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Combinatorics of Compositions

Meghann M. Gibson

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by

MEGHANN MORIAH GIBSON

(Under the Direction of Hua Wang)

ABSTRACT

Integer compositions and related enumeration problems have been extensively studied. The cyclic analogues of such questions, however, have significantly fewer results. In this thesis, we follow the cyclic construction of Flajolet and Soria to obtain generating functions for cyclic compositions and n-color cyclic compositions with various restrictions. With these generating functions we present some statistics and asymptotic formulas for the number of compositions and parts in such compositions. Combinatorial explanations are also provided for many of the enumerative observations presented.

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COMBINATORICS OF COMPOSITIONS

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DEDICATION

To the One who gets me up in the morning. To the One who gives me strength to get through the day and persevere through rough times. To the One who brings me peace in the midst of the stress, and joy in times of sadness. Thank you Jesus for being so good.

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CHAPTER 1

INTRODUCTION

In 1893, the mathematician Percy Alexander MacMahon was one of the first to publish any work on the subject of *integer compositions*. He introduced integer compositions, or simply compositions, in the context of integer partitions. Simply put, a composition is a list of positive integers in any order that sum to a given positive integer [5, pg.1-2]. Here, we study the cyclic and colored generalizations of this concept and related enumerative problems.

1.1 Compositions

Definition 1. (Composition) A composition of a positive integer ℓ is the sequence $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_k$ such that $\sum_{i=1}^{k} \sigma_i = \ell$, where $\sigma_i \in \mathbb{Z}_+$ for all $1 \leq i \leq k$.

Each σ_i is called a *part* in the composition, and it is important to note that, unlike partitions, the ordering of the parts in a composition of ℓ matters. As examples, the compositions for the positive integers 3 and 4 are given below:

Example 1.1.

All the compositions of $\ell = 3$ are

$$3 \qquad 1,2 \qquad 2,1 \qquad 1,1,1.$$

Example 1.2.

All the compositions of $\ell = 4$ are

$$4 \qquad 3,1 \qquad 1,3 \qquad 2,2 \qquad 2,1,1 \qquad 1,2,1 \qquad 1,1,2 \qquad 1,1,1,1.$$

Let $\mathscr{C}_\ell$ be the set of all compositions of ℓ, and let $|\mathscr{C}_\ell|$ be the number of all such compositions of ℓ. Figure 1.1 gives $|\mathscr{C}_\ell|$ for each ℓ from 1 to 8.

Figure 1.1: Number of Compositions for $1 \geq \ell \geq 8$

ℓ	$\lvert\mathscr{C}_\ell\rvert$
1	1
2	2
3	4
4	8
5	16
6	32
7	64
8	128

We can see from Figure 1.1 that for any given positive integer ℓ, the number of compositions of ℓ seems to be $2^{\ell-1}$. This is indeed the case and is well known, and for completeness we give a short proof of this fact below.

Fact 1.3. *The number of compositions of a given positive integer ℓ is $2^{\ell-1}$.*

Proof. Consider the positive integer ℓ on a number line with tick marks at every positive integer between 0 and ℓ. Note there exists $\ell - 1$ possible tick marks that could be used as a marker to break the distance between 0 and ℓ into smaller, nonzero parts. This is because parts of size zero are not considered in compositions. Since each of the $\ell - 1$ tick marks is either a marker or not a marker, the number of possible ways to create different combinations of markers on a number line from 0 to ℓ is $\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{\ell-1 \text{ times}}$. Because the different markers on the number line refer to different possible integer compositions of ℓ, it follows that the number of integer compositions of ℓ is $2^{\ell-1}$. $\square$

1.2 Color Compositions

A colored analogue of compositions, called *n-color compositions*, may have first been introduced in [1]. This type of integer composition adds a color component to each part of a composition.

Definition 2. (n-Color Composition) An n-color composition of a positive integer ℓ is a composition of ℓ where each part of size n has n possible colors.

For instance, given the composition $1,2$, the part of size 1 has one possible color, which we label 1_1. Also the part of size 2 has two possible colors, which label as 2_1 and 2_2. Using this notation, we give all the n-color compositions of 3 and 4 in the following examples:

Example 1.4.

All the n-color compositions of $\ell = 3$ are

$$3_1 \qquad 3_2 \qquad 3_3 \qquad 1_1,2_1 \qquad 1_1,2_2 \qquad 2_1,1_1 \qquad 2_2,1_1 \qquad 1_1,1_1,1_1.$$

Example 1.5.

All the n-color compositions of $\ell = 4$ are

$$4_1, \quad 4_2, \quad 4_3, \quad 4_4, \quad 3_1,1_1, \quad 3_2,1_1, \quad 3_3,1_1, \quad 1_1,3_1, \quad 1_1,3_2, \quad 1_1,3_3, \quad 2_1,2_1,$$

$$2_2,2_1, \quad 2_1,2_2, \quad 2_2,2_2, \quad 2_1,1_1,1_1, \quad 2_2,1_1,1_1, \quad 1_1,2_1,1_1, \quad 1_1,2_2,1_1,$$

$$1_1,1_1,2_1, \quad 1_1,1_1,2_2, \quad 1_1,1_1,1_1,1_1.$$

In general, we will use $\mathscr{N}\mathscr{C}_\ell$ to denote the set of all n-color compositions of ℓ. Using the notation $|\mathscr{N}\mathscr{C}_\ell|$ for the total number of n-color compositions of ℓ, we recall the fact below.

Fact 1.6.

$$|\mathscr{N}\mathscr{C}_\ell| = \tau^\ell + \sigma^\ell,$$

where $\tau = \frac{2}{3-\sqrt{5}}$ *and* $\sigma = \frac{1}{\tau} = \frac{2}{3+\sqrt{5}}$ *(OEIS A088305).*

This fact stems from the asymptotic formula that can be developed for the total number of n-color compositions. It is interesting to note that $(\tau)^2 = \varphi$, where φ is the of the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. This also implies that $1+\varphi = \tau$ and $\frac{1}{1+\varphi} = \sigma$. We will use this fact later in our exploration of different types n-color compositions. Using methods found in [6], we can also represent these compositions as tiles with spots used to mark the color of each part. In any given tile the bold lines represent the separation between parts. The non-bold lines divide a part of size n into n different colors, and a black dot is used to mark the color of the part in the given composition. In Figure 1.2, the eight n-color compositions of 3 are given in spotted tiling representation.

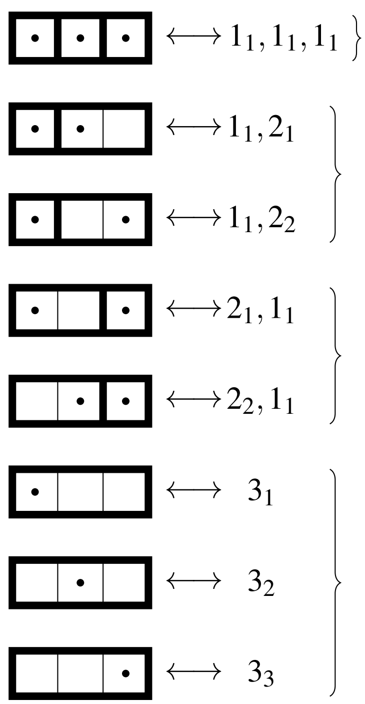


Figure 1.2: n-color compositions of 3.

Following [1], another series of studies on n-color compositions develop various bijections that establish the combinatorial connection between n-color compositions and other objects [2, 6].

1.3 Generating Functions

Within the study of compositions, it is important to identify the positive integers that form the counting sequence for the number of compositions, or parts of compositions, in question. In order to obtain such a sequence, we often use *generating functions*.

Definition 3. (Generating Function) A generating function is a formal power series $\sum_{n=0}^{\infty} A_n x^n$ whose coefficients A_n form an integer sequence for $n \geq 0$.

First, as an example, we will construct the generating function for the number of compositions of ℓ.

Example 1.7.

Consider the series $(x+x^2+x^3+\cdots+x^m+\ldots)$. Since $\sum\limits_{i=0}^{\infty} x^i = (1+x+x^2+x^3+\ldots) = \frac{1}{1-x}$ for $|x| < 1$, by the properties of geometric series we observe that

$$(x+x^2+x^3+\cdots+x^m+\ldots) = \frac{x}{1-x},$$

where we assume that the conditions on x that are necessary for convergence are met. Because the composition of ℓ can be of any length, we consider the sum of this generating function from 1 to infinity in order to account for every possible composition of ℓ. Thus the generating function for the number of compositions of ℓ, denoted $C(x)$, is given by

$$\begin{aligned} C(x) &= \sum_{k=1}^{\infty} \left(\frac{x}{1-x} \right)^k \\ &= \frac{\frac{x}{1-x}}{1 - \frac{x}{1-x}} \\ &= \frac{x}{1-2x}. \end{aligned}$$

Next, we will construct the generating function for the number of parts in all compositions of ℓ that are divisible by $m \in \mathbb{Z}^+$. In order to develop this generating function, we first introduce a little more notation. Let $\mathscr{P}$ be a condition that restricts either the composition type or the parts of the composition in some way. In this case let $\mathscr{P} \leftrightarrow (\equiv 0 \pmod{m})$. Then $cp_{\mathscr{P}}(\ell)$ denotes the number of parts in the compositions of ℓ that are divisible by m. Using this notation, we will construct the generating function $CP_{\mathscr{P}}(x)$.

Example 1.8.

Consider the series $(x + x^2 + x^3 + \cdots + yx^m + \cdots + yx^{2m} + \ldots)$, where the variable y is inserted to mark all the parts divisible by m. Since
$\sum_{i=0}^{\infty} x^i = (1 + x + x^2 + x^3 + \ldots) = \frac{1}{1-x}$ and $x + x^2 + \ldots x^{m-1} = \frac{x - x^m}{1-x}$, observe that

$$\begin{aligned}
(x + x^2 \cdots + yx^m + \ldots) &= (x^1 + x^{m+1} + x^{2m+1} + \ldots) \ + (x^2 + x^{m+2} + + x^{2m+2} + \ldots) \\
&\quad + \cdots + y(x^m + x^{m+m} + x^{2m+m} + \ldots), \\
&= \frac{x}{1-x^m} + \frac{x^2}{1-x^m} + \cdots + \frac{yx^m}{1-x^m} \\
&= \frac{x + x^2 + \cdots + x^{m-1}}{1-x^m} + \frac{yx^m}{1-x^m} \\
&= \frac{\frac{x-x^m}{1-x}}{1-x^m} + \frac{yx^m}{1-x^m} \\
&= \frac{x(1-x^{m-1}) + yx^m(1-x)}{(1-x)(1-x^m)}
\end{aligned}$$

Thus $\sum_{k=1}^{\infty} \left(\frac{x(1-x^{m-1}) + yx^m(1-x)}{(1-x)(1-x^m)} \right)^k$ is a bivariate generating function of the composition of ℓ with parts divisible by m labeled by y,and

$$\begin{aligned}
\sum_{k=1}^{\infty} \left(\frac{x(1-x^{m-1}) + yx^m(1-x)}{(1-x)(1-x^m)} \right)^k &= \frac{\frac{x(1-x^{m-1}) + yx^m(1-x)}{(1-x)(1-x^m)}}{1 - \frac{x(1-x^{m-1}) + yx^m(1-x)}{(1-x)(1-x^m)}} \\
&= \frac{x - x^m + yx^m - yx^{m+1}}{1 - 2x + x^{m+1} - yx^m + yx^{m+1}},
\end{aligned}$$

where the coefficient $x^\ell y^s$ gives the number of compositions of ℓ with s parts divisible by m. By taking the partial derivative with respect to y and then setting $y = 1$, we find that the generating function for the number of parts of the compositions of ℓ divisible by m, which is denoted using the statement $\mathscr{P}$, is

$$\begin{aligned}
CP_{\mathscr{P}}(x) &= \frac{\partial}{\partial y}\Big(\frac{x - x^m + yx^m - yx^{m+1}}{1 - 2x + x^{m+1} - yx^m + yx^{m+1}}\Big)\Big|_{y=1} \\
&= \frac{(1-2x-x^m+2x^{m+1})(x^m - x^{m+1})}{(1-2x-x^m+2x^{m+1})^2} - \frac{(x-x^{m+1})(-x^m+x^{m+1})}{(1-2x-x^m+2x^{m+1})^2} \\
&= \frac{(x^m - 2x^{m+1} + x^{m+2} - x^{2m} + 2x^{2m+1} - x^{2m+2})}{((1-x^m)(1-2x))^2} \\
&= \frac{x^m(1-x)^2}{(1-x^m)(1-2x)^2}.
\end{aligned}$$

Generating functions and their many uses are examined thoroughly from a discrete and analytic perspective in [16]. Following the strategy outlined in [3], we focus our attention on constructing generating functions in order to identify the counting sequences for compositions or for the number of parts.

1.4 Primitive Compositions

Primitive compositions first appear in [11] and are used in [3] for constructing generating functions of certain compositions. These types of compositions further deconstruct any given composition into smaller, nonrepeating sections of parts.

Definition 4. (Primitive Compositions) A primitive composition of a positive integer ℓ is a composition of ℓ that is not composed of repeated copies of shorter compositions.

For instance, consider the n-color compositions of 4 broken up into primitive and nonprimitive n-color compositions.

Example 1.9.

Primitive n-Color Compositions of 4:

$$4_1 \quad 4_2 \quad 4_3 \quad 4_4 \quad 3_1,1_1 \quad 3_2,1_1 \quad 3_3,1_1 \quad 1_1,3_1 \quad 1_1,3_2 \quad 1_1,3_3 \quad 2_2,2_1$$

$$2_1,2_2 \quad 2_1,1_1,1_1 \quad 2_2,1_1,1_1 \quad 1_1,2_1,1_1 \quad 1_1,2_2,1_1$$

$$1_1,1_1,2_1 \quad 1_1,1_1,2_2.$$

Nonprimitive n-Color Compositions of 4:

$$2_1,2_1 \quad 2_2,2_2, \quad 1_1,1_1,1_1,1_1.$$

Note that every composition type is composed of d copies of a primitive n-color composition in that same type for some $d \in \mathbb{Z}^+$. Following [3], the process of representing a composition as repeated copies of its primitive components will be used frequently in this thesis when constructing generating functions for cyclic compositions.

1.5 Cyclic Compositions

Cyclic compositions were first considered in [14] and enumerated via generating functions in [3]. In a sense they are the same as normal compositions except for their orientation, which is about a circle as opposed to a line in the case of normal compositions.

Definition 5. (Cyclic Compositions) Cyclic compositions form a set of equivalence classes of compositions in which any cyclic shift of a composition is considered to be the same cyclic composition.

Consider, for example, a cyclic composition of 10 corresponding to the given equivalence class and pictured in Figure 1.3.

Example 1.10.

The cyclic composition of $10 = 2,2,1,2,2,1 = 2,1,2,2,1,2 = 1,2,2,1,2,2$ is portrayed in Figure 1.3.

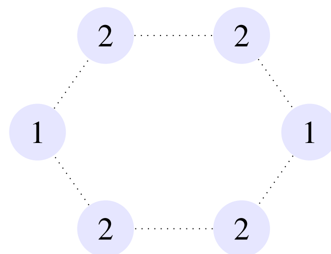


Figure 1.3: A cyclic composition of 10.

As one would imagine, the enumeration of various objects in cyclic compositions dramatically differs from that in compositions. For the remainder of this thesis we let $\mathscr{CC}_\ell$ denote the set of cyclic compositions of ℓ, and we use $cc(\ell)$ and $CC(x)$ to denote its corresponding cardinality and generating function. Some interesting properties of cyclic compositions were discussed in [10].

1.6 *n*-Color Cyclic Compositions

Another composition type that is used extensively in this thesis is *n-color cyclic compositions*. These compositions are a combination of the unique properties that define both cyclic compositions and n-color compositions. They are the same in their structure as regular n-color compositions, but their orientation is about a circle.

Definition 6. (n-Color Cyclic Compositions) The set of n-color cyclic compositions contains the equivalence classes of n-color compositions in which any cyclic shift of a composition is considered to be the same n-color cyclic composition.

As an example, we will develop a n-color cyclic composition of 15 where the color of each part is represented as a known color.

Example 1.11.

