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Homological Constructions over a Ring of Characteristic 2

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HOMOLOGICAL CONSTRUCTIONS OVER A RING OF CHARACTERISTIC 2

by

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(Under the Direction of Saeed Nasseh)

ABSTRACT

We study various homological constructions over a ring R of characteristic 2. We construct chain complexes over a field K of characteristic 2 using polynomial rings and partial derivatives. We also provide a link from the homology of these chain complexes to the simplicial homology of simplicial complexes. We end by showing how to construct all finitely-generated commutative differential graded R-algebras using polynomial rings and partial derivatives.

INDEX WORDS: Classification of differential graded algebras, Characteristic 2, Homological algebra, Chain complexes, Simplicial complexes, Gröbner bases

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DEDICATION

This thesis is dedicated to my father and my mother.

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CHAPTER 1

INTRODUCTION

In this thesis, we study various homological constructions over a ring R of characteristic 2. The reason we specialize to characteristic 2 is merely for simplicity. We believe that all of our results can be generalized to any characteristic.

In the second chapter, we introduce some preliminary material. Section 2.1 deals with the theory of Gröbner bases. The main references we used for this section are [1] and [2]. Section 2.2 deals with graded rings and modules. We used [2], [4], and [5] as references for this section. Section 2.3 deals with homological algebra. Our main reference in this section is [5], but we also used [3], [4], and [6] as well. Section 2.4 deals with simplicial complexes and simplicial homology. We use [7] as our reference here.

In the third chapter, we study some homological constructions over a field K of characteristic 2. In section 3.1, we construct some chain complexes over K using the polynomial ring $K[x_1, \ldots, x_n]$ and the differential $d := \sum_{\lambda=1}^{n} \partial_{x_\lambda}$. We then use the theory of Gröbner basis to show how $K[x_1, \ldots, x_n]/I$ can be equipped with a differential so that it becomes a chain complex over K. In section 3.2, we study differential graded K-algebras. In particular, we classify which of the chain complexes we constructed in section 3.1 can be realized as differential graded K-algebras. In section 3.3, we do some basic homology computations. In section 3.4, we give a topological interpretation of these homologies. Namely, we show how these homologies are linked to simplicial homology.

In the fourth and final chapter, we study some homological constructions over a ring R of characteristic 2. In contrast to the third chapter, which has more of a topological flavor, this chapter has more of an algebraic flavor. In this chapter, we classify all finitely-genered commutative differential graded R-algebras. We show how Koszul complexes and blowup algebras can be interpreted as differential graded R-algebras under this classification. We end this chapter with some basic homology computations.

CHAPTER 2

PRELIMINARY MATERIAL

We begin by introducing some preliminary material.

2.1 Polynomial rings over a field and Gröbner bases

Throughout this section, let K be a field and let S denote the polynomial ring $K[x_1, \ldots, x_n]$. We state all of our lemmas, propositions, and theorems without proof. All of the proofs can be found in [1] and [2].

Monomials and Polynomials in S

A **monomial** m in S is a product in S of the form

$$m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where all of the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers. Sometimes we will use the notation x^α to denote a monomial, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers. Note that $x^\alpha = 1$ when $\alpha = (0, \ldots, 0)$. If $m = x^\alpha$ is a monomial in S then the **degree** of m, denoted $\deg(m)$ or $|x^\alpha|$, is the sum $\alpha_1 + \cdots + \alpha_n$. We say that the monomial is **squarefree** if either $\alpha_\lambda = 1$ or $\alpha_\lambda = 0$ for all $\lambda = 1, \ldots, n$.

A **polynomial** f in S is a finite linear combination of monomials. We will write a polynomial f in the form

$$f = \sum_\alpha a_\alpha x^\alpha, \quad a_\alpha \in K,$$

where the sum is over a finite number of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$. We call a_α the **coefficient** of the monomial x^α. If $a_\alpha \neq 0$, then we call $a_\alpha x^\alpha$ a **term** of f, and we call x^α a **monomial** of f.

Remark 2.1. *If we replace the field K with a ring R, then the same terminology applies to the polynomial ring $R[x_1, \ldots, x_n]$.*

Monomial Orderings on S

A **monomial ordering** on S is a total ordering $>$ on $\mathbb{Z}^n_{\geq 0}$, or equivalently, a total ordering on the set of monomials x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$, satisfying

$$x^\alpha > x^\beta \implies x^\gamma x^\alpha > x^\gamma x^\beta,$$

for all $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$. We say $>$ is a **global monomial ordering** if $x^\alpha > 1$ for all $\alpha \neq 0$.

Lemma 2.1.1. *Let $>$ be a monomial ordering, then the following conditions are equivalent.*

1. *$>$ is a well-ordering, i.e. every nonempty set of monomials has a smallest element, or equivalently, every decreasing sequence*

   $$x^{\alpha(1)} > x^{\alpha(2)} > x^{\alpha(3)} > \cdots$$

   eventually terminates.

2. *$x_i > 1$ for $i = 1, \ldots, n$.*

3. *$>$ is global.*

4. *$\alpha \geq_{nat} \beta$ and $\alpha \neq \beta$ implies $x^\alpha > x^\beta$, where $\geq_{nat}$ is a partial order on $\mathbb{Z}^n_{\geq 0}$ defined by*

   $$(\alpha_1, \ldots, \alpha_n) \geq_{nat} (\beta_1, \ldots, \beta_n) \text{ if and only if } \alpha_i \geq \beta_i \text{ for all } i.$$

Remark 2.2. *Throughout this thesis, we will only be dealing with global monomial orderings. Thus, whenever we introduce a monomial ordering, we will always assume that it is a global monomial ordering.*

Examples of Monomial Orderings

We now describe some important examples of monomial orderings: Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$.

1. (Lexicographical ordering): We say $x^\alpha >_{lp} x^\beta$ if there exists $1 \leq i \leq n$ such that $\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i$.

2. (Degree reverse lexicographical ordering) We say $x^\alpha >_{dp} x^\beta$ if $|x^\alpha| > |x^\beta|$ or $|x^\alpha| = |x^\beta|$ and there exists $1 \leq i \leq n$ such that $\alpha_n = \beta_n, \ldots, \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i$.

3. (Degree lexicographical ordering) We say $x^\alpha >_{Dp} x^\beta$ if $|x^\alpha| > |x^\beta|$ or $|x^\alpha| = |x^\beta|$ and there exists $1 \leq i \leq n$ such that $\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i$.

Multidegree, Leading Coefficients, Leading Monomials, and Leading Terms

Let $f = \sum_\alpha c_\alpha x^\alpha$ be a nonzero polynomial in $K[x_1, \ldots, x_n]$ and let $>$ be a monomial ordering.

