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Combinatorial Game Theory: An Introduction to Tree Topplers

John S. Ryals Jr.

COMBINATORIAL GAME THEORY: AN INTRODUCTION TO TREE TOPPLERS

by

JOHN S. RYALS, JR.

(Under the Direction of Hua Wang)

ABSTRACT

The purpose of this thesis is to introduce a new game, Tree Topplers, into the field of Combinatorial Game Theory. Before covering the actual material, a brief background of Combinatorial Game Theory is presented, including how to assign advantage values to combinatorial games, as well as information on another, related game known as Domineering. *Please note that this document contains color images so please keep that in mind when printing.*

Key Words: combinatorial game theory, tree topplers, domineering, hackenbush

2009 *Mathematics Subject Classification*: 91A46

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DEDICATION

I would like to dedicate this thesis to Mrs. Pam Champion, my 11^{th} grade Pre-Calculus teacher. I only see her on occasion, but it was because of her that I realized how much I liked math and her teaching helped me, though not intentionally, to form my approach to mathematics: treat a problem like a puzzle. Like a game. And it is because of that that I have chosen Game Theory as the area for my topic. Her classes were hard and made me decide to dual enroll in college classes rather than take her Calculus class, but I can say without a doubt that she's the reason why I went for a degree in math in the first place.

ACKNOWLEDGMENTS

First off, I wish to acknowledge Dr. Hua Wang, who has put up with my erratic methodology in writing this thesis. His guidance has kept me on track, even when I was the worst procrastinator ever, and I would probably still be two pages in if it were not for him and his optimism. Next, I'd like to thank Dr. Stefan Wagner for providing me with a counter-example of an idea in this thesis that I was not sure was true or not. I would also like to acknowledge my friends and family who have all had my back and helped keep me confident that I could get to this point in my education. And finally, I would further like to acknowledge my other committee members, Dr. Colton Magnant and Dr. Goran Lesaja, for taking the time from their busy schedules to assit in the final steps of my degree. Thank you all.

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INTRODUCTION TO GAME THEORY

1.1 Overview of Game Theory

The branch of mathematics known as Combinatornial Game Theory may be widely known by name, but actual knowledge of the subject is not as common. Thus, before we can really get into the main subject of Tree Topplers, we need to lay the groundwork for Combinatorial Game Theory itself. Game Theory is defined as "the branch of mathematics concerned with the analysis of strategies for dealing with competitive situations where the outcome of a participants choice of action depends critically on the actions of other participants" [2]. Specifically, Combinatorial Game Theory involves the study of sequential games with perfect information, that is, all players know everything that can happen from a given position with no randomness. But what is a game? For the purposes of this paper, a game is defined with the following attributes:

- There are two players, known as Left and Right.
- There are finitely many positions, including a starting position.
- There are rules that specify the moves either player can make from a given position.
- Left and Right alternate making moves.
- Both players have access to all information at any given time.
- There is no randomness to moves made, such as rolling a die.
- A player that is unable to move loses.
- A player will always lose once the ending condition is met.

Some of the most recognizeable games that could be considered under these rules, for example, are Chess and Checkers. However, for the sake of simplifying the concepts needed for this study, we will look at another game known as Hackenbush. In Hackenbush, a figure is drawn using vertices and line segments and connect to a final line called the ground. Players take turns deleting one of their lines. Classically, Left and Right take on the colors b**L**ue and **R**ed respectively. If at any time a path cannot be drawn from the ground to a line segment, that segment is also deleted. This allows for more strategic plays as a player can delete an opponent's move during their turn [1]. The following is an example of a Hackenbush game:

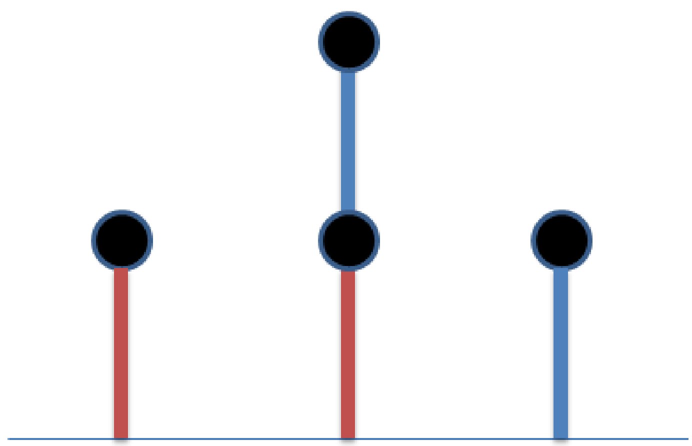

Figure 1.1: An example of Hackenbush.

Now, stop and think about how Left and Right would play this game logically. Should Left go first, he has two moves: the line on the right and the line on top of the red line. However, the latter move is the better move to make since Right can take his middle piece, effectively removing Left's piece with it. Likewise, Right should take his middle piece if he moves first for that exact reason. This is what it means to make optimal plays. Also, one thing that should be noted here is, with every move a player makes, the resulting gameboard becomes a subgame of the original, effectively making it a game in and of itself.

1.2 Scoring

There are ways to assign values to games in terms of the advantage a player has, assuming optimal plays will be made. These values are determined by looking at the advantage Left has after a player has moved. For example, after Left has played, he has a moves advantage over Right, but after Right moves, Left has b moves advantage over Right. We take these values and write them in the form $\{a|b\}$. This form does not make any quantifiable sense at the moment, but that is because the actual value is determined by what a and b are.

Before going into how to find that value, let us first consider the case in which there are no legal moves for a player. Then that player's score is left blank in the notation. If both players have no legal moves, the result is a zero-game.

Definition 1. *A* ***zero-game*** *is a game that scores* $\{\quad|\quad\} = 0$*, essentially making it such that the first player to move loses, assuming all moves made are optimal.*

For Hackenbush, the simplest form of a zero game equates to an empty board at the beginning. Thus it is obvious the first player to move has no legal move and loses automatically. Likewise, if we were to add one bLue line, Left would have 0 moves left after his move and Right would have no legal move giving Left a clear 1 move advantage, written $\{0|\quad\} = 1$. This trend continues in such a manner that $\{n|\quad\} = n + 1$, where n is the number of remaining moves Left has after an optimal move.

However, what if we add a Red line instead? As stated before, these values are applied with respect to the advantage of the Left player. Thus adding one line for Right puts Left at a one move disadvantage, or a (-1) advantage. So adding one Red line results in $\{\quad|0\} = -1$. Adding two Red lines would then be $\{\quad|-1\} = -2$. And so on to a general form of $\{\quad|-n\} = -(n+1)$, where n is the number of remaining moves Right has after an optimal move.

