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A Partition Function Connected with the Göllnitz-Gordon Identities

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by

NICOLAS ALLEN SMOOT

(Under the Direction of Andrew Sills)

ABSTRACT

Nearly a century ago, the mathematicians Hardy and Ramanujan established their celebrated circle method to give a remarkable asymptotic expression for the unrestricted partition function. Following later improvements by Rademacher, the method was utilized by Niven, Lehner, Iseki, and others to develop rapidly convergent series representations of various restricted partition functions. Following in this tradition, we use the circle method to develop formulæ for counting the restricted classes of partitions that arise in the Göllnitz–Gordon identities. We then show that our results are strongly supported by numerical tests. As a side note, we also derive and compare the asymptotic behavior of our formulæ.

Key Words: Integer Partitions, Circle Method

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DEDICATION

This thesis is dedicated with generosity and love to the perpetual students of academia—to those who struggled, who dropped out, who were told that they would never do anything important, who faced dreadful odds in the hope of making the smallest contribution to knowledge.

Know that you share this little honor with Galois, with Ramanujan, with Noether, with Einstein, and with so many other great scientists and artists. You can be great. Do not let yourself be told anything else.

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LIST OF SYMBOLS

The following are used throughout the body of this paper:

$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{Z}$	Set of integers
$\mathbb{N}$	Set of natural numbers, i.e. positive integers
$SL(2,\mathbb{Z})$	Group of 2×2 matrices with entries in $\mathbb{Z}$ and determinant 1, under matrix multiplication
φ	Golden ratio, $(1+\sqrt{5})/2$
δ_S	Silver ratio, $1+\sqrt{2}$

- $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j) = (1-a)(1-aq)(1-aq^2)...$
- Given $a, b \in \mathbb{Z}$, $a \mid b$ denotes that a divides b. Otherwise, $a \nmid b$.
- Given functions f, g, over $x \in \mathbb{R}$, $f(x) = O(g(x))$ as $x \to \infty$ means that there exists some x_0 such that $|f(x)| \leq Cg(x)$ for all $x \geq x_0$, for some number C that does not vary with x.
- Given functions f, g, $f(x) \sim g(x)$ means that $f/g \to 1$ as $x \to \infty$.

CHAPTER 1

INTRODUCTION

Let $n \in \mathbb{N}$. An integer partition of n is a representation of n as a sum of positive integers, called parts. The ordering of the parts is irrelevant.

As an example, consider the number 5, which has 7 partitions:

$$
\begin{aligned}
&5, \\
&4+1, \\
&3+2, \\
&3+1+1, \\
&2+2+1, \\
&2+1+1+1, \\
&1+1+1+1+1.
\end{aligned}
$$

Define the partition function $p(n)$ to be the number of partitions of n. For example, we have $p(5) = 7$. On inspection of individual cases, $p(n)$ is a rapidly increasing arithmetic function, with no obvious formulaic structure.

In 1918 the mathematicians G.H. Hardy and Srinivasa Ramanujan developed their celebrated circle method in order to give a remarkable asymptotic expression for $p(n)$ [8]. Their formula was capable of calculating the value of $p(n)$ to a degree of precision hitherto considered unthinkable.

This result was as stunning as it was useful, given the comparable pessimism of the mathematical community at the time. Barely 22 years before, the prime number theorem, providing asymptotic formulæ for the prime counting function, had been proven [25, Chapter 6]. But the proof came only after a solid century of contributions by some of the finest minds in the history of mathematics; even then, the resulting formulæ were so weak that they were effectively useless for direct computation [9, Chapter 1]. The discovery of an arithmetic function that could admit a much stronger asymptotic representation—together with the method used to develop such a representation—was as unexpected as it was innovative.

Almost immediately after its successful use, the applicability of the circle method was recognized, leading to its development into one of the most basic tools in analytic number theory, applied to Diophantine equations, bounds for the weak and strong forms of Waring's problem, and even approaches to the Goldbach and twin prime problems [27].

Notably, the method has continued to contribute to partition theory. By 1936, Hardy and Ramanujan's work had been carefully refined by Hans Rademacher, who first modified Hardy and Ramanujan's formula into a rapidly convergent series representation for $p(n)$ [16], and later developed a brilliant modification of the method itself [17]. We give Rademacher's completed form of the formula below.

Theorem 1.1. *Let $n \in \mathbb{N}$. Then*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k}\; A(n,k) \frac{d}{dx} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)}\right)}{\sqrt{x - \frac{1}{24}}} \right) \Bigg|_{x=n}, \tag{1.1}$$

where

$$A(n,k) = \sum_{\substack{0 \le h < k, \\ (h,k)=1}} \omega(h,k) e^{-2\pi i n h/k}, \tag{1.2}$$

and $\omega(h,k)$ is a certain 24kth root of unity.

The precise value of $\omega(h,k)$ is given in Chapter 3.

This astonishing formula for so simple an arithmetic function is achieved primarily through the remarkable properties of the generating function for $p(n)$. One can show that, for $|q| < 1$,

$$F(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}. \tag{1.3}$$

Formally, this can be easily recognized through a geometric expansion of each factor on the right hand side of (1.3). Notice that by convention, $p(0) = 1$. This is generally the case, even for more restrictive partition functions.

The key idea of the circle method is to use the residue theorem of Cauchy to extract the coefficient of q^n, by effectively integrating the right hand side of (1.3). With the right change of variables, this infinite product can be expressed in terms of a modular form, which can be adjusted to become a more elementary function, much more suitable to integration. The resulting integration yields (1.1).

Soon after Rademacher's work, formulæ were developed for enumerating partitions in which the given parts are restricted by various arithmetic progressions. For example, Hua and Hagis developed a formula to count the partitions of n in which each part is odd [10],[7].

Slightly earlier, through Rademacher's tutelage, Ivan Niven developed a formula to count partitions of n such that each part is $\pm 1 \pmod 6$ [15]. While Niven's formula required a more careful approach to the theory of modular forms than the unrestricted partition function, the applicability of the circle method itself was unchanged.

As a third example, of particular interest to us, Rademacher's doctoral student Joseph Lehner developed formulæ to count partitions in which parts are $\pm a \pmod 5$, with a fixed at either 1 or 2 [12].

The significance of the three examples given above lies in the fact that each class of partitions is associated with a certain identity. For example, due to an identity of Euler, Hua and Hagis' formula also counts the number of partitions of n in which the given parts are distinct [9, Chapter 19]. Similarly, an identity of Schur imposes that Niven's result also describes the number of partitions of n into distinct parts congruent to $\pm 1 \pmod 3$ [22].

Very notably, Lehner's classes of partitions are associated with the Rogers–Ramanujan identities [20], which are considered to be among the most beautiful results in modern mathematics [19, pp. xxxiv]. We give the identities here, for later reference, as identities both in q–series and in partitions.

For this and later reference, let $|q| < 1$, and define the infinite q–Pochhammer symbol $(a;q)_\infty$ as

$$(a;q)_\infty = \prod_{j=0}^{\infty} \left(1 - aq^j\right). \tag{1.4}$$

In particular,

$$(q^a;q^m)_\infty = \prod_{j=0}^{\infty} \left(1 - q^{mj+a}\right). \tag{1.5}$$

This symbol is a helpful shorthand when infinite products of this type we describe become common and rather clumsy to directly handle.

Theorem 1.2 (First Rogers–Ramanujan Identity)**.** *Let $q \in \mathbb{C}$ with $|q| < 1$. Then*

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q;q)_m} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}. \tag{1.6}$$

This identity has an associated partition–theoretic interpretation:

Theorem 1.3. *Given an integer n, the number of partitions of n in which parts are non-repeating and non-consecutive is equal to the number of partitions of n in which parts are congruent to $\pm 1 \pmod 5$.*

Theorem 1.4 (Second Rogers–Ramanujan Identity)**.** *Let $q \in \mathbb{C}$ with $|q| < 1$. Then*

$$\sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q;q)_m} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.7}$$

As a partition identity:

Theorem 1.5. *Given an integer n, the number of partitions of n in which parts are non-repeating, non-consecutive, and with smallest part ≥ 2 is equal to the number of partitions of n in which parts are congruent to $\pm 2 \pmod 5$.*

Later, Livingood and Iseki developed more general approaches to partition formulæ [13],[11]. More recently, Bringmann and Ono have developed formulæ to extract coefficients from any harmonic Maas forms with weights $\leq 1/2$. Their formulæ may be used to describe a large variety of q-series, along with any associated class of partitions in which the given parts are described by a set of symmetric arithmetic progressions, encompassing nearly all earlier results on this matter [4].

Beautiful and interesting relationships between q-series similar to the Rogers–Ramanujan identities abound. The variety of such identities can be glimpsed in Slater's enormous list [24]. Many of these identities have a partition–theoretic interpretation, so that formulæ for the associated Fourier coefficients might be useful. The formulæ for several of these identities have already been accounted for—for example, Livingood's work encompasses the earlier results of Lehner, Niven, Hua and Hagis, with many others, as special cases. Iseki's results, in turn, encompass Livingood's work.

However, other such identities have associated partition functions that cannot directly be included in such earlier work. On the other hand, Bringmann and Ono's formulæ can be used to derive all such partition functions, but a direct application of the circle method in the tradition of the earlier researchers has never been done.

We are here interested in an identity between q-series whose Fourier coefficients have "escaped the net," so to speak, of Hua, Hagis, Niven, Lehner, and their kin. The identities have a partition–theoretic interpretation, named after the mathematicians Göllnitz and Gordon, who independently discovered them in 1961 [5] and 1965 [6], respectively, although Slater includes the q-series identities in her earlier work [24, pp. 155, Equations (36), (34)] (and Ramanujan seems to have been aware of the q-series identities much earlier still [2, pp. 36–37, Entries (1.7.11), (1.7.12)]).

It should be noted that in 1972 Subramanyasastri claimed a result [26] using the classic circle method, which would encompass as a special case the Fourier coefficients calculated in this paper. However, he gives neither a full definition, nor a complete proof.

This class of q-series and partitions is especially interesting, in that the associated identities have a similar structure to those of the Rogers–Ramanujan identities. We provide both the q-series identities and the partition identities below.

Theorem 1.6 (First Ramanujan–Slater Identity). *Let $q \in \mathbb{C}$ with $|q| < 1$. Then*

$$\sum_{m=0}^{\infty} \frac{q^{m^2}(-q;q^2)_m}{(q^2;q^2)_m} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}. \tag{1.8}$$

As a partition identity:

Theorem 1.7 (First Göllnitz–Gordon Identity). *Given an integer n, the number of partitions of n in which parts are non-repeating and non-consecutive, and with any two even parts differing by at least 4, is equal to the number of partitions of n in which parts are congruent to $\pm 1, 4 \pmod 8$.*

Theorem 1.8 (Second Ramanujan–Slater Identity). *Let $q \in \mathbb{C}$ with $|q| < 1$. Then*

$$\sum_{m=0}^{\infty} \frac{q^{m^2+2m}(-q;q^2)_m}{(q^2;q^2)_m} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}. \tag{1.9}$$

As a partition identity:

Theorem 1.9 (Second Göllnitz–Gordon Identity). *Given an integer n, the number of partitions of n in which parts are non-repeating and non-consecutive, with any two* *even parts differing by at least 4, and with smallest part ≥ 3 is equal to the number of partitions of n in which parts are congruent to $\pm 3 \pmod 8$.*

Notice that the right-hand side of (1.8) serves as the generating function for the class of partitions in the first Göllnitz–Gordon identities with parts congruent to $4, \pm 1 \pmod 8$. A similar interpretation holds relating (1.9) to the second Göllnitz–Gordon identity. These generating functions turn out to be the easiest expressions from which to extract coefficients.

Definition 1.10. Fix $a = 1$ or 3. A Göllnitz–Gordon partition of type a is composed of even parts of the form $4 \pmod 8$, and odd parts of the form $\pm a \pmod 8$. Let $g_a(n)$ represent the number of type–a Göllnitz–Gordon partitions of n, and let $F_a(q)$ denote the generating function for $g_a(n)$, for any $q \in \mathbb{C}$ with $|q| < 1$. By convention, let $g_a(0) = 1$.

We can then define:

$$
\begin{aligned}
F_a(q) &= \sum_{k=0}^{\infty} g_a(k) q^k &(1.10)\\
&= \prod_{m=0}^{\infty} (1 - q^{8m+a})^{-1} (1 - q^{8m+4})^{-1} (1 - q^{8m+8-a})^{-1} &(1.11)\\
&= \frac{1}{(q^a;q^8)_\infty (q^4;q^8)_\infty (q^{8-a};q^8)_\infty}. &(1.12)
\end{aligned}
$$

We will extract the coefficient of q^n from $F_a(q)$ in a manner analogous to that described in the derivation of (1.1) above. We wish to evaluate the contour integral representation for $g_a(n)$ provided by the residue theorem of Cauchy (Chapter 2). At first sight, such an integral seems to be very much inaccessible to computation. However, we can take advantage of certain functional equations that $F_a(q)$ is subject to—in particular, the fact that $F_a(q)$ is essentially a modular form with respect to the the subgroup $H(8)$ of the modular group (Chapter 3). We can then approximate $F_a(q)$ as a more elementary function, more suitable to integration, for q near any point on the unit circle (Chapter 5). Such an approximation necessitates the presence of certain error terms in the integration, but through the use of certain exponential sum estimations (Chapter 4) we can force such terms to become arbitrarily small. We finish the integration with the use of modified Bessel functions (Chapter 6).

As we work towards the complete formula for $g_a(n)$, we will introduce each detailed step by sketching the analogous process for the derivation of $p(n)$, as demonstrated in Sills [23].

Once the integration is complete, we will then conduct numerical tests of our results using Mathematica (Chapter 7). As with the original work of Hardy and Ramanujan, our formulæ are remarkably efficient, and converge to the correct answer very quickly. The code for our Mathematica calculations is included in Appendix B.

As an interesting consequence of our formulæ, we shall parallel an approach Lehner developed [12] with the formulæ associated with the Rogers–Ramanujan identities, by examining the asymptotic behavior of $g_a(n)$, and especially of $\frac{g_1(n)}{g_3(n)}$ as $n \to \infty$ (Appendix A). Such asymptotics bear a surprising connection with the silver ratio, analogous to the relationship between the Rogers–Ramanujan partition functions and the golden ratio.

CHAPTER 2

CONTOUR INTEGRAL REPRESENTATION

In this chapter we will work on setting up the necessary machinery of complex analysis in order to apply the circle method. To begin, we will see how the residue theorem can be used to find $g_a(n)$ through a certain integration over $\mathbb{C}$. We will also develop an appropriate contour over which we may take our integral. We note that by interchanging $F_a(q)$ with $F(q)$, we may utilize the same manipulations to develop a theoretical formula for $p(n)$ [23]. Moreover, the contour developed by Rademacher, discussed in Section 2.2, is identical to that used in the derivation of $p(n)$. We may easily adapt this machinery to develop similar formulæ for a large variety of restricted partition functions.

2.1 Cauchy's Integral Theorem

We begin with a classical theorem of complex analysis, due to Cauchy [25, Chapter 3]. Recall that a holomorphic function $f(q)$ with a pole of order n at $q = w$ has an expansion of the form

$$f(q) = \frac{A_{-n}}{(q-w)^n} + \frac{A_{-n+1}}{(q-w)^{n-1}} + ... + \frac{A_{-1}}{q-w} + G(q) \tag{2.1}$$

with $A_k \in \mathbb{C}$ for all k, and $G(q)$ a holomorphic function in some neighborhood of w.

Theorem 2.1. *Let f be a holomorphic function in an open set containing a closed contour $\mathcal{C}$ and its interior, except for a pole at w. Then*

$$A_{-1} = \frac{1}{2\pi i} \oint_{\mathcal{C}} f(q)dq. \tag{2.2}$$

This formula gives us a means of calculating—at least in principle—the value of $g_a(n)$. Hereafter, n will represent an arbitrary but fixed positive integer. If we divide the generating function of $g_a(n)$ by q^{n+1}, we have

$$\frac{F_a(q)}{q^{n+1}} = \frac{1}{q^{n+1}} + \frac{g_a(1)}{q^n} + \frac{g_a(2)}{q^{n-1}} + ... + \frac{g_a(n)}{q} + g_a(n+1) + g_a(n+2)q + ... \tag{2.3}$$

$$= \frac{1}{q^{n+1}} + \frac{g_a(1)}{q^n} + \frac{g_a(2)}{q^{n-1}} + ... + \frac{g_a(n)}{q} + G(q), \tag{2.4}$$

with $G(q)$ a holomorphic function.

Now $F_a(q)$ is holomorphic on the interior of the unit circle, so $\frac{F_a(q)}{q^{n+1}}$ is holomorphic in the unit circle, except for $q = 0$.

Theorem 2.2.

$$g_a(n) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{F_a(q)}{q^{n+1}} dq. \tag{2.5}$$

Of course, 2.5 is not immediately helpful. Surely,

$$\frac{F_a(q)}{q^{n+1}} = \frac{1}{q^{n+1}} \prod_{m=0}^{\infty} (1 - q^{8m+a})^{-1}(1 - q^{8m+4})^{-1}(1 - q^{8m+8-a})^{-1} \tag{2.6}$$

is anything but an easy function to integrate. In particular, we note that $F_a(q)$ becomes undefined if q is anywhere on the unit circle. This is most apparent if q is a non-octic root of unity, i.e. $q = e^{2\pi ih/k}, 8 \nmid k$.

Moreover, the most common factors of $F_a(q)$ contain singularities in which k is small. For example, the factor $(1-q)$ divides every factor in (2.6), while $(1-q^2)$ is a less common factor, and $(1-q^4)$ is rarer still. This implies that the singularity at the root of unity 1 somehow holds more information than the singularity at -1, which holds more information than those at $\pm i$, etc.

This strange behavior at roots of unity is far from a disadvantage. As q approaches $e^{2\pi ih/k}$, $F_a(q)$ can be approximated as a finite product of much more elementary functions. Indeed, understanding the behavior of $F_a(q)$ near lower-order singularities will provide information for the bulk of the formula for $g_a(n)$, while higher-order singularities will provide more detailed adjustments.

Before we consider the behavior of F_a in the limit as $q \to e^{2\pi i h/k}$, we will first consider the closed curve $\mathcal{C}$ that our contour integral will be taken over.

2.2 Rademacher's Contour

Anticipating that the behavior of $F_a(q)$ is most important near the roots of unity with relatively small degree, we will now find an appropriate contour to integrate over.

Hardy and Ramanujan's initial work involved simply integrating over a circle centered at 0, with radius just less than 1 [8]. However, Rademacher recognized that the necessary integration implied in Section 2.1 could be made much easier by constructing a contour more suitable to manipulation near any given root of unity [17]. We will outline his construction of such a contour in this section, in a form slightly modified by Andrew Sills [23]. While the motivation for so intricate a contour is not obvious, later steps in our calculation will favor it to the more straightforward curves.

Recall the classic Farey sequence of degree N:

Definition 2.3. Let $N \in \mathbb{N}$. The Farey sequence of degree N, $\mathcal{F}_N$, is the set of all rational fractions h/k such that $0 \leq h < k \leq N$, with $h = 0$ only when $k = 1$. For our purposes, we will order $\mathcal{F}_N$ by $(\leq)$.

The Farey sequence of degree 5, for example, is

$$\mathcal{F}_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \right\}. \tag{2.7}$$

An excellent introduction to the Farey sequence can be found in Hardy and Wright [9, Chapter 3].

Definition 2.4. For a given $h/k \in \mathcal{F}_N$, define the Ford circle $C(h,k)$ as the curve given by

$$\left| \tau - \left(\frac{h}{k} + \frac{i}{2k^2} \right) \right| = \frac{1}{2k^2}. \tag{2.8}$$

Given the set of Ford circles corresponding to the Farey sequence of degree N, let $\gamma(h,k)$ be defined as the upper arc of $C(h,k)$ from

$$\tau_I(h,k) = \frac{h}{k} - \frac{k_p}{k(k^2+k_p^2)} + \frac{1}{k^2+k_p^2} i$$

to

$$\tau_T(h,k) = \frac{h}{k} + \frac{k_s}{k(k^2+k_s^2)} + \frac{1}{k^2+k_s^2} i,$$

with h_p/k_p and h_s/k_s the immediate predecessor and successor (respectively) of $h/k \in \mathcal{F}_N$ (let $0_p/1_p = (N-1)/N$; similarly, let $(N-1)_s/N_s = 0/1$).

It is a trivial geometrical exercise [3, Chapter 5] to verify the following lemma:

Lemma 2.5. *Any two Ford circles in $\mathcal{F}_N$ are either tangent to one another, or disjoint.*

Definition 2.6. The Rademacher path of order N, $P(N)$, is the union of all upper arcs $\gamma(h,k)$ from $\tau = i$ to $\tau = i+1$:

$$P(N) = \bigcup_{h/k \in \mathcal{F}_N} \gamma(h,k). \tag{2.9}$$

Since every Ford circle is almost entirely in the upper half of the τ plane $\mathbb{H}$ (with a single point tangent to $\mathbb{R}$), and the points within the upper arcs $\gamma(h,k)$ have nonzero imaginary part, therefore $\gamma(h,k)$ lies entirely in $\mathbb{H}$ for every $h/k \in \mathcal{F}_N$, and $P(N)$ is a curve that lies entirely in $\mathbb{H}$. See Figure 2.1.

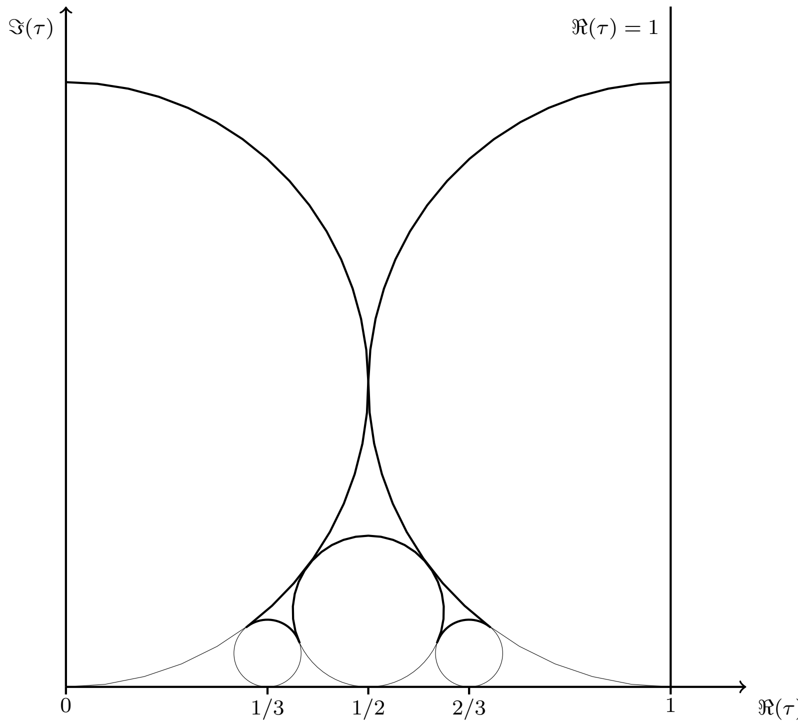


Figure 2.1: Ford circles $C(h,k)$ for $h/k \in \mathcal{F}_3$, with $P(3)$ highlighted.

