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Generalized Weibull and Inverse Weibull Distributions with Applications

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GENERALIZED WEIBULL AND INVERSE WEIBULL DISTRIBUTIONS WITH APPLICATIONS

by

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ABSTRACT

In this thesis, new classes of Weibull and inverse Weibull distributions including the generalized new modified Weibull (GNMW), gamma-generalized inverse Weibull (GGIW), the weighted proportional inverse Weibull (WPIW) and inverse new modified Weibull (INMW) distributions are introduced. The GNMW contains several submodels including the new modified Weibull (NMW), generalized modified Weibull (GMW), modified Weibull (MW), Weibull (W) and exponential (E) distributions, just to mention a few. The class of WPIW distributions contains several models such as: length-biased, hazard and reverse hazard proportional inverse Weibull, proportional inverse Weibull, inverse Weibull, inverse exponential, inverse Rayleigh, and Fréchet distributions as special cases. Included in the GGIW distribution are the submodels: gamma-generalized inverse Weibull, gamma-generalized Fréchet, gamma-generalized inverse Rayleigh, gamma-generalized inverse exponential, inverse Weibull, inverse Rayleigh, inverse exponential, Fréchet distributions. The INMW distribution contains several sub-models including inverse Weibull, inverse new modified exponential, inverse new modified Rayleigh, new modified Fréchet, inverse modified Weibull, inverse Rayleigh and inverse exponential distributions as special cases. Properties of these distributions including the behavior of the hazard function, moments, coefficients of variation, skewness, and kurtosis, $s-$entropy, distribution of order statistic and Fisher information are presented. Estimates of the parameters of the models via method of maximum likelihood (ML) are presented. Extensive simulation study is conducted and numerical examples are given.

Key Words: Proportional Inverse Weibull Distribution, Generalized Inverse Weibull Distribution, Weighted distribution, New modified Weibull distribution

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CHAPTER 1

INTRODUCTION

1.1 Inverse Weibull Distribution

The inverse Weibull (IW) distribution can be readily applied to modeling processes in reliability, ecology, medicine, branching processes and biological studies. The properties and applications of IW distribution in several areas can be seen in the literature (Keller [25], Calabria and Pulcini [6], [7], [8], Johnson [23], Khan et al. [26]). A random variable X has an IW distribution if the probability density function (pdf) is given by

$$f(x;\alpha,\beta) = \beta\alpha^{-\beta}x^{-\beta-1}exp[-(\alpha x)^{-\beta}], \qquad x \geq 0,\, \alpha > 0,\, \beta > 0. \tag{1.1}$$

If $\beta = 1$, the IW pdf becomes inverse exponential pdf, and when $\beta = 2$, the IW pdf is referred to as the inverse Raleigh pdf. The IW cumulative distribution function (cdf) is given by

$$F(x;\alpha,\beta) = exp[-(\alpha x)^{-\beta}], \qquad \alpha > 0,\, \beta > 0, \tag{1.2}$$

where α and β are the scale and shape parameters, respectively. If $\alpha = 1$, we have the Fréchet distribution function.

Motivated by various applications of inverse Weibull and weighted distributions, (Oluyede [38], Patil and Rao [39]) to biased samples in several areas including reliability, exponential tilting (weighting) in finance and actuarial sciences, we construct and present the statistical properties of new classes of distributions including generalized-gamma inverse Weibull (GGIW), weighted proportional inverse Weibull (WPIW), inverse new modified Weibull (INMW) distributions, and apply the proposed models to real lifetime data in order to demonstrate their usefulness. Generalized new modified Weibull (GNMW) distribution was also presented and studied.

1.2 Some Basic Utility Notions

In this section, some basic utility notions and definitions are presented. The n^{th}-order derivative of the gamma function is given by:

$$\Gamma^{(n)}(s) = \int_0^\infty z^{s-1}(\log z)^n \exp(-z)\, dz. \tag{1.3}$$

This derivative is used in this thesis. The lower and upper incomplete gamma functions are given by

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\, dt \quad \text{and} \quad \Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\, dt, \tag{1.4}$$

respectively. Also, $(1-z)^{b-1} = \sum_{j=0}^\infty \frac{(-1)^j\Gamma(b)}{\Gamma(b-j)\Gamma(j+1)} z^j$, for real non-integer $b > 0$ and $|z| < 1$. The digamma function is given by

$$\Psi(b) \;=\; \frac{d}{db}\ln\Gamma(b) = \frac{\Gamma'(b)}{\Gamma(b)}. \tag{1.5}$$

1.3 Outline of Thesis

This thesis is organized as follows. In Chapter 2, the gamma-generalized inverse Weibull (GGIW) distribution is presented. In Chapter 3, probability weighted moments and the weighted exponentiated or proportional inverse Weibull (WPIW) distribution are developed. Some statistical properties, including mode, hazard and reverse hazard functions are presented. Glaser's Lemma [17] is applied to the WPIW distribution to determine the behavior of the hazard function. Distribution of functions of WPIW random variables, moments, entropy measures and Fisher information are also given in Chapter 3, followed by concluding remarks. In Chapter 4, we present the WIW distribution, the special case of WPIW distribution, and some of its statistical properties. Estimation of parameters of WIW distribution via method of maximum likelihood is also given. Statistical properties of the generalized or exponentiated new modified Weibull distribution are developed in Chapter 5. In Chapter 6, we present the inverse new modified Weibull, as well as its sub-models. Hazard and reverse hazard functions, moments, $\nu-$entropy, Renyi entropy, order statistics, estimation of parameters, Fisher information, asymptotic confidence intervals and applications are presented.

CHAPTER 2

GAMMA-GENERALIZED INVERSE WEIBULL DISTRIBUTION

The inverse Weibull distribution has been used to model degradation of mechanical components such as pistons, crankshafts of diesel engines, as well as breakdown of insulating fluid to mention just a few areas. The usefulness and applications of inverse Weibull (IW) distribution in various areas including reliability, and branching processes can be seen in Keller et al. [25] and in references therein. The authors used the distribution to describe the degradation phenomena of mechanical components such as pistons, crank shaft of diesel engines. In this note, we generalize the inverse exponentiated Weibull distribution via the use of the gamma distribution function.

There are several generalizations of distribution including those of Eugene et al. [13] dealing with the beta-normal distribution, as well results on the moments of the beta-normal distribution given by Gupta and Nadarajah [20]. Famoye et al. [14] discussed and presented results on the beta-Weibull distribution. Nadarajah [34] studied the exponentiated beta distribution. Kong, Lee and Sepanski [24] presented results on the beta-gamma distribution.

In this chapter, we present and analyze the gamma-exponentiated or generalized inverse Weibull (GEIW or GGIW) distribution. First, we discuss the inverse Weibull distribution. The inverse Weibull (IW) cumulative distribution function (cdf) is given by

$$F(x,\alpha,\beta) = exp\Big[-(\alpha(x-x_0))^{-\beta}\Big], \qquad x \geq 0,\ \alpha > 0,\ \beta > 0,$$

where α, x_0 and β are the scale, location and shape parameters respectively. Often the parameter x_0 is called the minimum life or guarantee time. When $\alpha = 1$ and $x = x_0 + \alpha$, then $F(\alpha + x_0; 1; \beta) = F(\alpha + x_0; 1) = e^{-1} = 0.3679$. This value is in fact the characteristic life of the distribution. In what follows, we assume that $x_0 = 0$, and the IW distribution function becomes

$$F(x,\alpha,\beta) = exp[-(\alpha x)^{-\beta}], \qquad x \geq 0,\, \alpha > 0,\, \beta > 0.$$

The IW probability density function (pdf) is given by

$$f(x,\alpha,\beta) = \beta\alpha^{-\beta}x^{-\beta-1}exp(-(\alpha x)^{-\beta}), \qquad x \geq 0,\, \alpha > 0,\, \beta > 0.$$

The quantile function is $Q_F(y) = \{\frac{-\log(y)}{\alpha}\}^{-1/\beta}$. Note that when $\alpha = 1$, we have the Fréchet distribution function. Also, the IW probability density function (pdf) $f(x)$, satisfies:

$$xf(x,\alpha,\beta) = \beta F(x,\alpha,\beta)(-\ln(F(x,\alpha,\beta)), \qquad x \geq 0,\, \alpha > 0,\, \beta > 0.$$

Zografos and Balakrishnan [57] defined the gamma-generator, presented below, (when $\lambda = 1$) with pdf $g(x)$ and cdf $G(x)$ (for $\delta > 0$) as follows:

$$g(x) = \frac{1}{\Gamma(\delta)\lambda^{\delta}}[-\log(\overline{F}(x))]^{\delta-1}(1-F(x))^{(1/\lambda)-1}f(x), \tag{2.1}$$

and

$$G(x) = \frac{1}{\Gamma(\delta)\lambda^{\delta}}\int_0^{-\log(\overline{F}(x))} t^{\delta-1}e^{-t/\lambda}dt = \frac{\gamma(\delta, -\lambda^{-1}\log(\overline{F}(x)))}{\Gamma(\delta)},$$

respectively, where $F(x)$ is a baseline cdf, $g(x) = dG(x)/dx$, $\Gamma(\delta) = \int_0^{\infty} t^{\delta-1}e^{-t}dt$ is the gamma function, and $\gamma(z,\delta) = \int_0^z t^{\delta-1}e^{-t}dt$ denotes the incomplete gamma function. The corresponding hazard rate function (hrf) is

$$h_G(x) = \frac{[-\log(1-F(x))]^{\delta-1}f(x)(1-F(x))^{(1/\lambda)-1}}{\lambda^{\delta}(\Gamma(\delta) - \gamma(-\lambda^{-1}\log(1-F(x)),\delta))}.$$

When $\lambda = 1$, the distribution which of a special case of the family of distributions given in equation (2.1) is referred to as the Zografos and Balakrishnan-G family of distributions. Also, when $\lambda = 1$. Ristić and Balakrishnan [45] proposed an alternative gamma-generator defined by the cdf and pdf

$$G_2(x) = 1 - \frac{1}{\Gamma(\delta)\lambda^{\delta}}\int_0^{-\log F(x)} t^{\delta-1}e^{-t/\lambda}dt, \quad x \in \mathbf{R},\, \delta > 0,$$

and

$$g_2(x) = \frac{1}{\Gamma(\delta)\lambda^\delta}[-\log(F(x))]^{\delta-1}(F(x))^{(1/\lambda)-1}f(x), \quad (2.2)$$

respectively.

In this chapter, we presented a generalization of the IW distribution via the family given in equation (2.1). Zografos and Balakrishnan [57] motivated the ZB-G model as follows. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be lower record values from a sequence of independent and identically distributed (i.i.d.) random variables from a population with pdf $f(x)$. Then, the pdf of the n^{th} upper record value is given by equation (2.1), when $\lambda = 1$. A logarithmic transformation of the parent distribution F transforms the random variable X with density (2.1) to a gamma distribution. That is, if X has the density (2.1), then the random variable $Y = -\log[1-F(X)]$ has a gamma distribution $GAM(\delta; 1)$ with density $k(y;\delta) = \frac{1}{\Gamma(\delta)}y^{\delta-1}e^{-y}$, $y > 0$. The opposite is also true, if Y has a gamma $GAM(\delta; 1)$ distribution, then the random variable $X = G^{-1}(1-e^{-Y})$ has a ZB-G distribution (Zografos and Balakrishnan [57]). In addition to the motivations provided by Zografos and Balakrishnan [57], we are interested in the generalization of the inverse Weibull distribution via the gamma-generator and establishing the relationship between the distributions in equations (2.1) and (2.2), and weighted distributions in general.

Weighted distribution provides an approach to dealing with model specification and data interpretation problems. It adjusts the probabilities of actual occurrence of events to arrive at a specification of the probabilities when those events are recorded. Fisher [15] introduced the concept of weighted distribution, in order to study the effect of ascertainment upon estimation of frequencies. Rao [43] unified the concept of weighted distribution and used it to identify various sampling situations. Cox [12] and Zelen [56] introduced weighted distribution to present length biased sampling. Patil [39] used weighted distribution as stochastic models in the study of harvesting and predation. The usefulness and applications of weighted distribution to biased samples in various areas including medicine, ecology, reliability, and branching processes can also be seen in Nanda and Jain [37], Gupta and Keating [19], Oluyede [38] and in references therein.

Suppose Y is a non-negative random variable with its natural pdf $f(y;\underline{\theta})$, where $\underline{\theta}$ is a vector of parameters, then the pdf of the weighted random variable Y^w is given by:

$$f^w(y;\underline{\theta},\underline{\beta}) = \frac{w(y,\underline{\beta})f(y;\underline{\theta})}{\omega},$$

where the weight function $w(y,\underline{\beta})$ is a non-negative function, that may depend on the vector of parameters $\underline{\beta}$, and $0 < \omega = E(w(Y,\underline{\beta})) < \infty$ is a normalizing constant. A general class of weight function $w(y)$ is defined as follows:

$$w(y) = y^k e^{ly} F^i(y)\overline{F}^j(y).$$

Setting $k = 0$; $k = j = i = 0$; $l = i = j = 0$; $k = l = 0$; $i \to j-1$; $j = n-i$; $k = l = i = 0$ and $k = l = j = 0$ in this weight function, one at a time, implies probability weighted moments, moment-generating functions, moments, order statistics, proportional hazards and proportional reversed hazards, respectively, where $F(y) = P(Y \leq y)$ and $\overline{F}(y) = 1 - F(y)$. If $w(y) = y$, then $Y^* = Y^w$ is called the size-biased version of Y.

Ristić and Balakrishnan [45] provided motivations for the new family of distributions given in equation (2.2) when $\lambda = 1$, that is for $n \in N$, equation (2.2) is, the pdf of the n^{th} lower record value of a sequence of i.i.d. variables from a population with density $f(x)$. Ristić and Balakrishnan [45] used the exponentiated exponential (EE) distribution with cdf $F(x) = (1 - e^{-\beta x})^{\alpha}$, where $\alpha > 0$ and $\beta > 0$, and $\lambda = 1$ in equation (2.2) to obtain and study the gamma-exponentiated exponential (GEE) model. See references therein for additional results on the GEE model. Pinho et al. [42] presented results on the gamma-exponentiated Weibull distribution. In this note, we obtain a natural extension for the IW distribution, which we refer to as the gamma-generalized inverse Weibull (GGIW) distribution. Note that if $\lambda = 1$ and $\delta = n + 1$, in equation (2.1), we obtain the cdf and pdf of the upper record values U given by

$$G_U(u) = \frac{1}{n!} \int_0^{-\log(1-F(u))} y^n e^{-y} dy,$$

and

$$g_U(u) = f(u)[-\log(1 - F(u))]^n / n!.$$

Similarly, from equation (2.2), the pdf of the lower record values is given by

$$g_L(t) = f(t)[-\log(F(t))]^n / n!.$$

2.1 GGIW Distribution, Series Expansion and Sub-models

In this section, the GGIW distribution and some of its sub-models are presented. First, consider the generalized or exponentiated inverse Weibull (GIW or EIW) distribution given by

$$F_{_{GIW}}(x, \eta, \beta) = \exp[-\eta x^{-\beta}], \quad x \geq 0, \, \alpha > 0, \, \beta > 0, \, \theta > 0.$$

where $\eta = \theta \alpha^{-\beta}$. By inserting the GIW distribution in equation (2.1), we obtain the cdf of the GGIW distribution as follows:

$$G_{GGIW}(x) = \frac{1}{\Gamma(\delta)\lambda^\delta} \int_0^{-\log[1-e^{-\eta x^{-\beta}}]} t^{\delta-1} e^{-t/\lambda} dt = \frac{\gamma(-\lambda^{-1}\log(1 - e^{-\eta x^{-\beta}}), \delta)}{\Gamma(\delta)},$$

where $x > 0$, $\eta > 0$, $\beta > 0$, $\lambda > 0$, $\delta > 0$, and $\gamma(x, \delta) = \int_0^x t^{\delta-1} e^{-t} dt$ is the lower incomplete gamma function. The quantile function is obtained by solving the equation

$$G(Q_G(y)) = y, \quad 0 < y < 1. \tag{2.3}$$

From equation (2.3) the quantile function is

$$Q_G(y) = \eta^{-1/\beta}\left[-\log\left(1-\exp(-\lambda\gamma^{-1}(\Gamma(\delta)y,\delta))\right)\right]^{1/\beta}.$$

The GGIW pdf is given by

$$\begin{aligned} g_{_{GGIW}}(x) &= \frac{\eta\beta x^{-\beta-1}e^{-\eta x^{-\beta}}}{\Gamma(\delta)\lambda^{\delta}} \\ &\times \ [-\log(1-e^{-\eta x^{-\beta}})]^{\delta-1}[1-e^{-\eta x^{-\beta}}]^{(1/\lambda)-1}. \end{aligned}$$

If a random variable X has the GGIW density, we write $X \sim GGIW(\eta,\beta,\lambda,\delta)$. Let $y = \exp[-\eta x^{-\beta}]$, and $\psi = 1/\lambda$, then using the series representation $-\log(1-y) = \sum_{i=0}^{\infty}\frac{y^{i+1}}{i+1}$, we have

$$\left[-\log(1-y)\right]^{\delta-1} = y^{\delta-1}\left[\sum_{m=1}^{\infty}\binom{\delta-1}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right],$$

and applying the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty}a_s y^s\right)^m = \sum_{s=0}^{\infty}b_{s,m}y^s,$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, (Gradshteyn and Ryzhik [18]), the GGIW pdf can be written as

$$\begin{aligned} g_{_{GGIW}}(x) &= \frac{\eta\beta x^{-\beta-1}}{\Gamma(\delta)\lambda^{\delta}}y^{\delta}\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}y^{m+s}\sum_{k=0}^{\infty}\binom{\psi-1}{k}(-1)^k y^k \\ &= \frac{\eta\beta x^{-\beta-1}}{\Gamma(\delta)\lambda^{\delta}}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\delta-1}{m}\binom{\psi-1}{k}(-1)^k b_{s,m}y^{\delta+m+s+k} \\ &= \frac{1}{\Gamma(\delta)\lambda^{\delta}}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\delta-1}{m}\binom{\psi-1}{k}(-1)^k b_{s,m} \\ &\times \ \eta\beta x^{-\beta-1}e^{-\eta(\delta+m+s+k)x^{-\beta}} \\ &= \frac{1}{\Gamma(\delta)\lambda^{\delta}}\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}\binom{\delta-1}{m}\binom{\psi-1}{k}(-1)^k\frac{b_{s,m}}{\delta+m+s+k} \\ &\times \ \eta(\delta+m+s+k)\beta x^{-\beta-1}e^{-\eta(\delta+m+s+k)x^{-\beta}}, \end{aligned}$$

where $f(x;\alpha,\beta,\eta(\delta+m+s+k))$ is the generalized inverse Weibull pdf with parameters $\eta(\delta+m+s+k)$, and β. Let $C=\{(m,s,k)\in \mathbf{Z}_{+}^{\mathbf{3}}\}$, then the weights in the GGIW pdf above are

$$w_{\nu}=\frac{\psi^{\delta}}{\Gamma(\delta)}(-1)^{k}\binom{\delta-1}{m}\binom{\psi-1}{k}\frac{b_{m,s}}{\delta+m+s+k},$$

and the GGIW pdf can be written as

$$g_{_{GGIW}}(x)=\sum_{\nu\in C}w_{\nu}f(x;\alpha,\beta,\eta(\delta+m+s+k)).$$

It follows therefore that the GGIW density is a linear combination of the generalized or exponentiated inverse Weibull densities. The statistical and mathematical properties can be readily obtained from those of the generalized inverse Weibull distribution. Note that $g_{_{GGIW}}(x)$ is a weighted pdf with weight function

$$w(x)=[-\log(1-F(x))]^{\delta-1}[1-F(x)]^{\frac{1}{\lambda}-1},$$

that is,

$$\begin{aligned} g_{_{GGIW}}(x) &= \frac{[-\log(1-F(x))]^{\delta-1}[1-F(x)]^{\frac{1}{\lambda}-1}}{\lambda^{\delta}\Gamma(\delta)}f(x) \\ &= \frac{w(x)f(x)}{E_F(w(X))}, \end{aligned}$$

where $0<E_F\{[-\log(1-F(x))]^{\delta-1}[1-F(x)]^{\frac{1}{\lambda}-1}\}=\lambda^{\delta}\Gamma(\delta)<\infty$, is the normalizing constant. Similarly,

$$g_2(x) = \frac{[-\log(F(X))]^{\delta-1}[F(X)]^{\frac{1}{\lambda}-1}}{\lambda^{\delta}\Gamma(\delta)}f(x)=\frac{w(x)f(x)}{E_F(w(X))},$$

where $0<E_F(w(X))=E_F([-\log(F(X))]^{\delta-1}[F(X)]^{\frac{1}{\lambda}-1})=\lambda^{\delta}\Gamma(\delta)<\infty$.

Graphs of the GGIW pdf for five combinations of the values of the parameters are given in Figure 2.1. The graphs are asymmetric and right skewed. For some combinations of the GGIW model parameter values the graph of the pdf can be decreasing.

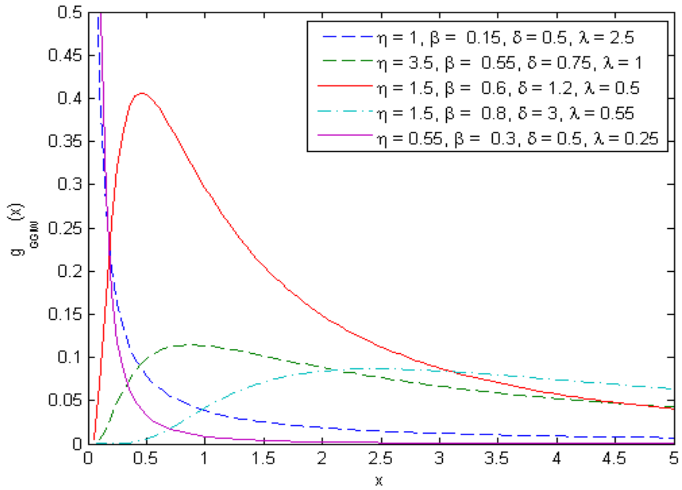

Figure 2.1: Plots of GGIW pdf for selected values of the parameters

2.1.1 Some Sub-models of the GGIW Distribution

Some of the sub-models of the GGIW distribution are listed below:

- If $\lambda = 1$, we obtain the gamma-generalized inverse Weibull distribution via the ZB-G (ZB-IW) distribution. Also, with $\lambda = \beta = 1$, we have the ZB-inverse exponential (ZB-IE) distribution. Similarly, if $\lambda = 1$ and $\beta = 2$, we obtain the ZB-inverse Rayleigh (ZB-IR) distribution.

- If $\eta = 1$, we get the gamma-generalized Fréchet (GGF) distribution.

- When $\beta = 1$, we have the gamma-generalized inverse exponential (GGIE) distribution.

- If $\beta = 2$, we obtain the gamma-generalized inverse Rayleigh (GGIR) distribution.

- When $\lambda = \delta = 1$, we have the inverse Weibull (IW) distribution.

- If $\beta = 2$, and $\lambda = \delta = 1$, we obtain the inverse Rayleigh (IR) distribution.

- When $\lambda = \delta = \beta = 1$, we get the Inverse exponential (IE) distribution.

- When $\lambda = \eta = \delta = 1$, we obtain Fréchet (F) distribution.

2.1.2 Hazard and Reverse Hazard Functions

Let X be a continuous random variable with distribution function F, and probability density function (pdf) f, then the hazard function, reverse hazard function and mean residual life functions are given by $h_F(x) = f(x)/\overline{F}(x)$, $\tau_F(x) = f(x)/F(x)$, and $\delta_F(x) = \int_x^\infty \overline{F}(u)du/\overline{F}(x)$, respectively. The functions $h_F(x)$, $\delta_F(x)$, and $\overline{F}(x)$ are equivalent (Shaked and Shanthikumar [50]). The hazard and reverse hazard functions of the GGIW distribution are

$$h_G(x) = \frac{\eta\beta x^{-\beta-1}e^{-\eta x^{-\beta}}(-\log(1-e^{-\eta x^{-\beta}}))^{\delta-1}[1-e^{-\eta x^{-\beta}}]^{\lambda^{-1}-1}}{\lambda^{\delta}(\Gamma(\delta)-\gamma(-\lambda^{-1}\log(1-e^{-\eta x^{-\beta}}),\delta))},$$

and

$$\tau_G(x) = \frac{\eta\beta x^{-\beta-1}e^{-\eta x^{-\beta}}(-\log(1-e^{-\eta x^{-\beta}}))^{\delta-1}[1-e^{-\eta x^{-\beta}}]^{\lambda^{-1}-1}}{\lambda^{\delta}(\gamma(-\lambda^{-1}\log(1-e^{-\eta x^{-\beta}}),\delta))},$$

for $x \geq 0$, $\eta > 0$, $\beta > 0$, $\lambda > 0$, $\delta > 0$, respectively.

Plots of the GGIW hazard rate function for selected values of the parameters are given in Figure 2.2. The graphs of the hazard rate function for the five combinations of the parameter values are uni-modal and upside down bathtub shaped.

2.2 Moments and Moment Generating Function

In this section, we obtain moments and moment generating function of the GGIW distribution. Let $\eta^* = \eta(\delta + m + s + k)$, and $Y \sim GIW(\beta, \eta^*)$. Note that from $Y \sim GIW(\beta, \eta^*)$, the j^{th} moment of the random variable Y is

$$E(Y^j) = (\eta^*)^{j/\beta}\Gamma(1 - j\beta^{-1}), \quad \beta > j,$$

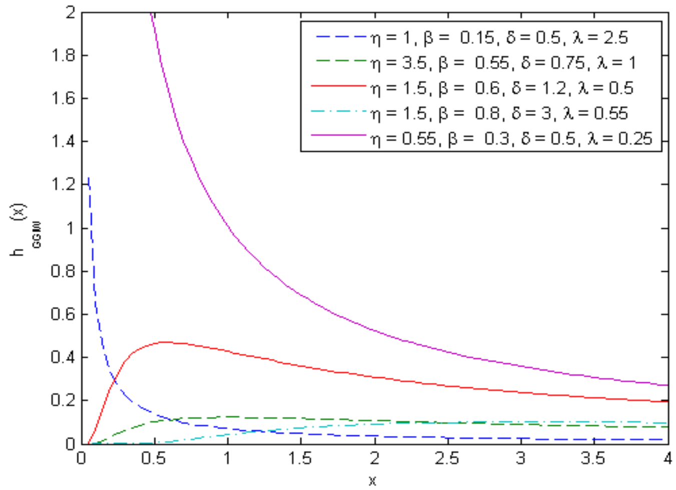

Figure 2.2: Plots of GGIW hazard function for selected values of the parameters.

so that the j^{th} raw moment of GGIW distribution is given by:

$$E(X^j) = \sum_{\nu \in C} w_\nu E(Y^j).$$

The moment generating function (MGF), for $|t| < 1$, is given by:

$$\begin{aligned} M_X(t) &= \sum_{\nu \in C} w_\nu M_Y(t) \\ &= \sum_{\nu \in C} \sum_{i=0}^{\infty} w_\nu \frac{t^i}{i!} (\eta^*)^{i/\beta} \Gamma(1 - i\beta^{-1}). \end{aligned}$$

Theorem 2.1.

$$E\{[-\log(1 - F(X))]^r [(1 - F(X))^s]\} = \frac{\lambda^r \Gamma(r + \delta)}{(s\lambda + 1)^\delta \Gamma(\delta)}, \quad \beta > j.$$

If $s = 0$,

$$E[-\log(1 - F(X))^r] = \frac{\lambda^r \Gamma(r + \delta)}{\Gamma(\delta)},$$

and similarly, if $r = 0$,

$$E[(1 - F(X))^s] = [s\lambda + 1]^{-\delta}.$$

Proof:

$$\begin{aligned} E\{[-\log(1-F(X))]^r[(1-F(X))^s]\} &= \int_0^\infty \frac{f(x)[-\log(1-F(x))]^{r+\delta-1}}{\lambda^\delta\Gamma(\delta)} \\ &\times [1-F(x)]^{s+(1/\lambda)-1}dx \\ &= \frac{\lambda^r\Gamma(r+\delta)}{(s\lambda+1)^\delta\Gamma(\delta)}. \end{aligned}$$

If $s=0$, we have

$$\begin{aligned} E[-\log(1-F(X))^r] &= \frac{\lambda^{r+\delta}\Gamma(r+\delta)}{\lambda^\delta\Gamma(\delta)}\int_0^\infty \frac{f(x)}{\lambda^{r+\delta}\Gamma(r+\delta)} \\ &\times [-\log(1-F(x))]^{r+\delta-1}[1-F(x)]^{(1/\lambda)-1}dx \\ &= \frac{\lambda^{r+\delta}\Gamma(r+\delta)}{\lambda^\delta\Gamma(\delta)}. \end{aligned}$$

Let $\lambda^* = s + \frac{1}{\lambda}$, then with $r=0$, we obtain

$$\begin{aligned} E[(1-F(X))^s] &= \int_0^\infty \frac{(\lambda^*)^\delta f(x)}{\Gamma(\delta)}[-\log(1-F(x))]^{\delta-1}[1-F(x)]^{\lambda^*-1}dx \\ &\times \left(\frac{1}{\lambda\lambda^*}\right)^\delta = [s\lambda+1]^{-\delta}. \end{aligned}$$

2.3 Order Statistics and Rényi Entropy

Order Statistics play an important role in probability and statistics. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. In this section, we present the distribution of the order statistics, and Rényi entropy for the GGIW distribution.

2.3.1 Order Statistics

Let $X_1, X_2, ..., X_n$ be independent and identically distributed GGIW random variables. The pdf of the i^{th} order statistic from the GGIW pdf $g_{GGIW}(x) = g(x)$ is given by

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-i} \\
&= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[G(x)]^{i+j-1} \\
&= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left[\frac{\gamma(-\lambda^{-1}\log(1-\overline{F}(x),\delta))}{\Gamma(\delta)}\right]^{i+j-1}.
\end{aligned}$$

Using the fact that $\gamma(x,\delta)=\sum_{m=0}^{\infty}\frac{(-1)^m x^{m+\delta}}{(m+\delta)m!}$, and setting $c_m=(-1)^m/((m+\delta)m!)$, we have

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\frac{(-1)^j}{[\Gamma(\delta)]^{i+j-1}} \\
&\times\ [-\lambda^{-1}\log(\overline{F}(x))]^{\delta(i+j-1)}\left[\sum_{m=0}^{\infty}\frac{(-1)^m(-\lambda^{-1}\log(\overline{F}(x)))^m}{(m+\delta)m!}\right]^{i+j-1} \\
&= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^j}{[\Gamma(\delta)]^{i+j-1}}[-\lambda^{-1}\log(\overline{F}(x))]^{\delta(i+j-1)} \\
&\times\ \sum_{m=0}^{\infty}d_{m,i+j-1}(-\lambda^{-1}\log(\overline{F}(x)))^m,
\end{aligned}$$

where $d_0=c_0^{(i+j-1)}$, $d_{m,i+j-1}=(mc_0)^{-1}\sum_{l=1}^{m}[(i+j-1)l-m+l]c_l d_{m-l,i+j-1}$. It follows therefore that

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j-1}} \\
&\times\ [-\lambda^{-1}\log(\overline{F}(x))]^{\delta(i+j-1)+m} \\
&= \frac{n![-\log(\overline{F}(x))]^{\delta-1}[\overline{F}(x)]^{\psi-1}f(x)}{(i-1)!(n-i)!\Gamma(\delta)\lambda^{\delta}}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j-1}} \\
&\times\ [-\lambda^{-1}\log(\overline{F}(x))]^{\delta(i+j-1)+m} \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j}} \\
&\times\ \frac{[-\log(\overline{F}(x))]^{\delta(i+j-1)+m+\delta-1}[\overline{F}(x)]^{\psi-1}f(x)}{\lambda^{i+j}}.
\end{aligned}$$

That is,

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,n-i+j}}{[\Gamma(\delta)]^{i+j}}\frac{1}{\lambda^{\delta(i+j)+m}}\\
&\times [-\log(\overline{F}(x))]^{\delta(i+j)+m-1}[\overline{F}(x)]^{\psi-1}f(x)\\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}\Gamma(\delta(i+j)+m)}{[\Gamma(\delta)]^{i+j}}\\
&\times \frac{[-\log(\overline{F}(x))]^{\delta(i+j)+m-1}[\overline{F}(x)]^{\psi-1}f(x)}{\Gamma(\delta(i+j)+m)\lambda^{\delta(i+j)+m}}\\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j d_{m,i+j-1}\Gamma(\delta(i+j)+m)}{[\Gamma(\delta)]^{i+j}}\\
&\times g(x;\eta,\beta,\lambda,\delta^*),
\end{aligned}$$

where $g(x;\eta,\beta,\lambda,\delta^*)$ is the GGIW pdf with parameters η,β, λ, and shape parameter $\delta^*=\delta(i+j)+m$. It follows therefore that

$$\begin{aligned}
E(X_{i:n}^j) &= \frac{n!}{(i-1)!(n-i)!\Gamma(\delta)}\sum_{\nu\in C}\sum_{j=0}^{n-i}\sum_{m=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j w_\nu d_{m,i+j-1}}{[\Gamma(\delta)]^{i+j}}\\
&\times \Gamma(\delta(i+j)+m)(\eta^*)^{j/\beta}\Gamma(1-j\beta^{-1}),
\end{aligned}$$

for $j<\beta$. These moments are often used in several areas including reliability, engineering, biometry, insurance and quality control for the prediction of future failures times from a set of past or previous failures.