Now with the groundwork out of the way, we can start using the values of subgames to determine the value of an overall game. Take the following Hackenbush game for example:

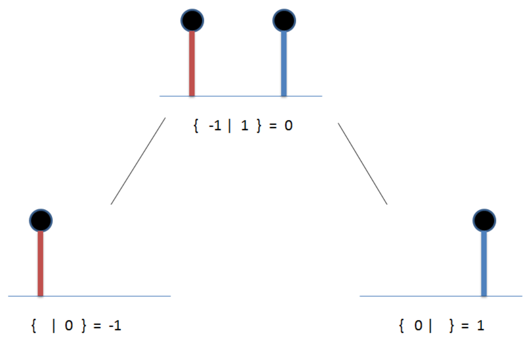

Figure 1.2: A simple Hackenbush game tree with values.

After Left moves, Right has one move. From that game, we clearly have a $\{\ \ |0\} = -1$ situation. Conversely, if Right moves first, we have $\{0|\ \ \} = 1$. This results in the overall game having a value of $\{-1|1\} = 0$. This makes sense as well, since we equally added one independent move for both players to an empty board, meaning advantage did not change. But what about this next game?

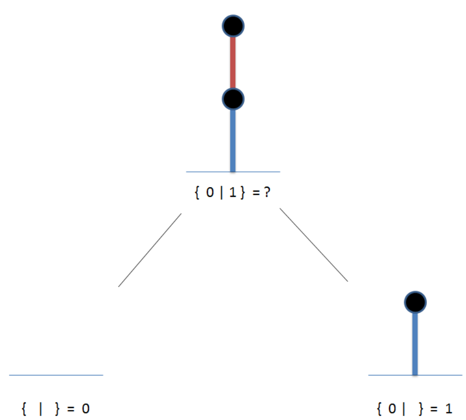

Figure 1.3: What could this game equal?

Now both players have been given one line each. But if Left moves, Right's only move is eliminated and if Right moves, Left still has a move. It is not so clear what the value of this game is given the scoring rules already introduced. There is another rule, known as the **Simplicity Rule**, that can determine the value of a game such as this.

Theorem 1.2.1. - ***The Simplicity Rule*** *[1] - For a combinatorial game of value* $\{a|b\} = G$, *G is the simplest number such that* $a < G < b$. *That is,*

$$G = \tfrac{2p+1}{2^{n+1}} = \{\tfrac{p}{2^n}|\tfrac{p+1}{2^n}\}.$$

Note:A formal proof of The Simplicity Rule will not be provided in this thesis and can be found in [1].

To simplify, the value of a game is the number between a and b with the lowest power of 2 as the denominator. Therefore, for our example above, we get $\{0|1\} = \frac{1}{2}$. Likewise, we can obtain values such as $\{\frac{3}{8}|\frac{3}{4}\} = \frac{1}{2}$ and so on. With this, we can finally obtain a value for our original example:

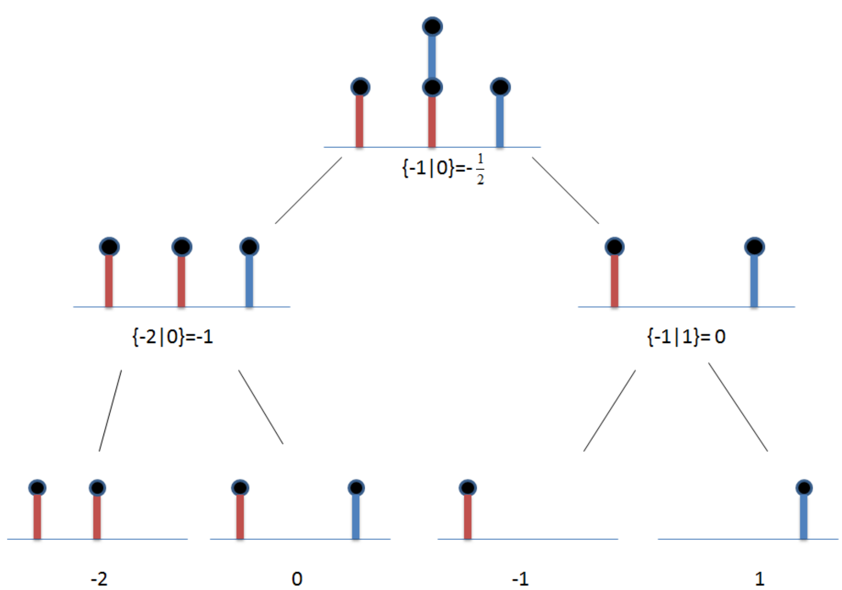

Figure 1.4: The game tree of the first Hackenbush example.

We can also obtain the value of this game by another means. We can look at the values of each individual, non-interacting game board and express the value as a sum:

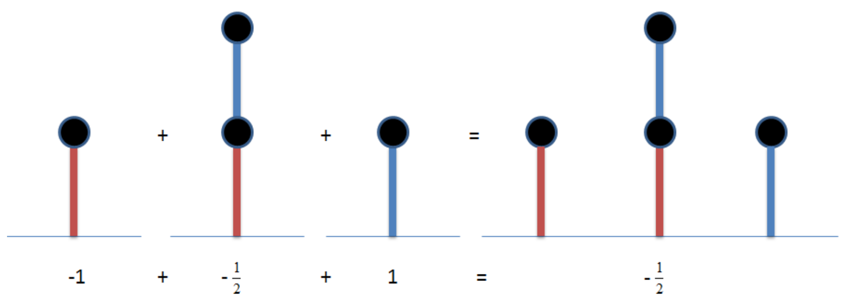

Figure 1.5: The original example broken up as a sum of games.

We can trivially find that a single red line has a value of -1 and a single blue line has a value of 1. Through similar steps to the game in Figure 1.3, we can find the middle game's value to be $-\frac{1}{2}$. Thus, we get a final value of $(-1)+(-\frac{1}{2})+(1) = -\frac{1}{2}$.

There are also other special games of infinitesimal, or extremely small, value.

Definition 2. *A* ****-game*** *(pronounced star game) is an infinitesimal game that scores* $\{0|0\} = *$*, essentially resulting the first player to move winning, assuming all moves made are optimal.*

Say, for example, Hackenbush had another line type that was green, which is claimable by either player. Then we get the following game which results in a value of $\{0|0\} = *$ (See Figure 1.6). Building onto this concept, we also have results like $\{n|-n\} = n*$, where $n* = n + *$. It is also worth noting that * has the property such that $* + * = 0$.

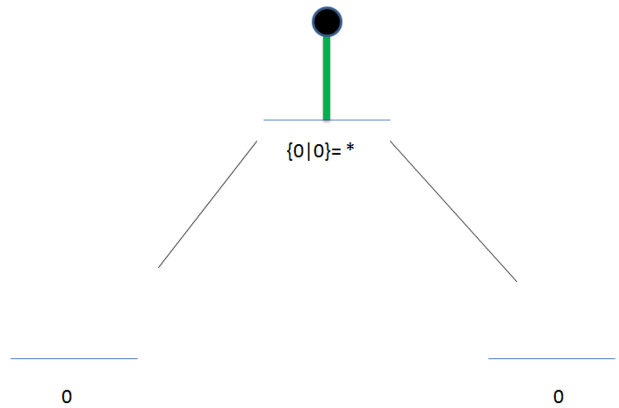

Figure 1.6: Example of a *-game with a green line.