So if we define $q = e^{2\pi i \tau}$, then the corresponding preimage of $P(N)$ remains inside of the unit circle, enclosing the origin. We may justifiably define $\mathcal{C} = P(N)$ for any $N \in \mathbb{N}$.

We can now give a precise definition of what it means for part of our contour to be "near" a root of unity: q is close to $e^{2\pi i h/k}$ if q lies on the preimage of $\gamma(h,k)$.

More specifically, we will make one more change of variables, so that on $\gamma(h,k)$ we have

$$\tau = \frac{h}{k} + \frac{iz}{k}, \tag{2.10}$$

with $\Re(z) > 0$. Going from the τ plane to the z plane, the corresponding Ford circle (with the upper arc $\gamma(h,k)$) is mapped to the circle

$$K_k^{(-)} : \left| z - \frac{1}{2k} \right| = \frac{1}{2k}. \tag{2.11}$$

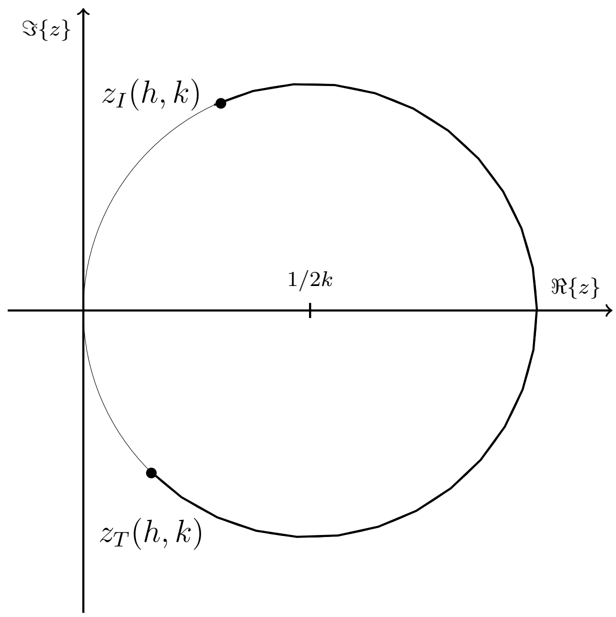


Figure 2.2: $K_k^{(-)}$ with the highlighted path from $z_I(h,k)$ to $z_T(h,k)$. Notice that the $(-)$ indicates that the path is taken clockwise.

Notice that $K_k^{(-)}$ itself has no dependence on h. Rather, the information regarding the fraction h/k is contained wholly in the initial and terminal points of the map of $\gamma(h,k)$. Indeed, we have

$$\tau_I(h,k) \mapsto z_I(h,k) = \frac{k}{k^2 + k_p^2} + \frac{k_p}{k^2 + k_p^2} i, \tag{2.12}$$

$$\tau_T(h,k) \mapsto z_T(h,k) = \frac{k}{k^2 + k_s^2} - \frac{k_s}{k^2 + k_s^2} i. \tag{2.13}$$

We end this chapter with one simple but extremely important geometrical lemma.

Lemma 2.7. *Let $N \in \mathbb{N}$ be given, and $h/k \in \mathcal{F}_N$ and let $z_I(h,k)$, $z_T(h,k)$ be the images of $\tau_I(h,k)$, $\tau_T(h,k)$, respectively, from $C(h,k)$ to $K_k^{(-)}$. Then for any z on the chord connecting $z_I(h,k)$ to $z_T(h,k)$, we have*

$$|z| = O\left(N^{-1}\right). \tag{2.14}$$

Proof. Let z lie on the chord of $K_k^{(-)}$ connecting $z_I(h,k)$ to $z_T(h,k)$. Then we have

$$|z| \leq \max\{|z_I(h,k)|, |z_T(h,k)|\} \tag{2.15}$$

$$\leq \max\left\{\sqrt{\frac{1}{k^2+k_p^2}}, \sqrt{\frac{1}{k^2+k_s^2}}\right\}. \tag{2.16}$$

We also have:

$$2(k^2+k_p^2) = k^2+k_p^2+(k^2+k_p^2) \tag{2.17}$$

$$\geq k^2+k_p^2+2kk_p \tag{2.18}$$

$$= (k+k_p)^2. \tag{2.19}$$

Finally, as a natural consequence of the Farey sequence of order N [9],[18], for any two consequtive fractions $h_p/k_p, h/k \in \mathcal{F}_N$, we have

$$k + k_p \geq N. \tag{2.20}$$

Therefore, we have

$$2(k^2+k_p^2) \geq N^2, \tag{2.21}$$

and,

$$\sqrt{\frac{1}{k^2+k_p^2}} \leq \frac{\sqrt{2}}{N} = O\left(N^{-1}\right). \tag{2.22}$$

Of course, an identical relationship holds for $\sqrt{\frac{1}{k^2+k_s^2}}$ by the same reasoning.

□

CHAPTER 3

TRANSFORMATION EQUATIONS

3.1 Introduction

3.1.1 Modular Forms

Before we take a close look at F_a, it is necessary to introduce some useful functions and their corresponding properties. Henceforth, let $q = \exp(2\pi i\tau)$.

The theory of modular forms, or functions that are nearly modular, is at times immensely technical. Even the precise definition of the term "modular form" can vary depending on the reference given, or even depending on the sections of a given reference (See, for example, the discussion in Chapter 6 of Apostol [3]).

For our purposes, we define modularity of a function as follows:

Definition 3.1. A function $\phi(\tau)$ is *modular* if it is holomorphic for $\tau \in \mathbb{H}$ and

$$\phi\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \omega(a, b, c, d)\phi(\tau), \tag{3.1}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ with $c > 0$. Here k is usually referred to as the *weight* of ϕ, and ω is a root of unity depending on a, b, c, d.

We will be interested in functions that satisfy a similar transformation equation to (3.1), except that the right hand side will sometimes include additional exponential factors, which may depend in part on τ. For this reason, the functions we work with may be thought of as "nearly" modular.

The appeal of working with functions that are modular or nearly modular is that, while ϕ itself may be an intimidating mathematical function, $(c\tau + d)^k\omega$ is a string of elementary factors. A common powerful technique to extract information from ϕ (such as Fourier coefficients) is to modify the argument of ϕ by a transformation in $SL(2,\mathbb{Z})$. By adjusting the argument, we may be able to force ϕ to approach 1, transferring all relevant information about ϕ to the elementary factor $(c\tau+d)^k\omega$, which is almost always easier to study than ϕ itself.

We will be especially interested in functions of weight 0 which are modular with respect to certain specific subgroups of $SL(2,\mathbb{Z})$, if not with $SL(2,\mathbb{Z})$ itself. We say that such a function is modular with respect to the associated subgroup of $SL(2,\mathbb{Z})$.

We will find that our generating function, with a given exponential factor, is exactly modular with respect to the subgroup $H(8)$, defined in Section 3.4.

More precise definitions are given in Apostol [3, Chapter 6] and Rademacher [18, Chapters 8, 9, 15]. We give an important example of an unambiguously modular form in the Dedekind eta function:

Definition 3.2. Let $\tau \in \mathbb{H}$. Then we define Dedekind's eta function by the following:

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{m=1}^{\infty}(1-e^{2\pi i m\tau}) \tag{3.2}$$

$$= q^{1/24}(q;q)_\infty. \tag{3.3}$$

The unrestricted partition function may be developed by manipulating this function alone. We give the relevant transformation equation for η here:

Theorem 3.3. *Let* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ *with* $c > 0$. *Then for* $\tau \in \mathbb{H}$ *we have*

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a,b,c,d)\sqrt{\frac{c\tau+d}{i}}\,\eta(\tau), \tag{3.4}$$

with

$$\epsilon(a,b,c,d) = \begin{cases} \left(\frac{d}{c}\right) i^{(1-c)/2}\exp\big((\pi i/12)(bd(1-c^2)+c(a+d))\big), & 2\nmid c \\ \left(\frac{c}{d}\right)\exp\big((\pi i d/4)+(\pi i/12)(ac(1-d^2)+d(b-c))\big), & 2\nmid d \end{cases}, \tag{3.5}$$

and $\left(\frac{m}{n}\right)$ *the Legendre–Jacobi character.*

We emphasize that we are interested in the principal values of the square roots from the above theorem forward.

We also note that Rademacher discovered a more succinct representation of $\epsilon(a,b,c,d)$ [17], in the form of

$$\epsilon(a,b,c,d) = \exp\left(\pi i \left(\frac{a+d}{12c} + s(-d,c)\right)\right), \tag{3.6}$$

with

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right). \tag{3.7}$$

We will make primary use of (3.5) for our theoretical calculations, especially regarding the estimations in Chapter 4. However, (3.6), (3.7) is also a useful expression, especially in matters of computation, being easier to input into a computer than (3.5). We return to this matter in Appendix B.

It may be shown that the roots of unity $\omega(h,k)$ alluded to in (1.2) have the form $\omega(h,k) = \exp(\pi i s(h,k))$.

Our problem will also necessitate the study of additional, more intricate forms. In particular, we need Jacobi's theta functions.

Definition 3.4. Let $v \in \mathbb{C}$ and $\tau \in \mathbb{H}$. Then for our purposes, we define Jacobi's first theta function as

$$\vartheta_1(v|\tau) = 2e^{\pi i\tau/4}\sin(\pi v)\prod_{m=1}^{\infty}(1-e^{2\pi i m\tau})(1-e^{2\pi i m\tau+2\pi i v})(1-e^{2\pi i m\tau-2\pi i v}) \tag{3.8}$$

$$= (q;q)_\infty (e^{2\pi i v}q;q)_\infty (e^{-2\pi i v}q;q)_\infty, \tag{3.9}$$

and Jacobi's Fourth Theta Function as

$$\vartheta_4(v|\tau) = \prod_{m=1}^{\infty}(1 - e^{2\pi i m\tau})(1 - e^{\pi i\tau(2m-1)+2\pi i v})(1 - e^{\pi i\tau(2m-1)-2\pi i v}) \tag{3.10}$$

$$=(q;q)_\infty(e^{\pi i(2v-\tau)}q;q)_\infty(e^{-\pi i(2v+\tau)}q;q)_\infty. \tag{3.11}$$

Now ϑ_1 is usually referred to as a Jacobi form, as it satisfies a slightly more intricate transformation equation than that of η. We may still take advantage of the transformation equation, albeit with slightly more trouble than with η:

Theorem 3.5. *Let* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ *with* $c > 0$. *Then for* $v \in \mathbb{C}$ *and* $\tau \in \mathbb{H}$ *we have*

$$\vartheta_1\left(\frac{v}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) = -i\epsilon(a,b,c,d)^3\sqrt{\frac{c\tau+d}{i}}e^{\pi i c v^2/(c\tau+d)}\vartheta_1(v|\tau), \tag{3.12}$$

with $\epsilon(a,b,c,d)$ *defined as in Theorem 3.3.*

Finally, we give an expression for Ramanujan's theta function, which is closely related to Jacobi's functions.

Definition 3.6. We define Ramanujan's theta function as

$$f(a,b) = \prod_{m=1}^{\infty}(1 + a^m b^{m-1})(1 + a^{m-1}b^m)(1 - a^m b^m) \tag{3.13}$$

$$=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{3.14}$$

As is hinted by the notation, there are four theta functions associated with Jacobi's name (as expressed in [18, Chapter 10]). In fact, the functions ϑ_u for $u = 1,2,3,4$, along with Ramanujan's theta function, are usually defined in terms of certain infinite series. We give the functions as infinite products to emphasize their connection with F_a. Such product representations are justified through Jacobi's Triple Product Identity [18, Chapter 12].

These functions will become especially useful for our manipulation of F_a itself, which turns out to be a certain quotient of eta and theta functions. In particular, the transformation equations are of immense value to us. We will also require additional information on the behavior of ϑ_1, as well as its connections to ϑ_4 and f.

Lemma 3.7. *Let $v \in \mathbb{C}$, $\tau \in \mathbb{H}$, $q = e^{2\pi i\tau}$, and $N \in \mathbb{N}$. Then the following apply:*

1. $\vartheta_1(v+1|\tau) = -\vartheta_1(v|\tau)$

2. $\vartheta_1(v+N\tau|\tau) = (-1)^N \exp\left(-\pi i N(2v+N\tau)\right)\vartheta_1(v|\tau)$

3. $\vartheta_1\left(v+\frac{\tau}{2}\middle|\tau\right) = i\exp\left(\frac{-\pi i}{4}(4v+\tau)\right)\vartheta_4(v|\tau)$

4. $f(-q^\alpha, -q^\beta) = -ie^{\frac{\pi i\tau}{4}(3\alpha-\beta)}\vartheta_1(\alpha\tau|(\alpha+\beta)\tau)$.

See Rademacher [18, Chapters 9, 10] for a proof of these properties, as well as the transformation formulæ in Theorems 3.3, 3.5.

3.1.2 Prelude: Transforming $F(q)$

We pause to discuss the use of $\eta(\tau)$ for the unrestricted partition function. The process sketched here is taken from Sills [23].

Letting $q = \exp(2\pi i\tau)$, the right-hand side of (1.3) becomes

$$F(q) = e^{-\pi i\tau/12}\frac{1}{\eta(\tau)}.$$

With the machinery of Chapter 2, we have

$$p(n) = \frac{1}{2\pi i}\oint\limits_{\mathcal{C}} \frac{F(q)}{q^{n+1}}dq = \int\limits_{P(N)} F\left(\exp(2\pi i\tau)\right)\exp(-2\pi i n\tau)d\tau,$$

for some large natural number N.

We we begin by examining this integral near a specific root of unity. Let $\tau = (h+iz)/k$, with $0 \le h < k$ and $(h,k) = 1$. Taking advantage of Theorem 3.3, we can use the following matrix from $SL(2,\mathbb{Z})$:

$$\begin{pmatrix} h & -\frac{1}{k}(hH+1) \\ k & -H \end{pmatrix},$$

where H is the inverse of $h \pmod{k}$.

Using this matrix, we may modify our expression of $F(q)$ by the following:

$$F(q) = \omega(h,k)e^{\pi(z^{-1}-z)/12k}\sqrt{z}F\left(\exp(2\pi i(iz^{-1}+H)/k)\right).$$

Because we are interested in the behavior of $F(q)$ near the roots of unity, and $q = \exp(2\pi i(h+iz)/k)$, it is important to know the behavior of $F(q)$ with z close to 0. Our matrix has transformed q to the variable $\exp(2\pi i(iz^{-1}+H)/k)$, which approaches 0 as $z \to 0$, forcing $F\left(\exp(2\pi i(iz^{-1}+H)/k)\right)$ to approach 1.

This means that all of the relevant information of $F(q)$ near the root of unity associated with h/k is transferred to the string of elementary factors

$$\omega(h,k)e^{\pi(z^{-1}-z)/12k}\sqrt{z},$$

and the remaining problem lies in estimating our original integral near $e^{2\pi ih/k}$ in terms of the much simpler integral

$$\frac{i}{k}e^{-2\pi inh/k}\omega(h,k)\int_{z_I(h,k)}^{z_T(h,k)} e^{2n\pi z/k}e^{\pi(z^{-1}-z)/12k}\sqrt{z}\,dz.$$

Taking such a contribution for every $h/k \in \mathcal{F}(N)$, we may then estimate $p(n)$ as a series of such integrals.

3.1.3 Transformations on $F_a(q)$

We will parallel the work outlined in 3.1.2, but with $F_a(q)$ rather than with $F(q)$. Because of the more intricate form of $F_a(q)$, we will have to use the Jacobi theta functions as well as the eta function.

We start with Ramanujan's theta function, which we can express with powers of q in the following way:

$$f(-q^\alpha, -q^\beta) = (q^\alpha; q^{\alpha+\beta})_\infty (q^\beta; q^{\alpha+\beta})_\infty (q^{\alpha+\beta}; q^{\alpha+\beta})_\infty \tag{3.15}$$

We then have

$$F_a(q) = \frac{1}{(q^a; q^8)_\infty (q^4; q^8)_\infty (q^{8-a}; q^8)_\infty} = \frac{(q^8; q^8)_\infty^2}{(q^4; q^4)_\infty f(-q^a, -q^{8-a})}. \tag{3.16}$$

Moreover, by Lemma 3.6.4,

$$f(-q^\alpha, -q^\beta) = -ie^{\frac{\pi i \tau}{4}(3\alpha-\beta)} \vartheta_1(\alpha\tau | (\alpha+\beta)\tau). \tag{3.17}$$

Also, by (3.3),

$$(q^\alpha; q^\alpha)_\infty = e^{-\frac{\alpha \pi i \tau}{12}} \eta(\alpha\tau). \tag{3.18}$$

Therefore, we can view F_a as a quotient of eta and theta functions:

Theorem 3.8. *Let $q = e^{2\pi i \tau}$, with $\tau \in \mathbb{H}$. Then*

$$F_a(q) = i \exp\left(\pi i \tau (1-a)\right) \frac{\eta(8\tau)^2}{\eta(4\tau)\vartheta_1(a\tau|8\tau)}. \tag{3.19}$$

To recognize the importance of expressing F_a in this form, we will consider the transformation equations on η and ϑ_1. Let N be some large positive integer, $\tau = \frac{h}{k} + \frac{iz}{k}$, with $0 < h < k$, $(h,k) = 1$, $k < N$, and $\Re(z) > 0$, as defined in Chapter 2. The value of h/k will depend on what singularity that our q is near, i.e. what upper arc $\gamma(h,k)$ that τ lies on.

On each upper arc, we may exploit the transformation properties of η, ϑ_1. The object is do express

$$F_a(q) = \Lambda(z)\Psi(y), \tag{3.20}$$

with $\Lambda(z)$ some elementary function in z, $\Psi(y)$ a modified quotient of eta and theta functions, and $y = e^{2\pi i\tau'}$, where τ' is related to τ by a specific transformation over the modular group.

There is one final complication that we must consider. Because the restrictions imposed on Göllnitz–Gordon partitions involve arithmetic progressions $\pmod 8$, the relation of k to 8 will carry some significance. Indeed, the most useful modular transformation to apply to τ will depend on the divisibility properties of k with respect to 8. We must therefore consider four different cases, depending on whether $(k,8) = 8, 4, 2, 1$.

3.2 Transformations

3.2.1 $GCD(k,8) = 8$

The simplest transformation formula relevant to our problem occurs for $(k,8) = 8$. Let H be defined as the negative inverse of h modulo $16k$:

$$hH \equiv -1 \pmod{16k}. \tag{3.21}$$

Notice that since $8|k$ by hypothesis, and $(h,k) = 1$, therefore $(h,16k) = (h,k) = 1$, so that H exists. Then the following are elements of $SL(2,\mathbb{Z})$:

$$\begin{pmatrix} h & -\frac{8}{k}(hH+1) \\ \frac{k}{8} & -H \end{pmatrix}, \tag{3.22}$$

$$\begin{pmatrix} h & -\frac{4}{k}(hH+1) \\ \frac{k}{4} & -H \end{pmatrix}, \tag{3.23}$$

We will allow

$$\tau' = \frac{H}{k} + \frac{iz^{-1}}{k}. \tag{3.24}$$

Applying (3.22) to $8\tau'$, we have

$$\frac{8h\tau' - \frac{8}{k}(hH+1)}{8\frac{k}{8}\tau' - H} = 8\tau.$$

Similarly, applying (3.23) to $4\tau'$, we have

$$\frac{4h\tau' - \frac{4}{k}(hH+1)}{4\frac{k}{4}\tau' - H} = 4\tau. \tag{3.25}$$

Therefore, we will transform $\eta(8\tau)$ to $\eta(8\tau')$, using (3.22). Similarly, we transform $\eta(4\tau)$ to $\eta(4\tau')$ using (3.23).

Invoking these transformations, we must contend with the roots of unity associated with (3.5). As a shorthand, we will refer to the roots of unity as the following:

$$\epsilon(8,8) = \epsilon\left(h, \frac{-8}{k}(hH+1), \frac{k}{8}, -H\right), \tag{3.26}$$

$$\epsilon(8,4) = \epsilon\left(h, \frac{-4}{k}(hH+1), \frac{k}{4}, -H\right). \tag{3.27}$$

This allows that

$$\frac{\eta(8\tau)^2}{\eta(4\tau)} = \frac{1}{z^{1/2}}\frac{\epsilon(8,8)^2}{\epsilon(8,4)}\frac{\eta(8\tau')^2}{\eta(4\tau')}. \tag{3.28}$$

Handling ϑ_1 turns out to be more difficult, due to the presence of a second complex variable. We will mimic our work with $\eta(8\tau)$, using (3.22), and setting

$$v = a\tau iz^{-1} = \frac{a(hiz^{-1}-1)}{k}. \tag{3.29}$$

This gives us

$$\vartheta_1(a\tau|8\tau) = \vartheta_1\left(\frac{v}{iz^{-1}}\middle|8\tau\right) \tag{3.30}$$

$$= -i\epsilon(8,8)^3\frac{1}{z^{1/2}}e^{z\pi a^2(hiz^{-1}-1)^2/8k}\vartheta_1\left(v|8\tau'\right). \tag{3.31}$$

While τ is successfully transformed, it is essential to interpret (3.29) properly. Recall that $hH \equiv -1 \pmod{16k}$. We can therefore write

$$-1 = hH + 16kM, \tag{3.32}$$

with $M \in \mathbb{Z}$. We then have

$$v = ah\tau' + 16aM. \tag{3.33}$$

If we also take advantage of Lemma 3.6.1, then with (3.33) we have

$$\vartheta_1\left(v|8\tau'\right) = \vartheta_1\left(ah\tau' + 16aM|8\tau'\right) = \vartheta_1\left(ah\tau'|8\tau'\right). \tag{3.34}$$

This is definitely an improvement, but we may go further still. Again considering that $(k,8) = 8$ and $(h,k) = 1$, and $a = 1,3$, we also have $ah \equiv 1,3,5,7 \pmod 8$. We may therefore write

$$\vartheta_1(ah\tau'|8\tau') = \vartheta_1(b\tau' + 8N\tau'|8\tau'), \tag{3.35}$$

with b the least positive residue of ah modulo 8.