2.3.2 Rényi Entropy

Rényi entropy is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^\infty[g(x;\eta,\beta,\lambda,\delta)]^v dx\right), v\neq 1, v>0.$$

Rényi entropy tends to Shannon entropy as $v\to 1$. Note that

$$\begin{aligned}
\int_0^\infty g^v(x)dx &= \left(\frac{\eta\beta}{\lambda^\delta\Gamma(\delta)}\right)^v\int_0^\infty x^{-v\beta-v}e^{-v\eta x^{-\beta}}[1-e^{-\eta x^{-\beta}}]^{\frac{v}{\lambda}-1}\\
&\times [-\log(1-e^{-\eta x^{-\beta}})]^{v\delta-v}dx.
\end{aligned}$$

Let $y = e^{-\eta x^{-\beta}}$, then using the same results as in section 2.1, we have for $\delta > 1$, and v/λ a natural number,

$$\begin{aligned}
\int_0^{\infty} g^v(x)dx &= \left(\frac{\eta\beta}{\lambda^\delta\Gamma(\delta)}\right)^v \sum_{m=1}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v\delta - v}{m}\binom{(v/\lambda)-1}{k} b_{s,m} \\
&\times \int_0^{\infty} x^{-v\beta - v} e^{-\eta(v\delta+m+s+k)x^{-\beta}}dx \\
&= \frac{\eta^v\beta^{v-1}\Gamma(v+\frac{1}{\beta}(v-1))}{(\lambda^\delta\Gamma(\delta))^v}\cdot\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v\delta-v}{m}\binom{(\frac{v}{\lambda})-1}{k} \\
&\times b_{s,m}[\eta(v\delta+m+s+k)]^{\frac{1}{\beta}(1-v)-v}.
\end{aligned}$$

Consequently, Rényi entropy is given by

$$\begin{aligned}
I_R(v) &= \left(\frac{1}{1-v}\right)\log\left[\frac{\eta^v\beta^{v-1}\Gamma(v+\frac{1}{\beta}(v-1))}{(\lambda^\delta\Gamma(\delta))^v}\right. \\
&\times \left.\sum_{m=0}^{\infty}\sum_{s,k=0}^{\infty}(-1)^k\binom{v\delta-v}{m}\binom{(\frac{v}{\lambda})-1}{k}b_{s,m}[\eta(v\delta+m+s+k)]^{\frac{1}{\beta}(1-v)-v}\right],
\end{aligned}$$

for $v > 0$, $v \neq 1$.

2.4 Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a random sample from the GGIW distribution. The likelihood function is given by

$$\begin{aligned}
L(\eta,\beta,\lambda,\delta) &= \frac{(\eta\beta)^n}{[\lambda^\delta\Gamma(\delta)]^n}e^{-\eta\sum_{i=1}^n x_i^{-\beta}}\prod_{i=1}^n\left\{x_i^{-\beta-1}\right. \\
&\times \left.\left[-\log\left(1-e^{-\eta x_i^{-\beta}}\right)\right]^{\delta-1}\left[1-e^{-\eta x_i^{-\beta}}\right]^{(1/\lambda)-1}\right\}.
\end{aligned}$$

Now, the log-likelihood function denoted by ℓ is given by

$$\begin{aligned}
\ell &= \log[L(\eta,\beta,\lambda,\delta)] \\
&= n\log(\eta) + n\log(\beta) - n\log(\Gamma(\delta)) - n\delta\log(\lambda) \\
&+ (-\beta-1)\sum_{i=1}^{n}\log(x_i) - \eta\sum_{i=1}^{n}x_i^{-\beta} \\
&+ (\delta-1)\sum_{i=1}^{n}\log\left[-\log\left(1-e^{-\eta x_i^{-\beta}}\right)\right] \\
&+ \left(\frac{1}{\lambda}-1\right)\sum_{i=1}^{n}\log\left(1-e^{-\eta x_i^{-\beta}}\right).
\end{aligned}$$

The entries of the score function are given by

$$\begin{aligned}
\frac{\partial\ell}{\partial\beta} &= \frac{n}{\beta} - \sum_{i=1}^{n}\log(x_i) + \eta\sum_{i=1}^{n}x_i^{-\beta}\log(x_i) \\
&- (\delta-1)\sum_{i=1}^{n}\frac{\eta x_i^{-\beta}e^{-\eta x_i^{-\beta}}\log(x_i)}{(1-e^{-\eta x_i^{-\beta}})\log(1-e^{-\eta x_i^{-\beta}})} \\
&- \left(\frac{1}{\lambda}-1\right)\sum_{i=1}^{n}\frac{\eta x_i^{-\beta}e^{-\eta x_i^{-\beta}}\log(x_i)}{(1-e^{-\eta x_i^{-\beta}})},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\ell}{\partial\eta} &= \frac{n}{\eta} - \sum_{i=1}^{n}x_i^{-\beta} + (\delta-1)\sum_{i=1}^{n}\frac{x_i^{-\beta}e^{-\eta x_i^{-\beta}}}{(1-e^{-\eta x_i^{-\beta}})\log(1-e^{-\eta x_i^{-\beta}})} \\
&+ \left(\frac{1}{\lambda}-1\right)\sum_{i=1}^{n}\frac{x_i^{-\beta}e^{-\eta x_i^{-\beta}}}{(1-e^{-\eta x_i^{-\beta}})},
\end{aligned}$$

$$\frac{\partial\ell}{\partial\delta} = -\frac{n\Gamma'(\delta)}{\Gamma(\delta)} - n\log(\lambda) + \sum_{i=1}^{n}\log\left(-\log\left(1-e^{-\eta x_i^{-\beta}}\right)\right),$$

and

$$\frac{\partial\ell}{\partial\lambda} = -\frac{n\delta}{\lambda} - \frac{1}{\lambda^2}\sum_{i=1}^{n}\log\left(1-e^{-\eta x_i^{-\beta}}\right).$$

The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters $\eta,\beta,\lambda,\delta$ must be found by using iterative methods. The maximum likelihood estimates of the parameters, denoted by

$\hat{\mathbf{\Theta}} = (\hat{\eta}, \hat{\beta}, \hat{\lambda}, \hat{\delta})$ is obtained by solving the nonlinear equation $(\frac{\partial \ell}{\partial \eta}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \delta})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix (FIM) is given by $\mathbf{I}(\mathbf{\Theta}) = [\mathbf{I}_{\theta_i,\theta_j}]_{4X4} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3, 4$, can be numerically obtained by MATHLAB, R or MAPLE software. The total Fisher information matrix $n\mathbf{I}(\mathbf{\Theta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Delta}}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \bigg|_{\mathbf{\Delta}=\hat{\mathbf{\Delta}}} \right]_{4X4}, \quad i, j = 1, 2, 3, 4. \tag{2.4}$$

For a given set of observations, the matrix given in equation (2.4) is obtained after the convergence of the Newton-Raphson procedure in MATHLAB or R software. Elements of the observed information matrix of the GGIW distribution can be readily obtain from the second and mixed partial derivatives given below:

$$\begin{aligned}
\frac{\partial^2 \ln g_{GGIW}(x; \eta, \beta, \lambda, \delta)}{\partial \eta^2} \quad = \quad & -\frac{e^{-2\eta x^{-\beta}} \left(\frac{1}{\lambda} - 1\right) x^{-2\beta}}{\left(1 - e^{-\eta x^{-\beta}}\right)^2} - \frac{e^{-\eta x^{-\beta}} \left(\frac{1}{\lambda} - 1\right) x^{-2\beta}}{1 - e^{-\eta x^{-\beta}}} \\
& -\frac{1}{\eta^2} - \frac{(\delta - 1) e^{-2\eta x^{-\beta}} x^{-2\beta}}{\left(1 - e^{-\eta x^{-\beta}}\right)^2 \ln^2\left(1 - e^{-\eta x^{-\beta}}\right)} \\
& -\frac{(\delta - 1) e^{-2\eta x^{-\beta}} x^{-2\beta}}{\left(1 - e^{-\eta x^{-\beta}}\right)^2 \ln\left(1 - e^{-\eta x^{-\beta}}\right)} \\
& -\frac{(\delta - 1) e^{-\eta x^{-\beta}} x^{-2\beta}}{\left(1 - e^{-\eta x^{-\beta}}\right) \ln\left(1 - e^{-\eta x^{-\beta}}\right)},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\eta\partial\beta} &= x^{-\beta}\ln(x) - \frac{\eta\left(\frac{1}{\lambda}-1\right)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2} \\
&\quad + \frac{\eta\left(\frac{1}{\lambda}-1\right)e^{-\eta x^{-\beta}}x^{-2\beta}\ln(x)}{1-e^{-\eta x^{-\beta}}} \\
&\quad - \frac{\left(\frac{1}{\lambda}-1\right)e^{-\eta x^{-\beta}}x^{-\beta}\ln(x)}{1-e^{-\eta x^{-\beta}}} \\
&\quad + \frac{\eta(\delta-1)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2\ln^2\left(1-e^{-\eta x^{-\beta}}\right)} \\
&\quad - \frac{\eta(\delta-1)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2\ln\left(1-e^{-\eta x^{-\beta}}\right)} \\
&\quad + \frac{\eta(\delta-1)e^{-\eta x^{-\beta}}x^{-2\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)} \\
&\quad - \frac{(\delta-1)e^{-\eta x^{-\beta}}x^{-\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)},
\end{aligned}$$

$$\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\eta\partial\lambda} = \frac{e^{-\eta x^{-\beta}}x^{-\beta}}{\lambda^2\left(1-e^{-\eta x^{-\beta}}\right)},$$

$$\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\eta\partial\delta} = \frac{e^{-\eta x^{-\beta}}x^{-\beta}}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)},$$

$$\begin{aligned}
\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\beta^2} &= -\frac{1}{\beta^2} - \frac{\eta^2\left(\frac{1}{\lambda}-1\right)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln^2(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2} \\
&\quad - \frac{\eta^2\left(\frac{1}{\lambda}-1\right)e^{-\eta x^{-\beta}}x^{-2\beta}\ln^2(x)}{1-e^{-\eta x^{-\beta}}} \\
&\quad - \eta x^{-\beta}\ln^2(x) + \frac{\left(\frac{1}{\lambda}-1\right)e^{-\eta x^{-\beta}}x^{-\beta}\ln^2(x)}{1-e^{-\eta x^{-\beta}}} \\
&\quad - \frac{\eta^2(\delta-1)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln^2(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2\ln^2\left(1-e^{-\eta x^{-\beta}}\right)} \\
&\quad - \frac{\eta^2(\delta-1)e^{-2\eta x^{-\beta}}x^{-2\beta}\ln^2(x)}{\left(1-e^{-\eta x^{-\beta}}\right)^2\ln\left(1-e^{-\eta x^{-\beta}}\right)} \\
&\quad - \frac{\eta^2(\delta-1)e^{-\eta x^{-\beta}}x^{-2\beta}\ln^2(x)}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)} \\
&\quad + \frac{(\delta-1)e^{-\eta x^{-\beta}}x^{-\beta}\ln^2(x)}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)},
\end{aligned}$$

$$\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\beta\partial\lambda} = \frac{\eta e^{-\eta x^{-\beta}} x^{-\beta}\ln(x)}{\lambda^2\left(1-e^{-\eta x^{-\beta}}\right)},$$

$$\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\beta\partial\delta} = -\frac{\eta e^{-\eta x^{-\beta}} x^{-\beta}\ln(x)}{\left(1-e^{-\eta x^{-\beta}}\right)\ln\left(1-e^{-\eta x^{-\beta}}\right)},$$

$$\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\lambda^2} = \frac{\delta}{\lambda^2} + \frac{2\ln\left(1-e^{-\eta x^{-\beta}}\right)}{\lambda^3},$$

$$\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\lambda\partial\delta} = -\frac{1}{\lambda},$$

and

$$\frac{\partial^2 \ln g_{GGIW}(x;\eta,\beta,\lambda,\delta)}{\partial\delta^2} = -\Psi'(\delta).$$

2.4.1 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the GGIW distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\mathbf{\Theta}} = (\hat{\eta}, \hat{\beta}, \hat{\lambda}, \hat{\delta})$ be the maximum likelihood estimate of $\mathbf{\Theta} = (\eta, \beta, \lambda, \delta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) \xrightarrow{d} N_4(\underline{0}, I^{-1}(\mathbf{\Theta}))$, where $I(\mathbf{\Theta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Theta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$, that is $J(\hat{\mathbf{\Theta}})$. The multivariate normal distribution $N_4(\underline{\mathbf{0}}, J(\hat{\mathbf{\Theta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0,0,0,0)^T$ can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. A large sample $100(1-\alpha)\%$ confidence intervals for η, β, λ, and δ are:

$$\widehat{\eta} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\eta\eta}(\widehat{\mathbf{\Theta}})},\ \widehat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\beta\beta}(\widehat{\mathbf{\Theta}})},\ \widehat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\lambda\lambda}(\widehat{\mathbf{\Theta}})},\ \text{and}\ \widehat{\delta} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\delta\delta}(\widehat{\mathbf{\Theta}})},$$

respectively, where $I_{\eta\eta}^{-1}(\hat{\mathbf{\Theta}})$, $I_{\beta\beta}^{-1}(\hat{\mathbf{\Theta}})$, $I_{\lambda\lambda}^{-1}(\hat{\mathbf{\Theta}})$, and $I_{\delta\delta}^{-1}(\hat{\mathbf{\Theta}})$ are the diagonal elements of $I_n^{-1}(\hat{\mathbf{\Theta}})$, and $Z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}^{th}$ percentile of a standard normal distribution.

The maximum likelihood estimates (MLEs) of the GGIW parameters η, β, λ, and δ are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\Theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Tables 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6. The values of the Kolmogorov-Smirnov statistic, $KS = \max_{1\leq i\leq n}\{G(x_i) - \frac{i-1}{n}, \frac{i}{n} - G(x_i)\}$ are also presented in Tables 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6. The GGIW distribution is fitted to the data sets and compared to the fits for the GGIE, GGIR, GIW, IW and ZB-inverse exponential distributions.

We can use the likelihood ratio (LR) test to compare the fit of the GGIW distribution with its sub-models for a given data set. For example, to test $\lambda = \delta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{\eta}, \hat{\beta}, \hat{\lambda}, \hat{\delta})) - \ln(L(\tilde{\eta}, \tilde{\beta}, 1, 1))]$, where $\hat{\eta}$, $\hat{\beta}$, $\hat{\lambda}$, and $\hat{\delta}$, are the unrestricted estimates, and $\tilde{\eta}$, and $\tilde{\beta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 2 degrees of freedom.

2.5 Applications

In this section, we present examples to illustrate the flexibility of the GGIW distribution and its sub-models for data modeling. Estimates of the parameters of GGIW distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), and Kolmogorov-Smirnov statistic (KS) are given in Table 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6. Plots of the fitted densities and the histogram of the data are given in Figures 2.3, 2.4, 2.5, 2.6, 2.7, and 2.8. Probability plots (Chambers et al. [10]) are also presented in Figures 2.3, 2.4, 2.5, 2.6, 2.7, and 2.8. For the probability plot, we plotted $G_{GGIW}(x_{(j)};\hat{\eta},\hat{\beta},\hat{\lambda},\hat{\delta})$ against $\dfrac{j-0.375}{n+0.25}$, $j=1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. We also computed a measure of closeness of each plot to the diagonal line. This measure of closeness is given by the sum of squares $SS=\sum_{j=1}^{n}\left[G_{GGIW}(x_{(j)};\hat{\eta},\hat{\beta},\hat{\lambda},\hat{\delta})-\left(\dfrac{j-0.375}{n+0.25}\right)\right]^2$.

In the first example, we consider a real life data set given by Lawless [30]. The data represents the fatigue failure times of ball bearings: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

Table 2.1: Estimates of models for ball bearings data

	Estimates				Statistics					
Model	η	β	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GGIW($\eta,\beta,\lambda,\delta$)	49.0531	7.8745	0.6250	46.0324	227.1	235.1	237.3	239.6	0.1304	0.0261
	140.28	0.8175	0.1723	12.6340						
GGIE($\eta,1,\lambda,\delta$)	0.2745	1	0.05187	104.98	226.8	232.8	234.0	236.2	0.087	0.0247
	1.6121		0.05727	226.71						
GIW($\eta,\beta,1,1$)	1240.49	1.8344	1	1	231.6	235.6	236.2	237.8	0.3478	0.8565
	1231.6	0.2692								
IE($\eta,1,1,1$)	55.0595	1	1	1	243.5	245.5	245.6	246.6	0.5652	2.7320
	11.4807									
ZBIE($\eta,1,1,\delta$)	194.37	1	1	0.3013	239.9	243.9	244.5	246.2	0.7391	5.0725
	144.05			0.2288						

The LR test statistic, for the first example, of the hypothesis H_0: $GGIE$ against H_a: $GGIW$, is $w=0.3$. The p-value is 0.5839. Therefore, we do not have enough

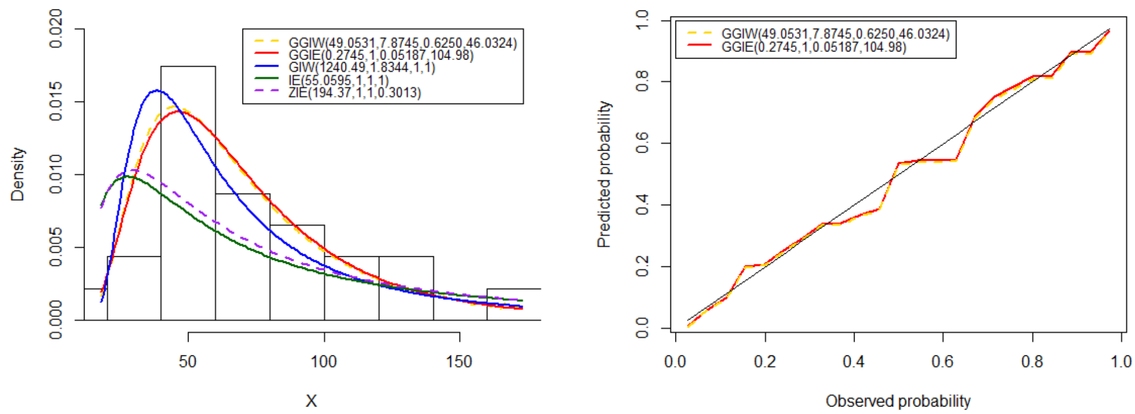

Figure 2.3: Fitted density and probability plots for Lawless ball bearing data

evidence to reject H_0 in favor of H_a. The value of KS also supports the GGIE distribution as a "better" or "superior" fit for the data.

For the second example, we consider the data consisting of the waiting times between 65 consecutive eruptions of the Kiama Blowhole. These data can be obtained at http://www.statsci.org/data/oz/kiama.html, and are given below: 83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

The LR test statistic of the hypothesis H_0: $GGIE$ against H_a: $GGIW$, is $w =$ 3.9. The p-value is 0.0482. The p-value is marginally significant. However, the value of KS statistic also supports GGIE as a "better" model for the Kiama Blowhole data set.

The third data set from Bjerkedal [5] represents the survival time, in days, of guinea pigs injected with different doses of tubercle bacilli. It is known that guinea pigs have high susceptibility of human tuberculosis. The data set consists of 72 observations.

Table 2.2: Estimates of models for Kiama Blowhole data

	Estimates				Statistics					
Model	η	β	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GGIW$(\eta,\beta,\lambda,\delta)$	94.7927	0.7390	0.06037	0.1546	584.7	592.7	593.4	601.3	0.5938	9.8538
	33.8942	0.1665	0.1105	0.05647						
GGIE$(\eta,1,\lambda,\delta)$	0.03231	1	0.1052	64.4217	587.6	593.6	594	600.1	0.1094	0.1268
	0.1437		0.07093	84.9128						
GIW$(\eta,\beta,1,1)$	48.258	1.3239	1	1	591	595	595.2	599.3	0.2656	1.6718
	17.1671	0.1286								
IE$(\eta,1,1,1)$	20.4134	1	1	1	598.4	600.4	600.4	602.5	0.3438	3.7762
ZBIE$(\eta,1,1,\delta)$	110.73	1	1	0.1968	592.9	596.9	597.1	601.2	0.5781	11.4178
	86.5896			0.1608						

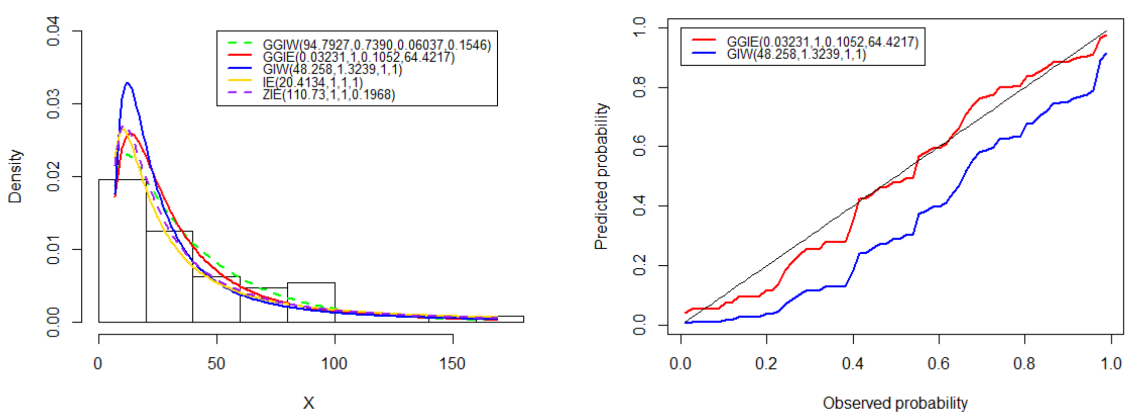

Figure 2.4: Fitted density and probability plots for Kiama Blowhole data

For the Bjerkedal data, the LR test statistic of the hypothesis H_0: $GGIE$ against H_a: $GGIW$, is $w = 0.1$. The p-value is 0.7518. Therefore, we do not have enough evidence to reject H_0 in favor of H_a. Also, the value of KS statistics is smaller for GGIE model. We can conclude, that GGIE is a “superior” fit for this data.

Table 2.3: Estimates of models for Bjerkedal data

	Estimates				Statistics					
Model	η	β	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GGIW($\eta, \beta, \lambda, \delta$)	6.7266	0.3096	0.03433	5.8272	780.5	788.5	789.1	797.6	0.1944	0.7453
	32.6026	0.7888	0.1637	41.1586						
GGIE($\eta, 1, \lambda, \delta$)	0.05157	1	0.06965	104.94	780.6	786.6	787.0	793.5	0.0972	0.1771
	0.3388		0.06418	190.19						
GIW($\eta, \beta, 1, 1$)	283.84	1.4148	1	1	791.3	795.3	795.5	799.9	0.3333	3.0557
	125.63	0.1173								
IE($\eta, 1, 1, 1$)	60.0975	1	1	1	805.3	807.3	807.4	809.6	0.4444	6.2891
	7.0826									
ZBIE($\eta, 1, 1, \delta$)	230.68	1	1	0.279	797	801	801.2	805.6	0.625	13.0313
	130.53			0.1622						

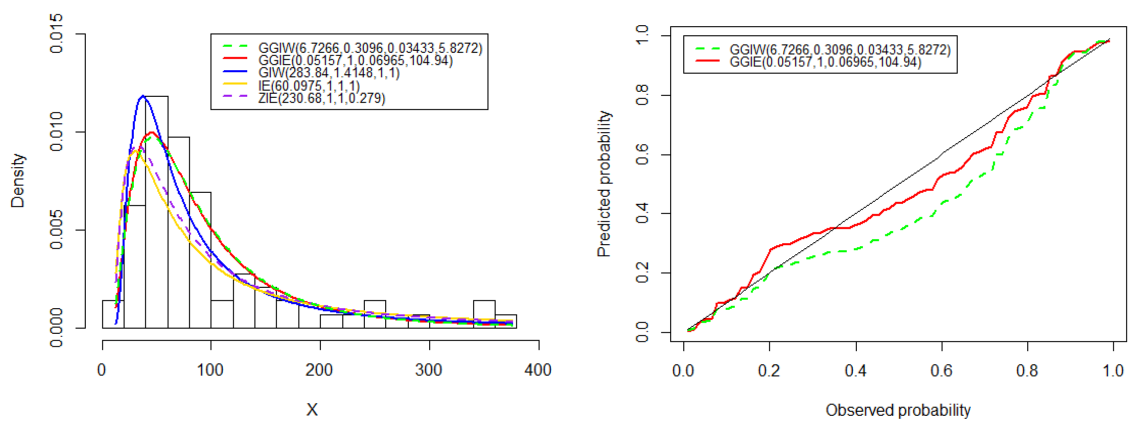

Figure 2.5: Fitted density and probability plots for Bjerkedal (pigs) data

Our next example consists of price of 428 new vehicles for the 2004 year, the data was published in the Kiplinger's Personal Finance magazine, December 2003.

The value of KS statistic for $GGIW$ distribution supports this model as a "better" or "superior" fit for the vehicles price data.

Table 2.4: Estimates of models for car prices data

	Estimates				Statistics					
Model	η	β	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GGIW$(\eta, \beta, \lambda, \delta)$	0.001651	6.7706	0.8001	16.9713	1488	1496	1496.1	1512.3	0.0701	0.7962
	0.1277	1.0087	0.6555	22.3023						
GGIE$(\eta, 1, \lambda, \delta)$	1.5848	1	0.1511	5.8679	1488	1494.9	1494.9	1507	0.1215	2.6045
	2.0889		0.5504	8.4821						
GIW$(\eta, \beta, 1, 1)$	6.7735	2.3166	1	1	1506.5	1510.5	1510.5	1518.6	0.2477	14.2982
	0.485	0.08417								
IE$(\eta, 1, 1, 1)$	2.5838	1	1	1	1856.8	158.8	1858.9	1862.9	0.5584	55.7895
	0.1249									
ZBIE$(\eta, 1, 1, \delta)$	8.6363	1	1	0.3176	1789.1	1793.1	1793.2	1801.3	0.715	96.3835
	1.419			0.05316						

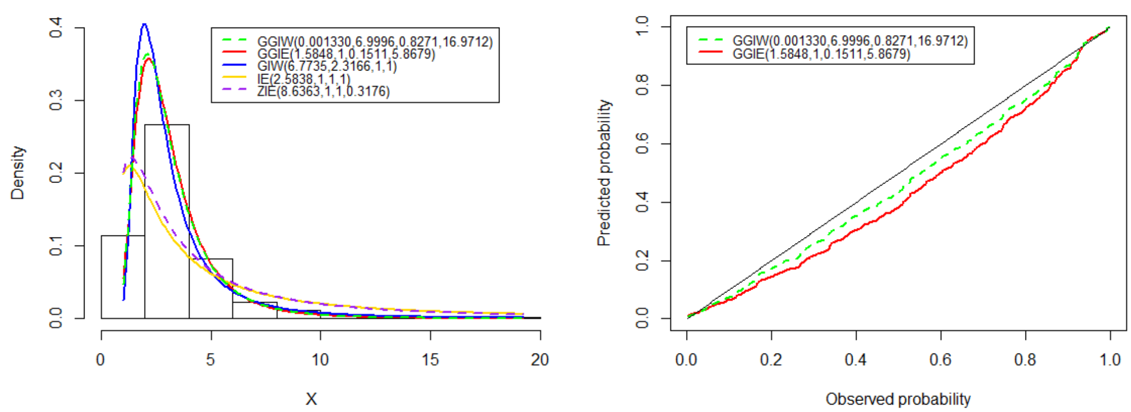

Figure 2.6: Fitted density and probability plots for car prices data

The analysis of strength data reported by Badar and Priest [2] is our next example. The data represent the strength measured in GPA, for single carbon fibers with the gauge length of 10 mm. The data is given below: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Table 2.5: Estimates of models for Badar and Priest data

	Estimates				Statistics					
Model	η	β	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GGIW($\eta,\beta,\lambda,\delta$)	91.7233	2.9637	0.1793	0.421	112.5	120.5	121.2	129	0.3968	5.017
	251.4	1.5683	0.1718	0.8707						
GGIE($\eta,1,\lambda,\delta$)	3.5313	1	0.02909	12.8967	112.6	118.6	119	125	0.1429	0.197
	2.5734		0.005114	13.1457						
GIW($\eta,\beta,1,1$)	230.45	5.4338	1	1	117.8	121.8	122	126.1	0.254	1.961
	110.89	0.5078								
IE($\eta,1,1,1$)	2.9424	1	1	1	266.8	268.8	268.9	271	0.746	11.476
	0.3707									
ZBIE($\eta,1,1,\delta$)	9.1117	1	1	0.3397	254.9	258.9	259.1	263.2	0.873	17.297
	3.2772			0.1238						

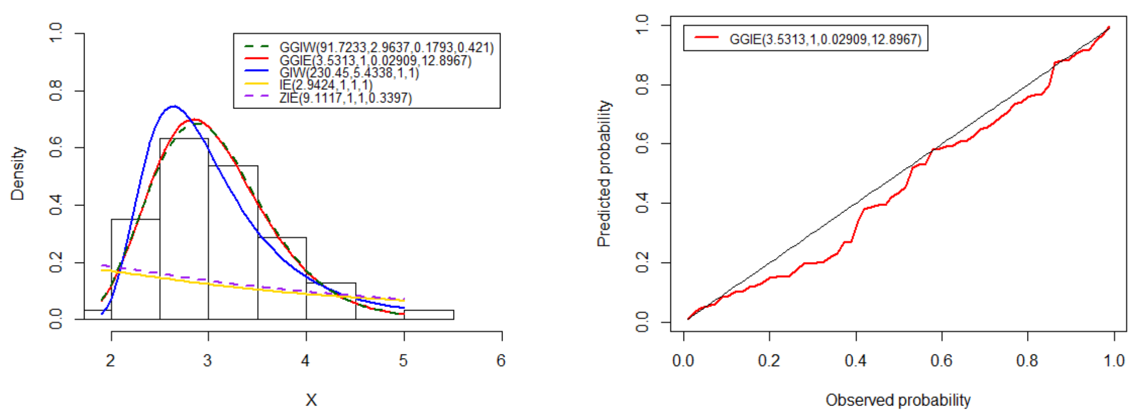

Figure 2.7: Fitted density and probability plots for Badar and Priest data set

The value of KS statistic supports the $GGIE$ distribution as a "better" or "superior" fit for the data. The values of the statistics AIC, AICC and BIC shows that sub-model $GGIE$ is a "better" fit for this data. Also, the value SS given in the table is the smallest for this model.

The next data set from Santos de Gusmao et al. [46] represents survival times in days, of cancer patients, who were exposed to radiotherapy.

Table 2.6: Estimates of models for survival times data

	Estimates				Statistics					
Model	η	β	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GGIW($\eta,\beta,\lambda,\delta$)	94.9603	0.5422	0.06966	0.2712	555.2	563.2	564.3	570.1	0.5714	5.683
	21.4187	0.1129	0.09447	0.06496						
GGIE($\eta,1,\lambda,\delta$)	0.00612	1	0.09829	104.8	555.3	561.3	562	566.6	0.119	0.128
	0.0297		0.05162	102.01						
GIW($\eta,\beta,1,1$)	57.3085	0.8616	1	1	570.3	574.3	574.6	577.8	0.3571	2.319
	21.6976	0.08549								
IE($\eta,1,1,1$)	98.4543	1	1	1	572.9	574.9	575	576.6	0.2857	1.294
	15.1918									
ZBIE($\eta,1,1,\delta$)	48.8908	1	1	1.6259	572.4	576.4	576.8	579.9	1	13.956
	40.2288			0.8051						

For the survival times data set, the goodness-of-fit statistic, KS supports the $GGIE$ distribution as a "superior" fit for the data. The values of the statistics AIC, AICC and BIC shows that sub-model $GGIE$ is a "better" fit for this data. Also, the value SS given in the table is smaller for this model.