Furthermore, there are two more infinitesimal games.

Definition 3. *An* $\uparrow$***-game*** *(pronounced "up game") is a positive infinitesimal game where the score is* $\{0|*\}$*, which favors the Left player [7].*

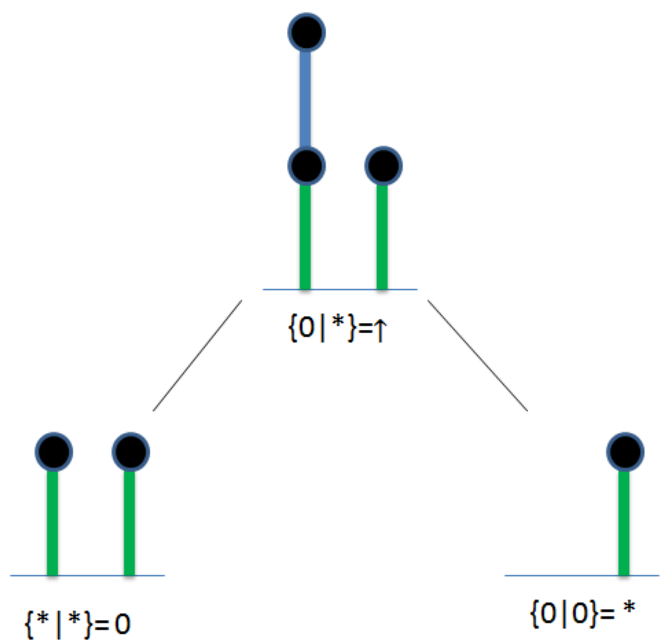

Figure 1.7: An example of an up-game.

The negative version of an up-game is called a down-game and is defined as follows:

Definition 4. *An* $\downarrow$***-game*** *(pronounced "down game") is a negetive infinitesimal game where the score is* $\{*|0\}$ *which favors the Right player [7].*

With the relation between up and down games, now is a good time to mention the relation between the inverses of games. With every game, there is a way to reverse every move and, as as result, negates the value the game originally had. With Hackenbush, this is obtained by replacing every red line with a blue line and vice-versa, like in Figure 1.8.

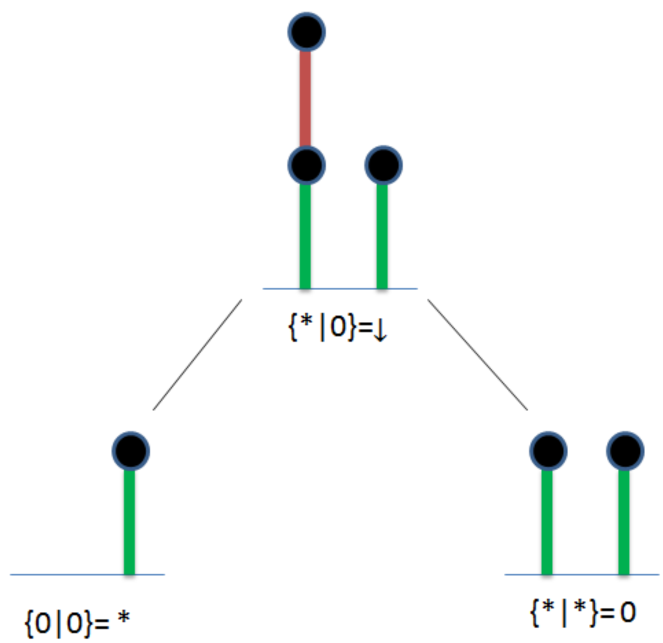

Figure 1.8: A down-game, showing the negation relation.

CHAPTER 2

DOMINEERING

Before getting into Tree Topplers, we must first briefly discuss a couple of topics, the first of which is another game known as Domineering. The premise of the game is simple; each player takes turns placing a domino, made of two tiles, on a tiled game board (similar to a chess board), with Left placing their piece vertically and Right placing their piece horizontally. Pieces are not allowed to overlap and cannot be played outside the boundries of the board. The first one that is unable to play loses. As an example, observe the following game where blue is Left's vertical move and red is Right's horizontal move:

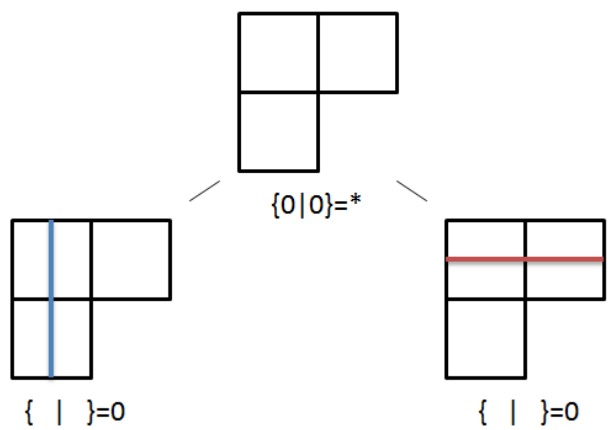

Figure 2.1: A sample game of Domineering with moves included.

One problem with looking at Domineering in this form is it is harder to visually distinguish the subgame. Therefore, when drawing game trees of Domineering, whenever a domino is places onto the board, the covered cells are removed from the drawing. See Figure 2.2 for an example.

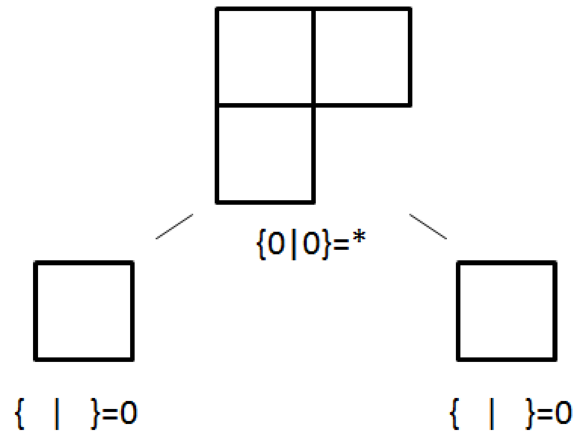

Figure 2.2: A sample game of Domineering eliminating moves.

In that form, we can more easily see that the two games resulting from Left's and Right's individual moves actually result in the same game, thus having the same value.

Now, there is one concept that needs to be addressed with Domineering. Whereas with all our Hackenbush games, we had $a \leq b$ for all games of value $\{a|b\}$, it is possible for games in Domineering to play out such that $a > b$, as follows:

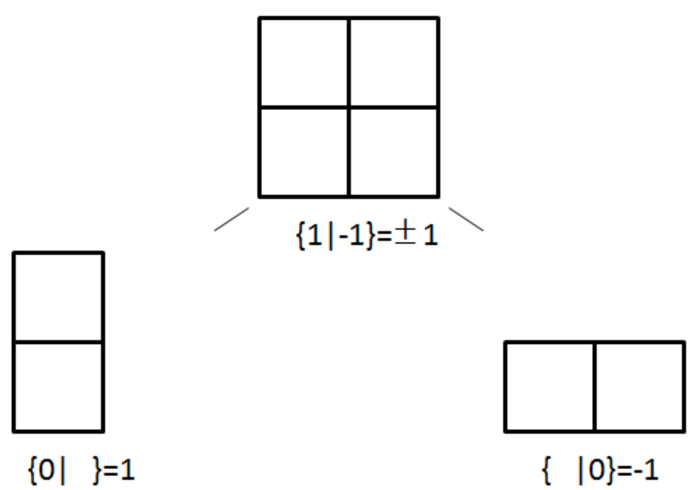

Figure 2.3: A Domineering game with value ± 1.