We now make use of Lemma 3.6.2, and attain

$$\vartheta_1(ah\tau'|8\tau') = (-1)^N \exp(-\pi i N(2b\tau' + 8N\tau'))\vartheta_1(b\tau'|8\tau'). \tag{3.36}$$

Combining (3.31), (3.34), (3.36), and inverting, we have the following:

$$\frac{1}{\vartheta_1(a\tau|8\tau)} = i\frac{(-1)^N}{\epsilon(8,8)^3} z^{1/2} e^{-z\pi a^2(hiz^{-1}-1)^2/8k} \exp(\pi i N(2b\tau'+8N\tau'))\frac{1}{\vartheta_1(b\tau'|8\tau')}. \tag{3.37}$$

We now have sufficient information, in (3.28), (3.37), to reassemble the transformed generating function.

$$F_a(q) = i\exp(\pi i\tau(1-a))\frac{\eta(8\tau)^2}{\eta(4\tau)\vartheta_1(a\tau|8\tau)} \tag{3.38}$$

$$= i\exp(\pi i\tau(1-a))\frac{1}{z^{1/2}}\frac{\epsilon(8,8)^2}{\epsilon(8,4)}\frac{\eta(8\tau')^2}{\eta(4\tau')}\frac{1}{\vartheta_1(a\tau|8\tau)} \tag{3.39}$$

$$\begin{aligned} &= \exp(\pi i\tau(1-a))\frac{1}{\epsilon(8,4)\epsilon(8,8)} e^{-z\pi a^2(hiz^{-1}-1)^2/8k}(-1)^{N-1} \\ &\quad \times \exp(\pi i N(2b\tau' + 8N\tau'))\frac{\eta(8\tau')^2}{\eta(4\tau')\vartheta_1(b\tau'|8\tau')}. \end{aligned} \tag{3.40}$$

Here $b = 1, 3, 5, 7$. However, noting from (3.17) that

$$f(-q^\alpha, -q^\beta) = f(-q^\beta, -q^\alpha), \tag{3.41}$$

we may define $F_5(q) = F_3(q), F_7(q) = F_1(q)$. We therefore have

$$\begin{aligned} F_a(q) = \frac{i(-1)^N}{\epsilon(8,8)\epsilon(8,4)} \times \exp\Big(&\pi i\tau(1-a) - z\pi a^2(hiz^{-1}-1)^2/8k \\ &+ \pi i N(2b\tau' + 8N\tau') + \pi i\tau'(a-1)\Big) F_b(y), \end{aligned} \tag{3.42}$$

with $y = \exp(2\pi i \tau')$.

Here, referring to Section 3.1, we have $F_a(q) = \Lambda(z)\Psi_8(y)$ with $\Psi_8(y) = F_b(y)$, and $\Lambda(z)$ an exponential polynomial in $1/z$. Remembering that

$$N = \left\lfloor \frac{ah}{8} \right\rfloor = \frac{ah-b}{8}, \tag{3.43}$$

and that

$$a^2 - 4a + 3 = (a-1)(a-3) = 0, \tag{3.44}$$

we may collect and reorganize the coefficients of 1, z, and $1/z$ in the exponential of (3.42). Doing so gives the following transformation formula:

$$F_a(q) = \omega_8(h,k) \exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8}{z} + z(4a-5)\right)\right) F_b(y), \tag{3.45}$$

where

$$\omega_8(h,k) = \frac{i(-1)^{\lfloor \frac{ah}{8} \rfloor}}{\epsilon(8,8)\epsilon(8,4)} \exp\left(\frac{\pi i}{8k}(h(5-4a) - H((b-4)^2-8))\right). \tag{3.46}$$

It should be noted that work by Iseki [11] strongly suggests that $\omega_8(h,k)$ can be written in a more reduced form, making use of certain Dedekind and semi-Dedekind sums. We shall not pursue the matter here.

3.2.2 $GCD(k,8) = 4$

The result of the previous section emphasises that $F_a(q)$ is modular, at least with respect to a subgroup of the modular group, and up to a shift of a. While such a property does not carry over exactly to the remaining 3 cases, it is only necessary that $F_a(q) = f(z)\Psi(y)$, with $\Psi(y)$ a suitable quotient of eta and theta functions.

An important consideration for the following cases is that we will have to be more restrictive over the possible values of h and H. In the case of $(k, 8) = 4$, for example, we could define H simply as the negative inverse of h again. However, since $k/4$ is odd, it is possible (and more helpful) to let $2H$ be the negative inverse of h:

$$2hH \equiv -1 \pmod{k/4}. \tag{3.47}$$

Then the following are elements of $SL(2, \mathbb{Z})$:

$$\begin{pmatrix} 2h & -\frac{4}{k}(2hH+1) \\ \frac{k}{4} & -H \end{pmatrix}, \tag{3.48}$$

$$\begin{pmatrix} h & -\frac{4}{k}(2hH+1) \\ \frac{k}{4} & -2H \end{pmatrix}. \tag{3.49}$$

We will allow

$$\tau' = \frac{H}{k} + \frac{iz^{-1}}{2k}. \tag{3.50}$$

Applying (3.48) to $4\tau'$, we have

$$\frac{8h\tau' - \frac{4}{k}(2hH+1)}{4\frac{k}{4}\tau' - H} = 8\tau. \tag{3.51}$$

Similarly, applying (3.49) to $8\tau'$, we have

$$\frac{8h\tau' - \frac{4}{k}(2hH+1)}{8\frac{k}{4}\tau' - 2H} = 4\tau. \tag{3.52}$$

Therefore, we will have

$$\frac{\eta(8\tau)^2}{\eta(4\tau)} = \frac{1}{2z^{1/2}} \frac{\epsilon(4,8)^2}{\epsilon(4,4)} \frac{\eta(4\tau')^2}{\eta(8\tau')}, \tag{3.53}$$

with

$$\epsilon(4,8) = \epsilon\left(2h, \frac{-4}{k}(2hH+1), \frac{k}{4}, -H\right), \tag{3.54}$$

$$\epsilon(4,4) = \epsilon\left(h, \frac{-4}{k}(2hH+1), \frac{k}{4}, -2H\right). \tag{3.55}$$

Once again, ϑ_1 requires the most work by far. The initial transformation through (3.48) gives us

$$\vartheta_1(a\tau|8\tau) = \vartheta_1\left(\frac{v}{iz^{-1}/2}\middle|8\tau\right) \tag{3.56}$$

$$= -i\epsilon(4,8)^3 \frac{1}{(2z)^{1/2}} e^{z\pi a^2(hiz^{-1}-1)^2/8k} \vartheta_1(v|4\tau'), \tag{3.57}$$

with

$$v = \frac{a(hiz^{-1}-1)}{2k}. \tag{3.58}$$

Now remembering that $2hH \equiv -1 \pmod{k/4}$, we let

$$-1 = 2hH + Mk/4, \tag{3.59}$$

with $M \in \mathbb{Z}$, and rewrite

$$v = \frac{a(hiz^{-1}-1)}{2k} = \frac{a}{2k}(hiz^{-1} + 2hH + Mk/4) = ah\tau' + \frac{aM}{8}. \tag{3.60}$$

This gives us

$$\vartheta_1(v|4\tau') = \vartheta_1\left(ah\tau' + \frac{aM}{8}\middle|4\tau'\right). \tag{3.61}$$

We may now allow (as before) $b \equiv ah \pmod 4$, letting $ah = 4N + b$, so that (3.61), together with Lemma 3.6.2, gives

$$\vartheta_1\left(v|4\tau'\right) = (-1)^N \exp(-\pi i N(2\tau'(2N+b)+aM/4)) \times \vartheta_1\left(b\tau' + \frac{aM}{8}\middle|4\tau'\right). \quad (3.62)$$

To this point, our work has paralleled the case for $(k, 8) = 8$. However, the first argument of ϑ_1 contains a second term in the form of $aM/8$. Since $2hH+1$ is odd, $aM/8$ is no integer, so we may not easily filter this term out.

We now shift from ϑ_1 to ϑ_4, by the identity from Lemma 3.6.3:

$$\vartheta_1(v|\tau) = i\exp(-\pi i\tau/4 - \pi i v)\vartheta_4(v|\tau).$$

Write

$$\begin{aligned}\vartheta_1\left(b\tau' + \frac{aM}{8}\middle|4\tau'\right) =& \vartheta_1\left((b-2)\tau' + \frac{aM}{8} + 2\tau'\middle|4\tau'\right) & (3.63)\\ =& i\exp(-\pi i((b-1)\tau' + aM/8)) & (3.64)\\ & \times \vartheta_4\left((b-2)\tau' + \frac{aM}{8}\middle|4\tau'\right). & (3.65)\end{aligned}$$

We now express ϑ_4 as an infinite product:

$$\begin{aligned}\vartheta_4\left((b-2)\tau' + \frac{aM}{8}\middle|4\tau'\right) &= \prod_{m=1}^{\infty}(1-y^{4m})(1-\rho_4 y^{4m-4+b})(1-\rho_4^{-1}y^{4m-b}) & (3.66)\\ &= (y^4;y^4)_\infty(\rho_4 y^b;y^4)_\infty(\rho_4^{-1}y^{4-b};y^4)_\infty, & (3.67)\end{aligned}$$

with

$$\rho_4 = \exp\left(\pi i a(2hH+1)/k\right), \quad (3.68)$$

Combining (3.53), (3.57), (3.62), (3.65), (3.67), we have

$$F_a(a) = i\exp(\pi i\tau(1-a))\frac{1}{2z^{1/2}}\frac{\epsilon(4,8)^2}{\epsilon(4,4)} \times i\frac{(2z)^{1/2}}{\epsilon(4,8)^3}$$
$$\times \exp(-z\pi a^2(hiz^{-1}-1)^2/8k)$$
$$\times (-1)^N \exp(\pi i N(2\tau'(2N+b)+aM/4))$$
$$\times -i\exp(\pi i((b-1)\tau' - aM/8)) \times \Psi_4(y), \quad (3.69)$$

with

$$\Psi_4(q) = \frac{\eta(4\tau')^2}{\eta(8\tau')\vartheta_4\left((b-2)\tau' + \frac{aM}{8}\middle|4\tau'\right)} \quad (3.70)$$
$$= \frac{(q^4;q^4)_\infty^2}{(q^8;q^8)_\infty f(-\rho_4 q^b; -\rho_4^{-1}q^{4-b})}. \quad (3.71)$$

Reducing (3.69), we have

$$F_a(q) = \frac{1}{\sqrt{2}}\frac{i(-1)^N}{\epsilon(4,8)\epsilon(4,4)}\exp\Big(\pi i\tau(1-a) - z\pi a^2(hiz^{-1}-1)^2/8k$$
$$+ \pi i N(2\tau'(2N+b)+aM/4) + \pi i((b-1)\tau' - aM/8)\Big)\Psi_4(y). \quad (3.72)$$

We must once again collect the coefficients of 1, z, and $1/z$ in the exponential of (3.72). In doing so, we have the following:

$$F_a(q) = \frac{1}{\sqrt{2}}\omega_4(h,k)\exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(4a-5)\right)\right)\Psi_4(y), \quad (3.73)$$

where we define $\Psi_4(y)$ by (3.71), and

$$\omega_4(h,k) = \frac{i(-1)^{\lfloor\frac{ah}{4}\rfloor}}{\epsilon(4,8)\epsilon(4,4)}$$
$$\times \exp\left(\frac{\pi i}{4k}(h - H - h(4a-3)(hH+1) + a(2hH+1)(b-2)\right). \quad (3.74)$$

3.2.3 $GCD(k,8)=2$

Let $4hH \equiv -1 \pmod{k/2}$ and consider the matrices

$$\begin{pmatrix} 4h & -\frac{2}{k}(4hH+1) \\ \frac{k}{2} & -H \end{pmatrix}, \tag{3.75}$$

$$\begin{pmatrix} 2h & -\frac{2}{k}(4hH+1) \\ \frac{k}{2} & -2H \end{pmatrix}. \tag{3.76}$$

Both matrices are elements of $SL(2,\mathbb{Z})$. We will allow

$$\tau' = \frac{H}{k} + \frac{iz^{-1}}{4k}. \tag{3.77}$$

Applying (3.75) to $2\tau'$, we have

$$\frac{8h\tau' - \frac{2}{k}(4hH+1)}{2\frac{k}{2}\tau' - H} = 8\tau. \tag{3.78}$$

Similarly, applying (3.76) to $4\tau'$, we have

$$\frac{8h\tau' - \frac{2}{k}(4hH+1)}{4\frac{k}{2}\tau' - 2H} = 4\tau. \tag{3.79}$$

Therefore, we will have

$$\frac{\eta(8\tau)^2}{\eta(4\tau)} = \frac{1}{2\sqrt{2}z^{1/2}} \frac{\epsilon(2,8)^2}{\epsilon(2,4)} \frac{\eta(2\tau')^2}{\eta(4\tau')}. \tag{3.80}$$

Once again, ϑ_1 requires the most work by far. The initial transformation through (3.75) gives us

$$\vartheta_1(a\tau|8\tau) = \vartheta_1\left(\frac{v}{iz^{-1}/4}\middle|8\tau\right) \tag{3.81}$$

$$= -i\epsilon(2,8)^3 \frac{1}{2z^{1/2}} e^{z\pi a^2(hiz^{-1}-1)^2/8k} \vartheta_1(v|2\tau'), \tag{3.82}$$

with

$$v = \frac{a(hiz^{-1} - 1)}{4k}. \tag{3.83}$$

Now remembering that $4hH \equiv -1 \pmod{k/2}$, we let

$$-1 = 4hH + Mk/2, \tag{3.84}$$

with $M \in \mathbb{Z}$, and rewrite

$$v = \frac{a(hiz^{-1} - 1)}{4k} = \frac{a}{4k}(hiz^{-1} + 4hH + Mk/2) = ah\tau' + \frac{aM}{8}. \tag{3.85}$$

This gives us

$$\vartheta_1\left(v|2\tau'\right) = \vartheta_1\left(ah\tau' + \frac{aM}{8}\middle|2\tau'\right). \tag{3.86}$$

Notice that both a and h are odd. We may therefore write $ah = 2N + 1$, so that with Lemma 6.3.2,

$$\vartheta_1\left(v|2\tau'\right) = (-1)^N \exp(-\pi i N(2\tau' + aM/4 + 2\tau' N))\vartheta_1\left(\tau' + \frac{aM}{4}\middle|2\tau'\right). \tag{3.87}$$

We now shift from ϑ_1 to ϑ_4, by Theorem 3.6.3. Write

$$\vartheta_1\left(\tau' + \frac{aM}{8}\middle|2\tau'\right) = i\exp(\frac{-\pi i}{8}(12\tau' + aM/8))\vartheta_4\left(\frac{aM}{8}\middle|2\tau'\right). \tag{3.88}$$

We now express ϑ_4 as an infinite product:

$$\vartheta_4\left(\frac{aM}{8}\middle|2\tau'\right) = \prod_{m=1}^{\infty}(1 - y^{2m})(1 - \rho_2 y^{2m-1})(1 - \rho_2^{-1} y^{2m-1}) \tag{3.89}$$

$$= (y^2; y^2)_\infty(\rho_2 y; y^2)_\infty(\rho_2^{-1} y; y^2)_\infty, \tag{3.90}$$

with

$$\rho_2 = \exp\left(\pi i a(4hH+1)/2k\right). \tag{3.91}$$

Combining (3.80), (3.82), (3.87), (3.88), (3.90), we have

$$\begin{aligned}F_a(q) = \frac{1}{\sqrt{2}}\frac{i(-1)^N}{\epsilon(2,8)\epsilon(2,4)}\exp\Bigg(&\pi i\tau(1-a) - z\pi a^2(hiz^{-1}-1)^2/4k \\ &+ \pi i N(2\tau' + aM/4 + 2\tau' N) + \frac{-\pi i}{8}(12\tau' + aM/8)\Bigg)\Psi_2(y), \end{aligned} \tag{3.92}$$

with

$$\Psi_2(q) = \frac{\eta(2\tau')^2}{\eta(4\tau')\vartheta_4\left(\frac{aM}{8}\middle|2\tau'\right)} \tag{3.93}$$

$$= \frac{(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty f(-\rho_2 q; -\rho_2^{-1} q)}. \tag{3.94}$$

We must once again collect the coefficients of 1, z, and $1/z$ in the exponential of (3.92). In doing so, we have the following:

$$F_a(q) = \frac{1}{\sqrt{2}}\omega_2(h,k)\exp\left(\frac{\pi}{8k}\left(z(4a-5)\right)\right)\Psi_2(y), \tag{3.95}$$

with $\Psi_2(q)$ defined by 3.94, and

$$\omega_2(h,k) = \frac{i(-1)^{\lfloor \frac{ah}{2} \rfloor}}{\epsilon(2,8)\epsilon(2,4)}\exp\left(\frac{\pi i}{4k}\left(1-(4a-3)(2hH+1)\right)\right). \tag{3.96}$$

3.2.4 $GCD(k,8) = 1$

Let $8hH \equiv -1 \pmod{k}$ and consider the matrices

$$\begin{pmatrix} 8h & -\frac{1}{k}(8hH+1) \\ k & -H \end{pmatrix}, \tag{3.97}$$

$$\begin{pmatrix} 4h & -\frac{1}{k}(8hH+1) \\ k & -2H \end{pmatrix}. \tag{3.98}$$

Both matrices are elements of $SL(2,\mathbb{Z})$. We will allow

$$\tau' = \frac{H}{k} + \frac{iz^{-1}}{8k}. \tag{3.99}$$

Applying (3.97) to τ', we have

$$\frac{8h\tau' - \frac{1}{k}(8hH+1)}{k\tau' - H} = 8\tau. \tag{3.100}$$

Similarly, applying (3.98) to $2\tau'$, we have

$$\frac{8h\tau' - \frac{1}{k}(8hH+1)}{2k\tau' - 2H} = 4\tau. \tag{3.101}$$

Therefore, we will have

$$\frac{\eta(8\tau)^2}{\eta(4\tau)} = \frac{1}{4z^{1/2}} \frac{\epsilon(1,8)^2}{\epsilon(1,4)} \frac{\eta(\tau')^2}{\eta(2\tau')}. \tag{3.102}$$

Returning to ϑ_1, we have

$$\vartheta_1(a\tau|8\tau) = \vartheta_1\left(\frac{v}{iz^{-1}/8}\middle|8\tau\right) \tag{3.103}$$

$$= -i\epsilon(1,8)^3 \frac{1}{2\sqrt{2}z^{1/2}} e^{8\pi kzv^2} \vartheta_1\left(\frac{a(hiz^{-1}-1)}{8k}\middle|\tau'\right), \tag{3.104}$$

with

$$v = \frac{a}{8k}(hiz^{-1} - 1). \tag{3.105}$$

Remembering that $8hH \equiv -1 \pmod{k}$, we write

$$-1 = 8hH + Mk, \tag{3.106}$$

so that

$$v = \frac{a(hiz^{-1} - 1)}{8k} = \frac{a}{8k}(hiz^{-1} + 8hH + Mk) = ah\tau' + \frac{aM}{8}. \tag{3.107}$$

Therefore,

$$\vartheta_1(v|\tau') = \vartheta_1\left(ah\tau' + \frac{aM}{8}\middle|\tau'\right). \tag{3.108}$$

Recognizing from Lemma 3.6.1 that we may extract $ah\tau'$ altogether from our first variable, and recognizing that $(-1)^{ah} = (-1)^h$, we have

$$\vartheta_1(v|\tau') = (-1)^h \exp(-\pi iah(ah\tau' + aM/4))\vartheta_1\left(\frac{aM}{8}\middle|\tau'\right). \tag{3.109}$$

We may now write $\vartheta_1\left(\frac{aM}{8}\middle|\tau'\right)$ in the product form we defined it by in 3.1:

$$\vartheta_1\left(\frac{aM}{8}\middle|\tau'\right) = 2e^{\pi i\tau'/4}\sin(\pi aM/8)\prod_{m=1}^{\infty}(1 - e^{2\pi im\tau'})(1 - e^{2\pi im\tau' + 2\pi iaM/8}) \times (1 - e^{2\pi im\tau' - 2\pi iaM/8}). \tag{3.110}$$

Let

$$\rho_1 = \exp(-2\pi iaM/8) = \exp\left(\pi ia(8hH + 1)/4k\right), \tag{3.111}$$

and

$$y = \exp\left(2\pi i\left(\frac{H}{k}+\frac{iz^{-1}}{8k}\right)\right). \tag{3.112}$$

Then we have

$$\vartheta_1\left(\frac{aM}{8}\middle|\tau'\right) = 2e^{\pi i\tau'/4}\sin(\pi aM/8)\prod_{m=1}^{\infty}(1-y^m)(1-\rho_1 y^m)(1-\rho_1^{-1}y^m) \tag{3.113}$$

$$= 2e^{\pi i\tau'/4}\sin(\pi aM/8)(y;y)_\infty(\rho_1 y;y)_\infty(\rho_1^{-1}y;y)_\infty. \tag{3.114}$$

Examining the sine function, let $aM = 8N + c$, with c the least positive residue of $aM \pmod 8$. Then

$$\sin\left(\frac{\pi aM}{8}\right) = (-1)^N \sin\left(\frac{\pi c}{8}\right). \tag{3.115}$$

Notice that $\sin\left(\frac{\pi c}{8}\right) > 0$. We know that since

$$M = -\frac{1}{k}(8hH+1), \tag{3.116}$$

and since $(k, 8) = 1$, therefore

$$c \equiv -ak^{-1} \pmod 8. \tag{3.117}$$

Moreover, k is odd, so $k \equiv 1, 3, 5, 7 \pmod 8$; and in each of these cases, $k^{-1} = k$. So

$$\sin\left(\frac{\pi c}{8}\right) = \left|\sin\left(\frac{\pi c}{8}\right)\right| \tag{3.118}$$

$$= \left|\sin\left(\frac{-\pi c}{8}\right)\right| \tag{3.119}$$

$$= \left|\sin\left(\frac{-\pi ak}{8}\right)\right| \tag{3.120}$$

$$= \left|\sin\left(\frac{\pi ak}{8}\right)\right|. \tag{3.121}$$

Combining (3.102), (3.104), (3.108), (3.109), (3.110), (3.114), (3.115), we have:

$$F_a(q) = \frac{1}{2\sqrt{2}} \frac{(-1)^{N+h-1}}{\epsilon(1,8)\epsilon(1,4)} \exp\Big(\pi i\tau(1-a) - 8\pi kz(a(hiz^{-1}-1)/8k)^2 + \pi ah(ah\tau' + aM/4 - \pi i\tau'/4)\Big) \left|\sin\left(\frac{\pi ak}{8}\right)\right| \Psi_1(y), \quad (3.122)$$

where

$$\Psi_1(y) = \frac{\eta(\tau')^2}{\eta(2\tau')} \frac{1}{(y;y)_\infty(\rho_1 y;y)_\infty(\rho_1^{-1}y;y)_\infty} \quad (3.123)$$

$$= \frac{(y;y)_\infty}{(y^2;y^2)_\infty(\rho_1 y;y)_\infty(\rho_1^{-1}y;y)_\infty}. \quad (3.124)$$

Collecting coefficients of 1, z, and $1/z$ in the exponential of (3.122), we have

$$F_a(q) = \frac{1}{2\sqrt{2}}\omega_1(h,k)\left|\csc\left(\frac{\pi ak}{8}\right)\right| \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(4a-5)\right)\right)\Psi_1(y), \quad (3.125)$$

where $\Psi_1(y)$ is defined by (3.124), and

$$\omega_1(h,k) = \frac{(-1)^{\lfloor\frac{-a(8hH+1)}{8k}\rfloor+h-1}}{\epsilon(1,8)\epsilon(1,4)} \exp\left(\frac{\pi i}{4k}(4h(1-a+hH(3-4a)) - H)\right). \quad (3.126)$$

3.3 Complete Transformation Equations

We summarize Section 3.2 by giving the complete transformations in all four cases below.