Consider the following n-color cyclic composition of 15:

$$\begin{aligned} & 1_1, 2_2, 3_3, 4_4, 5_5 \\ = {} & 5_5, 1_1, 2_2, 3_3, 4_4 \\ = {} & 4_4, 5_5, 1_1, 2_2, 3_3 \\ = {} & 3_3, 4_4, 5_5, 1_1, 2_2 \\ = {} & 2_2, 3_3, 4_4, 5_5, 1_1. \end{aligned}$$

Let the color type 1 be associated with the color blue, the color type 2 be associated with green, the color type 3 be associated with red, the color type 4 be associated with orange, and the color type 5 be associated with purple. It follows that

$$15 = 1, 2, 3, 4, 5 = 5, 1, 2, 3, 4 = 4, 5, 1, 2, 3 = 3, 4, 5, 1, 2 = 2, 3, 4, 5, 1,$$

and is portrayed in Figure 1.4.

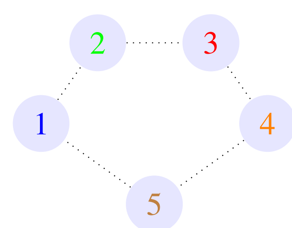


Figure 1.4: An n-color cyclic composition of 15.

It is important to note that the parts of an n-color cyclic composition of size n still have n possible colors. We let $\mathscr{NCC}_\ell$ denote the set of n-color cyclic compositions of ℓ, and we use $ncc(\ell)$ and $NCC(x)$ to denote its cardinality and generating function.

1.7 Our Results

For the rest of this thesis, we will focus on and explain various results involving cyclic compositions and n-color cyclic compositions. Below is a chapter-by-chapter breakdown of our work.

- Integer compositions and related enumeration problems have been extensively studied. The cyclic analogues of such questions, however, have significantly fewer results. In Chapter 2, we follow the cyclic construction found in [3] to obtain generating functions of parts under modular conditions in cyclic compositions. Recall from Section 1.2 that the set containing cyclic compositions is $\mathscr{C}\mathscr{C}_\ell$. Thus $CCP_{i,m}(x)$ and $ccp(i;m;\ell)$ are used to denote the generating function and its corresponding cardinality. Using other generating functions related to cyclic compositions, we present some statistics and asymptotic formulas for the parts in cyclic compositions. A combinatorial observation of this enumerative question is also provided.

- In Chapter 3, we explore the combinatorics of n-color cyclic compositions, a type of composition that has received significantly less attention than its linear counterpart. Using methods found in [3], we construct generating functions for the total number of n-color cyclic compositions and for the total number of parts in all n-color cyclic compositions. We then present various bijections, asymptotic formulas related to the number of such compositions, and/or combinatorial arguments for these generating functions.

- In Chapter 4, we continue to focus on n-color cyclic compositions and, using similar methods to those found in Chapter 2 and Chapter 3, we construct the generating function for the total number of parts under modular, ($\equiv i \pmod{m}$), conditions in all n-color cyclic compositions. Our notation for this section will follow the pattern used in the preceding chapters. The set containing these types of compositions is denoted as $\mathscr{N}\mathscr{C}\mathscr{C}\mathscr{P}_\ell(i;m)$, where $NCCP_{i,m}(x)$ and $nccp(i;m;\ell)$ are used to denote the generating function and its corresponding cardinality. We then give the proof for this generating function, construct a table from its corresponding integer sequence, and give a combinatorial argument for properties observed within the table.

- In Chapter 5, we further restrict the parts of n-color cyclic compositions and consider the number of compositions given two different restrictions on the parts. These restrictions include “parts no greater than h” and “parts divisible by j”. In order to simplify notation, we choose to define the given restriction to be the condition $\mathscr{P}$. Then we denote the set containing that type of composition as $\mathscr{NCCP}_{\mathscr{P}\ell}$, where $NCCP_{\mathscr{P}}(x)$ and $nccp_{\mathscr{P}}(\ell)$ are used to denote the generating function and its corresponding cardinality. We then give the generating functions for these compositions, present asymptotic formulas related to the number of compositions, construct a table corresponding to the integer sequence of each composition, and explore patterns found within the table.

CHAPTER 2

RESTRICTED PARTS IN CYCLIC COMPOSITIONS

2.1 Introduction

This chapter is based on [8], which has been submitted for publication. Cyclic compositions were first considered in [14], and enumerated via generating functions in [3]. Although similar in nature, the enumeration of various objects in cyclic compositions differs substantially from that in compositions.

As a concept in Combinatorial Number Theory, compositions have been extensively studied by both Combinatorialists and Number Theorists. Much of such work focuses on the enumeration of parts or sub-word patterns under various restrictions. A nice summary of known results can be found in [5] and the references therein. In the recent work [7], the number of parts under modular conditions in all compositions of ℓ was studied, and interesting combinatorial relations with the number of sub-word patterns in all compositions of ℓ were found. Among other observations, the following was presented as a consequence of previously known results. Recall from Section 1.1 that $\mathscr{C}_\ell$ is the set containing all compositions of ℓ, and recall from Section 1.3 that $\mathscr{P}$ denotes any given condition that restricts either the composition type or the parts of the composition in some way. In this case we let $\mathscr{P}$ be the statement "parts equal to k." Then $cp_{\mathscr{P}}(\ell)$ denotes the number of parts equal to k in all compositions of ℓ.

Theorem 2.1 ([7]).

$$cp_{\mathscr{P}}(\ell) = \begin{cases} (\ell-k+3)2^{\ell-k-2} & \textit{if } \ell > k, \\ 1 & \textit{if } \ell = k. \end{cases}$$

The number of parts j such that $j \equiv i \mod m$ in all compositions of ℓ follows from Theorem 2.1 above and is denoted by $cp(i;m;\ell)$ for $i = 1, \ldots, m$. It is important to note that when dealing with congruence classes, we will use m instead of 0 for the corresponding congruent class.

Theorem 2.2 ([7])**.**

$$cp(i;m;\ell) = \left\lfloor \frac{2^{\ell+m-i-2}\Big((\ell-i+3)(2^m-1)-m\Big)}{(2^m-1)^2} \right\rceil ,$$

where $\lfloor x \rceil$ *is the integer nearest to* x *.*

It is a known result that $2^{\ell-2}(\ell+1)$ is the number of parts in all compositions of ℓ [15, p. 120, Ex. 23]. Theorem 2.2 provides a direct tool to study the statistics of the parts under modular conditions.

Unlike regular compositions, there has been surprisingly little work done to answer similar questions regarding cyclic compositions. We make some modest progress towards filling this gap in this chapter. The tool that is critical to our study is developed in [3], where the construction of cycles of combinatorial structures is examined analytically.

Recall from Section 1.5 that $\mathscr{C}\mathscr{C}_\ell$ denotes the set of cyclic compositions of ℓ. Also let $TCCP(x)$ denote the generating function for the number of parts in all compositions in $\mathscr{C}\mathscr{C}_\ell$, where $tccp(\ell)$ is its corresponding cardinality for each ℓ. Direct application of the cyclic construction in [3] yields the bivariate generating function for cyclic compositions

$$CC(x,u) = \sum_{s\geq 1} \frac{\varphi(s)}{s} \log\left(\frac{1}{1-u^s\frac{x^s}{1-x^s}}\right), \tag{2.1}$$

where the coefficient of $x^\ell u^k$ gives the number of cyclic compositions of ℓ with k parts with φ representing the Euler totient function. Hence the generating function for $tccp(\ell)$ is

$$TCCP(x) = \frac{\partial}{\partial u} CC(x,u)\bigg|_{u=1} = \varphi(s)\sum_{s\geq 1}\frac{x^s}{1-2x^s}.$$

Consequently we have the following.

Proposition 2.3.

$$tccp(\ell) = \frac{1}{2}\sum_{s|\ell} \varphi(s)2^{\ell/s}.$$

Let $ccp(i;m;\ell)$ $(1 \le i \le m)$ denote the number of parts j such that $j \equiv i \mod m$ in all compositions in $\mathscr{C}\mathscr{C}_\ell$. Examining $ccp(i;m;\ell)$ and the corresponding generating function is a bit more involved. In this chapter we show that the generating function for the number of parts in all cyclic compositions that are congruent to $(i \mod m)$ is

$$CCP_{i;m}(x) = \sum_{s\ge 1}\left(\varphi(s)\frac{x^{si}(1-x^s)}{(1-2x^s)(1-x^{sm})}\right), \tag{2.2}$$

and we provide a detailed study of Equation (2.2) in Section 2.2. Based on our findings, some statistical behaviors of the parts in all compositions in $\mathscr{C}\mathscr{C}_\ell$ are presented and justified in Section 2.3. In Section 2.4, we comment on an interesting combinatorial observation which arises from our study.

2.2 Construction of the generating function of $ccp(i;m;\ell)$

Recall from Section 1.7 that $\mathscr{C}\mathscr{C}_\ell$ is the set of cyclic compositions of ℓ, and $ccp(i;m;\ell)$ and $CC_{i;m}(x)$ are the corresponding cardinality and generating function for the number of parts in all cyclic compositions congruent to $i \mod m$. The generating function,

$$CCP_{i;m}(x) = \sum_{s\ge 1}\left(\varphi(s)\frac{x^{si}(1-x^s)}{(1-2x^s)(1-x^{sm})}\right),$$

which is given in (2.2), is constructed directly through the "cycle construction" shown in [3]. In this section we summarize the formation of Equation (2.2).

First consider the series $x+x^2+x^3+\dots$ which is the generating function for the number of regular compositions. Multiplying each part by y that is congruent to $i \mod m$ yields

$$x+x^2+\cdots+yx^i+x^{i+1}+\cdots+yx^{i+m}+\dots.$$

This forms the generating function for the total number of compositions, where parts congruent to $i \mod m$ (that is, parts in the same equivalence class as i) are labeled using the variable y. Further multiplying each term by u to mark all the parts gives us

$$\begin{aligned}
&ux+ux^2+\dots+yux^i+ux^{i+1}+\dots+yux^{i+m}+\dots\\
=&u(x+x^2+\dots)+u(y-1)\left(x^i+x^{i+m}+\dots\right)\\
=&\frac{ux}{1-x}+\frac{(y-1)ux^i}{1-x^m}.
\end{aligned}$$

Consequently we have the multivariable generating function of compositions

$$\begin{aligned}
C(x,u,y) &= \sum_{k=1}^{\infty}\left(\frac{ux}{1-x}+\frac{(y-1)ux^i}{1-x^m}\right)^k\\
&= \frac{\frac{ux}{1-x}+\frac{(y-1)ux^i}{1-x^m}}{1-\frac{ux}{1-x}+\frac{(y-1)ux^i}{1-x^m}}\\
&= \frac{ux(1-x^m)+ux^i(y-1)(1-x)}{(1-x)(1-x^m)-(ux(1-x^m)+ux^i(y-1)(1-x))},
\end{aligned}$$

where the coefficient of $x^\ell u^r y^t$ is the number of compositions of ℓ with r parts, t of which are congruent to $i \mod m$.

Recall from Section 1.4 that a primitive composition is a composition that is not composed of repeated copies of shorter compositions. Since every composition is composed of d copies of a primitive composition for some $d \in \mathbb{Z}^+$, we can construct the generating function for the total number of primitive compositions by using the multivariate generating function for compositions. Let

$$PC(x,u,y) = \sum_{n,r,t} pc(\ell,r,t)x^\ell u^r y^t$$

denote the generating function for primitive compositions, where the coefficient $pc(\ell,r,t)$ is the number of primitive compositions of ℓ with r parts, t of which are congruent to $i \mod m$. Using the relationship between compositions and primitive compositions, we have

$$C(x,u,y) = \sum_{d\geq 1} PC(x^d,u^d,y^d).$$

We derive $PC(x,u,y)$ implicitly using a Möbius inversion to yield

$$PC(x,u,y) = \sum_{k\geq 1} \mu(d) C(x^d, u^d, y^d)$$

where $\mu(d)$ is the Möbius μ function. We now let

$$PCC(x,u,y) = \sum_{n,r,t} pcc(\ell,r,t) x^\ell u^r y^t$$

denote the generating function for primitive cyclic compositions, where the coefficient $pcc(\ell,r,t)$ is the number of primitive cyclic compositions of ℓ with r parts, t of which are congruent to $i \mod m$. First note that each primitive cyclic composition with r parts has r unique primitive composition representations. Thus, there is a one-to-r relationship between primitive cyclic compositions and primitive compositions. Consequently

$$pcc(\ell,r,t) x^\ell u^r y^t = \frac{pc(\ell,r,t)}{r} x^\ell u^r y^t = \int_0^u pc(n,r,t) x^\ell w^{r-1} y^t \, dw,$$

and we have

$$\begin{aligned}
PCC(x,u,y) &= \int_0^u \frac{PC(x,w,y)}{w} \, dw \\
&= \int_0^u \frac{1}{w} \sum_{d\geq 1} \mu(d) \, C(x^d, w^d, y^d) \, dw \\
&= \int_0^u \frac{1}{w} \sum_{d\geq 1} \mu(d) \, \frac{w^d x^d (1-x^{md}) + w^d x^{di} (y^d - 1)(1-x^d)}{(1-x^d)(1-x^{md}) - (w^d x^d (1-x^{md}) + w^d x^{di} (y^d-1)(1-x^d))} \, dw \\
&= \sum_{d\geq 1} \mu(d) \int_0^u \frac{w^{d-1}(x^d (1-x^{md}) + x^{di} (y^d - 1)(1-x^d))}{(1-x^d)(1-x^{md}) - w^d (x^d (1-x^{md}) + x^{di} (y^d-1)(1-x^d))} \, dw.
\end{aligned}$$

Integrating through substitution with

$$\beta = (1-x^d)(1-x^{md}) - w^d (x^d (1-x^{md}) + x^{di} (y^d - 1)(1-x^d)),$$

and

$$d\beta = -w^{d-1} (x^d (1-x^{md}) + x^{di} (y^d - 1)(1-x^d)) \, dw$$

and setting $u = 1$, we have

$$PCC(x,y) = PCC(x,u,y)\Big|_{u=1} = \left(\int_0^u \frac{PC(x,w,y)}{w}\,dw\right)\Big|_{u=1}$$
$$= \left(\sum_{d\geq 1}\frac{\mu(d)}{d}\log\left(\frac{(1-x^d)(1-x^{md})}{(1-x^d)(1-x^{md})-u^d(x^d(1-x^{md})+x^{di}(y^d-1)(1-x^d))}\right)\right)\Big|_{u=1}$$
$$= \sum_{d\geq 1}\frac{\mu(d)}{d}\log\left(\frac{(1-x^d)(1-x^{md})}{(1-x^d)(1-x^{md})-(x^d(1-x^{md})+x^{di}(y^d-1)(1-x^d))}\right).$$

Since every Cyclic Composition is composed of q adjacent copies of primitive compositions for some $q \in \mathbb{Z}^+$, the bivariate generating functions for cyclic compositions, which is denoted by $CC(x,y)$, is

$$CC(x,y) = \sum_{q\geq 1} PCC(x^q,y^q)$$
$$= \sum_{q\geq 1}\sum_{d\geq 1}\frac{\mu(d)}{d}\log\left(\frac{(1-x^{qd})(1-x^{mqd})}{(1-x^{qd})(1-x^{mqd})-(x^d(1-x^{mqd})+x^{di}(y^{qd}-1)(1-x^{qd}))}\right).$$

Using the variable substitution, $s = qd$ and given the identity $\sum_{d|s}\frac{\mu(d)}{d} = \frac{\varphi(s)}{s}$, where $\varphi(s)$ is the Euler totient function, $CC(x,y)$ becomes

$$CC(x,y) = PCC(x^q,y^q)$$
$$= \sum_{s\geq 1}\sum_{d|s}\frac{\mu(d)}{d}\log\left(\frac{(1-x^{qd})(1-x^{mqd})}{(1-x^{qd})(1-x^{mqd})-(x^d(1-x^{mqd})+x^{di}(y^{qd}-1)(1-x^{qd}))}\right)$$
$$= \sum_{s\geq 1}\frac{\varphi(s)}{s}\log\left(\frac{(1-x^s)(1-x^{sm})}{(1-x^s)(1-x^{sm})-(x^s(1-x^{sm})+x^{si}(y^s-1)(1-x^s))}\right).$$

Here the coefficient of $x^\ell y^t$ is the number of cyclic compositions of ℓ with t parts that are congruent to $i \mod m$. Taking the partial derivative of $CC(x,y)$ with respect to y and setting $y = 1$ yields the generating function, denoted $CCP_{i;m}(x)$, for the number of parts congruent to $i \mod m$ in all cyclic compositions in $\mathscr{C}\mathscr{C}_\ell$.

$$
\begin{aligned}
CCP_{i;m}(x) &= \frac{\partial}{\partial y}\Big(CC(x,y)\Big)\Big|_{y=1} \\
&= \sum_{s\geq 1} \frac{\varphi(s)}{s}\left(\frac{\partial}{\partial y}\left(\log\left(\frac{(1-x^s)(1-x^{sm})}{(1-x^s)(1-x^{sm})-(x^s(1-x^{sm})+x^{si}(y^s-1)(1-x^s))}\right)\right)\right)\Big|_{y=1} \\
&= \sum_{s\geq 1}\left(\varphi(s)\frac{x^{si}(1-x^s)}{(1-2x^s)(1-x^{sm})}\right),
\end{aligned}
$$

which is the generating function given in (2.2).