1. The **multidegree** of f is

$$\text{multdeg}(f) = \max(\alpha \in \mathbb{Z}^n_{\geq 0} \mid c_\alpha \neq 0).$$

2. The **leading coefficient** of f is

$$\text{LC}(f) = c_{\text{multdeg}(f)} \in K.$$

3. The **leading monomial** of f is

$$\text{LM}(f) = x^{\text{multdeg}(f)}.$$

4. The **leading term** of f is

$$\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f).$$

Example 2.3. *Let* $f = 4xy^2z + 4z^2 - 5x^3 + 7x^2z^2$*. With respect to lexicographical ordering we have*

$$\mathit{multdeg}(f) = (3,0,0)$$
$$\mathit{LC}(f) = -5$$
$$\mathit{LM}(f) = x^3$$
$$\mathit{LT}(f) = -5x^3.$$

With respect to degree reverse lexicographical ordering we have

$$\mathit{multdeg}(f) = (1,2,1)$$
$$\mathit{LC}(f) = 4$$
$$\mathit{LM}(f) = xy^2z$$
$$\mathit{LT}(f) = 4xy^2z.$$

Monomial Ideals

An ideal $I \subseteq S$ is a called a **monomial ideal** if it is generated by monomials. It is called a **squarefree monomial ideal** if it is generated by squarefree monomials.

Example 2.4. *An example of a monomial ideal is given by*

$$I = \langle x^4y^2, x^3y^4, x^2y^5 \rangle \subseteq K[x,y].$$

A nontrivial example of a monomial ideal is given by

$$J = \langle f_1, f_2, f_3, f_4 \rangle = \langle x^2 + x^2y^3, -x^2y^3 + y^3, x^4, y^6 \rangle.$$

It is easy to see that $J \subset \langle x^2, y^3 \rangle$*. For the reverse inclusion, note that*

$$x^2 = f_1 - x^2f_2 - y^3f_3$$
$$y^3 = y^3f_1 + f_2 - x^2f_4.$$

So $\langle x^2, y^3 \rangle \subset J$*. Therefore* $J = \langle x^2, y^3 \rangle$*.*

LEAD TERM IDEAL

Throughout the rest of section 2.1, fix a monomial ordering on S.

Lead Term Ideal

Let I be a nonzero ideal in S.

1. We denote by LT(I) the set of lead terms of nonzero elements of I. Thus,

$$\text{LT}(I) = \{cx^\alpha \mid \text{there exists } f \in I \setminus \{0\} \text{ with } \text{LT}(f) = cx^\alpha\}.$$

2. We denote by $\langle \text{LT}(I) \rangle$ the ideal generated by the elements of LT(I).

GRÖBNER BASIS

Let I be a nonzero ideal in S. The **reduced Gröbner basis** for I is a subset $G = \{g_1, \ldots, g_t\}$ of polynomials in S such that

1. $\langle \text{LT}(g_1), \ldots, \text{LT}(g_t) \rangle = \langle \text{LT}(I) \rangle$,

2. LC$(g) = 1$ for all $g \in G$,

3. For all $g \in G$, no monomial of g lies in $\langle \text{LT}(G \setminus \{g\} \rangle$.

Remark 2.5. *If a monomial ordering is fixed, then every ideal has a unique reduced Gröbner basis.*

Given a polynomial f in S, there are unique polynomials $\pi(f) \in I$ and $f^G \in S$ such that $f = \pi(f) + f^G$ and no term of f^G is divisible by any of LT$(g_1), \ldots,$ LT(g_t). We call f^G the **normal form of f with respect to** G. It follows from uniqueness of f^G and $\pi(f)$ that taking the normal form of a polynomial is a K-linear map $-^G : S \to S$:

$$c_1 f_1^G + c_2 f_2^G = (c_1 f_1 + c_2 f_2)^G \tag{2.1}$$

for all $c_1, c_2 \in K$ and $f_1, f_2 \in S$. Another important property of $-^G$ is that it preserves homogeneity.

2.2 Graded Rings and Modules

Graded Rings

A **graded ring** R is a ring together with a direct sum decomposition

$$R = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} R_i,$$

where the R_i are abelian groups such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}_{\geq 0}$. The R_i are called **homogeneous components** of R and the elements of R_i are called **homogeneous elements** of **degree** i. If r is a homogeneous element in R, then we denote the degree of r by $\deg(r)$. When we say "Let R be a graded ring", it is understood that the homogeneous components of R are denoted R_i.

Example 2.6. *An important example of a graded ring is a ring R endowed with the* ***trivial grading****: The homogoneneous components of R being $R_0 := R$ and $R_i := 0$ for all $i > 0$. When we say "Let R be a ring", then we will assume that R is trivially graded.*

Weighted Polynomial Rings

Let R be a ring and let $w := (w_1, \ldots, w_n)$ be an n-tuple of positive integers. We define the **weighted polynomial ring** $R[x_1, \ldots, x_n]_w$ with respect to the **weight** w to be the polynomial ring $R[x_1, \ldots, x_n]$ endowed with the unique grading such that $\deg(x_\lambda) = w_\lambda$ for all $\lambda = 1, \ldots, n$. We define the **weighted degree** of a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, denoted $\deg_w(m)$, to be

$$\deg_w(m) := \sum_{\lambda=1}^{n} w_\lambda \alpha_\lambda.$$

This grading gives S_w the structure of a graded ring, where the homogeneous components are given by

$$(S_w)_i := \text{Span}_R\langle m \in S_w \mid m \text{ is monomial of weighted degree } i\rangle.$$

Example 2.7. *Let K be a field and let S_w denote the weighted polynomial ring $K[x, y, z]_{(1,2,3)}$. The first few homogeneous components of S_w start out as*

$$\begin{aligned}
(S_w)_0 &= K \\
(S_w)_1 &= Kx \\
(S_w)_2 &= Kx^2 + Ky \\
(S_w)_3 &= Kx^3 + Kxy + Kz \\
&\ \vdots
\end{aligned}$$

GRADED R-MODULES

Let R be a graded ring. An R-module M, together with a direct sum decomposition

$$M = \bigoplus_{i\in\mathbb{Z}} M_i$$

into abelian groups M_i is called a **graded R-module** if $R_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. The M_i are called **homogeneous components** of M and the elements of M_i are called **homogeneous** of **degree** i. If m is a homogeneous element in M, then we denote the degree of m as $\deg(m)$. When we say "Let M be a graded R-module", then it is understood that the the homogeneous components of M are denoted by M_i.

Example 2.8. *If M is a graded R-module, then for $j \in \mathbb{Z}$, we define the j-**th twist** or the j-**th shift** of M to be the graded R-module*

$$M(j) := \bigoplus_{i\in\mathbb{Z}} M(j)_i$$

where $M(j)_i := M_{i+j}$.

Graded R-Submodules

Let R be graded ring and let M be a graded R-module. A submodule $N \subseteq M$ is called a **graded** (or **homogeneous**) R-submodule of M if it is generated by homogeneous elements.