2.6 Concluding Remarks

A new class of generalized inverse Weibull distribution called the gamma-generalized inverse Weibull distribution is proposed and studied in details. The GGIW distri-

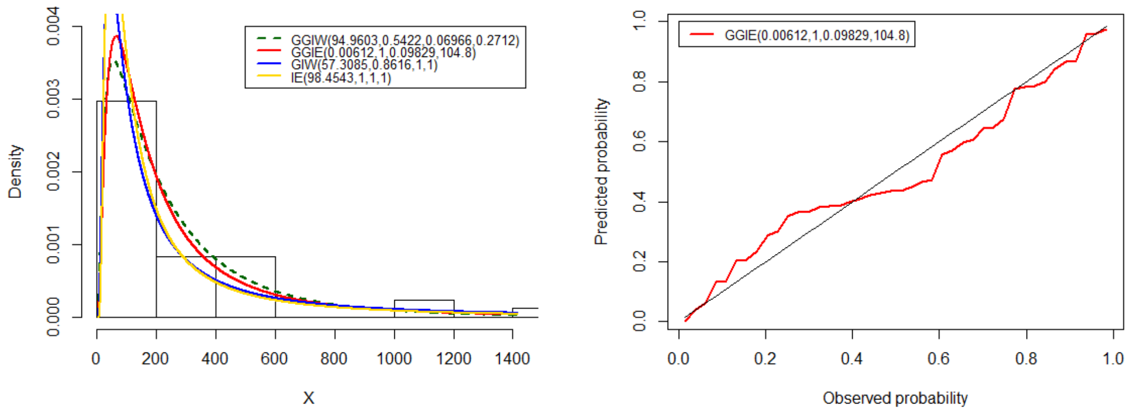

Figure 2.8: Fitted density and probability plots for survival times data

bution has the GGIE, GIR, IW, IE, IR, ZB-GIW, ZB-GIE, ZB-GIR and Fréchet distributions as special cases. The density of this new class of distributions can be expressed as a linear combination of GIW density functions. The GGIW distribution possesses hazard function with flexible behavior. We also obtain closed form expressions for the moments, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. Finally, the GGIW model is fitted to real data sets to illustrate the usefulness of the distribution.

CHAPTER 3

WEIGHTED PROPORTIONAL INVERSE WEIBULL DISTRIBUTION

3.1 Weighted Distribution

Weighted distribution can be used to deal with model specification and data interpretation problems. Patil and Rao [39] used weighted distributions as stochastic models in the study of harvesting and predation.

Let Y be a non-negative random variable with its natural pdf $f(y;\theta)$, where θ is a parameter in the parameter space Θ, then the pdf of the weighted random variable Y^w is given by:

$$f^w(y;\theta,\beta) = \frac{w(y,\beta)f(y;\theta)}{\omega},$$

where the weight function $w(y,\beta)$ is a positive function, that may depend on the parameter β, and $0 < \omega = E(w(Y,\beta)) < \infty$ is a normalizing constant. A general class of weight functions $w(y)$ is defined as follows:

$$w(y) = y^k e^{\ell y} F^i(y) \overline{F}^j(y).$$

Setting $k = 0$; $k = j = i = 0$; $\ell = i = j = 0$; $k = \ell = 0$; $i \to i-1$; $j = n-i$; $k = \ell = i = 0$ and $k = \ell = j = 0$ in this weight function, one at a time, implies probability weighted moments, moment-generating functions, moments, order statistics, proportional hazards and proportional reversed hazards, respectively, where $F(y) = P(Y \leq y)$ and $\overline{F}(y) = 1 - F(y)$. If $w(y) = y$, then $Y^* = Y^w$ is called the size-biased version of Y.

3.2 Weighted Proportional Inverse Weibull Distribution

In this section, probability weighted moments (PWMs) of the proportional inverse Weibull distribution and WPIW distribution are presented. The mode, hazard and reverse hazard functions are given. The proportional inverse Weibull (PIW) distribution has a cdf given by

$$G(x;\alpha,\beta,\gamma) = [F(x)]^{\gamma} = \exp[-\gamma(\alpha x)^{-\beta}],$$

for $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $x \geq 0$. Let $\alpha^{-\beta}\gamma = \theta$, then PIW cdf reduces to

$$G(x;\theta,\beta) = \exp[-\theta x^{-\beta}],$$

for $\theta > 0$, $\beta > 0$, and $x \geq 0$. The corresponding pdf is given by

$$g(x;\theta,\beta) = \theta\beta x^{-\beta-1}\exp[-\theta x^{-\beta}],$$

for $\theta > 0$, $\beta > 0$, and $x \geq 0$.

3.2.1 Probability Weighted Moments

The PWMs of the PIW distribution are given by

$$E[X^k G^l(X)\overline{G}^m(X)] = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)} E[X^k G^{l+j}(X)],$$

where

$$E[X^k G^{l+j}(X)] = \int_0^{\infty} x^k e^{-\theta x^{-\beta}(j+l)} \theta\beta x^{-\beta-1} e^{-\theta x^{-\beta}} dx.$$

Now, we make the substitution $u = \theta x^{-\beta}(l+j+1)$, so that

$$\begin{aligned} E[X^k G^{l+j}(X)] &= \theta^{\frac{k}{\beta}}(l+j+1)^{\frac{k}{\beta}-1}\int_0^{\infty} u^{-\frac{k}{\beta}} e^{-u} du \\ &= \theta^{\frac{k}{\beta}}(l+j+1)^{\frac{k}{\beta}-1}\Gamma\left(1-\frac{k}{\beta}\right). \end{aligned}$$

Therefore, PWMs of the PIW distribution are

$$\begin{aligned} E[X^k G^l(X)\overline{G}^m(X)] &= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)} \\ &\quad \cdot\theta^{\frac{k}{\beta}}(l+j+1)^{\frac{k}{\beta}-1}\Gamma\left(1-\frac{k}{\beta}\right), \quad \text{for } \beta > k. \end{aligned}$$

Remark: Special cases

1. When $l = m = 0$, the k^{th} noncentral moments of the PIW distribution is

$$E[X^k] = \theta^{\frac{k}{\beta}}\Gamma\left(1 - \frac{k}{\beta}\right), \quad \text{for } \beta > k.$$

2. When $l = k = 0$, we have

$$E[\overline{G}^m(X)] = \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)} \cdot (j+1)^{-1},$$

3. When $l = 0$, we have

$$E[X^k\overline{G}^m(X)] = \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)} \cdot \theta^{\frac{k}{\beta}}(j+1)^{\frac{k}{\beta}-1}\Gamma\left(1 - \frac{k}{\beta}\right),$$

for $\beta > k$.

4. When $m = 0$, we obtain

$$E[X^kG^l(X)] = \theta^{\frac{k}{\beta}}(l+1)^{\frac{k}{\beta}-1}\Gamma\left(1 - \frac{k}{\beta}\right), \quad \beta > k.$$

5. When $k = m = 0$, we have $E[G^l(X)] = (l+1)^{-1}$.

6. When $l \to i-1$, $m \to n-i$, we have

$$E[X^kG^{i-1}(X)\overline{G}^{n-i}(X)] = \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(n-i+1)}{\Gamma(n-i+1-j)\Gamma(j+1)} \cdot\theta^{\frac{k}{\beta}}(j+i)^{\frac{k}{\beta}-1}\Gamma\left(1 - \frac{k}{\beta}\right), \quad \beta > k.$$

7. When $k = 0$, we obtain

$$E[G^l(X)\overline{G}^m(X)] = \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(m+1)}{\Gamma(m+1-j)\Gamma(j+1)} \cdot (l+j+1)^{-1}.$$

3.2.2 Weighted Proportional Inverse Weibull Distribution

In this section, we present the WPIW distribution and some of its properties. We are particularly interested in studying the statistical properties of the WPIW distribution with the weight function $w(x) = x^k G^l_{_{PIW}}(x)$, compared to those with the weight function when $l = 0$, as well as the parent PIW, and IW distributions. The WPIW pdf is

$$\begin{aligned} g_{_{WPIW}}(x) &= \frac{x^k G^l(x)\overline{G}^m(x)g(x)}{E[X^k G^l(X)\overline{G}^m(X)]} \\ &= \frac{x^{k-\beta-1}\theta\beta\left(1-e^{-\theta x^{-\beta}}\right)^m e^{-\theta(l+1)x^{-\beta}}}{\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(m+1)\,\gamma^{\frac{k}{\beta}}(l+j+1)^{\frac{k}{\beta}-1}}{\Gamma(m+1-j)\Gamma(j+1)}\Gamma\left(1-\frac{k}{\beta}\right)}, \end{aligned}$$

for $\beta > k$. When $m = 0$, the corresponding WPIW pdf and cdf are given by

$$\begin{aligned} g_{_{WPIW}}(x) &= \frac{x^k G^l(x)g(x)}{E[X^k G^l(X)]} \\ &= \frac{\beta\left(\theta(l+1)\right)^{1-\frac{k}{\beta}} x^{k-\beta-1}e^{-\theta(l+1)x^{-\beta}}}{\Gamma\left(1-\dfrac{k}{\beta}\right)}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} G_{_{WPIW}}(x;\theta,\beta,k,l) &= \int_0^x \frac{\beta\left(\theta(l+1)\right)^{1-\frac{k}{\beta}} x^{k-\beta-1}e^{-\theta(l+1)x^{-\beta}}}{\Gamma\left(1-\dfrac{k}{\beta}\right)} \\ &= \frac{\Gamma\left(1-\frac{k}{\beta},\theta(l+1)x^{-\beta}\right)}{\Gamma\left(1-\frac{k}{\beta}\right)}, \quad \text{for } \beta > k, \end{aligned}$$

respectively, where we have made the substitution $u = \theta(l+1)x^{-\beta}$, and $\Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}\,dt$ is the upper incomplete gamma function. Graphs of the WPIW pdfs for selected values of the parameters θ, β, k and l show different shapes of the curves depending on the values of the parameters.

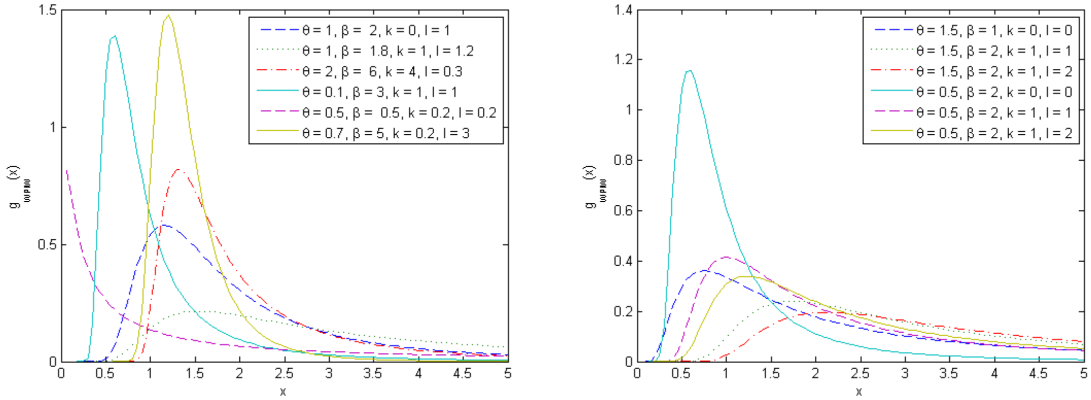

Figure 3.1: Pdfs of the WPIW Distribution

Mode of the WPIW Distribution

Consider the WPIW pdf given in equation (3.1). Note that

$$\begin{aligned} \ln(g_{WPIW}(x)) &= \left(1-\frac{k}{\beta}\right)(\ln(\theta)+\ln(l+1))+\ln(\beta)+(k-\beta-1)\ln(x) \\ &\quad -\theta(l+1)x^{-\beta}-\ln\left(\Gamma\left(1-\frac{k}{\beta}\right)\right). \end{aligned} \tag{3.2}$$

Differentiating eguation (3.2) with respect to x, we obtain

$$\frac{\partial \ln g_{WPIW}(x;\theta,\beta,k,l)}{\partial x} = \frac{k-\beta-1}{x}+\frac{\theta\beta(l+1)x^{-\beta}}{x}. \tag{3.3}$$

Now, set equation (3.3) equal 0 and solve for x, to get

$$x_0 = \left(\frac{\theta\beta(l+1)}{1+\beta-k}\right)^{\frac{1}{\beta}}.$$

Note, that

$$\frac{\partial^2 \ln g_{WPIW}(x;\theta,\beta,k,l)}{\partial x^2} = -\frac{k-\beta-1}{x^2}-\frac{\beta(\beta+1)\theta(l+1)x^{-\beta}}{x^2} < 0.$$

When $0 < x < \left(\frac{\theta\beta(l+1)}{1+\beta-k}\right)^{\frac{1}{\beta}}$, $\frac{\partial \ln g(x;\theta,\beta,k,l)}{\partial x} > 0$, so $g_{WPIW}(x;\theta,\beta,k,l)$ is increasing, and when $x > \left(\frac{\theta\beta(l+1)}{1+\beta-k}\right)^{\frac{1}{\beta}}$, $g_{WPIW}(x;\theta,\beta,k,l)$ is decreasing, therefore $g_{WPIW}(x;\theta,\beta,k,l)$ achieves a maximum when $x_0 = \left(\frac{\theta\beta(l+1)}{1+\beta-k}\right)^{\frac{1}{\beta}}$, so that x_0 is the mode of WPIW distribution.

Hazard and Reverse Hazard Functions

The hazard function of the WPIW distribution is given by

$$\lambda_{G_{WPIW}}(x;\theta,\beta,k,l) \;=\; \frac{\beta\left(\theta(l+1)\right)^{1-\frac{k}{\beta}}\, x^{k-\beta-1}e^{-\theta(l+1)x^{-\beta}}}{\Gamma\left(1-\dfrac{k}{\beta}\right)-\Gamma\left(1-\dfrac{k}{\beta},\theta(l+1)x^{-\beta}\right)},$$

for $\theta > 0$, $\beta > k$, $k \geq 0$, $l \geq 0$, and $x \geq 0$. The behavior of the hazard function of the WPIW distribution is established via Glaser's Lemma [17]. Note that

$$\begin{aligned}\eta_{G_{WPIW}}(x;\theta,\beta,k,l) &= -\frac{g'_{WPIW}(x;\theta,\beta,k,l)}{g_{WPIW}(x;\theta,\beta,k,l)}\\ &= (\beta+1-k)x^{-1}-\theta\beta(l+1)x^{-\beta-1},\end{aligned}$$

and

$$\eta'_{G_{WPIW}}(x) = (k-\beta-1)x^{-2}+\theta\beta(\beta+1)(l+1)x^{-\beta-2}.$$

Now, $\eta'_{G_{WPIW}}(x) = 0$ implies $x_0^* = \left(\frac{\beta\gamma(l+1)(\beta+1)}{\beta+1-k}\right)^{\frac{1}{\beta}}$, for $\theta > 0$, $\beta > k$, $k \geq 0$, $l \geq 0$ and $k \neq \beta+1$. Note that, when $0 < x < x_0^*$, $\eta'_{G_{WPIW}}(x) > 0$, $\eta'_{G_{WPIW}}(x_0^*) = 0$ and when $x > x_0^*$, $\eta'_{G_{WPIW}}(x) < 0$. Consequently, WPIW hazard function has an *upside down bathtub shape*. The graphs of the hazard function show upside down bathtub shape for the selected values of the parameters [Figure 3.2]. The reverse hazard function is

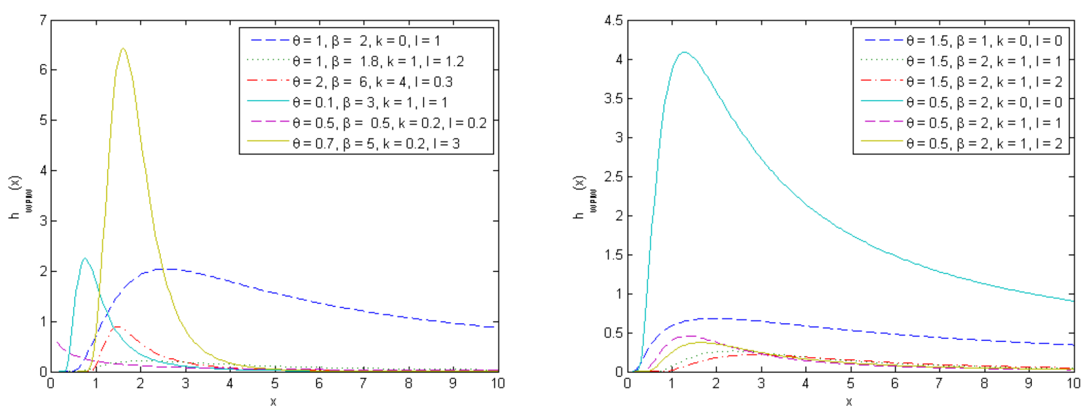

Figure 3.2: Graphs of Hazard Functions of the WPIW Distribution

given by

$$\tau_{G_{WPIW}}(x;\theta,\beta,k,l) = \frac{\beta\left(\theta(l+1)\right)^{1-\frac{k}{\beta}} x^{k-\beta-1} e^{-\theta(l+1)x^{-\beta}}}{\Gamma\left(1-\dfrac{k}{\beta},\theta(l+1)x^{-\beta}\right)},$$

for $\theta > 0$, $\beta > k$, $k \geq 0$, $l \geq 0$, and $x \geq 0$.

3.3 Distribution of Functions of Random Variables

In this section, distributions of functions of the PIW and WPIW random variables are presented. Consider the PIW pdf:

$$g(x;\theta,\beta) = \theta\beta x^{\beta-1} e^{-\theta x^{\beta}}, \quad x > 0,\ \theta > 0,\ \beta > 0.$$

1. Let $Y = \theta X^{-\beta}$. Then the pdf of Y is

$$\begin{aligned} g_1(y;\theta,\beta) &= \theta\beta x^{-\beta-1} e^{-\theta x^{\beta}} (\theta\beta)^{-1} x^{\beta+1} \\ &= e^{-\theta x^{-\beta}} = e^{-y},\ \ y > 0, \end{aligned}$$

that is, if $X \sim PIW(\theta,\beta)$, then $Y = \theta X^{-\beta} \sim EXP(1)$, unit exponential distribution.

2. Let $X \sim WPIW(\theta,\beta,k,l)$, that is

$$g_{WPIW}(x;\theta,\beta,k,l) = \frac{\beta(\theta(l+1))^{1-\frac{k}{\beta}} x^{k-\beta-1} e^{-\theta(l+1)x^{\beta}}}{\Gamma\left(1-\frac{k}{\beta}\right)},$$

and $Y = \theta X^{-\beta}$, then the resulting pdf of Y is given by

$$g_2(y;\theta,\beta,k,l) = \frac{(l+1)^{1-\frac{k}{\beta}} e^{-y(l+1)}}{\Gamma\left(1-\frac{k}{\beta}\right)}, \quad \beta > k,\ l \geq 0, \text{ and } y > 0.$$

Note, that if $l = k = 0$, then $g_1(y) = g_2(y)$, $y > 0$.

3. Now, let $Z = \theta(l+1)X^{-\beta}$, where $X \sim WPIW(\theta,\beta,k,l)$. The pdf of the random variable Z is given by

$$g(z;\theta,\beta,k,l) = \frac{(\theta(l+1)z)^{-\frac{k}{\beta}} (\theta(l+1))^{\frac{k}{\beta}} e^{-z}}{\Gamma\left(1-\frac{k}{\beta}\right)} = \frac{z^{1-\frac{k}{\beta}-1} e^{-z}}{\Gamma\left(1-\frac{k}{\beta}\right)},$$

for $z > 0$, $\beta > k$. Thus, if $X \sim WPIW(\theta,\beta,k,l)$, then $Z \sim GAM(1-\frac{k}{\beta},1)$.

3.4 Moments, Entropy and Fisher Information

In this section, we present the moments and related functions as well as entropies and Fisher information for the WPIW distribution. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

3.4.1 Moments and Moment Generating Function

The c^{th} non-central moment of the WPIW distribution is given by

$$\begin{aligned} E(X^c) &= \int_0^\infty x^c \cdot g_{WPIW}(x;\theta,\beta,k,l)\,dx \\ &= \int_0^\infty \frac{\beta(\theta(l+1))^{1-\frac{k}{\beta}} \cdot x^c \cdot x^{k-\beta-1} e^{-\theta(l+1)x^{-\beta}}}{\Gamma\left(1-\dfrac{k}{\beta}\right)}\,dx. \end{aligned}$$

Making the substitution $u = \theta x^{-\beta}(l+1)$, $du = -\theta\beta(l+1)x^{-\beta-1}dx$, so that $x = \left(\dfrac{\theta(l+1)}{u}\right)^{\frac{1}{\beta}}$, we obtain

$$\begin{aligned} E(X^c) &= \int_0^\infty \frac{(\theta(l+1))^{\frac{c}{\beta}} \cdot u^{-\frac{c+k}{\beta}} exp[-u]}{\Gamma\left(1-\dfrac{k}{\beta}\right)}\,dx \\ &= \frac{(\theta(l+1))^{\frac{c}{\beta}}\Gamma\left(1-\dfrac{c+k}{\beta}\right)}{\Gamma\left(1-\dfrac{k}{\beta}\right)}, \end{aligned}$$

where $\beta > c+k$. Let $\delta_c = \Gamma\left(1-\dfrac{c+k}{\beta}\right)$, then the mean and variance are $\mu_X = \frac{(\theta(l+1))^{\frac{1}{\beta}}\delta_1}{\delta_0}$, and $\sigma_X^2 = \dfrac{(\theta(l+1))^{\frac{2}{\beta}}(\delta_0\delta_2-\delta_1^2)}{\delta_0^2}$, respectively. The coefficient of variation (CV) is $CV = \dfrac{\sqrt{\delta_0\delta_2-\delta_1^2}}{\delta_1}$.

The coefficient of skewness (CS) is given by $CS = \dfrac{2\delta_1^3 - 3\delta_0\delta_1\delta_2 + \delta_0^2\delta_3}{[\delta_0\delta_2 - \delta_1^2]^{\frac{3}{2}}}$, and the coefficient of kurtosis (CK) is $CK = \dfrac{\delta_0^3\delta_4 - 4\delta_0^2\delta_1\delta_3 + 6\delta_0\delta_1^2\delta_2 - 3\delta_1^4}{[\delta_0\delta_2 - \delta_1^2]^2}$.

The graphs of CV, CS and CK versus β and k are given in Figures 3.3, 3.4 and 3.5, 3.6, respectively. We can see decreasing coefficients (CV, CS, CK) for increasing values of β, and fixed k. For fixed values of β and increasing k we observe an increasing behavior of coefficients (CV, CS, CK). The moment generat-

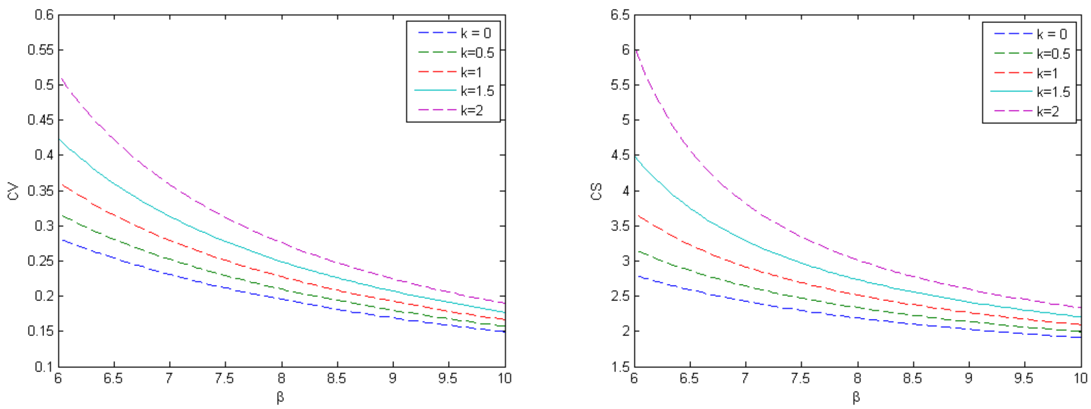

Figure 3.3: Graphs of CV and CS versus β for WPIW Distribution

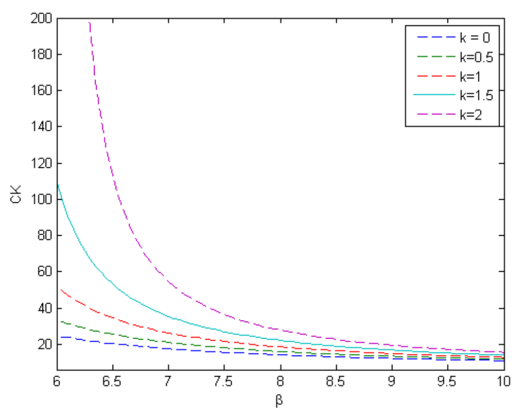

Figure 3.4: Graphs of CK versus β for WPIW Distribution

ing function (MGF) of the WPIW distribution is $M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} E\left[X^j\right]$, where

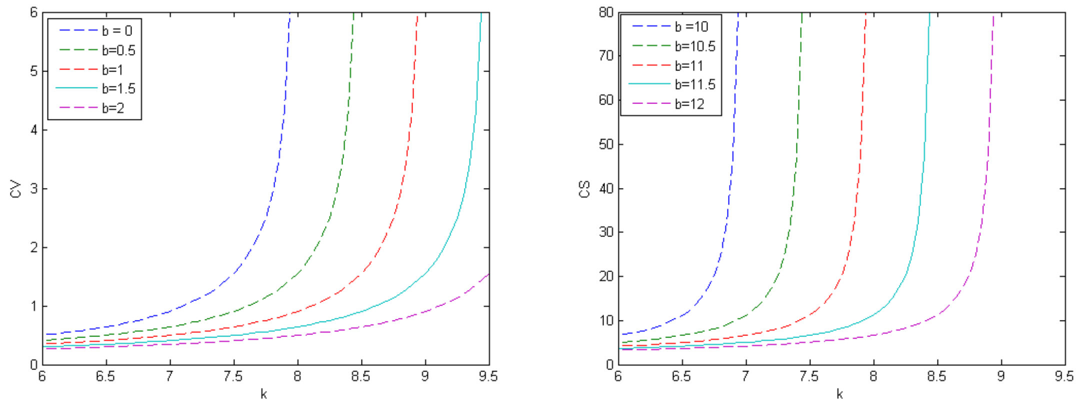

Figure 3.5: Graphs of CV and CS versus k for WPIW Distribution

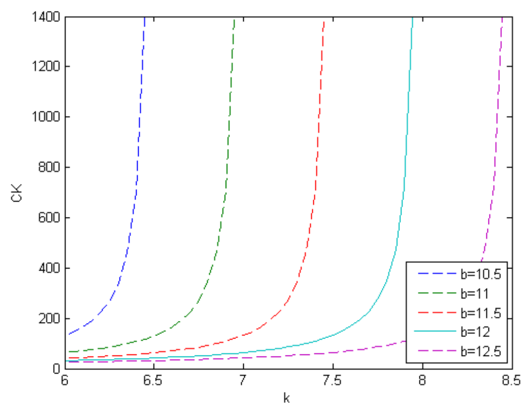

Figure 3.6: Graphs of CK versus k for WPIW Distribution

$E\left[X^j\right] = \dfrac{(\theta(l+1))^{\frac{j}{\beta}}\Gamma\left(1-\dfrac{j+k}{\beta}\right)}{\Gamma\left(1-\dfrac{k}{\beta}\right)}$, $\beta > k$. Table 3.1 gives the mode, mean, standard deviation (STD), CV, CS and CK for some values of the parameters θ, β, k and l. For fixed β , ($\beta = 9$), we can see from Table 3.1, as k increases, the values of CV, CS and CK increase. We can also see that as θ increases, the values for mode and mean increase, for the selected values of the model parameters.

Table 3.1: Mode, Mean, STD, Coefficients of Variation, Skewness and Kurtosis

θ	β	k	l	Mode	Mean	STD	CV	CS	CK
1.0	5.0	0.0	1.0	6.9882712	1.3373488	0.1764993	0.3657341	3.5350716	48.0915121
1.0	6.0	1.0	1.0	8.9020365	1.3465304	0.1652203	0.3621262	3.6848057	51.6558037
1.0	8.0	1.0	2.0	18.5065299	1.2901375	0.0681465	0.2275525	2.5155915	18.5192490
1.0	11.0	2.0	2.0	26.7673174	1.2136238	0.0330265	0.1644585	2.1565125	13.2835147
1.0	14.0	6.0	4.0	59.8325980	1.2757688	0.0418564	0.1823705	2.5536743	17.5403973
0.5	14.0	6.0	4.0	29.9162990	1.2141430	0.0379103	0.1823705	2.5536743	17.5403973
0.1	9.0	0.0	0.0	0.6968373	0.8344695	0.0171375	0.1690772	2.0279664	12.2288967
0.2	9.0	0.2	0.5	2.0952099	0.9470688	0.0229521	0.1731845	2.0702410	12.6652064
0.3	9.0	0.9	1.0	4.2250670	1.0405926	0.0321038	0.1896393	2.2388765	14.5472995
0.5	9.0	1.0	1.0	7.0504293	1.1042834	0.0369783	0.1922976	2.2660506	14.8729807
0.8	9.0	1.5	1.0	11.3525578	1.1800099	0.0475716	0.2070113	2.4164086	16.7971337
1.0	9.0	2.0	2.0	21.4299142	1.2854874	0.0643989	0.2246083	2.5967769	19.4039748
1.2	9.0	2.5	1.0	17.2673114	1.2765594	0.0735293	0.2460281	2.8186110	23.1258866
2.0	9.0	3.0	1.5	36.2503940	1.4135825	0.1063104	0.2726620	3.1006528	28.8496151
2.3	9.0	3.5	0.4	23.5382773	1.3784479	0.1219454	0.3066580	3.4760961	38.7047540
4.0	9.0	4.0	1.0	59.0026131	1.5686032	0.1961385	0.3515103	4.0104143	59.2887795
4.1	9.0	4.5	1.6	79.3847704	1.6756038	0.2890137	0.4132986	4.8546026	124.7159488

3.4.2 Shannon Entropy

Shannon entropy [51] for WPIW distribution is given by

$$\begin{aligned} H(g_{_{WPIW}}) &= -\int_0^\infty \log\left(\frac{\beta\left(\theta(l+1)\right)^{1-\frac{k}{\beta}}x^{k-\beta-1}e^{-\theta(l+1)x^{-\beta}}}{\Gamma\left(1-\dfrac{k}{\beta}\right)}\right) \\ &\times \quad \left(\frac{\beta\left(\theta(l+1)\right)^{1-\frac{k}{\beta}}x^{k-\beta-1}e^{-\theta(l+1)x^{-\beta}}}{\Gamma\left(1-\dfrac{k}{\beta}\right)}\right)dx \\ &= -[A+B+C], \end{aligned}$$

where A, B and C are obtained below:

$$\begin{aligned} A &= \log\left(\frac{\beta(\theta(l+1))^{1-\frac{k}{\beta}}}{\delta_0}\right)\int_0^\infty \frac{\beta(\theta(l+1))^{1-\frac{k}{\beta}}x^{k-\beta-1}e^{-\theta(l+1)x^{-\beta}}}{\delta_0}\,dx \\ &= \log(\beta)+\left(1-\frac{k}{\beta}\right)\log(\theta(l+1))-\log(\delta_0), \end{aligned}$$

$$\begin{aligned} B &= \int_0^\infty \frac{\beta(\theta(l+1))^{1-\frac{k}{\beta}}x^{k-\beta-1}e^{-\theta(l+1)x^{-\beta}}}{\delta_0}\cdot\left(-\theta(l+1)x^{-\beta}\right)\,dx \\ &= -\frac{\beta(\theta(l+1))^{2-\frac{k}{\beta}}}{\delta_0}\int_0^\infty e^{-\theta(l+1)x^{-\beta}}x^{k-2\beta-1}\,dx. \end{aligned}$$

Let $u=\theta(l+1)x^{-\beta}$, then $du=-\beta\theta(l+1)x^{-\beta-1}\,dx$, $x=\left(\frac{\theta(l+1)}{u}\right)^{\frac{1}{\beta}}$, and we obtain $B=-\frac{\Gamma\left(2-\frac{k}{\beta}\right)}{\delta_0}$. Also,

$$\begin{aligned} C &= \int_0^\infty \frac{\beta(\theta(l+1))^{1-\frac{k}{\beta}}x^{k-\beta-1}e^{-\theta(l+1)x^{-\beta}}}{\delta_0}\cdot(k-\beta-1)\log(x)\,dx, \\ &= -\frac{(k-\beta-1)}{\beta\delta_0}\int_0^\infty\left[\log(u)-\log(\theta(l+1))\right]e^{-u}u^{-\frac{k}{\beta}}\,du, \end{aligned}$$

where $u=\theta(l+1)x^{-\beta}$. Using the fact that $\Gamma^{(n)}(t)=\int_0^\infty \log^n(x)x^{t-1}\exp(-x)\,dx$, the integral becomes

$$C=-\frac{(k-\beta-1)\delta_0'}{\beta\delta_0}+\frac{(k-\beta-1)\delta_0\log(\theta(l+1))}{\beta\delta_0}.$$

Consequently, Shannon entropy for WPIW distribution is given by

$$H(g_{WPIW})=\frac{\beta\Gamma\left(2-\frac{k}{\beta}\right)+(k-\beta-1)\delta_0'-\beta\delta_0\log\left(\frac{\beta}{\delta_0}\right)+\delta_0\log(\theta(l+1))}{\beta\delta_0}.$$

When $k=l=0$, we obtain Shannon entropy for the IW distribution.