2.3 Some statistics of the parts in $\mathscr{C}\mathscr{C}_\ell$

Evaluating Equation (2.2) at $m = 10$ with $1 \leq i \leq 10$ yields the following Table 4.1 of values for $ccp(i;m;\ell)$:

A number of interesting observations immediately follow:

- As $\ell \to \infty$,
$$\frac{ccp(i;m;\ell+1)}{ccp(i;m;\ell)} \to 2.$$
That is, going down a column in the table by one step doubles the value of the next entry.

- As $\ell \to \infty$,
$$\frac{ccp(i+1;m;\ell)}{ccp(i;m;\ell)} \to \frac{1}{2}.$$
That is, moving right in a row in the table by one step halves the value of the next entry.

- As $\ell \to \infty$,
$$\frac{ccp(i+1;m;\ell+1)}{ccp(i;m;\ell)} \to 1.$$
That is, the values along the diagonal or sub-diagonals are asymptotically the same.

To see the reasoning behind these observations, we start with the following formula for $cc(i;m;\ell)$.

ℓ \ i	1	2	3	4	5	6	7	8	9	10=m
1	1	0	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0	0
3	4	1	1	0	0	0	0	0	0	0
4	7	3	1	1	0	0	0	0	0	0
5	12	4	2	1	1	0	0	0	0	0
6	22	11	5	2	1	1	0	0	0	0
7	38	16	8	4	2	1	1	0	0	0
8	74	36	17	9	4	2	1	1	0	0
9	138	66	34	16	8	4	2	1	1	0
10	272	136	66	33	17	8	4	2	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$	$7.03 \cdot 10^{13}$	$3.52 \cdot 10^{13}$	$1.76 \cdot 10^{13}$	$8.79 \cdot 10^{12}$	$4.39 \cdot 10^{12}$	$2.20 \cdot 10^{12}$	$1.10 \cdot 10^{12}$	$5.49 \cdot 10^{11}$
51	$5.62 \cdot 10^{14}$	$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$	$7.03 \cdot 10^{13}$	$3.52 \cdot 10^{13}$	$1.76 \cdot 10^{13}$	$8.79 \cdot 10^{12}$	$4.39 \cdot 10^{12}$	$2.20 \cdot 10^{12}$	$1.10 \cdot 10^{12}$
52	$1.12 \cdot 10^{15}$	$5.62 \cdot 10^{14}$	$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$	$7.03 \cdot 10^{13}$	$3.52 \cdot 10^{13}$	$1.76 \cdot 10^{13}$	$8.79 \cdot 10^{12}$	$4.39 \cdot 10^{12}$	$2.20 \cdot 10^{12}$
53	$2.25 \cdot 10^{15}$	$1.12 \cdot 10^{15}$	$5.62 \cdot 10^{14}$	$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$	$7.03 \cdot 10^{13}$	$3.52 \cdot 10^{13}$	$1.76 \cdot 10^{13}$	$8.79 \cdot 10^{12}$	$4.39 \cdot 10^{12}$
54	$4.50 \cdot 10^{15}$	$2.25 \cdot 10^{15}$	$1.12 \cdot 10^{15}$	$5.62 \cdot 10^{14}$	$2.81 \cdot 10^{14}$	$1.41 \cdot 10^{14}$	$7.03 \cdot 10^{13}$	$3.52 \cdot 10^{13}$	$1.76 \cdot 10^{13}$	$8.79 \cdot 10^{12}$

Table 2.1: Values of $ccp(i;m;\ell)$ for $m = 10$.

Theorem 2.4.

$$ccp(i;m;\ell) \to \frac{2^{\ell+m-i-1}}{2^m - 1}$$

as $\ell \to \infty$.

Proof. First rewrite (2.2) as $CCP_{i;m}(x) = \sum_{s\geq 1} \varphi(s)A(x^s)$, where

$$A(x) = x^i \cdot \frac{1-x}{(1-2x)(1-x^m)}.$$

Next, we will consider the partial fraction decomposition of $\frac{1-x}{(1-2x)(1-x^m)}$. Observe that

$$\frac{1-x}{(1-2x)(1-x^m)} = \frac{1}{(1-2x)(1+x+\ldots+x^{m-1})} = \frac{a_0}{1-2x} + \sum_{j=1}^{m-1} \frac{a_j}{x-w_j}$$

with constants $a_0, a_1, \ldots, a_{m-1}$, where w_j for $1 \leq j \leq m-1$ are the complex roots of $x^m = 1$. Through solving the partial fraction decomposition by multiplying both sides by $(1-2x)(1-x^m)$ and plugging in $x = \frac{1}{2}$, we have

$$\begin{aligned}
1 - \frac{1}{2} &= a_0 \left(1 - \left(\frac{1}{2}\right)^m\right) \\
\tfrac{1}{2} &= a_0 \left(\frac{2^m - 1}{2^m}\right) \\
a_0 &= \left(\frac{1}{2}\right)\left(\frac{2^m}{2^m - 1}\right) \\
a_0 &= \frac{2^{m-1}}{2^m - 1}.
\end{aligned}$$

Since $x = \frac{1}{2}$ is a distinct root of the denominator of smallest magnitude, it follows that the coefficient of x^ℓ in $A(x)$ is asymptotically $\frac{2^{\ell+m-i-1}}{2^m-1}$ (contributed from $x^i \cdot \frac{a_0}{1-2x}$) as $n \to \infty$. Thus considering $ccp(i;m;\ell)$, the coefficient of x^ℓ in $\sum_{s\geq 1} \varphi(s)A(x^s)$, we have

$$ccp(i;m;\ell) \sim \frac{1}{2^m - 1}\left(\sum_{s|\ell} \varphi(s) 2^{\frac{\ell}{s}+m-i-1}\right). \tag{2.3}$$

By considering the $s = 1$ term of $\sum_{s\geq 1} \varphi(s)A(x^s)$, we see that

$$\frac{1}{2^m - 1}\left(\sum_{s|\ell} \phi(s) 2^{\frac{\ell}{s}+m-i-1}\right) \geq \frac{2^{\ell+m-i-1}}{2^m - 1}.$$

However, we also know that

$$\begin{aligned}
\frac{1}{2^m - 1}\left(\sum_{s|\ell} \phi(s) 2^{\frac{\ell}{s}+m-i-1}\right) &= \sum_{s|n} \phi(s) \frac{2^{m-i-1}}{2^m - 1} 2^{\frac{\ell}{s}} \\
&= \frac{2^{m-i-1}}{2^m - 1} \sum_{s|\ell} \phi(s) 2^{\frac{\ell}{s}} \\
&= \frac{2^{m-i-1}}{2^m - 1}\left(2^\ell + \sum_{s|\ell, s\neq 1} \phi(s) 2^{\frac{\ell}{s}}\right) \\
&\leq \frac{2^{m-i-1}}{2^m - 1}\left(2^\ell + (\ell - 1) 2^{\frac{\ell}{2}}\right) \\
&\to \frac{2^{\ell+m-i-1}}{2^m - 1}
\end{aligned}$$

for large ℓ. Thus by definition, $\frac{1}{2^m-1}\left(\sum_{s|\ell}\phi(s)2^{\frac{\ell}{s}+m-i-1}\right) \to \frac{2^{\ell+m-i-1}}{2^m-1}$. Therefore, it follows that

$$ccp(i;m;\ell) \to \frac{2^{\ell+m-i-1}}{2^m-1}$$

as $\ell \to \infty$. □

Similarly, it is easy to see from Proposition 2.3 in the introduction, that $tccp(\ell) \sim 2^{\ell-1}$ as $\ell \to \infty$. Consequently we have

Corollary 2.5.

$$\frac{ccp(i;m;\ell)}{tpcc(\ell)} \to \frac{2^{m-i}}{2^m-1}$$

as $\ell \to \infty$.

Among other things, Theorem 2.4 and Corollary 2.5 imply that for large ℓ:

- $ccp(i;m;\ell+1) \to 2ccp(i;m;\ell)$;
- $ccp(i+1;m;\ell) \to \frac{1}{2}ccp(i;m;\ell)$;
- $ccp(i+1;m;\ell+1) \to ccp(i;m;\ell)$.

These observations are somewhat surprising, but they follow directly from the generating function.

2.4 A combinatorial observation

We conclude this chapter by exploring an interesting observation related to $CCP_{i;m}(x)$. Recall that

$$CCP_{i;m}(x) = \sum_{s\geq 1}\varphi(s)A(x^s), \tag{2.4}$$

where

$$A(x) = x^i \cdot \frac{1-x}{(1-2x)(1-x^m)}.$$

First, we make the following claim:

Proposition 2.6. *The function*

$$A(x) = x^i \cdot \frac{1-x}{(1-2x)(1-x^m)}$$

is the generating function for the number of compositions with last part congruent to i $\bmod m$*.*

Proof. Let A_ℓ denote the set of compositions of ℓ with last part congruent to $i \mod m$. We will show that

$$|A_\ell| = \sum_{j=1}^{m} |A_{\ell-j}| + |A_{\ell-m}|.$$

The generating function as claimed immediately follows from combining the above generating functions on the right.

Consider the compositions of ℓ with last part congruent to $i \mod m$ in the following different cases and apply the corresponding operations:

1) If the last part x is greater than m, we simply replace x with $x-m$ yielding a composition in $A_{\ell-m}$.

2) If the last part x is exactly i, consider the second to last part $y \equiv j \mod m$ for $j = 1, 2, \ldots, m$. In this case replacing x and y with the single part $y - j + i$, where k is a positive integer, yields a composition in $A_{\ell-j}$.

It is not hard to see that this map is a bijection between A_n and $\left(\cup_{j=1}^{m} A_{n-j}\right) \cup A'_{n-m}$, where A'_{n-m} and A_{n-m} are two copies of the same set. Thus the generating function A_n is equivalent to the generating function for $\left(\cup_{j=1}^{m} A_{n-j}\right) \cup A_{n-m}$, which means

$$|A_\ell| = \sum_{j=1}^{m} |A_{\ell-j}| + |A_{\ell-m}|.$$

□

This fact inspires a combinatorial argument that establishes $CC_{i;m}(x)$ directly from $A(x)$. We will briefly discuss the justification for this idea in the rest of this section.

For a cyclic composition with some part congruent to $i \mod m$, "cutting" the composition right after this part and orienting this new composition on a line instead of a circle produces a composition with the last part congruent to $i \mod m$. Such regular compositions are counted by the generating function $A(x)$, by the previous proposition. Now we consider all the regular compositions of ℓ of length k formed from cutting cyclic compositions at every part congruent to $i \mod m$. This process of "cutting" can be broken down into the following cases:

- If every regular composition, with its corresponding last part congruent to $i \mod m$, corresponds to at least one such "cutting" of some cyclic composition, then it is counted by the $s = 1$ term of (2.4).

- If the result of the first "cutting" yields a regular, nonprimitive composition composed of two identical copies of compositions of length $\frac{k}{2}$, then further cutting this composition right after the part congruent to $i \mod m$, which is located in the middle of this composition, yields two compositions of length $\frac{k}{2}$. However, since the compositions in exactly one of these groups of compositions was already counted by the first case, this second case is counted with only one copy of the generating function $A(x^2)$, which is the $s = 2$ term of (2.4) and it counts a pair of compositions ending with parts congruent to $i \mod m$.

- If the resulting regular composition is made of three identical copies of compositions of length $\frac{k}{3}$, then further cutting this composition at the parts congruent to $i \mod m$ yields three compositions of length $\frac{k}{3}$. However, since the compositions in exactly one of these groups was already counted by the first case, only two of the groups of compositions with last part congruent to $i \mod m$ need to be counted. Thus this case is counted by two copies of the generating function $A(x^3)$, which is the $s=3$ term of (2.4) and it counts a triple of compositions ending with parts congruent to $i \mod m$.

- If the resulting regular composition is made of four identical copies of compositions of length $\frac{k}{4}$, then further cutting this composition at the parts congruent to $i \mod m$ yields four compositions of length $\frac{k}{4}$. However exactly two of these groups of compositions with last part congruent to $i \mod m$ have already been counted; one group was counted by the first case and another group was counted by the second case. Therefore this case, is counted by two copies of the generating function $A(x^4)$, which is the $s=4$ term of (2.4) and it counts such a 4-tuple of compositions ending with parts congruent to $i \mod m$.

- In general, if the resulting regular composition is made of s identical copies of compositions of length $\frac{k}{s}$, then further cutting this composition at the parts congruent to $i \mod m$ yields s compositions of length $\frac{k}{s}$. Among these s parts congruent to $i \mod m$, $s-\varphi(s)$ are already counted in previous cases. The other $\varphi(s)$ such groups of compositions, each of which have a last part congruent to $i \mod m$, are counted by $\varphi(s)$ copies of the generating function $A(x^s)$, which is the s^{th} term of (2.4).

The above explanation presents the following idea. Consider any cyclic composition of n of length k, made up of d copies of a primitive composition containing p parts congruent to $i \mod m$ for some $d, p \in \mathbb{Z}^+$. Then there are $d \cdot p$ total parts congruent to $i \mod m$. Of the $\frac{k}{d}$ compositions corresponding to this one cyclic composition, p of them have last part congruent to $i \mod m$. For any divisor s of d, consider the composition formed by s copies of the primitive composition to be the cyclic composition in question ending in one of the p parts congruent to $i \mod m$. Each of these compositions will be counted by $A(x^s)$. Now summing over all these compositions and multiplying by $\varphi(s)$ for those counted in $A(x^s)$, we obtain

$$\sum_{s|d} \varphi(s)p = d \cdot p,$$

which is exactly the number of parts congruent to $i \mod m$ in this composition. Thus summing over all cyclic compositions of ℓ we see the relationship between $GPCC(x)$ and $A(x)$.

Example 2.7.

Consider $\ell = 6$, $i = 1$ and $m = 3$. For the purpose of illustration, we will use subscripts to denote the location of an entry in a cyclic composition. That is, a cyclic composition of 6 with all parts of size 1 can be "cut" into a composition $1_1 1_2 1_3 1_4 1_5 1_6$ or $1_6 1_1 1_2 1_3 1_4 1_5$ depending on the location of the cut. Also, for each case the corresponding figures will denote the number of cuts by s, and the specific cuts the generating function counts by a blue oval.

For each part j in a cyclic composition of 6 such that $j \equiv 1 \mod 3$, "cutting" all the cyclic compositions of 6 at each part j yields the following compositions of 6:

(a) $1_1 1_2 1_3 1_4 1_5 1_6$, $1_6 1_1 1_2 1_3 1_4 1_5$, $1_5 1_6 1_1 1_2 1_3 1_4$, $1_4 1_5 1_6 1_1 1_2 1_3$, $1_3 1_4 1_5 1_6 1_1 1_2$, $1_2 1_3 1_4 1_5 1_6 1_1$

(b) 11121, 11211, 12111, 21111, 2211, 1221, 1131, 1311, 3111, 231, 321, 411, 141, 114, 24

(c) $2 1_2 2 1_4$, $2 1_4 2 1_2$

- All the compositions of case (b), together with $1_1 1_2 1_3 1_4 1_5 1_6$ and $2 1_2 2 1_4$, provide us exactly the set of compositions of 6 that ends with one ($s = 1$) part that is in the same equivalence class as 1. They are counted by the coefficient of x^6 in $\varphi(1)A(x)$, which is $1 \cdot 17 = 17$.

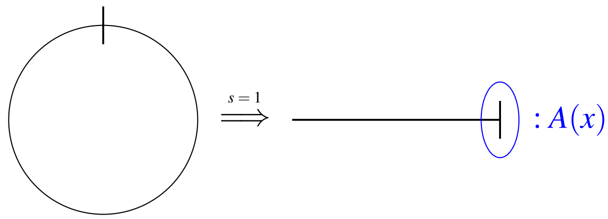


Figure 2.1: Compositions with circled cuts counted by $A(x)$

- Since 2121 and 111111 can both be considered as repeating two ($s = 2$) copies of compositions of 3 (i.e., 21 and 111), they can be further cut in the middle, yielding 21;21 and 111;111 and counting $21_4 21_2$ and $1_4 1_5 1_6 1_1 1_2 1_3$. This is the coefficient of x^6 in $\varphi(2)A(x^2)$, which is $1 \cdot 2 = 2$.