Example 2.9. *Let K be a field, let S_w denote the weighted polynoimal ring $K[x, y, z]_{(5,6,15)}$, and let $I = \langle y^5 - z^2, x^3 - z, x^6 - y^5 \rangle$ be an ideal S_w. Then I is a homogeneous ideal in S_w.*

Remark 2.10. *Let R be a graded ring, and let I be a homogeneous ideal in R. Then the quotient R/I has an induced structure as a graded ring, where the homogeneous component of R/I is*

$$(R/I)_i := (R_i + I)/I \cong R_i/(I \cap R_i)$$

Homomorphisms of Graded R-Modules

Let M and N be graded R-modules. An R-module homomorphism $\varphi\colon M \to N$ is called **homogeneous** (or **graded**) of degree j if $\varphi(M_i) \subset N_{i+j}$ for all $i \in \mathbb{Z}$. If φ is homogeneous of degree zero then we will simply say φ is **homogeneous**.

Example 2.11. *Let R denote the polynomial ring $K[x, y, z, t]$ with its natural grading. Then the matrix*

$$U := \begin{pmatrix} x+y+z & w^2 - x^2 & x^3 \\ 1 & x & xy + z^2 \end{pmatrix}$$

defines an R-module homomorphism $U\colon R(-1) \oplus R(-2) \oplus R(-3) \to R \oplus R(-1)$ which is graded of degree zero.

GRADED R-ALGEBRAS

Let R be a graded ring and let A be an R-algebra. We say A is a **graded R-algebra** if A is graded as a ring and $A_0 = R$.

Remark 2.12. *More general definitions of graded R-algebras can be found throughout the literature, but this definition we shall use in this thesis.*

Example 2.13. *Let Q be an ideal in R. The **blowup algebra of** Q **in** R is the R-algebra*

$$B_Q(R) := R + tQ + t^2Q^2 + t^3Q^3 + \cdots \cong R \oplus Q \oplus Q^2 \oplus Q^3 \oplus \cdots$$

where multiplication in $B_Q(A)$ is induced by the multiplication map $Q^i \times Q^j \to Q^{i+j}$.

Homomorphisms of Graded R-Algebras

Let A and A' be graded R-algebras. We say $\varphi\colon A \to A'$ is an R-algebra homomorphism if

1. φ is a homomorphism when viewed as a map of R-modules. In other words,

   $$\varphi(r_1a_1 + r_2a_2) = r_1\varphi(a_1) + r_2\varphi(a_2)$$

   for all $r_1, r_2 \in R$ and $a_1, a_2 \in A$.

2. φ preserves the algebra structure. In other words

   $$\varphi(ab) = \varphi(a)\varphi(b)$$

   for all $a, b \in A$.

Moreover, we say φ is **graded** if φ is a graded homomorphism when viewed as a map of graded R-modules.

Finitely-Generated Graded R-Algebras

A graded R-algebra A is said to be **finitely-generated** if it is finitely-generated as an R-algebra. The next proposition gives a classification of all finitely-generated commutative R-algebras.

Proposition 2.14. *Every finitely-generated commutative graded R-algebra is isomorphic to a quotient of a weighted polynomial ring.*

Proof. Let A be a finitely-generated commutative R-algebra with generators $a_1, \dots, a_n$. Then for each $\lambda = 1, \dots, n$ we have $a_\lambda \in A_{w_\lambda}$, where $w_\lambda \in \mathbb{Z}_{\geq 0}$. Let S_w denote the weighted polynomial ring $R[x_1, \dots, x_n]_{(w_1,\dots,w_n)}$ and let $\varphi\colon S_w \to A$ be the unique morphism of graded R-algebras such that $\varphi(x_\lambda) = a_\lambda$ for all $\lambda = 1, \dots, n$. Then Ker(φ) is easily checked to be a homogeneous ideal of S_w, and moreover A is isomorphic to $S_w/\text{Ker}(\varphi)$ as graded R-algebras. □

Algorithmic Computations in the K-algebra S/I using Gröbner Basis

Let I be a homogeneous ideal in the polynomial ring $S = K[x_1, \dots, x_n]$. Then S/I is a graded K-algebra, where the homogeneous component S_i is the K-vector space of all homogeneous polynomials $f \in S$ of degree i. Now fix a monomial ordering and let G be the reduced Gröbner basis of I with respect to this ordering. Define

$$S_I := \text{Span}_K(x^\alpha \mid x^\alpha \notin \langle \text{LT}(I) \rangle)$$

There is an obvious decompostion of S_I into K-vector spaces $(S_I)_i$, where

$$(S_I)_i = \text{Span}_K(x^\alpha \mid x^\alpha \notin \langle \text{LT}(I) \rangle \text{ and } \deg(x^\alpha) = i).$$

In fact, S/I and S_I are isomorphic as graded K-modules. The isomorphism is given by mapping $\overline{f} \in S/I$ to $f^G \in S_I$. Indeed, $-^G$ is a K-linear map which preserves homogeneity. This makes S/I isomorphic to S_I as graded K-modules. Using this isomorphism, we can carry multiplication from S/I over to S_I to turn S_I into a graded K-algebra: For $f_1, f_2 \in S_I$, we define multiplication as

$$f_1 \cdot f_2 = (f_1 f_2)^G. \tag{2.2}$$

Defining multilpication in this way makes S_I isomorphic to S/I as graded K-algebras. For computational purposes, it is easier to work with S_I rather than S/I.

Example 2.15. *Consider $S = K[x, y]$ and $I = \langle xy^2 + y^3, x^3 + x^2y\rangle$. Then*

$$G \ = \ \{xy^2 + \ y^3 \ , \ x^3 \ + \ x^2y\}$$

is the reduced Grobner basis with respect to graded reverse lexicographical ordering. In particular LT$(I) = \langle xy^2, x^3\rangle$. We write the first few homogeneous terms of S_I:

$$\begin{aligned}
&(S_I)_0 = K \\
&(S_I)_1 = Kx + Ky \\
&(S_I)_2 = Kx^2 + Kxy + Ky^2 \\
&(S_I)_3 = Kx^2y + Ky^3 \\
&(S_I)_4 = Ky^4 \\
&(S_I)_5 = Ky^5 \\
&\quad\ \ \vdots
\end{aligned}$$

Next, we multiply some elements together in S_I in the multiplication table below

$\cdot$	x	y	y^3
x^2y	y^4	y^4	y^6
x^2	x^2y	x^2y	y^5
x	x^2	xy	y^4

Example 2.16. *Consider $S = K[x, y]$ and $I = \langle xy + y^2, x^3\rangle$. We first use Singular to compute the reduced Grobner basis G of I with respect to graded reverse lexicographical ordering. We obtain $G = \{xy + y^2, x^3, y^4\}$. Now we write down the first few homogeneous*

components of I, S/I and S_I

$$\begin{array}{lll}
I_0 = 0 & (S/I)_0 = K & (S_I)_0 = K \\
I_1 = 0 & (S/I)_1 = K\overline{x} + K\overline{y} & (S_I)_1 = Kx + Ky \\
I_2 = Kg_1 & (S/I)_2 = K\overline{x}^2 + K\overline{y}^2 & (S_I)_2 = Kx^2 + Ky^2 \\
I_3 = Kxg_1 + Kyg_1 + Kg_2 & (S/I)_3 = K\overline{y}^3 & (S_I)_3 = Ky^3 \\
I_4 = S_4 & (S/I)_4 = 0 & (S_I)_4 = 0 \\
\vdots & \vdots & \vdots
\end{array}$$

2.3 Homological Algebra

Throughout this section, let R be a ring.