Notice that this game was assigned the value ± 1. This is an example of a **switch game**, as the value can switch depending on who makes the first move. In this example, the first player to move gains a 1-move advantage. The way this breaks down is as follows:

For a game $\{y|z\}$ such that $y > z$,

$$\{y|z\} = a + \{x| - x\} = a \pm x,$$

where $a = \frac{y+z}{2}$ and $x = \frac{y-z}{2}$.

Now, keeping in mind that, for Domineering, negating the value involves turning the board by 90° , we can can look at an example that is a bit more complicated.

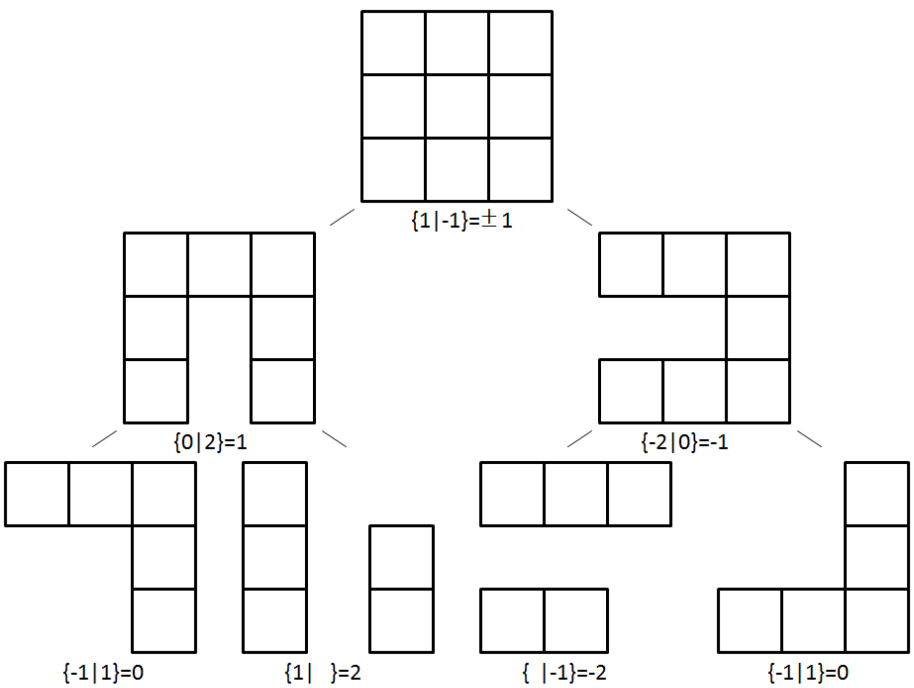

Figure 2.4: Domineering played on a 3x3 board.

Notice that, from the starting position, the optimal move for either player involves taking the center square, thereby leaving the other player with only one available move to take. The best reason for this move, however, is that it reserves two more moves for that player to use later. This strategy can be applied to bigger boards to reserve a move against an edge since the other player cannot play in that space. From there, the only available moves for either player has equal impact on the game's value. Notice, however, that every game on the left side of the tree is the negative of each game on the right as they all are a 90° rotation of another. Also, the second game from the left can quickly have its value calculated as the sum of two games that clearly have a value of 1 making its value 2, meaning its corresponding negative have a value of -2. The value for the game on the bottom left can quickly have its value calculated as 0. This shows the value of the previous game being 1. Finding all the values on the left side allows immediate results in finding the values of the games on the right, since they are all negations of the left side, ending in a value of $\{1|-1\} = \pm 1$.

Domineering has been around for fair amount of time [6]. As such, many values have already been found for many game. Several examples may be seen in Figure 2.5.

There has also been a lot of research into the game, such as who wins on various sizes of rectangular boards, that is, a board of size $m \times n$. While this can be a subject of interest, it does not give any specific values to the games. At best, a lot of results that are known simply boil down to whether vertical or horizontal always wins, or if the first or second player always wins. Beyond that, some other boards only have the known result that, for instance, horizontal always wins if they go first [4]. However, for the purposes of this thesis, we will not be going that in-depth into the subject of Domineering as just a basic understanding is necessary for Tree Topplers. If further reading is desired, there are manybooks and online articles on the subject, such as [4].

