}\log(x_i) \\
&+ (-\beta-1)\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta})+(\alpha-1)\sum_{i=1}^{n}\log\left[-\log\left(1-(1+\lambda x_i^{-\delta})\right)\right] \\
&+ \left(\frac{1}{\theta}-1\right)\sum_{i=1}^{n}\log\left[1-(1+\lambda x_i^{-\delta})\right]-n\alpha\log(\theta)-n\log(\Gamma(\alpha)). \qquad (6.40)
\end{aligned}$$

The entries of the score function are given by

$$\begin{aligned}
\frac{\partial\ell}{\partial\lambda} &= \frac{n}{\lambda}+(-\beta-1)\sum_{i=1}^{n}\frac{x_i^{-\delta}}{1+\lambda x_i^{-\delta}} \\
&+ (\alpha-1)\sum_{i=1}^{n}\frac{\beta(1+\lambda x_i^{-\delta})^{-\beta-1}x_i^{-\delta}}{(1-(1+\lambda x_i^{-\delta})^{-\beta})\log(1-(1+\lambda x_i^{-\delta})^{-\beta})} \\
&+ \left(\frac{1}{\theta}-1\right)\sum_{i=1}^{n}\frac{\beta(1+\lambda x_i^{-\delta})^{-\beta-1}x_i^{-\delta}}{\log(1-(1+\lambda x_i^{-\delta})^{-\beta})}, \qquad (6.41)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\ell}{\partial\beta} &= \frac{n}{\beta}-\sum_{i=1}^{n}\log(1+\lambda x_i^{-\delta}) \\
&+ (\alpha-1)\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-\beta}\log(1+\lambda x_i^{-\delta})(-1)}{(1-(1+\lambda x_i^{-\delta})^{-\beta})(\log(1-(1+\lambda x_i^{-\delta})^{-\beta}))} \\
&+ \left(\frac{1}{\theta}-1\right)\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-\beta}\log(1+\lambda x_i^{-\delta})}{[1-(1+\lambda x_i^{-\delta})^{-\beta}]}, \qquad (6.42)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\ell}{\partial\delta} &= \frac{n}{\delta}-\sum_{i=1}^{n}\log(x_i)+(-\beta-1)\sum_{i=1}^{n}\frac{\lambda x_i^{-\delta}\log(x_i^{-\delta})(-1)}{1+\lambda x_i^{-\delta}} \\
&+ (\alpha-1)\sum_{i=1}^{n}\frac{\lambda x_i^{-\delta}\log(\lambda x_i^{-\delta})}{(1-(1+\lambda x_i^{-\delta})^{-\beta})(\log(1-(1+\lambda x_i^{-\delta})^{-\beta}))}, \qquad (6.43)
\end{aligned}$$

$$\frac{\partial \ell}{\partial \alpha} = -\frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - n\log(\alpha) + \sum_{i=1}^{n} \log\left(-\log\left(1-(1+\lambda x_i^{-\delta})^{-\beta}\right)\right), \tag{6.44}$$

and

$$\frac{\partial \ell}{\partial \theta} = -\frac{n\alpha}{\theta} - \frac{1}{\theta^2}\sum_{i=1}^{n} \log\left(-\log\left(1-(1+\lambda x_i^{-\delta})^{-\beta}\right)\right). \tag{6.45}$$

The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters $\alpha, \beta, \theta, \lambda, \delta$ must be found by using iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \delta})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $\mathbf{I}(\mathbf{\Delta}) = [\mathbf{I}_{\theta_i,\theta_j}]_{5X5} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3, 4, 5$, can be numerically obtained by MATLAB or MAPLE software. The total Fisher information matrix $n\mathbf{I}(\mathbf{\Delta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Delta}}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j}\bigg|_{\mathbf{\Delta}=\hat{\mathbf{\Delta}}} \right]_{5X5}, \quad i, j = 1, 2, 3, 4, 5. \tag{6.46}$$

For a given set of observations, the matrix given in equation (6.46) is obtained after the convergence of the Newton-Raphson procedure in MATLAB software.

6.5.2 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the GD distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\mathbf{\Delta}} = (\hat{\lambda}, \hat{\beta}, \hat{\delta}, \hat{\theta}, \hat{\alpha})$ be the maximum likelihood estimate of $\mathbf{\Delta} = (\lambda, \beta, \delta, \theta, \alpha)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Delta}} - \mathbf{\Delta}) \xrightarrow{d} N_5(\underline{0}, I^{-1}(\mathbf{\Delta}))$, where $I(\mathbf{\Delta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Delta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Delta}}$, that is $J(\hat{\mathbf{\Delta}})$. The multivariate normal distribution $N_5(\underline{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{0} = (0,0,0,0,0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. A large sample $100(1-\eta)\%$ confidence intervals for $\lambda, \beta, \delta, \theta$ and α are:

$$\widehat{\lambda} \pm Z_{\frac{\eta}{2}}\sqrt{I_{\lambda\lambda}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\beta} \pm Z_{\frac{\eta}{2}}\sqrt{I_{\beta\beta}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\delta} \pm Z_{\frac{\eta}{2}}\sqrt{I_{\delta\delta}^{-1}(\widehat{\mathbf{\Delta}})}$$

$$\widehat{\theta} \pm Z_{\frac{\eta}{2}}\sqrt{I_{\theta\theta}^{-1}(\widehat{\mathbf{\Delta}})}, \text{ and } \quad \widehat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{I_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Delta}})}$$

respectively, where $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the GD distribution with its sub-models for a given data set. For example, to test $\theta = \alpha = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{\lambda}, \hat{\beta}, \hat{\delta}, \hat{\theta}, \hat{\alpha})) - \ln(L(\tilde{\lambda}, \tilde{\beta}, \tilde{\delta}, 1, 1))]$, where $\hat{\lambda}$, $\hat{\beta}$, $\hat{\delta}$, $\hat{\theta}$ and $\hat{\alpha}$, are the unrestricted estimates, and $\tilde{\lambda}$, $\tilde{\beta}$, and $\tilde{\delta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 2 degrees of freedom.