Chain Complexes over R

A **chain complex** (A, d) **over** R, or simply R-**complex** , is a sequence of R-modules A_i and morphisms $d_i \colon A_i \to A_{i-1}$

$$(A, d) := \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. The condition $d_i \circ d_{i+1} = 0$ is equivalent to the condition $\mathrm{Ker}(d_i) \supset \mathrm{Im}(d_{i+1})$. Thus we are able to define the i**th homology of the chain complex** (A, d) to be

$$H_i(A, d) := \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

Let (A, d) and (A', d') be two chain complexes. A **chain map** $\varphi : (A, d) \to (A', d')$ is a sequence of R-module homomoprhisms $\varphi_i : A_i \to A'_i$ such that $d'_i\varphi_i = \varphi_{i-1}d'_i$ for all $i \in \mathbb{Z}$. We can view a chain map visually as illustrated in the diagram below:

$$\begin{array}{ccccccccc}
(A,d) := \cdots & \longrightarrow & A_{i+1} & \xrightarrow{d_{i+1}} & A_i & \xrightarrow{d_i} & A_{i-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varphi_{i+1}} & & \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i-1}} & & \\
(A',d') := \cdots & \longrightarrow & A'_{i+1} & \xrightarrow{d'_{i+1}} & A'_i & \xrightarrow{d'_i} & A'_{i-1} & \longrightarrow & \cdots
\end{array}$$

Simplifying Notation

To simplify notation in what follows, we think of R as a trivially graded ring. If (A, d) is an R-complex, then we think of (A, d) as a graded R-module A together with a graded endomorphism $d : A \to A$ of degree -1 such that $d^2 = 0$. We think of d_i as being the restriction of d to A_i and we often refer to d as the **differential**.

An element in Ker(d) is called a **cycle** of (A, d) and an element in Im(d) is called a **boundary** of (A, d). We define the **homology** of (A, d) to be

$$H(A, d) := \mathrm{Ker}(d)/\mathrm{Im}(d)$$

Note that $H(A, d) = \bigoplus_{i\in\mathbb{Z}} H_i(A, d)$. We sometimes write $H(A)$ instead of $H(A, d)$ if the differential is understood from context.

Let (A, d) and (A', d') be chain complexes. A chain map $\varphi\colon (A, d) \to (A', d')$ can be thought of as a homogeneous homomorphism of graded R-modules such that $\varphi d = d'\varphi$.

Homotopy Equivalence

Let φ and ψ be chain maps of chain complexes (A, d) and (A', d'). We say φ is **homotopic** to ψ if there is a graded homomorphism $h\colon A \to A'$ of degree 1 such that $\varphi - \psi = d'h + hd$.

Proposition 2.17. *Let φ and ψ be chain maps of chain complexes (A, d) and (A', d'). Then φ and ψ induce the same map on homology.*

Proof. The proof is straightforward and can be found in [5]. $\square$

Exact Sequences of Chain Complexes

Let (A, d), (A', d'), and (A'', d'') be chain complexes and let $\varphi\colon (A', d') \to (A, d)$ and $\psi\colon (A, d) \to (A'', d'')$ be chain maps. Then we say that

$$0 \longrightarrow A' \xrightarrow{\varphi} A \longrightarrow A'' \longrightarrow 0$$

is a **short exact sequence** of chain complexes if the following diagram is commutative with exact rows:

$$\begin{tikzcd}
 & \vdots \arrow[d, "d'_{i+2}"] & \vdots \arrow[d, "d_{i+2}"] & \vdots \arrow[d, "d''_{i+2}"] & \\
0 \arrow[r] & A'_{i+1} \arrow[r, "\varphi_{i+1}"] \arrow[d, "d'_{i+1}"] & A_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[d, "d_{i+1}"] & A''_{i+1} \arrow[r] \arrow[d, "d''_{i+1}"] & 0 \\
0 \arrow[r] & A'_{i} \arrow[r, "\varphi_{i}"] \arrow[d, "d'_{i}"] & A_{i} \arrow[r, "\psi_{i}"] \arrow[d, "d_{i}"] & A''_{i} \arrow[r] \arrow[d, "d''_{i}"] & 0 \\
0 \arrow[r] & A'_{i-1} \arrow[r, "\varphi_{i-1}"] \arrow[d, "d'_{i-1}"] & A_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[d, "d_{i-1}"] & A''_{i-1} \arrow[r] \arrow[d, "d''_{i-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Given such a short exact sequence, we get induced maps $\varphi_i\colon H_i(A') \to H_i(A)$ and $\psi_i\colon H_i(A) \to H_i(A'')$, and **connecting homomorphisms** $\gamma_i\colon H_i(A'') \to H_{i-1}(A')$ which give rise to a long exact sequence in homology:

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{i+1}(A'') \arrow[dll, "\gamma_{i+1}"] \\
H_i(A') \arrow[r, "\varphi_i"] & H_i(A) \arrow[r, "\psi_i"] & H_i(A'') \arrow[dll, "\gamma_i"] \\
H_{i-1}(A') \arrow[r, "\varphi_{i-1}"] & H_{i-1}(A) \arrow[r, "\psi_{i-1}"] & \cdots
\end{tikzcd}$$

DIFFERENTIAL GRADED ALGEBRAS

A **differential graded** R**-algebra** is a chain complex (A, d) such that A is a graded R-algebra and such that the differential d satisfies the **Leibniz law** with respect to this algebra structure:

$$d(ab) = d(a)b + (-1)^{\deg(a)} ad(b). \tag{2.3}$$

for all $a, b \in A$. We say that the differential graded R-algebra is **commutative** if

$$ab = (-1)^{\deg(a)\deg(b)} ba$$

Homomorphisms of Differential Graded R-Algebras

Let (A, d) and (A', d') be differential graded R-algebras. We say $\varphi\colon (A, d) \to (A', d')$ is **homomorphism of differential graded** R-algebras if φ is both a chain map and an R-algebra homomorphism.