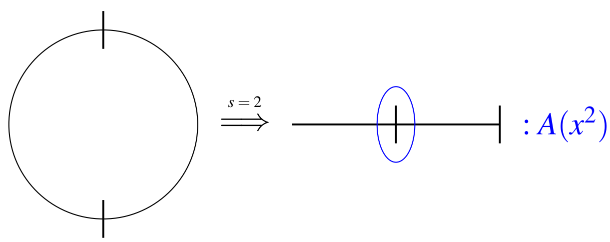


Figure 2.2: Compositions with circled cuts counted by $A(x^2)$

- Furthermore, 111111 can also be considered as repeating three ($s = 6$) copies of compositions of 2 (i.e, 11). Thus it can be further cut to obtain 11;11;11, which produces the cuts $1_5 1_6 1_1 1_2 1_3 1_4$ and $1_3 1_4 1_5 1_6 1_1 1_2$. This repeated cutting is counted through the coefficient of x^6 in $\varphi(3)A(x^3)$, which is $2 \cdot 1 = 2$.

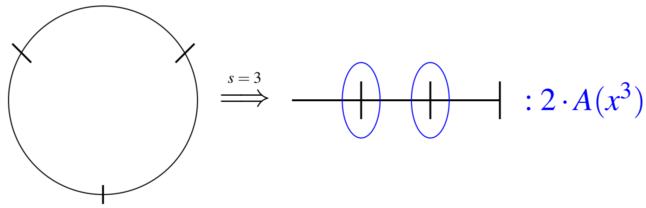


Figure 2.3: Compositions with circled cuts counted by $A(x^3)$

- Lastly, 111111 can be considered as repeating six ($s = 6$) copies of compositions of 1, it can be further cut to obtain $1;1;1;1;1;1$, which produces the cuts $1_6 1_1 1_2 1_3 1_4 1_5$ and $1_2 1_3 1_4 1_5 1_6 1_1$. The results are counted through the coefficient of x^6 in $\varphi(6)A(x^6)$, which is $2 \cdot 1 = 2$

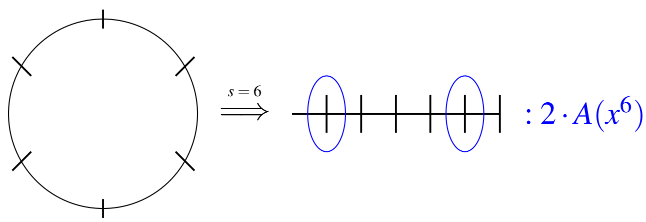


Figure 2.4: Compositions with circled cuts counted by $A(x^6)$

Thus

$$CCP_{i;m}(x) = [x^6]\left(\sum_{s|6} \varphi(s)A(x^s)\right) = 17+2+2+2 = 23,$$

which is exactly the number of parts that are congruent to $1 \mod 3$ in all cyclic compositions of 6.

CHAPTER 3

N-COLOR CYCLIC COMPOSITIONS

3.1 Introduction

This chapter is based on [4]. Following [1], a series of studies have been presented on n-color compositions, including various connections between the counting sequences for n-color compositions other integer sequences [2, 6]. Much of the study on compositions focuses on the enumeration of compositions, parts, or sub-word patterns under various restrictions. A nice summary of known results can be found in [5] and the references therein. In this chapter we introduce and explore such questions for n-color cyclic compositions. Let $\mathscr{NCC}_\ell$ and $\mathscr{NCCP}_\ell$ be, respectively, the set of n-color cyclic compositions of ℓ and the set of all parts in each n-color cyclic composition of ℓ. We use $ncc(\ell)$, $nccp(\ell)$, $NCC(x)$, and $NCCP(x)$ to denote the corresponding cardinalities and generating functions. For instance, $ncc(\ell)$ is the total number of n-color compositions of ℓ and

$$NCC(x) = \sum_{\ell \geq 0} ncc(\ell) \cdot x^\ell.$$

Essentially following the analytic tools developed in [3], we find that the generating functions $NCC(x)$ and $NCCP(x)$, whose proofs we postpone to Section 3.5, are

$$NCC(x) = \sum_{s \geq 1} \frac{\varphi(s)}{s} \log\left(\frac{(1-x^s)^2}{(1-3x^s+x^{2s})}\right), \tag{3.1}$$

and

$$NCCP(x) = \sum_{s \geq 1} \varphi(s) \left(\frac{x^s}{1-3x^s+x^{2s}}\right), \tag{3.2}$$

where $\varphi(s)$ is the Euler totient function. From these generating functions, we obtain patterns and discuss the combinatorics associated with these enumeration problems. In Section 3.2 we first note a formula of $ncc(\ell)$, which leads to interesting connections to the alternate Lucas numbers and the number of states in a dynamic storage allocation system. Combinatorial proofs are presented for these observations. In Section 3.4 we consider $nccp(\ell)$. In addition to providing its formula, we explore its connection to regular n-color compositions using combinatorial methods. In Section 3.5, we summarize the proof of (3.1). Then we briefly discuss the how to obtain (3.2) through the same approach.

3.2 Alternate Lucas numbers and n-color cyclic compositions

The sequence $ncc(\ell)$ generates $1, 3, 6, 13, 25, 58, 121, 283, \ldots$, which is the sequence A032198 from the Online Encyclopedia of Integer Sequences (OEIS) [13]. In fact, the formula

$$ncc(\ell) = \frac{1}{\ell} \sum_{d|\ell} \varphi\left(\frac{\ell}{d}\right) \cdot A_d \tag{3.3}$$

where $\{A_d\}_n$, given by A004146 in OEIS, is the sequence for the alternate Lucas numbers that enumerates the number of spanning subtrees of wheel graphs or d-helm graphs with $d+1$ vertices citeoeis. We examine this observation and prove (3.3) combinatorially in this section.

Before examining the combinatorial connections between the alternate Lucas numbers and $ncc(\ell)$, we first take a closer look at A_d and the combinatorial objects that A_d enumerates. As listed in A004146, A_d is the number of spanning trees of a wheel W_d on $d+1$ vertices or a d-helm graph on $d+1$ vertices. Note that the wheel W_d is formed by taking a cycle C_d with a vertex in the center that connects to every vertex on the cycle. A d-helm graph is then simply a W_d with an additional pendant edge at each of the d vertices on the cycle. See Figure 3.1 for an example of the wheel and a d-helm graph, pictured from left to right.

Example 3.1.

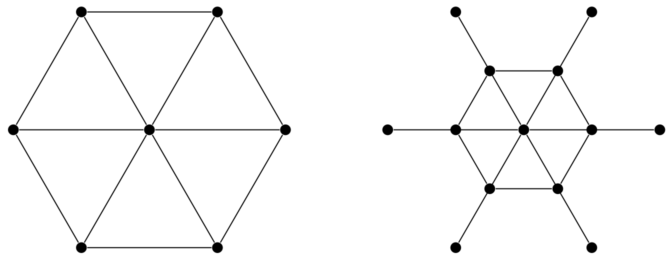


Figure 3.1: The wheel W_6 and the 6-helm graph on 7 vertices

A *spanning tree* of a graph G is a connected, acyclic subgraph of G that contains every vertex of G. Because of this, a bijection between the spanning trees of W_d and the spanning trees of d-helm graphs can be readily established through adding d pendant edges to any given wheel W_d. In the rest of this section, we will focus our attention on the wheel and its connection to $ncc(\ell)$. For our purposes the wheels on 2 and 3 vertices are specially depicted in Figure 3.2 from left to right.

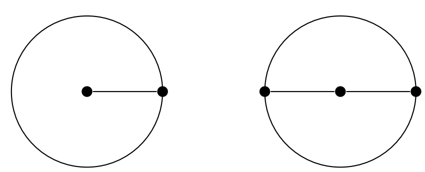


Figure 3.2: The wheel graphs on 2 and 3 vertices

In a wheel we call all the edges containing the center vertex *spokes* and the other edges *arcs*. It is easy to see that every spanning tree of the wheel W_d contains at least one spoke. We now describe a function f that maps each spanning tree of W_d to a n-color cyclic composition of d using the following conditions:

- Choose a spanning tree containing s spokes $e_1, e_2, e_3, \ldots, e_s$ with a clockwise ordering, and let a_i be the number of arcs between e_i and e_{i+1}, with $e_{s+1} = e_1$.

- The spanning tree, as a subgraph, must contain exactly $a_i - 1$ arcs between e_i and e_{i+1} for each i. This is because the graph can have at most one arc missing in order for the subgraph to be connected and spanning, but at least one arc must be missing for the subgraph to be acyclic.

- For any i, let the c_i^{th} arc be missing from the spanning tree, where c_i is the i^{th} arc between e_i and e_{i+1} when counting in the clockwise direction. Then our n-color cyclic composition is

$$(a_1)_{c_1} + (a_2)_{c_2} + \ldots + (a_s)_{c_s}$$

  wrapped around a circle.

The process of relating this type of spanning tree to an n-color cyclic composition using the function f is given in the following example:

Example 3.2.

Consider a spanning tree T (bold faced edges) of W_6 mapped to the corresponding n-color cyclic composition in Figure 3.3.

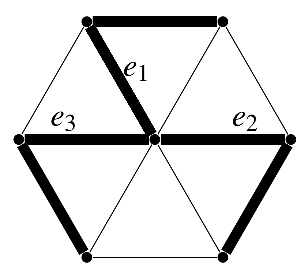


Figure 3.3: A spanning tree T such that $f(T)$: $2_2 + 3_2 + 1_1 = 6$.

Remark 3.3. *It is easy to see that the function described above is not a bijection. In fact, exactly d spanning trees (generated from rotating the current spanning tree) of W_d are mapped to the same n-color cyclic composition of d.*

We are now ready to further explore the combinatorial relationship between the n-color cyclic compositions and spanning trees of wheels with a proof of Equation 3.3.

Theorem 3.4. *For a positive integer ℓ, we have*

$$ncc(\ell) = \frac{1}{\ell}\sum_{d|\ell}\varphi\left(\frac{\ell}{d}\right)A_d. \tag{3.4}$$

Proof. Recall that $\mathscr{NCC}_\ell$ is the set of all n-color cyclic compositions of ℓ. For each d dividing ℓ, let $\mathscr{A}_d$ be the set of spanning trees of W_d with $d+1$ vertices, and let $\mathscr{B}_d$ be the set of those subtrees where all of the edges are identically colored by color 1, color 2, ..., or color $\varphi(\frac{\ell}{d})$. By letting $A_d = |\mathscr{A}_d|$ and $B_d = |\mathscr{B}_d|$, we have $B_d = \varphi(\frac{\ell}{d})A_d$. Define a map $g : \bigcup_{d|\ell}\mathscr{B}_d \longrightarrow \mathscr{NCC}_\ell$ in the following way:

- Let $T \in \mathscr{B}_d$ such that T has s spokes.
- First generate $f(T)$, an n-color cyclic composition of d of the following form:

$$(a_1)_{c_1} + (a_2)_{c_2} + \ldots + (a_s)_{c_s}.$$

- Then, $g(T)$ is the n-color cyclic composition with $\frac{sn}{d}$ parts that occur $f(T)$ $\frac{\ell}{d}$ times.

It is easy to see that g, like f, is a well defined map that maps multiple spanning trees to the same n-color cyclic composition, as we saw as an example in Figure 3.3. Therefore, we introduce notation that allows us to identify this property for all $T \in \mathscr{B}_d$.

- For a tree $T \in \mathscr{B}_d$, let $R(T)$ be the rotation of the tree T by $\frac{2\pi}{d}$. Note that $g(R^i(T)) = g(T)$ for all positive integers i.
- Recall for an n-color cyclic composition C, if r is the smallest positive integer such that C is just $\frac{\ell}{r}$ copies of those first r parts, then C is composed of primitive compositions of length r. Then $R^r(T) = T$ and $R^i(T) \neq T$ for any $1 \leq i < r$, since we suppose r to be the length for the smallest pattern contained in C.

Since $r|\ell$, for any T such that $g(T) = C$, there must be some s such that $T \in \mathscr{B}_{rs}$. Now note that each such tree T can be colored one of $\varphi(\frac{\ell}{rs})$ colors in $\bigcup_{rs|\ell} \mathscr{B}_{rs}$, and the same is true for $R(T), R^2(T), \ldots, R^{r-1}(T)$. Thus, for each s such that $(rs)|\ell$, there are $r\varphi(\frac{\ell}{rs})$ trees that are mapped to C. We then have

$$|g^{-1}(C)| = \sum_{rs|\ell} r\varphi\left(\frac{\ell}{rs}\right) = r\sum_{s|\frac{\ell}{r}} \varphi\left(\frac{\left(\frac{\ell}{r}\right)}{s}\right) = r\frac{\ell}{r} = \ell.$$

In other words, there are always exactly ℓ unique spanning subtrees in $\bigcup_{d|\ell} \mathscr{B}_d$ which map to C given any n-color cyclic composition C in $\mathscr{NCC}_\ell$. Thus,

$$\ell \cdot ncc_\ell = \left|\bigcup_{d|\ell} \mathscr{B}_d\right| = \sum_{d|\ell} B_d = \sum_{d|\ell} \varphi\left(\frac{\ell}{d}\right) A_d.$$

Dividing through by ℓ yields

$$ncc(\ell) = \frac{1}{\ell}\sum_{d|\ell} \varphi\left(\frac{\ell}{d}\right) A_d.$$

□

3.3 Dynamic storage allocation systems and n-color cyclic compositions

Using OEIS, we found that A032198 also appears to be exactly one less than A005594, which counts the number of states of a dynamic storage allocation system in circular arenas [12, 13]. This type of storage system partitions a storage space into cells, each of which can be either busy or idle. Several consecutive cells can form a block to be used for storage, which in turn defines such cells as busy. It is only free cells that have the potential to be idle because they are single-cell blocks, not blocks containing two or more cells. However single-cell blocks can be designated as busy also. For these types of systems contained within circular arenas, each state corresponds to an allocation system composed of a certain number of cells on a circle such that there can be both busy blocks of any size and idle blocks only of size 1. In mathematical terms, A005594 enumerates the number of cyclic compositions with two colors for parts of size 1 and one color for all other parts. We provide a combinatorial proof for this connection in this section.

First, let a busy storage block of size $k \geq 1$ be denoted by k, and let an idle block be denoted by $1'$. Consider the following example of all such states of a dynamic storage allocation system for $\ell = 3$ not including the identical copies of each state produced when the allocation system is rotated.

Example 3.5.

All unique states of such an allocation system with 3 cells are

$$1'1'1',\ 11'1',\ 111',\ 111,\ 21',\ 21,\ 3.$$

These expressions can be easily interpreted as colored cyclic compositions where parts of size 1 have two possible colors, either 1 or $1'$ as busy or idle respectively, and parts of size greater than 1 have only one color, busy. Let $\mathscr{DCC}_\ell$ be the set of such n-color cyclic compositions of ℓ, and let $dcc(\ell)$ be its cardinality. We will provide a combinatorial proof to the following theorem by establishing a bijection between $\mathscr{NCC}_\ell$ and

$$\mathscr{DCC}_\ell - \left\{ \underbrace{1'1'\ldots 1'}_{\ell\ 1'\text{s}} \right\}.$$

Theorem 3.6. *For any positive ℓ, we have*

$$ncc(\ell) = dcc(\ell) - 1.$$

Proof. With the understanding that all compositions under consideration are n-color cyclic compositions, we present our bijective map through a recursive manner that considers one part a time. Given a part in a composition in $\mathscr{NCC}_\ell$, we consider the following options:

1) If the part is 1_1, then it is mapped to a 1 as part of a composition in $\mathscr{DCC}_\ell$. This 1 corresponds to a busy storage block of size 1 in the dynamical storage system;

2) If the part is k_k, then it is mapped to k as a part of a composition in $\mathscr{DCC}_\ell$, where k_k corresponds a busy storage block of size k in the dynamical storage system;

3) If the part is k_{k-i} for some $1 \le i \le k-1$, then it is mapped to a part $k-i$ followed by $\underbrace{1' \dots 1'}_{i\ 1'\text{s}}$, which corresponds to a busy storage block of size $k-i$ followed by i idle storage cells in the dynamical storage system.