Theorem 3.9. *Let* $(k,8) = d$*, with* $d = 8,4,2,1,$ *and let* $0 \le h < k$ *with* $(h,k) = 1$*. We define* $\frac{8}{d}H$ *to be the negative inverse of* $h \pmod{k/d}$*, and*

$$q = \exp\left(2\pi i\left(\frac{h}{k} + i\frac{z}{k}\right)\right), \quad (3.127)$$

$$y = \exp\left(2\pi i\left(\frac{H}{k} + i\frac{dz^{-1}}{8k}\right)\right). \tag{3.128}$$

If $d = 8$, *then*

$$F_a(q) = \omega_8(h,k)\exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8}{z} + z(4a-5)\right)\right)F_b(y), \tag{3.129}$$

with $b \equiv ah \pmod 8 = 1, 3, 5, 7,$ *and*

$$\omega_8(h,k) = \frac{i(-1)^{\lfloor \frac{ah}{8} \rfloor}}{\epsilon(8,8)\epsilon(8,4)}\exp\left(\frac{\pi i}{8k}(h(5-4a) - H((b-4)^2-8))\right). \tag{3.130}$$

If $d = 4$, *then*

$$F_a(q) = \frac{1}{\sqrt{2}}\omega_4(h,k)\exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(4a-5)\right)\right)\Psi_4(y), \tag{3.131}$$

with

$$\begin{aligned}\omega_4(h,k) = {} & \frac{i(-1)^{\lfloor \frac{ah}{4} \rfloor}}{\epsilon(4,8)\epsilon(4,4)} \\ & \times \exp\left(\frac{\pi i}{4k}(h - H - h(4a-3)(hH+1) + a(2hH+1)(b-2)\right).\end{aligned} \tag{3.132}$$

If $d = 2$, *then*

$$F_a(q) = \frac{1}{\sqrt{2}}\omega_2(h,k)\exp\left(\frac{\pi}{8k}\left(z(4a-5)\right)\right)\Psi_2(y), \tag{3.133}$$

with

$$\omega_2(h,k) = \frac{i(-1)^{\lfloor \frac{ah}{2} \rfloor}}{\epsilon(2,8)\epsilon(2,4)}\exp\left(\frac{\pi i}{4k}\left(1 - (4a-3)(2hH+1)\right)\right). \tag{3.134}$$

If $d = 1$, *then*

$$F_a(q) = \frac{1}{2\sqrt{2}}\omega_1(h,k)\left|\csc\left(\frac{\pi a k}{8}\right)\right|\exp\left(\frac{\pi}{8k}\left(\frac{1}{4z}+z(4a-5)\right)\right)\Psi_1(y), \qquad (3.135)$$

with

$$\omega_1(h,k) = \frac{(-1)^{\lfloor\frac{-a(8hH+1)}{8k}\rfloor+h-1}}{\epsilon(1,8)\epsilon(1,4)}\exp\left(\frac{\pi i}{4k}(4h(1-a+hH(3-4a))-H)\right). \qquad (3.136)$$

We define Ψ_d *for* $d = 4, 2, 1$ *by (3.71), (3.94), (3.124), respectively, as q-series, with* $\Psi_d(0) = 1$*.*

3.4 Note on the Modularity of $F_a(q)$

It is noteworthy that, while we do not achieve modularity in the cases that $(k, 8) = 4, 2, 1$, we nearly do with $(k, 8) = 8$. This point is worth expanding on. Given a matrix $V = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, we have $\alpha\delta = 1 + \beta\gamma$. Clearly, $\alpha\delta \equiv 1 \pmod{\gamma}$, and $(\alpha, \gamma) = 1$. Therefore, we may think of any matrix in $SL(2,\mathbb{Z})$ as having the form

$$V = \begin{pmatrix} h & -\frac{1}{k}(hH+1) \\ k & -H \end{pmatrix}, \qquad (3.137)$$

with $hH \equiv -1 \pmod{k}$, and $(h, k) = 1$ (though in this more general case, h may exceed k).

Given $(k, 8) = 8$, we have

$$\tau = \frac{h}{k} + i\frac{z}{k} \qquad (3.138)$$

and

$$\tau' = \frac{H}{k} + i\frac{z^{-1}}{k}. \qquad (3.139)$$

This yields

$$z = \frac{1}{i}(k\tau - h), \tag{3.140}$$

$$\frac{1}{z} = \frac{1}{i}(k\tau' - H). \tag{3.141}$$

In this manner, we may rewrite (3.45) as

$$F_a\left(2\pi i\tau\right) = \omega_8(h,k)\exp\left(\frac{\pi i}{8k}(((b-4)^2-8)h+(4a-5)H)\right) \times \exp\left(\frac{-\pi i}{8}(((b-4)^2-8)\tau+(4a-5)\tau')\right)F_b\left(\exp(2\pi i\tau')\right). \tag{3.142}$$

Now, supposing that V has a more specific form $\pmod 8$:

$$V = \begin{pmatrix} h & -\frac{1}{k}(hH+1) \\ k & -H \end{pmatrix} \equiv \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \pmod 8, \tag{3.143}$$

we have $h \equiv 1 \pmod 8$, ensuring that $b \equiv a \pmod 8$, i.e. that $b = a$. With this in mind, we simplify (3.142) as

$$F_a\left(2\pi i\tau\right) = \omega_8(h,k)\exp\left(\frac{-\pi i}{8k}(4a-5)(h-H)\right) \times \exp\left(\frac{\pi i}{8}(4a-5)(\tau-\tau')\right)F_a\left(\exp(2\pi i\tau')\right), \tag{3.144}$$

or

$$F_a\left(2\pi i\tau\right)\exp\left(\frac{-\pi i}{8}(4a-5)\tau\right) = \omega_8(h,k)\exp\left(\frac{-\pi i}{8k}(4a-5)(h-H)\right) \times F_a\left(\exp(2\pi i\tau')\right)\exp\left(\frac{-\pi i}{8}(4a-5)\tau'\right). \tag{3.145}$$

Since $\omega_8(h,k)\exp\left(\frac{-\pi i}{8k}(4a-5)(h-H)\right)$ is a root of unity, we find the following:

Theorem 3.10. *For* $\tau \in \mathbb{C}$ *with* $\Im(\tau) > 0$, *the function*

$$F_a\left(2\pi i\tau\right)\exp\left(\frac{-\pi i}{8}(4a-5)\tau\right) \tag{3.146}$$

is a modular form of weight 0 *with respect to the subgroup*

$$H(8) = \left\{V \in SL(2,\mathbb{Z}) : V \equiv \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \pmod 8\right\} \leq SL(2,\mathbb{Z}). \tag{3.147}$$

CHAPTER 4

IMPORTANT ESTIMATIONS

4.1 Kloosterman Sums

The transformation equations of $F_a(q)$ discussed in the previous chapter will allow us more control over the integral equation that we will evaluate in Chapter 5. However, there is a second, and perhaps more difficult, matter that we must also contend with, in the form of estimating certain exponential sums.

This is an immensely important technical question, with applications not only to partition theory, but throughout the theory of numbers.

Definition 4.1. Let the integers u, v, q be given, and for any $0 \leq h < q$, let H be given with $hH \equiv -1 \pmod{q}$. Define the complete Kloosterman sum S by

$$S = \sum_{\substack{0 \leq h < q, \\ (h,q)=1}} \exp\left(\frac{2\pi i}{q}(uh + vH)\right). \tag{4.1}$$

A Kloosterman sum is incomplete if the given sum does not extend over all h with $0 \leq h < q$ and $(h, q) = 1$.

We will find that our ability to closely estimate S is a technical lynchpin on which our calculation rests. A quick inspection of S gives us

$$|S| = O\left(\varphi(q)\right) = O(q). \tag{4.2}$$

However, such an estimate will not be very useful to us, and we will work to achieve something stronger; in particular, we make use of an ingenious estimate due to Hans Salié:

Lemma 4.2. *Given $\epsilon > 0$, and integers $u, v, \lambda, \Lambda, q$, $hH \equiv -1 \pmod{q}$, with $\Lambda > 0$ a positive divisor of q, and*

$$S = \sum_{\substack{0 \le h < q, \\ (h,q)=1 \\ h \equiv \lambda \pmod{\Lambda}}} \exp\left(\frac{2\pi i}{q}(uh + vH)\right), \tag{4.3}$$

we have

$$|S| = O(q^{2/3+\epsilon}(u,q)^{1/3}). \tag{4.4}$$

The proof of this extraordinary result is in Salié's 1931 paper, "Zur Abschätzung der Fourierkoeffizienten ganzer Modulformen" [21].

We must now work to estimate sums based around the roots of unity $\omega_d(h,k)$ that we calculated in the previous chapter. We will actually work to estimate

$$\sum_{\substack{0 \le h < k, \\ (h,k)=1}} \omega_d(h,k) e^{2\pi i h/k}. \tag{4.5}$$

Theorem 4.3. *For* $d = 8, 4, 2, 1$,

$$\left| \sum_{\substack{0 \le h < k, \\ (h,k)=1}} \omega_d(h,k) e^{2\pi i h/k} \right| = O\left(k^{2/3+\epsilon} n^{1/3}\right). \tag{4.6}$$

The proof of this theorem will take up the remainder of this chapter. We note that this problem of estimating exponential sums is not as immediate a problem in the case of deriving a formula for the unrestricted partition function, $p(n)$. The reason for this is that, in transforming the generating function $F(q)$ for $p(n)$, a factor of $\sqrt{z}$ emerges, which can be used to better control the contribution of the associated exponential sums [23], [18]. However, because $F_a(q)$ has weight 0, no such power of z emerges from the transformations in the previous chapter, and more precise esimations will be needed.

4.2 Estimations

4.2.1 $\omega_8(h,k)$

Recall that

$$\omega_8(h,k) = \frac{i(-1)^{\lfloor \frac{ah}{8} \rfloor}}{\epsilon(8,8)\epsilon(8,4)} \exp\left(\frac{\pi i}{8k}(h(5-4a) - H((b-4)^2-8))\right), \tag{4.7}$$

so that

Remembering that $\epsilon(8,8), \epsilon(8,4)$ are determined by the first and second matrices, respectively, in the case of $(k,8)=8$. Since $-H$ is necessarily odd, we can determine the value of $\epsilon(8,8)\epsilon(8,4)$, by the formula in Section 3.1:

$$\epsilon(8,8)\epsilon(8,4) = (-1)^{\frac{H^2-1}{8}} \exp\left(\frac{-\pi ik}{32}\right) \exp\left(\frac{\pi i}{8k}\left(h(k^2/4) + H(k^2/4)\right)\right). \tag{4.8}$$

Simplifying $\omega_8(h,k)$ and combining with $e^{2\pi ih/k}$, this gives us

$$\begin{aligned}\omega_8(h,k)e^{2\pi ih/k} &= i(-1)^{\lfloor \frac{ah}{8} \rfloor + \frac{H^2-1}{8}} \exp\left(\frac{\pi ik}{32}\right) \\ &\quad \times \exp\left(\frac{-2\pi i}{16k}(h(16n+4a-5+k^2/4) - H((b-4)^2-8-k^2/4))\right).\end{aligned} \tag{4.9}$$

Let $u = 16n+4a-5+k^2/4$ and $v = (b-4)^2-8-k^2/4$. Notice that the value of v depends on the congruence of ah, whereas the value of u does not.

We wish to estimate

$$\left|\sum_{\substack{0\le h<k,\\(h,k)=1}} \omega_8(h,k)e^{2\pi i h/k}\right|$$
$$= \left|\sum_{\substack{0\le h<k,\\(h,k)=1}} i(-1)^{\lfloor\frac{ah}{8}\rfloor+\frac{H^2-1}{8}} \exp\left(\frac{\pi i k}{32}\right)\exp\left(\frac{-2\pi i}{16k}(hu-Hv)\right)\right|. \tag{4.10}$$

We now separate our sum into terms, depending on the congruence properties of ah: our previous sum is

$$\le \sum_{j=1}^{8}\left|\sum_{\substack{0\le h<k,\\(h,k)=1\\ ah\equiv(2j-1)\pmod{16}}} i(-1)^{\lfloor\frac{ah}{8}\rfloor+\frac{H^2-1}{8}} \exp\left(\frac{\pi i k}{32}\right)\exp\left(\frac{-2\pi i}{16k}(hu-Hv)\right)\right| \tag{4.11}$$

Notice that since i and $\exp\left(\frac{\pi i k}{32}\right)$ are fixed throughout our sum, we may factor them out of our summation.

Next, we recognize that in each case $ah\equiv 2j-1 \pmod{16}$, $\frac{H^2-1}{8}$ will have a definite parity, which will fix the alternation in sign that depends on H.

Moreover, if $ah\equiv 1,3,5,7 \pmod{16}$, then $\lfloor\frac{ah}{8}\rfloor$ is even; if $ah\equiv 9,11,13,15 \pmod{16}$, then $\lfloor\frac{ah}{8}\rfloor$ is odd. In either case, for a specific value of ah modulo 16, the alternation induced by $\lfloor\frac{ah}{8}\rfloor$ is fixed. We may therefore ignore the alternating sign in front of each sum, and focus on estimating the sum of the exponentials themselves.

Finally, for each value of $ah \pmod 8$, the value of v is fixed. Each sum is an incomplete Kloosterman sum, in which only h and H vary in the exponential.

We now take advantage Salié's estimate, provided we also understand that, given $8|q$,

$$\left|\sum_{\substack{0\le h<q,\\ (h,q)=1\\ h\equiv\lambda \pmod{\Lambda}}} \exp\left(\frac{2\pi i}{16q}(uh+vH)\right)\right|$$

$$= O\left(\left|\sum_{\substack{0\le h<16q,\\ (h,q)=1\\ h\equiv\lambda \pmod{\Lambda}}} \exp\left(\frac{2\pi i}{16q}(uh+vH)\right)\right|\right). \quad (4.12)$$

We may now write

$$\left|\sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_8(h,k)e^{2\pi i h/k}\right|$$

$$\le \left|8i\exp\left(\frac{\pi i k}{32}\right)\right| \sum_{j=1}^{8} \left|\sum_{\substack{0\le h<k,\\ (h,k)=1\\ ah\equiv(2j-1) \pmod{16}}} \exp\left(\frac{-2\pi i}{16k}(hu-Hv)\right)\right| \quad (4.13)$$

$$=O\left((16k)^{2/3+\epsilon}(16n+4a-5,16k)^{1/3}\right) \quad (4.14)$$

$$=O\left(k^{2/3+\epsilon}(16n+4a-5,k)^{1/3}\right). \quad (4.15)$$

We finally recognize that since $(16n+4a-5,k) = O(16n+4a-5) = O(n)$, that therefore

$$\left|\sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_8(h,k)e^{2\pi i h/k}\right| = O\left(k^{2/3+\epsilon}n^{1/3}\right). \quad (4.16)$$

4.2.2 $\omega_4(h,k)$

Recall that

$$\omega_4(h,k) = \frac{i(-1)^{\lfloor \frac{ah}{4} \rfloor}}{\epsilon(4,8)\epsilon(4,4)} \times \exp\left(\frac{\pi i}{4k}(h - H - h(4a-3)(hH+1) + a(2hH+1)(b-2)\right). \quad (4.17)$$

We have

$$\frac{1}{\epsilon(4,8)\epsilon(4,4)} = (-1)^{(k-4)(k-12)/128} \exp\left(\frac{\pi i}{4k}(h(-k^2/4) + H(k^2/4))\right). \quad (4.18)$$

To simplify $\omega_4(h,k)$, recall that $2hH \equiv -1 \pmod{k/4}$, i.e. $hH \equiv -2^{-1} \pmod{k/4}$. We will let $c \equiv -2^{-1} \pmod{k/4}$, and $hH = c + \beta k/4$. Then this gives us

$$\omega_4(h,k) = i(-1)^{(k-4)(k-12)/128 + \lfloor \frac{ah}{4} \rfloor} \exp\left(\frac{\pi i}{4k} a(b-2)(2c+1)\right) \times \exp\left(\frac{\pi i \beta}{16}(2a(b-2) - h(4a-3))\right) \times \exp\left(\frac{\pi i}{4k}(h(1-(4a-3)(c+1) - k^2/4) - H(k^2/4+1)\right). \quad (4.19)$$

Finally, we multiply by $e^{-2\pi i n h/k}$:

$$\omega_4(h,k)e^{-2\pi i n h/k} = i(-1)^{(k-4)(k-12)/128 + \lfloor \frac{ah}{4} \rfloor} \exp\left(\frac{\pi i}{4k} a(b-2)(2c+1)\right) \times \exp\left(\frac{\pi i \beta}{16}(2a(b-2) - h(4a-3))\right) \times \exp\left(\frac{-\pi i}{4k}(h(16n + (4a-3)(c+1) - 1 + k^2/4) + H(k^2/4+1)\right). \quad (4.20)$$

For convenience, let $u = 2(16n + (4a-3)(c+1) - 1 + k^2/4)$, $v = -(k^2/4+1)$. We then have

$$\omega_4(h,k)e^{-2\pi inh/k} = i(-1)^{(k-4)(k-12)/128+\lfloor \frac{ah}{4} \rfloor} \exp\left(\frac{\pi i}{4k}a(b-2)(2c+1)\right)$$
$$\times \exp\left(\frac{2\pi i\beta}{32}(2a(b-2)-h(4a-3))\right)\exp\left(\frac{-\pi i}{8k}(hu-2Hv)\right). \quad (4.21)$$

Once again, we may factor i out of our sum. Moreover, since we fix k for the given sum, we need only pay attention to the alternation with respect to $\lfloor \frac{ah}{4} \rfloor$.

Since c depends only on k, therefore $\exp\left(\frac{\pi i}{4k}a(b-2)(2c+1)\right)$ will vary only as $ah \equiv b \pmod 4$ varies.

Finally, we may examine that value of $\beta\,(2a(b-2)-h(4a-3)) \pmod{32}$; since the expression depends entirely on $h \pmod{32}$, we need only fix

$$\exp\left(\frac{2\pi i\beta}{32}(2a(b-2)-h(4a-3))\right)$$

by fixing $ah \pmod{32}$.

Therefore, the alternating sign and extraneous root of unity may be disregarded after fixing k and varying the possible congruence conditions of $h \pmod{32}$;

Therefore, we may estimate our sum:

$$\left|\sum_{\substack{0\le h<k,\\(h,k)=1}}\omega_4(h,k)e^{-2\pi inh/k}\right|$$

$$\le\sum_{j=1}^{16}\left|\sum_{\substack{0\le h<k,\\(h,k)=1\\ah\equiv(2j-1)\pmod{32}}}\exp\left(\frac{-2\pi i}{16k}(hu-2Hv)\right)\right| \tag{4.22}$$

$$=O\left(\left|\sum_{\substack{0\le h<k,\\(h,k)=1}}\exp\left(\frac{-\pi i}{16k}(hu-2Hv)\right)\right|\right) \tag{4.23}$$

$$=O\left((16k)^{2/3+\epsilon}(u,16k)^{1/3}\right) \tag{4.24}$$

$$=O\left(k^{2/3+\epsilon}n^{1/3}\right). \tag{4.25}$$

4.2.3 $\omega_2(h,k)$

Recall that

$$\omega_2(h,k)=\frac{i(-1)^{\lfloor\frac{ah}{2}\rfloor}}{\epsilon(2,8)\epsilon(2,4)}\exp\left(\frac{\pi i}{4k}\left(1-(4a-3)(2hH+1)\right)\right). \tag{4.26}$$

We calculate that

$$\frac{1}{\epsilon(2,8)\epsilon(2,4)}=(-1)^{\frac{(k-6)(k-2)}{32}}\exp\left(\frac{\pi ik}{4}(-2h+H)\right). \tag{4.27}$$

We will let $c\equiv-2^{-1}\pmod{k/2}$, and $2hH=c+\beta k/2$. Then combining with our expression above, we have

$$\omega_2(h,k)=i(-1)^{\lfloor\frac{ah}{2}\rfloor+\frac{(k-6)(k-2)}{32}}\exp\left(\frac{\pi i}{4k}(1-(4a-3)(c+1))\right)$$
$$\times\exp\left(\frac{\pi i}{8}(3-4a)\beta\right)\exp\left(\frac{\pi i}{4k}\left(h(-2k^2)+H(k^2)\right)\right). \tag{4.28}$$

Combining with $e^{-2\pi inh/k}$,

$$\omega_2(h,k)e^{-2\pi inh/k} = i(-1)^{\lfloor \frac{ah}{2} \rfloor + \frac{(k-6)(k-2)}{32}} \exp\left(\frac{\pi i}{4k}(1-(4a-3)(c+1))\right) \times \exp\left(\frac{\pi i}{8}(3-4a)\beta\right) \exp\left(\frac{-\pi i}{4k}\left(h(8n+2k^2) - H(k^2)\right)\right). \quad (4.29)$$

Again letting $u = 8n + 2k^2$ and $v = k^2/4$, we have

$$\omega_2(h,k)e^{-2\pi inh/k} = i(-1)^{\lfloor \frac{ah}{2} \rfloor + \frac{(k-6)(k-2)}{32}} \exp\left(\frac{\pi i}{4k}(1-(4a-3)(c+1))\right) \times \exp\left(\frac{\pi i}{8}(3-4a)\beta\right) \exp\left(\frac{-\pi i}{4k}(hu - 4Hv)\right). \quad (4.30)$$

Again, $i(-1)^{\frac{(k-6)(k-2)}{32}}$ will not vary with h; nor will $\exp\left(\frac{\pi i}{4k}(1-(4a-3)(c+1))\right)$. On the other hand, $\exp\left(\frac{\pi i}{8}(3-4a)\beta\right)$ will vary with the value of $\beta \pmod{16}$, which will depend on the value of $h \pmod{16}$. So we have

$$\left| \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_2(h,k)e^{-2\pi inh/k} \right|$$

$$\le \exp\left(\frac{\pi i}{8}(3-4a)\beta\right) \sum_{j=1}^{8} \left| \sum_{\substack{0\le h<k,\\ (h,k)=1\\ h\equiv(2j-1) \pmod{16}}} \exp\left(\frac{-2\pi i}{8k}(hu-4Hv)\right) \right| \quad (4.31)$$

$$= O\left(\left| \sum_{\substack{0\le h<k,\\ (h,k)=1}} \exp\left(\frac{-\pi i}{8k}(hu-4Hv)\right) \right| \right) \quad (4.32)$$

$$= O\left((8k)^{2/3+\epsilon}(u,8k)^{1/3}\right) \quad (4.33)$$

$$= O\left(k^{2/3+\epsilon} n^{1/3}\right). \quad (4.34)$$

4.2.4 $\omega_1(h,k)$

Recall that

$$\omega_1(h,k) = \frac{(-1)^{\lfloor \frac{-a(8hH+1)}{8k} \rfloor + h - 1}}{\epsilon(1,8)\epsilon(1,4)} \exp\left(\frac{\pi i}{4k}(4h(1-a+hH(3-4a))-H)\right). \quad (4.35)$$

We begin with evaluating and inverting $\epsilon(1,8)\epsilon(1,4)$:

$$\frac{1}{\epsilon(1,8)\epsilon(1,4)} = (-1)^{\frac{(k-3)(k-1)}{8}} \exp\left(\frac{-\pi i k}{4}(4h-H)\right). \quad (4.36)$$

Since $8hH \equiv -1 \pmod{k}$, we will let $hH \equiv c \equiv -8^{-1} \pmod{k}$. We therefore have $hH = c + \beta k$, $8hH + 1 = 8c + 8\beta k + 1$.