6.6 Applications

In this section, we present examples to illustrate the flexibility of the GD distribution and its sub-models, as well as the gamma-exponentiated Weibull (GEW) distribution [30] for data modeling. The pdf of GEW distribution is given by

$$g_{GEW}(x) = \frac{k\alpha^\delta}{\lambda\Gamma(\delta)}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(\frac{x}{\lambda})^k}\left[1 - e^{-(\frac{x}{\lambda})^k}\right]^{\alpha-1}\left[-\log\left(1 - e^{-(\frac{x}{\lambda})^k}\right)\right]^{\delta-1}, \quad (6.47)$$

for $\alpha, \delta, \lambda, k > 0$.

The maximum likelihood estimates (MLEs) of the GD parameters λ, β, δ, θ, and α are computed by maximizing the objective function via the subroutine NLMIXED

Table 6.1: GD Estimation for Baseball Player Salary Data

	Estimates					Statistics					
Model	λ	β	δ	α	θ	$-2\log L$	AIC	$AICC$	BIC	SS	KS
GD	0.000016	98.9836	7.354	0.6932	12.0803	2913.9	2923.9	2924.0	2947.4	9.1210	0.1657
	(0.000004127)	(0.05025)	(0.05141)	(0.08185)	(1.031)						
ZB-D	0.000809	99.564	1.5113	3.1336	1	3201.7	3209.7	3209.8	3228.6	6.5741	0.1216
	(0.000469)	(0.000067)	(0.1338)	(0.5981)	-						
ZB-BurrIII	1	26.3597	1.146	0.05352	1	3345.7	3351.7	3351.8	3365.9	5.6495	0.1528
	-	(0.3903)	(0.03028)	(0.001913)	-						
ZB-Fisk	0.00873	1	1.9669	5.4913	1	3221.5	3227.5	3227.5	3241.6	6.3699	0.1262
	(0.005112)	-	(0.1193)	(0.6208)	-						
D	0.00973	83.1324	1.0131	1	1	3225.5	3231.5	3231.5	3245.6	6.1089	0.1324
	(0.001811)	(15.3394)	(0.02966)	-	-						
	k	α	δ	λ							
GEW	0.5861	5.2885	0.1684	0.06826		3294.5	3302.5	3302.5	3321.3	6.0177	0.1457
	(0.009715)	(0.5144)	(0.004691)	(0.002001)							

in SAS. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\mathbf{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Table 6.1, 6.2 and 6.3. Also, presented are values of the Kolmogorov-Smirnov statistic, $KS = max_{1\leq i\leq n}\{G_{GD}(x_i) - \frac{i-1}{n}, \frac{i}{n} - G_{GD}(x_i)\}$, and the sum of squares $SS = \sum_{j=1}^{n}\left[G_{GD}(x_{(j)}; \hat{\lambda}, \hat{\beta}, \hat{\delta}, \hat{\theta}, \hat{\alpha}) - \left(\frac{j-0.375}{n+0.25}\right)\right]^2$. These statistics are used to compare the distributions presented in these tables. Plots of the fitted densities and the histogram of the data are given in Figures 6.3, 6.4 and 6.5. Probability plots (Chambers et al. [1]) are also presented in Figures 6.3, 6.4 and 6.5. For the probability plot, we plotted $G_{GD}(x_{(j)}; \hat{\lambda}, \hat{\beta}, \hat{\delta}, \hat{\theta}, \hat{\alpha})$ against $\frac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data.

The first example consists of the salaries of 818 professional baseball players for the year 2009 (USA TODAY). Initial value for GD model in SAS code are $\lambda = 0.0001, \beta = 100, \delta = 1, \alpha = 0.1, \theta = 15$. Estimates of the parameters of GD distribution and its related sub-models (standard error in parentheses), AIC, AICC, BIC, KS and SS for baseball player salary data are given in Table 6.1. The estimated covariance matrix for GD distribution is given by:

$$\begin{pmatrix} 1.70E-11 & 1.67E-07 & 1.08E-07 & -3.2E-07 & 3.43E-06 \\ 1.67E-07 & 0.002525 & 0.001457 & -0.00368 & 0.05181 \\ 1.08E-07 & 0.001457 & 0.002643 & -0.00252 & 0.02977 \\ -3.16E-07 & -0.00368 & -0.00252 & 0.006699 & -0.07547 \\ 3.43E-06 & 0.05181 & 0.02977 & -0.07547 & 1.063 \end{pmatrix}$$

The 95% asymptotic confidence intervals are: $\lambda \in 0.000016 \pm 1.96(0.000004127), \beta \in 98.9836 \pm 1.96(0.05025), \delta \in 7.354 \pm 1.96(0.05141), \alpha \in 0.6932 \pm 1.96(0.08185)$ and $\theta \in 12.0803 \pm 1.96(1.031)$.

Plots of the fitted densities and the histogram, observed probability vs predicted probability, and empirical survival function for the baseball player salary data are given in Figure 6.3.