This process is illustrated in Figure 3.4

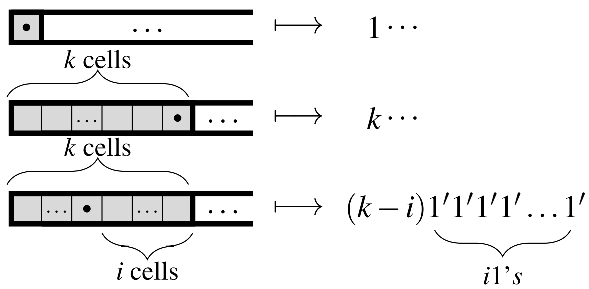


Figure 3.4: Outline of Bijection Construction

Notice that the state of a dynamical storage system in which all n spaces are both of size 1 and idle does not have a corresponding n-colored composition of ℓ. Under this construction, this type of storage space would be analogous to a composition of zero, which, by the convention we use here, is not counted. Thus, no composition from $\mathscr{NCC}_\ell$ will be mapped to $\underbrace{1'1'\dots1'}_{\ell\ 1'\text{s}}$ as the image of any part will always generate at least one busy block. We finish the proof by defining the inverse of our map. Starting from a composition in $\mathscr{DCC}_\ell - \left\{\underbrace{1'1'\dots1'}_{\ell\ 1'\text{s}}\right\}$, we pick such a part m in the state that is not an idle block since we are guaranteed to have at least one busy block (i.e., not all cells are $1'$s). Note that the part m will serve as the "start" of the state and every element that may have come before it is now shifted, following the pattern of cyclic compositions, until m is the first component of the state.

- If the part following m (clockwise to the right) is not a $1'$, then map it to a part m_m.

- If m is followed by some $1'$s, let k be the number of $1'$s before there is another part j. Then $m\underbrace{1'1'\ldots1'}_{k\ 1'\text{s}}$ is mapped to $(m+k)_m$.

$\square$

In order to illustrate this bijection, we will display n-color cyclic compositions according to the method used in [5], which represents n-color compositions as a series of spotted tilings. Recall from Section 1.2 that this method of illustration places each part of a color composition on a tile with bold lines to separate the different tiles and dots in the middle of the tiles to mark the specific color for each part. Figure 3.5 presents an n-color cyclic composition of 10, namely $2_2 3_2 1_1 4_1$, first represented using spotted tilings and subsequently mapped to a state of a dynamic storage allocation system with 10 cells.

Example 3.7.

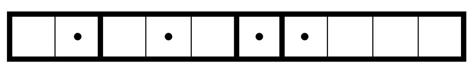
$2_2 3_2 1_1 4_1 \longmapsto$  $\longmapsto \quad 221'111'1'1'$

Figure 3.5: Bijection between $2_2 3_2 1_1 4_1$ and $221'111'1'1'$.

From this example, we see the parts from the n-color cyclic composition $2_2 3_2 1_1 4_1$ correspond to the blocks in the state $221'111'1'1'$ in the following way:

- 2_2 is mapped to 2;
- 3_2 is mapped to $21'$;
- 1_1 is mapped to 1;
- 4_1 is mapped to $11'1'1'$.

Similarly, using the construction for the inverse of the bijection outlined in the proof, we give the following example of the same state in Figure 3.5 mapped to the same n-color cyclic composition.

Example 3.8.

$$221'111'1'1' \longmapsto 2_2(2+1)_2 1_1(1+3)_1 \longmapsto 2_2 3_2 1_1 4_1$$

Figure 3.6: Bijection between $2_2 3_2 1_1 4_1$ and $221'111'1'1'$.

It is important to note that the state $221'111'1'1'$ has 8 cyclic shifts that comprise the same state. However, according to the parameters of our bijection, since the states cannot begin with a $1'$, there are only 4 cyclic shifts that must be considered. It is these four cyclic shifts of the state $221'111'1'1'$ that correspond uniquely to one of the four cyclic shifts of $2_2 3_2 1_1 4_1$.

3.4 Parts in n-Color Cyclic Compositions

Through (3.2) it is easy to verify that

$$nccp(\ell) = \sum_{s|\ell} \varphi(s)[\tau^{\ell/s} + \sigma^{\ell/s}] \tag{3.5}$$

where $\tau = \frac{2}{3-\sqrt{5}}$ and $\sigma = \frac{1}{\tau} = \frac{2}{3+\sqrt{5}}$. Recall from Fact (1.6), given in Section 1.2, that

$$\{\tau^m + \sigma^m\}_m,$$

listed as A088305 in OEIS, is exactly the counting sequence of the non-cyclic n-color compositions [13]. We now provide a combinatorial proof for Equation (3.5) that also explains this interesting connection.

Theorem 3.9. *For positive integer ℓ, we have*

$$nccp(\ell) = \sum_{s|\ell} \varphi(s)[\tau^{\ell/s} + \sigma^{\ell/s}],$$

where $\tau = \frac{2}{3-\sqrt{5}}$ and $\sigma = \frac{1}{\tau} = \frac{2}{3+\sqrt{5}}$.

Proof. Note that the number of non-cyclic n-color compositions of m, which is given by the sequence A088305 in OEIS, is $a(m) = \tau^m + \sigma^m$ [13]. We will show that

$$nccp(\ell) = \sum_{s|\ell} \varphi(s) \cdot a\left(\frac{\ell}{s}\right).$$

For the rest of this proof, the weight of a composition w will be denoted $weight(w)$ while the number of parts of w will be denoted $|w|$, where $weight(w)$ is the sum of the parts in a composition of w .

First, we make the following observation. For $s|\ell$, note that $a(\ell/s)$ counts the number of (non-cyclic) n-color compositions, which consist of s copies of a composition of weight $\frac{\ell}{s}$. Let $(w)^d$ be a n-color cyclic composition with d copies of the primitive root w and with $weight(w^d) = \ell$ for some $d \in \mathbb{Z}^+$. Then, there are $|w|$ non-cyclic n-color compositions which, when wrapped around a circle, give you $(w)^d$, since every cyclic shift of w will yield a new non-cyclic n-color composition. Thus, the number of non-cyclic n-color compositions which map to $(w)^d$ and are counted by $a(\ell)$ is just $|w|$, because the number of parts in w is the only thing that determines the number of non-cyclic n-color compositions, not the d copies of w.

Note that $weight(w) = \frac{\ell}{d}$ since there are d copies of $(w)^d$ with $weight(w)^d = \ell$. Consider s such that $s|\frac{\ell}{d}$. Similarly, because the number of parts in w is the only thing that contributes to the number of non-cyclic n-color compositions, there are only $|w|$ such non-cyclic n-color compositions of $(w)^s$ counted by $a(\frac{\ell}{s})$. Hence, in $\sum_{s|\ell} \varphi(s) a\left(\frac{\ell}{s}\right)$, the number of non-cyclic n-color compositions which map to $(w)^d$ is

$$\sum_{s|\frac{\ell}{d}} \varphi(s) a\left(\frac{\ell}{s}\right) = \sum_{s|\frac{\ell}{d}} \varphi(s)|w| = |w| \sum_{s|\frac{\ell}{d}} \varphi(s) = |w|\frac{\ell}{d}.$$

This is exactly the number of parts in $(w)^d$. Then for all compositions μ of ℓ, this means that

$$nccp(\ell) = |\mu|\frac{\ell}{d} = |\mu| \sum_{s|\frac{\ell}{d}} \varphi(s) = \sum_{s|\frac{\ell}{d}} \varphi(s) a\left(\frac{\ell}{s}\right).$$

$\square$

3.5 Generating functions

Recall that $\mathscr{NCC}_\ell$ is the set of n-color cyclic compositions of ℓ, and $ncc(\ell)$ and $NCC(x)$ are the corresponding cardinality and generating function. The generating function

$$NCC(x) = \sum_{s\geq 1} \frac{\varphi(s)}{s} \log\left(\frac{(1-x^s)^2}{(1-3x^s+x^{2s})}\right),$$

which is also given in Equation (3.1), can be derived directly through the "cycle construction" shown in [3]. In this section we first summarize the proof of Equation (3.1). Then we discuss how to obtain Equation (3.2) through the same approach.

First consider the series

$$x+2x^2+3x^3+\dots$$

for the number of ways to have a part of each size in the set of n-color compositions. Multiplying each term by u to mark all the parts, we have

$$\begin{aligned}
&u(x+2x^2+\cdots+ix^i+\dots)\\
=&ux\,\frac{d}{dx}(x+x^2+\cdots+x^i+\dots)\\
=&ux\,\frac{d}{dx}\left(\frac{x}{1-x}\right)\\
=&\left(\frac{ux}{(1-x)^2}\right).
\end{aligned}$$

Consequently we have the bivariate generating function for n-color compositions

$$\begin{aligned} NC(x,u) &= \sum_{k=1}^{\infty}\left(\frac{ux}{(1-x)^2}\right)^k \\ &= \frac{\frac{ux}{(1-x)^2}}{1-\frac{ux}{(1-x)^2}} \\ &= \frac{ux}{(1-x)^2-ux}, \end{aligned}$$

where the coefficient of $x^\ell u^r$ is the number of n-color compositions of ℓ with r parts. Note that every n-color composition is composed of d copies of a primitive n-color composition for some $d \in \mathbb{Z}^+$. By letting

$$PNC(x,u) = \sum_{\ell,r} pnc(\ell,r)x^\ell u^r$$

denote the generating function for primitive n-color compositions we have

$$NC(x,u) = \sum_{d\geq 1} PNC(x^d,u^d).$$

Note that the coefficient $pnc(\ell,r)$ is the number of primitive n-color compositions of ℓ with r parts. Then, using Möbius inversion, we implicitly find $PNC(x,u)$ to be

$$PNC(x,u) = \sum_{t\geq 1}\mu(t)NC(x^t,u^t),$$

where $\mu(t)$ is the Möbius μ function. We now let

$$PNCC(x,u) = \sum_{\ell,r} pncc(\ell,r)x^\ell u^r$$

denote the generating function for primitive n-color cyclic compositions, where the coefficient $pncc(\ell,r)$ is the number of primitive n-color cyclic compositions of ℓ with r parts. First note that each composition in $\mathscr{PNCC}_\ell$ with r parts has r unique primitive n-color composition representations. Thus, there is a one-to-r relationship between primitive n-color cyclic compositions and primitive n-color compositions. Consequently,

$$pncc(\ell,r)x^{\ell}u^{r} = \frac{pnc(\ell,r)}{r}x^{\ell}u^{r} = \int_0^u pnc(\ell,r)x^{\ell}w^{r-1}dw,$$

which means that

$$\begin{aligned}
PNCC(x,u) &= \left(\int_0^u \frac{PNC(x,w)}{w}\,dw\right)\\
&= \left(\int_0^u \frac{1}{w}\sum_{t\geq 1}\mu(t)\,NC(x^t,w^t)\,dw\right)\\
&= \left(\sum_{t\geq 1}\int_0^u \frac{w^{t-1}x^t}{(1-x^t)^2-w^tx^t}\,dw\right).
\end{aligned}$$

Integrating through substitution with

$$\beta = (1-x^t)^2 - w^tx^t, \text{ and } d\beta = -tw^{t-1}x^t\,dw$$

yields

$$\begin{aligned}
PNCC(x,u) &= \left(\sum_{t\geq 1}\int_0^u \frac{w^{t-1}x^t}{(1-x^t)^2-w^tx^t}\right)\\
&= \sum_{t\geq 1}\frac{\mu(t)}{t}\left[-\log\left((1-x^t)^2-w^tx^t\right)\right]\Big|_0^u\\
&= \sum_{t\geq 1}\frac{\mu(t)}{t}\log\left(\frac{(1-x^t)^2}{(1-x^t)^2-u^tx^t}\right).
\end{aligned}$$

Since every n-color cyclic composition is composed of q adjacent copies of n-color cyclic primitive compositions, the bivariate generating function for n-color cyclic compositions, which we denote $NCC(x,u)$, can be constructed using $PNCC(x^q,u^q)$. Given the identity $\sum_{t|s}\frac{\mu(t)}{t} = \frac{\varphi(s)}{s}$ and using the variable substitution $s = qt$, we have the bivariate generating function for n-color cyclic compositions as

$$
\begin{aligned}
NCC(x,u) &= \sum_{q\geq 1} PNCC(x^q,u^q) \\
&= \sum_{q\geq 1}\left(\sum_{t\geq 1}\frac{\mu(t)}{t}\log\left(\frac{(1-x^{qt})^2}{(1-x^{qt})^2-u^{qt}x^{qt}}\right)\right) \\
&= \sum_{s\geq 1}\sum_{t|s}\frac{\mu(t)}{t}\log\left(\frac{(1-x^{qt})^2}{(1-x^{qt})^2-u^{qt}x^{qt}}\right) \\
&= \sum_{s\geq 1}\frac{\varphi(s)}{s}\log\left(\frac{(1-x^s)^2}{1-2x^s+x^{2s}-u^s x^s}\right).
\end{aligned}
$$

Here, the coefficient of $x^{\ell}u^r$ is the number of n-color cyclic compositions of ℓ with r parts. Letting $u=1$ leads to Equation (3.1).

From the above discussion we can see that taking the partial derivative of the bivariate generating function $NCC(x,u)$ and then setting $u=1$ will yield the generating function for the total number of parts in all n-color cyclic compositions of ℓ. Therefore

$$
\begin{aligned}
NCCP(x) =& \frac{\partial}{\partial u}\Big(NCC(x,u)\Big)\bigg|_{u=1} \\
=& \sum_{s\geq 1}\frac{\varphi(s)}{s}\left(\frac{\partial}{\partial u}\log\left(\frac{(1-x^s)^2}{(1-x^s)^2-u^s x^s}\right)\right)\bigg|_{u=1} \\
=& \sum_{s\geq 1}\varphi(s)\left(\frac{x^s}{1-3x^s+x^{2s}}\right),
\end{aligned}
$$

which is the same as the generating function in Equation (3.2).

CHAPTER 4

PARTS UNDER MODULAR RESTRICTIONS

4.1 Introduction

This chapter is based on [4]. One focus within the study on compositions is the enumeration of parts and sub-word patterns under various restrictions. A summary of current results is contained in [5] and its references. In this chapter we consider the enumeration of parts under modular conditions in n-color cyclic compositions. Recall from Section 1.7 that $\mathscr{NCCP}_\ell(i,m)$ is the set of parts j such that $j \equiv i \mod m$ in all n-color cyclic compositions of ℓ, and $nccp(i;m;\ell)$ and $NCCP_{i,m}(x)$ denote the corresponding cardinality and generating function. Following the analytic tools developed in [3], we have that the generating function for $nccp(i;m;\ell)$ is

$$NCCP_{i,m}(x) = \sum_{s\geq 1} \varphi(s)\left(\frac{(x^{si}(1-x^s)^2)(i+(m-i)x^{sm})}{(1-3x^s+x^{2s})(1-x^{sm})^2}\right), \tag{4.1}$$

where $\varphi(s)$ is the Euler totient function.

The formula for $nccp(i;m;\ell)$ is studied in Section 4.2, where some interesting observations on the statistical behavior of this counting sequence are presented. We then conclude this chapter by giving a brief overview of the method used to construct $NCCP_{i,m}(x)$ in Section 4.3.

4.2 Parts Under Modular Restrictions in n-Color Cyclic Compositions

Counting the number of parts under certain equivalence classes has proven to be an interesting topic that relates to other combinatorial objects, such as subword patterns. From Equation (4.1) we generate Table 4.1, which contains values of $nccp(i,m;\ell)$ for $m = 10$.