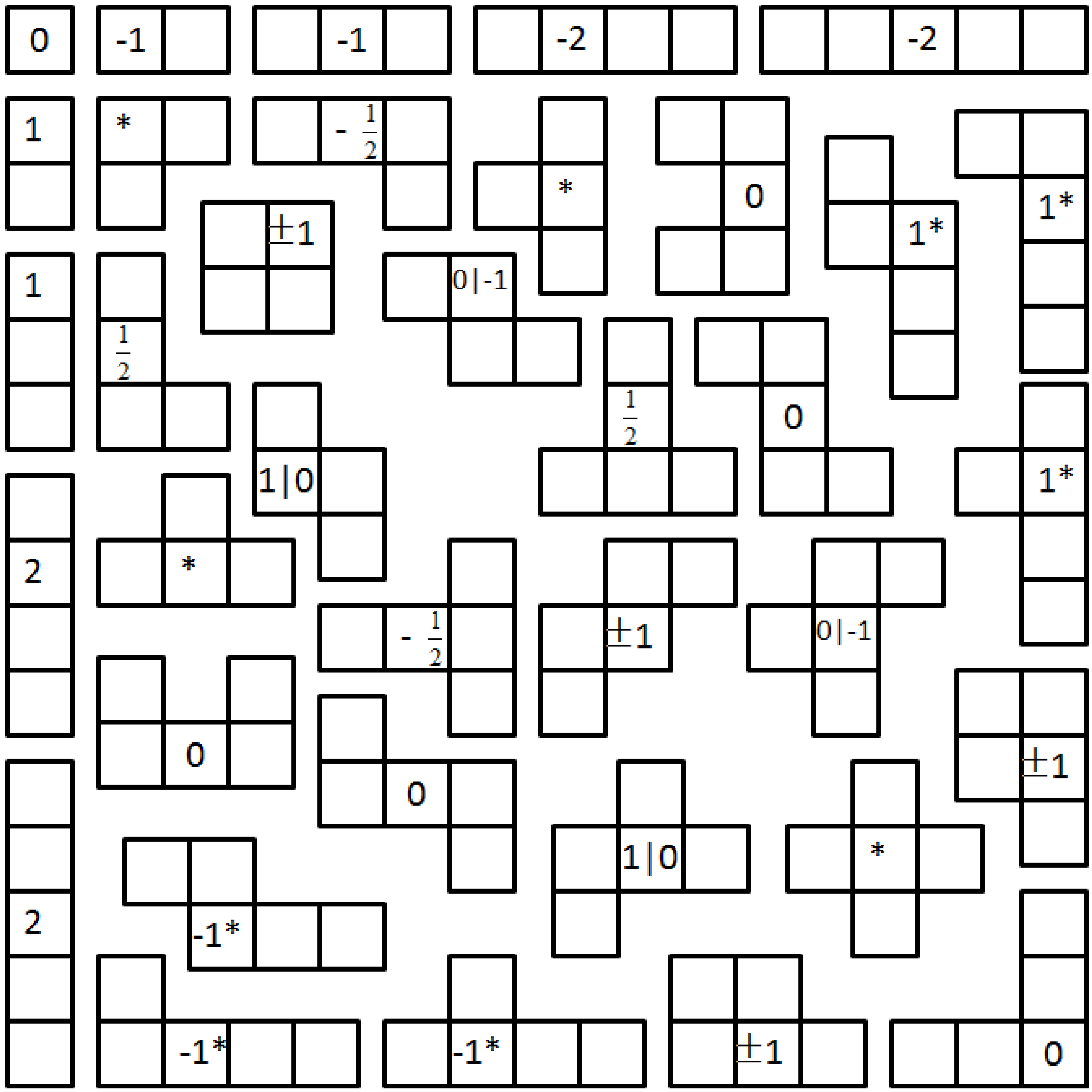

Figure 2.5: Several Domineering games with their corresponding values.[1]

CHAPTER 3

TREE TOPPLERS

3.1 Young Tableau and Hook Length

Our second set of topics before getting into Tree Topplers are the Young tableau and hook length. A Young tableau can be formed by taking a partition of a positive integer and filling out a tableau of squares with left justified rows of length equal to the partitions in decreasing order [3].

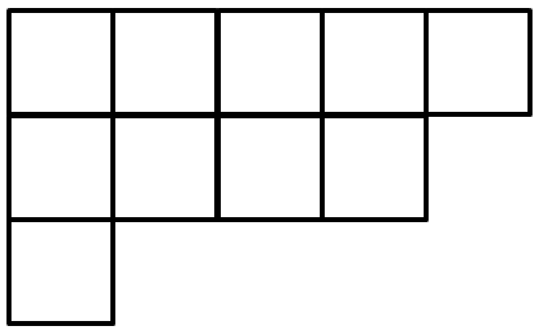

Figure 3.1: A partition of (5,4,1) would have 5 squares on the top row, then 4 in the middle, and 1 on the bottom row, all aligned on the left side.

A standard Young tableau of a partition of n has distinct integers from 1 to n such that each row and column form increasing sequences.

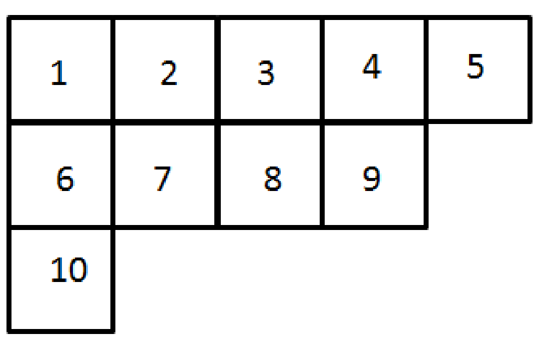

Figure 3.2: A standard Young tableau of partition (5,4,1).

Related to the Young tableau is the concept of the hook and its hook length. Let a Young tableau have a shape denoted by λ. A hook, $H_\lambda(i,j)$ on a Young tableau is the subset of cells on the tableau starts at the (i,j) and continues right and down from there until the column and row terminate. The hook length of $H_\lambda(i,j)$, denoted $h_\lambda(i,j)$, is the total number of cells in $H_\lambda(i,j)$.

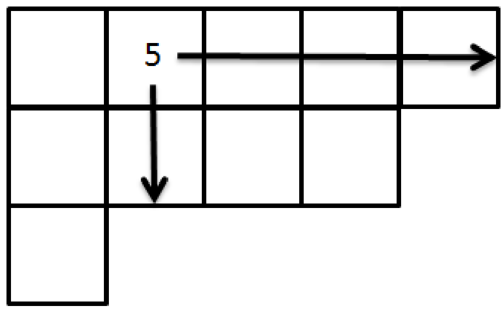

Figure 3.3: A visual representation of $H_\lambda(1,2)$ filled with its hook length of 5.

The number of standard Young tableaus of a shape λ, denoted d_λ can be calculated by $d_\lambda = \frac{n!}{\prod h_\lambda(i,j)}$. The easiest was to obtain $\prod h_\lambda(i,j)$ would be to fill out the tableau with all the corresponding hook lengths, like in Figure 3.4.

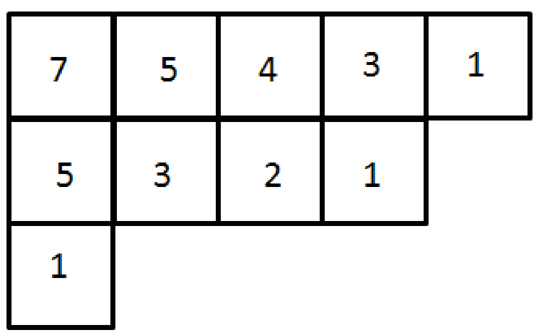

Figure 3.4: The same tableau filled with the hook lengths of each cell.

Using this as an example, we can see that for a tableau of shape $\lambda = (5,4,1)$ we get $d_\lambda = \frac{10!}{7 \cdot 5 \cdot 5 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \cdot 1} = 288$.

This concept has since been generalized to binary trees to the effect of the equation not changing at all [5].

3.2 Introduction to Tree Topplers

The inspiration for Tree Topplers came from the concept presented in the previous section where hook length for a Young tableau was generalized for binary tree. Using Domineering as a basis, Tree Topplers began with the premise of looking at Domineering as a rooted binary tree such that every square is a vertex and every vertex of adjacent squares are joined by an edge, with a vertical connection slanting from right to left and a horizontal connection slanting from left to right.

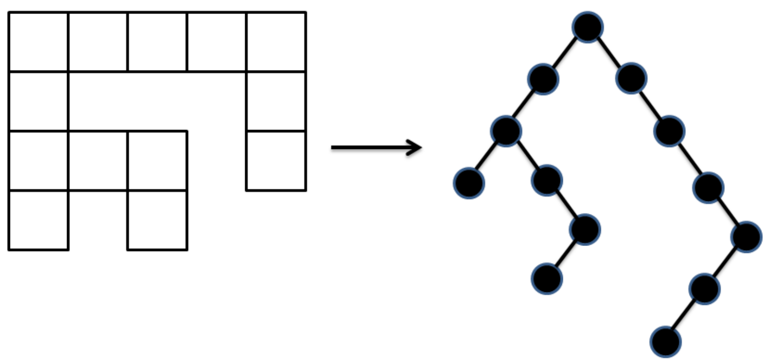

Figure 3.5: A Domineering game with its equivalent Tree Topper game.