The LR test statistic of the hypothesis H_0: ZB-D against H_a: GD, H_0: ZB-BurrIII against H_a: GD and H_0: ZB-Fisk against H_a: GD are 287.8 (p-value < 0.0001), 431.8 (p-value < 0.0001) and 307.6 (p-value < 0.0001). We can conclude that there is a significant difference between GD and ZB-D, ZB-BurrIII and ZB-Fisk distributions.

The second example consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes [28]. Initial value for GD model in SAS code are $\lambda = 1.2, \beta = 14, \delta = 1, \alpha = 0.9, \theta = 0.01$.

Table 6.2: GD Estimation for Air Conditioning System Data

	Estimates					Statistics					
Model	λ	β	δ	α	θ	$-2\log L$	AIC	$AICC$	BIC	SS	KS
GD	2.1816	31.0783	0.538	0.1856	0.05384	2065.1	2075.1	2075.4	2091.2	0.0334	0.0401
	(0.8567)	(7.1966)	(0.05068)	(0.0188)	(0.034)						
ZB-D	10.611	14.8939	0.9507	0.1885	1	2084.7	2092.7	2092.9	2105.7	0.5144	0.0982
	(1.9869)	(1.0488)	(0.0375)	(0.0194)	-						
ZB-BurrIII	1	51.7658	0.7498	0.2579	1	2101.5	2107.5	2107.6	2117.2	0.3876	0.0786
	-	(1.0861)	(0.0327)	(0.02906)	-						
ZB-Fisk	102.03	1	1.2902	1.2013	1	2078.5	2084.5	2084.6	2094.2	0.0797	0.0467
	(51.2069)	-	(0.07275)	(0.1991)	-						
D	118.02	1.1792	1.2873	1	1	2077.4	2083.4	2083.5	2093.1	0.0687	0.0421
	(71.3654)	(0.2375)	(0.09666)	-	-						
	k	α	δ	λ							
GEW	0.2651	1.3363	0.05339	0.0007		2338.4	2346.4	2346.6	2359.3	8.4196	0.3863
	(0.001118)	(0.4333)	(0.003965)	(0.000001897)							

Estimates of the parameters of GD distribution and its related sub-models (standard error in parentheses), AIC, AICC, BIC, KS and SS for air conditioning system data are given in Table 6.2. The estimated covariance matrix for GD model is given by:

$$\begin{pmatrix} 0.7339 & -6.1407 & 0.02873 & -0.00537 & 0.006472 \\ -6.1407 & 51.7908 & -0.2362 & 0.04324 & -0.05223 \\ 0.02873 & -0.2362 & 0.002568 & 0.00014 & 0.001453 \\ -0.00537 & 0.04324 & 0.00014 & 0.000353 & 0.000194 \\ 0.006472 & -0.05223 & 0.001453 & 0.000194 & 0.001156 \end{pmatrix}$$

Plots of the fitted densities and the histogram, observed probability vs predicted probability, and empirical survival function for the air conditioning system data are given in Figure 6.4.

The LR test statistic of the hypothesis H_0: ZB-D against H_a: GD, H_0: ZB-

BurrIII against H_a: GD and H_0: ZB-Fisk against H_a: GD are 19.6 (p-value < 0.0001), 36.4 (p-value < 0.0001) and 13.4 (p-value $= 0.0012$). We can conclude that there is a significant difference between GD and ZB-D, ZB-BurrIII and ZB-Fisk distributions. The values of the statistics KS as well value of SS are smallest for the GD distribution. We conclude that the GD distribution provides the best fit for the air conditioning system data.

Table 6.3: GD Estimation for Remission Times Data

	Estimates					Statistics					
Model	λ	β	δ	α	θ	$-2\log L$	AIC	$AICC$	BIC	SS	KS
GD	36.5904	4.6432	1.6783	0.1763	0.827	818.9	828.9	829.4	843.2	0.0153	0.0345
	(20.0777)	(0.6152)	(0.2154)	(0.02128)	(0.3848)						
ZB-D	14.4218	5.6739	1.4787	0.1932	1	821.4	829.4	829.7	840.8	0.0422	0.0457
	(4.9839)	(0.5272)	(0.1038)	(0.02313)	-						
ZB-BurrIII	1	14.0701	0.955	0.3086	1	846.8	852.8	853.0	861.4	0.3641	0.0945
	-	(0.8515)	(0.054)	(0.04013)	-						
ZB-Fisk	78.8617	1	1.9597	0.6595	1	819.6	825.6	825.8	834.1	0.0205	0.0396
	(47.7597)	-	(0.1721)	(0.1125)	-						
D	0.04426	56.2014	0.7713	1	1	884.2	890.2	890.4	898.8	0.9490	0.1382
	(0.02063)	(25.1544)	(0.04672)	-	-						
	k	α	δ	λ							
GEW	0.9718	1.8349	0.03244	1.9013		1154.7	1162.7	1163.0	1174.1	25.4662	0.6689
	(0.008578)	(0.9136)	(0.002901)	(0.002509)							

The third example represents the remission times (in months) of a random sample of 128 bladder cancer patients [21]. Initial value for GD model in SAS code are $\lambda = 7, \beta = 1, \delta = 0.84, \alpha = 0.21, \theta = 0.3$. Estimates of the parameters of GD distribution and its related sub-models (standard error in parentheses), AIC, AICC, BIC and SS for remission times data are given in Table 6.3. The estimated covariance matrix for GD model is given by:

$$\begin{pmatrix} 403.11 & -11.7265 & 3.5502 & -0.1583 & 2.8776 \\ -11.7265 & 0.3785 & -0.1238 & 0.002601 & -0.1345 \\ 3.5502 & -0.1238 & 0.0464 & -0.00018 & 0.06253 \\ -0.1583 & 0.002601 & -0.00018 & 0.000453 & 0.000548 \\ 2.8776 & -0.1345 & 0.06253 & 0.000548 & 0.148 \end{pmatrix}$$

Plots of the fitted densities and the histogram, observed probability vs predicted probability, and empirical survival function for the remission times data are given in Figure 6.5.