ℓ \ i	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	2	2	0	0	0	0	0	0	0	0
3	6	2	3	0	0	0	0	0	0	0
4	11	8	3	4	0	0	0	0	0	0
5	25	16	9	4	5	0	0	0	0	0
6	62	48	27	12	5	6	0	0	0	0
7	150	110	63	32	15	6	7	0	0	0
8	391	298	168	88	40	18	7	8	0	0
9	999	758	438	220	105	48	21	8	9	0
10	2613	1998	1140	580	280	126	56	24	9	10
11	6786	5168	2961	1508	720	330	147	64	27	10
12	17805	13604	7788	3968	1890	870	385	168	72	30
13	46413	35434	20308	10336	4935	2262	1008	440	189	80
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	$1.35 \cdot 10^{20}$	$1.03 \cdot 10^{20}$	$5.92 \cdot 10^{19}$	$3.02 \cdot 10^{19}$	$1.44 \cdot 10^{19}$	$6.60 \cdot 10^{18}$	$2.94 \cdot 10^{18}$	$1.28 \cdot 10^{18}$	$5.52 \cdot 10^{17}$	$2.34 \cdot 10^{17}$
51	$3.54 \cdot 10^{20}$	$2.71 \cdot 10^{20}$	$1.55 \cdot 10^{20}$	$7.90 \cdot 10^{19}$	$3.77 \cdot 10^{19}$	$1.73 \cdot 10^{19}$	$7.70 \cdot 10^{18}$	$3.36 \cdot 10^{18}$	$1.44 \cdot 10^{18}$	$6.13 \cdot 10^{17}$
52	$9.28 \cdot 10^{20}$	$7.09 \cdot 10^{20}$	$2.07 \cdot 10^{20}$	$2.08 \cdot 10^{20}$	$9.87 \cdot 10^{19}$	$4.52 \cdot 10^{19}$	$2.02 \cdot 10^{19}$	$8.80 \cdot 10^{18}$	$3.78 \cdot 10^{18}$	$1.61 \cdot 10^{18}$
53	$2.43 \cdot 10^{21}$	$1.86 \cdot 10^{21}$	$1.06 \cdot 10^{21}$	$5.41 \cdot 10^{20}$	$2.58 \cdot 10^{20}$	$1.18 \cdot 10^{20}$	$5.28 \cdot 10^{19}$	$2.30 \cdot 10^{19}$	$9.90 \cdot 10^{18}$	$4.20 \cdot 10^{18}$
54	$6.36 \cdot 10^{21}$	$4.86 \cdot 10^{21}$	$2.78 \cdot 10^{21}$	$1.42 \cdot 10^{21}$	$6.77 \cdot 10^{20}$	$3.10 \cdot 10^{20}$	$1.38 \cdot 10^{20}$	$6.03 \cdot 10^{19}$	$2.59 \cdot 10^{19}$	$1.10 \cdot 10^{19}$

Table 4.1: Values of $nccp(i;m;\ell)$ for $m = 10$.

From studying Table 4.1, an interesting patterns arises. The sequence along the diagonals behaves like an arithmetic progression. That is,

$$nccp(i+1;m;\ell+1) - nccp(i;m;\ell) \sim nccp(i;m;\ell) - nccp(i-1;m;\ell-1)$$

as $\ell \to \infty$. For justification of this observation, we provide the following combinatorial proof for the proposition below.

Proposition 4.1. *For given i and (large) m, as $\ell \to \infty$,*

$$\frac{nccp(i+1;m;\ell+1)}{nccp(i;m;\ell)} \to \frac{i+1}{i}.$$

Proof. First, note that there is a simple bijection that maps each part of size k in a composition of ℓ to a part of size $k+1$ in some composition of $\ell+1$. This means that

$$\frac{ncp(k+1;\ell+1)}{ncp(k;\ell)} \to \frac{k+1}{k},$$

as $\ell \to \infty$, where $ncp(k;\ell)$ denotes the number of parts of size k in all compositions of ℓ. This bijection extends naturally to the cyclic case. For n-color cyclic compositions, each part of size k has k colors and each part of size $k+1$ has $k+1$ colors. Consequently, for large enough m, it follows that k of such parts of size k in $\mathscr{NCCP}_\ell(i,m)$ are mapped to $k+1$ of such parts of size $k+1$ in $\mathscr{NCCP}_{\ell+1}(i+1,m)$, where $k \equiv i \pmod{m}$. Using the same reasoning found in Theorem 2.2, we form the asymptotic formula for $NCCP_{i,m}(x)$ by letting $s=1$. This means that the majority of the parts counted by $nccp(i;m;\ell)$ are of size i, and not a multiple of m added to i. It follows for n-color cyclic compositions, i of such parts of size i in $\mathscr{NCCP}_\ell(i,m)$ are mapped to $i+1$ of such parts of size $i+1$ in $\mathscr{NCCP}_{\ell+1}(i+1,m)$ when m is large. We therefore conclude that

$$\frac{nccp(i+1;m;\ell+1)}{nccp(i;m;\ell)} \to \frac{i+1}{i},$$

as $\ell \to \infty$. □

4.3 Proof of $NCCP_{i;m}(x)$

Recall that $\mathscr{NCCP}_\ell(i,m)$ is the set of all parts j in n-color cyclic compositions of ℓ with the condition that $j \equiv i \mod m$, where $nccp(i;m;\ell)$ and $NCCP_{i,m}(x)$ are the corresponding cardinality and generating function for the set. The generating function

$$NCCP_{i,m}(x) = \sum_{s \geq 1} \varphi(s) \left(\frac{(x^{si}(1-x^s)^2)(i+(m-i)x^{sm})}{(1-3x^s+x^{2s})(1-x^{sm})^2}, \right),$$

given in Equation (4.1), can also be derived directly through the "cycle construction" shown in [3].

First consider the series

$$x+2x^2+3x^3+\dots$$

that generates the number of ways to have a part of each size in the set of n-color compositions. Multiplying each term that is congruent to $i \mod m$ by a y to mark those parts yields

$$x+2x^2+\cdots+iyx^i+(i+1)x^{i+1}+\cdots+(i+m)yx^{i+m}+\dots.$$

This can also be written as,

$$(x+2x^2+\cdots+ix^i+\dots)+(i(y-1)x^i+(i+m)(y-1)x^{i+m}+\dots).$$

Further multiplying each term by u to mark all the parts, we have

$$\begin{aligned}
&u(x+2x^2+\cdots+ix^i+\dots)+u(i(y-1)x^i+(m+i)(y-1)x^{m+i}+\dots)\\
=&u[x(1+2x+\cdots+ix^{i-1}+\dots)+(y-1)x(ix^{i-1}+(m+i)x^{m+i-1}+\dots)]\\
=&u\left[x\,\frac{d}{dx}\left(\frac{x}{1-x}\right)+(y-1)x\,\frac{d}{dx}\left(\frac{x^i}{1-x^m}\right)\right]\\
=&u\left[\left(\frac{x}{(1-x)^2}\right)+(y-1)x\left(\frac{ix^{i-1}-ix^{m+i-1}+mx^{m+i-1}}{(1-x^m)^2}\right)\right]\\
=&\frac{ux}{(1-x)^2}+\frac{ui(y-1)x^i+u(m-i)(y-1)x^{m+i}}{(1-x^m)^2}.
\end{aligned}$$

Consequently we have the multivariable generating function of n-color compositions

$$\begin{aligned}
NC(x,u,y)&=\sum_{k=1}^{\infty}\left(\frac{ux}{(1-x)^2}+\frac{ui(y-1)x^i+u(m-i)(y-1)x^{m+i}}{(1-x^m)^2}\right)^k\\
&=\frac{\frac{ux}{(1-x)^2}+\frac{ui(y-1)x^i+u(m-i)(y-1)x^{m+i}}{(1-x^m)^2}}{1-\left(\frac{ux}{(1-x)^2}+\frac{ui(y-1)x^i+u(m-i)(y-1)x^{m+i}}{(1-x^m)^2}\right)}\\
&=\frac{ux(1-x^m)^2+(ui(y-1)x^i+u(m-i)(y-1)x^{m+i})(1-x)^2}{(1-x)^2(1-x^m)^2-(ux(1-x^m)^2+(ui(y-1)x^i+u(m-i)(y-1)x^{m+i})(1-x)^2)},
\end{aligned}$$

where the coefficient of $x^\ell u^s y^t$ is the number of n-color compositions of ℓ with s parts, t of which are congruent to $i \mod m$. Recall from Section 1.4 that every n-color composition is composed of d copies, for some $d \in \mathbb{Z}^+$, of a primitive n-color composition. By letting

$$PNC(x,u) = \sum_{\ell,r} pnc(\ell,r)x^\ell u^r$$

denote the generating function for primitive n-color compositions (where the coefficient $pnc(\ell,r)$ is the number of primitive n-color compositions of ℓ with r parts), we have

$$NC(x,u,y) = \sum_{d\geq 1} PNC(x^d,u^d,y^d).$$

Then $PNC(x,u,y)$ can be implicitly derived using Möbius inversion so that

$$PNC(x,u,y) = \sum_{d\geq 1} \mu(d) NC(x^d,u^d,y^d)$$

where $\mu(d)$ is the Möbius μ function.

We now let

$$PNCC(x,u,y) = \sum_{n,r,t} pncc(\ell,r,t)x^\ell u^r y^t$$

denote the generating function for primitive n-color cyclic compositions, where the coefficient $pncc(\ell,r,t)$ is the number of primitive n-color cyclic compositions of ℓ with r parts, t of which are congruent to $i \mod m$. Recall $\mathcal{NCC}_\ell$ is the set containing all n-color cyclic compositions of ℓ. First note that each composition in $\mathcal{NCC}_\ell$ with r parts has r unique primitive n-color composition representations. Thus there is a one-to-r relationship between primitive cyclic n-color compositions and primitive n-color compositions. Consequently

$$pncc(\ell,r,t)x^\ell u^r y^t = \frac{pnc(\ell,r,t)}{r} x^\ell u^r y^t = \int_0^u pnc(\ell,r,t)x^n w^{r-1} y^t \, dw,$$

and we have

$$\begin{aligned}
PNCC(x,u,y) &= \int_0^u \frac{PNC(x,w,y)}{w} \, dw \\
&= \int_0^u \frac{1}{w} \sum_{d\geq 1} \mu(d)\, NC(x^d,w^d,y^d)\, dw \\
&= \int_0^u \sum_{d\geq 1} \frac{\mu(d)\, w^{d-1}[x^d(1-x^{md})^2 + (i(y^d-1)x^{id} + (m-i)(y^d-1)x^{(m+i)d})(1-x^d)^2] \; dw}{(1-x^d)^2(1-x^{md})^2 - w^d x^d (1-x^{md})^2 - w^d(i(y^d-1)x^{id} - w^d(m-i)(y^d-1)x^{(m+i)d})(1-x^d)^2} \\
&= \sum_{d\geq 1} \int_0^u \frac{\mu(d)\, w^{d-1}[x^d(1-x^{md})^2 + (i(y^d-1)x^{id} + (m-i)(y^d-1)x^{(m+i)d})(1-x^d)^2] \; dw}{(1-x^d)^2(1-x^{md})^2 - w^d[x^d(1-x^{md})^2 + (i(y^d-1)x^{id} + (m-i)(y^d-1)x^{(m+i)d})(1-x^d)^2]}.
\end{aligned}$$

Integrating through substitution with

$$\beta = (1-x^d)^2(1-x^{md})^2 - w^d[x^d(1-x^{md})^2 + (i(y^d-1)x^{id} + (m-i)(y^d-1)x^{(m+i)d})(1-x^d)^2],$$

and

$$\frac{d\beta}{dw} = -dw^{d-1}[x^d(1-x^{md})^2 + (i(y^d-1)x^{id} + (m-i)(y^d-1)x^{(m+i)d})(1-x^d)^2],$$

and evaluating u at 1, we have

$$PNCC(x,y) = PNCC(x,u,y)\Big|_{u=1} = \left(\int_0^u \frac{PNC(x,w,y)}{w}\,dw\right)\Big|_{u=1}$$
$$= \sum_{d\geq 1} \frac{\mu(d)}{d} \log\left(\frac{(1-x^d)^2(1-x^{md})^2}{(1-x^d)^2(1-x^{md})^2 - u^d[x^d(1-x^{md})^2 + (i(y^d-1)x^{id} + (m-i)(y^d-1)x^{(m+i)d})(1-x^d)^2]}\right)\Bigg|_{u=1}$$
$$= \sum_{d\geq 1} \frac{\mu(d)}{d} \log\left(\frac{(1-x^d)^2(1-x^{md})^2}{(1-x^d)^2(1-x^{md})^2 - [x^d(1-x^{md})^2 + (i(y^d-1)x^{id} + (m-i)(y^d-1)x^{(m+i)d})(1-x^d)^2]}\right).$$

Since every n-color cyclic composition is composed of q adjacent copies, for some $q \in \mathbb{Z}^+$, of n-color cyclic primitive compositions, the bivariate generating functions for n-color cyclic compositions, which we denote $NCC(x,y)$, can be constructed using $PNCC(x^q, y^q)$. Given the identity $\sum_{d|s} \frac{\mu(d)}{d} = \frac{\varphi(s)}{s}$ and using the variable substitution $s = qd$, we have the bivariate generating function for cyclic compositions as

$$NCC(x,y) = \sum_{q\geq 1} PNCC(x^q, y^q)$$
$$= \sum_{s\geq 1}\sum_{d|s} \frac{\mu(d)}{d} \log\left(\frac{(1-x^{qd})^2(1-x^{mqd})^2}{(1-x^{qd})^2(1-x^{mqd})^2 - [x^{qd}(1-x^{mqd})^2 + (i(y^{qd}-1)x^{iqd} + (m-i)(y^{qd}-1)x^{(m+i)qd})(1-x^{qd})^2]}\right)$$
$$= \sum_{s\geq 1} \frac{\varphi(s)}{s} \log\left(\frac{(1-x^s)^2(1-x^{sm})^2}{(1-x^s)^2(1-x^{sm})^2 - [x^s(1-x^{sm})^2 + (i(y^s-1)x^{si} + (m-i)(y^s-1)x^{(m+i)s})(1-x^s)^2]}\right).$$

Here the coefficient of $x^\ell y^t$ is the number of n-color cyclic compositions of ℓ with t parts that are congruent to $i \mod m$. By taking the partial derivative of $NCC(x,y)$ with respect to y and then setting $y = 1$, we have that the generating function for the total number of parts congruent to $i \mod m$ in all n-color cyclic compositions, denoted $NCCP(x)$, is

$$NCCP(x) = \frac{\partial}{\partial y}\Big(NCC(x,y)\Big)\Big|_{y=1}$$
$$= \sum_{s\geq 1} \frac{\varphi(s)}{s} \left(\frac{\partial}{\partial y}\left(\log \frac{(1-x^s)^2(1-x^{sm})^2}{(1-x^s)^2(1-x^{sm})^2 - [x^s(1-x^{sm})^2 + (i(y^s-1)x^{si} + (m-i)(y^s-1)x^{(m+i)s})(1-x^s)^2]}\right)\right)\Bigg|_{y=1}$$
$$= \sum_{s\geq 1} \frac{\varphi(s)}{s} \left(\frac{(sy^{s-1}x^{si}(1-x^s)^2)(i+(m-i)x^{sm})}{(1-x^s)^2(1-x^{sm})^2 - [x^s(1-x^{sm})^2 + (i(y^s-1)x^{si} + (m-i)(y^s-1)x^{(m+i)s})(1-x^s)^2]}\right)\Bigg|_{y=1}$$
$$= \sum_{s\geq 1} \frac{\varphi(s)}{s} \left(\frac{(sx^{si}(1-x^s)^2)(i+(m-i)x^{sm})}{(1-x^s)^2(1-x^{sm})^2 - x^s(1-x^{sm})^2}\right)$$
$$= \sum_{s\geq 1} \varphi(s) \left(\frac{(x^{si}(1-x^s)^2)(i+(m-i)x^{sm})}{(1-3x^s+x^{2s})(1-x^{sm})^2}\right),$$

which is exactly Equation (4.1).

CHAPTER 5

N-COLOR CYCLIC COMPOSITIONS: FURTHER RESTRICTIONS ON PARTS

5.1 Introduction

This chapter is based on [4]. In this last chapter, we examine observations resulting from the generating functions for the number of n-color cyclic compositions with two different types of restrictions on parts, namely with parts no greater than a certain size (h) and with parts that are multiples of a given number (j). We first present the generating functions that quantify these restrictions and discuss related consequences and observations in Sections 5.2 and 5.3. Lastly, in Section 5.4, we summarize the proof of Equation (5.1). Then we briefly discuss how to obtain Equation (5.2) through the same approach.

For clarity's sake, we use the notation previously introduced in Section 1.3 to better present the generating functions for the two different types of restrictions we explore on the parts of $\mathscr{N}\mathscr{C}\mathscr{C}_\ell$. Just as in Section 1.3, let $\mathscr{P}$ denote a general condition that is imposed upon the parts of the n-color cyclic compositions, with $\mathscr{P} \leftrightarrow (\leq h)$ denoting the constraint that there are no parts of size larger than h and $\mathscr{P} \leftrightarrow (\equiv 0 \pmod{j})$ denoting the constraint that each part has to be divisible by j. Using methods similar to those used previously in Section 4.3, we find that the generating functions for the number of such constrained n-color cyclic compositions are

$$NCC_{(\leq h)}(x) = \sum_{s\geq 1} \frac{\varphi(s)}{s} \log\left(\frac{(1-x^s)^2}{(1-x^s)^2 - x^s(1-x^{hs} - hx^{hs}(1-x^s))}\right), \tag{5.1}$$

and

$$NCC_{(\equiv 0 \pmod{j})}(x) = \sum_{s\geq 1} \frac{\varphi(s)}{s} \log\left(\frac{(1-x^{js})^2}{1-(2+j)x^{js}+x^{2js}}\right). \tag{5.2}$$

Note that the coefficient of x^ℓ in Equation (5.1) is the number of n-color cyclic compositions of ℓ with part size at most h, which we will denote by $ncc_{(\leq h)}(\ell)$. Also, the coefficient of x^ℓ in Equation (5.2) is the number of n-color cyclic compositions of ℓ with parts divisible by j, which we denote by $ncc_{(\equiv 0 \pmod{j})}(\ell)$. Similar to the results given in Chapter 4, the generating functions given in this section provide us with interesting information on these counting sequences, leading to some intriguing observations.