Differential Graded A-Modules

Let (A, d) be a differential graded R-algebra. A **differential graded** A**-module** (M, d) is a chain complex (M, d) such that M is an A-module and such that the differential d satisfies the **Leibniz law** with respect to the algebra structure in A:

$$d(am) = d(a)m + (-1)^{\deg(a)} ad(m). \tag{2.4}$$

for all $a \in A$ and $m \in M$.

Obtaining a Differential Graded A-Module from a Chain Complex

Let (A, d_A) be a differential graded R-algebra and let (B, d_B) be a chain complex. Then $A \otimes_R B$ is an A-module and a graded R-module whose homogeneous component in degree

k is

$$(A \otimes_R B)_k := \bigoplus_{i+j=k} A_i \otimes_R B_j.$$

We define a differential d on $A \otimes_R B$ by first definining it on the elementary tensors as

$$d(a \otimes b) := d_A(a) \otimes b + (-1)^{\deg(a)} a \otimes d_B(b),$$

for all $a \in A$ and $b \in B$, and then extending it R-linearly everywhere else.

A straightforward calculation shows that $d^2 = 0$ and that the differential satisfies Leibniz law (2.4). Moreover, if B is a differential graded R-algebra, then $A \otimes_R B$ can realized as a differential graded A-algebra and a differential graded B-algebra. Multiplication in $A \otimes_R B$ is defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'.$$

for all $a, a' \in A$ and $b, b' \in B$.

Remark 2.18. *In particular, if M is an R-module endowed with the trivial grading, then $(A \otimes_R M, d)$ is a differential graded A-module where the homogeneous componenet of degree k in $A \otimes_R M$ is $(A \otimes_R M)_k := A_k \otimes_R M$, and d acts on elementary tensors as $d(a \otimes m) = d(a) \otimes m$.*

Exterior Algebras and Koszul Complexes

Exterior Algebras

Let M an R-module and let $k \geq 2$. The kth **exterior power** of M, denoted $\Lambda^k(M)$, is the R-module $M^{\otimes k}/J_k$ where J_k is the submodule of $M^{\otimes k}$ spanned by all $m_1 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for $i \neq j$. For any $m_1, \ldots, m_k \in M$, the coset of $m_1 \otimes \cdots \otimes m_k$ in $\Lambda^k(M)$ is denoted $m_1 \wedge \cdots \wedge m_k$. For completeness, we set $\Lambda^0(M) = R$ and $\Lambda^1(M) = M$. A general element in $\Lambda^k(M)$ will be denoted as ω or η.

Since $M^{\otimes k}$ is spanned by tensors $m_1 \otimes \cdots \otimes m_k$, the quotient module $M^{\otimes k}/J_k = \Lambda^k(M)$ is spanned by their images $m_1 \wedge \cdots \wedge m_k$. That is, any $\omega \in \Lambda^k(M)$ is a finite R-linear combination

$$\omega = \sum r_{i_1,\dots,i_k} m_{i_1} \wedge \cdots \wedge m_{i_k},$$

where there coefficients $r_{i_1,\dots,i_k}$ are in R and the m_i's are in M. We call $m_1 \wedge \cdots \wedge m_k$ an **elementary wedge product**. We define the **exterior algebra** of M to be

$$\Lambda(M) := \bigoplus_{k \geq 0} \Lambda^k(M),$$

where the multiplication rule given by the wedge product. The exterior algebra of M is a graded R-algebra, where the degree k homogeneous component is $\Lambda^k(M)$. If R does not have characteristic 2, then the exterior algebra of M is **skew commutative**. This means that if ω_1 and ω_2 are homogeneous elements, then

$$\omega_1 \wedge \omega_2 = (-1)^{\deg(\omega_1)\deg(\omega_2)} \omega_2 \wedge \omega_1.$$

The construction of $\Lambda(M)$ is functioral in M. This means that if N is another R-module and $\varphi\colon M \to N$ is an R-module homomorphism. Then φ induces a graded R-algebra homomorphism $\wedge\varphi\colon \Lambda(M) \to \Lambda(N)$, where $\wedge\varphi$ takes the elementary wedge product $m_1 \wedge \cdots \wedge m_k$ in $\Lambda(M)$ and maps it to the wedge product $\varphi(m_1) \wedge \cdots \wedge \varphi(m_k)$ in $\Lambda(N)$. We will write $\wedge^k\varphi$ to be the induced R-module homomorphism from $\Lambda^k(M)$ to $\Lambda^k(N)$. In particular, if N is free of rank n, then $\Lambda^n(N) \cong R$, and if $\varphi\colon N \to N$ is an R-module homomorphism, then $\wedge^n\varphi$ is multiplication by the determinant of any matrix representing φ.

Koszul Complexes

Let R be a ring, M an R-module, and $\varphi\colon M \to R$ an R-module homomorphism. The assignment

$$(m_1, \dots, m_k) \mapsto \sum_{i=1}^{k} (-1)^{i+1} \varphi(m_i) m_1 \wedge \cdots \wedge \widehat{m}_i \wedge \cdots \wedge m_k$$

defines an alternating n-linear map $M^k \to \Lambda^{k-1}(M)$. By the universal property of the kth exterior power, there exists an R-linear map $d_\varphi^{(k)} : \Lambda^k(M) \to \Lambda^{k-1}(M)$ with

$$d_\varphi^{(k)}(m_1 \wedge \cdots \wedge m_k) = \sum_{i=1}^{n} (-1)^{i+1} \varphi(m_i) m_1 \wedge \cdots \wedge \widehat{m}_i \wedge \cdots \wedge m_k$$

for all $m_1, \dots, m_k \in L$. The collection of the maps $d_\varphi^{(k)}$ defines a graded R-homomorphism

$$d_\varphi\colon \Lambda(M) \to \Lambda(M)$$

of degree -1. A straightforward calculation shows that d_φ gives $\Lambda(M)$ the structure of a differential graded R-algebra. This differential graded R-algebra is called the **Koszul complex** of φ and is denoted $\mathcal{K}_\bullet(\varphi)$. The **dual Koszul complex** of φ, denoted $\mathcal{K}^\bullet(\varphi)$, is the chain complex over R whose underlying graded R-module is $\mathrm{Hom}_R(\mathcal{K}_\bullet(\varphi), R)$ and whose differential is $d^\star$, where $d^\star$ is obtained by applying the functor $\mathrm{Hom}_R(-, R)$ to d.