The LR test statistic of the hypothesis H_0: ZB-D against H_a: GD and H_0: ZB-BurrIII against H_a: GD are 2.5 (p-value = 0.1138) and 27.9 (p-value $<$ 0.0001). We can conclude that there is a significant difference between GD and ZB-BurrIII distributions. GD distribution gives smaller AIC, AICC, BIC, SS and KS values. The GD distribution provides the best fit for the remission time data.

The fourth example represents the poverty rate of 533 districts with more than 15,000 students in 2009. Estimates and statistics are given in Table 6.4.

Plots of the fitted densities and the histogram, observed probability vs predicted probability, and empirical survival function for the poverty rate data are given in Figure 6.6.

In this example, ZB-Burr III distribution has smaller AIC, AICC and BIC values. Dagum model has smallest SS and KS value. We conclude that the Dagum and ZB-Burr III sub-models provide good fits for this data.

Table 6.4: GD Estimation for Poverty Rate Data

	Estimates					Statistics					
Model	λ	β	δ	α	θ	$-2\log L$	AIC	$AICC$	BIC	SS	KS
GD	11.2183	0.008726	8.6879	26.0407	1.0012	3833.3	3843.3	3843.4	3864.7	1.5034	0.1006
	(0.195)	(0.001009)	(0.03426)	(1.4376)	(0.05551)						
ZB-D	1.4224	0.05819	8.6002	26.0351	1	3832.8	3840.8	3840.9	3857.9	1.4312	0.0979
	(0.3187)	(0.000937)	(0.07815)	(0.2144)	-						
ZB-BurrIII	1	0.07221	8.631	26.2914	1	3833.6	3839.6	3839.7	3852.5	1.3499	0.0951
	-	(0.008338)	(0.04403)	(0.2035)	-						
ZB-Fisk	350.03	1	2.5291	1.6577	1	3846.3	3852.3	3852.3	3865.1	1.1908	0.0808
	(83.6114)	-	(0.07197)	(0.1602)	-						
D	250.03	1.9862	2.3418	1	1	3838.2	3844.2	3844.2	3857.0	1.0088	0.0763
	(73.788)	(0.2337)	(0.07534)	-	-						
	k	α	δ	λ							
GEW	0.6504	1.0122	0.04372	0.2043		4515.8	4523.8	4523.9	4540.9	15.7331	0.2588
	(0.003773)	(0.2143)	(0.001904)	(0.000456)							

6.7 Concluding Remarks

A new class of generalized Dagum distribution called the gamma-Dagum distribution is proposed and studied. The GD distribution has the GB, GF, ED and D distributions as special cases. The density of this new class of distributions can be expressed as a linear combination of Dagum density functions. The GD distribution possesses hazard function with flexible behavior. We also obtain closed form expressions for the moments, mean and median deviations, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. Finally, the GD model is fitted to real data sets to illustrate the usefulness of the distribution.

6.8 Areas of Further Research

Bayesian analysis of model parameters by specifying prior distributions will be studied in the future. Also, inference for McLLog and GD distributions using censored data are going to be investigated. In particular, we will consider type I right censored and type II doubly censored data.