However, while at first glance it seems as though Tree Topplers is nothing but a restricted version of Domineering, there are in fact games that are exclusive to each particular game (See Figure 3.6).

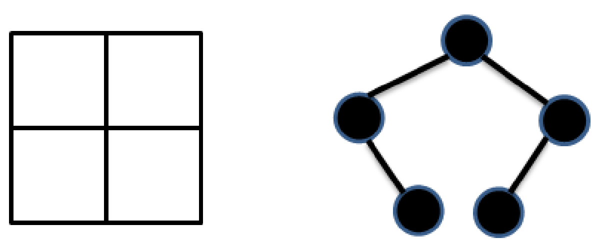

Figure 3.6: Examples of games exclusive to Domineering and Tree Topplers respectively.

For the Domineering game in Figure 3.6, if we were to convert it into a graph, we would end up with a cycle graph C_4. As for the Tree Toppler game, it cannot directly be converted to a tableau since the bottom two vertices would cause the corresponding cells to overlap.

Before moving on, we need to clarify some terms that will be used intermittently. For starters, the premise of Tree Topplers plays very similar to Domineering. Converting from Domineering to Tree Topplers, we can define our analog of what a piece is.

Definition 5. *A **piece** in Tree Topplers refers to two vertices joined by one edge.*

Furthermore, the following definitions give name to certain parts of a Tree Topplers game board. Figure 3.7 on the next page will give a visual representation of each.

Definition 6. *A **contested vertex** in this game refers to a vertex that is in pieces that may be taken by either players.*

Definition 7. *A **free vertex** in this game refers to a vertex that is in pieces that may only be taken by a specific player.*

Definition 8. *We call a piece an **extension** of another piece if the two pieces share a vertex and both pieces belong to the same player.*

Definition 9. *We call a set of pieces a **leg** if the the following conditions are met:*

- *Every piece in the set shares at least one vertex with another piece in the set.*
- *Exactly one vertex in the set is a contested vertex.*
- *Exactly one vertex in the set is a leaf.*
- *The maximum degree of all free vertices in the set is 2.*

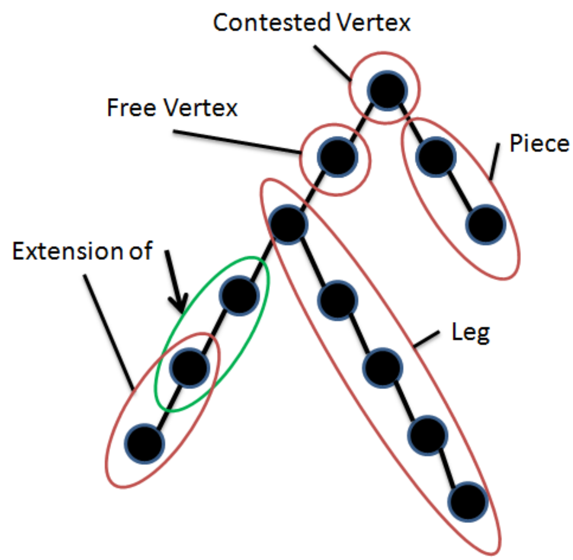

Figure 3.7: A Tree Topplers game with the sections labelled.

3.3 Gameplay

Gameplay follows by players alternatingly taking their pieces from the board, with Left taking pieces that slant from right to left and Right taking the opposite. The first player unable to remove a piece loses. Take the following simple game tree for example. Do note that, when a piece is removed, all edges connecting to that piece, colored green in following figures, are suddenly useless, and therefore are removed from successive plays. For example:

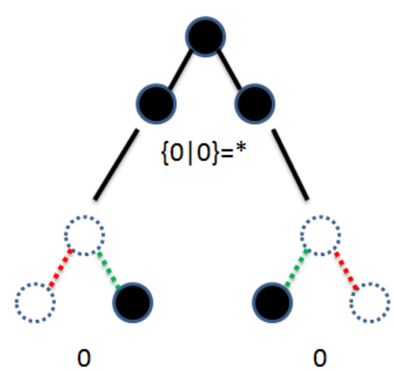

Figure 3.8: Example game tree of Tree Topplers, marking removed pieces for clarity.

As with Domineering, various games from Tree Topplers can be a switch game as well, such as the following:

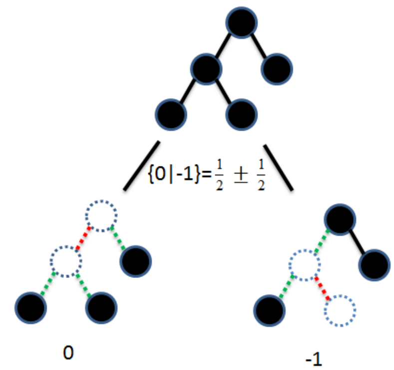

Figure 3.9: A switch game in Tree Topplers.

Other properties used in finding the value of this example include negating the game by performing a horizontal flip on the entire tree. However, while a vertical flip can also have the same effect, it violates the general structure of the game as the root would then be at the bottom. Many different values can be found just through experimentation alone:

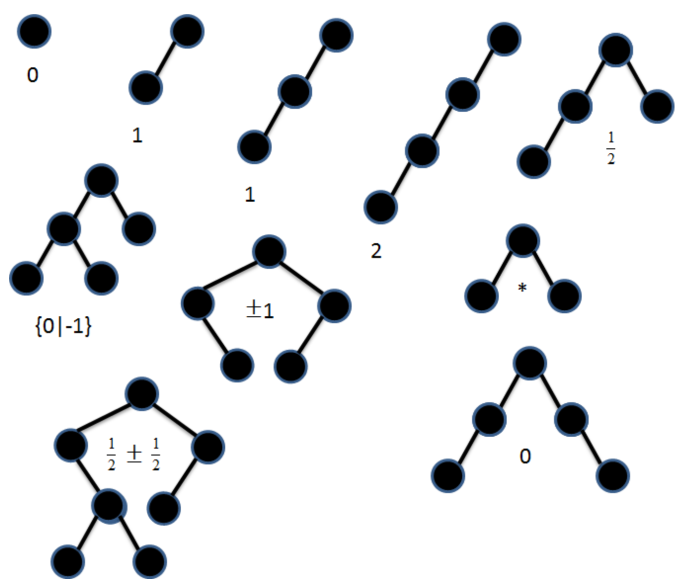

Figure 3.10: Various Tree Topplers games and their values.

One particular Tree Toppler game from Figure 3.11 is of interest due to the way the game plays out.