Example 2.19. *Suppose R has characteristic 2. Let $\varphi\colon S_1 := \bigoplus_{\lambda=1}^{n} Rx_\lambda \to R$ be the unique R-linear map such that $\varphi(x_\lambda) = r_\lambda \in R$ for all $\lambda = 1, \dots, n$. Then $\Lambda(S_1)$ is isomorphic to $S/\langle x_1^2, \dots, x_n^2 \rangle$ as graded R-algebras. Using this isomorphism, we give $S/\langle x_1^2, \dots, x_n^2 \rangle$ the structure of a differential graded R-algebra by carrying over the differential d_φ for $\Lambda(S_1)$ to the differential d for $S/\langle x_1^2, \dots, x_n^2 \rangle$, where $d = \sum_{\lambda=1}^{n} r_\lambda \partial_{x_\lambda}$. We denote this Koszul complex as $\mathcal{K}(r_1, \dots, r_n)$.*

2.4 Simplicial Complexes

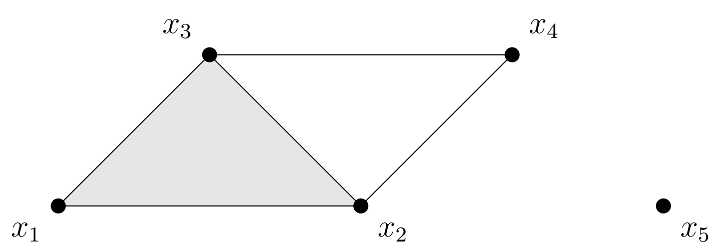
A **simplicial complex** Δ on the set $\{x_1, \dots, x_n\}$ is a collection of subsets of $\{x_1, \dots, x_n\}$ such that

1. The simplicial complex Δ contains all singletons: $\{x_\lambda\} \in \Delta$ for all $\lambda = 1, \ldots, n$.

2. The simplicial complex Δ is closed under containment: if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$.

An element of a simplicial complex is called a **face** or **simplex**, and a simplex of Δ not properly contained in another simplex of Δ is called a **facet**. A simplex $\sigma \in \Delta$ of cardinality $i+1$ is called an i-dimensional face or an i-face of Δ. The empty set $\emptyset$, is the unique face of dimension -1, as long as Δ is not the **void complex** $\{\}$ consisting of no subsets of $\{1, \ldots, n\}$. The **dimension** of Δ, denoted $\dim(\Delta)$, is defined to be the maximum of the dimensions of its faces (or $-\infty$ if $\Delta = \{\}$).

Example 2.20. *The simplicial complex Δ on $\{x_1, x_2, x_3, x_4, x_5\}$ consisting of all subsets of $\{x_1, x_2, x_3\}$, $\{x_2, x_4\}$, $\{x_3, x_4\}$, and $\{x_4\}$ is pictured below*



Simplicial Homology

Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$ and let K be a field of characteristic 2. For $i \in \mathbb{Z}$, let

$$S_i(\Delta) := \operatorname{Span}_K(\sigma \in \Delta \mid \dim(\sigma) = i) \quad \text{and} \quad S(\Delta) := \bigoplus_{i \in \mathbb{Z}} S_i(\Delta).$$

Then $S(\Delta)$ is a graded K-module. Let $\partial\colon S(\Delta) \to S(\Delta)$ be the unique graded endomorphism of degree -1 such that

$$\partial(\sigma) = \sum_{\lambda \in \sigma} \sigma \backslash \{\lambda\}.$$

for all $\sigma \in \Delta$. By a direct calculation, we have $\partial^2 = 0$, and so $(S(\Delta), \partial)$ forms a K-complex; it is called the **reduced chain complex of Δ over K**. The ith homology of $(S(\Delta), \partial)$ is called the **ith reduced homology** of Δ over K, and is denoted $\widetilde{H}_i(\Delta, K)$.

Example 2.21. *For Δ as in Example (2.20), we have*

$$\begin{aligned}
S_2(\Delta) &= \{\{x_1, x_2, x_3\}\} \\
S_1(\Delta) &= \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\} \\
S_0(\Delta) &= \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\} \\
S_{-1}(\Delta) &= \{\emptyset\}
\end{aligned}$$

Choosing bases for the $S_i(\Delta)$ as suggested by the ordering of the faces listed above, the chain complex for Δ becomes

$$0 \longrightarrow K \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} K^5 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}} K^5 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}} K \longrightarrow 0$$

For example, $\partial_2(e_{\{1,2,3\}}) = e_{\{2,3\}} + e_{\{1,3\}} + e_{\{1,2\}}$, which we identify with the vector $(1, 1, 1, 0, 0)$. The mapping ∂_1 has rank 3, so $\widetilde{H}_0(\Delta; K) \cong \widetilde{H}_1(\Delta; K) \cong K$ and the other homology groups are 0. Intuitively, $\widetilde{H}_0(\Delta; K) \cong K$ corresponds to the fact that Δ consists of two connected components and $\widetilde{H}_1(\Delta; K) \cong K$ corresponds to the fact that Δ contains a triangle which is not the boundary of an element of Δ, i.e. it contains a ”hole”.

CHAPTER 3

HOMOLOGICAL CONSTRUCTIONS OVER A FIELD OF CHARACTERISTIC 2

Throughout this chapter, let K be a field of characteristic 2, S denote the polynomial ring $K[x_1, \ldots, x_n]$, I be a homogeneous ideal in S, and let $G = \{g_1, \ldots, g_r\}$ be the reduced Gröbner basis for I with respect to a fixed monomial ordering.

3.1 Constructing K-Complexes

Let $d\colon S \to S$ be the graded K-linear map of degree -1 given by $d := \sum_{k=1}^{n} \partial_{x_k}$. Since K has characteristic 2, we have $d^2 = 0$. Indeed, it suffices to show that $d^2(m) = 0$ for all monomials m in S. So let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S. Then

$$\begin{aligned} d^2(m) &= \left(\sum_{k=1}^{n} \partial_{x_k} \right)^2 (x_1^{\alpha_1} \cdots x_n^{\alpha_n}) \\ &= \left(\sum_{k=1}^{n} \partial_{x_k}^2 \right) (x_1^{\alpha_1} \cdots x_n^{\alpha_n}) \\ &= \sum_{k=1}^{\infty} \alpha_k(\alpha_k - 1) x_k^{\alpha_k - 2} \\ &= 0. \end{aligned}$$

Thus the differential d gives the graded K-module S the structure of a K-complex.

Construction of (S_I, d)

Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S. We denote

$$[m]_o = \{1 \leq \lambda \leq n \mid \alpha_\lambda \text{ is odd}\} \quad \text{and} \quad [m]_e = \{1 \leq \mu \leq n \mid \alpha_\mu \text{ is even}\}$$

Using this notation, we can express the differential in another way:

$$d(m) = \sum_{\lambda \in [m]_o} x_\lambda^{-1} m.$$

This makes it clear that the differential maps S_I into S_I. Indeed, if m is not in LT(I), then every term $x_\lambda^{-1}m$ of $d(m)$ is not in LT(I) either. Thus the differential d gives the graded K-module S_I the structure of a chain complex over K.