3.4.3 Renyi Entropy

Renyi entropy [44] generalizes Shannon entropy. Renyi entropy of order t, where $t>0$ and $t\neq 1$ is given by

$$H_R(g)=\frac{1}{1-t}\log\left[\int_0^\infty g^t(x)\,dx\right].$$

Note that

$$\int_0^\infty g^t_{_{WPIW}}(x)\,dx \;=\; \int_0^\infty \left[\frac{\beta(\theta(l+1))^{1-\frac{k}{\beta}} x^{k-\beta-1} e^{-\theta(l+1)x^{-\beta}}}{\delta_0}\right]^t dx.$$

Let $u = \theta\,(l+1)\,t\,x^{-\beta}$, then the integral becomes

$$\int_0^\infty g^t_{_{WPIW}}(x)\,dx \;=\; \frac{\beta^{t-1}(\theta(l+1))^{\frac{1-t}{\beta}}}{\delta_0^t} t^{\frac{t(k-\beta-1)+1}{\beta}} \Gamma\left(\frac{t(\beta+1-k)-1}{\beta}\right).$$

Renyi entropy for WPIW distribution reduces to

$$\begin{aligned} H_R(g_{_{WPIW}}) \;&=\; \log(\beta) + \frac{1}{\beta}\log(\theta(l+1)) + \frac{t(k-\beta-1)+1}{\beta(1-t)}\log(t) \\ &\quad + \frac{1}{1-t}\log\Gamma\left(\frac{t(\beta+1-k)-1}{\beta}\right) - \frac{t}{1-t}\log(\delta_0). \end{aligned}$$

When $k = l = 0$, we obtain Renyi entropy for the IW distribution.

3.4.4 s- Entropy

Recall that s-entropy is a one parameter generalization of Shannon entropy and is defined by

$$H_s(g) = \frac{1}{s-1}\left[1 - \int_0^\infty g^s(x)\,dx\right], \quad \text{for } s \neq 1.$$

Consequently, s-entropy for WPIW distribution is given by

$$\begin{aligned} H_s(g_{_{WPIW}}) \;&=\; \frac{1}{s-1}\left[1 - \frac{(\beta)^{s-1}(\theta(l+1))^{\frac{1-s}{\beta}} s^{\frac{s(k-\beta-1)+1}{\beta}}}{\delta_0^s}\right. \\ &\quad \times \left.\Gamma\left(\frac{s(\beta+1-k)-1}{\beta}\right)\right], \quad \text{for } s \neq 1. \end{aligned}$$

Note, that $s-$entropy for the sub-models can be readily obtained.

3.4.5 Fisher Information

Let $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3, \theta_4) = (\theta, \beta, k, l)$. If $\ln(g(X, \mathbf{\Theta}))$ is twice differentiate with respect to $\mathbf{\Theta}$, and under certain regularity conditions [31], Fisher Information (FIM) is the

4×4 matrix whose elements are:

$$I(\mathbf{\Theta}) = -E_{\mathbf{\Theta}}\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln(g(X,\mathbf{\Theta}))\right].$$

Fisher information matrix (FIM) for WPIW distribution is given by:

$$I(\mathbf{\Theta}) = I(\theta,\beta,k,l) = \begin{bmatrix} I_{\theta\theta} & I_{\theta\beta} & I_{\theta k} & I_{\theta l} \\ I_{\theta\beta} & I_{\beta\beta} & I_{\beta k} & I_{\beta l} \\ I_{\theta k} & I_{\beta k} & I_{kk} & I_{kl} \\ I_{\theta l} & I_{\beta l} & I_{kl} & I_{ll} \end{bmatrix},$$

where the entries of the $I(\theta,\beta,k,l)$ are given below. We have the following partial derivatives of $\ln[g(x;\theta,\beta,k,l)]$ with respect to the parameters:

$$\frac{\partial \ln g_{WPIW}(x;\theta,\beta,k,l)}{\partial\theta} = \frac{1-\frac{k}{\beta}}{\theta} - (l+1)x^{-\beta}, \tag{3.4}$$

$$\begin{aligned}\frac{\partial \ln g_{WPIW}(x;\theta,\beta,k,l)}{\partial\beta} &= \frac{k\ln(\theta(l+1))}{\beta^2} + \theta(l+1)x^{-\beta}\ln(x) \\ &+\frac{1}{\beta} - \ln(x) - \frac{\Psi\left(1-\frac{k}{\beta}\right)k}{\beta^2},\end{aligned} \tag{3.5}$$

$$\frac{\partial \ln g_{WPIW}(x;\theta,\beta,k,l)}{\partial k} = \ln(x) - \frac{\ln(\theta(l+1))}{\beta} + \frac{\Psi\left(1-\frac{k}{\beta}\right)}{\beta^2}, \tag{3.6}$$

and

$$\frac{\partial \ln g_{WPIW}(x;\theta,\beta,k,l)}{\partial l} = \frac{1-\frac{k}{\beta}}{l+1} - \theta x^{-\beta}. \tag{3.7}$$

We differentiate (3.4) with respect to θ,β,k, and l, we obtain:

$$\frac{\partial^2 \ln g_{WPIW}(x;\theta,\beta,k,l)}{\partial\theta^2} = -\frac{1-\frac{k}{\beta}}{\theta^2},$$

$$\frac{\partial^2 \ln g_{WPIW}(x;\theta,\beta,k,l)}{\partial\theta\partial\beta} = \frac{k}{\theta\beta^2} + (l+1)x^{-\beta}\ln(x),$$

$$\frac{\partial^2 \ln g_{_{WPIW}}(x;\theta,\beta,k,l)}{\partial\theta\partial k} = -\frac{1}{\theta\beta}, \quad \frac{\partial^2 \ln g_{_{WPIW}}(x;\theta,\beta,k,l)}{\partial\theta\partial l} = -x^{-\beta}.$$

Differentiating (3.5) with respect to β, k, and l, we get:

$$\begin{aligned}\frac{\partial^2 \ln g_{_{WPIW}}(x;\theta,\beta,k,l)}{\partial\beta^2} &= -\frac{2k\ln(\theta(l+1))}{\beta^3} - \theta(l+1)x^{-\beta}\ln^2(x) \\ &\quad -2\theta(l+1)x^{-\beta}\ln(x) - \frac{1}{\beta^2} \\ &\quad -\frac{\Psi\left(1,\, 1-\frac{k}{\beta}\right)k^2}{\beta^4} + \frac{2\Psi(1-\frac{k}{\beta})k}{\beta^3},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ln g_{_{WPIW}}(x;\theta,\beta,k,l)}{\partial\beta\partial k} &= \frac{\ln(\theta(l+1))}{\beta^2} + \frac{\Psi'\left(1-\frac{k}{\beta}\right)k}{\beta^3} \\ &\quad -\frac{\Psi(1-\frac{k}{\beta})}{\beta^2},\end{aligned}$$

$$\frac{\partial^2 \ln g_{_{WPIW}}(x;\theta,\beta,k,l)}{\partial\beta\partial l} = \frac{k}{(l+1)\beta^2} + \theta x^{-\beta}\ln(x).$$

Taking the derivative of (3.6) with respect to k and l we get:

$$\frac{\partial^2 \ln g_{_{WPIW}}(x;\theta,\beta,k,l)}{\partial k^2} = -\frac{\Psi'\left(1-\frac{k}{\beta}\right)}{\beta^2}, \quad \frac{\partial^2 \ln g_{_{WPIW}}(x;\theta,\beta,k,l)}{\partial k\partial l} = -\frac{1}{\beta(l+1)},$$

and differentiating (3.7) with respect to l, we obtain:

$$\frac{\partial^2 \ln g_{_{WPIW}}(x;\theta,\beta,k,l)}{\partial l^2} = -\frac{1-\frac{k}{\beta}}{(l+1)^2}.$$

Now, we compute the following expectations: $E\left[X^{-\beta}\right]$, $E\left[X^{-\beta}\ln(X)\right]$, and $E\left[X^{-\beta}\ln^2(X)\right]$ in order to obtain FIM $I(\theta,\beta,k,l)$.

$$\begin{aligned}E\left[X^{-\beta}\right] &= \int_0^\infty x^{-\beta} g_{_{WPIW}}(x)dx \\ &= \int_0^\infty \frac{x^{k-2\beta-1}[\theta(l+1)]^{1-\frac{k}{\beta}}\beta e^{-\theta(l+1)x^{-\beta}}\,dx}{\Gamma\left(1-\frac{k}{\beta}\right)}.\end{aligned}$$

Let $u = \theta x^{-\beta}(l+1)$, $du = -\theta\beta(l+1)x^{-\beta-1}\,dx$ and $x = \left(\dfrac{\theta(l+1)}{u}\right)^{\frac{1}{\beta}}$. The integral becomes:

$$\begin{aligned} E\left[X^{-\beta}\right] &= \int_0^\infty \left(\frac{\theta(l+1)}{u}\right)^{\frac{k}{\beta}-1} \frac{(\theta(l+1))^{-\frac{k}{\beta}} e^{-u}\,du}{\Gamma\left(1-\frac{k}{\beta}\right)} \\ &= \frac{(\theta(l+1))^{-1}\,\Gamma\left(2-\frac{k}{\beta}\right)}{\Gamma\left(1-\frac{k}{\beta}\right)}. \end{aligned}$$

$$\begin{aligned} E\left[X^{-\beta}\ln(X)\right] &= \int_0^\infty x^{-\beta}\ln(x) g_{WPIW}(x)dx \\ &= \int_0^\infty \frac{x^{k-2\beta-1}[\theta(l+1)]^{1-\frac{k}{\beta}}\beta\ln(x)\,e^{-\theta(l+1)x^{-\beta}}\,dx}{\Gamma\left(1-\frac{k}{\beta}\right)}. \end{aligned}$$

Making the same substitution $u = \theta x^{-\beta}(l+1)$, we obtain:

$$\begin{aligned} E\left[X^{-\beta}\ln(X)\right] &= \int_0^\infty \left(\frac{\theta(l+1)}{u}\right)^{\frac{k}{\beta}-1} \frac{(\theta(l+1))^{-\frac{k}{\beta}} e^{-u}\,du}{\Gamma\left(1-\frac{k}{\beta}\right)} \\ &\times \ln\left(\left(\frac{\theta(l+1)}{u}\right)^{\frac{1}{\beta}}\right) du \\ &= \frac{(\theta(l+1))^{-1}\ln\left((\theta(l+1))^{\frac{1}{\beta}}\right)\Gamma\left(2-\frac{k}{\beta}\right)}{\Gamma\left(1-\frac{k}{\beta}\right)} \\ &\quad + \frac{(\theta(l+1))^{-1}\,\Gamma'\left(2-\frac{k}{\beta}\right)}{\beta\Gamma\left(1-\frac{k}{\beta}\right)}. \end{aligned}$$

$$\begin{aligned} E\left[X^{-\beta}\ln^2(X)\right] &= \int_0^\infty x^{-\beta}\ln^2(x) g_{WPIW}(x)dx \\ &= \int_0^\infty \frac{x^{k-2\beta-1}[\theta(l+1)]^{1-\frac{k}{\beta}}\beta\ln^2(x)\,e^{-\theta(l+1)x^{-\beta}}\,dx}{\Gamma\left(1-\frac{k}{\beta}\right)}. \end{aligned}$$

Making the same substitution one more time we get:

$$\begin{aligned} E\left[X^{-\beta}\ln^2(X)\right] &= \int_0^\infty \left(\frac{\theta(l+1)}{u}\right)^{\frac{k}{\beta}-1} \frac{(\theta(l+1))^{-\frac{k}{\beta}} e^{-u}\, du}{\Gamma\left(1-\frac{k}{\beta}\right)} \\ &\times\ \ln^2\left(\left(\frac{\theta(l+1)}{u}\right)^{\frac{1}{\beta}}\right) du \\ &= \frac{(\theta(l+1))^{-1}}{\Gamma\left(1-\frac{k}{\beta}\right)} \int_0^\infty u^{2-\frac{k}{\beta}-1} e^{-u} \left[\ln^2\left((\theta(l+1))^{\frac{1}{\beta}}\right)\right. \\ &\times\ \left. -\frac{2}{\beta}\ln(u)\ln\left((\theta(l+1))^{\frac{1}{\beta}}\right) + \frac{1}{\beta^2}\ln^2(u)\right] du \\ &= \frac{(\theta(l+1))^{-1}\ln^2\left((\theta(l+1))^{\frac{1}{\beta}}\right)\Gamma\left(2-\frac{k}{\beta}\right)}{\Gamma\left(1-\frac{k}{\beta}\right)} \\ &\quad - \frac{(\theta(l+1))^{-1}\,2\ln\left((\theta(l+1))^{\frac{1}{\beta}}\right)\Gamma'\left(2-\frac{k}{\beta}\right)}{\beta\Gamma\left(1-\frac{k}{\beta}\right)} \\ &\quad + \frac{(\theta(l+1))^{-1}\,\Gamma^{(2)}\left(2-\frac{k}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{k}{\beta}\right)}. \end{aligned}$$

Now, the entries of FIM are given below:

$$\begin{aligned} I_{\beta\beta} &= \frac{1}{\beta^2} + \frac{2k\ln(\theta(l+1))}{\beta^3} + \frac{k^2\Psi\left(1, 1-\frac{k}{\beta}\right)}{\beta^4} - \frac{2k\Psi\left(1-\frac{k}{\beta}\right)}{\beta^3} \\ &\quad + \frac{\ln^2(\theta(l+1))\Gamma\left(2-\frac{k}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{k}{\beta}\right)} - \frac{2\ln(\theta(l+1))\Gamma'\left(2-\frac{k}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{k}{\beta}\right)} \\ &\quad + \frac{\Gamma^{(2)}\left(2-\frac{k}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{k}{\beta}\right)} + \frac{\ln(\theta(l+1))\Gamma\left(2-\frac{k}{\beta}\right)}{\beta\Gamma\left(1-\frac{k}{\beta}\right)} + \frac{2\Gamma'\left(2-\frac{k}{\beta}\right)}{\beta\Gamma\left(1-\frac{k}{\beta}\right)}, \end{aligned}$$

$$I_{\theta\theta} = \frac{1-\frac{k}{\beta}}{\theta^2}, \quad I_{kk} = \frac{\Psi'\left(1-\frac{k}{\beta}\right)}{\beta^2}, \quad I_{ll} = \frac{1-\frac{k}{\beta}}{(l+1)^2},$$

$$I_{\theta\beta} = -\frac{k}{\beta^2\theta} - \frac{\ln(\theta(l+1))\Gamma\left(2-\frac{k}{\beta}\right)}{\theta\beta\Gamma\left(1-\frac{k}{\beta}\right)} - \frac{\Gamma'\left(2-\frac{k}{\beta}\right)}{\theta\beta\Gamma\left(1-\frac{k}{\beta}\right)},$$

$$I_{\theta k} = \frac{1}{\theta\beta}, \quad I_{\theta l} = \frac{\Gamma\left(2-\frac{k}{\beta}\right)}{\theta(l+1)\Gamma\left(1-\frac{k}{\beta}\right)}, \; I_{kl} = \frac{1}{\beta(l+1)},$$

$$I_{\beta k} = \frac{\Psi(1-\frac{k}{\beta}) - \ln(\theta(l+1))}{\beta^2} - \frac{\Psi\left(1,\,1-\frac{k}{\beta}\right)k}{\beta^3}, \text{ and}$$

$$I_{\beta l} = -\frac{k}{\beta^2(l+1)} - \frac{\ln(\theta(l+1))\Gamma\left(2-\frac{k}{\beta}\right)}{\beta(l+1)\Gamma\left(1-\frac{k}{\beta}\right)} - \frac{\Gamma'\left(2-\frac{k}{\beta}\right)}{\beta(l+1)\Gamma\left(1-\frac{k}{\beta}\right)}.$$

3.5 Concluding Remarks

Statistical properties of the weighted proportional inverse Weibull distribution and its sub-models including the pdf, cdf, moment, hazard function, reverse hazard function, coefficients of variation, skewness and kurtosis, Fisher information, Shannon entropy, Renyi entropy and s-entropy are given. Estimation of the parameters of the model are also presented. The proposed class of distributions contains a large number of distributions with potential applications to several areas of probability and statistics, finance, economics, and medicine.

CHAPTER 4

THE LENGTH-BIASED AND PROPORTIONAL REVERSE HAZARD INVERSE WEIBULL DISTRIBUTIONS

4.1 Weighted Inverse Weibull Distribution

In this chapter, we present a special case of the WPIW distribution, and some of its properties. We are particularly interested in the distribution obtained via the weight function $w(x) = xG_{IW}(x)$. When $m = 0$, and $k = l = 1$, the WIW pdf and cdf reduces to

$$
\begin{aligned}
g_{WIW}(x) &= \frac{xG(x)g(x)}{E[XG(X)]} \\
&= \frac{\beta\,(2\theta)^{1-\frac{1}{\beta}}\,x^{-\beta}e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\dfrac{1}{\beta}\right)}, \qquad (4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
G_{WIW}(x;\theta,\beta) &= \int_0^x \frac{\beta\,(2\theta)^{1-\frac{1}{\beta}}\,y^{-\beta}e^{-2\theta y^{-\beta}}}{\Gamma\left(1-\dfrac{1}{\beta}\right)}dy \\
&= \frac{\Gamma\left(1-\frac{1}{\beta}, 2\theta x^{-\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}, \quad \text{for } \beta > 1, \qquad (4.2)
\end{aligned}
$$

respectively, where $\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt$ is the upper incomplete gamma function. Graphs of the WIW pdfs for selected values of the parameters θ and β show different shapes of the curves depending on the values of the parameters. If the distribution of the a random variable is given by equation (4.2), we write $X \sim WIW(\theta,\beta) \equiv WIW(\theta,\, \beta,\, 1,\, 1)$.

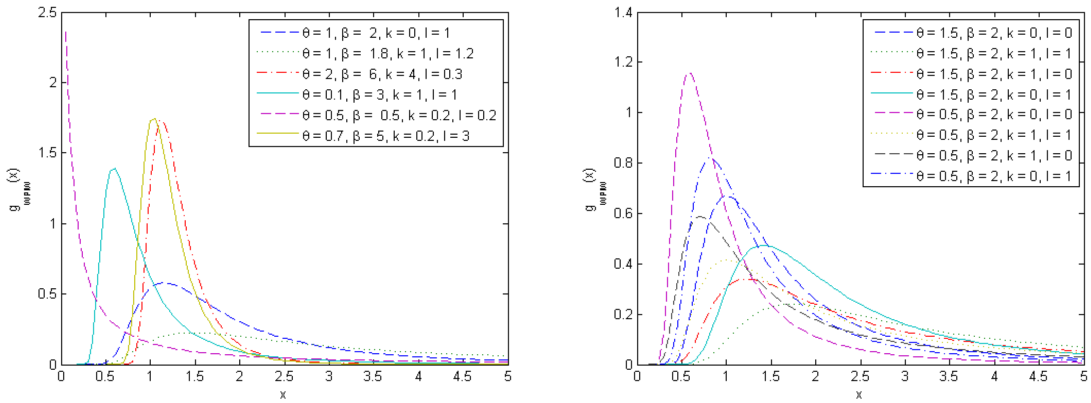

Figure 4.1: Pdfs of the WIW Distribution

Mode of the WIW Distribution

Consider the WIW pdf given in equation (4.1). Note that

$$\ln(g_{WIW}(x)) = \left(1-\frac{1}{\beta}\right)\ln(2\theta)+\ln(\beta)-\beta\ln(x) -2\theta x^{-\beta}-\ln\left(\Gamma\left(1-\frac{1}{\beta}\right)\right). \quad (4.3)$$

Differentiating equation (4.3) with respect to x, we obtain

$$\frac{\partial \ln g_{WIW}(x;\theta,\beta)}{\partial x} = \frac{-\beta}{x}+\frac{2\theta\beta x^{-\beta}}{x}. \quad (4.4)$$

Now, set equation (4.4) equal 0 and solve for x, to get

$$x_0 = (2\theta)^{\frac{1}{\beta}}.$$

Note, that

$$\frac{\partial^2 \ln g_{WIW}(x;\theta,\beta)}{\partial x^2} = -\frac{-\beta}{x^2}-\frac{2\beta(\beta+1)\theta x^{-\beta}}{x^2} < 0.$$

When $0 < x < (2\theta)^{\frac{1}{\beta}}$, $\frac{\partial \ln g(x;\theta,\beta)}{\partial x} > 0$, so $g_{WIW}(x;\theta,\beta)$ is increasing, and when $x > (2\theta)^{\frac{1}{\beta}}$, $g_{WIW}(x;\theta,\beta,1,1)$ is decreasing, therefore $g_{WIW}(x;\theta,\beta,1,1)$ achieves a maximum when $x_0 = (2\theta)^{\frac{1}{\beta}}$, so that x_0 is the mode of WIW distribution.

Hazard and Reverse Hazard Functions

The hazard function of the WIW distribution is given by

$$\lambda_{G_{WIW}}(x;\theta,\beta) \;=\; \frac{\beta\,(2\theta)^{1-\frac{1}{\beta}}\,x^{-\beta}e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)-\Gamma\left(1-\frac{1}{\beta},2\theta x^{-\beta}\right)},$$

for $\theta > 0$, $\beta > 1$, and $x \geq 0$. The behavior of the hazard function of the WIW distribution is established via Glaser's Lemma [17]. Note that

$$\begin{aligned}\eta_{G_{WIW}}(x;\theta,\beta) &= -\frac{g'_{WIW}(x;\theta,\beta)}{g_{WIW}(x;\theta,\beta)}\\ &= (\beta)x^{-1}-2\theta\beta x^{-\beta-1},\end{aligned}$$

and

$$\eta'_{G_{WIW}}(x) = (-\beta)x^{-2}+2\theta\beta(\beta+1)x^{-\beta-2}.$$

Now, $\eta'_{G_{WIW}}(x) = 0$ implies $x_0^* = (2\gamma(\beta+1))^{\frac{1}{\beta}}$, for $\theta > 0$, $\beta > 1$. Note that, when $0 < x < x_0^*$, $\eta'_{G_{WIW}}(x) > 0$, $\eta'_{G_{WIW}}(x_0^*) = 0$ and when $x > x_0^*$, $\eta'_{G_{WIW}}(x) < 0$. Consequently, WIW hazard function has an *upside down bathtub shape.* The graphs of the hazard function show upside down bathtub shape for the selected values of the parameters. The reverse hazard function is given by

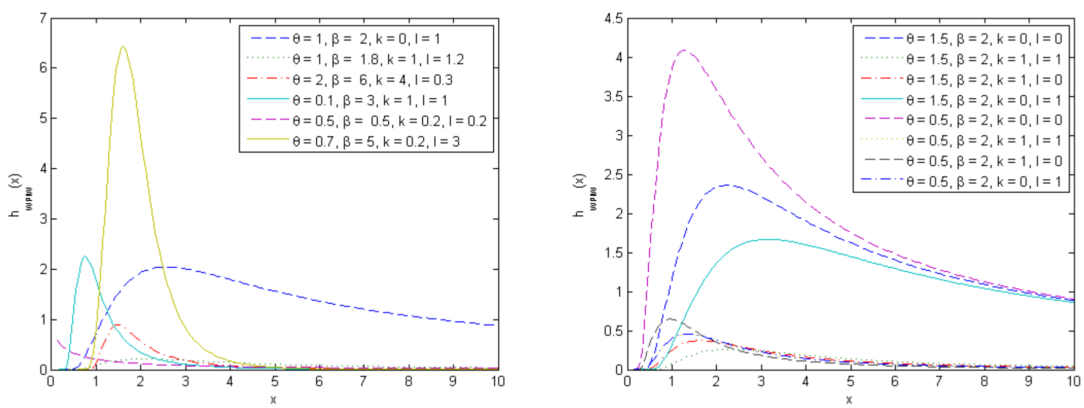

Figure 4.2: Graphs of Hazard Functions of the WIW Distribution

$$\tau_{G_{WIW}}(x;\theta,\beta) = \frac{\beta\,(2\theta)^{1-\frac{k}{\beta}}\,x^{-\beta}e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\frac{1}{\beta}, 2\theta x^{-\beta}\right)},$$

for $\theta > 0$, $\beta > 1$, and $x \geq 0$.

4.2 Distribution of Functions of Random Variables

In this section, distributions of functions of the IW and WIW random variables are presented. Consider the IW pdf:

$$g(x;\theta,\beta) = \theta\beta x^{\beta-1}e^{-\theta x^{\beta}}, \quad x > 0,\ \theta > 0,\ \beta > 0.$$

1. Let $Y = \theta X^{-\beta}$. Then the pdf of Y is

$$\begin{aligned} g_1(y;\theta,\beta) &= \theta\beta x^{-\beta-1}e^{-\theta x^{\beta}}(\theta\beta)^{-1}x^{\beta+1} \\ &= e^{-\theta x^{-\beta}} = e^{-y},\ \ y > 0, \end{aligned}$$

that is, if $X \sim IW(\theta,\beta)$, then $Y = \theta X^{-\beta} \sim EXP(1)$, unit exponential distribution.

2. Let $X \sim WIW(\theta,\beta)$, that is

$$g_{WIW}(x;\theta,\beta) = \frac{\beta(2\theta)^{1-\frac{1}{\beta}}x^{-\beta}e^{-2\theta x^{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)},$$

and $Y = \theta X^{-\beta}$, then the resulting pdf of Y is given by

$$g_2(y;\beta) = \frac{2^{1-\frac{1}{\beta}}e^{-2y}}{\Gamma\left(1-\frac{1}{\beta}\right)}, \quad \beta > 1, \text{ and } y > 0.$$

3. Now, let $Z = 2\theta X^{-\beta}$, where $X \sim WIW(\theta,\beta)$. The pdf of the random variable Z is given by

$$g(z;\theta,\beta) = \frac{(2\theta z)^{-\frac{1}{\beta}}(2\theta)^{\frac{1}{\beta}}e^{-z}}{\Gamma\left(1-\frac{1}{\beta}\right)} = \frac{z^{1-\frac{1}{\beta}-1}e^{-z}}{\Gamma\left(1-\frac{1}{\beta}\right)},$$

for $z > 0$, $\beta > 1$. Thus, if $X \sim WIW(\theta,\beta)$, then $Z \sim GAM(1-\frac{1}{\beta}, 1)$.

4.3 Moments, Entropy and Fisher Information

In this section, we present the moments and related functions as well as entropies and Fisher information for the WIW distribution. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

4.3.1 Moments and Moment Generating Function

The c^{th} non-central moment of the WIW distribution is given by

$$E(X^c) = \int_0^\infty \frac{\beta(2\theta)^{1-\frac{1}{\beta}} \cdot x^c \cdot x^{-\beta} e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)} dx.$$

Making the substitution $u = 2\theta x^{-\beta}$, $du = -2\theta\beta x^{-\beta-1}dx$, so that $x = \left(\frac{2\theta}{u}\right)^{\frac{1}{\beta}}$, we obtain

$$\begin{aligned} E(X^c) &= \int_0^\infty \frac{(2\theta)^{\frac{c}{\beta}} \cdot u^{-\frac{c+1}{\beta}} exp[-u]}{\Gamma\left(1-\frac{1}{\beta}\right)} dx \\ &= \frac{(2\theta)^{\frac{c}{\beta}} \Gamma\left(1-\frac{c+1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}, \end{aligned}$$

where $\beta > c+1$. Let $\delta_c = \Gamma\left(1-\frac{c+1}{\beta}\right)$, then the mean and variance are $\mu_X = \frac{(2\theta)^{\frac{1}{\beta}}\delta_1}{\delta_0}$, and $\sigma_X^2 = \frac{(2\theta)^{\frac{2}{\beta}}(\delta_0\delta_2 - \delta_1^2)}{\delta_0^2}$, respectively.

The coefficient of variation (CV) is $CV = \frac{\sqrt{\delta_0\delta_2 - \delta_1^2}}{\delta_1}$. The coefficient of skewness (CS) is given by $CS = \frac{2\delta_1^3 - 3\delta_0\delta_1\delta_2 + \delta_0^2\delta_3}{[\delta_0\delta_2 - \delta_1^2]^{\frac{3}{2}}}$, and the coefficient of kurtosis (CK) is $CK = \frac{\delta_0^3\delta_4 - 4\delta_0^2\delta_1\delta_3 + 6\delta_0\delta_1^2\delta_2 - 3\delta_1^4}{[\delta_0\delta_2 - \delta_1^2]^2}$.

The graphs of CV, CS and CK versus β are given in Figures 4.3 and 4.4. We can see decreasing coefficients (CV, CS, CK) for increasing values of β.

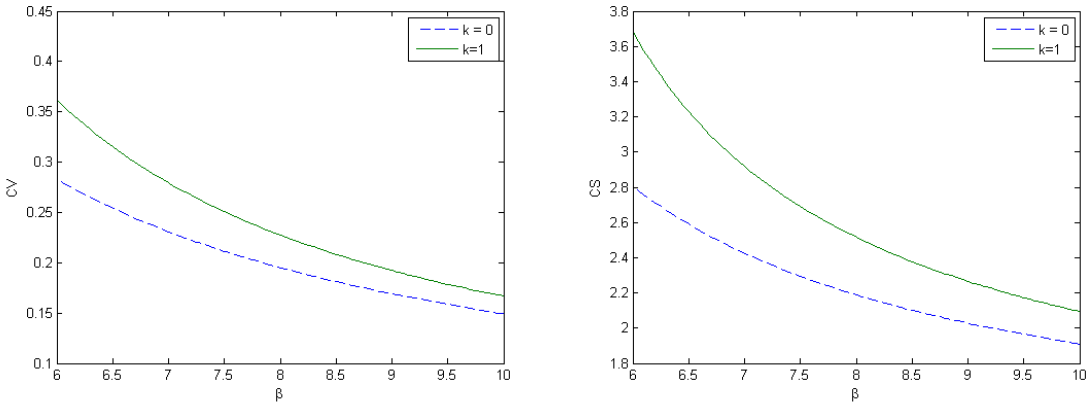

Figure 4.3: Graphs of CV and CS for WIW Distribution

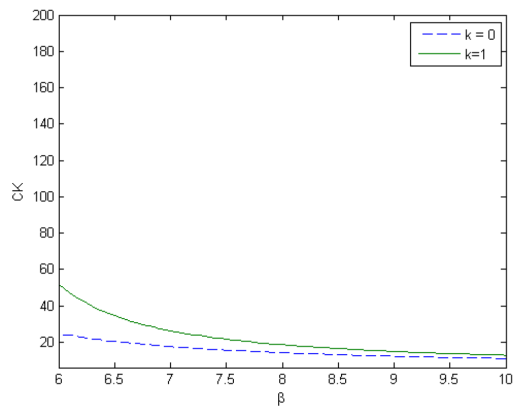

Figure 4.4: Graphs of CK for WIW Distribution

The moment generating function (MGF) of the WIW distribution is $M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} E\left[X^j\right]$, where $E\left[X^j\right] = \dfrac{(2\theta)^{\frac{j}{\beta}}\Gamma\left(1-\dfrac{j+1}{\beta}\right)}{\Gamma\left(1-\dfrac{1}{\beta}\right)}$, $\beta > j+1$. Table 4.1 gives the mode, mean, standard deviation (STD), CV, CS and CK for some values of the parameters θ, β, k and l, $(k, l = 0\,\text{or}\ 1)$. We can see from Table 4.1, as β increases,

Table 4.1: Mode, Mean, STD, Coefficients of Variation, Skewness and Kurtosis

θ	β	k	l	Mode	Mean	STD	CV	CS	CK
1.0	5.0	0.0	1.0	6.9882712	1.3373488	0.1764993	0.3657341	3.5350716	48.0915121
0.1	9.0	0.0	0.0	0.6968373	0.8344695	0.0171375	0.1690772	2.0279664	12.2288967
0.5	9.0	1.0	1.0	7.0504293	1.1042834	0.0369783	0.1922976	2.2660506	14.8729807
0.5	6.0	0.0	0.0	2.1690601	1.0056349	0.0634625	0.2827681	2.8055664	24.6781193
1.0	6.0	0.0	0.0	4.3381202	1.1287870	0.0799578	0.2827681	2.8055664	24.6781193
1.5	6.0	0.0	0.0	6.5071802	1.2077041	0.0915288	0.2827681	2.8055664	24.6781193
0.5	6.0	0.0	1.0	4.3381202	1.1287870	0.0799578	0.2827681	2.8055664	24.6781193
1.0	6.0	0.0	1.0	8.6762403	1.2670206	0.1007405	0.2827681	2.8055664	24.6781193
1.5	6.0	0.0	1.0	13.0143605	1.3556021	0.1153191	0.2827681	2.8055664	24.6781193
0.5	6.0	1.0	1.0	4.4510183	1.1996222	0.1311354	0.3621262	3.6848057	51.6558037
1.0	6.0	1.0	1.0	8.9020365	1.3465304	0.1652203	0.3621262	3.6848057	51.6558037
1.5	6.0	1.0	1.0	13.3530548	1.4406706	0.1891300	0.3621262	3.6848057	51.6558037
0.5	6.0	1.0	0.0	2.2255091	1.0687418	0.1040822	0.3621262	3.6848057	51.6558037
1.0	6.0	1.0	0.0	4.4510183	1.1996222	0.1311354	0.3621262	3.6848057	51.6558037
1.5	6.0	1.0	0.0	6.6765274	1.2834916	0.1501126	0.3621262	3.6848057	51.6558037

the values of CV, CS and CK decrease. We can also see that as θ increases, the values for mode and mean increase, for the selected values of the model parameters.