Combining and simplifying, we have

$$\omega_1(h,k) = i(-1)^{\lfloor \frac{-a(8hH+1)}{8k} \rfloor + h + hH - 1 - c + \frac{(k-3)(k-1)}{8}} \times \exp\left(\frac{\pi i}{4k}\left(4h(1-a+c(3-4a)-k^2)+H(k^2-1)\right)\right). \quad (4.37)$$

Multiply by $e^{-2\pi i n h/k}$ and let $u = 8(2n+a-1+c(4a-3)+k^2)$, $v = \frac{k^2-1}{4}$:

$$\omega_1(h,k) = i(-1)^{\lfloor \frac{-a(8hH+1)}{8k} \rfloor + h + hH - 1 - c + \frac{(k-3)(k-1)}{8}} \exp\left(\frac{-\pi i}{8k}(hu - 8Hv)\right). \quad (4.38)$$

Again, $i(-1)^{-1-c+\frac{(k-3)(k-1)}{8}}$ will not vary with h. We may examine the congruence of $h \pmod 8$ to fix the remaining alternating sign. We have the following:

$$\left|\sum_{\substack{0\le h<k,\\(h,k)=1}} \omega_1(h,k)e^{-2\pi inh/k}\right|$$

$$\le \sum_{j=1}^{4}\left|\sum_{\substack{0\le h<k,\\(h,k)=1\\h\equiv(2j-1)\pmod 8}} \exp\left(\frac{-2\pi i}{16k}(hu-8Hv)\right)\right| \tag{4.39}$$

$$=O\left(\left|\sum_{\substack{0\le h<k,\\(h,k)=1}} \exp\left(\frac{-2\pi i}{16k}(hu-8Hv)\right)\right|\right) \tag{4.40}$$

$$=O\left((16k)^{2/3+\epsilon}(u,16k)^{1/3}\right) \tag{4.41}$$

$$=O\left(k^{2/3+\epsilon}n^{1/3}\right). \tag{4.42}$$

CHAPTER 5

INTEGRATION

5.1 Heuristics

Recall our results from Chapter 2: Section 1 established that

$$g_a(n) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{F_a(q)}{q^{n+1}} dq,$$

while in Section 2 we described a contour that will prove useful for integration. We will now begin the integration proper.

Let N be some large positive integer, and let the corresponding Rademacher curve $P(N)$ be given. Then we have the following:

$$g_a(n) = \frac{1}{2\pi i} \oint_{P(N)} \frac{F_a(q)}{q^{n+1}} dq = \sum_{k=1}^{N} \sum_{\substack{0 \le h < k, \\ (h,k)=1}} \frac{1}{2\pi i} \int_{\gamma(h,k)} \frac{F_a(q)}{q^{n+1}} dq. \tag{5.1}$$

In Chapter 3, we gave transformation equations for $F_a(q)$ depending on the divisibility properties of k. We now separate our integral into the corresponding cases:

$$g_a(n) = g_a^{(8)}(n) + g_a^{(4)}(n) + g_a^{(2)}(n) + g_a^{(1)}(n), \tag{5.2}$$

with

$$g_a^{(d)}(n) = \sum_{\substack{(k,8)=d, \\ k \le N}} \sum_{\substack{0 \le h < k, \\ (h,k)=1}} \frac{1}{2\pi i} \int_{\gamma(h,k)} \frac{F_a(q)}{q^{n+1}} dq. \tag{5.3}$$

We now see the justification of the change of variables $q = \exp\left(2\pi i \left(\frac{h}{k} + \frac{iz}{k}\right)\right)$. We will transform the integrand by the results in Chapter 3. For a substantial portion of our integrand, we will let $z \in \mathbb{C}$ with $\Re(z) > 0$, modify our path of integration, and take z close to 0. As $z \to 0$, $q \to e^{2\pi ih/k}$, and $y \to 0$. We will utilize the corresponding arc $\gamma(h,k)$ and show that $y \to 0$ rapidly enough that the functions $F_b(y), \Psi_d(y) \to 1$. Those portions of the integral that remain will constitute a finite string of elementary exponential factors—nontrivial, but far more accessible to integration. For such contributions, we will modify our contour in z to encompass the whole of the corresponding circle $K_k^{(-)}$, and appeal to the theory of Bessel functions to finish the integration.

Remembering from Lemma 2.7 that $|z_I(h,k)|, |z_T(h,k)| = O(N^{-1})$, and N being the upper bound on the denominators of h/k for the singularities $e^{2\pi ih/k}$ considered, we will ultimately take $N \to \infty$.

5.2 In the Case of $p(n)$

We begin by sketching the method of solving for $p(n)$ [23]. As the summary in 3.1.2, our formula for $p(n)$ has the form

$$\begin{aligned} p(n) &= \sum_{k=1}^{N} \sum_{\substack{0 \le h < k, \\ (h,k)=1}} \frac{i}{k} e^{-2\pi inh/k} \omega(h,k) \int_{z_I(h,k)}^{z_T(h,k)} e^{2n\pi z/k} e^{\pi(z^{-1}-z)/12k} \sqrt{z}\ F(y)\ dz \\ &= \sum_{k=1}^{N} \frac{i}{k} \sum_{\substack{0 \le h < k, \\ (h,k)=1}} e^{-2\pi inh/k} \omega(h,k) \left(I(h,k) + I^*(h,k) \right), \end{aligned}$$

where $y = \exp(2\pi i(iz^{-1} + H)/k)$, and

$$I(h,k) = \int_{z_I(h,k)}^{z_T(h,k)} e^{2n\pi z/k} e^{\pi(z^{-1}-z)/12k} \sqrt{z}\ dz,$$

and

$$I^*(h,k) = \int_{z_I(h,k)}^{z_T(h,k)} e^{2n\pi z/k} e^{\pi(z^{-1}-z)/12k} \sqrt{z}\,(F(y)-1)\,dz.$$

We then wish to show that $I^*(h,k)$ can be forced to become arbitrarily small, through deforming our path of integration to a chord connecting $z_I(h,k)$ to $z_T(h,k)$. Due to the final result in Chapter 2, and through manipulating the integrand of $I^*(h,k)$, we may show that $|I^*(h,k)| = O(N^{-3/2})$.

We may then manipulate $I(h,k)$ through altering the path of integration to include the whole of $K_k^{(-)}$. Doing so gives us the following estimation of $p(n)$:

$$p(n) = \sum_{k=1}^{N} \frac{i}{k} \sum_{\substack{0\le h<k, \\ (h,k)=1}} e^{-2\pi i n h/k} \omega(h,k) \oint_{K_k^{(-)}} e^{2n\pi z/k} e^{\pi(z^{-1}-z)/12k} \sqrt{z}\ dz + O(N^{-1/2}),$$

with the error term on the order of $N^{-1/2}$ after taking into account an estimation of the sums of roots of unity attached to $I^*(h,k)$.

5.3 Integration

Our method to simplify the formula for $g_a(n)$ will closely follow the sketch of the derivation of $p(n)$. The key complication to take into account is the division of the integral of $g_a(n)$ into cases depending on the divisibility of k in the nearby root of unity $e^{2\pi i h/k}$. In each case, we make use of the corresponding transformation derived in Chapter 3.

However, in each case we will extract a portion of our integral analogous to that of $I^*(h,k)$ above, which we will refer to as $I_d^{(0)}(h,k)$, and which will also be shown not to contribute at all to the final formula.

We rewrite

$$g_a^{(d)}(n) = \sum_{\substack{(k,8)=d \\ k\leq N}} \frac{i}{k} \sum_{\substack{0\leq h<k, \\ (h,k)=1}} e^{-2\pi inh/k} \int_{z_I(h,k)}^{z_T(h,k)} F_a\left(\exp\left(2\pi i\left(\frac{h}{k}+\frac{iz}{k}\right)\right)\right) e^{2\pi nz/k}dz, \quad (5.4)$$

with $z_I(h,k), z_T(h,k)$ the initial and terminal points, respectively, of $\gamma(h,k)$ in terms of z.

5.3.1 $g_a^{(8)}(n)$

Applying Theorem 3.2, we have

$$g_a^{(8)}(n) = \sum_{\substack{(k,8)=8 \\ k\leq N}} \frac{i}{k} \sum_{\substack{0\leq h<k, \\ (h,k)=1}} \omega_8(h,k) e^{-2\pi inh/k}$$
$$\times \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8}{z}+z(16n+4a-5)\right)\right) F_b(y)dz. \quad (5.5)$$

We now expand $F_b(y) = \sum_{j=0}^{\infty} g_b(j)y^j$ as the generating function for $g_b(n)$. Recall that we defined $g_b(0)=1$, i.e. the constant term in $F_b(q)$ is necessarily 1. Therefore, we may write

$$\int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8}{z}+z(16n+4a-5)\right)\right) F_b(y)dz$$
$$= \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8}{z}+z(16n+4a-5)\right)\right) dz \quad (5.6)$$
$$+ \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8}{z}+z(16n+4a-5)\right)\right) \sum_{j=1}^{\infty} g_b(j)y^j dz. \quad (5.7)$$

Hereafter,

$$I_8^{(1)}(h,k) = \int\limits_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8}{z} + z(16n+4a-5)\right)\right) dz, \tag{5.8}$$

and

$$I_8^{(0)}(h,k) = \int\limits_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8}{z} + z(16n+4a-5)\right)\right) \sum_{j=1}^{\infty} g_b(j) y^j dz. \tag{5.9}$$

Therefore,

$$g_a^{(8)}(n) = \sum_{\substack{(k,8)=8 \\ k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k, \\ (h,k)=1}} \omega_8(h,k) e^{-2\pi i n h/k} \left(I_8^{(1)}(h,k) + I_8^{(0)}(h,k)\right). \tag{5.10}$$

Lemma 5.1.

$$\left|I_8^{(0)}(h,k)\right| = O\left(\exp(3n\pi) N^{-1}\right). \tag{5.11}$$

Proof. We may interchange the summation with the integration. Also, remembering that $y = \exp\left(2\pi i\left(\frac{H}{k} + \frac{iz^{-1}}{k}\right)\right)$,

$$\begin{aligned} &I_8^{(0)}(h,k) \\ =&\sum_{j=1}^{\infty} g_b(j) e^{2\pi i H j/k} \\ &\times \int\limits_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8}{z} + z(16n+4a-5)\right)\right) e^{-2\pi z^{-1}/k} dz \qquad (5.12) \\ =&\sum_{j=1}^{\infty} g_b(j) e^{2\pi i H j/k} \\ &\times \int\limits_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8-16j}{z} + z(16n+4a-5)\right)\right) dz. \qquad (5.13) \end{aligned}$$

Notice that the coefficient of $1/z$ in the exponent of the integrand is now always negative.

Taking advantage of the fact that $\Re(1/z) \geq k$ and $\Re(z) \leq 1/k$, we now examine the magnitude of the integrand:

$$\left|\exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8-16j}{z}+z(16n+4a-5)\right)\right)\right| \tag{5.14}$$

$$=\exp\left(\frac{\pi}{8k}((b-4)^2-8-16j)\Re(1/z)+\frac{\pi}{8k}(16n+4a-5)\Re(z)\right) \tag{5.15}$$

$$\leq \exp\left(\frac{\pi(1-16j)}{8}+\frac{\pi(16n+4a-5)}{8k^2}\right) \tag{5.16}$$

$$=\exp(-\pi(2j-1/8)+3n\pi) \leq \exp(\pi j+3n\pi). \tag{5.17}$$

We therefore have

$$|I_8^{(0)}(h,k)|$$

$$\leq \sum_{j=1}^{\infty} g_b(j)|e^{2\pi i Hj/k}|$$

$$\times \int_{z_I(h,k)}^{z_T(h,k)} \left|\exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8-16j}{z}+z(16n+4a-5)\right)\right)\right| dz \tag{5.18}$$

$$\leq \sum_{j=1}^{\infty} g_b(j)\exp(-\pi j+3n\pi)\int_{z_I(h,k)}^{z_T(h,k)} dz. \tag{5.19}$$

Recall that we are integrating along the circle $K_k^{(-)}$ in the z-plane. We will now now deform our contour so that it is a chord connecting z_I and z_T along $K_k^{(-)}$.

Recognizing from our discussion in Chapter 2 that the length of such a chord is bounded above by a constant multiple of N^{-1}, we have

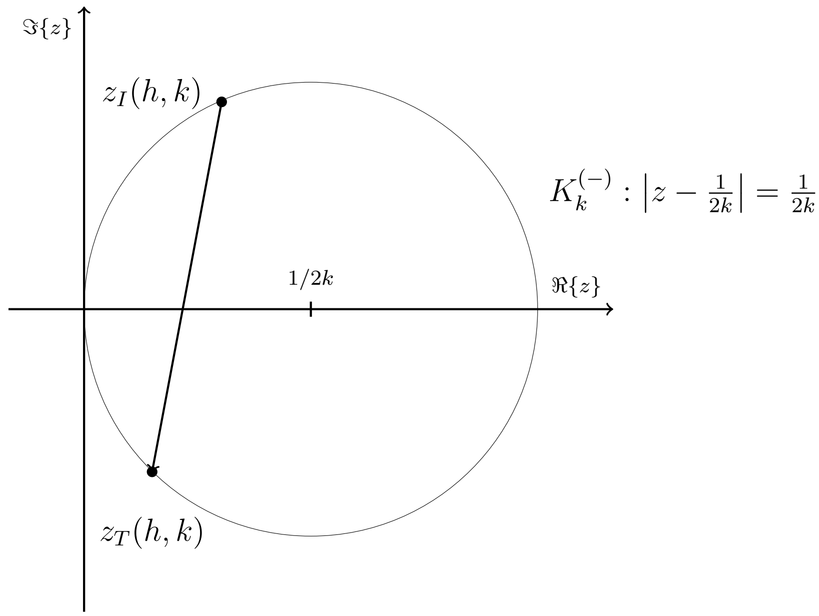


Figure 5.1: $K_k^{(-)}$ with the chord connecting $z_I(h,k)$ to $z_T(h,k)$.

$$|I_8^{(0)}(h,k)| = O\left(\sum_{j=1}^{\infty} g_b(j)\exp(-\pi j + 3n\pi)N^{-1}\right) \tag{5.20}$$

$$= O\left(\exp(3n\pi)N^{-1}\sum_{j=1}^{\infty} g_b(j)\exp(-\pi j)\right) \tag{5.21}$$

$$= O(\exp(3n\pi)N^{-1}). \tag{5.22}$$

□

We must now estimate not only $I_8^{(0)}(h,k)$, but the entire summation that makes use of $I_8^{(0)}(h,k)$.

Lemma 5.2. *Let $\epsilon > 0$. Then*

$$\left|\sum_{\substack{(k,8)=8\\k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\(h,k)=1}} \omega_8(h,k)e^{-2\pi inh/k} I_8^{(0)}(h,k)\right| = O\big(e^{3n\pi}n^{1/3}N^{-1/3+\epsilon}\big). \tag{5.23}$$

Proof. We begin by taking the previous result into account:

$$\left| \sum_{\substack{(k,8)=8\\k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\(h,k)=1}} \omega_8(h,k) e^{-2\pi i n h/k} I_8^{(0)}(h,k) \right|$$

$$\le \sum_{\substack{(k,8)=8\\k\le N}} \frac{1}{kN} e^{3n\pi} \left| \sum_{\substack{0\le h<k,\\(h,k)=1}} \omega_8(h,k) e^{-2\pi i n h/k} \right|. \tag{5.24}$$

With the result from Chapter 4, we know that

$$\left| \sum_{\substack{0\le h<k,\\(h,k)=1}} \omega_8(h,k) e^{-2\pi i n h/k} \right| = O\left(k^{2/3+\epsilon} n^{1/3}\right). \tag{5.25}$$

This gives us

$$\left| \sum_{\substack{(k,8)=8\\k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\(h,k)=1}} \omega_8(h,k) e^{-2\pi i n h/k} I_8^{(0)}(h,k) \right|$$

$$= O\left(\left| \sum_{\substack{(k,8)=8\\k\le N}} \frac{1}{kN} e^{3n\pi} k^{2/3+\epsilon} n^{1/3} \right| \right) \tag{5.26}$$

$$= O\left(\left| e^{3n\pi} n^{1/3} N^{-1} \sum_{k=1}^{N} \frac{k^{2/3+\epsilon}}{k} \right| \right). \tag{5.27}$$

Recognizing that

$$\sum_{k=1}^{N} \frac{k^{2/3+\epsilon}}{k} = \sum_{k=1}^{N} \frac{k^{2/3+2\epsilon}}{k^{1+\epsilon}} \le \sum_{k=1}^{N} \frac{N^{2/3+2\epsilon}}{k^{1+\epsilon}} = N^{2/3+2\epsilon} \sum_{k=1}^{N} \frac{1}{k^{1+\epsilon}}, \tag{5.28}$$

that $\sum_{k=1}^{N} \frac{1}{k^{1+\epsilon}}$ is bounded above as N gets large, and finally noting that we may replace 2ϵ with ϵ, we now have

$$O\left(\left|e^{3n\pi}n^{1/3}N^{-1}\sum_{k=1}^{N}\frac{k^{2/3+\epsilon}}{k}\right|\right)=O(e^{3n\pi}n^{1/3}N^{-1/3+\epsilon}), \qquad (5.29)$$

and the proof is completed.

□

We now have

$$g_a^{(8)}(n)=\sum_{\substack{(k,8)=8\\ k\leq N}}\frac{i}{k}\sum_{\substack{0\leq h<k,\\ (h,k)=1}}\omega_8(h,k)e^{-2\pi inh/k}I_8^{(1)}(h,k)+O\left(e^{3n\pi}n^{1/3}N^{-1/3+\epsilon}\right). \qquad (5.30)$$

Notice that a substantial portion of the original integral has been absorbed into the error term, which will become negligible when N is taken to be arbitrarily large. Almost everything that remains will contribute to the final form of $g_a^{(8)}(n)$. Notice that the method we applied to dispose of large parts of our integrand can be used to dispose of a portion of $I_8^{(1)}(h,k)$. Recall that

$$I_8^{(1)}(h,k)=\int\limits_{z_I(h,k)}^{z_T(h,k)}\exp\left(\frac{\pi}{8k}\left(\frac{(b-4)^2-8}{z}+z(16n+4a-5)\right)\right)dz, \qquad (5.31)$$

with $b\equiv ah \pmod 8$. If $b=1,7$, then the coefficient of $1/z$ in the exponent is 1. However, if $b=3,5$, then the coefficient is -7. If we suppose $b=3,5$ for the moment, then by almost identical reasoning of Lemma 5.1, we have

$$\left|I_8^{(1)}(h,k)\right|=O\left(\exp(3n\pi)N^{-1}\right). \qquad (5.32)$$

With Lemma 5.2, we therefore have

$$g_a^{(8)}(n) = \sum_{\substack{(k,8)=8\\ k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\ (h,k)=1\\ ah\equiv \pm 1 \pmod 8}} \omega_8(h,k) e^{-2\pi i n h/k} I_8^{(1)}(h,k)$$

$$+ O\left(e^{3n\pi} n^{1/3} N^{-1/3+\epsilon}\right). \quad (5.33)$$

Our object now will be to put the remainder into a form approachable from the theory of Bessel functions.