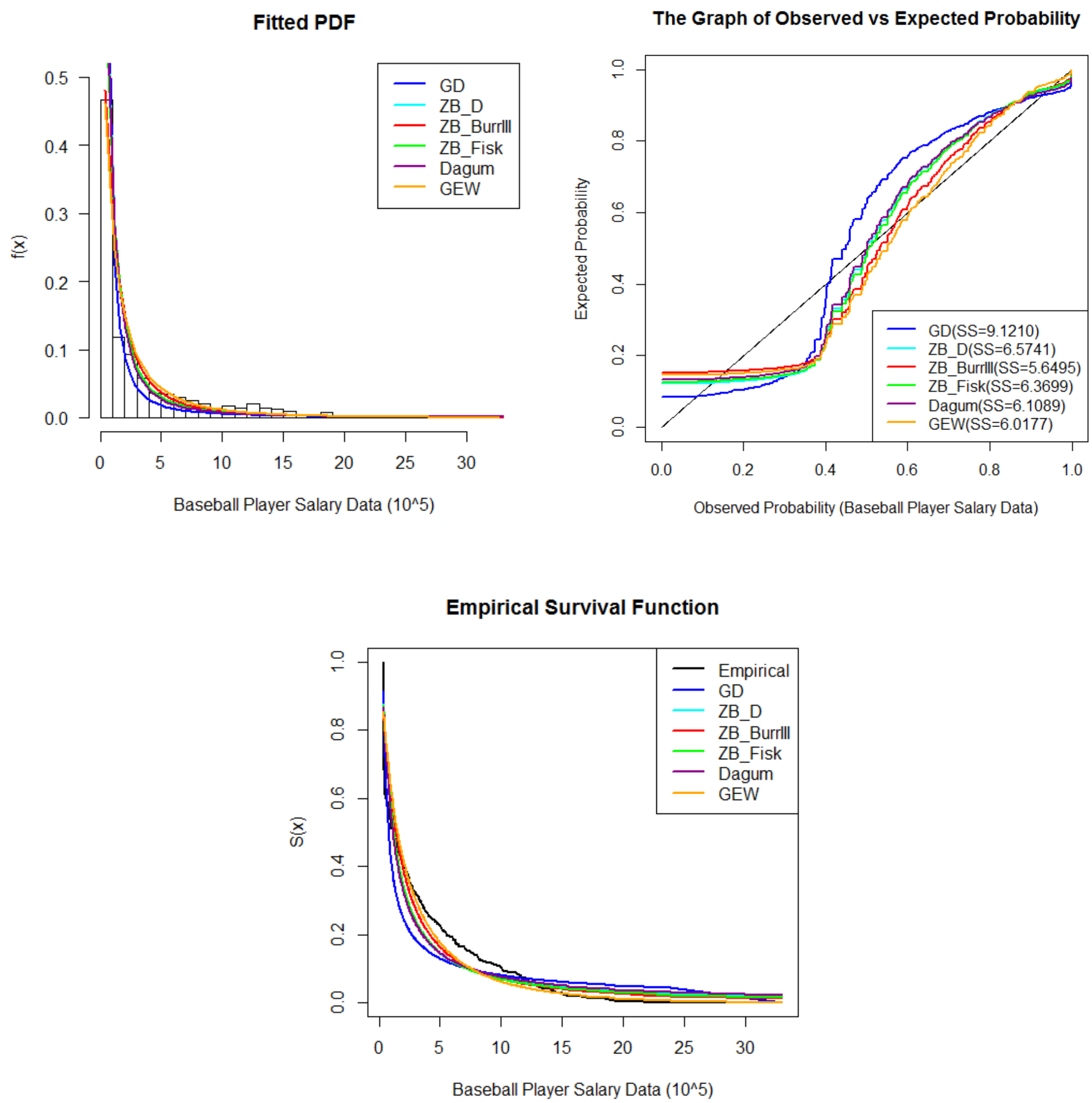

Figure 6.3: GD Fitted Densities, Observed Probabilities and Empirical Survival Curves for Baseball Player Salary Data

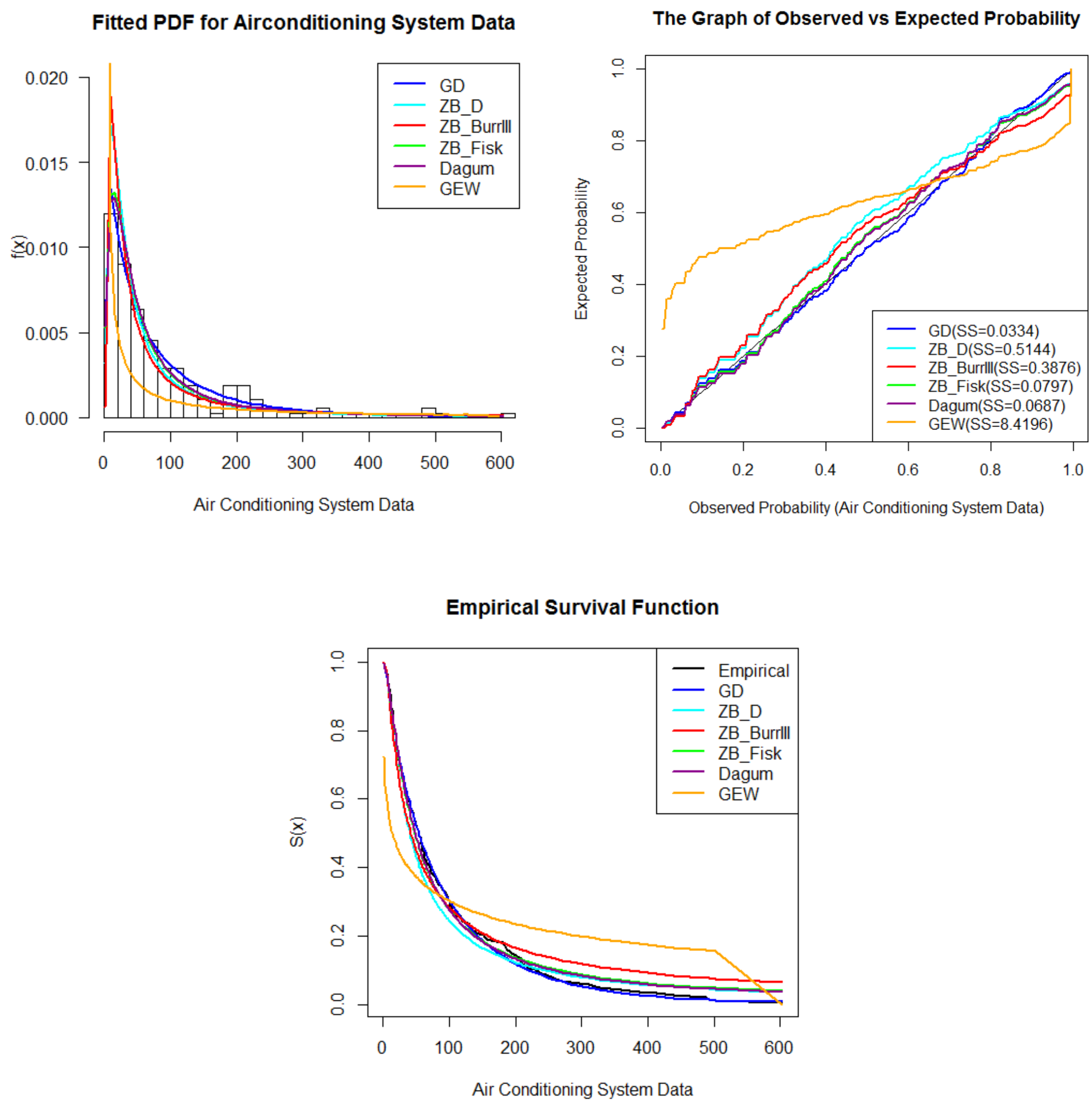

Figure 6.4: GD Fitted Densities, Observed Probabilities and Empirical Survival Curves for Air Conditioning System Data