4.3.2 Shannon Entropy

Shannon entropy [51] for WIW distribution is given by

$$\begin{aligned} H(g_{WIW}) &= -\int_0^\infty \log\left(\frac{\beta\,(2\theta)^{1-\frac{1}{\beta}}\,x^{-\beta}e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\dfrac{1}{\beta}\right)}\right) \\ &\times \left(\frac{\beta\,(2\theta)^{1-\frac{1}{\beta}}\,x^{-\beta}e^{-2\theta x^{-\beta}}}{\Gamma\left(1-\dfrac{1}{\beta}\right)}\right) dx \\ &= -[A+B+C], \end{aligned}$$

where A, B and C are obtained below:

$$\begin{aligned} A &= \log\left(\frac{\beta(2\theta)^{1-\frac{1}{\beta}}}{\delta_0}\right)\int_0^\infty \frac{\beta(2\theta)^{1-\frac{1}{\beta}}x^{-\beta}e^{-2\theta x^{-\beta}}}{\delta_0}\,dx \\ &= \log(\beta) + \left(1-\frac{1}{\beta}\right)\log(2\theta) - \log(\delta_0), \end{aligned}$$

$$\begin{aligned} B &= \int_0^\infty \frac{\beta(2\theta))^{1-\frac{1}{\beta}}x^{-\beta}e^{-2\theta x^{-\beta}}}{\delta_0}\cdot\left(-2\theta x^{-\beta}\right)\,dx \\ &= -\frac{\beta(2\theta)^{2-\frac{1}{\beta}}}{\delta_0}\int_0^\infty e^{-2\theta x^{-\beta}}x^{-2\beta}\,dx. \end{aligned}$$

Let $u = 2\theta x^{-\beta}$, then $du = -2\beta\theta x^{-\beta-1}\,dx$, $x = \left(\frac{2\theta}{u}\right)^{\frac{1}{\beta}}$, and we obtain

$$B = -\frac{\Gamma\left(2-\dfrac{1}{\beta}\right)}{\delta_0}.$$

Also,

$$\begin{aligned} C &= \int_0^\infty \frac{\beta(2\theta)^{1-\frac{1}{\beta}}x^{-\beta}e^{-2\theta x^{-\beta}}}{\delta_0}\cdot(-\beta)\log(x)\,dx, \\ &= -\frac{1}{\delta_0}\int_0^\infty \left[\log(2\theta)-\log(u)\right]e^{-u}u^{-\frac{1}{\beta}}\,du, \end{aligned}$$

where $u = 2\theta x^{-\beta}$. Using the fact that $\Gamma^{(n)}(t) = \int_0^\infty \log^n(x) x^{t-1} \exp(-x)\, dx$, the integral becomes

$$C = \frac{\delta_0'}{\delta_0} - \frac{\delta_0 \log(2\theta)}{\delta_0}.$$

Consequently, Shannon entropy for WIW distribution is given by

$$H(g_{WIW}) = \frac{\beta\Gamma\left(2 - \frac{1}{\beta}\right) - \beta\delta_0' - \beta\delta_0 \log\left(\frac{\beta}{\delta_0}\right) + \delta_0 \log(2\theta)}{\beta\delta_0}.$$

4.3.3 Renyi Entropy

Renyi entropy [44] generalizes Shannon entropy. Renyi entropy of order t, where $t > 0$ and $t \neq 1$ is given by

$$H_R(g) = \frac{1}{1-t} \log\left[\int_0^\infty g^t(x)\, dx\right].$$

Note that

$$\int_0^\infty g^t_{WIW}(x)\, dx = \int_0^\infty \left[\frac{\beta(2\theta)^{1-\frac{1}{\beta}} x^{-\beta} e^{-2\theta x^{-\beta}}}{\delta_0}\right]^t dx.$$

Let $u = 2\,\theta\, t\, x^{-\beta}$, then the integral becomes

$$\int_0^\infty g^t_{WIW}(x)\, dx = \frac{\beta^{t-1}(2\theta)^{\frac{1-t}{\beta}}}{\delta_0^t} t^{\frac{-\beta t+1}{\beta}} \Gamma\left(\frac{\beta t - 1}{\beta}\right).$$

Renyi entropy for WIW distribution reduces to

$$\begin{aligned} H_R(g_{WIW}) &= \log(\beta) + \frac{1}{\beta}\log(2\theta) + \frac{-\beta t + 1}{\beta(1-t)}\log(t) \\ &\quad + \frac{1}{1-t}\log\Gamma\left(\frac{\beta t - 1}{\beta}\right) - \frac{t}{1-t}\log(\delta_0), \end{aligned}$$

for $t > 0$, and $t \neq 1$.

4.3.4 Fisher Information

Let $\Theta = (\theta, \beta)$. Fisher information matrix (FIM) for WIW distribution is given by:

$$I(\Theta) = I(\theta, \beta) = \begin{bmatrix} I_{\theta\theta} & I_{\theta\beta} \\ I_{\theta\beta} & I_{\beta\beta} \end{bmatrix},$$

where the entries of the $I(\theta, \beta)$ are given below. The FI for the WIW distribution, that X contains about the parameters $\Theta = (\theta, \beta)$ is obtained below. We have the following partial derivatives of $\ln\left[g(x; \theta, \beta)\right]$ with respect to the parameters:

$$\frac{\partial \ln g_{WIW}(x; \theta, \beta)}{\partial \theta} = \frac{1 - \frac{1}{\beta}}{\theta} - 2x^{-\beta}, \tag{4.5}$$

$$\begin{aligned} \frac{\partial \ln g_{WIW}(x; \theta, \beta)}{\partial \beta} &= \frac{\ln(2\theta)}{\beta^2} + 2\theta x^{-\beta} \ln(x) \\ &+ \frac{1}{\beta} - \ln(x) - \frac{\Psi\left(1 - \frac{1}{\beta}\right)}{\beta^2}, \end{aligned} \tag{4.6}$$

We differentiate (4.5) with respect to θ and β, we obtain:

$$\frac{\partial^2 \ln g_{WIW}(x; \theta, \beta)}{\partial \theta^2} = -\frac{1 - \frac{1}{\beta}}{\theta^2},$$

$$\frac{\partial^2 \ln g_{WIW}(x; \theta, \beta)}{\partial \theta \partial \beta} = \frac{1}{\theta \beta^2} + 2x^{-\beta} \ln(x),$$

Differentiating (4.6) with respect to β, we get:

$$\begin{aligned} \frac{\partial^2 \ln g_{WIW}(x; \theta, \beta)}{\partial \beta^2} &= -\frac{2\ln(2\theta)}{\beta^3} - 2\theta x^{-\beta} \ln^2(x) \\ &-4\theta x^{-\beta} \ln(x) - \frac{1}{\beta^2} \\ &-\frac{\Psi\left(1,\, 1 - \frac{1}{\beta}\right)}{\beta^4} + \frac{2\Psi(1 - \frac{1}{\beta})}{\beta^3}. \end{aligned}$$

Now, we compute the following expectations: $E\left[X^{-\beta}\right]$, $E\left[X^{-\beta}\ln(X)\right]$, $E\left[X^{-\beta}\ln^2(X)\right]$ in order to obtain FIM $I(\theta, \beta)$.

$$E\left[X^{-\beta}\right] = \frac{(2\theta)^{-1}\,\Gamma\left(2 - \frac{1}{\beta}\right)}{\Gamma\left(1 - \frac{1}{\beta}\right)}.$$

$$E\left[X^{-\beta}\ln(X)\right] = \frac{(2\theta)^{-1}\ln\left((2\theta)^{\frac{1}{\beta}}\right)\Gamma\left(2-\frac{1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} + \frac{(2\theta)^{-1}\Gamma'\left(2-\frac{1}{\beta}\right)}{\beta\Gamma\left(1-\frac{1}{\beta}\right)}.$$

$$\begin{aligned} E\left[X^{-\beta}\ln^2(X)\right] &= \frac{(2\theta)^{-1}\ln^2\left((2\theta)^{\frac{1}{\beta}}\right)\Gamma\left(2-\frac{1}{\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)} \\ &\quad -\frac{(2\theta)^{-1}2\ln\left((2\theta)^{\frac{1}{\beta}}\right)\Gamma'\left(2-\frac{1}{\beta}\right)}{\beta\Gamma\left(1-\frac{1}{\beta}\right)} \\ &\quad +\frac{(2\theta)^{-1}\Gamma^{(2)}\left(2-\frac{1}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{1}{\beta}\right)}. \end{aligned}$$

Now, the entries of FIM are given below:

$$\begin{aligned} I_{\beta\beta} &= \frac{1}{\beta^2} + \frac{2\ln(2\theta)}{\beta^3} + \frac{\Psi\left(1,1-\frac{1}{\beta}\right)}{\beta^4} - \frac{2\Psi\left(1-\frac{1}{\beta}\right)}{\beta^3} \\ &\quad +\frac{\ln^2(2\theta)\Gamma\left(2-\frac{1}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{1}{\beta}\right)} - \frac{2\ln(2\theta)\Gamma'\left(2-\frac{1}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{1}{\beta}\right)} \\ &\quad +\frac{\Gamma^{(2)}\left(2-\frac{1}{\beta}\right)}{\beta^2\Gamma\left(1-\frac{1}{\beta}\right)} + \frac{\ln(2\theta)\Gamma\left(2-\frac{1}{\beta}\right)}{\beta\Gamma\left(1-\frac{1}{\beta}\right)} + \frac{2\Gamma'\left(2-\frac{1}{\beta}\right)}{\beta\Gamma\left(1-\frac{1}{\beta}\right)}, \end{aligned}$$

$$I_{\theta\theta} = \frac{1-\frac{1}{\beta}}{\theta^2},$$

$$I_{\theta\beta} = -\frac{1}{\beta^2\theta} - \frac{\ln(2\theta)\Gamma\left(2-\frac{1}{\beta}\right)}{\theta\beta\Gamma\left(1-\frac{1}{\beta}\right)} - \frac{\Gamma'\left(2-\frac{1}{\beta}\right)}{\theta\beta\Gamma\left(1-\frac{1}{\beta}\right)}.$$

4.4 Estimation of Parameters of the Length-Biased and Proportional Reverse Hazard Inverse Weibull Distributions

In this section, we obtain estimates of the parameters for the WIW distribution. Method of maximum likelihood (ML) estimation is presented. Asymptotic confidence intervals and likelihood ratio test are also given.

Maximum Likelihood Estimation

Let $x_1, x_2, \cdots, x_n$ be a random sample from a WIW distribution and $\Theta = (\theta, \beta)$. The log-likelihood function is

$$\begin{aligned}\ln L &= l(\Theta) = n\ln(\beta) + \left(n - \frac{n}{\beta}\right)\ln(2\theta) - n\ln\Gamma\left(1 - \frac{1}{\beta}\right) \\ &\quad -\beta\sum_{i=1}^{n}\ln(x_i) - 2\theta\sum_{i=1}^{n}x_i^{-\beta}.\end{aligned}$$

The normal equations are

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta} &= \left(1 - \frac{1}{\hat{\beta}}\right)\frac{n}{\hat{\theta}} - 2\sum_{i=1}^{n}x_i^{-\hat{\beta}} = 0, \\ \text{and} & \\ \frac{\partial \ln L}{\partial \beta} &= \frac{n}{\hat{\beta}} + \frac{n}{\hat{\beta}^2}\ln(2\hat{\theta}) - \sum_{i=1}^{n}\ln(x_i) - \frac{n}{\hat{\beta}^2}\Psi\left(1 - \frac{1}{\hat{\beta}}\right) \\ &\quad +2\hat{\theta}\sum_{i=1}^{n}\left(x_i^{-\hat{\beta}}\ln(x_i)\right) = 0. \qquad (4.7)\end{aligned}$$

Therefore, if β is known, we can obtain an estimate for $\hat{\theta}$ from (4.7)

$$\hat{\theta} = \frac{n(\hat{\beta} - 1)}{2\hat{\beta}\sum_{i=1}^{n}x_i^{-\hat{\beta}}},$$

and if θ is known, we can find an estimate for β using Newton's method. When all parameters are unknown, numerical methods must be used to obtain the MLE $\hat{\theta}$, and $\hat{\beta}$ of the parameters θ, and β, respectively, since the system of equations does not admit any closed form solutions.

Asymptotic Confidence Intervals and Likelihood Ratio Test

The multivariate normal distribution with covariance matrix $I(\Theta)$, where $\Theta = (\theta, \beta)$, can be used to obtain confidence intervals and confidence regions for the parameters of the WIW distribution. The approximate $100(1-\delta)\%$ two-sided confidence intervals for the parameters θ, and β are given by

$$\hat{\theta} \pm z_{\delta/2}\left[\widehat{Var(\hat{\theta})}\right]^{1/2} \quad \text{and} \quad \hat{\beta} \pm z_{\delta/2}\left[\widehat{Var(\hat{\beta})}\right]^{1/2}.$$

respectively, where $Var(\cdot)$ are the diagonal elements of $I^{-1}(\hat{\Theta})$ or $J^{-1}(\hat{\Theta})$, corresponding to each parameter, $J(\Theta) = \left[-\frac{\partial^2 l(\Theta)}{\partial\theta_i \partial\theta_j}|_{\Theta=\hat{\Theta}} \right]_{2x2}$ is the observed information matrix and $z_{\delta/2}$ is the upper $\frac{\delta}{2}^{th}$ percentile of the standard normal distribution.

The likelihood ratio (LR) statistic for testing $\theta = 1$ is given by

$$w = 2[\ln(L(\hat{\theta},\, \hat{\beta},\, 1,\, 1)) - \ln(L(1,\, \tilde{\beta},\, 1,\, 1))],$$

where $\hat{\theta}$, and $\hat{\beta}$ are the unrestricted estimates, and $\tilde{\beta}$ is restricted estimate. The LR test reject the null hypothesis if $w > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of a χ^2 distribution with 1 degrees of freedom.

4.4.1 Simulation Study

Various simulation were conducted for different sample sizes (n=50, 100, 200, 300, 500, 1000) to study the performance of IW and WIW distributions. We simulated 1000 samples for Model 1 with the true values of the parameters $\theta = 0.5$, $\beta = 3$, $k = 1$, and $l = 1$, Model 2 with the true values of the parameters $\theta = 1.2$, $\beta = 4.6$, $k = 1$, and $l = 1$, and Model 3 with the true values of the parameters $\theta = 1$, $\beta = 2$, $k = 1$, and $l = 1$. From the results of the simulations presented in Tables 4.2, 4.4 and 4.6, we can see that average bias for the parameters are very small, it is negative for the parameter l in the Model 1, the average bias for the parameters β and k in the Model 2 is negative also. The root mean squared errors (RMSEs) decreases as the sample size n increases. Also, as the sample size gets larger the mean estimates of the parameters gets closer to the true parameter values. When $k = 0$ or $l = 0$, the simulation results are presented in Tables 4.3, 4.5 and 4.7.

4.4.2 Applications

In this section, we present examples to illustrate the flexibility of the WIW distribution and its sub-models for data modeling. The first data set from Bjerkedal [5] represents the survival time, in days, of guinea pigs injected with different doses of tubercle bacilli. It is known that guinea pigs have high susceptibility of human tuberculosis. The data set consists of 72 observations. For the second example, the data is a subset of the breast feeding study from the National Longitudinal Survey of Youth, the complete data set is available in [27]. The data set considered here consists of the times to weaning 927 children of white-race mothers who choose to breast feed their children. The duration of the breast feeding was measured in weeks. Estimates of the parameters of WIW distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC) are given in Table 4.8 for the first data set and in Table 4.9 for the second data set. Plots of the fitted densities and histogram of the data are given in Figures 4.5 and 4.6. Probability plots (Chambers et al. [10]) are also presented in figures 4.5 and 4.6. For the probability plot, we plotted for example,

$$G_{WIW}(x_{(j)}) = \frac{\Gamma(1-\frac{1}{\hat{\beta}}, 2\hat{\theta}x_{(j)}^{-\hat{\beta}})}{\Gamma(1-\frac{1}{\hat{\beta}})}$$

against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. We also computed a measure of closeness of each plot to the diagonal line. This measure of closeness is given by the sum of squares

$$SS = \sum_{j=1}^{n}\left[G_{WIW}(x_{(j)}) - \left(\frac{j-0.375}{n+0.25}\right)\right]^2.$$

For the first dataset, the LR test statistic of the hypothesis H_0: WIW (1, β, 1, 1) against H_a: $WIW(\theta, \beta, 1, 1)$, is $w = 950.7 - 801.7 = 149.0$. The p-value is

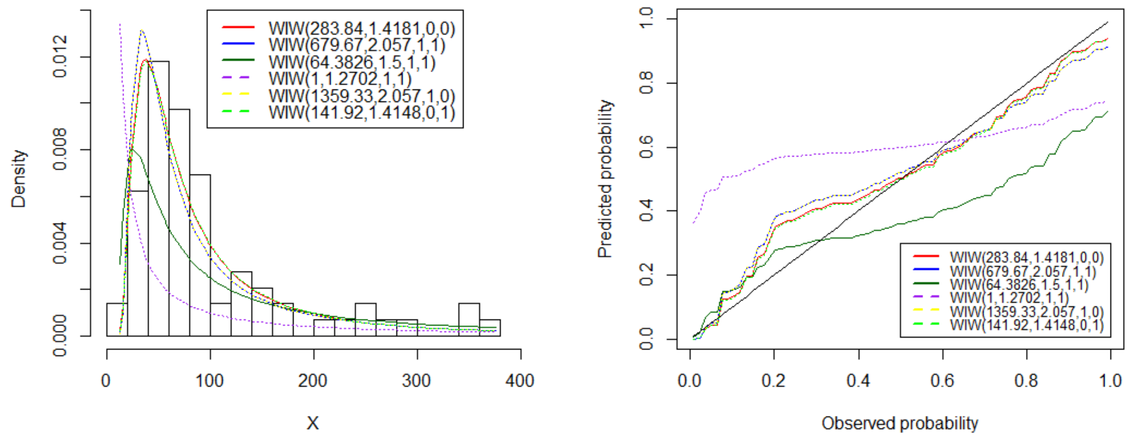

Figure 4.5: Fitted density and probability plots for guinea pigs survival time

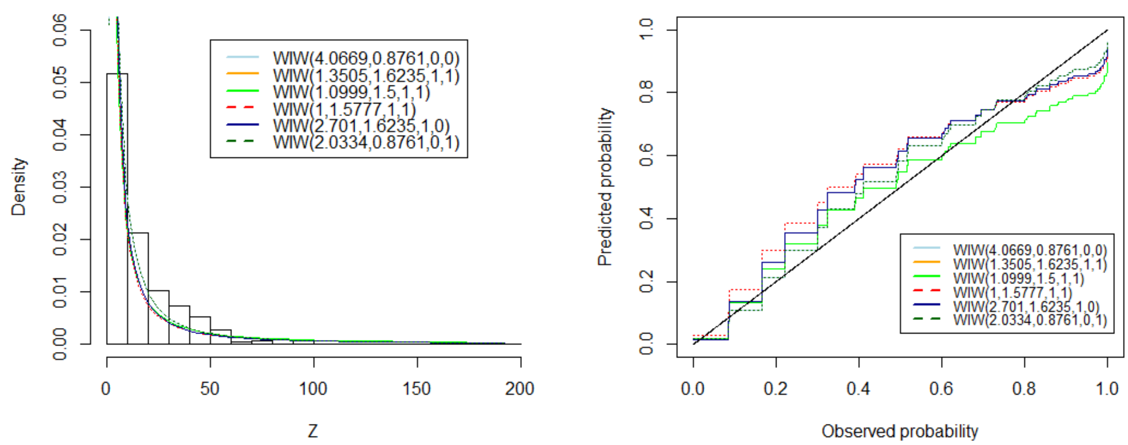

Figure 4.6: Fitted density and probability plots for breast feeding data

$2.86\text{x}10^{-34} < 0.001$. Therefore, we reject H_0 in favor of H_a: $WIW(\theta, \beta, 1, 1)$. The test statistic for the hypothesis H_0: WIW (θ, 1.5, 1, 1) against H_a: $WIW(\theta, \beta, 1, 1)$, is $w = 836.7 - 801.7 = 35.0$. The p-value is $3.29\text{x}10^{-9} < 0.001$. Therefore, we reject H_0 in favor of H_a: $WIW(\theta, \beta, 1, 1)$. The values of the statistics AIC, AICC and BIC shows that sub-model WIW $(\theta, \beta, 0, 1)$ is a "better" fit for this data. Also, the value of SS given in the probability plot is smallest for this model. For the second dataset, the LR test statistic of the hypothesis H_0: WIW (θ, 1.5, 1, 1) against H_a: WIW (θ, β, $1, 1$), is $w = 7311.3 - 7268.6 = 42.7$. The p-value is < 0.0001. Therefore, we reject H_0 in favor of H_a: WIW (θ, β, $1, 1$). According to the values of the statistics AIC, AICC and BIC shows that models WIW $(\theta, \beta, 0, 1)$ and WIW $(\theta, \beta, 0, 0)$ provides good fits for the second dataset. Also, the value of SS corresponded to the WIW $(\theta, \beta, 0, 1)$ model is the smallest.

Table 4.2: Simulation Results for Model 1: Mean Estimates, Average Bias and RMSEs

		IW			WIW		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	0.5056	0.00562	0.00946	0.5275	0.02754	0.03327
	β	3.0745	0.07454	0.14025	3.399	0.39902	1.11667
	k	-			1.3346	0.33464	1.54536
	l	-			0.8742	-0.12578	0.11911
100	θ	0.502	0.00199	0.00486	0.5163	0.01626	0.02223
	β	3.0488	0.04880	0.06619	3.2935	0.29346	0.79699
	k	-			1.256	0.25604	1.16051
	l	-			0.8947	-0.10532	0.10025
200	θ	0.5007	0.00074	0.00229	0.5142	0.01418	0.01441
	β	3.0247	0.02471	0.02917	3.153	0.15299	0.46398
	k	-			1.121	0.12101	0.71501
	l	-			0.9492	-0.05076	0.06618
300	θ	0.5003	0.00032	0.00158	0.509	0.00899	0.01089
	β	3.0178	0.01777	0.02126	3.1196	0.11964	0.33067
	k	-			1.1025	0.10249	0.53677
	l	-			0.9609	-0.03905	0.05251
500	θ	0.5006	0.00057	0.00089	0.508	0.008043	0.0063
	β	3.0022	0.00222	0.01074	3.0569	0.056878	0.17724
	k	-			1.0396	0.039614	0.29899
	l	-			0.9797	-0.020335	0.03299
1000	θ	0.5003	0.00033	0.00043	0.5062	0.006174	0.00306
	β	3.0015	0.00151	0.00549	3.0171	0.017092	0.08441
	k	-			1.006	0.006035	0.14517
	l	-			0.9946	-0.005418	0.01981

Table 4.3: Simulation Results for Model 1: Mean Estimates, Average Bias and RMSEs of the parameters θ, β with $k = 0$ and $l = 0$

		WIW, l=0			WIW, k=0		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	0.5331	0.03308	0.10193	0.5129	0.012869	0.00624
	β	3.37	0.37002	1.10579	3.0906	0.090562	0.13476
	k	1.2905	0.29051	1.52578			
	l				0.9751	-0.024869	0.00216
100	θ	0.5015	0.00146	0.0737	0.5068	0.006829	0.003235
	β	3.3378	0.33783	0.83619	3.0373	0.037291	0.058164
	k	1.3033	0.30325	1.18555			
	l				0.9806	-0.019356	0.001479
200	θ	0.5247	0.02473	0.05581	0.5059	0.005931	0.001602
	β	3.1388	0.1388	0.45679	3.0108	0.010799	0.02751
	k	1.1122	0.11225	0.71587			
	l				0.9868	-0.013191	0.000619
300	θ	0.5189	0.018929	0.03914	0.503	0.003011	0.000948
	β	3.0963	0.096287	0.28324	3.0114	0.011392	0.018291
	k	1.0711	0.071086	0.47277			
	l				0.9894	-0.010597	0.000454
500	θ	0.5144	0.014358	0.02661	0.5035	0.003529021	0.000574
	β	3.054	0.054024	0.18532	3.0053	0.005339821	0.011493
	k	1.0414	0.041424	0.30938			
	l				0.9936	-0.006393745	0.000269
1000	θ	0.5022	0.002201	0.01326	0.5003	0.00028	0.000251493
	β	3.0374	0.037408	0.08778	3.0104	0.010412	0.005642621
	k	1.0329	0.032876	0.15123			
	l				0.9978	-0.002225	0.000099492

Table 4.4: Simulation Results for Model 2: Mean Estimates, Average Bias and RMSEs

		IW			WIW		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	1.2233	0.02328	0.03193	1.2827	0.08274	0.06
	β	4.7348	0.13483	0.28005	4.5406	-0.05935	0.29298
	k	-			0.744	-0.25596	0.67972
	l	-			1.0309	0.03094	0.01757
100	θ	1.2133	0.013254	0.01538	1.2441	0.0441	0.0313
	β	4.6599	0.059884	0.1311	4.5775	-0.02254	0.23071
	k	-			0.871	-0.12903	0.55897
	l	-			1.0147	0.01472	0.01124
200	θ	1.2074	0.007447	0.007283	1.2298	0.02985	0.01594
	β	4.6297	0.029682	0.063478	4.5723	-0.027728	0.20737
	k	-			0.8929	-0.10713	0.4495
	l	-			1.0114	0.011388	0.00733
300	θ	1.2051	0.005126	0.005201	1.2283	0.028268	0.01215
	β	4.6167	0.016657	0.043962	4.5588	-0.04115	0.19049
	k	-			0.8995	-0.1005	0.41477
	l	-			1.0134	0.01339	0.00657
500	θ	1.2027	0.002675	0.002887	1.2224	0.0224	0.00886
	β	4.6136	0.013564	0.026183	4.554	-0.046	0.16117
	k	-			0.9149	-0.085144	0.35118
	l	-			1.0121	0.0121	0.00531
1000	θ	1.2014	0.001405606	0.00142	1.2142	0.014229	0.00501
	β	4.6049	0.004931619	0.01304	4.5746	-0.025379	0.11855
	k	-			0.9455	-0.054508	0.24502
	l	-			1.0054	0.005373	0.00371

Table 4.5: Simulation Results for Model 2: Mean Estimates, Average Bias and RMSEs of the parameters θ, β with $k = 0$ and $l = 0$

		WIW, l=0			WIW, k=0		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	1.3143	0.11427	0.11366	1.2421	0.04215	0.02991
	β	4.5703	-0.02975	0.27167	4.7382	0.13819	0.32755
	k	0.8014	-0.19859	0.62554			
	l				1.0124	0.01239	0.0067
100	θ	1.2879	0.08786	0.07876	1.2125	0.01245	0.01009
	β	4.5211	-0.07886	0.25823	4.6621	0.062093	0.14815
	k	0.8228	-0.17723	0.54967			
	l				1.004	0.003983	0.00231
200	θ	1.2649	0.0649	0.06273	1.2091	0.009139	0.005537
	β	4.565	-0.03501	0.21171	4.624	0.023984	0.066618
	k	0.8899	-0.1101	0.47792			
	l				1.0036	0.003556	0.001258
300	θ	1.2409	0.040854	0.04763	1.2075	0.007451	0.00329
	β	4.5706	-0.029358	0.18307	4.6225	0.022453	0.042093
	k	0.9202	-0.079816	0.40167			
	l				1.0026	0.0026	0.000769
500	θ	1.2402	0.040234	0.04053	1.2051	0.005124	0.001949
	β	4.5664	-0.033586	0.15896	4.6102	0.010249	0.025118
	k	0.9222	-0.077767	0.35341			
	l				1.0009	0.000931	0.000502
1000	θ	1.2286	0.028571	0.0278	1.2023	0.002313918	0.000905
	β	4.5766	-0.023371	0.12162	4.6069	0.006890177	0.011887
	k	0.9456	-0.054446	0.256			
	l				1.0009	0.000889	0.00034

Table 4.6: Simulation Results for Model 3: Mean Estimates, Average Bias and RMSEs

		IW			WIW		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	1.0161	0.01612	0.02519	1.08660	0.08661	0.08435
	β	2.0592	0.05924	0.05569	2.15480	0.15476	0.43824
	k	-			1.08300	0.08303	0.60451
	l	-			1.03870	0.03874	0.03482
100	θ	1.0066	0.00657	0.01123	1.04380	0.04379	0.04818
	β	2.0217	0.02166	0.02607	2.12930	0.12935	0.32394
	k	-			1.08870	0.08873	0.45989
	l	-			1.01550	0.01554	0.02062
200	θ	1.0033	0.00329	0.00620	1.02320	0.02141	0.02522
	β	2.012	0.01205	0.01326	2.09120	0.09122	0.19365
	k	-			1.07590	0.07591	0.26830
	l	-			1.00580	0.00356	0.01040
300	θ	1.0031	0.00312	0.00394	1.02140	0.02323	0.01710
	β	2.0112	0.01122	0.00805	2.04670	0.04673	0.13609
	k	-			1.03140	0.03141	0.19059
	l	-			1.00360	0.00579	0.00650
500	θ	1.0017	0.00231	0.00229	1.00650	0.00646	0.01062
	β	2.0039	0.00387	0.00493	2.05690	0.05695	0.09171
	k	-			1.05270	0.05274	0.12723
	l	-			1.00190	0.00190	0.00452
1000	θ	1.0023	0.00166	0.00107	1.00440	0.00440	0.00530
	β	2.0015	0.00146	0.00230	2.01720	0.01717	0.04023
	k	-			1.01340	0.01344	0.05693
	l	-			1.00090	0.00095	0.00180

Table 4.7: Simulation Results for Model 3: Mean Estimates, Average Bias and RMSEs of the parameters θ, β with $k = 0$ and $l = 0$

		WIW, l=0			WIW, k=0		
Sample size, n	Parameter	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
50	θ	2.138	0.13796	0.44436	1.0263	0.02631	0.02051
	β	1.1749	0.17487	0.35592	2.0569	0.05695	0.05692
	k	1.0979	0.09785	0.62271			
	l				1.0061	0.00613	0.00222
100	θ	1.0692	0.06921	0.19982	1.0114	0.01141	0.00838
	β	2.1335	0.13347	0.31485	2.0298	0.02983	0.02488
	k	1.0648	0.06476	0.43609			
	l				1.0052	0.00522	0.00098
200	θ	1.0361	0.03610	0.12421	1.0061	0.00606	0.00398
	β	2.0993	0.09933	0.21118	2.0105	0.01054	0.01279
	k	1.0836	0.08358	0.29572			
	l				1.0051	0.00509	0.00046
300	θ	1.028	0.02799	0.07702	1.0053	0.00529	0.00267
	β	2.0589	0.05889	0.14201	2.0131	0.01306	0.00845
	k	1.0485	0.04847	0.19445			
	l				1.0035	0.00345	0.00033
500	θ	1.0214	0.02137	0.04729	1.0042	0.00421	0.00163
	β	2.0309	0.03092	0.08440	2.0072	0.00722	0.00519
	k	1.029	0.02904	0.11847			
	l				1.0015	0.00145	0.00027
1000	θ	1.0014	0.00139	0.02373	1.002	0.00205	0.00072
	β	2.0309	0.03093	0.04344	2.0023	0.00229	0.00253
	k	1.0245	0.02448	0.06076			
	l				1.0004	0.00038	0.00023

Table 4.8: Estimates of models for Bjerkedal data

	Estimates					Statistics			
Model	θ	β	k	l	$-2\log L$	AIC	$AICC$	BIC	SS
WIW(θ, β,0,0)	283.84	1.4181	0	0	791.3	795.3	795.5	799.9	0.2572
	125.63	0.1173							
WIW(θ, β,1,1)	679.67	2.057	1	1	801.7	805.7	805.8	810.2	0.42460
	302.43	0.1095							
WIW(θ,1.5,1,1)	64.3826	1.5	1	1	836.7	838.7	838.8	841	2.3415
	13.142								
WIW(1,β,1,1)	1	1.2702	1	1	950.7	952.7	952.8	955	4.1629
		0.03014							
WIW(θ, β,1,0)	1359.33	2.057	1	0	801.7	805.7	805.8	810.2	0.42462
	604.86	0.1095							
WIW(θ, β,0,1)	141.92	1.4148	0	1	791.3	795.3	795.5	799.9	0.2453
	62.8164	0.1173							

Table 4.9: Estimates of models for the breast feeding data

	Estimates					Statistics			
Model	θ	β	k	l	$-2\log L$	AIC	$AICC$	BIC	SS
WIW(θ, β,0,0)	4.0669	0.8761	0	0	7134.3	7138.3	7138.3	7148	2.6436
	(0.1619)	(0.02136)							
WIW(θ, β,1,1)	1.3505	1.6235	1	1	7268.6	7272.6	7272.6	7282.3	5.9571
	(0.08192)	(0.02008)							
WIW(θ,1.5,1,1)	1.0999	1.5	1	1	7311.3	7313.3	7313.3	7318.1	4.8026
	(0.06257)								
WIW(1,β,1,1)	1	1.5777	1	1	7290.7	7292.7	7292.7	7297.6	7.9303
		(0.01685)							
WIW(θ, β,1,0)	2.701	1.6235	1	0	7268.6	7272.6	7272.6	7282.3	5.9571
	(0.1638)	(0.02008)							
WIW(θ, β,0,1)	2.0334	0.8761	0	1	7134.3	7138.3	7138.3	7148	2.6440
	(0.08097)	(0.02136)							

CHAPTER 5

GENERALIZED NEW MODIFIED WEIBULL DISTRIBUTIONS

5.1 Introduction

In this chapter, we generalize the new modified Weibull distribution, proposed by Almalki and Yuan [1] via the introduction of the resilience parameter δ into the new modified Weibull distribution to obtain a generalized new modified Weibull distribution. Weibull distribution is widely used for modeling data in a wide variety of areas including reliability, engineering, survival analysis and renewal theory. Weibull distribution fails to accommodate non-monotonic hazard rate functions such as unimodal and bath-tub shapes. The addition of one or more parameters to a distribution makes it more richer and quite flexible for modeling data. For a baseline cumulative distribution function $F(x)$, the exponentiated version $G(x) = [F(x)]^{\delta}$ is different and flexible enough to accommodate both monotone as well as non-monotone hazard rate functions. Exponentiated distributions are indeed different from the baseline cdf $F(x)$, for example, if the $F(x) = 1 - \exp(-\lambda x)$ and $G(x) = [1 - \exp(-\lambda x)]^{\delta}$, the exponential distribution ($\delta = 1$) has constant hazard rate λ, however, the exponentiated exponential distribution $G(x)$ has increasing hazard rate if $\delta > 1$, constant hazard rate if $\delta = 1$, and decreasing hazard rate if $\delta < 1$.