We now return to the original Rademacher contour of $I_8^{(1)}(h,k)$, along a portion of $K_k^{(-)}$. The brilliance of the contour becomes clear once it is realized that $\Re(1/z) = k$, i.e. is a constant, provided we remain along $K_k^{(-)}$ (and avoid $z = 0$, of course). We wish to make use of the whole of $K_k^{(-)}$, so we will make adjustments to the contour as follows:

$$I_8^{(1)}(h,k) = \left(\oint_{K_k^{(-)}} - \int_0^{z_I(h,k)} - \int_{z_T(h,k)}^{0} \right) \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) dz. \quad (5.34)$$

Notice that $\int_0^{z_I(h,k)}$ and $\int_{z_T(h,k)}^{0}$ are improper: the integrand is not defined at $z = 0$. We interpret these integrals as limits in which a variable approaches 0. We will now show that $\int_0^{z_I(h,k)}$ and $\int_{z_T(h,k)}^{0}$ will not contribute anything of importance:

Lemma 5.3.

$$\left| \int_{z_T(h,k)}^{0} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) dz \right|,$$

$$\left| \int_0^{z_I(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) dz \right|$$

$$= O\big(\exp(3n\pi) N^{-1}\big). \quad (5.35)$$

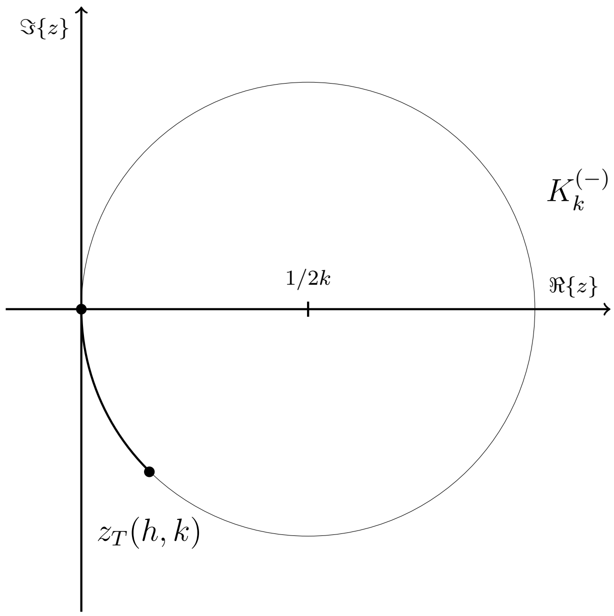


Figure 5.2: $K_k^{(-)}$ with the highlighted path from $z_T(h,k)$ to 0.

Proof. We will keep on $K_k^{(-)}$ for these estimations. Since the estimation is almost identical in either case, we will work with the integral $\int\limits_{z_T(h,k)}^{0}$. We begin by estimating the integrand of the integral:

$$\left|\exp\left(\frac{\pi}{8k}\left(\frac{1}{z}+z(16n+4a-5)\right)\right)\right|$$
$$=\exp\left(\frac{\pi}{8k}\left(\Re(1/z)+\Re(z)(16n+4a-5)\right)\right) \tag{5.36}$$
$$\leq \exp\left(\frac{\pi}{8k}\left(k+\frac{16n+4a-5}{k}\right)\right) \tag{5.37}$$
$$\leq \exp\left(\frac{\pi}{8}+\frac{\pi(16n+4a-5)}{8k^2}\right) \tag{5.38}$$
$$\leq \exp(3n\pi). \tag{5.39}$$

We now estimate the path of integration:

The chord connecting 0 with $z_T(h,k)$ can be no longer than the diameter of $K^{(-)}$, so the length along the arc from 0 to $z_T(h,k)$ can be no longer than $|z_T(h,k)|\frac{\pi}{2}$. Since $|z_T(h,k)| < \sqrt{2}/N$, we have a path length that is $O(N^{-1})$. This gives us

$$
\begin{aligned}
&\left| \int_{z_T(h,k)}^{0} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) dz \right| \\
&\le \int_{z_T(h,k)}^{0} \left| \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right)\right| dz && (5.40) \\
&\le \exp(3n\pi) \int_{z_T(h,k)}^{0} dz && (5.41) \\
&= O\left(\exp(3n\pi) N^{-1}\right). && (5.42)
\end{aligned}
$$

The case for $\int_{0}^{z_I(h,k)}$ is virtually identical.

□

As a consequence of the previous Lemmas 5.1, 5.2, and 5.3, we have

Theorem 5.4.

$$
\begin{aligned}
g_a^{(8)}(n) = &\sum_{\substack{(k,8)=8,\\ k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_8(h,k) e^{-2\pi i n h/k} \\
&\times \oint_{K_k^{(-)}} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) dz + O\left(e^{3n\pi} n^{1/3} N^{-1/3+\epsilon}\right). \quad (5.43)
\end{aligned}
$$

We will save the remaining integral for (5.43) for the next chapter.

5.3.2 $g_a^{(4)}(n)$

Beginning with the opening expression 5.4 for $g_a^{(d)}(n)$ for $d = 4$, and remembering Chapter 3, we have

$$g_a^{(4)}(n) = \sum_{\substack{(k,8)=8 \\ k \le N}} \frac{i}{k\sqrt{2}} \sum_{\substack{0 \le h < k, \\ (h,k)=1}} \omega_4(h,k) e^{-2\pi i n h/k}$$
$$\times \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) \Psi_4(y) dz. \quad (5.44)$$

Since this expression has similar structure to $g_a^{(8)}(n)$, we may utilize many of the same techniques and estimations. We will let

$$\Psi_4(q) = \sum_{j=0}^{\infty} \psi_4(j) q^j \quad (5.45)$$

be the q-expansion of $\Psi_4(q)$, with $\psi_4(0) = 1$. Then write

$$\begin{aligned}
&\int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) \Psi_4(y) dz \\
=& \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) dz \\
&+ \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) \sum_{j=1}^{\infty} \psi_4(j) y^j dz \quad (5.46) \\
=& I_4^{(1)}(h,k) + I_4^{(0)}(h,k), \quad (5.47)
\end{aligned}$$

analogous to the integrals we dealt with in 4.2.1. In particular, $I_4^{(1)}(h,k) = I_8^{(1)}(h,k)$.

Lemma 5.5.

$$\left| I_4^{(0)}(h,k) \right| = O\left(\exp(3n\pi) N^{-1}\right). \quad (5.48)$$

Proof. We write

$$I_4^{(0)}(h,k) \tag{5.49}$$

$$= \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) \sum_{j=1}^{\infty} \psi_4(j) y^j dz \tag{5.50}$$

$$= \sum_{j=1}^{\infty} \psi_4(j) e^{2\pi i H j/k} \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) e^{-2\pi j z^{-1}/2k} dz \tag{5.51}$$

$$= \sum_{j=1}^{\infty} \psi_4(j) e^{2\pi i H j/k} \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1-8j}{z} + z(16n+4a-5)\right)\right) dz. \tag{5.52}$$

Notice that the coefficient of $1/z$ in the exponent of the integrand is now always negative.

Taking advantage of the fact that $\Re(1/z) \geq k$ and $\Re(z) \leq 1/k$, we now examine the magnitude of the integrand:

$$\left|\exp\left(\frac{\pi}{8k}\left(\frac{(1-8j}{z} + z(16n+4a-5)\right)\right)\right| \tag{5.53}$$

$$= \exp\left(\frac{\pi}{8k}(1-8j)\Re(1/z) + \frac{\pi}{8k}(16n+4a-5)\Re(z)\right) \tag{5.54}$$

$$\leq \exp\left(\frac{\pi(1-8j)}{8} + \frac{\pi(16n+4a-5)}{8k^2}\right) \tag{5.55}$$

$$= \exp(-\pi j + 3n\pi). \tag{5.56}$$

We therefore have

$$
\begin{aligned}
&|I_4^{(0)}(h,k)| \\
&= \left| \int\limits_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) \sum_{j=1}^{\infty} \psi_4(j) y^j dz \right| && (5.57)\\
&\leq \sum_{j=1}^{\infty} |\psi_4(j)||e^{2\pi i H j/k}| \\
&\quad \times \int\limits_{z_I(h,k)}^{z_T(h,k)} \left| \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) e^{-2\pi j z^{-1}/2k} \right| dz && (5.58)\\
&= \sum_{j=1}^{\infty} |\psi_4(j)| \int\limits_{z_I(h,k)}^{z_T(h,k)} \left| \exp\left(\frac{\pi}{8k}\left(\frac{1-8j}{z} + z(16n+4a-5)\right)\right) \right| dz && (5.59)\\
&\leq \sum_{j=1}^{\infty} |\psi_4(j)| \exp(-\pi j + 3n\pi) \int\limits_{z_I(h,k)}^{z_T(h,k)} dz. && (5.60)
\end{aligned}
$$

Remembering from Chapter 2 that we may deform our path from $z_I(h,k)$ to $z_T(h,k)$ into the corresponding chord of $K_k^{(-)}$, and that such a chord is $O(N^{-1})$, we have

$$
|I_4^{(0)}(h,k)| = O\left(e^{3n\pi} N^{-1} \sum_{j=1}^{\infty} |\psi_4(j)| e^{-\pi j}\right). \tag{5.61}
$$

Notice that since $\Psi_4(q) = \sum_{j=0}^{\infty} \psi_4(j) q^j$ is a q-series, it is absolutely convergent, which gives us

$$
O\left(e^{3n\pi} N^{-1} \sum_{j=1}^{\infty} |\psi_4(j)| e^{-\pi j}\right) = O\left(\exp(3n\pi) N^{-1}\right), \tag{5.62}
$$

and the proof is completed.

□

In parallel to the case of $(k,8) = 8$, we now show that the entire portion of our formula containing $I_4^{(0)}(h,k)$ will not contribute:

Lemma 5.6. *Let $\epsilon > 0$. Then*

$$\left|\sum_{\substack{(k,8)=4,\\ k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_4(h,k)e^{-2\pi inh/k} I_4^{(0)}(h,k)\right| = O\big(e^{3n\pi}n^{1/3}N^{-1/3+\epsilon}\big). \tag{5.63}$$

Proof. We begin by taking the previous result into account:

$$\begin{aligned}&\left|\sum_{\substack{(k,8)=4,\\ k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_4(h,k)e^{-2\pi inh/k} I_4^{(0)}(h,k)\right| \\ &\le \sum_{\substack{(k,8)=4,\\ k\le N}} \frac{1}{kN}e^{3n\pi}\left|\sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_4(h,k)e^{-2\pi inh/k}\right|.\end{aligned} \tag{5.64}$$

With the result from Chapter 4, we know that

$$\left|\sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_4(h,k)e^{-2\pi inh/k}\right| = O\left(k^{2/3+\epsilon}n^{1/3}\right). \tag{5.65}$$

This gives us

$$\left|\sum_{\substack{(k,8)=4,\\ k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_4(h,k)e^{-2\pi inh/k} I_4^{(0)}(h,k)\right|$$

$$= O\left(\left|\sum_{\substack{(k,8)=4,\\ k\le N}} \frac{1}{kN}e^{3n\pi}k^{2/3+\epsilon}n^{1/3}\right|\right) \tag{5.66}$$

$$= O\left(\left|e^{3n\pi}n^{1/3}N^{-1}\sum_{k=1}^{N}\frac{k^{2/3+\epsilon}}{k}\right|\right). \tag{5.67}$$

We proceed in a manner identical to the proof of Lemma 5.2.

□

We now have

$$g_a^{(4)}(n) = \frac{1}{\sqrt{2}} \sum_{\substack{(k,8)=4,\\ k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_4(h,k) e^{-2\pi inh/k} I_4^{(1)}(h,k)$$

$$+ O\left(e^{3n\pi} n^{1/3} N^{-1/3+\epsilon}\right). \quad (5.68)$$

Notice that $I_4^{(1)}(h,k)$ has the same form as $I_8^{(1)}(h,k)$. We may therefore apply Lemma 5.3, in conjunction with (5.34). We then have the following:

Theorem 5.7.

$$g_a^{(4)}(n) = \frac{1}{\sqrt{2}} \sum_{\substack{(k,8)=4,\\ k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_4(h,k) e^{-2\pi inh/k}$$

$$\times \oint_{K_k^{(-)}} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) dz + O\left(e^{3n\pi} n^{1/3} N^{-1/3+\epsilon}\right). \quad (5.69)$$

5.3.3 $g_a^{(2)}(n)$

Again referring to Equation 5.4 and Chapter 3, we have

$$g_a^{(2)}(n) = \sum_{\substack{(k,8)=2,\\ k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_2(h,k) e^{-2\pi inh/k}$$

$$\times \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(z(16n+4a-5)\right)\right) \Psi_2(y) dz. \quad (5.70)$$

Expanding,

$$g_a^{(2)}(n) = \sum_{\substack{(k,8)=2\\k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\(h,k)=1}} \omega_2(h,k)e^{-2\pi inh/k}$$

$$\times \sum_{j=0}^{\infty} \psi_2(j)e^{2\pi iHj/k} \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(z(16n+4a-5)\right)\right) e^{-2\pi jz^{-1}/4k}dz \quad (5.71)$$

$$= \sum_{\substack{(k,8)=2\\k\le N}} \frac{i}{k} \sum_{\substack{0\le h<k,\\(h,k)=1}} \omega_2(h,k)e^{-2\pi inh/k}$$

$$\times \sum_{j=0}^{\infty} \psi_2(j)e^{2\pi iHj/k} \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{-4j}{z} + z(16n+4a-5)\right)\right) dz. \quad (5.72)$$

Critically, the coefficient of $1/z$ in the exponential of the integrand is never positive. Therefore, we may immediately apply the reasoning of Lemmas 5.1 and 5.5:

$$\left|\exp\left(\frac{\pi}{8k}\left(\frac{(-4j}{z} + z(16n+4a-5)\right)\right)\right| \quad (5.73)$$

$$= \exp\left(\frac{\pi}{8k}(-4j)\Re(1/z) + \frac{\pi}{8k}(16n+4a-5)\Re(z)\right) \quad (5.74)$$

$$\le \exp\left(\frac{\pi(-4j)}{8} + \frac{\pi(16n+4a-5)}{8k^2}\right) \quad (5.75)$$

$$= \exp(-\pi j/2 + 3n\pi). \quad (5.76)$$

Therefore,

$$\begin{aligned}
&\left| \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(z(16n+4a-5)\right)\right)\Psi_2(y)dz \right| \\
&= \left| \sum_{j=0}^{\infty} \psi_2(j) e^{2\pi i H j/k} \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{-4j}{z} + z(16n+4a-5)\right)\right) dz \right| &(5.77)\\
&\leq \sum_{j=1}^{\infty} |\psi_2(j)||e^{2\pi i H j/k}| \int_{z_I(h,k)}^{z_T(h,k)} \left|\exp\left(\frac{\pi}{8k}\left(\frac{-4j}{z} + z(16n+4a-5)\right)\right)\right| dz &(5.78)\\
&\leq \sum_{j=1}^{\infty} |\psi_4(j)| \exp(-\pi j/2 + 3n\pi) \int_{z_I(h,k)}^{z_T(h,k)} dz. &(5.79)
\end{aligned}$$

Finally altering our contour, and remembering that $\Psi_2(y)$ is absolutely convergent, we have

$$\left| \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{-4j}{z} + z(16n+4a-5)\right)\right)\Psi_2(y)dz \right| = O\left(\exp(3n\pi)N^{-1}\right). \quad (5.80)$$

We now take note that

$$\left|g_a^{(2)}(n)\right| = O\left(\sum_{\substack{(k,8)=2,\\ k\leq N}} \frac{1}{kN} e^{3n\pi} \left| \sum_{\substack{0\leq h<k,\\ (h,k)=1}} \omega_2(h,k) e^{-2\pi i n h/k} \right| \right). \quad (5.81)$$

Returning to the previous chapter:

$$\left| \sum_{\substack{0\leq h<k,\\ (h,k)=1}} \omega_2(h,k) e^{-2\pi i n h/k} \right| = O\left(k^{2/3+\epsilon} n^{1/3}\right). \quad (5.82)$$

Therefore,

$$
\begin{aligned}
&|g_a^{(2)}(n)| \\
=&\left| \sum_{\substack{(k,8)=2,\\ k\leq N}} \frac{i}{k} \sum_{\substack{0\leq h<k,\\ (h,k)=1}} \omega_2(h,k) e^{-2\pi i n h/k} \right. \\
&\left. \times \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{-4j}{z} + z(16n+4a-5)\right)\right) \Psi_2(y)dz \right| \\
=&O\left(\left| \sum_{\substack{(k,8)=4,\\ k\leq N}} \frac{1}{kN} e^{3n\pi} k^{2/3+\epsilon} n^{1/3} \right|\right) && (5.83) \\
=&O\left(\left| e^{3n\pi} n^{1/3} N^{-1} \sum_{k=1}^{N} \frac{k^{2/3+\epsilon}}{k} \right|\right). && (5.84)
\end{aligned}
$$

We proceed in a manner identical to the proof of Lemma 5.2, to achieve the following:

Theorem 5.8.

$$g_a^{(2)}(n) = O\left(e^{3n\pi} n^{1/3} N^{-1/3+\epsilon}\right). \tag{5.85}$$

5.3.4 $g_a^{(1)}(n)$

Again referring to Equation 5.4 and Chapter 3, we have

$$g_a^{(1)}(n) = \sum_{\substack{(k,8)=1,\\ k\leq N}} \frac{i}{k2\sqrt{2}} \left|\csc\left(\frac{\pi a k}{8}\right)\right| \sum_{\substack{0\leq h<k,\\ (h,k)=1}} \omega_1(h,k) e^{-2\pi i n h/k}$$

$$\times \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) \Psi_1(y) dz \tag{5.86}$$

$$= \sum_{\substack{(k,8)=1,\\ k\leq N}} \frac{i}{k2\sqrt{2}} \left|\csc\left(\frac{\pi a k}{8}\right)\right| \sum_{\substack{0\leq h<k,\\ (h,k)=1}} \omega_1(h,k) e^{-2\pi i n h/k} \tag{5.87}$$

$$\times \left(I_1^{(1)}(h,k) + I_1^{(0)}(h,k)\right), \tag{5.88}$$

with

$$I_1^{(1)}(h,k) = \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) dz, \tag{5.89}$$

and

$$I_1^{(0)}(h,k) = \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) \sum_{j=1}^{\infty} \psi_1(j) y^j dz. \tag{5.90}$$

Notice that the coefficient of $1/z$ in the exponent of $I_1^{(1)}(h,k)$ is positive, indicating that, unlike in the previous case, there will be a nontrivial contribution. However, contrary to the cases $(k,8) = 8, 4$, that same coefficient is $1/4$ rather than 1. This will not substantially affect the application of the same techniques employed in the previous cases, although it will slightly affect results in the sequel chapter.

Lemma 5.9.

$$\left|I_1^{(0)}(h,k)\right| = O\left(\exp(3n\pi)N^{-1}\right). \tag{5.91}$$

Proof. We write

$$I_1^{(0)}(h,k) \tag{5.92}$$

$$= \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) \sum_{j=1}^{\infty} \psi_1(j) y^j dz \tag{5.93}$$

$$= \sum_{j=1}^{\infty} \psi_1(j) e^{2\pi i H j/k} \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) e^{-2\pi j z^{-1}/8k} dz \tag{5.94}$$

$$= \sum_{j=1}^{\infty} \psi_1(j) e^{2\pi i H j/k} \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{-2j+1/4}{z} + z(16n+4a-5)\right)\right) dz. \tag{5.95}$$

Taking advantage of the fact that $\Re(1/z) \geq k$ and $\Re(z) \leq 1/k$, we now examine the magnitude of the integrand:

$$\left|\exp\left(\frac{\pi}{8k}\left(\frac{-2j+1/4}{z} + z(16n+4a-5)\right)\right)\right| \tag{5.96}$$

$$= \exp\left(\frac{\pi}{8k}(-2j+1/4)\Re(1/z) + \frac{\pi}{8k}(16n+4a-5)\Re(z)\right) \tag{5.97}$$

$$\leq \exp\left(\frac{\pi(-2j+1/4)}{8} + \frac{\pi(16n+4a-5)}{8k^2}\right) \tag{5.98}$$

$$\leq \exp(-\pi j/4 + 3n\pi). \tag{5.99}$$

We therefore have

$$
\begin{aligned}
&|I_1^{(0)}(h,k)| \\
&= \left| \int_{z_I(h,k)}^{z_T(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) \sum_{j=1}^{\infty} \psi_1(j) y^j dz \right| && (5.100) \\
&\leq \sum_{j=1}^{\infty} |\psi_1(j)| |e^{2\pi i H j/k}| \\
&\quad \times \int_{z_I(h,k)}^{z_T(h,k)} \left| \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) e^{-2\pi j z^{-1}/8k} \right| dz && (5.101) \\
&= \sum_{j=1}^{\infty} |\psi_1(j)| \int_{z_I(h,k)}^{z_T(h,k)} \left| \exp\left(\frac{\pi}{8k}\left(\frac{-2j+1/4}{z} + z(16n+4a-5)\right)\right) \right| dz && (5.102) \\
&\leq \sum_{j=1}^{\infty} |\psi_1(j)| \exp(-\pi j/4 + 3n\pi) \int_{z_I(h,k)}^{z_T(h,k)} dz. && (5.103)
\end{aligned}
$$

Deforming our path from $z_I(h,k)$ to $z_T(h,k)$ once again into the corresponding chord of $K_k^{(-)}$, we have

$$
|I_1^{(0)}(h,k)| = O\left(e^{3n\pi} N^{-1} \sum_{j=1}^{\infty} |\psi_1(j)| e^{-\pi j/4}\right). \tag{5.104}
$$

Notice that since $\Psi_1(y)$ is absolutely convergent, which gives us

$$
O\left(e^{3n\pi} N^{-1} \sum_{j=1}^{\infty} |\psi_1(j)| e^{-\pi j}\right) = O\left(\exp(3n\pi) N^{-1}\right), \tag{5.105}
$$

and the proof is completed.

□

We will again show that no part of $g_a^{(1)}(n)$ associated with $I_1^{(0)}(h,k)$ will contribute to our final formula:

Lemma 5.10. *Let $\epsilon > 0$. Then*

$$\left| \sum_{\substack{(k,8)=1,\\ k\le N}} \frac{i}{k} \left|\csc\left(\frac{\pi a k}{8}\right)\right| \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_1(h,k) e^{-2\pi i n h/k} I_1^{(0)}(h,k) \right| = O\big(e^{3n\pi} n^{1/3} N^{-1/3+\epsilon}\big). \quad (5.106)$$

Proof. We begin by taking the previous result into account:

$$\left| \sum_{\substack{(k,8)=1,\\ k\le N}} \frac{i}{k} \left|\csc\left(\frac{\pi a k}{8}\right)\right| \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_1(h,k) e^{-2\pi i n h/k} I_1^{(0)}(h,k) \right|$$
$$\le \sum_{\substack{(k,8)=1,\\ k\le N}} \frac{1}{kN} e^{3n\pi} \left|\csc\left(\frac{\pi a k}{8}\right)\right| \left| \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_1(h,k) e^{-2\pi i n h/k} \right|. \quad (5.107)$$

With the result from Chapter 4, we know that

$$\left| \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_1(h,k) e^{-2\pi i n h/k} \right| = O\left(k^{2/3+\epsilon} n^{1/3}\right). \quad (5.108)$$

This gives us

$$\left| \sum_{\substack{(k,8)=1,\\ k\le N}} \frac{i}{k} \left|\csc\left(\frac{\pi a k}{8}\right)\right| \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_1(h,k) e^{-2\pi i n h/k} I_1^{(0)}(h,k) \right|$$
$$= O\left(\left| \sum_{\substack{(k,8)=1,\\ k\le N}} \frac{1}{kN} \left|\csc\left(\frac{\pi a k}{8}\right)\right| e^{3n\pi} k^{2/3+\epsilon} n^{1/3} \right| \right) \quad (5.109)$$
$$= O\left(\left| e^{3n\pi} n^{1/3} N^{-1} \sum_{k=1}^{N} \left|\csc\left(\frac{\pi a k}{8}\right)\right| \frac{k^{2/3+\epsilon}}{k} \right| \right). \quad (5.110)$$

We must quickly make note of the behavior of the cosecant term. Since $(k,8)=1$, $\pi ak/8$ will never become an integer multiple of π. Moreover, we know that by the periodicity of $\csc(x)$ that

$$\left|\csc\left(\frac{\pi ak}{8}\right)\right| \in \{|\csc(\pi am/8)| : m = 1,3,5,7\}. \tag{5.111}$$

We may therefore easily bound $\left|\csc\left(\frac{\pi ak}{8}\right)\right|$ by the maximum of its four possible defined values.