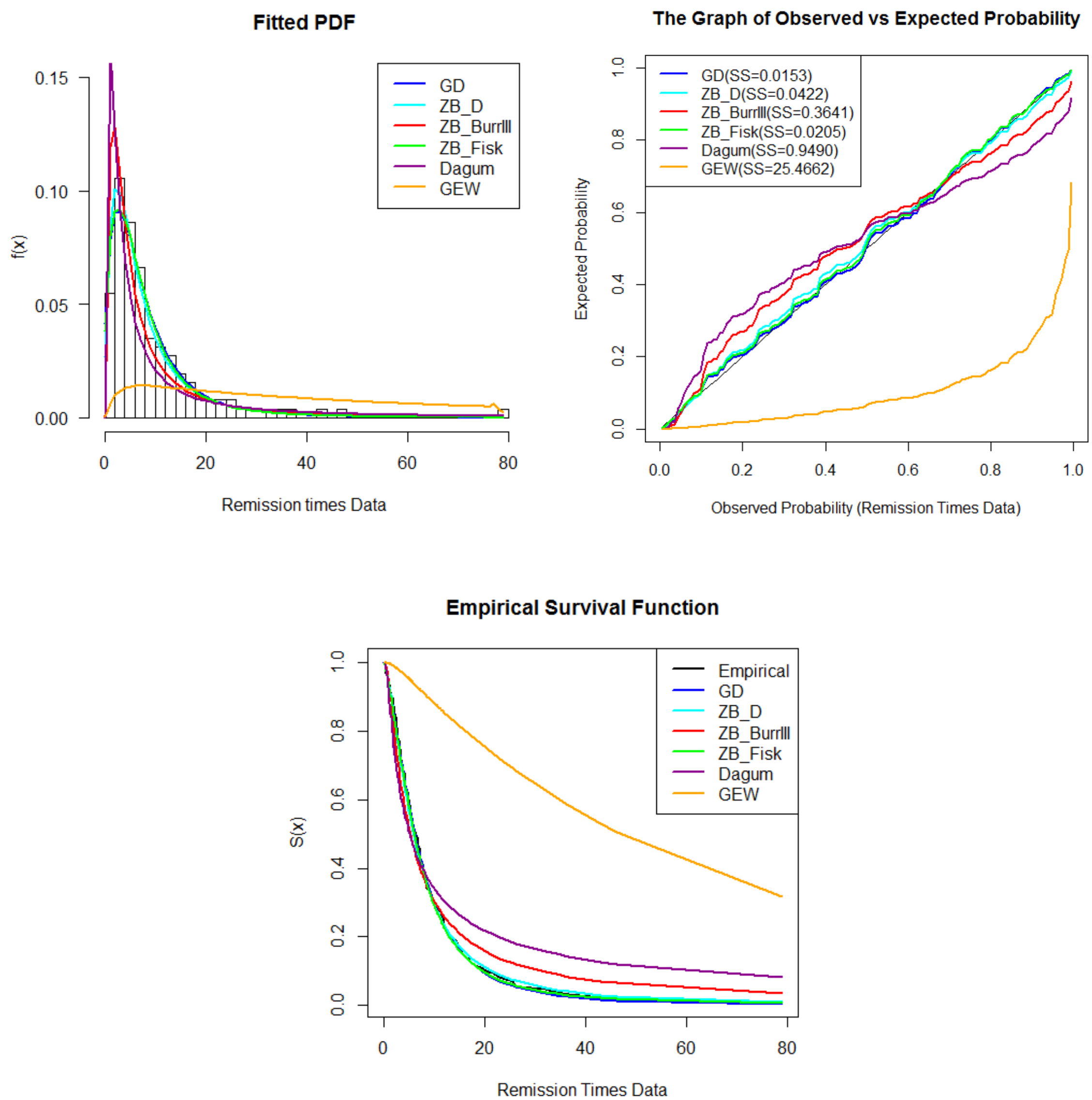

Figure 6.5: GD Fitted Densities, Observed Probabilities and Empirical Survival Curves for Remission Times Data