There are several generalizations of the Weibull distribution including those of Famoye et al. [14] dealing with results on the beta-Weibull distribution. Nadarajah [35] presented results on the modified Weibull distribution. In this chapter, we present and study the mathematical properties of the exponentiated new modified Weibull distribution. This class of distributions is flexible in accommodating all forms of hazard rate functions and contains several well known and new sub-models such as Weibull, Rayleigh, exponentiated Weibull (Mudholkar et al. [32]), generalized Rayleigh (Kundu and Rakab [28]), exponentiated exponential (Gupta and Kundu

[21], [22]), modified Weibull (Lai et al. [29]), and a host of other, some of which are presented in section 2 of this chapter. A host of researchers have also developed several parameter modified Weibull and flexible Weibull distributions over the years. The two parameter Weibull extensions include those of Bebbington et al. [4]. The three parameter Weibull extensions include those by Xie et al. [54], Mudholkar and Srivastava [33], some of these extensions enable the accommodation of bathtub shape hazard rate function. The four parameter generalizations include the additive Weibull distribution of Xie and Lai [53], beta-Weibull proposed by Famoye et al. [14] and exponentiated Weibull by Choudhury [11]. The five parameter modified Weibull include those introduced by Phani [41], beta modified Weibull by Silva et al. [48], and Nadarajah et al. [36]. Additional results on the generalization of the Weibull distribution include work by Singla et al. [49], as well as Almalki and Yuan [1], where results on a new modified Weibull distribution was presented.

In this chapter, we present and analyze the generalized or exponentiated new modified Weibull (GNMW) distribution. This new distribution is flexible in accommodating all the forms of the hazard rate function that can be used in a a variety of problems in modeling and for testing goodness-of-fit of several sub-models including the new modified Weibull (NMW), exponentiated modified Weibull (EMW), modified Weibull (MW), Weibull (W), and a host of other sub-models. In general, the exponentiated NMW distribution is

$$F_{GNMW}(x) = [G_{NMW}(x)]^{\delta},$$

$G_{NMW}(x)$ is a baseline NMW cdf, and $\delta > 0$, with the corresponding pdf given by

$$f_{GNMW}(x) = \delta[G_{NMW}(x)]^{\delta-1} g_{NMW}(x).$$

Note that, for large values of x, and for $\delta > 1 (< 1)$, the multiplicative factor $\delta[G_{NMW}(x)]^{\delta-1} > 1 (< 1)$, respectively. The reverse statement holds for smaller values of x. Consequently, this implies that the ordinary moments of $f_{GNMW}(x)$ are larger (smaller) than those of $g_{NMW}(x)$ when $\delta > 1 (< 1)$. The cumulative distribution function (cdf) of GNMW distribution is given by

$$F(x) = [1 - e^{-\alpha x^{\theta} - \beta x^{\gamma} e^{\lambda x}}]^{\delta}, \quad \alpha,\, \theta,\, \beta,\, \gamma,\, \lambda,\, \delta \text{ and } x \geq 0.$$

The corresponding pdf is given by

$$\begin{aligned} f(x;\alpha,\theta,\beta,\gamma,\lambda,\delta) &= \delta\{1 - e^{-\alpha x^{\theta} - \beta x^{\gamma} e^{\lambda x}}\}^{\delta-1} e^{-\alpha x^{\theta} - \beta x^{\gamma} e^{\lambda x}} \\ &\times (\alpha\theta x^{\theta-1} + \beta(\gamma + \lambda x) x^{\gamma-1} e^{\lambda x}) \\ &= \delta[F_{NMW}(x;\alpha,\theta,\beta,\gamma,\lambda)]^{\delta-1} S_{MW}(x;\beta,\gamma,\lambda) \\ &\times (h_W(x;\alpha,\theta) + h_{MW}(x;\beta,\gamma,\lambda)), \end{aligned}$$

where $F_{NMW}(x;\alpha,\theta,\beta,\gamma,\lambda)$, $S_{MW}(x;\beta,\gamma,\lambda)$, $h_W(x;\alpha,\theta)$, and $h_{MW}(x;\beta,\gamma,\lambda)$ are the cdf of the new modified Weibull distribution, survival function of the modified Weibull distribution, hazard rate function of the Weibull distribution and hazard rate of the modified Weibull distribution, respectively. Note that the parameters γ, θ, and δ controls the shape of the distribution. The parameters α, and β, controls the scale of the distribution and the parameter λ is an accelerating factor in the imperfection time and works as a factor of fragility in the survival of the individual when the time increases.

5.2 GNMW Distribution and Sub-models

In this section, the properties of the GNMW distribution and some of its sub-models are presented. The survival function of the GNMW distribution is given by

$$S(x) = 1 - \{1 - e^{-\alpha x^{\theta} - \beta x^{\gamma} e^{\lambda x}}\}^{\delta}, \quad \alpha,\, \theta,\, \beta,\, \gamma,\, \lambda,\, \delta \text{ and } x \geq 0.$$

We can simulate from the GNMW distribution by solving the nonlinear equation

$$\alpha y^{\theta} - \beta y^{\gamma} e^{\lambda y} - \log(1 - u^{1/\delta}) = 0,$$

where u is uniform on $(0,1)$. Note that when b is a positive real non-integer and $|z|<1$, we have the series representation

$$(1-z)^{b-1} = \sum_{j=0}^{b} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\Gamma(j+1)} z^j,$$

which is used to obtain the series expansion of the GNMW cdf and pdf. The series expansion of the GNMW cdf and pdf are given by

$$\begin{aligned} F(x) &= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\delta+1)}{\Gamma(\delta+1-j)\Gamma(j+1)} e^{-j\alpha x^{\theta} - j\beta x^{\gamma} e^{\lambda x}} \\ &= \sum_{j=0}^{\infty} \omega_j \overline{F}_{NMW}(x; j\alpha, j\beta, \gamma, \lambda), \end{aligned}$$

and

$$f(x) = \sum_{j=0}^{\infty} \frac{(-1)^{j+1} \Gamma(\delta+1)}{\Gamma(\delta+1-j)\Gamma(j+1)} f_{NMW}(x; j\alpha, j\beta, \gamma, \lambda),$$

where $\omega_j = \frac{(-1)^j \Gamma(\delta+1)}{\Gamma(\delta+1-j)\Gamma(j+1)}$, and $\overline{F}(x) = S(x) = 1 - F(x)$.

5.2.1 Some New and Known Sub-models

There are several sub-models of the GNMW distribution including some well known distributions such as Weibull (W), Rayleigh (R), exponential (E) and extreme value (EV) distributions. Some of the sub-models of the GNMW distribution are listed in Table 5.1. The sub-models of the GNMW distribution include the generalized new modified Rayleigh (GNMR), generalized new modified exponential (GNME), generalized new exponential modified Weibull (GNEMW), generalized new exponential modified exponential (GNEME), generalized Sarhan and Zaindin modified Weibull (GS-ZMW), generalized linear failure rate (GLFR), generalized extreme value (GEV), generalized additive Weibull (GAW), generalized modified Weibull (GMW), generalized Weibull (GW), generalized Rayleigh (GR), generalized exponential (GE), new modified Weibull (NMW), additive Weibull (AW), modified Weibull (MW), and linear failure rate (LFR) distributions.

Table 5.1: Sub-models of the GNM Weibull Distribution

Model	α	β	γ	θ	λ	δ	$F(x)$	$Reference$
GNMR	-	-	2	2	-	-	$[1-e^{-\alpha x^2-\beta x^2 e^{\lambda x}}]^\delta$	New
GNME	-	-	1	1	-	-	$[1-e^{-\alpha x-\beta x e^{\lambda x}}]^\delta$	New
GNEMW	-	-	1	-	-	-	$[1-e^{-\alpha x^\theta-\beta x e^{\lambda x}}]^\delta$	New
GNEME	-	-	1	1	-	-	$[1-e^{-\alpha x-\beta x e^{\lambda x}}]^\delta$	New
GS-ZMW	-	-	-	1	0	-	$[1-e^{-\alpha x-\beta x^\gamma}]^\delta$	New
GLFR	-	-	2	1	0	-	$[1-e^{-\alpha x-\beta x^2}]^\delta$	New
GEV	0	1	0	0	-	-	$[1-e^{-e^{\lambda x}}]^\delta$	New
GAW	-	-	-	-	0	-	$[1-e^{-\alpha x^\theta-\beta x^\gamma}]^\delta$	New
GMW	0	-	-	0	-	-	$[1-e^{-\beta x^\gamma e^{\lambda x}}]^\delta$	Carrasco et al. (2008)
GW	-	0	0	-	0	-	$[1-e^{-\alpha x^\theta}]^\delta$	Mudholkar et al. (1995, 1996)
GR	-	0	0	2	0	-	$[1-e^{-\alpha x^2}]^\delta$	Kundu and Rekab (2005)
GE	-	0	0	1	0	-	$[1-e^{-\alpha x}]^\delta$	Gupta and Kundu (1999)
NMW	-	-	-	-	-	1	$1-e^{-\alpha x^\theta-\beta x^\gamma e^{\lambda x}}$	Almalki and Yuan (2013)
AW	-	-	-	-	0	1	$1-e^{-\alpha x^\theta-\beta x^\gamma}$	Xie and Li (1995)
MW	0	-	-	0	-	1	$1-e^{-\beta x^\gamma e^{\lambda x}}$	Li, Xie and Murthy (2003)
S-ZMW	-	-	-	1	0	1	$1-e^{-\alpha x-\beta x^\gamma}$	Sarhan and Zaindin (2008, 2009)
LFR	-	-	2	1	0	1	$1-e^{-\alpha x-\beta x^2}$	Bain (1974)
EV	0	1	0	0	-	1	$1-e^{-e^{\lambda x}}$	Bain (1974)
W	-	0	0	-	0	1	$1-e^{-\alpha x^\theta}$	Weibull (1951)
R	-	0	0	2	0	1	$1-e^{-\alpha x^2}$	Bain (1974)
E	-	0	0	1	0	1	$1-e^{-\alpha x}$	Bain (1974)

Graphs of the pdf of GNMW distribution are given in the Figures 5.1 and 5.2

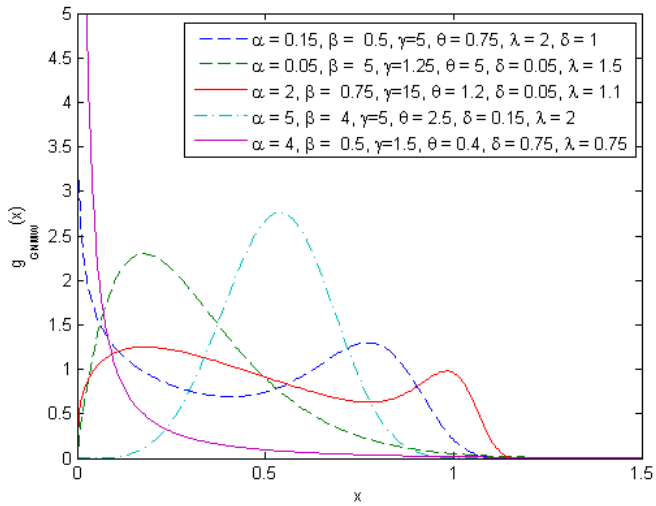

Figure 5.1: Plot of the pdf of GNMW distribution

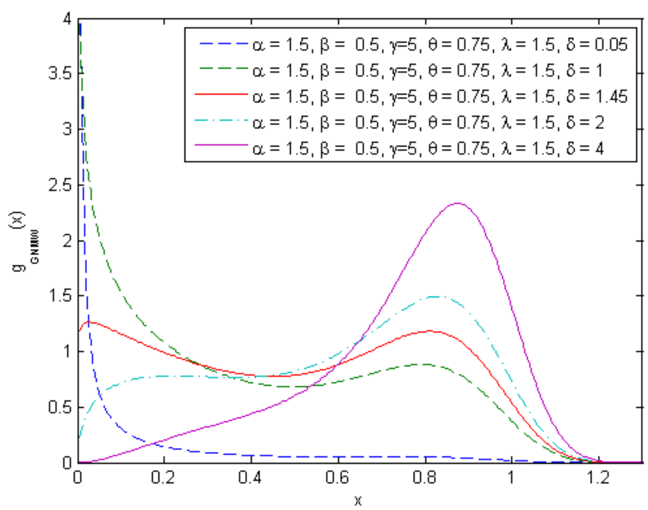

Figure 5.2: Plot of the pdf of GNMW distribution

for selected values of the parameters. The plots show that the GNMW pdf can take various shapes including decreasing, unimodal, bimodal or right skewed among several other possible shapes as seen in the figures. The distribution can also have positive and negative asymmetry.

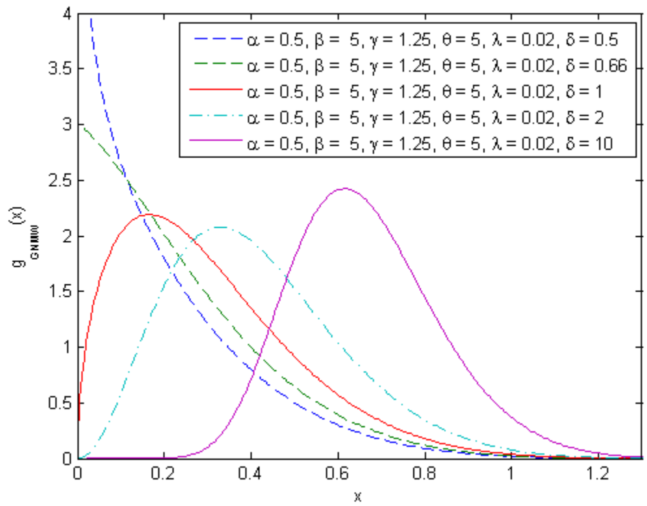

Figure 5.3: Graph of pdf of GNMW distribution

5.2.2 Hazard and Reverse Hazard Functions

In this section, we present the hazard rate and reverse hazard functions of the GNMW distribution and its sub-models. Graphs of the hazard rate function are also presented for selected values of the model parameters. The hazard and reverse hazard functions are

$$h_F(x) = \frac{\delta\{1-e^{-\alpha x^{\theta}-\beta x^{\gamma}e^{\lambda x}}\}^{\delta-1}e^{-\alpha x^{\theta}-\beta x^{\gamma}e^{\lambda x}}(\alpha\theta x^{\theta-1}+\beta(\gamma+\lambda x)x^{\gamma-1}e^{\lambda x})}{1-\{1-e^{-\alpha x^{\theta}-\beta x^{\gamma}e^{\lambda x}}\}^{\delta}},$$

and

$$\tau_F(x) = \frac{e^{-\alpha x^{\theta}-\beta x^{\gamma}e^{\lambda x}}(\alpha\theta x^{\theta-1}+\beta(\gamma+\lambda x)x^{\gamma-1}e^{\lambda x})}{1-e^{-\alpha x^{\theta}-\beta x^{\gamma}e^{\lambda x}}},$$

respectively. Plots of the hazard rate function for different combinations of the parameter values are given in Figure 5.4. The plot shows various shapes including monotonically increasing, monotonically decreasing, and bathtub shapes among others for five combinations of the values of the parameters. This flexibility makes the GNMW hazard rate function suitable for both monotonic and non-monotonic empirical hazard behaviors that are likely to be encountered in practice and real life

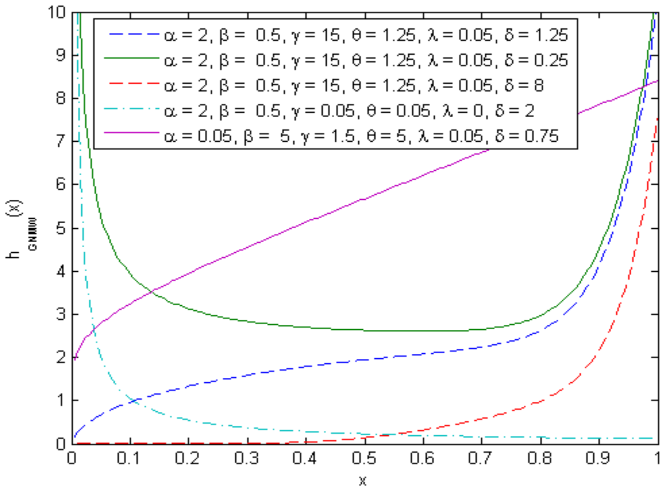

Figure 5.4: Graph of hazard function of the GNMW distribution

situations.

5.3 Moments and Moment Generating Function

In this section, we obtain moments for the GNMW distribution. The r^{th} raw moment is obtained as follows, for δ real non-integer :

$$
\begin{aligned}
E(X^r) &= \int_0^\infty r x^{r-1}[1-e^{-\alpha x^\theta - \beta x^\gamma e^{\lambda x}}]^\delta dx \\
&= \int_0^\infty r x^{r-1} \sum_{j=0}^\infty \frac{\Gamma(\delta+1)(-1)^j}{\Gamma(\delta-j+1)j!} e^{j(-\alpha x^\theta - \beta x^\gamma e^{\lambda x})} dx \\
&= \sum_{j=0}^\infty \frac{r\Gamma(\delta+1)(-1)^j}{\Gamma(\delta-j+1)j!} \int_0^\infty x^{r-1} \sum_{n=0}^\infty \frac{(-\beta j)^n}{n!} x^{n\gamma} e^{n\lambda x} e^{j(-\alpha x^\theta)} dx \\
&= \sum_{j=0}^\infty \frac{r\Gamma(\delta+1)(-1)^j}{\Gamma(\delta-j+1)j!} \int_0^\infty \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-\beta j)^n}{n!} \frac{(n\lambda)^m}{m!} x^{n\gamma+m+r-1} e^{j(-\alpha x^\theta)} dx \\
&= \sum_{j,n,m=0}^\infty \frac{r\Gamma(\delta+1)(-1)^j(-\beta j)^n (n\lambda)^m}{\theta\Gamma(\delta-j+1)j!n!m!} (j\alpha)^{-(r+n\gamma+m)} \Gamma\left(\frac{r+n\gamma+m}{\theta}\right),
\end{aligned}
$$

for $r = 1, 2, ...,$ where $\Gamma(.)$ is the gamma function.

5.3.1 Moment Generating Function

The moment generating function of the GNMW distribution (MGF) of the GNMW distribution is given below. Recall the Taylor's series expansion of the function e^{tx}, that is $e^{tx} = \sum_{j=0}^{\infty} \frac{(tx)^j}{j!}$, so the MGF of the GNMW distribution is given by

$$\begin{aligned} M_X(t) &= E\left(\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right) \\ &= \sum_{k,j,n,m=0}^{\infty} \frac{t^k}{k!} \frac{r\Gamma(\delta+1)(-1)^j(-\beta j)^n(n\lambda)^m}{\theta\Gamma(\delta-j+1)j!n!m!}(j\alpha)^{-(k+n\gamma+m)}\Gamma\left(\frac{k+n\gamma+m}{\theta}\right). \end{aligned}$$

5.4 Order Statistics

Order statistics play an important role in probability and statistics. In this section, we present the distribution of the order statistic for the GNMW distribution. The pdf of the i^{th} order statistic from the GNMW pdf $f(x)$ is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!f(x)}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i} \\ &= \frac{n!f(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^{i+j-1}, \end{aligned}$$

by using the binomial expansion $(1-F(x))^{n-i} = \sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k F(x)^k$, so that

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)}\sum_{k=0}^{n-i}\binom{n-i}{k}\frac{(-1)^k}{k+i}(k+i)F^{k+i-1}(x)f(x) \\ &= \sum_{k=0}^{n-i}\omega_{i,k}f_{i+k}(x), \end{aligned}$$

where $f_{i+k}(x)$ is the pdf of the GNMW distribution with parameters α, θ, β, γ, λ, and $\delta(i+k)$, $B(.,.)$ is the beta function and the weights $\omega_{i,k}$ are given by

$$\omega_{i,k} = (-1)^k\binom{i+k-1}{k}\binom{n}{i+k}.$$

Consequently, the s^{th} moment of the i^{th} order statistics is given by

$$\begin{aligned} E(X_{i:n}^s) &= \sum_{k=0}^{n-i} \sum_{s,j,n,m=0}^{\infty} \omega_{i,k} \frac{s\Gamma(\delta(i+k)-1)(-1)^j(-\beta j)^n(n\lambda)^m}{\theta\Gamma(\delta(i+k)-j+1)j!n!m!} \\ &\times \ (j\alpha)^{-(r+n\gamma+m)}\Gamma\left(\frac{s+n\gamma+m}{\theta}\right). \end{aligned}$$

These moments are often used in several areas including reliability, insurance and quality control for the prediction of future failures times from a set of past or previous failures.

5.5 Maximum Likelihood Estimation

Let $x = (x_1, x_2, ..., x_n)$ be a random sample from the GNMW distribution with unknown parameters $\Delta = (\alpha, \beta, \theta, \lambda, \gamma, \delta)^T$. The likelihood function is given by

$$\begin{aligned} L(\alpha,\theta,\beta,\gamma,\lambda,\delta) &= \delta^n \prod_{i=1}^{n} \Big[\{1-e^{-\alpha x_i^\theta - \beta x_i^\gamma e^{\lambda x_i}}\}^{\delta-1} e^{-\alpha x_i^\theta - \beta x_i^\gamma e^{\lambda x_i}} \\ &\times \ (\alpha\theta x_i^{\theta-1} + \beta(\gamma+\lambda x_i) x_i^{\gamma-1} e^{\lambda x_i}) \Big]. \end{aligned}$$

Now, the log likelihood function denoted by ℓ is given by

$$\begin{aligned} \ell_n = \log[L(\alpha,\theta,\beta,\gamma,\lambda,\delta)] &= n\log(\delta) - (\delta-1)\sum_{i=1}^{n} \log(1-e^{-\alpha x_i^\theta - \beta x_i^\gamma e^{\lambda x_i}}) \\ &- \ \alpha \sum_{i=1}^{n} x_i^\theta - \beta \sum_{i=1}^{n} x_i^\gamma e^{\lambda x_i} \\ &+ \ \sum_{i=1}^{n} \log(\alpha\theta x_i^{\theta-1} + \beta(\gamma+\lambda x_i) x_i^{\gamma-1} e^{\lambda x_i}). \end{aligned}$$

Let $k(x_i; \alpha, \theta, \beta, \gamma, \lambda) = \alpha\theta x_i^{\theta-1} + \beta(\gamma+\lambda x_i) x_i^{\gamma-1} e^{\lambda x_i}$. The entries of the score function $U(\Theta) = \left(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \lambda}, \frac{\partial \ell_n}{\partial \gamma}, \frac{\partial \ell_n}{\partial \delta}\right)^T$ are given by

$$\frac{\partial \ell_n}{\partial \alpha} = \sum_{i=1}^{n} \frac{\theta x_i^{\theta-1}}{k(x_i;\alpha,\theta,\beta,\gamma,\lambda)} - \sum_{i=1}^{n} x_i^\theta + (\delta-1)\sum_{i=1}^{n} \frac{x_i^\theta e^{-\alpha x_i^\theta - \beta e^{\lambda x_i} x_i^\gamma}}{1-e^{-\alpha x_i^\theta - \beta e^{\lambda x_i} x_i^\gamma}},$$

$$\frac{\partial \ell_n}{\partial \beta} = (\delta-1)\sum_{i=1}^{n}\frac{x_i e^{\lambda x_i} e^{-\alpha x_i^{\theta}-\beta x_i^{\gamma} e^{\lambda x_i}}}{1-e^{-\alpha x_i^{\theta}-\beta x_i^{\gamma} e^{\lambda x_i}}} - \sum_{i=1}^{n} x_i^{\gamma} e^{\lambda x_i} + \sum_{i=1}^{n}\frac{(\gamma+\lambda x_i) x_i^{\gamma-1} e^{\lambda x_i}}{k(x_i;\alpha,\theta,\beta,\gamma,\lambda)},$$

$$\frac{\partial \ell_n}{\partial \theta} = (\delta-1)\sum_{i=1}^{n}\frac{-\alpha x_i^{\theta}\log(\alpha x_i) e^{-\alpha x_i^{\theta}-\beta x_i^{\gamma} e^{\lambda x_i}}}{1-e^{-\alpha x_i^{\theta}-\beta x_i^{\gamma} e^{\lambda X_i}}} - \alpha\sum_{i=1}^{n} x^{\theta}\log(x_i) + \sum_{i=1}^{n}\frac{\alpha x_i^{\theta-1}(1+\log(x_i))}{k(x_i;\alpha,\theta,\beta,\gamma,\lambda)},$$

$$\frac{\partial \ell_n}{\partial \lambda} = (\delta-1)\sum_{i=1}^{n}\frac{\beta x_i^{\gamma+1} e^{\lambda x_i} e^{-\alpha x_i^{\theta}-\beta x_i^{\gamma} e^{\lambda x_i}}}{1-e^{-\alpha x_i^{\theta}-\beta x_i^{\gamma} e^{\lambda x_i}}} + \sum_{i=1}^{n}\frac{\beta(1+\gamma+\lambda x_i) x_i^{\gamma} e^{\lambda x_i}}{k(x_i;\alpha,\theta,\beta,\gamma,\lambda)} - \beta\sum_{i=1}^{n} x_i^{\gamma+1} e^{\lambda x_i},$$

$$\frac{\partial \ell_n}{\partial \gamma} = (\delta-1)\sum_{i=1}^{n}\frac{e^{-\alpha x_i^{\theta}-\beta x_i^{\gamma} e^{\lambda x_i}}\beta e^{\lambda x_i} x_i^{\gamma}\log(x_i)}{1-e^{-\alpha x_i^{\theta}-\beta x_i^{\gamma} e^{\lambda x_i}}} + \sum_{i=1}^{n}\frac{\beta x_i e^{\lambda x_i}[(\gamma+\lambda x_i)\log(x_i)+1]}{k(x_i;\alpha,\theta,\beta,\gamma,\lambda)} - \sum_{i=1}^{n}\beta e^{\lambda x_i} x_i^{\gamma}\log(x_i),$$

and

$$\frac{\partial \ell_n}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^{n}\log(1-e^{\alpha x_i^{\theta}-\beta x_i^{\gamma} e^{\lambda x_i}}). \tag{5.1}$$

The equations above are not in closed form and the values of the parameters $\alpha, \beta, \theta, \lambda, \gamma, \delta$ can be found by using iterative methods. The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters $\alpha, \beta, \theta, \lambda, \gamma, \delta$ must be found by using iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda}, \hat{\gamma}, \hat{\delta})$ is obtained by solving the nonlinear equation $(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\gamma}, \frac{\partial\ell}{\partial\delta})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $\mathbf{I}(\boldsymbol{\Delta}) = [\mathbf{I}_{\theta_i,\theta_j}]_{6X6} = E(-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j})$, $i,j = 1,2,3,4,5,6$, can be numerically obtained by MATHLAB or MAPLE software. The total Fisher information matrix $\mathbf{I}(\boldsymbol{\Delta})$ can be approximated by

$$\mathbf{J}_n(\hat{\boldsymbol{\Delta}}) \approx \left[-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}\bigg|_{\boldsymbol{\Delta}=\hat{\boldsymbol{\Delta}}} \right]_{6X6}, \quad i,j = 1,2,3,4,5,6. \tag{5.2}$$