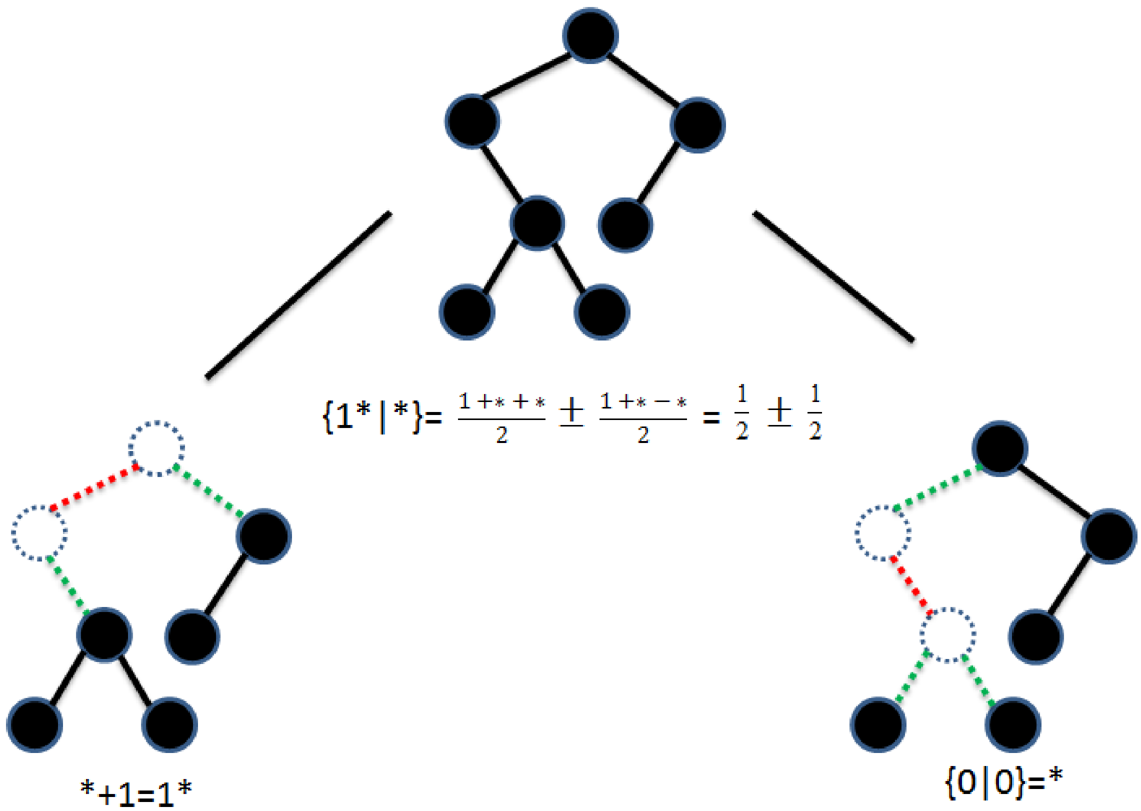

Figure 3.11: Examples of games excluside to Domineering and Tree Topplers respectively.

This game is of interest as it shows an example of a game here that, not only results in a switch game, but also demonstrates various operations such as the addition of games and switch games involving * values. In fact, as noted earlier, in order to find the value of this game, we must take advantage of the fact that $* + * = 0$ when applying the formula for switch games.

3.4 Observations

Through working with this game, several observations have been made, documented here in the form of theorems and their respective proofs.

Lemma 3.4.1. *Taking a piece with a contested vertex always yields a more favorable result than taking a piece with only free vertices.*

Proof. Let there be two moves for left such that M_0 is a move with no contested vertices and M_1 is a move with at least 1 contested vertex. For M_0, Right has no way to interact with this move. However, Right can interact with M_1 by taking a piece containing one of the contested vertices. Therefore, Left should make a priority to the take M_1 over M_0. □

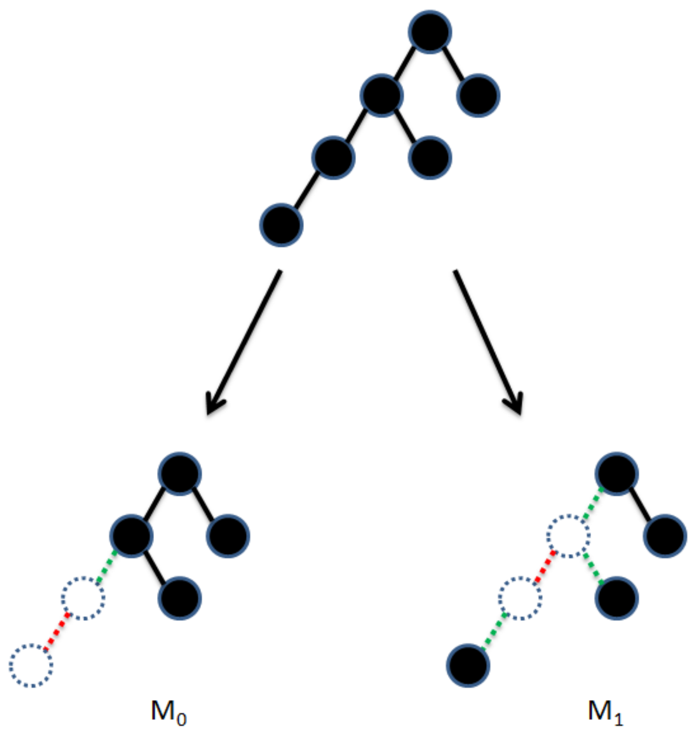

Figure 3.12: The difference in choices of taking zero, one, or two contested vertices.

Remark 1. *In regards to Lemma 3.4.1, it is natural to think that it is always a better move to take a piece that has the largest number of contested vertices. However, the Figure 3.13 shows that this is not always the case.*

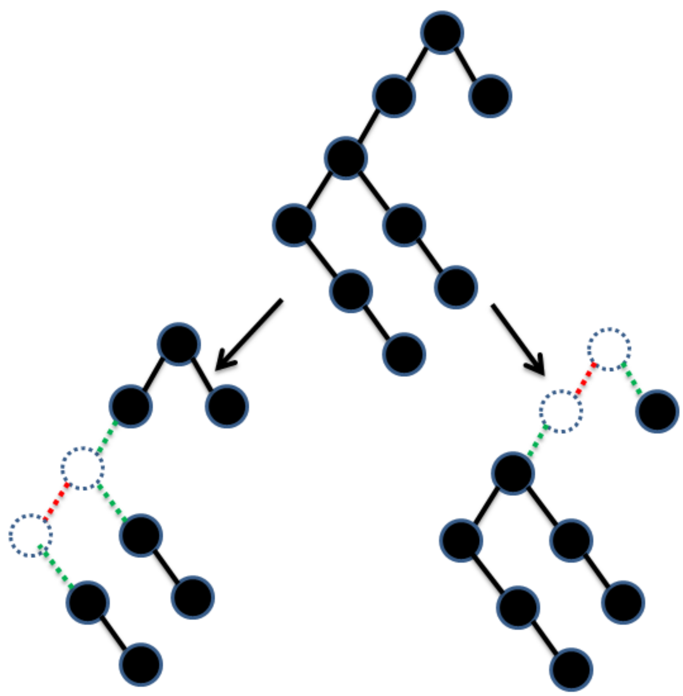

Figure 3.13: An example where taking one contested vertex yields a better result over two.