CONSTRUCTION OF $(S/I, \overline{d})$

Definition 3.1. *We say I is d-**stable** if d maps I into I.*

Suppose I is d-stable. Then the differential $d\colon S \to S$ induces a graded linear map of degree -1, denoted $\overline{d}\colon S/I \to S/I$, where

$$\overline{d}(\overline{f}) = \overline{d(f)} \text{ for all } f \in S.$$

Indeed, the map $\overline{d}$ is well-defined since d is I-stable. Moreover, the differential $\overline{d}$ gives S/I the structure of a differential graded K-algebra since it inherits all of the necessary properties from d. For instance, to see that $\overline{d}$ satisfies Leibniz law, let $\overline{f}_1$ and $\overline{f}_2$ be in S/I. Then

$$\begin{aligned}
\overline{d}(\overline{f_1 f_2}) &= \overline{d(f_1 f_2)} \\
&= \overline{d(f_1)f_2 + f_1 d(f_2)} \\
&= \overline{d(f_1)f_2} + \overline{f_1 d(f_2)} \\
&= \overline{d}(\overline{f_1})\overline{f_2} + \overline{f_1}\overline{d}(\overline{f_2}).
\end{aligned}$$

Thus if I is d-stable, then the differential $\overline{d}$ gives the graded K-algebra S/I the structure of a differential graded K-algebra.

CONSTRUCTION OF $(I, \underline{d})$

Our final construction involves the graded K-module I. Let $\underline{d}\colon I \to I$ be the graded K-linear map of degree -1 given by

$$\underline{d}(f) := \pi(d(f)) = d(f) + d(f)^G$$

for all $f \in I$. Then $\underline{d}^2 = 0$. Indeed, for all $f \in I$, we have

$$\begin{aligned}\underline{d}(\underline{d}(f)) &= \underline{d}(d(f) + d(f)^G) \\ &= d(d(f) + d(f)^G) + d(d(f) + d(f)^G)^G \\ &= d(d(f)^G) + d(d(f)^G)^G \\ &= d(d(f)^G) + d(d(f)^G) \\ &= 0,\end{aligned}$$

where $d(d(f)^G)^G = d(d(f)^G)$ since every term in $d(d(f)^G)$ is not in I. Thus the differential $\underline{d}$ gives the graded K-module I the structure of a chain complex over K.

3.2 Differential Graded K-Algebras

Since d is defined in terms of partial derivatives, it is clear that d satisfies Leibniz law. Thus (S, d) is more than just a chain complex over K; it is a differential graded K-algebra. Since S_I is a graded K-algebra, it is natural wonder if (S_I, d) is also a differential graded K-algebra. A quick counterexample shows that this is not necessarily the case:

Example 3.2. *Consider $S = K[x]$ and $I = \langle x^5 \rangle$. Then*

$$\begin{aligned}d(x \cdot x^4) &= d((x^5)^G) \\ &= d(0) \\ &= 0,\end{aligned}$$

but

$$\begin{aligned}d(x) \cdot x^4 + x \cdot d(x^4) &= 1 \cdot x^4 + x \cdot 0 \\ &= (x^4)^G + 0^G \\ &= x^4,\end{aligned}$$

so $d(x \cdot x^5) \neq d(x) \cdot x^4 + x \cdot d(x^4)$.

Thus, in order for (S_I, d) to be a differential graded K-algebra, we need a condition on I to be satisfied. This is given in the following theorem.

Theorem 3.3. *(S_I, d) is a differential graded K-algebra if and only if $d(g) = 0$ for all $g \in G$.*

Proof. Assume that $d(g) = 0$ for all $g \in G$. We first prove that $d(f^G) = d(f)^G$ for all $f \in S$. Let $f \in S$. From the division algorithm, we have $f = g_1q_1 + \cdots + g_rq_r + f^G$ for some $q_1, \ldots, q_r \in S$. Thus

$$\begin{aligned} d(f) &= d(g_1q_1 + \cdots + g_rq_r + f^G) \\ &= d(g_1q_1) + \cdots + d(g_rq_r) + d(f^G) \\ &= g_1d(q_1) + \cdots + g_rd(q_r) + d(f^G). \end{aligned}$$

Since $g_1d(h_1) + \cdots + g_rd(h_r) \in I$ and no term of $d(f^G)$ is divisible by any element of LT(I), it follows from uniqueness of normal forms that $d(f^G) = d(f)^G$.

Now we show that this implies that (S_I, d) is a differential graded K-algebra. Let $f_1, f_2 \in S_I$. Then

$$\begin{aligned} d(f_1 \cdot f_2) &= d((f_1f_2)^G) \\ &= (d(f_1f_2))^G \\ &= (d(f_1)f_2 + f_1d(f_2))^G \\ &= (d(f_1)f_2)^G + (f_1d(f_2))^G \\ &= d(f_1) \cdot f_2 + f_1 \cdot d(f_2). \end{aligned}$$

Therefore (S_I, d) is a differential graded K-algebra.

Now we prove the converse. Assume (S_I, d) is a differential graded K-algebra. Let $g \in G$ and let m be the lead term of g. We may assume g is not a constant (otherwise we'd clearly have $d(g) = 0$). Thus, there exists some x_λ such that x_λ divides m. Then on the one hand, we have

$$\begin{aligned} d(x_\lambda \cdot x_\lambda^{-1} m) &= d(m^G) \\ &= d(g+m) \\ &= d(g) + d(m), \end{aligned}$$

since $m^G = g + m$. On the other hand, we have

$$\begin{aligned} d(x_\lambda) \cdot x_\lambda^{-1} m + x_\lambda \cdot d(x_\lambda^{-1} m) &= (x_\lambda^{-1} m)^G + (x_\lambda d(x_\lambda^{-1} m))^G \\ &= x_\lambda^{-1} m + (x_\lambda d(x_\lambda^{-1} m))^G \\ &= x_\lambda^{-1} m + (x_\lambda (x_\lambda^{-2} m + x_\lambda^{-1} d(m)))^G \\ &= x_\lambda^{-1} m + (x_\lambda^{-1} m + d(m))^G \\ &= x_\lambda^{-1} m + (x_\lambda^{-1} m)^G + d(m)^G \\ &= x_\lambda^{-1} m + x_\lambda^{-1} m + d(m)^G \\ &= d(m), \end{aligned}$$

since $(x_\lambda^{-1} m)^G = x_\lambda^{-1} m$ and $d(m)^G = d(m)$ (every term of $d(m)$ does not lie in $\langle \mathrm{LT}(G) \rangle$). Since (S_I, d) is a differential graded K-algebra, we must have $d(g) = 0$. This establishes this theorem. $\square$

Remark 3.4. *We should note that the identity*

$$x_\lambda d(x_\lambda^{-1} m) = x_\lambda (x_\lambda^{-2} m + x_\lambda^{-1} d(m)) = x_\lambda^{-1} m + d(m)$$

follows since d satisfies Leibniz law not just in S, but also in $S[x_1^{-1}, \ldots, x_n^{-1}]$. Again, this is because d is defined in terms of partial derivatives.