For a given set of observations, the matrix given in equation (5.2) is obtained after the convergence of the Newton-Raphson procedure in MATHLAB software. The elements of the observed Fisher information matrix can be readily obtained from the mixed and partial derivatives of the $\ln f(x)$. In this case, they are:

$$\begin{aligned}\frac{\partial^2 \ln f(x)}{\partial\alpha^2} &= -\frac{(\delta-1)e^{-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta}}x^{2\theta}}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2} - \frac{(\delta-1)e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{2\theta}}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}} \\ &\quad -\frac{\theta^2 x^{-2+2\theta}}{(\beta\gamma x^{-1+\gamma}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{-1+\theta})^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ln f(x)}{\partial\alpha\partial\beta} &= -\frac{(\delta-1)e^{\lambda x-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta}}x^{\gamma+\theta}}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2} - \frac{(\delta-1)e^{\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{\gamma+\theta}}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}} \\ &\quad -\frac{\theta x^{\theta-1}(\gamma x^{\gamma-1}+e^{\lambda x}\lambda x^{\gamma}}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1})^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ln f(x)}{\partial\alpha\partial\gamma} &= -\frac{\beta(\delta-1)e^{\lambda x-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta}}x^{\gamma+\theta}\ln(x)}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2} \\ &\quad -\frac{\beta(\delta-1)e^{\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{\gamma+\theta}\ln(x)}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}} \\ &\quad -\frac{\theta x^{\theta-1}(\beta x^{\gamma-1}+\beta\gamma x^{\gamma-1}\ln(x)+\beta e^{\lambda x}\lambda x^{\gamma}\ln(x))}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1})^2},\end{aligned}$$

$$\frac{\partial^2 \ln f(x)}{\partial\alpha\partial\theta} = -\frac{\alpha(\delta-1)e^{-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta}}x^{2\theta}\ln(x)}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2} - \frac{\alpha(\delta-1)e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{2\theta}\ln(x)}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}}$$
$$+\frac{\theta x^{\theta-1}\ln(x)}{\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1}} - \frac{\theta x^{\theta-1}(\alpha x^{\theta-1}+\alpha\theta x^{\theta-1}\ln(x)}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1})^2}$$
$$+\frac{x^{\theta-1}}{\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1}} + \frac{(\delta-1)e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{\theta}\ln(x)}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}}$$
$$-x^{\theta}\ln(x),$$

$$\frac{\partial^2 \ln f(x)}{\partial\alpha\partial\lambda} = -\frac{\beta(\delta-1)e^{\lambda x-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta}}x^{\gamma+\theta+1}}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2} - \frac{\beta(\delta-1)e^{\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{\gamma+\theta+1}}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}}$$
$$-\frac{\theta x^{\theta-1}(\beta e^{\lambda x}x^{\gamma}+\beta e^{\lambda x}\lambda x^{\gamma+1})}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1})^2},$$

$$\frac{\partial^2 \ln f(x)}{\partial\beta^2} = -\frac{(\delta-1)e^{2\lambda x-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta}}x^{2\gamma}}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2} - \frac{(\delta-1)e^{2\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{2\gamma}}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}}$$
$$-\frac{(\gamma x^{\gamma-1}+e^{\lambda x}\lambda x^{\gamma})^2}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1})^2},$$

$$\frac{\partial^2 \ln f(x)}{\partial\beta\partial\gamma} = -\frac{(\gamma x^{\gamma-1}+e^{\lambda x}\lambda x^{\gamma})(\beta x^{\gamma-1}+\beta\gamma x^{\gamma-1}\ln(x)+\beta e^{\lambda x}\lambda x^{\gamma}\ln(x))}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1})^2}$$
$$-\frac{\beta(\delta-1)e^{2\lambda x-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta}}x^{2\gamma}\ln(x)}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2}$$
$$-\frac{\beta(\delta-1)e^{2\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{2\gamma}\ln(x)}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}}$$
$$+\frac{x^{\gamma+1}+\gamma x^{\gamma-1}\ln(x)+e^{\lambda x}\lambda x^{\gamma}\ln(x)}{\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1}}$$
$$-e^{\lambda x}x^{\gamma}\ln(x) + \frac{(\delta-1)e^{\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{\gamma}\ln(x)}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}},$$

$$\frac{\partial^2 L}{\partial\beta\partial\theta} = -\frac{\alpha(\delta-1)e^{\lambda x-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta})x^{\gamma+\theta}}\ln(x)}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2}$$
$$-\frac{\alpha(\delta-1)e^{\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{\gamma+\theta}\ln(x)}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}}$$
$$-\frac{(\gamma x^{\gamma-1}+e^{\lambda x}\lambda x^{\gamma})(\alpha x^{\theta-1}+\alpha\theta x^{\theta-1}\ln(x))}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1})^2},$$

$$\frac{\partial^2 \ln f(x)}{\partial\beta\partial\lambda} = -\frac{\beta(\delta-1)e^{\lambda x-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta}}x^{\gamma+\theta+1}}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2}$$
$$-\frac{\beta(\delta-1)e^{\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{\gamma+\theta+1}}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}}$$
$$-\frac{\theta x^{\theta-1}(\beta e^{\lambda x}x^{\gamma}+\beta e^{\lambda x}\lambda x^{\gamma+1}}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta})^2},$$

$$\frac{\partial^2 \ln f(x)}{\partial\gamma^2} = -\beta e^{\lambda x}x^{\gamma}\ln(x)^2+\frac{\beta(\delta-1)e^{\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{\gamma}\ln^2(x)}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}}$$
$$-\frac{\beta^2(\delta-1)e^{2\lambda x-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta}}x^{2\gamma}\ln^2(x)}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2}$$
$$-\frac{\beta^2(\delta-1)e^{2\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{2\gamma}\ln^2(x)}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}}$$
$$-\frac{(\beta x^{\gamma-1}+\beta\gamma x^{\gamma}\ln(x)+\beta e^{\lambda x}\lambda x^{\gamma}\ln(x))^2}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1})^2}$$
$$+\frac{2\beta x^{\gamma-1}\ln(x)+\beta\gamma x^{\gamma-1}\ln(x)^2+\beta e^{\lambda x}\lambda x^{\gamma}\ln(x)^2}{\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1}},$$

$$\frac{\partial^2 \ln f(x)}{\partial\gamma\partial\theta} = -\frac{(\beta x^{\gamma-1}+\beta\gamma x^{\gamma-1}\ln(x)+\beta e^{\lambda x}\lambda x^{\gamma}\ln(x))(\alpha x^{\theta-1}+\alpha\theta x^{\theta-1}\ln(x))}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^{\gamma}+\alpha\theta x^{\theta-1})^2}$$
$$-\frac{(\alpha\beta(\delta-1)e^{\lambda x-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}x^{\gamma+\theta}\ln^2(x)}{1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}}}$$
$$-\frac{(\alpha\beta(\delta-1)e^{\lambda x-2\beta e^{\lambda x}x^{\gamma}-2\alpha x^{\theta}}x^{\gamma+\theta}\ln^2(x)}{(1-e^{-\beta e^{\lambda x}x^{\gamma}-\alpha x^{\theta}})^2},$$

$$\frac{\partial^2 \ln f(x)}{\partial\gamma\partial\lambda} = -\beta e^{\lambda x}x^{\gamma+1}\ln(x) - \frac{\beta^2(\delta-1)e^{2\lambda x-2\beta e^{\lambda x}x^\gamma-2\alpha x^\theta}x^{2\gamma+1}\ln(x)}{(1-e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta})^2}$$
$$+\frac{\beta(\delta-1)e^{\lambda x-\beta e^{\lambda x}x^\gamma-\alpha x^\theta}x^\gamma(x-\beta e^{\lambda x}x^{\gamma+1})\ln(x))}{1-e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta}}$$
$$-\frac{\beta^2 e^{\lambda x}x^{2\gamma-1}(1+\lambda x)(1+\gamma\ln(x)+e^{\lambda x}\lambda x\ln(x))}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^\gamma+\alpha\theta x^{\theta-1})^2}$$
$$+\frac{\beta e^{\lambda x}x^\gamma\ln(x)+\beta e^{\lambda x}\lambda x^{\gamma+1}\ln(x)}{\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^\gamma+\alpha\theta x^{\theta-1}},$$

$$\frac{\partial^2 L}{\partial\theta^2} = -\alpha x^\theta\ln^2(x) + \frac{\alpha(\delta-1)e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta}x^\theta\ln^2(x)}{1-e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta}}$$
$$-\frac{\alpha^2(\delta-1)e^{-2\beta e^{\lambda x}x^\gamma-2\alpha x^\theta}x^{2\theta}\ln^2(x)}{(1-e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta})^2}$$
$$-\frac{\alpha^2(\delta-1)e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta}x^{2\theta}\ln^2(x)}{1-e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta}}$$
$$-\frac{(\alpha x^{\theta-1}+\alpha\theta x^{theta-1}\ln(x))^2}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^\gamma+\alpha\theta x^{\theta-1})^2}$$
$$+\frac{2\alpha x^{\theta-1}\ln(x)+\alpha\theta x^{\theta-1}\ln^2(x)}{\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^\gamma+\alpha\theta x^{\theta-1}},$$

$$\frac{\partial^2 \ln f(x)}{\partial\theta\partial\lambda} = -\frac{\alpha\beta(\delta-1)e^{\lambda x-2\beta e^{\lambda x}x^\gamma-2\alpha x^\theta}x^{\gamma+\theta+1}\ln(x)}{(1-e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta})^2}$$
$$-\frac{\alpha\beta(\delta-1)e^{\lambda x-\beta e^{\lambda x}x^\gamma-\alpha x^\theta}x^(\gamma+\theta+1)\ln(x)}{1-e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta}}$$
$$-\frac{(\beta e^{\lambda x}x^\gamma+\beta e^{\lambda x}\lambda x^{\gamma-1})(\alpha x^{\theta-1}+\alpha\theta x^{\theta-1}\ln(x))}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^\gamma+\alpha\theta x^{\theta-1})^2},$$

and

$$\frac{\partial^2 \ln f(x)}{\partial\lambda^2} = -\beta e^{\lambda x}x^{\gamma+2} - \frac{\beta^2(\delta-1)e^{2\lambda x-2\beta e^{\lambda x}x^\gamma-2\alpha x^\theta}x^{2+2\gamma}}{(1-e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta})^2}$$
$$+\frac{\beta(\delta+1)e^{\lambda x-\beta e^{\lambda x}x^\gamma-\alpha x^\theta}x^{\gamma+1}(x-\beta e^{\lambda x}x^{\gamma+1}}{1-e^{-\beta e^{\lambda x}x^\gamma-\alpha x^\theta}}$$
$$-\frac{(\beta e^{\lambda x}x^\gamma+\beta e^{\lambda x}\lambda x^{\gamma+1})^2}{(\beta\gamma x^{\gamma-1}+\beta e^{\lambda x}\lambda x^\gamma+\alpha\theta x^{\theta-1})^2}$$
$$+\frac{2\beta e^{\lambda x}x^{\gamma+1}+\beta e^{\lambda x}\lambda x^{\gamma+2}}{\beta\gamma x^{\gamma+1}+\beta e^{\lambda x}\lambda x^\gamma+\alpha\theta x^{\theta+1}}.$$

5.5.1 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the GNMW distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\mathbf{\Delta}} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda}, \hat{\gamma}, \hat{\delta})$ be the maximum likelihood estimate of $\mathbf{\Delta} = (\alpha, \beta, \theta, \lambda, \gamma, \delta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Delta}} - \mathbf{\Delta}) \xrightarrow{d} N_6(\underline{0}, I^{-1}(\mathbf{\Delta}))$, where $I(\mathbf{\Delta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Delta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Delta}}$, that is $J(\hat{\mathbf{\Delta}})$. The multivariate normal distribution $N_5(\underline{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{0} = (0,0,0,0,0,0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. A large sample $100(1-\eta)\%$ confidence intervals for $\alpha, \beta, \theta, \lambda, \gamma$ and δ are:

$$\widehat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Delta}})},\ \widehat{\beta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\beta\beta}^{-1}(\widehat{\mathbf{\Delta}})},\ \widehat{\theta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\theta\theta}^{-1}(\widehat{\mathbf{\Delta}})},$$

$$\widehat{\lambda} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\widehat{\mathbf{\Delta}})},\ \widehat{\gamma} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\gamma\gamma}^{-1}(\widehat{\mathbf{\Delta}})},\ \text{and}\ \widehat{\delta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\delta\delta}^{-1}(\widehat{\mathbf{\Delta}})},$$

respectively, where $I_{\alpha\alpha}^{-1}(\hat{\mathbf{\Delta}})$, $I_{\beta\beta}^{-1}(\hat{\mathbf{\Delta}})$, $I_{\theta\theta}^{-1}(\hat{\mathbf{\Delta}})$, $I_{\lambda\lambda}^{-1}(\hat{\mathbf{\Delta}})$, $I_{\gamma\gamma}^{-1}(\hat{\mathbf{\Delta}})$, and $I_{\delta\delta}^{-1}(\hat{\mathbf{\Delta}})$ are the diagonal elements of $I_n^{-1}(\hat{\mathbf{\Delta}})$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

5.6 Applications

In this section, we present examples to illustrate the flexibility of the GNMW distribution and its sub-models for data modeling. Estimates of the parameters of GGIW distribution (standard error in parentheses), AIC, AICC, BIC, and KS are given in Tables 5.2, and 5.3. Plots of the fitted densities and the histogram of the data are given in Figures 5.5 and 5.7. Probability plots (Chambers et al. [10]) are also presented in Figures 5.6 and 5.8. We computed a measure of closeness of each plot to the diagonal line. This measure of closeness is given by the sum of squares $SS = \sum_{j=1}^{n} \left[G_{GGIW}(x_{(j)}; \hat{\eta}, \hat{\beta}, \hat{\lambda}, \hat{\delta}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2$.

For the first example, we used a phosphorus concentration in leaves data set from Fonseca and Françca [16], they studied the soil fertility influence and the characterization of the biologic fixation of N2 for the Dimorphandra wilsonii rizz growth. For 128 plants, they made measures of the phosphorus concentration in the leaves. The data is given below: 0.22, 0.17, 0.11, 0.10, 0.15, 0.06, 0.05, 0.07, 0.12, 0.09, 0.23, 0.25, 0.23, 0.24, 0.20, 0.08, 0.11, 0.12, 0.10, 0.06, 0.20, 0.17, 0.20, 0.11, 0.16, 0.09, 0.10, 0.12, 0.12, 0.10, 0.09, 0.17, 0.19, 0.21, 0.18, 0.26, 0.19, 0.17, 0.18, 0.20, 0.24, 0.19, 0.21, 0.22, 0.17, 0.08, 0.08, 0.06, 0.09, 0.22, 0.23, 0.22, 0.19, 0.27, 0.16, 0.28, 0.11, 0.10, 0.20, 0.12, 0.15, 0.08, 0.12, 0.09, 0.14, 0.07, 0.09, 0.05, 0.06, 0.11, 0.16, 0.20, 0.25, 0.16, 0.13, 0.11, 0.11, 0.11, 0.08, 0.22, 0.11, 0.13, 0.12, 0.15, 0.12, 0.11, 0.11, 0.15, 0.10, 0.15, 0.17, 0.14, 0.12, 0.18, 0.14, 0.18, 0.13, 0.12, 0.14, 0.09, 0.10, 0.13, 0.09, 0.11, 0.11, 0.14, 0.07, 0.07, 0.19, 0.17, 0.18, 0.16, 0.19, 0.15, 0.07, 0.09, 0.17, 0.10, 0.08, 0.15, 0.21, 0.16, 0.08, 0.10, 0.06, 0.08, 0.12, 0.13.

For the phosphorus concentration data, the LR test statistic of H_0 : AW against H_a : GNMW is $w = 12.2$. The p-value = 0.00224. The GNMW distribution significantly better than the AW distribution. The value of KS statistic shows that GNMW distribution is a “better” fit.

The second data is plasma concentration of indomethicin. This data set, taken from the R base package. It is located in the Indometh object. The data consists of plasma concentrations of indomethicin (mcg/ml) and has 119 observations: 0.22, 0.17, 0.11, 0.10, 0.15, 0.06, 0.05, 0.07, 0.12, 0.09, 0.23, 0.25, 0.23, 0.24, 0.20, 0.08, 0.11, 0.12, 0.10, 0.06, 0.20, 0.17, 0.20, 0.11, 0.16, 0.09, 0.10, 0.12, 0.12, 0.10, 0.09, 0.17, 0.19, 0.21, 0.18, 0.26, 0.19, 0.17, 0.18, 0.20, 0.24, 0.19, 0.21, 0.22, 0.17, 0.08,

Table 5.2: Estimates of models for phosphorus concentration in leaves data

	Estimates						Statistics					
Model	α	β	γ	θ	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GNMW$(\alpha,\beta,\gamma,\theta,\lambda,\delta)$	13.8436	4.187	3.8906	0.6148	16.6245	38.1357	-401.8	-389.8	-389.1	-372.7	0.0781	0.0643
	(2.6227)	(62.9352)	(6.4428)	(0.2453)	(26.1428)	(53.7177)						
AW$(\alpha,\beta,\gamma,\theta,0,1)$	89.9057	89.9057	2.8185	2.8185	0	1	-389.6	-381.6	-381.3	-370.2	0.1172	0.1858
	(30.1153)	(30.1153)	(0.4372)	(0.4372)								
W$(0,\beta,\gamma,1,0,1)$	0	179.81	2.8185	1	0	1	-389.6	-385.6	-385.5	-379.9	0.1172	0.1559
		(60.2293)	(0.1919)									
R$(\alpha,0,1,2,0,1)$	43.9439	0	1	2	0	1	-368.1	-366.1	-366.1	-363.3	0.1484	0.5912
	(3.8841)											
E$(\alpha,0,1,1,0,1)$	7.1032	0	1	1	0	1	-245.9	-243.9	-243.9	-241.0	0.3359	4.3305
	(0.6278)											

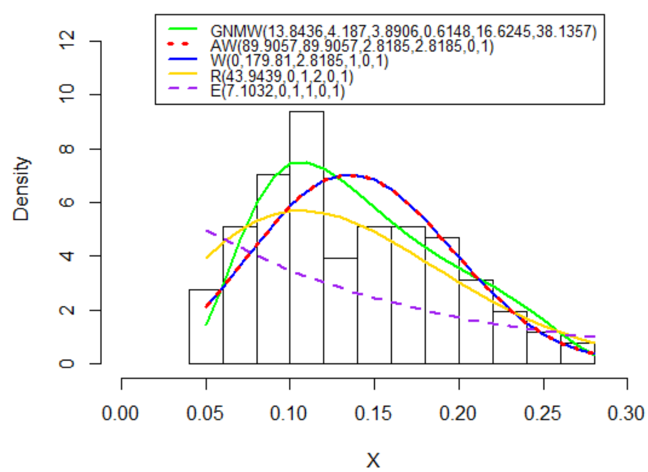

Figure 5.5: Fitted density plot for phosphorus concentration in leaves data

0.08, 0.06, 0.09, 0.22, 0.23, 0.22, 0.19, 0.27, 0.16, 0.28, 0.11, 0.10, 0.20, 0.12, 0.15, 0.08, 0.12, 0.09, 0.14, 0.07, 0.09, 0.05, 0.06, 0.11, 0.16, 0.20, 0.25, 0.16, 0.13, 0.11,

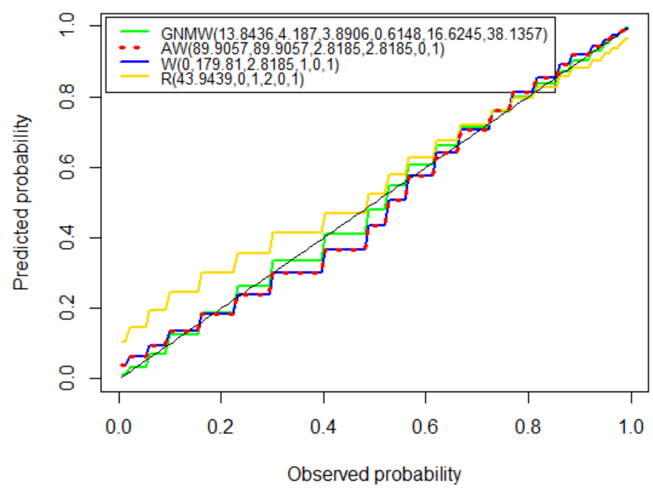

Figure 5.6: Probability plot for phosphorus concentration in leaves data

0.11, 0.11, 0.08, 0.22, 0.11, 0.13, 0.12, 0.15, 0.12, 0.11, 0.11, 0.15, 0.10, 0.15, 0.17, 0.14, 0.12, 0.18, 0.14, 0.18, 0.13, 0.12, 0.14, 0.09, 0.10, 0.13, 0.09, 0.11, 0.11, 0.14, 0.07, 0.07, 0.19, 0.17, 0.18, 0.16, 0.19, 0.15, 0.07, 0.09, 0.17, 0.10, 0.08, 0.15, 0.21, 0.16, 0.08, 0.10, 0.06, 0.08, 0.12, 0.13.

For the plasma concentration data, the LR test statistic of H_0 : W against H_a : GNMW is $w = 15.8$. The p-value = 0.0032. The GNMW distribution significantly better than the W distribution.

5.7 Concluding Remarks

We have presented a new class of generalized modified Weibull distributions called the generalized new modified Weibull (GNMW) distribution. The GNMW distribution has several distributions such as the GNMR, GNME, GNAW, GNAE, GLFR, LFR, GMW, GME, MW, ME, Weibull, Rayleigh and exponential distributions as special cases. The density of this new class of distributions can be expressed as a

Table 5.3: Estimates of models for plasma concentration of indomethicin data

	Estimates						Statistics					
Model	α	β	γ	θ	λ	δ	$-2\log L$	AIC	$AICC$	BIC	KS	SS
GNMW$(\alpha,\beta,\gamma,\theta,\lambda,\delta)$	8.499	0.04943	3.1808	0.1249	0.3079	1099.6	46.7	58.7	60.1	71.8	0.1515	0.3967
	(1.8695)	(0.135)	(5.1702)	(0.03553)	(2.608)	(2051.13)						
W$(0,\beta,\gamma,0,1)$	0	1.6857	0.9546	1	0	1	62.5	66.5	66.7	70.9	0.1364	0.2263
		(0.2078)	(0.09035)									
R$(\alpha,0,1,2,0,1)$	1.3435	0	1	2	0	1	148.9	150.9	151.0	153.1	0.3939	3.3319
	(0.1654)											
E$(\alpha,0,1,1,0,1)$	1.6897	0	1	1	0	1	62.8	64.8	64.8	66.9	0.1515	0.2884
	(0.2080)											

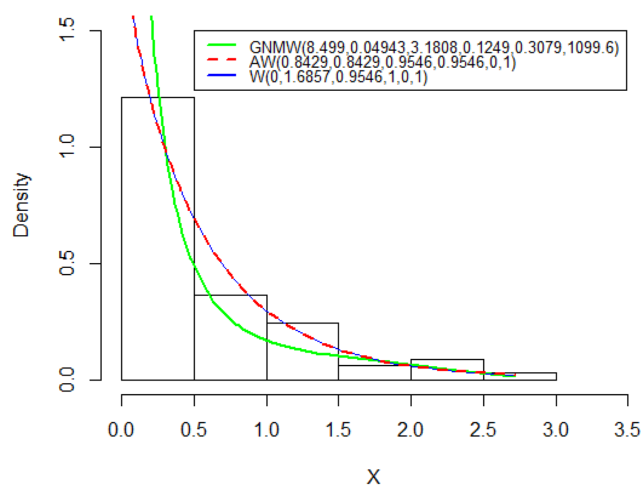

Figure 5.7: Fitted density plot for plasma concentration of indomethicin data

linear combination of NMW density functions. The GNMW distribution possesses hazard function with flexible behavior. We also obtain closed form expressions for the moments, distribution of order statistic and entropy. Maximum likelihood estimation

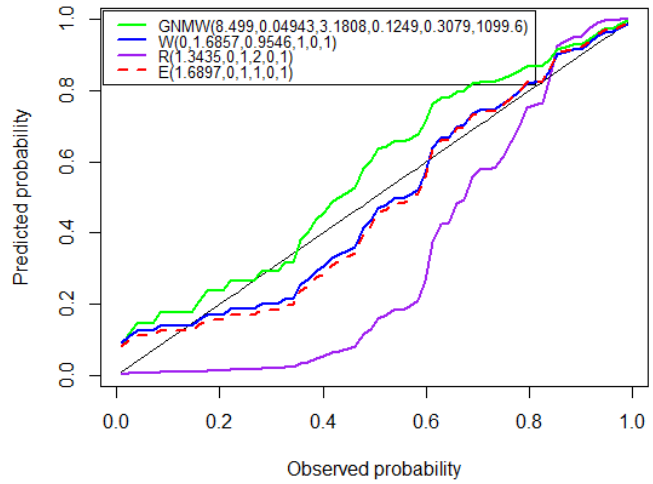

Figure 5.8: Probability plot for plasma concentration of indomethicin data

technique is used to estimate the model parameters. Finally, the GNMW model is fitted to real data sets to illustrate the usefulness of this class of distributions.

CHAPTER 6

INVERSE NEW MODIFIED WEIBULL DISTRIBUTIONS

6.1 Definition - The Model

Almalki and Yuan [1] proposed the new modified Weibull (NMW) distribution that includes several well known extensions or generalizations of the Weibull distribution. The cumulative distribution function (cdf)of the NMW distribution is defined by

$$F_{NMW}(y) = 1 - e^{-\alpha y^{\theta} - \beta y^{\gamma} e^{\lambda y}}, \quad y > 0,$$

where θ and γ are shape parameters, α and β are scale parameters, and λ is an acceleration parameter. The corresponding probability density function (pdf) is given by

$$f_{NMW}(y) = e^{-\alpha y^{\theta} - \beta y^{\gamma} e^{\lambda y}} (\alpha \theta y^{\theta - 1} + \beta(\gamma + \lambda y) y^{\gamma - 1} e^{\lambda y}),$$

for $\alpha, \beta, \gamma, \theta, \lambda$ non-negative and $y > 0$.

In this chapter, we present the mathematical and statistical properties of the inverse new modified Weibull (INMW) distribution. If $Y \sim$ NMW $(\alpha, \beta, \gamma, \theta, \lambda)$ and $X = \dfrac{1}{Y}$, then $g_X(x) = f_Y(g^{-1}(x)) \cdot \left|\dfrac{\partial g^{-1}(x)}{\partial x}\right|$, and thus, $X \sim$ INMW $(\alpha, \beta, \gamma, \theta, \lambda)$. The INMW pdf is given by

$$\begin{aligned} g_{INMW}(x) &= e^{-\alpha x^{-\theta} - \beta x^{-\gamma} e^{\lambda x^{-1}}} (\alpha \theta x^{-\theta - 1} + \beta(\gamma + \lambda x^{-1}) x^{-\gamma - 1} e^{\lambda x^{-1}}) \\ &= G_{IW}(x; \alpha, \theta) G_{IMW}(x; \beta, \gamma, \lambda)(r_{IW}(x; \alpha, \theta) + r_{IMW}(x; \beta, \gamma, \lambda)), \end{aligned}$$

for $\alpha, \beta, \gamma, \theta, \lambda$ non-negative and $x > 0$, where $G_{IW}(x; \alpha, \theta)$, $r_{IW}(x; \alpha, \theta)$, and $G_{IMW}(x; \beta, \gamma, \lambda)$, $r_{IMW}(x; \beta, \gamma, \lambda)$ are cdf, reverse hazard functions of the inverse Weibull and inverse modified Weibull distributions, respectively. The INMW cdf is given by

$$G_{INMW}(x) = e^{-\alpha x^{-\theta} - \beta x^{-\gamma} e^{\lambda x^{-1}}} = G_{IW}(x; \alpha, \theta) G_{IMW}(x; \beta, \gamma, \lambda),$$

for $\alpha, \beta, \gamma, \theta, \lambda$ non-negative and $x > 0$. Thus, the INMW cdf is the product of the IW and IMW cdf's.

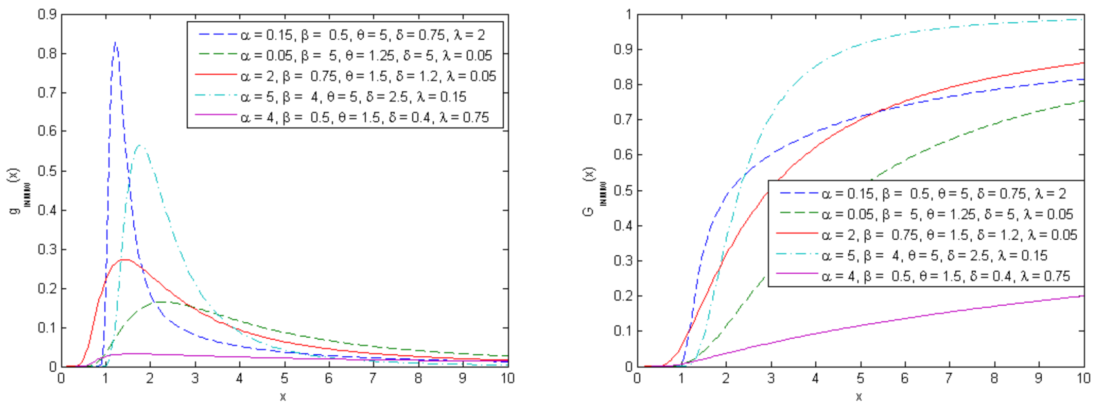

Figure 6.1: Plot of the pdf and cdf of INMW distribution

The plots of the pdf for the selected values of the parameters are right skewed. The INMW distribution is asymmetric.

6.1.1 Sub-Models

In this section, we present several sub-models of the INMW distribution:

- if we set $\theta = 1$ and $\beta = 0$, the INMW distribution reduces to Inverse Exponential (IE) distribution;
- if we set $\theta = 2$ and $\beta = 0$, we get Inverse Rayleigh (IR) distribution;
- if we set $\alpha = 0$, $\beta = 1$, and $\lambda = 0$, we obtain the Fréchet (F) distribution;
- if we set $\beta = 0$, the INMW distribution reduces to the Inverse Weibull (IW) distribution;
- if we set $\alpha = 0$, the INMW distribution reduces to Inverse Modified Weibull (IMW) distribution;
- if we set $\lambda = 0$, we obtain Additive Inverse Weibull (AIW) distribution;

- if we set $\theta = \gamma = 2$, and $\lambda = 0$, we get the Additive Inverse Rayleigh (AIR) distribution;
- if we set $\theta = \gamma = 2$, we get the Inverse New Modified Rayleigh (INMR) distribution;
- if we set $\alpha = \beta = 1$, and $\lambda = 0$, we get the Additive Fréchet (AF) distribution;
- if we set $\alpha = \beta = 1$, we get the New Modified Fréchet (NMF) distribution;
- if we set $\theta = \gamma = 1$, and $\lambda = 0$, we get the Additive Inverse Exponential (AIE) distribution;
- if we set $\theta = \gamma = 1$, we get the Inverse New Modified Exponential (INME) distribution.

6.2 Some Properties

6.2.1 Hazard and Reverse Hazard Functions

The hazard function of the INMW distribution is

$$h_{G_{INMW}}(x) = \frac{g(x)}{\bar{G}(x)} = \frac{e^{-\alpha x^{-\theta} - \beta x^{-\gamma} e^{\lambda x^{-1}}}}{1 - e^{-\alpha x^{-\theta} - \beta x^{-\gamma} e^{\lambda x^{-1}}}} \cdot (\alpha\theta x^{-\theta-1} + \beta(\gamma + \lambda x^{-1}) x^{-\gamma-1} e^{\lambda x^{-1}}).$$

Plots of the hazard function for selected values of the parameters, given in figure 6.2, shows unimodal and upside-down bathtub shapes. The reverse hazard rate function of the INMW distribution is

$$\begin{aligned} r_{G_{INMW}}(x) &= \frac{g(x)}{G(x)} = \frac{e^{-\alpha x^{-\theta} - \beta x^{-\gamma} e^{\lambda x^{-1}}} (\alpha\theta x^{-\theta-1} + \beta(\gamma + \lambda x^{-1}) x^{-\gamma-1} e^{\lambda x^{-1}})}{e^{-\alpha x^{-\theta} - \beta x^{-\gamma} e^{\lambda x^{-1}}}} \\ &= \alpha\theta x^{-\theta-1} + \beta(\gamma + \lambda x^{-1}) x^{-\gamma-1} e^{\lambda x^{-1}} \\ &= r_{IW}(x; \alpha, \theta) \cdot r_{IMW}(x; \beta, \gamma, \lambda). \end{aligned} \quad (6.1)$$

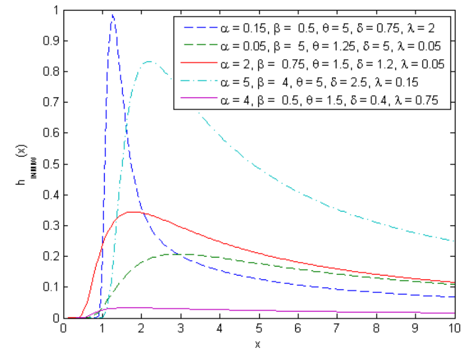

Figure 6.2: Plot of hazard functions of INMW distribution

6.2.2 Moments

The k^{th} non-central moment of the INMW distribution is:

$$E(X^k) = \int_0^\infty x^k dG_{_{INMW}}(x) = \int_0^\infty x^k d\left(e^{-\alpha x^{-\theta} - \beta x^{-\gamma} e^{\lambda x^{-1}}}\right).$$

Applying integration by parts we get:

$$E(X^k) = -\int_0^\infty k x^{k-1} e^{-\alpha x^{-\theta} - \beta x^{-\gamma} e^{\lambda x^{-1}}} dx.$$

Note that

$$e^{-\beta x^{-\gamma} e^{\lambda x^{-1}}} = \sum_{n,m=0}^{\infty} \frac{(-\beta)^n (n\lambda)^m x^{-\gamma n - m}}{n! m!}.$$

The integral reduces to:

$$E(X^k) = -\sum_{n,m=0}^{\infty} \frac{(-\beta)^n (n\lambda)^m}{n! m!} \int_0^\infty x^{k-\gamma n - m} e^{-\alpha x^{-\theta}} dx.$$

Consequently, the k^{th} moment of INMW is

$$E(X^k) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\beta)^n (n\lambda)^m}{n! m!} \frac{k}{\theta} \alpha^{\frac{k-\gamma n - m}{\theta}} \Gamma\left(\frac{k - \gamma n - m}{\theta}\right),$$

for $\alpha \geq 0$, $k > \gamma n + m$, $\theta > 0$.