Let $C = \max\{|\csc(\pi am/8)| : m = 1,3,5,7\}$. Then

$$\sum_{k=1}^{N}\left|\csc\left(\frac{\pi ak}{8}\right)\right|\frac{k^{2/3+\epsilon}}{k} \le C\sum_{k=1}^{N}\frac{k^{2/3+\epsilon}}{k} = O\left(\sum_{k=1}^{N}\frac{k^{2/3+\epsilon}}{k}\right). \tag{5.112}$$

With this minor obstacle sidestepped, we now have

$$\left|\sum_{\substack{(k,8)=1,\\ k\le N}}\frac{i}{k}\left|\csc\left(\frac{\pi ak}{8}\right)\right|\sum_{\substack{0\le h<k,\\ (h,k)=1}}\omega_1(h,k)e^{-2\pi inh/k}I_1^{(0)}(h,k)\right|$$
$$= O\left(\left|e^{3n\pi}n^{1/3}N^{-1}\sum_{k=1}^{N}\frac{k^{2/3+\epsilon}}{k}\right|\right). \tag{5.113}$$

We finish in the manner of Lemma 5.2.

□

We now have

$$g_a^{(1)}(n) = \frac{1}{2\sqrt{2}}\sum_{\substack{(k,8)=4,\\ k\le N}}\frac{i}{k}\left|\csc\left(\frac{\pi ak}{8}\right)\right|\sum_{\substack{0\le h<k,\\ (h,k)=1}}\omega_1(h,k)e^{-2\pi inh/k}I_1^{(1)}(h,k)$$
$$+ O\left(e^{3n\pi}n^{1/3}N^{-1/3+\epsilon}\right). \tag{5.114}$$

We must now apply an adjustment to our contour analogous to that of Lemma 5.3, with the knowledge that $I_1^{(1)}(h,k)$ has a slightly different form from $I_4^{(1)}(h,k)$ and $I_8^{(1)}(h,k)$.

Lemma 5.11.

$$\left| \int_{z_T(h,k)}^{0} \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) dz \right|,$$
$$\left| \int_{0}^{z_I(h,k)} \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) dz \right|$$
$$= O\big(\exp(3n\pi)N^{-1}\big). \quad (5.115)$$

Proof. We will keep on $K_k^{(-)}$ for these estimations, remembering that $\Re(1/z) = k$. Since the estimation is almost identical in either case, we will work with the integral $\int_{z_T(h,k)}^{0}$. We begin by estimating the integrand of the integral:

$$\left|\exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right)\right|$$
$$= \exp\left(\frac{\pi}{8k}\left(\Re(1/4z) + \Re(z)(16n+4a-5)\right)\right) \quad (5.116)$$
$$\leq \exp\left(\frac{\pi}{8k}\left(k/4 + \frac{16n+4a-5}{k}\right)\right) \quad (5.117)$$
$$\leq \exp\left(\frac{\pi}{32} + \frac{\pi(16n+4a-5)}{8k^2}\right) \quad (5.118)$$
$$\leq \exp(3n\pi). \quad (5.119)$$

We now estimate the path of integration:

The chord connecting 0 with $z_T(h,k)$ can be no longer than the diameter of $K^{(-)}$, so the length along the arc from 0 to $z_T(h,k)$ can be no longer than $|z_T(h,k)|\frac{\pi}{2}$. Since $|z_T(h,k)| < \sqrt{2}/N$, we have a path length that is $O(N^{-1})$. This gives us

$$\left| \int\limits_{z_T(h,k)}^{0} \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) dz \right|$$

$$\le \int\limits_{z_T(h,k)}^{0} \left| \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) \right| dz \tag{5.120}$$

$$\le \exp(3n\pi) \int\limits_{z_T(h,k)}^{0} dz \tag{5.121}$$

$$= O\left(\exp(3n\pi)N^{-1}\right). \tag{5.122}$$

The case for $\int\limits_{0}^{z_I(h,k)}$ is virtually identical.

□

We now have the following:

Theorem 5.12.

$$g_a^{(1)}(n) = \frac{1}{2\sqrt{2}} \sum_{\substack{(k,8)=1 \\ k\le N}} \frac{i}{k} \left| \csc\left(\frac{\pi a k}{8}\right) \right| \sum_{\substack{0\le h<k, \\ (h,k)=1}} \omega_1(h,k) e^{-2\pi i n h/k}$$
$$\times \oint_{K_k^{(-)}} \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) dz + O\left(e^{3n\pi} n^{1/3} N^{-1/3+\epsilon}\right). \tag{5.123}$$

CHAPTER 6

LIMIT PROCESS COMPLETED

Once we take (5.43), (5.69), (5.85), (5.123), and collect the error terms, we have

$$\begin{aligned} g_a(n) = g_a^{(8)}(n) + g_a^{(4)}(n) + g_a^{(2)}(n) + g_a^{(1)}(n) \\ = \frac{1}{2\sqrt{2}} \sum_{\substack{(k,8)=1 \\ k \leq N}} \frac{i}{k} \left| \csc\left(\frac{\pi a k}{8}\right) \right| \sum_{\substack{0 \leq h < k, \\ (h,k)=1}} \omega_1(h,k) e^{-2\pi i n h/k} \\ \times \oint_{K_k^{(-)}} \exp\left(\frac{\pi}{8k}\left(\frac{1}{4z} + z(16n+4a-5)\right)\right) dz \\ + \frac{1}{\sqrt{2}} \sum_{\substack{(k,8)=4 \\ k \leq N}} \frac{i}{k} \sum_{\substack{0 \leq h < k, \\ (h,k)=1}} \omega_4(h,k) e^{-2\pi i n h/k} \\ \times \oint_{K_k^{(-)}} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) dz \\ + \sum_{\substack{(k,8)=8 \\ k \leq N}} \frac{i}{k} \sum_{\substack{0 \leq h < k, \\ (h,k)=1}} \omega_8(h,k) e^{-2\pi i n h/k} \\ \times \oint_{K_k^{(-)}} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) dz \\ + O\left(e^{3n\pi} n^{1/3} N^{-1/3+\epsilon}\right). \quad (6.1) \end{aligned}$$

The remaining integrals can be put in terms of modified Bessel functions. We shall complete this step before finishing with the rather obvious limit process.

6.1 Bessel Functions

We briefly recall the formula for $p(n)$ as summarized in the beginning of Chapter 5:

$$p(n) = \sum_{k=1}^{N} \frac{i}{k} \sum_{\substack{0 \le h < k, \\ (h,k)=1}} e^{-2\pi inh/k} \omega(h,k) \oint_{K_k^{(-)}} e^{2n\pi z/k} e^{\pi(z^{-1}-z)/12k} \sqrt{z}\, dz$$
$$+ O(N^{-1/2}).$$

The remaining step to simplify what is left of the integration is to shift variables, and express our integral in terms of modified Bessel functions [23]. In particular, the function of interest here is

$$I_{3/2}(Z) = \sqrt{\frac{2Z}{\pi}} \frac{d}{dZ} \left(\frac{\sinh(Z)}{Z} \right).$$

With the resulting modification, and finally taking N to be arbitrarily large, we have

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k}\ A(n,k) \frac{d}{dx} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)}\right)}{\sqrt{x - \frac{1}{24}}} \right) \Bigg|_{x=n}.$$

In the case of $g_a(n)$, the necessary Bessel function is of integer order, but otherwise the process of modifying the given integrals is nearly identical to the process just summarized.

Lemma 6.1.

$$\oint_{K_k^{(-)}} \exp\left(\frac{\pi}{8k} \left(\frac{1}{z} + z(16n + 4a - 5) \right) \right) dz$$
$$= \frac{-2\pi i}{\sqrt{16n + 4a - 5}} I_1 \left(\frac{\pi\sqrt{16n + 4a - 5}}{4k} \right), \quad (6.2)$$

$$\oint_{K_k^{(-)}} \exp\left(\frac{\pi}{8k} \left(\frac{1}{4z} + z(16n + 4a - 5) \right) \right) dz$$
$$= \frac{-\pi i}{\sqrt{16n + 4a - 5}} I_1 \left(\frac{\pi\sqrt{16n + 4a - 5}}{8k} \right). \quad (6.3)$$

Proof. Beginning with the first integral, we change variables, first by $w = 1/z$, then by $w = 8kt/\pi$:

$$
\begin{aligned}
&\oint_{K_k^{(-)}} \exp\left(\frac{\pi}{8k}\left(\frac{1}{z} + z(16n+4a-5)\right)\right) dz \\
&= -\int_{k-\infty i}^{k+\infty i} w^{-2} \exp\left(\frac{\pi w}{8k} + \frac{\pi(16n+4a-5)}{8kw}\right) dw \qquad (6.4)\\
&= -\int_{\pi/8-\infty i}^{\pi/8+\infty i} \frac{\pi}{8k} t^{-2} \exp\left(t + \frac{\pi^2(16n+4a-5)}{64k^2 t}\right) dt. \qquad (6.5)
\end{aligned}
$$

Finally, letting $Z = \frac{\pi}{4k}\sqrt{16n+4a-5}$, we have

$$
\begin{aligned}
&= -\frac{\pi}{8k}\int_{\pi/8-\infty i}^{\pi/8+\infty i} t^{-2} \exp\left(t + \frac{Z^2}{4t}\right) dt \qquad (6.6)\\
&= -\frac{\pi}{8k}\frac{2\pi i}{Z/2}\left(\frac{Z/2}{2\pi i}\int_{\pi/8-\infty i}^{\pi/8+\infty i} t^{-2} \exp\left(t + \frac{Z^2}{4t}\right) dt\right) \qquad (6.7)\\
&= \frac{-2\pi i}{\sqrt{16n+4a-5}} I_1(Z). \qquad (6.8)
\end{aligned}
$$

The second relation may be proved almost identically to the first, except with the shift of variables $w = 32kt/\pi$.

□

6.2 Completed Formulæ

With the previous Lemma, we may write

$$
\begin{aligned}
g_a(n) = g_a^{(8)}(n) + g_a^{(4)}(n) + g_a^{(2)}(n) + g_a^{(1)}(n) \\
= \frac{\pi\sqrt{2}}{4\sqrt{16n+4a-5}} \sum_{\substack{(k,8)=1 \\ k \le N}} \frac{1}{k} \left| \csc\left(\frac{\pi a k}{8}\right) \right| \sum_{\substack{0 \le h < k, \\ (h,k)=1}} \omega_1(h,k) e^{-2\pi i n h/k} \\
\times I_1\left(\frac{\pi\sqrt{16n+4a-5}}{8k}\right) \\
+ \frac{\pi\sqrt{2}}{\sqrt{16n+4a-5}} \sum_{\substack{(k,8)=4 \\ k \le N}} \frac{1}{k} \sum_{\substack{0 \le h < k, \\ (h,k)=1}} \omega_4(h,k) e^{-2\pi i n h/k} \\
\times I_1\left(\frac{\pi\sqrt{16n+4a-5}}{4k}\right) \\
+ \frac{2\pi}{\sqrt{16n+4a-5}} \sum_{\substack{(k,8)=8 \\ k \le N}} \frac{1}{k} \sum_{\substack{0 \le h < k, \\ (h,k)=1}} \omega_8(h,k) e^{-2\pi i n h/k} \\
\times I_1\left(\frac{\pi\sqrt{16n+4a-5}}{4k}\right) \\
+ O\left(e^{3n\pi} n^{1/3} N^{-1/3+\epsilon}\right). \quad (6.9)
\end{aligned}
$$

Since the corresponding Bessel Functions have integer order, we have simplified our formulæ as far as possible.

The final step is now clear, and we have saved it until the end as an admittedly symbolic act.

Theorem 6.2. *Let $d = 1, 4$, and*

$$A_d(n,k) = \sum_{\substack{0\le h<k,\\ (h,k)=1}} \omega_d(h,k) e^{-2\pi i n h/k}, \tag{6.10}$$

and

$$A_8(n,k) = \sum_{\substack{0\le h<k,\\ (h,k)=1,\\ ah\equiv\pm 1 \pmod 8}} \omega_8(h,k) e^{-2\pi i n h/k}. \tag{6.11}$$

Then

$$\begin{aligned} g_a(n) = \frac{\pi\sqrt{2}}{4\sqrt{16n+4a-5}} \sum_{(k,8)=1} \left|\csc\left(\frac{\pi a k}{8}\right)\right| \frac{A_1(n,k)}{k} I_1\left(\frac{\pi\sqrt{16n+4a-5}}{8k}\right) \\ + \frac{\pi\sqrt{2}}{\sqrt{16n+4a-5}} \sum_{(k,8)=4} \frac{A_4(n,k)}{k} I_1\left(\frac{\pi\sqrt{16n+4a-5}}{4k}\right) \\ + \frac{2\pi}{\sqrt{16n+4a-5}} \sum_{(k,8)=8} \frac{A_8(n,k)}{k} I_1\left(\frac{\pi\sqrt{16n+4a-5}}{4k}\right). \end{aligned} \tag{6.12}$$

Proof. Take (6.9) and let $N \to \infty$.

□

We note that since $|z_I(h,k)|, |z_T(h,k)| \le \frac{\sqrt{2}}{N}$, $N \to \infty$ immediately implies that the endpoints along our z-contour approach 0, i.e., $q = \exp\left(2\pi i\left(\frac{h}{k} + \frac{iz}{k}\right)\right) \to \exp(2\pi i h/k)$, approaching the corresponding root of unity, confirming our initial heuristic approach in Chapters 2 and 5 that the roots of unity along the unit circle contained much of the information concerning the behavior and structure of $g_a(n)$.

CHAPTER 7

NUMERICAL TESTS

In the previous chapters we have developed and rigorously justified our formulæ for $g_a(n)$. However, it is not automatically obvious that such an equation can be effectively used. Hardy and Wright [9, Chapters 1, 22, Appendix], for example, list multiple different formulæ for the prime numbers which, although valid, are veridically useless. While we do not claim to have conducted a detailed study into the overall efficiency of our results, we may point to strong numerical evidence that our work is useful.

We have subjected (6.12) to numerical tests through Mathematica for the first 200 positive integers. Of course, since (6.12) is an infinite series over the variable k, we must truncate our formulæ. The results for the unrestricted partition function $p(n)$, developed by Hardy, Ramanujan, and Rademacher, would give adequate results if the series over k were truncated to a number of terms on the order of $\sqrt{n}$.

Suspecting (6.12) to behave in a similar fashion, we truncated the series over k to various multiples of $\sqrt{n}$, for $1 \leq n \leq 200$. We found that, by computing (6.12) truncated for $1 \leq k \leq 3\sqrt{n}$, we obtain a numerical result that differs from the correct value of $g_a(n)$ by an absolute error strictly less than 0.5. See Tables 7.1 and 7.2. Given that the correct value of $g_a(n)$ is always a nonnegative integer, we need only round our numerical result to achieve the exact answer.

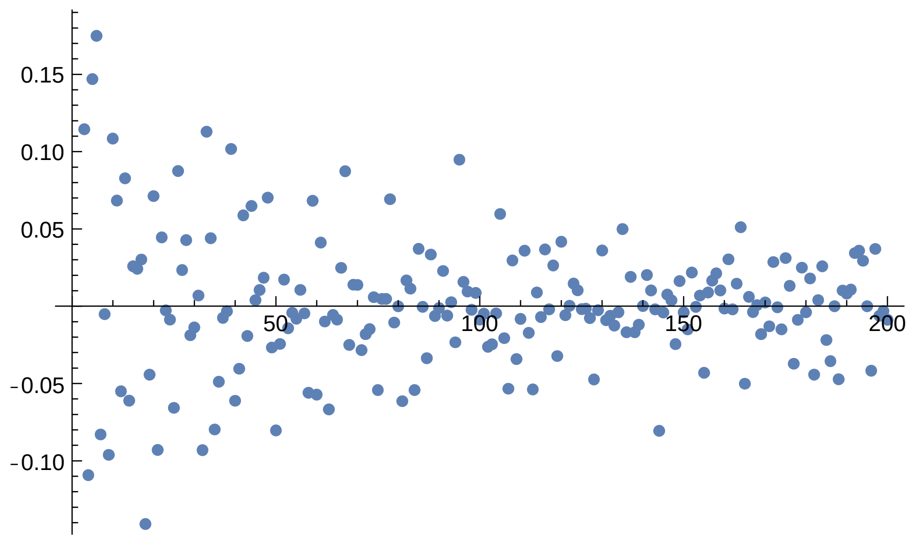
Moreover, upon examining the behavior of the error term as n grows, we find that the largest errors—slightly less than ± 0.3 for $a = 1$, and slightly more than ± 0.3 for $a = 3$—occur for small values of n. For larger n, the absolute error is much smaller: for $n > 100$, the largest absolute error for $a = 1$ is 0.0806. The largest absolute error for $a = 3$ is 0.0667. See Figures 7.1 and 7.2.

These results strongly suggest that summing over the first $3\sqrt{n}$ terms of (6.12) and rounding is sufficient to give the correct value for $g_a(n)$ for any value of n.

n	$g_1(n)$	Eqn. (6.12), $1 \leq k \leq 3\sqrt{n}$	Absolute Error
1	1	0.7784305652	0.2215694348
2	1	0.7196351376	0.2803648624
3	1	1.114485490	0.114485490
4	2	1.890769460	0.109230540
5	2	2.146945231	0.146945231
6	2	2.174897898	0.174897898
7	3	2.917027886	0.082972114
8	4	3.994864237	0.005135763
9	5	4.903833678	0.096166322
10	5	5.108441112	0.108441112
20	26	26.07125673	0.07125673
30	92	91.98629238	0.01370762
40	288	287.9388309	0.0611691
50	783	782.9196301	0.0803699
60	1989	1988.942843	0.057157
70	4695	4695.013781	0.013781
80	10570	10569.99993	0.00007
90	22705	22704.99878	0.00122
100	47091	47090.99132	0.00868
110	94450	94449.99175	0.00825
120	184376	184376.0417	0.0417
130	350845	350845.0360	0.0360
140	653257	653257.0001	0.0001
150	1191854	1191853.996	0.004
160	2135922	2135921.998	0.002
170	3764251	3764251.002	0.002
180	6534755	6534754.996	0.004
190	11185460	11185460.01	0.001
200	18900623	18900622.99	0.001

Table 7.1: $g_1(n)$ compared to (6.12) truncated for k, with $a = 1$. See Appendix B for relevant Mathematica code.

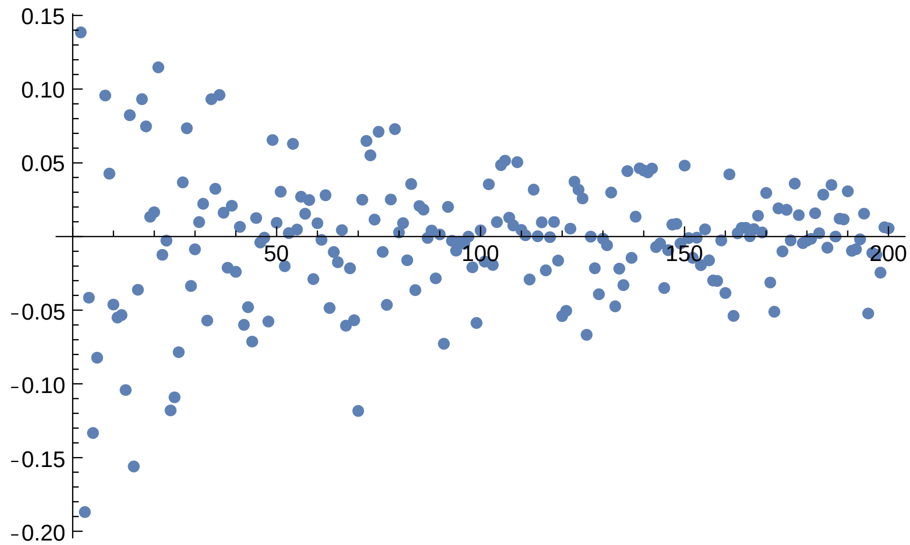
Figure 7.1: Graph of the difference between (6.12), with $1 \leq k \leq 3\sqrt{n}$, and the correct value of $g_1(n)$, $1 \leq n \leq 200$. See Appendix B for relevant Mathematica code.



n	$g_3(n)$	Eqn. (6.12), $a = 3$, $1 \leq k \leq 3\sqrt{n}$	Absolute Error
1	0	0.2908871603	0.2908871603
2	0	0.1385488254	0.1385488254
3	1	0.8129880460	0.1870119540
4	1	0.9584818018	0.0415181982
5	1	0.8666320258	0.1333679742
6	1	0.9177374697	0.0822625303
7	1	1.323340028	0.323340028
8	2	2.095679009	0.095679009
9	2	2.042654099	0.042654099
10	2	1.953812941	0.046187059
20	12	12.01649403	0.01649403
30	40	39.99130288	0.00869712
40	127	126.9760443	0.0239557
50	338	338.0093175	0.0093175
60	865	865.0090307	0.0090307
70	2023	2022.881670	0.118330
80	4560	4560.002784	0.002784
90	9754	9754.001400	0.001400
100	20223	20223.00416	0.00416
110	40461	40461.00444	0.00444
120	78939	78938.94603	0.05397
130	149955	149954.9985	0.0015
140	279016	279016.0448	0.0448
150	508454	508454.0481	0.0481
160	910572	910571.9617	0.0383
170	1603268	1603268.030	0.030
180	2781541	2781540.997	0.003
190	4757566	4757566.031	0.031
200	8034534	8034534.006	0.006

Table 7.2: $g_3(n)$ compared to (6.12) truncated for k, with $a = 3$. See Appendix B for relevant Mathematica code.