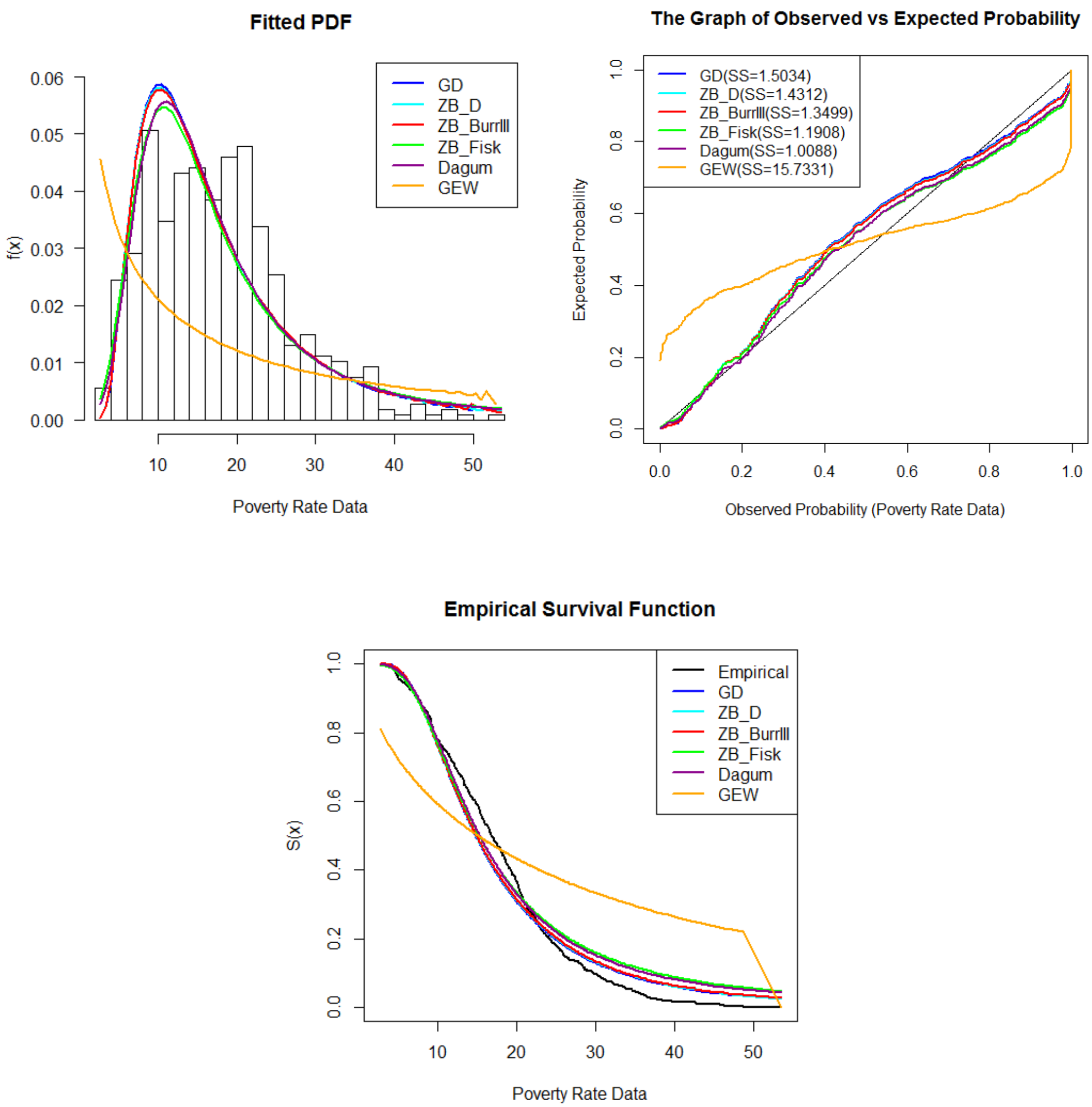

Figure 6.6: GD Fitted Densities, Observed Probabilities and Empirical Survival Curves for Poverty Rate Data

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Appendix A

R CODE FOR DISTRIBUTIONS

A.1 The Exponentiated Kumaraswamy-Dagum Distribution

```
# define EKD pdf
g=function(alpha,lambda,delta,phi,theta,x){
y=alpha*lambda*delta*phi*theta*((x)^(-delta-1))*((1+lambda*(x^(-delta)))^(-alpha-
1)) *((1-((1+lambda*(x^(-delta)))^(-alpha)))^(phi-1))
*((1-((1-((1+lambda*(x^(-delta)))^(-alpha)))^(phi)))^(theta-1))
return(y)
}
# define EKD cdf
G=function(alpha,lambda,delta,phi,theta,x){
y=(1-((1-((1+lambda*(x^(-delta)))^(-alpha)))^(phi)))^(theta)
return(y)
}
# define EKD hazard
h=function(alpha,lambda,delta,phi,theta,x){
y=g(alpha,lambda,delta,phi,theta,x)/(1-G(alpha,lambda,delta,phi,theta,x))
return(y)
}
# define EKD quantile
quantile=function(alpha,lambda,delta,phi,theta,q){
((lambda)^(1/delta))*((((1-((1-((q)^(1/theta)))^(1/phi)))^(-1/alpha))-1)^(-1/delta))
}
# define EKD moments.
```

```
# note: k<delta
moments=function(alpha,lambda,delta,phi,theta,k){
f=function(alpha,lambda,delta,phi,theta,k,x){(x^k)*(g(alpha,lambda,delta,phi,theta,x))}
y=integrate(f,lower=0,upper=Inf,subdivisions=100000,alpha=alpha,lambda=lambda,
delta=delta,phi=phi,theta=theta,k=k)
return(y)
}
# define EKD I(a)
Ia=function(alpha,lambda,delta,phi,theta,a){
n=length(a)
y=0
for(i in 1:n){
y[i]=integrate(function(alpha,lambda,delta,phi,theta,x){x*g(alpha,lambda,delta,phi,theta,x)},
lower=0,upper=a[i],subdivisions=100000,alpha=alpha,lambda=lambda,delta=delta,phi=phi,
theta=theta)$value
}
return(y)
}
# define EKD bonferroni
# note: p is between (0,1)
bonferroni=function(alpha,lambda,delta,phi,theta,p){
q=quantile(alpha,lambda,delta,phi,theta,p)
mu=moments(alpha,lambda,delta,phi,theta,1)$value
y=(Ia(alpha,lambda,delta,phi,theta,q))/(p*mu)
return(y)
}
```

```
# define EKD lorenz
# note: p is between (0,1)
lorenz=function(alpha,lambda,delta,phi,theta,p){
q=quantile(alpha,lambda,delta,phi,theta,p)
mu=moments(alpha,lambda,delta,phi,theta,1)$value
y=(Ia(alpha,lambda,delta,phi,theta,q))/(mu)
return(y)
}
```