5.2 n-color cyclic compositions with parts at most h

By definition of Equation (3.1) and (5.1), we have

$$ncc_{(\leq h)}(\ell) \to ncc(\ell)$$

as $h \to \infty$. Recall from Equation (5.1) we know that as $\ell \to \infty$

$$ncc(\ell) = \Theta\left(\frac{1}{\ell}\left(\frac{2}{3-\sqrt{5}}\right)^{\ell-1}\right).$$

We will show, in what follows, that (5.1) also leads to the same conclusion.

Proposition 5.1. *For large ℓ we have*

$$\lim_{h\to\infty} ncc_{(\leq h)}(\ell) = \Theta\left(\frac{1}{\ell}\left(\frac{2}{3-\sqrt{5}}\right)^{\ell-1}\right).$$

Proof. Let $F(x) = NCC_{(\leq h)}(x)$ and $f_\ell = ncc_{(\leq h)}(\ell)$. Consider

$$F'(x) =: G(x) = \sum_{\ell\geq 0} g_\ell x^\ell.$$

Then

$$g_\ell = (\ell+1) f_{\ell+1}. \tag{5.3}$$

We now examine the exponential growth rate of the sequence $\{g_\ell\}$. From the way $F(x)$ and $G(x)$ were defined above, we have that

$$\begin{aligned} F(x) &= \sum_{s\geq 1} \frac{\varphi(s)}{s} \log\left(\frac{(1-x^s)^2}{(1-x^s)^2 - x^s(1-x^{hs}-hx^s(1-x^s))}\right), \\ G(x) &= \sum_{s\geq 1} \varphi(s) \frac{x^{s-1}((2h^2+2h-1)x^{(h+1)s} - h^2x^{(h+2)s} - (h+1)x^{hs} + x^s + 1)}{(1-x^s)((h+1)x^{(h+1)s} - hx^{(h+2)s} + x^{2s} - 3x^s + 1)}. \end{aligned}$$

Thus, $G(x)$ is the sum of rational functions. Let

$$A(x) = (1-x)((h+1)x^{(h+1)} - hx^{(h+2)} + x^2 - 3x + 1),$$

and note that $A(x^i)$ would be the denominator for the $s = i$ term in $G(x)$. The exponential growth rate of the coefficients of $G(x)$ are completely determined by the roots closest to zero for the various $A(x^s)$. So for each s:

- We have $B(x^s) = \lim_{h\to\infty} A(x^s) = (1-x^s)(x^{2s} - 3x^s + 1)$ for all complex x with $|x| < \frac{1}{2}$.
- The roots of $B(x^s)$ are $x^s = 1$ and $x^s = \frac{3\pm\sqrt{5}}{2}$.
- Thus, the roots closest to zero are the solutions to $x^s = \frac{3-\sqrt{5}}{2}$.

Therefore, letting $\tau = \frac{2}{3-\sqrt{5}}$ and for some constants $c_0 \leq c_1 \leq \cdots \leq c_m$, we have

$$\lim_{h\to\infty} g_\ell = \Theta\left(\sum_{s|\ell} c_{\ell/s}\tau^{\ell/s}\right).$$

Note by the definition of τ and c_ℓ that the sequence $\{c_\ell\tau^\ell, c_{\ell/2}\tau^{\ell/2}, c_{\ell/3}\tau^{\ell/3}, \ldots\}$ forms a decreasing sequence as s increases. Using similar arguments to those used in Theorem 2.4, we know the term that results from $s = 1$ is the dominant term in the sum above. Therefore it follows that $g_\ell = \Theta\left(\tau^\ell\right)$. Hence

$$\lim_{h\to\infty} ncc_{(\leq h)}(\ell) = \lim_{h\to\infty} f_\ell = \Theta\left(\frac{\tau^{\ell-1}}{\ell}\right) = \Theta\left(\frac{1}{\ell}\left(\frac{2}{3-\sqrt{5}}\right)^{\ell-1}\right),$$

by (5.3). $\square$

As an example of the results that are produced from (5.1), we have Table 5.1, which gives values for $ncc_{(\leq h)}(\ell)$ with $1 \leq h \leq 12$.

ℓ \ h	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	3	3	3	3	3	3	3	3	3	3	3
3	1	3	6	6	6	6	6	6	6	6	6	6
4	1	6	9	13	13	13	13	13	13	13	13	13
5	1	7	16	20	25	25	25	25	25	25	25	25
6	1	14	35	47	52	58	58	58	58	58	58	58
7	1	19	61	93	108	114	121	121	121	121	121	121
8	1	36	132	210	250	268	275	283	283	283	283	283
9	1	59	271	455	560	608	629	637	646	646	646	646
10	1	108	579	1037	1302	1428	1484	1508	1517	1527	1527	1527
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	1	$2.25 \cdot 10^{13}$	$1.20 \cdot 10^{17}$	$2.26 \cdot 10^{18}$	$7.06 \cdot 10^{18}$	$1.13 \cdot 10^{19}$	$1.37 \cdot 10^{19}$	$1.49 \cdot 10^{19}$	$1.54 \cdot 10^{19}$	$1.57 \cdot 10^{19}$	$1.58 \cdot 10^{19}$	$1.58 \cdot 10^{19}$
51	1	$4.42 \cdot 10^{13}$	$2.79 \cdot 10^{17}$	$5.57 \cdot 10^{18}$	$1.78 \cdot 10^{19}$	$2.87 \cdot 10^{19}$	$3.51 \cdot 10^{19}$	$3.82 \cdot 10^{19}$	$3.96 \cdot 10^{19}$	$4.02 \cdot 10^{19}$	$4.05 \cdot 10^{19}$	$4.06 \cdot 10^{19}$
52	1	$8.66 \cdot 10^{13}$	$6.50 \cdot 10^{17}$	$1.38 \cdot 10^{19}$	$4.50 \cdot 10^{19}$	$7.32 \cdot 10^{19}$	$8.99 \cdot 10^{19}$	$9.80 \cdot 10^{19}$	$1.02 \cdot 10^{20}$	$1.03 \cdot 10^{20}$	$1.04 \cdot 10^{20}$	$1.04 \cdot 10^{20}$
53	1	$1.70 \cdot 10^{14}$	$1.51 \cdot 10^{18}$	$3.40 \cdot 10^{19}$	$1.14 \cdot 10^{20}$	$1.87 \cdot 10^{20}$	$2.30 \cdot 10^{20}$	$2.51 \cdot 10^{20}$	$2.61 \cdot 10^{20}$	$2.65 \cdot 10^{20}$	$2.67 \cdot 10^{20}$	$2.68 \cdot 10^{20}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
500	1	$6.55 \cdot 10^{147}$	$1.20 \cdot 10^{185}$	$6.74 \cdot 10^{197}$	$5.97 \cdot 10^{202}$	$6.42 \cdot 10^{204}$	$4.59 \cdot 10^{205}$	$1.06 \cdot 10^{206}$	$1.51 \cdot 10^{206}$	$1.75 \cdot 10^{206}$	$1.86 \cdot 10^{206}$	$1.91 \cdot 10^{206}$
501	1	$1.31 \cdot 10^{148}$	$2.85 \cdot 10^{185}$	$1.69 \cdot 10^{198}$	$1.53 \cdot 10^{203}$	$1.67 \cdot 10^{205}$	$1.20 \cdot 10^{206}$	$2.76 \cdot 10^{206}$	$3.93 \cdot 10^{206}$	$4.57 \cdot 10^{206}$	$4.86 \cdot 10^{206}$	$4.98 \cdot 10^{206}$
502	1	$2.61 \cdot 10^{148}$	$6.75 \cdot 10^{185}$	$4.26 \cdot 10^{198}$	$3.94 \cdot 10^{203}$	$4.32 \cdot 10^{205}$	$3.12 \cdot 10^{206}$	$7.21 \cdot 10^{206}$	$1.03 \cdot 10^{207}$	$1.19 \cdot 10^{207}$	$1.27 \cdot 10^{207}$	$1.30 \cdot 10^{207}$
503	1	$5.21 \cdot 10^{148}$	$1.60 \cdot 10^{186}$	$1.07 \cdot 10^{199}$	$1.01 \cdot 10^{204}$	$1.12 \cdot 10^{206}$	$8.13 \cdot 10^{206}$	$1.88 \cdot 10^{207}$	$2.68 \cdot 10^{207}$	$3.12 \cdot 10^{207}$	$3.32 \cdot 10^{207}$	$3.40 \cdot 10^{207}$

Table 5.1: Values of $ncc_{(\leq h)}(\ell)$

One of the interesting observations that can be made from Table 5.1 is $ncc_{(\leq h)}\ell$ converges quickly to $ncc(\ell)$ as $h \to \infty$ even when ℓ is large.

5.3 n-color cyclic compositions with parts divisible by j

Similarly, from (5.2) we generate Table 5.2, which gives values for $ncc_{(\equiv 0 \pmod j)}(\ell)$ for $1 \leq j \leq 10$.

ℓ \ j	1	2	3	4	5	6	7	8	9	10
$1 \cdot j$	1	1	1	1	1	1	1	1	1	1
$2 \cdot j$	3	7	12	18	25	33	42	52	63	75
$3 \cdot j$	6	18	38	68	110	166	238	328	438	570
$4 \cdot j$	13	52	138	298	565	978	1582	2428	3573	5080
$5 \cdot j$	25	146	507	1348	3029	6054	11095	19016	30897	48058
$6 \cdot j$	58	463	2042	6578	17350	39793	82278	$1.57 \cdot 10^5$	$2.81 \cdot 10^5$	$4.77 \cdot 10^5$
$7 \cdot j$	121	1442	8283	32644	$1.02 \cdot 10^5$	$2.68 \cdot 10^5$	$6.26 \cdot 10^5$	$1.33 \cdot 10^6$	$2.63 \cdot 10^6$	$4.87 \cdot 10^6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$50 \cdot j$	$1.58 \cdot 10^{19}$	$7.91 \cdot 10^{26}$	$2.11 \cdot 10^{32}$	$3.79 \cdot 10^{36}$	$1.25 \cdot 10^{40}$	$1.28 \cdot 10^{43}$	$5.49 \cdot 10^{45}$	$1.20 \cdot 10^{48}$	$1.55 \cdot 10^{50}$	$1.28 \cdot 10^{52}$
$51 \cdot j$	$4.06 \cdot 10^{19}$	$2.90 \cdot 10^{27}$	$9.90 \cdot 10^{32}$	$2.17 \cdot 10^{37}$	$8.43 \cdot 10^{40}$	$9.90 \cdot 10^{43}$	$4.79 \cdot 10^{46}$	$1.17 \cdot 10^{49}$	$1.65 \cdot 10^{51}$	$1.50 \cdot 10^{53}$
$52 \cdot j$	$1.04 \cdot 10^{20}$	$1.06 \cdot 10^{28}$	$4.65 \cdot 10^{33}$	$1.24 \cdot 10^{38}$	$5.67 \cdot 10^{41}$	$7.64 \cdot 10^{44}$	$4.17 \cdot 10^{47}$	$1.13 \cdot 10^{50}$	$1.77 \cdot 10^{52}$	$1.75 \cdot 10^{54}$
$53 \cdot j$	$2.68 \cdot 10^{20}$	$3.88 \cdot 10^{28}$	$2.19 \cdot 10^{34}$	$7.07 \cdot 10^{38}$	$3.81 \cdot 10^{42}$	$5.90 \cdot 10^{45}$	$3.64 \cdot 10^{48}$	$1.10 \cdot 10^{51}$	$1.89 \cdot 10^{53}$	$2.05 \cdot 10^{55}$

Table 5.2: Values of $ncc_{(\equiv 0 \pmod j)}(\ell)$

To introduce an interesting observation from Table 5.2, we first want to recall the notation for the difference operator on integer sequences, which is denoted as Δ. More nformation regarding difference operators can be found in [9]. If $A = \{a_n\}_n$ is an infinite sequence of positive integers, then $\Delta(A) = \{\Delta(a_n)\}_n$ is defined by

$$\Delta(a_n) = a_{n+1} - a_n.$$

For instance, a sequence A is a constant sequence if $\Delta(A)$ is the zero sequence, and a sequence B is an arithmetic sequence if $\Delta^2(B)$ is the zero sequence.

Now let $a_{i,j} = ncc_{(\equiv 0 \pmod{j})}(i \cdot j)$ and let A_i be the sequence $\{a_{i,j}\}_j$, where $j \in \mathbb{Z}^+$. From Tabel 5.2 we see that

$$\{0\}_j = \Delta^3(A_2) = \Delta^4(A_3) = \Delta^5(A_4) = \Delta^6(A_5) = \dots.$$

In general we will show the following.

Theorem 5.2. *Letting $A_i = \{ncc_{(\equiv 0 \pmod{j})}(i \cdot j)\}_j$, we have*

$$\Delta^{i+1}(A_i) = \{0\}_j.$$

In fact, this interesting observation also holds for non-cyclic n-color compositions. This means it is the n-color condition that results in the above conclusion, not the cyclic condition. To present the proof for this theorem we first let $a_{i,j}$ ($a_{i,j,v}$) denote the number of n-color cyclic compositions of $i \cdot j$ (with v parts), where each part is divisible by j. Similarly $b_{i,j}$ ($b_{i,j,v}$) denotes the number of n-color non-cyclic compositions of $i \cdot j$ (with v parts), where each part is divisible by j. First, consider the following lemma:

Lemma 5.3. *For any positive integer r, and for any sequence $\{c_{i,j}\}_j$ we have*

$$\Delta^r(b_{i,j,v}) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} b_{i,j+k,v},$$

$$\Delta^r(b_{i,j}) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} b_{i,j+k},$$

and

$$\Delta^r(b_{i,j} + c_{i,j}) = \Delta^r(b_{i,j}) + \Delta^r(c_{i,j}).$$

Proof. First, note that the third equation follows from the fact that Δ is a linear transformation. Next we will prove the second equation, and proof for the first equation can be constructed similarly.

We will derive a proof by induction on r. First consider the case $r = 1$. Note

$$\Delta^1(b_{i,j}) = \sum_{k=0}^{1} \binom{1}{k} (-1)^{1-k} b_{i,j+k} = \sum_{k=0}^{1} \binom{1}{k} (-1)^{1-k} b_{i,j+k} = b_{i,j+1} - b_{i,j},$$

which holds by the definition of Δ. Next we consider our inductive step. Suppose the result holds true for up to $r-1$, that is

$$\Delta^{r-1}(b_{i,j}) = \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^{r-1-k} b_{i,j+k}.$$

Then

$$\begin{aligned}
\Delta^r(b_{i,j}) &= \Delta(\Delta^{r-1}(b_{i,j})) \\
&= \Delta\left(\sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^{r-1-k} b_{i,j+k}\right) \\
&= \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^{r-1-k} (b_{i,j+k+1} - b_{i,j+k}) \\
&= \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^{r-1-k} b_{i,j+k+1} - \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^{r-1-k} b_{i,j+k} \\
&= b_{i,j+r} + \left(\sum_{k=1}^{r-1} \binom{r-1}{k-1} (-1)^{r-k} b_{i,j+k} + \sum_{k=1}^{r-1} \binom{r-1}{k} (-1)^{r-k} b_{i,j+k}\right) + (-1)^r b_{i,j} \\
&= b_{i,j+r} + \left(\sum_{k=1}^{r-1} \left[\binom{r-1}{k-1} + \binom{r-1}{k}\right] (-1)^{r-k} b_{i,j+k}\right) + (-1)^r b_{i,j}.
\end{aligned}$$

By noting $\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}$, we have

$$\begin{aligned}
\Delta^r(b_{i,j}) &= b_{i,j+r} + \left(\sum_{k=1}^{r-1} \binom{r}{k} (-1)^{r-k} b_{i,j+k}\right) + (-1)^r b_{i,j} \\
&= \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} b_{i,j+k},
\end{aligned}$$

as desired. $\square$

The next Lemma states that $a_{i,j,v}$ can be written as a linear combination of $b_{i',j',v'}$'s. The proof, which we skip, follows the same idea as the cyclic construction of generating functions.