Figure 7.2: Graph of the difference between (6.12), with $1 \leq k \leq 3\sqrt{n}$, and the correct value of $g_3(n)$, $1 \leq n \leq 200$. See Appendix B for relevant Mathematica code.



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Appendix A

ASYMPTOTICS AND RELATIONSHIP WITH THE SILVER RATIO

In, say, the theory of the distribution of primes, or more ambitious problems in additive number theory, such as the Goldbach conjecture, exact formulæ are exceedingly rare or cumbersome [9],[27]. These problems must usually be studied with much less reliable asymptotic formulæ. However, since we can achieve exact, convergent formulæ in the cases discussed in this paper, we have the advantage of deriving and studying the less reliable, but simpler, asymptotics, with relatively little work.

We begin with a class of partitions studied by Lehner in 1941 [12], already alluded to in Chapter 1.

Definition A.1. Fix $a = 1$ or 2. A Rogers–Ramanujan partition of type a is composed of parts of the form $\pm a \pmod 5$. Let $r_a(n)$ represent the number of type-a Rogers–Ramanujan partitions of n.

Lehner proved the following:

Theorem A.2 (Lehner)**.**

$$r_a(n) = \frac{\pi}{\sqrt{60n - 12a + 13}} \sum_{(k,5)=1} \left| \csc\left(\frac{\pi a k}{5}\right) \right| \frac{A_{5,1}(n,k)}{k} I_1\left(\frac{\pi\sqrt{60n-12a+13}}{15k}\right) + \frac{2\pi}{\sqrt{60n-12a+13}} \sum_{(k,5)=5} \frac{A_{5,5}(n,k)}{k} I_1\left(\frac{\pi\sqrt{60n-12a+13}}{15k}\right), \quad \text{(A.1)}$$

with $A_{5,1}(n,k), A_{5,5}(n,k)$ sums of roots of unity associated with (6.10), (6.11).

Notice the similarity to Theorem 6.2. Lehner was able to derive simpler asymptotic formulæ for $r_a(n)$, and in so doing, demonstrated a remarkable relationship between $r_a(n)$ and the golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$.

We will demonstrate that an analogous relationship holds between $g_a(n)$ and the silver ratio, $\delta_S = 1 + \sqrt{2}$, by developing their corresponding asymptotic formulæ.

The method by which we arrive at this relationship is almost identical to Lehner's method, so we do not claim this as a substantial original result. We only wish to make note of it, since it is a beautiful consequence of the more exact work done in the main body of our paper.

We will begin by noting a lemma whose proof can be found in Watson [28]:

Lemma A.3. *If* $|X| < 1$, *then*

$$I_1(X) = O(X). \tag{A.2}$$

If $|X| > 1$, *then*

$$I_1(X) \sim \frac{e^X}{\sqrt{2\pi X}}. \tag{A.3}$$

With this in mind, we now develop the asymptotics of $g_a(n)$.

Theorem A.4.

$$g_a(n) \sim \sqrt{\frac{1}{2}}\frac{\csc(\pi a/8)}{(16n+4a-5)^{3/4}} e^{\frac{\pi}{8}\sqrt{16n+4a-5}}. \tag{A.4}$$

Proof. Returning to Theorem 6.2, we have

$$\begin{aligned} g_a(n) = \frac{\pi\sqrt{2}}{4\sqrt{16n+4a-5}} &\sum_{(k,8)=1} \left|\csc\left(\frac{\pi a k}{8}\right)\right| \frac{A_1(n,k)}{k} I_1\left(\frac{\pi\sqrt{16n+4a-5}}{8k}\right) \\ &+ \frac{\pi\sqrt{2}}{\sqrt{16n+4a-5}} \sum_{(k,8)=4} \frac{A_4(n,k)}{k} I_1\left(\frac{\pi\sqrt{16n+4a-5}}{4k}\right) \\ &+ \frac{2\pi}{\sqrt{16n+4a-5}} \sum_{(k,8)=8} \frac{A_8(n,k)}{k} I_1\left(\frac{\pi\sqrt{16n+4a-5}}{4k}\right). \end{aligned} \tag{A.5}$$

For convenience, let $X = \frac{\pi\sqrt{16n+4a-5}}{8}$. Notice that $X > 1$. Anticipating that the most significant contribution to the value of $g_a(n)$ will be the very first term, i.e. $k = 1$, we will factor this term out of our formula:

$$g_a(n) = \frac{\pi\sqrt{2}}{4\sqrt{16n+4a-5}} \csc\left(\frac{\pi a}{8}\right) I_1(X)$$
$$\times \left(\sum_{(k,8)=1} \frac{A_1(n,k)}{k} \frac{|\csc(\pi a k/8)|}{\csc(\pi a/8)} \frac{I_1(X/k)}{I_1(X)} \right.$$
$$+ 4 \sum_{(k,8)=4} \frac{A_4(n,k)}{k} \sin\left(\frac{\pi a}{8}\right) \frac{I_1(2X/k)}{I_1(X)}$$
$$\left. + 4\sqrt{2} \sum_{(k,8)=8} \frac{A_8(n,k)}{k} \sin\left(\frac{\pi a}{8}\right) \frac{I_1(2X/k)}{I_1(X)} \right). \quad \text{(A.6)}$$

Focusing on the sum over k with $(k,8) = 1$, we will treat first the majority of the sum, in which $X/k < 1$. We then have $I_1(X/k) = O(X/k)$, and

$$\frac{I_1(X/k)}{I_1(X)} \sim O\left(\frac{X^{3/2}}{k} e^{-X}\right) = O\left(\frac{n^{3/4}}{k} e^{-\frac{\pi}{8}\sqrt{16n+4a-5}}\right). \quad \text{(A.7)}$$

Moreover, we remember from Chapter 4 that $|A_1(n,k)| = O\left(k^{2/3+\epsilon} n^{1/3}\right)$. So we have

$$\frac{A_1(n,k)}{k} \frac{|\csc(\pi a k/8)|}{\csc(\pi a/8)} \frac{I_1(X/k)}{I_1(X)} \sim O\left(n^{13/12} e^{-\frac{\pi}{8}\sqrt{16n+4a-5}} k^{-4/3+\epsilon}\right). \quad \text{(A.8)}$$

Our sum, then, is

$$\sum_{\substack{(k,8)=1,\\ X<k}} \frac{A_1(n,k)}{k} \frac{|\csc(\pi a k/8)|}{\csc(\pi a/8)} \frac{I_1(X/k)}{I_1(X)}$$
$$\sim O\left(n^{13/12} e^{-\frac{\pi}{8}\sqrt{16n+4a-5}} \sum_k k^{-4/3+\epsilon}\right). \quad \text{(A.9)}$$

But $X = \frac{\pi\sqrt{16n+4a-5}}{8} < k$, and therefore $n = O(k^2)$, and $n^{1/12} = O(k^{1/6})$. So we have

$$\sum_{\substack{(k,8)=1,\\ X<k}} \frac{A_1(n,k)}{k} \frac{|\csc(\pi a k/8)|}{\csc(\pi a/8)} \frac{I_1(X/k)}{I_1(X)}$$

$$\sim O\left(n e^{-\frac{\pi}{8}\sqrt{16n+4a-5}} \sum_k k^{-7/6+\epsilon} \right) \tag{A.10}$$

$$= O\left(n e^{-c_1\sqrt{n}} \right), \tag{A.11}$$

with constant $c_1 > 0$.

On the other hand, if $X/k > 1$, then we have

$$\frac{I_1(X/k)}{I_1(X)} \sim e^{-X(1-1/k)}\sqrt{k} = e^{-\frac{\pi\sqrt{16n+4a-5}}{8}(1-1/k)}\sqrt{k}. \tag{A.12}$$

We have

$$\frac{A_1(n,k)}{k} \frac{|\csc(\pi a k/8)|}{\csc(\pi a/8)} \frac{I_1(X/k)}{I_1(X)}$$

$$\sim \frac{A_1(n,k)}{\sqrt{k}} \frac{|\csc(\pi a k/8)|}{\csc(\pi a/8)} e^{-\frac{\pi\sqrt{16n+4a-5}}{8}(1-1/k)} \tag{A.13}$$

$$= O\left(e^{-\frac{\pi\sqrt{16n+4a-5}}{8}(2/3)} A_1(n,k) n^{-1/4} \right) \tag{A.14}$$

$$= O\left(n^{2/3+\epsilon} n^{-1/4} e^{-\frac{\pi\sqrt{16n+4a-5}}{8}(2/3)} \right) \tag{A.15}$$

$$= O\left(n e^{-c_2\sqrt{n}} \right), \tag{A.16}$$

with constant $c_2 > 0$.

Therefore

$$\sum_{(k,8)=1} \frac{A_1(n,k)}{k} \frac{|\csc(\pi a k/8)|}{\csc(\pi a/8)} \frac{I_1(X/k)}{I_1(X)} \sim O\left(n e^{-c_3\sqrt{n}} \right), \tag{A.17}$$

for some constant $c_3 > 0$.

Similarly,

$$\sum_{(k,8)=4} \frac{A_4(n,k)}{k} \sin\left(\frac{\pi a}{8}\right) \frac{I_1(2X/k)}{I_1(X)} \sim O\left(ne^{-c_4\sqrt{n}}\right), \tag{A.18}$$

$$\sum_{(k,8)=8} \frac{A_8(n,k)}{k} \sin\left(\frac{\pi a}{8}\right) \frac{I_1(2X/k)}{I_1(X)} \sim O\left(ne^{-c_5\sqrt{n}}\right), \tag{A.19}$$

with constants $c_4, c_5 > 0$. Finally,

$$\begin{aligned} &\sum_{(k,8)=1} \frac{A_1(n,k)}{k} \frac{|\csc(\pi ak/8)|}{\csc(\pi a/8)} \frac{I_1(X/k)}{I_1(X)} \\ &+4 \sum_{(k,8)=4} \frac{A_4(n,k)}{k} \sin\left(\frac{\pi a}{8}\right) \frac{I_1(2X/k)}{I_1(X)} \\ &+4\sqrt{2} \sum_{(k,8)=8} \frac{A_8(n,k)}{k} \sin\left(\frac{\pi a}{8}\right) \frac{I_1(2X/k)}{I_1(X)} \\ &\sim O\left(ne^{-c\sqrt{n}}\right), \end{aligned} \tag{A.20}$$

with constants $c > 0$.

On the other hand, examining our initial factor,

$$\begin{aligned} &\frac{\pi\sqrt{2}}{4\sqrt{16n+4a-5}} \csc\left(\frac{\pi a}{8}\right) I_1(X) \\ \sim&\frac{\pi\sqrt{2}}{4\sqrt{16n+4a-5}} \csc\left(\frac{\pi a}{8}\right) \times \frac{2\sqrt{2}}{\pi\sqrt{2}(16n+4a-5)^{1/4}} e^{\frac{\pi\sqrt{16n+4a-5}}{8}} &(A.21) \\ =&\sqrt{\frac{1}{2}} \frac{\csc(\pi a/8)}{(16n+4a-5)^{3/4}} e^{\frac{\pi}{8}\sqrt{16n+4a-5}}. &(A.22) \end{aligned}$$

Due to the positive exponential, (A.22) easily dominates (A.20) as $n \to \infty$, and the proof is completed.

□

This asymptotic formula (A.4) is more elegant, and agrees with more general work on the asymptotics of partition formulæ [1, Chapter 6]. The value of this formula to us is that we may very easily compare the behavior of $g_1(n)$ and $g_3(n)$. In doing so, we come upon this striking relationship.

Corollary A.5.

$$\frac{g_1(n)}{g_3(n)} \to \delta_S, \tag{A.23}$$

as $n \to \infty$.

Proof. Take the asymptotics of $g_1(n)$ and $g_3(n)$:

$$\frac{g_1(n)}{g_3(n)} \sim \frac{\sin(3\pi/8)}{\sin(\pi/8)} \left(\frac{16n+7}{16n-1}\right)^{3/4} e^{\frac{\pi}{8}\left(\sqrt{16n-1}-\sqrt{16n+7}\right)}. \tag{A.24}$$

By the half-angle formula,

$$\frac{\sin(3\pi/8)}{\sin(\pi/8)} = 1+\sqrt{2}, \tag{A.25}$$

while

$$\left(\frac{16n+7}{16n-1}\right)^{3/4} e^{\frac{\pi}{8}\left(\sqrt{16n-1}-\sqrt{16n+7}\right)} \to 1, \tag{A.26}$$

as $n \to \infty$.

□

This is especially remarkable, as Lehner proved the following with nearly identical methods [12]:

Theorem A.6 (Lehner)**.**

$$\frac{r_1(n)}{r_2(n)} \to \varphi, \tag{A.27}$$

as $n \to \infty$.

This provides the lovely perspective that the Göllnitz–Gordon identities (1.8), (1.9) bear a relationship to the Rogers–Ramanujan identities (1.6), (1.7) that is in some way analogous to the relationship between the silver ratio δ_S and the golden ratio φ.

Appendix B

MATHEMATICA CODE: SUPPLEMENT TO CHAPTER 7

Here we provide the code used to program and test (6.12), concatenated to a sum over $1 \leq k \leq N$.

B.1 Defining $g_a(n)$

We start by defining $g_a(n)$ itself, through the coefficients of the generating function defined by (3.16).

For $a = 1$ we define

```
G1 = Series[1/(QPochhammer[q^1, q^8] QPochhammer[q^7, q^8]
QPochhammer[q^4, q^8]), {q, 0, 200}].
```

For $a = 3$, we have

```
G3 = Series[1/(QPochhammer[q^3, q^8] QPochhammer[q^5, q^8]
QPochhammer[q^4, q^8]), {q, 0, 200}].
```

Notice that in both cases we have taken the first 200 values of n.

We may now compute $g_a(n)$ for any n from 1 to 200 simply by extracting the associated coefficient. For example, to find $g_1(43)$, we input

```
Coefficient[G1, q, 43],
```

and return

```
390.
```

B.2 Defining (6.12)

We now give our formula for $g_a(n)$ itself, (6.12). To assist our calculations, we make use of (3.7), which Rademacher showed to be equivalent to (3.5). To begin, we define $s(h,k)$:

```
s[h_, k_] := Sum[r/k ((h r)/k - Floor[(h r)/k] - 1/2), {r, 1, k - 1}]
```

Now, we start with developing the necessary numbers in the case of $(k,8) = 8$. We refer to H defined by (3.21) as $H8$. The second entry of (3.22) is defined as $M8$.

```
H8[h_, k_] := PowerMod[-h, -1, 16 k]


M8[h_, k_] := -8/k (h H8[h, k] + 1)
```

We now define the roots of unity, (3.26), (3.27):

```
e88[h_, k_] := Exp[\[Pi] I ((2 (h - H8[h, k]))/(3 k)
+ s[H8[h, k], k/8])]


e84[h_, k_] := Exp[\[Pi] I ((h - H8[h, k])/(3 k)
+ s[H8[h, k], k/4])]
```

We now define (3.43) and b as defined in (3.35):

```
N8[a_, h_] := Floor[(a h)/8]


b[a_, h_] := Mod[a h, 8]
```

We now have enough information to properly define the sum $A_8(n,k)$, defined in (3.130) and (6.11):

```
w8[a_, h_, k_] := Exp[(\[Pi] I)/(8 k) (h (5 - 4 a) - H8[h, k])]
```

```
A8[a_, k_, n_] := Sum[
  If[GCD[h, k] == 1, (I (-1)^N8[a, h])/(e88[h, k] e84[h, k])
     w8[a, h, k] Exp[(-2 \[Pi] I n h)/k]
        If[b[a, h] == 1 \[Or] b[a, h] == 7, 1, 0], 0], {h, 1, k - 1}]
```

We now work with the case for $(k, 8) = 4$, defining (3.132) and (6.10) for the case of $d = 4$:

```
H4[h_, k_] := PowerMod[-2 h, -1, k/4]
```

```
M4[h_, k_] := -4/k (2 h H4[h, k] + 1)
```

```
e44[h_, k_] := Exp[\[Pi] I ((h - 2 H4[h, k])/(3 k)
+ s[2 H4[h, k], k/4])]
```

```
e48[h_, k_] := Exp[\[Pi] I ((2 h - H4[h, k])/(3 k)
+ s[H4[h, k], k/4])]
```

```
N4[a_, h_] := Floor[(a h)/4]
```

```
b4[a_, h_] := Mod[a h, 4]
```

```
w4[a_, h_, k_] :=
 Exp[(\[Pi] I)/(4 k) (h - H4[h, k] - h (4 a - 3) (h H4[h, k] + 1)
+ a (2 h H4[h, k] + 1) (b4[a, h] - 2))]
```

```
A4[a_, k_, n_] :=
 Sum[If[GCD[h, k] == 1, (I (-1)^N4[a, h])/(e48[h, k] e44[h, k])
     w4[a, h, k] Exp[(-2 \[Pi] I n h)/k], 0], {h, 1, k - 1}]
```

We now do the same for $(k, 8) = 1$, defining (3.136) and (6.10) with $d = 1$.

Notice that here we use $N1$ to denote part of the alternating sign in (3.136).

We must also be careful in the case that $k = 1$, not least because, as shown in Appendix A, this term contributes the most to the correct answer. In that case, $A_1(n, 1) = 1$.

```
H1[h_, k_] := PowerMod[-8 h, -1, k]


M1[h_, k_] := -1/k (8 h H1[h, k] + 1)


e14[h_, k_] := Exp[\[Pi] I ((2 h - H1[h, k])/(6 k)
+ s[2 H1[h, k], k])]


e18[h_, k_] := Exp[\[Pi] I ((8 h - H1[h, k])/(12 k)
+ s[H1[h, k], k])]


N1[a_, h_, k_] := Floor[(a M1[h, k])/8]


w1[a_, h_, k_] := Exp[(\[Pi] I)/(4 k) (4 h (1 - a
+ h H1[h, k] (3 - 4 a)) - H1[h, k])]


B1[a_, h_, k_, n_] :=
 If[GCD[h, k] == 1, (-1)^(N1[a, h, k] + h - 1)/(e18[h, k] e14[h, k])
    w1[a, h, k] Exp[(-2 \[Pi] I n h)/k], 0]
```

```
A1[a_, k_, n_] := If[k == 1, 1, Sum[B1[a, h, k, n], {h, 1, k - 1}]]
```

Finally, with the relevant sums defined, we move on to the overall structure of (6.12)

```
p8[a_, n_, N_] := (2 \[Pi])/Sqrt[16 n + 4 a - 5]
   Sum[If[GCD[k, 8] == 8,
    A8[a, k, n]/k BesselI[1, (\[Pi] Sqrt[16 n + 4 a - 5])/(4 k)],
    0], {k, 1, N}]


p4[a_, n_, N_] := (\[Pi] Sqrt[2])/Sqrt[16 n + 4 a - 5]
   Sum[If[GCD[k, 8] == 4,
    A4[a, k, n]/k BesselI[1, (\[Pi] Sqrt[16 n + 4 a - 5])/(4 k)],
    0], {k, 1, N}]


p1[a_, n_, N_] := (\[Pi] Sqrt[2])/(4 Sqrt[16 n + 4 a - 5])
   Sum[If[GCD[k, 8] == 1,
    A1[a, k, n]/
     k Abs[Csc[(\[Pi] a k)/8]] BesselI[
      1, (\[Pi] Sqrt[16 n + 4 a - 5])/(8 k)], 0], {k, 1, N}]


p[a_, n_, N_] := p8[a, n, N] + p4[a, n, N] + p1[a, n, N]
```

Notice that we give our formula concatenated to a sum over $1 \leq k \leq N$, since we cannot expect Mathematica to carry out the complete series for any $n \in \mathbb{N}$.

B.3 Testing (6.12)

As a quick example, of how to test our formula in a single case, we test our formula for $a = 1$ and $n = 7$ with $N = 5$. Remembering that our result will not be an integer, and will have some minor imaginary contributions, owing to the roots of unity present in our sums (6.10) and (6.11), we will focus on the real part, and take a numerical approximation to, say, ten decimal places:

```
N[Re[p[1, 7, 5]], 10].
```

This input returns

```
2.884331031.
```

One may check, through the coefficient of q^7 in G1 (or through simple trial for so small a number), that

$$g_1(7) = 3. \tag{B.1}$$

To check our formula for $a = 1$ with all numbers $1 \leq n \leq 200$, with $1 \leq k \leq \lfloor 3\sqrt{n} \rfloor$, we construct the table

```
Table[ N[Re[p[1, n, Floor[3 Sqrt[n]]]], 10] -
   Coefficient[G1, q, n], {n, 1, 200}];
```

Following immediately with

```
Max[%]
```

and

```
Min[%%]
```

we return

```
0.174897898
```

and

```
-0.2803648624.
```

This gives us the largest and smallest errors for our formula: the largest absolute error is less than 0.3.

Moreover, we may check for $100 \leq n \leq 200$,

```
Table[ N[Re[p[1, n, Floor[3 Sqrt[n]]]], 10] -
   Coefficient[G1, q, n], {n, 100, 200}];
```

Here we find that the largest absolute error is less than 0.09.

Similarly, for $a = 3$ we may construct

```
Table[ N[Re[p[3, n, Floor[3 Sqrt[n]]]], 10] -
   Coefficient[G3, q, n], {n, 1, 200}];
```

In this case we find that applying

```
Max[%]
```

and

```
Min[%%]
```

we return

```
0.323340028
```

and

```
-0.1870119540.
```

The largest error slightly exceeds 0.3. However, again checking the error for $100 \leq n \leq 200$, similarly to the case for $a = 1$, we find that the largest absolute error is less than 0.07. This suggests that the errors may be relatively large (but not large enough to hinder our estimation) for relatively small values of n, before diminishing for larger values of n.

We can construct Table 7.1 with the following:

```
Grid[Table[ {n, N[Re[p[1, n, Floor[3 Sqrt[n]]]], 10],
   N[Re[p[1, n, Floor[3 Sqrt[n]]]], 10] - Coefficient[G1, q, n]}, {n,
   1, 200}]].
```

We may construct Table 7.2 in similar fashion. The interested reader is invited to substitute any desired number of terms in for $\lfloor 3\sqrt{n} \rfloor$, to check (6.12) to whatever level of precision is desired.

For a more visual representation of the size of our error term, we may construct Figures 7.1 with the following:

```
ListPlot[Table[
  N[Re[p[1, n, Floor[3 Sqrt[n]]]], 10] - Coefficient[G1, q, n], {n, 1,
    200}]].
```

Figure 7.2 may be constructed in a similar fashion.