Next, we have a theorem that can help quickly find values of games by adding to the shape in certain ways.

Theorem 3.4.2. *Let G be a game with at least one leg for Left. Then, adding two extensions to that leg increases the value of the game by 1.*

Before going into the proof of this theorem, let us first note that, for the sake of convenience, we will abuse notation somewhat. Let G be a game and M_n be a move for either player. Then $G - M_n = z$ refers the game G having the move M_n removed from it and resulting in a game with a value of z.

Proof. Let G be a game such that $G = \{a|b\}$ and G has at least one leg for Left with terminal vertex v. Let G' be a game resembling G except with two extensions extended from v, adding vertices w and x. Let Right's best move in G be M_R. Then $G - M_R = b$. Then, Right's best move in G' is M_R.

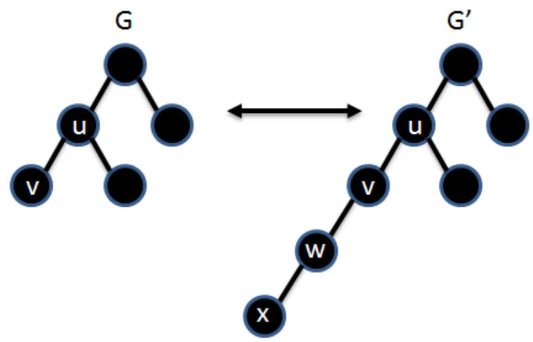

Figure 3.14: A general game for visualization purposes.

Case 1. Assume M_R includes a contested vertex u such that the path from u to v is a leg, called L. WLOG, let $|L| = 1$. Then $G' - M_R$ still contains the path wx. Thus $G' - M_R = b + 1$.

Case 2. Assume M_R does not include the contested vertex in L. Then the proof is trivial and $G' - M_R = b + 1$.

Now let Left's best move in G be M_L. Then $G - M_L = a$. We want to show that, if M'_L is Left's best move in G', then $M'_L = M_L$, that is, adding the two extensions does not change Left's best move.

Assume M'_L lies on the extension, that is M'_L removes the piece vw or wx. However, since u is a contested vertex, then by Lemma, 3.4.1, a piece including u would be preferable. Thus M'_L cannot exist on the extension. That means $M'_L = M_L$.

Case 1. Let $M_L \subset L$. Then M_L removes the piece uv. Let WX be the game consisting only of the piece wx. Then $G' - M_L = G - M_L + WX = a + 1$.

Case 2. Let $M_L \not\subset L$. Then the proof is trivial and $G' - L = a + 1$.

Thus, in general, $G' = \{a + 1|b + 1\} = \{a|b\} + 1$. $\square$

Remark 2. *It should be noted that there are cases where one extension can increase the value of a game by 1, but that does not always occur. Two extensions, however, will always increase the game value by 1. For example, see Figure 3.15.*

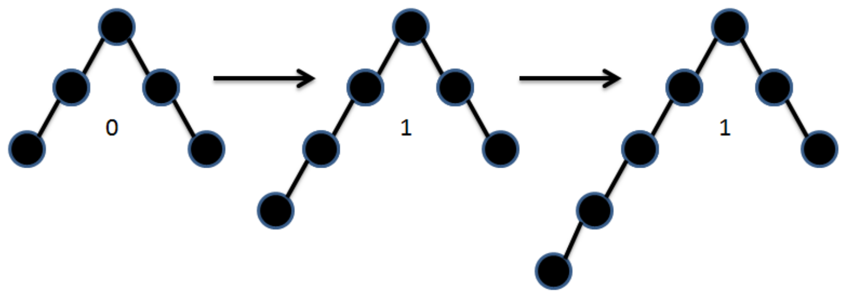

Figure 3.15: A Tree Topplers game showing how the value is affected by one and two extensions.

Naturally, this theorem can be reworked to apply to Right's moves as well.

Corollary 3.4.3. *Let G be a game with at least one leg for Right. Then, adding two extensions to that leg decreases the value of the game by 1.*

Proof. Since performing a horizontal flip on a Tree Topplers game negates the value, the proof is trivial. □

Remark 3. *Since Tree Topplers and Domineering have similar structures, this theorem and corollary can quickly be applied to Domineering as well by adding a set of two horizontal or vertical squares.*

One of the most basic structures of a Tree Topplers game is a game with two legs of equal length meeting at one contested vertex. The following theorem will show that there are only two possible values for games of this particular shape. See Figure 3.16 for examples.

Proposition 3.4.4. *Let $G(n)$ be a game with exactly one contested vertex and all pieces form two legs of equal length n. Then the following are true:*

(a) *If n is odd, then the result is a *-game.*

(b) *If n is even, then the result is a zero-game.*

Proof. To begin, we know $G(0) = 0$ and $G(1) = *$. Let a game $G(n) = \{a|b\}$. Let $m \in \mathbb{N}$. By Theorem 3.4.2, adding two extensions to Left increases the value by 1 and by Corollary 3.4.4, adding two extensions to Right decreases the value by 1. Since there are an equal number of extensions on either side, the increase/decrease in value is nullified.

(a) If n is even, $G(n) = G(0 + 2m) = G(0) = 0$.

(b) If n is odd, $G(n) = G(1 + 2m) = G(1) = *$. □

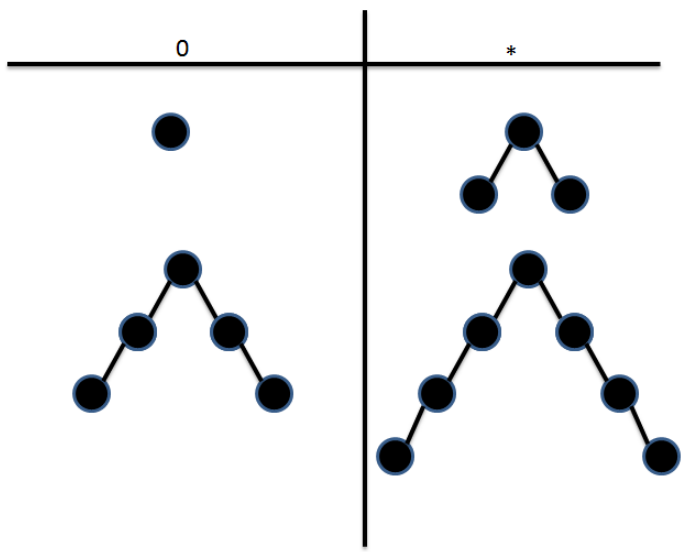

Figure 3.16: Equal-legged Tree Topplers games with their respective values.

3.5 Future Potential Topics

One topic that may be of interest to apply towards Tree Topplers harkens back to hook length. It is possible that there could be a relation between hook length values and Tree Topplers game values. Also, as opposed to Domineering, Tree Toppler games have an easier structure to interpret, potentially making studies of these games easier. As Tree Topplers and Domineering share similar attributes, it would seem relevent to try to apply results from Tree Topplers to Domineering. This is a new game and there are many things likely waiting to be discovered on it. Have fun.

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