A.2 The Log-Exponentiated Kumaraswamy-Dagum Distribution

```
# define Log-EKD pdf
LogEKD_pdf=function(alpha,lambda,delta,phi,theta,y){
alpha*lambda*delta*phi*theta*exp(-delta*y)*((1+lambda*exp(-delta*y))^(-alpha-1))*((1-
((1+lambda*exp(-delta*y))^(-alpha)))^(phi-1))*((1-((1-((1+lambda*exp(-delta*y))^(-
alpha)))^(phi)))^(theta-1))
}
# define Log-EKD cdf
LogEKD_cdf=function(alpha,lambda,delta,phi,theta,y){
(1-((1-((1+lambda*exp(-delta*y))^(-alpha)))^(phi)))^(theta)
}
# define Log-EKD hazard
LogEKD_hazard=function(alpha,lambda,delta,phi,theta,y){
LogEKD_pdf(alpha,lambda,delta,phi,theta,y)/(1-LogEKD_cdf(alpha,lambda,delta,phi,theta,y))
}
```

```
#define Log-EKD quantile
LogEKD_quantile=function(alpha,lambda,delta,phi,theta,q){
delta^(-1)*log(lambda)-(delta^(-1))*(log(((1-((1-q^(1/theta))^(1/phi)))^(-1/alpha))-1))
}
```

A.3 The McDonald Log-Logistic Distribution

```
# define McLLog pdf
# Note: beta(a,b)=integral_0^1 t^(a-1)(1-t)^(b-1)dt
McLLogg=function(a,b,c,lambda,delta,x){
(1/beta(a/c,b))*c*lambda*delta*(x^(-delta-1))*((1+lambda*(x^(-delta)))^(-a-1))
*((1-((1+lambda*((x)^(-delta)))^(-c)))^(b-1))
}
# define McLLog cdf
# Note: pbeta(x,a,b)=I_x(a,b)=B_x(a,b)/B(a,b)
McLLogG=function(a,b,c,lambda,delta,x){
pbeta((1+lambda*(x^(-delta)))^(-c),a/c,b)
}
# define McLLog Hazard
McLLogh=function(a,b,c,lambda,delta,x){
McLLogg(a,b,c,lambda,delta,x)/(1-McLLogG(a,b,c,lambda,delta,x))
}
# define McLLog Quantile
# Note: qbeta(pbeta(x,a,b),a,b)=x
McLLogquantile=function(a,b,c,lambda,delta,q){
```

```
((((qbeta(q,a/c,b))^(-1/c))-1)/lambda)^(-1/delta)
}
# define McLLog moments
# Note: s<delta
McLLogmoments=function(a,b,c,lambda,delta,s){
f=function(a,b,c,lambda,delta,s,x){(x^s)*McLLogg(a,b,c,lambda,delta,x)}
y=integrate(f,lower=0,upper=Inf,subdivisions=100000,a=a,b=b,c=c,
lambda=lambda,delta=delta,s=s)
return(y)
}
# define McLLog I(q)
McLLogIq=function(a,b,c,lambda,delta,q){
n=length(q)
y=0
for(i in 1:n){
y[i]=integrate(function(a,b,c,lambda,delta,x){x*McLLogg(a,b,c,lambda,delta,x)}
,lower=0,upper=q[i],subdivisions=100000,a=a,b=b,c=c,lambda=lambda,delta=delta)$value
}
return(y)
}
# define McLLog Bonferroni
McLLogBonferroni=function(a,b,c,lambda,delta,p){
q=McLLogquantile(a,b,c,lambda,delta,p)
mu=McLLogmoments(a,b,c,lambda,delta,1)$value
y=(McLLogIq(a,b,c,lambda,delta,q))/(p*mu)
return(y)
```

```
}
# define McLLog Lorenz
McLLogLorenz=function(a,b,c,lambda,delta,p){
q=McLLogquantile(a,b,c,lambda,delta,p)
mu=McLLogmoments(a,b,c,lambda,delta,1)$value
y=(McLLogIq(a,b,c,lambda,delta,q))/(mu)
return(y)
}
```

A.4 The Gamma-Dagum Distribution

```
# define GD pdf
GDg=function(lambda,beta,delta,alpha,theta,x){
lambda*beta*delta*(x^(-delta-1))/((gamma(alpha))*((theta)^(alpha)))
*((1+lambda*((x)^(-delta)))^(-beta-1))
*((-log(1-((1+lambda*((x)^(-delta)))^(-beta))))^(alpha-1))
*((1-((1+lambda*((x)^(-delta)))^(-beta)))^((1/theta)-1))
}
# define GD cdf
# note: pgamma(x,a)=integral_0^x t^(a-1)exp(-t)dt/gamma(a).
# incomplete gamma function: gamma(a,x)=integral_0^x t^(a-1)exp(-t)dt,
# i.e., pgamma(x,a)*gamma(a).
GDG=function(lambda,beta,delta,alpha,theta,x){
pgamma(-((theta)^(-1))*(log(1-((1+lambda*(x^(-delta)))^(-beta)))),alpha)
}
```

```
# define GD hazard
GDh=function(lambda,beta,delta,alpha,theta,x){
y=GDg(lambda,beta,delta,alpha,theta,x)/(1-GDG(lambda,beta,delta,alpha,theta,x))
return(y)
}
# define GD quantile
# note: qgamma(x,a)
GDquantile=function(lambda,beta,delta,alpha,theta,y){
u=qgamma(y,alpha)
y=((lambda)^(1/delta))*((((1-exp(-theta*u))^(-1/beta))-1)^(-1/delta))
return(y)
}
```