If $\lambda = 0$, we get the k^{th} moment of the AIW distribution, that is

$$E(X^k) = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \frac{k}{\theta} \alpha^{\frac{k-\gamma n}{\theta}} \Gamma\left(\frac{k - \gamma n}{\theta}\right),$$

for $\alpha \geq 0$, $k > \gamma n$, $\theta > 0$. If $\beta = 0$, we get the k^{th} moment of the IW distribution:

$$E(X^k) = \alpha^{\frac{k}{\theta}} \Gamma\left(1 - \frac{k}{\theta}\right),$$

for $\alpha \geq 0$, $\theta > k$.

6.2.3 Entropy

Entropy is a measure of unpredictability of information content.

ν-entropy

ν-entropy is given by

$$I_\nu(g) = \frac{1}{\nu - 1}\left[1 - \int_0^\infty g^\nu(x)dx\right], \; \nu \neq 1.$$

Note that, using Taylor series expansion:

$$\begin{aligned}
g^\nu(x) &= e^{-\nu\alpha-\nu\beta x^{-\gamma}e^{\lambda x^{-1}}}(\alpha\theta x^{-\theta-1} + (\lambda x^{-1} + \gamma)\beta x^{-\gamma-1}e^{\lambda x^{-1}})^\nu \\
&= e^{-\nu\alpha x^{-\theta}} \sum_{k=0}^{\infty} \frac{(-\nu\beta x^{-\gamma}e^{\lambda x^{-1}})^k}{k!} \\
&\quad * \sum_{j=0}^{\nu} \binom{\nu}{j} (\alpha\theta x^{-\theta-1})^{\nu-j} \cdot (\lambda x^{-1} + \gamma)^j \beta^j x^{-\gamma j - j} e^{j\lambda x^{-1}} \\
&= e^{-\nu\alpha x^{-\theta}} \sum_{k,s=0}^{\infty} \frac{(-\nu\beta)^k (k\lambda)^s}{k!s!} x^{-k\lambda - s} \\
&= \sum_{j=0}^{\nu} \sum_{n=0}^{j} \sum_{m=0}^{\infty} \binom{\nu}{j}\binom{j}{n} (\alpha\theta)^{\nu-j}\gamma^{j-n}\lambda^n \frac{\beta^j (j\lambda)^m}{m!} x^{(\theta+1)(j-\nu)-\gamma j - j - m - n}.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^\infty g^\nu(x)dx &= \sum_{j=0}^{\nu} \sum_{n=0}^{j} \sum_{m,k,s=0}^{\infty} \binom{\nu}{j}\binom{j}{n} (\alpha\theta)^{\nu-j}\gamma^{j-n}\lambda^n \frac{(-\nu\beta)^k (k\lambda)^s \beta^j (j\lambda)^m}{k!s!m!} \\
&\quad \cdot \int_0^\infty x^{-\nu\theta+\theta j-\nu+j-n-\gamma j-j-m-k\gamma-s} e^{-\nu\alpha x^{-\theta}} dx.
\end{aligned}$$

Consequently, ν-entropy reduces to

$$I_\nu(g) = \frac{1}{\nu - 1}\left[1 - S\right],$$

$\nu \neq 1$, $\nu > 0$, where

$$\begin{aligned} S &= \sum_{j=0}^{\nu}\sum_{n=0}^{j}\sum_{m,k,s=0}^{\infty}\binom{\nu}{j}\binom{j}{n}\Gamma\left[\tau\right] \\ &\quad \cdot\frac{(\alpha\theta)^{\nu-j}\gamma^{j-n}\lambda^n(-\nu\beta)^k(k\lambda)^s\beta^j(j\lambda)^m(\alpha\nu)^{-\tau}}{k!s!m!\theta}, \end{aligned} \tag{6.2}$$

and

$$\tau = \nu - j + \frac{1}{\theta}(\nu + n + \gamma j + m + kj + s + 1). \tag{6.3}$$

Renyi Entropy

Renyi Entropy is given by

$$I_R(g) = \frac{1}{1-\nu}\log\left[\int_0^\infty g^\nu(x)dx\right] = \frac{1}{1-\nu}\log\left[S\right],$$

$\nu \neq 1$, $\nu > 0$, where S and τ are given by equations (6.2) and (6.3) respectively.

6.2.4 Order Statistics

The distribution of the r^{th} order statistic, based on the sample size of n from the INMW distribution, is given by:

$$\begin{aligned} f_{r:n}(x) &= \frac{g(x)}{B(r, n-r+1)}\left[G(x)\right]^{r-1}\left[1 - G(x)\right]^{n-r} \\ &= \frac{g(x)}{B(r, n-r+1)}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j e^{-(r+j-1)\left(\alpha x^{-\theta} - \beta x^{-\gamma}e^{\lambda x^{-1}}\right)} \\ &= \sum_{j=0}^{n-r}\binom{n-r}{j}\frac{(-1)^j e^{-(r+j)\left(\alpha x^{-\theta} + \beta x^{-\gamma}e^{\lambda x^{-1}}\right)}}{B(r, n-r+1)} \\ &\quad \cdot\frac{r+j}{r+j}(\alpha\theta x^{-\theta-1} + \beta(\gamma + \lambda x^{-1})x^{-\gamma-1}e^{-\lambda x}) \end{aligned}$$

Now, we can rewrite the distribution of the r^{th} order statistic as

$$\begin{aligned} f_{r:n}(x) &= \sum_{j=0}^{n-r} \binom{n-r}{j} \frac{(-1)^j}{B(r,n-r+1)(r+j)} g(x;\alpha^*,\beta^*,\gamma,\theta,\lambda) \\ &= \sum_{j=0}^{n-r} w_{j,r} g(x;\alpha^*,\beta^*,\gamma,\theta,\lambda), \end{aligned}$$

where $\alpha^* = (r+j)\alpha$ and $\beta^* = (r+j)\beta$, and the weights

$$w_{j,r} = \binom{n-r}{j} \frac{(-1)^j}{B(r,n-r+1)(r+j)}.$$

Thus the pdf of the r^{th} order statistic from INMW distribution can be written as a linear combination of INMW densities with scale parameters α^* and β^*, shape parameters θ and γ, and acceleration parameter λ. The k^{th} moment of the r^{th} order statistic is given by

$$\begin{aligned} E\left[X_{r:n}^k\right] &= \sum_{n,m=0}^{\infty} \sum_{j=0}^{n-r} \binom{n-r}{j} \frac{(-1)^{n+j}(\beta(r+j))^n(n\lambda)^m}{n!m!B(r,n-r+1)(r+j)} \\ &\quad \cdot \frac{k}{\theta}(\alpha(r+j))^{\frac{k-\gamma n-m}{\theta}} \Gamma\left(\frac{k-\gamma n-m}{\theta}\right). \end{aligned}$$

These moments are used in several areas including reliability, insurance, and quality control for the prediction of future failure times from a set of past or previous failures.

6.3 Estimation of Parameters

Let x_1, $x_2 \cdots x_n$ be a random sample from a INMW distribution. The log-likelihood function is

$$\begin{aligned} \ln \mathcal{L} &= \sum_{i=1}^{n} \ln\left(\beta\left(\lambda x_i^{-1}+\gamma\right) x_i^{-\gamma-1} e^{\lambda x_i^{-1}} + \alpha\theta x_i^{-\theta-1}\right) \\ &\quad -\alpha \sum_{i=1}^{n} x_i^{-\theta} - \beta \sum_{i=1}^{n} x_i^{-\gamma} e^{\lambda x_i^{-1}}. \end{aligned}$$

The normal equations are

$$\frac{\partial \ln \mathcal{L}}{\partial \alpha} = \sum_{i=1}^{n} \frac{\theta x_i^{-\theta-1}}{r_{G_{INMW}}(x)} - \sum_{i=1}^{n} x_i^{-\theta} = 0,$$

$$\frac{\partial \ln \mathcal{L}}{\partial \beta} = \sum_{i=1}^{n} \frac{\left(\lambda x_i^{-1} + \gamma\right) x_i^{-\gamma-1} e^{\lambda x_i^{-1}}}{r_{G_{INMW}}(x)} - \sum_{i=1}^{n} x_i^{-\gamma} e^{\lambda x_i^{-1}} = 0,$$

$$\begin{aligned}\frac{\partial \ln \mathcal{L}}{\partial \gamma} &= \sum_{i=1}^{n} \frac{\beta x_i^{-\gamma-1} e^{\lambda x_i^{-1}} \left(1 - \left(\gamma + \lambda x_i^{-1}\right) \ln(x_i)\right)}{r_{G_{INMW}}(x)} \\ &\quad + \beta \sum_{i=1}^{n} x_i^{-\gamma} \ln(x_i) e^{\lambda x_i^{-1}} = 0,\end{aligned}$$

$$\frac{\partial \ln \mathcal{L}}{\partial \theta} = \sum_{i=1}^{n} \frac{\alpha x_i^{-\theta-1} \left(1 + \theta \ln(x_i)\right)}{r_{G_{INMW}}(x)} + \alpha \sum_{i=1}^{n} x_i^{-\theta} \ln(x_i) = 0,$$

and

$$\frac{\partial \ln \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{n} \frac{\beta x_i^{-\theta-2} e^{\lambda x_i^{-1}} \left(\gamma + \lambda x_i^{-1} + 1\right)}{r_{G_{INMW}}(x)} - \beta \sum_{i=1}^{n} x_i^{-\theta-1} e^{\lambda x_i^{-1}} = 0,$$

where $r_{G_{INMW}}(x)$ is the reverse hazard function of the inverse new modified Weibull distribution.

The maximum likelihood estimates can be obtained by solving the non-linear equations numerically for α, β, γ, θ and λ. The relatively large number of parameters can cause problems especially when the sample size is small. A good set of initial values is important.

6.4 Fisher Information

Let $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (\alpha, \beta, \gamma, \theta, \lambda)$. If $\ln(g(X, \mathbf{\Theta}))$ is twice differentiate with respect to $\mathbf{\Theta}$, and under certain regularity conditions [31], Fisher Information (FIM) is the 5×5 matrix whose elements are:

$$I(\mathbf{\Theta}) = -E_{\mathbf{\Theta}} \left[\frac{\partial^2}{\partial \theta_i \partial \theta_j} \ln(g(X, \mathbf{\Theta})) \right]. \qquad (6.4)$$

Note, that

$$\ln(g(x)) = -\alpha x^{-\theta} - \beta x^{-\gamma} e^{\lambda x^{-1}} + \ln\left(\alpha\theta x^{-\theta-1} + \beta x^{-\gamma-1}\left(\gamma + \lambda x^{-1}\right) e^{\lambda x^{-1}}\right).$$

We have the following mixed and partial derivatives of $\ln g(x)$:

$$\frac{\partial^2 \ln g(x)}{\partial \alpha^2} = -\frac{\theta^2 x^{-\theta-2}}{r^2_{INMW}(x)}, \tag{6.5}$$

$$\frac{\partial^2 \ln g(x)}{\partial \alpha \partial \beta} = -\frac{\theta e^{\lambda x^{-1}}\left(\gamma + \lambda x^{-1}\right) x^{-\theta-\gamma-2}}{r^2_{INMW}(x)}, \tag{6.6}$$

$$\frac{\partial^2 \ln g(x)}{\partial \alpha \partial \gamma} = -\frac{\theta x^{-\theta-1}\left(\beta e^{\lambda x^{-1}} x^{-\gamma-1} - \beta e^{\lambda x^{-1}}\left(\gamma + \lambda x^{-1}\right)\ln x\right)}{r^2_{INMW}(x)}, \tag{6.7}$$

$$\frac{\partial^2 \ln g(x)}{\partial \alpha \partial \theta} = x^{-\theta}\ln x + \frac{\left(1 - \theta\ln(x)\right) x^{-\theta-1}}{r_{INMW}(x)} - \frac{\alpha\theta x^{-2\theta-2}\left(1 - \theta\ln(x)\right)}{r^2_{INMW}(x)}, \tag{6.8}$$

$$\frac{\partial^2 \ln g(x)}{\partial \alpha \partial \lambda} = -\frac{\theta x^{-\theta-1}\left(\beta e^{\lambda x^{-1}} x^{-\gamma-2}\left(\lambda x^{-1} + \gamma + 1\right)\right)}{r^2_{INMW}(x)}, \tag{6.9}$$

$$\frac{\partial^2 \ln g(x)}{\partial \beta^2} = -\frac{e^{2\lambda x^{-1}}(\gamma + \lambda x^{-1}) x^{-2\gamma-2}}{r^2_{INMW}(x)}, \tag{6.10}$$

$$\begin{aligned}\frac{\partial^2 \ln g(x)}{\partial \beta \partial \gamma} &= \frac{e^{\lambda x^{-1}} x^{-\gamma-1}}{r_{INMW}(x)} + e^{\lambda x^{-1} x^{-\gamma}\ln(x)} - \frac{e^{\lambda x^{-1}}(\gamma + \lambda x^{-1}) x^{-\gamma-1}\ln(x)}{r_{INMW}(x)} \\ &\quad - \frac{\beta(\gamma + \lambda x^{-1}) e^{2\lambda x^{-1}}\left(1 - \left(\gamma + \lambda x^{-1}\right)\ln(x)\right) x^{-2\gamma-2}}{r^2_{INMW}(x)},\end{aligned} \tag{6.11}$$

$$\frac{\partial^2 \ln g(x)}{\partial \beta \partial \theta} = -\frac{e^{\lambda x^{-1}}\alpha(\gamma + \lambda x^{-1})\left(1 - \theta\ln(x)\right) x^{-\gamma-\theta-2}}{r^2_{INMW}(x)}, \tag{6.12}$$

$$\frac{\partial^2 \ln g(x)}{\partial \beta \partial \lambda} = -\frac{e^{\lambda x^{-1}}\beta\theta(\gamma + \lambda x^{-1} + 1) x^{-\gamma-\theta-3}}{r^2_{INMW}(x)}, \tag{6.13}$$

$$\frac{\partial^2 \ln g(x)}{\partial\gamma^2} = -\beta e^{\lambda x^{-1}} x^{-\gamma} \ln^2(x) - \frac{\left(\beta e^{\lambda x^{-1}} x^{-\gamma-1}\left[1-(\gamma+\lambda x^{-1})\ln(x)\right]\right)^2}{r^2_{INMW}(x)} + \frac{e^{\lambda x^{-1}}\beta\left((\gamma+\lambda x^{-1})\ln(x)-2\right)x^{-\gamma-1}\ln(x)}{r_{INMW}(x)}, \tag{6.14}$$

$$\frac{\partial^2 \ln g(x)}{\partial\gamma\partial\theta} = -\frac{\alpha\beta e^{\lambda x^{-1}}\left[1-(\gamma+\lambda x^{-1})\ln(x)\right]\left[1-\theta\ln(x)\right]x^{-\gamma-\theta-2}}{r_{INMW}(x)}, \tag{6.15}$$

$$\frac{\partial^2 \ln g(x)}{\partial\gamma\partial\lambda} = \beta e^{\lambda x^{-1}} x^{-\gamma-1}\ln(x) + \frac{\beta e^{\lambda x^{-1}} x^{-\gamma-2}\left[1-\ln(x)-(\gamma+\lambda x^{-1})\ln(x)\right]}{r_{INMW}(x)} - \frac{\beta^2 e^{2\lambda x^{-1}} x^{-2\gamma-3}\left[1+\gamma+\lambda x^{-1}\right]\left[1-(\gamma+\lambda x^{-1})\ln(x)\right]}{r_{INMW}(x)}, \tag{6.16}$$

$$\frac{\partial^2 \ln g(x)}{\partial\theta^2} = -\alpha x^{-\theta}\ln^2(x) - \frac{\left(\alpha x^{-\theta-1}(1-\ln(x))\right)^2}{r^2_{INMW}(x)} - \frac{\alpha x^{-\theta-1}\ln(x)\left(1-\ln(x)\right)}{r_{INMW}(x)}, \tag{6.17}$$

$$\frac{\partial^2 \ln g(x)}{\partial\theta\partial\lambda} = -\frac{\alpha\beta e^{\lambda x^{-1}} x^{-\theta-\gamma-3}\left[1-\theta\ln(x)\right]\left[1+\gamma+\lambda x^{-1}\right]}{r_{INMW}(x)} - \frac{\beta^2 e^{2\lambda x^{-1}} x^{-2\gamma-3}\left[1+\gamma+\lambda x^{-1}\right]\left[1-(\gamma+\lambda x^{-1})\ln(x)\right]}{r^2_{INMW}(x)}, \tag{6.18}$$

$$\frac{\partial^2 \ln g(x)}{\partial^2\lambda} = -\beta e^{\lambda x^{-1}} x^{-\gamma-2} - \frac{\left(\beta e^{\lambda x^{-1}} x^{-\gamma-2}(1+\gamma+\lambda x^{-1})\right)^2}{r^2_{INMW}(x)} - \frac{\beta e^{\lambda x^{-1}} x^{-\gamma-3}\left[2+\gamma+\lambda x^{-1}\right]}{r_{INMW}(x)}, \tag{6.19}$$

where $r_{INMW}(x)$ is the reverse hazard function of the INMW distribution.

6.4.1 Asymptotic Confidence Intervals

The expectations in the Fisher Information Matrix can be obtained numerically. Let $\hat{\boldsymbol{\Theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\theta}, \hat{\lambda})$ be the maximum likelihood estimate of $\boldsymbol{\Theta} = (\alpha, \beta, \gamma, \theta, \lambda)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) \xrightarrow{d} N_5(\underline{0}, I^{-1}(\mathbf{\Theta}))$, where $I(\mathbf{\Theta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Theta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$, that is $J(\hat{\mathbf{\Theta}})$. The multivariate normal distribution $N_5(\underline{\mathbf{0}}, J(\hat{\mathbf{\Theta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0,0,0,0,0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. A large sample $100(1-\eta)\%$ confidence intervals for $\alpha, \beta, \gamma, \theta$ and λ are:

$$\widehat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\alpha\alpha}(\widehat{\mathbf{\Theta}})}, \quad \widehat{\beta} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\beta\beta}(\widehat{\mathbf{\Theta}})}, \quad \widehat{\gamma} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\gamma\gamma}(\widehat{\mathbf{\Theta}})},$$

$$\widehat{\theta} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\theta\theta}(\widehat{\mathbf{\Theta}})}, \text{ and } \quad \widehat{\lambda} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\lambda\lambda}(\widehat{\mathbf{\Theta}})},$$

respectively, where $I^{-1}_{\alpha\alpha}(\hat{\mathbf{\Theta}})$, $I^{-1}_{\beta\beta}(\hat{\mathbf{\Theta}})$, $I^{-1}_{\gamma\gamma}(\hat{\mathbf{\Theta}})$, $I^{-1}_{\theta\theta}(\hat{\mathbf{\Theta}})$, and $I^{-1}_{\lambda\lambda}(\hat{\mathbf{\Theta}})$ are the diagonal elements of $I^{-1}_{n}(\hat{\mathbf{\Theta}})$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

The maximum likelihood estimates (MLEs) of the INMW parameters α, β, γ, θ, and λ are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\Theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Table 6.1.

6.5 Applications

In this section, we present a real data examples to illustrate the flexibility of INMW distribution. For the example, we used the data set from Bjerkedal [5], that represents the survival time of guinea pigs injected with different doses of tubercle bacilli. The values of the statistics: AIC, BIC, and KS show that the sub-model inverse Weibull distribution is a very "good" fit for the Bjerkedal data as was previously established.

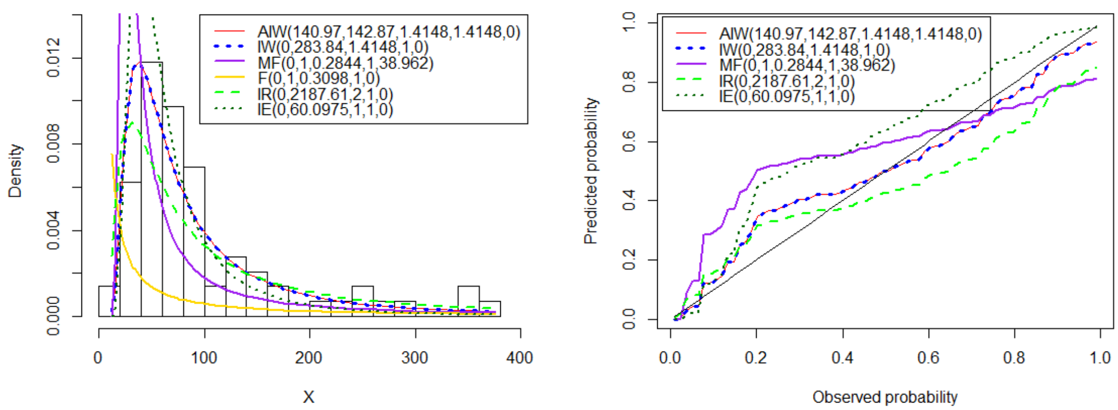

Figure 6.3: Fitted pdf and probability plots for guinea pigs data

6.6 Concluding Remarks

Mathematical properties of the IMNW distribution are presented. It is an important alternative model to several models discussed in the literature since it contains the IE, IR, IF, IW, IMW, AIW, AIR, AIMR, AIF, AIMF, AIE, and AIME distributions, among others, as special sub-models. Moments, $\nu-$entropy, Renyi entropy, and distribution of order statistics are presented. The pdf of the order statistic can be expressed as a linear combination of INMW pdfs. Estimates of parameters via method of maximum likelihood was obtained. We presented the mixed and second partial derivatives of $\ln g(x)$, from which the observed Fisher information can be easily obtained. The usefulness of the INMW distribution is illustrated via a real data.

Table 6.1: Estimates of models for Bjerkedal data

	Estimates					Statistics					
Model	α	β	γ	θ	λ	$-2\ln$	AIC	$AICC$	BIC	KS	SS
$AIW(\alpha,\beta,\gamma,\theta,0)$	140.97	142.87	1.4148	1.4148	0	791.3	799.3	799.9	808.4	0.1528	0.2453
	62.8163	62.8164	0.2022	0.2040							
$IW(0,\beta,\gamma,1,0)$	0	283.84	1.4148	1	0	791.3	795.3	795.5	799.9	0.1528	0.2454
		0.1173	125.63								
$MF(0,1,\gamma,1,\lambda)$	0	1	0.2844	1	38.962	858.1	862.1	862.3	866.7	0.3056	1.7604
			0.03603		2.8074						
$F(0,1,\gamma,1,0)$	0	1	0.3098	1	0	1026.5	1028.5	1028.6	1030.8	0.6528	9.4992
			0.03020								
$IR(0,\beta,2,1,0)$	0	2187.61	2	1	0	813.5	815.5	815.5	817.7	0.1806	0.8190
		257.81									
$IE(0,\beta,1,1,0)$	0	60.0975	1	1	0	805.3	807.3	807.4	809.6	0.25	1.2445
		7.0826									

6.7 Future Research

In the future, Bayesian techniques will be applied to estimate parameters of the proposed models. Estimates of the parameters of the proposed generalized Weibull and inverse Weibull distributions under type I censored and type II double censored data will also be developed. Exploration of the possibility of inclusion or incorporation of concomitant information into the new models will be conducted.

REFERENCES

[1] Almalki, S. J., and Yuan, J., *A new modified Weibull distribution*, Reliability Engineering and System Safety, 111, 164-170 (2013).

[2] Badar, M. G., and Priest, A. M., *Statistical aspects of fiber and bundle strength in hybrid composites*, Progress in Science and Engineering Composites, Hayashi, T., Kawata, K. and Umekawa, S. (eds.), ICCM-IV, Tokyo, 1129-1136 (1982).

[3] Bain, L. J., *Analysis for the linear failure rate life-testing distribution*, Technometrics, 16(4), 551-559 (1974).

[4] Bebbington, M., Lai, C. D., Zitikis, R., *A flexible Weibull extension,* Reliability Engineering and System Safety, 92, 719-726 (2007).

[5] Bjerkedal, T., *Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli*, American Journal of Hygiene, Vol. 72, 130-148 (1960).

[6] Calabria, R., Pulcini, G., *Confidence limits for reliability and tolerance limits in the inverse Weibull distribution*, Engineering and System Safety, 24, 77-85 (1989).

[7] Calabria, R., Pulcini, G., *On the maximum likelihood and least squares estimation in inverse Weibull distribution,* Statistica Applicata, 2, 53-66 (1990).

[8] Calabria, R., Pulcini, G., *Bayes 2-sample prediction for the inverse Weibull distribution,* Communications in Statistics-Theory Meth., 23(6), 1811-1824 (1994).

[9] Carrasco, J. M. F., Ortega, E. M. M. and Cordeiro, M. G., *A generalized modified Weibull distribution for lifetime modeling,* Computational Statistics and Data Analysis, 53, 450462 (2008).

[10] Chambers, J., Cleveland, W., Kleiner, B., and Tukey, P., *Graphical methods of data analysis*, Chapman and Hall (London) (1983).

[11] Choudhury, A., 2005. *A simple derivation of moments of the exponentiated Weibull distribution*, Metrika, 62, 17-22 (2005).

[12] Cox, D. R., *Renewal theory*, Barnes & Noble (New York) (1962).

[13] Eugene, N., Lee, C., and Famoye, F., *Beta normal distribution and its applications*, Communications in Statistics-Theory and Methods, 31(4), 497-512 (2002)

[14] Famoye, F., Lee, C., and Olumolade, O., *The beta-Weibull distribution*, Journal of Statistical Theory and Applications, 121-138 (2005)

[15] Fisher, R. A., *The effects of methods of ascertainment upon the estimation of frequencies*, Annals of Human Genetics, 6(1), 439-444 (1934).

[16] Fonseca, M. B. and Françca, M. G. C., *A influência da fertilidade do solo e caracterizaçcão da fixaç ão biológica de N2 para o crescimento de Dimorphandra wilsonii rizz,* Master's thesis, Universidade Federal de Minas Gerais (2007).

[17] Glaser, R. E., *Bathtub and related failure rate characterizations*, Journal of American Statistical Association, 75, 667-672 (1980).

[18] Gradshteyn, I. S., and Ryzhik, I. M., *Tables of integrals, Series and Products*, Academic Press (San Diego) (2000).

[19] Gupta, R. C., and Keating, J. P., *Relation for reliability measures under length biased sampling*, Scandinavian Journal of Statistics, 13, 49-56 (1985).

[20] Gupta, A. K., and Nadarajah, S., *On the moments of the beta normal distribution*, Communications in Statistics-Theory and Methods, 33 (2004).

[21] Gupta, R.D., Kundu, D., *Generalized exponential distributions.* Australian and New Zealand Journal of Statistics, 41, 173-188 (1999).

[22] Gupta, R.D., Kundu, D., *Exponentiated exponential distribution: an alternative to gamma and Weibull distributions*, Biometrical Journal, 43, 117-130 (2001).

[23] Johnson, N. L., Kotz, S., Balakrishnan, N., *Continuous univariate distributions-1*, Second Edition, John Wiley and Sons, Inc. (Hoboken, NJ) (1984).

[24] Kong, L., Lee, C., and Sepanski, J. H., *On the properties of beta-gamma distribution*, Journal of Modern Applied Statistical Methods, Vol. 6, No. 1, 187-211 (2007).

[25] Keller, A. Z., Giblin, M. T., Farnworth, N. R., *Reliability analysis of commercial vehicle engines*, Reliability Engineering, 10, 15-25, 89-102 (1985).

[26] Khan, M. S., Pasha, G. R., Pasha, A. H., *Theoretical analysis of inverse Weibull distribution*, Issue 2, Vol. 7, 30-38 (2008).

[27] Klein, J. P., Moeschberger, M. L., *Survival analysis: techniques for censoring and truncated Data*, 2nd ed., Springer-Verlag New York, Inc. (New York) (2003).

[28] Kundu, D., Rakab, M. Z., *Generalized rayleigh distribution: different methods of estimation*, Computational Statistics and Data Analysis, 49, 187-200 (2005).

[29] Lai, C. D., Xie, M., Murthy, D. N. P., *A modified Weibull distribution*, IEEE Transactions on Reliability, 52, 33-37 (2003).

[30] Lawless, J. F., *Statistical models and methods for lifetime data,* Wiley (Hoboken, NJ) (1982).

[31] Lehmann, E. L., *Theory of point estimation*, Springer-Verlag New York, Inc. (New York) (1998).

[32] Mudholkar, G. S., Srivastava, D. K., Friemer, M., *The exponentiated Weibull family: a reanalysis of the bus-motor-failure data*, Technometrics, 37, 436-445 (1995).

[33] Mudholkar, G. S., Srivastava, D. K., Kollia, G. D., *A generalization of the Weibull distribution with application to the analysis of survival data*, Journal of the American Statistical Association, 91, 1575-1583 (1996).

[34] Nadarajah, S., *Exponentiated beta distribution*, Computers and Mathematics with Applications, 49, 1029-1035 (2005).

[35] Nadarajah, S., *On the moments of the modified Weibull distribution*, Reliability Engineering and System Safety, 90, 114-117 (2005).

[36] Nadarajah, S., Kotz, S., *On some recent modifications of Weibull distribution*, IEEE Transactions Reliability, 54, 561-562 (2005).

[37] Nanda, K. A., and Jain, K., *Some weighted distribution results on univariate and bivariate cases*, Journal of Statistical Planning and Inference, 77(2), 169-180 (1999).

[38] Oluyede, B. O., *On Inequalities and selection of experiments for length biased distributions*, Probability in the Engineering and Informational Sciences, 13.2, 169-185 (1999).

[39] Patil, G. P., Rao, C. R., *Weighted distributions and size-biased sampling with applications to wildlife populations and human families*, Biometrics, 179-189 (1978).

[40] Pham, H., Lai, C. D., *On recent generalizations of the Weibull distribution,* IEEE Transactions on Reliability, 56, 454-458 (2007).

[41] Phani, K. K., *A new modified Weibull distribution function*, Communications of the American Ceramic Society, 70(8), 182-184 (1987).

[42] Pinho, L. G. B., Cordeiro, G. M., and Nobre, J. S., *The gamma-exponentiated Weibull distribution*, 11(4), 379-395 (2012).

[43] Rao, C. R., *On discrete distributions arising out of methods of ascertainment*, The Indian Journal of Statistics, 27(2), 320-332 (1965).

[44] Renyi, A., *On measures of information and entropy*, Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability, 547-561 (1961).

[45] Ristic, M. M., and Balakrishnan, N., *The gamma-exponentiated exponential distribution*, Journal Statist. Comp. and Simulation, 82(8), 1191-1206 (2011).

[46] Santos de Gusmao, F. R., Siqueira Junior, E. L., Santos, E., and Ortega, E. M. M., *Distribuio Weibull inversa: uma aplicao a dados mdicos com censuras*, 8 Encontro Regional de Matemtica Aplicada e Computacional (2008).

[47] Sarhan, A. M., and Zaindin, M., *New generalized Weibull distribution*, Applied Science, In Press, (2008).

[48] Silva, G. O., Ortega, E. M., and Cordeiro, G. M., *The beta modified Weibull distribution*, Lifetime Data Analysis, 16, 409-430 (2010).

[49] Singla, N., Jain, K., and Kumar, S. S., *The beta generalized Weibull distribution: properties and applications*, Reliability Engineering and System Safety, 102, 5-15 (2012).

[50] Shaked, M. and Shanthikumar, J. G., *Stochastic orders and their applications*, Academic Press (New York) (1994).

[51] Shannon, E., *A mathematical theory of communication*, The Bell System Technical Journal, 27(10), 379-423 (1948).

[52] Weibull, W., *A statistical distribution function of wide applicability*, Journal Appl. Mech.-Trans. ASME, 18(3), 293-297 (1951).

[53] Xie, M., Lai, C. D., *Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function*, Reliability Engineering and System Safety, 52, 87-93 (1995).

[54] Xie, M., Tang, Y., Goh, T. N., *A modified Weibull extension with bathtub failure rate function*, Reliability Engineering and System Safety, 76, 279-285 (2002).

[55] Zaindin, M., and Sarhan, A. M., *Parameters estimation of the modified Weibull distribution*, Applied Mathematical Sciences, Vol. 3, no. 11, 541 - 550, (2009).

[56] Zelen, M., *Problems in cell kinetics and early detection of disease*, Reliability and Biometry, 56(3), 701-726 (1974).

[57] Zografos, K. and Balakrishnan, N., *On families of beta- and generalized gamma-generated distribution and associated inference*, Stat. Method, 6, 344-362 (2009).