Lemma 5.4. *For all positive integers i, j, and v, we have*

$$a_{i,j,v} = \sum_{d|(i,v)} \frac{d}{v} \sum_{sd|(i,v)} \mu(s) b_{i/(sd),j,v/(sd)}$$

where $\mu(\cdot)$ is the number-theoretic mobius function.

Next, we will now show that

$$\Delta^{i+1}(b_{i,j,v}) = 0. \tag{5.4}$$

Although the above identity can be shown through induction and some algebra, we provide a combinatorial proof as follows:

Proof of (5.4). We first claim that

$$\frac{b_{i,j',v}}{b_{i,j,v}} = \frac{j'^v}{j^v}$$

for any j and j'. This is because, given any composition of $i \cdot j$ with v parts of the form $ij = x_1 j + x_2 j + \ldots + x_v j$, the sum $ij' = x_1 j' + x_2 j' + \ldots + x_v j'$ is a composition of $i \cdot j'$ with v parts. It is easy to see that there exists a bijection between such compositions of $i \cdot j$ (where all v parts are divisible by j), and compositions of $i \cdot j'$ (where all v parts are divisible by j'). When n-color compositions are considered, each composition of $i \cdot j$ corresponds to $\prod_{s=1}^{v}(x_s \cdot j)$ many n-color compositions, due to the number of choices of colors. Similarly, the composition of $i \cdot j'$ also corresponds to $\prod_{s=1}^{v}(x_s \cdot j')$ many n-color compositions. Thus every $\prod_{s=1}^{v}(x_s \cdot j)$ n-color compositions of $i \cdot j$ are mapped to their corresponding n-color composition of the $\prod_{s=1}^{v}(x_s \cdot j')$ n-color compositions of $i \cdot j'$. Since $b_{i,j,v}$ denotes the number of n-color compositions of $i \cdot j$ with v parts, it follows that

$$\frac{b_{i,j',v}}{b_{i,j,v}} = \frac{\prod_{s=1}^{v}(x_s \cdot j')}{\prod_{s=1}^{v}(x_s \cdot j)} = \frac{(j')^v (x_1 \cdot \cdots \cdot x_v)}{j^v (x_1 \cdot \cdots \cdot x_v)} = \frac{j'^v}{j^v}.$$

Now by the above claim and by Lemma 5.3, we know

$$\begin{aligned}\Delta^{i+1}(b_{i,j,v}) &= \sum_{k=0}^{i+1}\binom{i+1}{k}(-1)^{i+1-k}b_{i,j+k,v}\\ &= \sum_{k=0}^{i+1}\binom{i+1}{k}(-1)^{i+1-k}b_{i,j,v}\cdot\frac{(j+k)^v}{j^v}\\ &= \frac{b_{i,j,v}}{j^v}\sum_{k=0}^{i+1}\binom{i+1}{k}(-1)^{i+1-k}(j+k)^v.\end{aligned}$$

To prove (5.4) it suffices to show

$$\sum_{k=0}^{i+1}\binom{i+1}{k}(-1)^{i+1-k}(j+k)^v = 0,$$

which is equivalent to

$$\sum_{s=0}^{N}\binom{N}{s}(-1)^s(j+(N-s))^v = 0 \tag{5.5}$$

by letting $N = i+1$ and $s = N-k$. It is important to note that $N > i$ by definition of N.

We now show (5.5) by considering the following scenario: A group of $j+N$ students, j male and N female, pick a leader each day for v days. The number of ways to do this such that at most $N-s$ female students are picked as leaders is

$$\binom{N}{s}(j+(N-s))^v,$$

since this is equivalent to at least s female students never being picked as leaders. Hence, by the inclusion-exclusion principle, the number of ways to pick leaders for v days so that every female student is picked as the leader at least once is

$$\sum_{s=0}^{N}\binom{N}{s}(-1)^s(j+(N-s))^v.$$

But for every female student to serve as leader at least once is impossible since the number of days v is at most i and $N = i+1 > i \geq v$. This means there are more female students than there are number of days to be the leader. Hence

$$\sum_{s=0}^{N}\binom{N}{s}(-1)^s(j+(N-s))^v = 0,$$

which implies that

$$0 = \frac{b_{i,j,v}}{j^v} \sum_{k=0}^{i+1} \binom{i+1}{k} (-1)^{i+1-k} (j+k)^v = \Delta^{i+1}(b_{i,j,v}).$$

□

Consequently by Lemma 5.4 and since Δ is a linear transformation, we have that $0 = \Delta^{i+1}(a_{i,j,v})$. Taking the sum over all possible v we have $\Delta^{i+1}(b_{i,j}) = 0$ which implies

$$\Delta^{i+1}(a_{i,j}) = 0,$$

proving Theorem 5.2.

5.4 Summary of Generating functions

Recall that $\mathscr{NCC}_\ell$ is the set of n-color cyclic compositions of ℓ, and $ncc(\ell)$ and $NCC(x)$ are its corresponding cardinality and generating function. When we restrict the parts of the compositions to be no greater than h, we denote such a subset of $\mathscr{NCC}_\ell$ as $\mathscr{NCC}_{\mathscr{P}\ell}$, where in this case the condition $\mathscr{P}$ is defined as $\mathscr{P} \leftrightarrow (\leq h)$. In this section, we summarize the construction for the generating function 5.1, which we will denote using the condition $\mathscr{P}$. This means that $NCC_{\mathscr{P}}(x)$ and $ncc_{\mathscr{P}}(\ell)$ will denote the corresponding generating function and cardinality for $\mathscr{NCC}_{\mathscr{P}\ell}$.

Let $\mathscr{P}$ be the conditional statement such that $\mathscr{P} \leftrightarrow (\leq h)$. Then consider the series

$$x + 2x^2 + 3x^3 + \ldots + hx^h,$$

which generates the parts for such compositions with parts no greater than h. Further multiplying each term by u to mark all the parts yields

$$ux + 2ux^2 + 3ux^3 + \cdots + uhx^h = u\left[x \frac{d}{dx}\left(x + x^2 + x^3 + \cdots + x^h\right)\right].$$

Since the sum of a finite geometric series is $\sum\limits_{i=1}^{n} a_i x^i = \frac{a(1-x^n)}{1-x}$, it follows that

$$\begin{aligned}
&u\left[x\,\frac{d}{dx}\left(x+x^2+x^3+\cdots+x^h\right)\right]\\
=&u\left[x\,\frac{d}{dx}\left(\frac{x(1-x^h)}{1-x}\right)\right]\\
=&u\left[x\cdot\left(\frac{1-x^h-hx^h(1-x)}{(1-x)^2}\right)\right]\\
=&\frac{ux(1-x^h-hx^h(1-x))}{(1-x)^2}
\end{aligned}$$

Consequently the multivariable generating function of n-color compositions of ℓ with parts of size at most h is

$$\begin{aligned}
NC_{\mathscr{P}}(x,u) &= \sum_{k=1}^{\infty}\left(\frac{ux(1-x^h-hx^h(1-x))}{(1-x)^2}\right)^k\\
&= \frac{\frac{ux(1-x^h-hx^h(1-x))}{(1-x)^2}}{1-\frac{ux(1-x^h-hx^h(1-x))}{(1-x)^2}}\\
&= \frac{ux(1-x^h-hx^h(1-x))}{(1-x)^2-ux(1-x^h-hx^h(1-x))}.
\end{aligned}$$

where the coefficient of $x^\ell u^r$ is the number of n-color compositions of ℓ with r parts, each being no larger than h. Recall from Section 1.4 that a primitive composition is a composition that is not composed of repeated copies of shorter compositions. Also recall that all n-color compositions are comprised of d copies of primitive n-color compositions, for some $d \in \mathbb{Z}^+$. By letting

$$PNC_{\mathscr{P}}(x,u) = \sum_{\ell,r,t} pnc_{\mathscr{P}}(\ell,r)x^\ell u^r$$

denote the generating function for primitive n-color compositions, where the coefficient

$pnc_{\mathscr{P}}(\ell,r)$ is the number of primitive n-color compositions of ℓ with r parts, we have

$$NC_{\mathscr{P}}(x,u) = \sum_{d\geq 1} PNC_{\mathscr{P}}(x^d,u^d).$$

Then $PNC_{\mathscr{P}}(x,u)$ can be implicitly derived using Möebius inversion as

$$PNC_{\mathscr{P}}(x,u) = \sum_{d\geq 1} \mu(d) NC_{\mathscr{P}}(x^d,u^d),$$

where $\mu(d)$ is the Möebius μ function.

Now let

$$PNCC_{\mathscr{P}}(x,u) = \sum_{\ell,r} pncc_{\mathscr{P}}(\ell,r)x^\ell u^r$$

denote the generating function for primitive n-color cyclic compositions, where the coefficient $pcc_n(\ell,r)$ is the number of primitive n-color cyclic compositions of n with r parts. Note that each composition in $\mathscr{NCC}_{\mathscr{P}}$ with r parts has r unique primitive n-color composition representations. Thus there is a one-to-r relationship between primitive n-color cyclic compositions and primitive n-color compositions. Consequently

$$pncc_{\mathscr{P}}(\ell,r,t)x^\ell u^r = \frac{pnc_{\mathscr{P}}(\ell,r)}{r} x^\ell u^r = \int_0^u pnc_{\mathscr{P}}(\ell,r)x^\ell w^{r-1}\, dw,$$

and we have

$$\begin{aligned} PNCC_{\mathscr{P}}(x,u) &= \int_0^u \frac{PNC_{\mathscr{P}}(x,w)}{w}\, dw \\ &= \int_0^u \frac{1}{w} \sum_{d\geq 1} \mu(d)\, NC_{\leq h}(x^d,w^d)\, dw \\ &= \int_0^u \sum_{d\geq 1} \mu(d)\, \frac{w^{d-1}x^d(1-x^{hd}-hx^{hd}(1-x^d))}{(1-x^d)^2 - w^d x^d(1-x^{hd}-hx^{hd}(1-x^d))}\, dw \end{aligned}$$

Integrating through substitution with

$$\beta = (1-x^d)^2 - w^d x^d (1-x^{hd} - hx^{hd}(1-x^d)),$$

and

$$\frac{d\beta}{dw} = -d\ w^{d-1}x^d(1-x^{hd}-hx^{hd}(1-x^d)),$$

and evaluating u at 1, we have

$$PNCC_{\mathscr{P}}(x) = PNCC_{\mathscr{P}}(x,u)\Big|_{u=1} = \left(\int_0^u \frac{PNC_{\mathscr{P}}(x,w)}{w}\,dw\right)\Big|_{u=1}$$
$$= \left(\sum_{d\geq 1}\frac{\mu(d)}{d}\left[-\log\left((1-x^d)^2 - w^d x^d(1-x^{hd}-hx^{hd}(1-x^d))\right)\right]\Big|_0^u\right)\Big|_{u=1}$$
$$= \left(\sum_{d\geq 1}\frac{\mu(d)}{d}\log\left(\frac{(1-x^d)^2}{(1-x^d)^2-u^d x^d(1-x^{hd}-hx^{hd}(1-x^d))}\right)\right)\Big|_{u=1}$$
$$= \sum_{d\geq 1}\frac{\mu(d)}{d}\log\left(\frac{(1-x^d)^2}{(1-x^d)^2-x^d(1-x^{hd}-hx^{hd}(1-x^d))}\right).$$

Since every cyclic n-color composition is composed of q adjacent copies of cyclic n-color primitive compositions, for some $q \in \mathbb{Z}^+$, the generating function for all cyclic compositions in $\mathscr{NCC}_{\mathscr{P}}$ is

$$NCC_{\mathscr{P}}(x) = \sum_{q\geq 1} PNCC_{\mathscr{P}}(x^q)$$
$$= \sum_{q\geq 1}\sum_{d\geq 1}\frac{\mu(d)}{d}\log\left(\frac{(1-x^{qd})^2}{(1-x^{qd})^2-x^d(1-x^{hqd}-hx^{hqd}(1-x^{qd}))}\right).$$

Using the variable substitution, $s = qd$ and given the identity $\sum_{d|s}\frac{\mu(d)}{d} = \frac{\varphi(s)}{s}$, we have that the generating function for cyclic compositions with parts no greater than h is

$$NCC_{\mathscr{P}}(x) = \sum_{s\geq 1}\sum_{d|s}\frac{\mu(d)}{d}\log\left(\frac{(1-x^{dq})^2}{(1-x^{dq})^2-x^{dq}(1-x^{qhd}-hx^{qhd}(1-x^{dq}))}\right)$$
$$= \sum_{s\geq 1}\frac{\varphi(s)}{s}\log\left(\frac{(1-x^s)^2}{(1-x^s)^2-x^s(1-x^{hs}-hx^{hs}(1-x^s))}\right).$$

Next let $\mathscr{P}$ be the conditional statement such that $\mathscr{P} \leftrightarrow (\equiv 0 \pmod{j})$. Then consider the series

$$jx^j + 2jx^{2j} + 3jx^{3j} + \ldots,$$

which generates the parts for n-color compositions with parts divisible by j. Note that this initial series differs slightly from that given earlier because the part size is unbounded as long as each part size is divisible by j. Further multiplying each term in the generating function by u to mark all the parts results in

$$\begin{aligned}
&u[(jx^j + 2jx^{2j} + 3jx^{3j} + \ldots)]\\
=&ux\left[\frac{d}{dx}\left(\frac{x^j}{1-x^j}\right)\right]\\
=&ux\left[\left(\frac{(jx^{j-1} - jx^{2j-1}) + jx^{2j-1}}{(1-x^j)^2}\right)\right]\\
=&\frac{ujx^j}{(1-x^j)^2}.
\end{aligned}$$

Consequently we have the multivariable generating function of n-color compositions of ℓ with parts divisible by j as .

$$\begin{aligned}
NC_{\mathscr{P}}(x,u) &= \sum_{k=1}^{\infty}\left(\frac{ujx^j}{(1-x^j)^2}\right)^k\\
&= \frac{\frac{ujx^j}{(1-x^j)^2}}{1-\frac{ujx^j}{(1-x^j)^2}}\\
&= \frac{ujx^j}{(1-x^j)^2 - ujx^j}
\end{aligned}$$

where the coefficient of $x^\ell u^r$ is the number of n-color compositions of ℓ with r parts.

From this point, we can see that the process for determining (5.2) would be essentially the same as the process for determining (5.1). Through the same argument, we see that the generating function for the number of compositions with part sizes divisible by j is indeed (5.2).

CHAPTER 6

CONCLUDING REMARKS

In this thesis, we first studied the total number of parts congruent to $i \mod m$ in all cyclic compositions of ℓ. This was done by following the cyclic construction of the generating function illustrated in general by [3]. From the generating function we provided justification for some interesting behaviors of the asymptotic values of this counting sequence. We noted an intriguing relation between our generating function $NCCP_{i;m}(x)$ and the generating function of the number of compositions that end with a part congruent to $i \mod m$. Then we presented combinatorial reasoning for this observation, which is interesting in its own right as it provides a direct and combinatorial way of constructing the cyclic version of the generating function $NCCP_{i;m}(x)$.

After our discussion of parts of cyclic compositions, we began our inquiries into the structure of n-color cyclic compositions. The first two generating functions we constructed were the number of n-color cyclic compositions and the number of total parts of n-color cyclic compositions. Surprisingly, the generating function $NCC(x)$ was shown to relate to two combinatorial objects, namely, the number of spanning subtrees of a wheel graph and the number of states of a dynamic storage allocation system. Certain bijections and combinatorial arguments followed these observations. We also presented the asymptotic formula for $nccp(\ell)$. The presentation of these generating functions was followed by a brief explanation of their cyclic construction using methods from [3].

We then turned our attention again to a generating function for the number of parts under modular restrictions. Only this time, the parts under consideration were from the n-color cyclic compositions of ℓ. This generating function was followed by an example of its values given in table format. The distribution of numbers within the table gave way to interesting observations which were supported by asymptotic formulas and combinatorial arguments.

Lastly, we constructed two different generating functions for the number of n-color cyclic compositions of ℓ. The only difference between these two generating functions was the specific restriction imposed on the parts of these compositions. Similarly to other generating functions previously constructed, we gave, as an example, a table of values from each generating function. From the patterns within the table, we proposed certain conjectures and developed asymptotic formulas as a result of these patterns.

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