Example 3.5. *Going back to Example (2.15), where $S = K[x, y]$, $I = \langle xy^2 + y^3, x^3 + x^2 y \rangle$, and $G = \{xy^2 + y^3, x^3 + x^2 y\}$. We have $d(xy^2 + y^3) = d(x^3 + x^2 y) = 0$. Therefore Theorem (3.3) implies (S_I, d) is a differential graded K-algebra.*

Now we want to show that (S_I, d) is a differential graded K-algebra if and only if $(S/I, \overline{d})$ is a differential graded K-algebra, and moreover, they are isomorphic to each other. We start with the following lemma.

Lemma 3.2.1. *For all $g \in G = \{g_1, \ldots, g_r\}$, we have $d(g) = d(g)^G$.*

Proof. We prove this in the case $g = g_1$. The other cases can be proved using a similar argument. If $d(g_1) = 0$, then clearly we have $d(g_1) = d(g_1)^G$, so assume $d(g_1) \neq 0$. Since G is a Gröbner basis, $d(g_1) = d(g_1)^G$ if and only if no term of $d(g_1)$ belongs to $\langle \mathrm{LT}(G) \rangle$. Every term in $d(g_1)$ has the form $x_\lambda^{-1} m$ where m is some term of g. This term cannot belong to $\langle \mathrm{LT}(G) \rangle$, since if $x_\lambda^{-1} m \in \langle \mathrm{LT}(G) \rangle$, then $m \in \langle \mathrm{LT}(g_2), \ldots, \mathrm{LT}(g_r) \rangle$, and this contradicts the fact that G is a *reduced* Gröbner basis. □

Lemma 3.2.2. *I is d-stable if and only if $d(g) \in I$ for all $g \in G$.*

Proof. One direction is trivial, so we prove the other direction. Suppose $d(g) \in I$ for all $g \in G$ and let $f \in I$. Since G generates I, we can write $f = \sum_{\lambda=1}^{r} q_\lambda g_\lambda$ for some $q_1, \ldots, q_r \in S$. Thus, by Leibniz law, we have

$$\begin{aligned} d(f) &= d\left(\sum_{\lambda=1}^{r} q_\lambda g_\lambda\right) \\ &= \sum_{\lambda=1}^{r} d(q_\lambda g_\lambda) \\ &= \sum_{\lambda=1}^{r} (d(q_\lambda) g_\lambda + q_\lambda d(g_\lambda)) \in I. \end{aligned}$$

Thus, I is d-stable. □

Remark 3.6. *The same proof shows that if F is a generating set of I such that $d(f) \in I$ for all $f \in F$, then I is d-stable.*

Combining Lemma (3.2.1) and Proposition (3.2.2), we find that that $d(g) = 0$ for all $g \in G$ if and only if $d(g) \in I$ for all $g \in G$ if and only if I is d-stable. Combining this with

Theorem (3.3), we find that (S_I, d) is a differential graded K-algebra if and only if $(S/I, \overline{d})$ is a differential graded K-algebra. Now we will show that they are in fact isomorphic to each other.

Theorem 3.7. *Suppose I is d-stable. Then (S_I, d) is isomorphic to $(S/I, \overline{d})$ as differential graded K-algebras.*

Proof. Recall that S/I is isomorphic to S_I as graded K-algebras, where the isomorphism is given by mapping $\overline{f} \in S/I$ to $f^G \in S_I$. It remains to show that this isomorphism respects the differential graded algebra structure. In particular, we need to show that $d(f^G) = d(f)^G$ for all $f \in S$. This was already proven in Theorem (3.3). $\square$

MORE DIFFERENTIAL GRADED K-ALGEBRAS

Proposition 3.8. *Suppose I is d-stable and let g be a homogeneous polynomial such that $d(g) = 0$. Then $(S_{\langle I,g\rangle}, d)$ and $(S_{I:g}, d)$ are differential graded K-algebras.*

Proof. We just need to show that $\langle I, g\rangle$ and $I : g$ are both d-stable. Since $d(g) = 0$, it follows that $\langle I, g\rangle$ is d-stable. To prove that $I : g$ is d-stable, let $f \in I : g$. Then since $fg \in I$, $d(g) = 0$, and I is d-stable, it follows that

$$d(f)g = d(f)g + fd(g) = d(fg) \in I$$

Therefore $d(f) \in I : g$, which implies that $I : g$ is d-stable. $\square$

Example 3.9. *Consider $S = K[x, y, z]$, $g = x^2y + x^2z$, and $I = \langle f_1, f_2, f_3\rangle$ where*

$$f_1 = xy + xz + yz$$

$$f_2 = x^4y + x^5$$

$$f_3 = y^3 + y^2z$$

Then $d(f_1) = d(f_2) = d(f_3) = 0 \in I$ implies (S_I, d) is a differential graded K-algebra. The reduced Gröbner basis for I with respect to graded lexicographical ordering is $G = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, where

$$\begin{aligned}
g_1 &= xy + xz + yz \\
g_2 &= y^3 + y^2 z \\
g_3 &= y^2 z^2 \\
g_4 &= xz^4 + yz^4 \\
g_5 &= x^5 + x^4 z + x^3 z^2 + x^2 z^3 \\
g_6 &= x^4 z^2.
\end{aligned}$$

Since $d(g) = 0$, we know that $(S_{\langle I,g\rangle}, d)$ and $(S_{I:g}, d)$ are also differential graded K-algebras. The reduced Gröbner basis for $I : g$ with respect to graded lexicographical ordering is $G'' = \{g_1'', g_2'', g_3''\}$, where

$$\begin{aligned}
g_1'' &= y + z \\
g_2'' &= z^2 \\
g_3'' &= x^3 + x^2 z
\end{aligned}$$

and the reduced Gröbner basis for $\langle I, g\rangle$ with respect to graded lexicographical ordering is $G' = \{g_1', g_2', g_3', g_4', g_5'\}$, where

$$\begin{aligned}
g_1' &= xy + xz + yz \\
g_2' &= y^3 + y^2 z \\
g_3' &= xz^2 + yz^2 \\
g_4' &= y^2 z^2 \\
g_5' &= x^5 + x^4 z + x^3 z^2 + x^2 z^3
\end{aligned}$$

3.3 Homology Computations

In this section we record some basic homology computations.

Proposition 3.10. *Suppose I is d-stable. Then $H(S_I) = 0$.*

Proof. Let f be a homogeneous polynomial in S_I such that $d(f) = 0$. Then for any $x_\lambda \in (S_I)_1$, we have

$$d(x_\lambda f) = d(x_\lambda)f + x_\lambda d(f) = f.$$

Therefore $\text{Ker}(d) = \text{Im}(d)$, hence $H(S_I) = 0$. □

Proposition 3.11. *The differential d induces isomorphisms $H_i(I) \cong H_{i-1}(S_I)$ for all $i > 0$.*

Proof. For all $f \in S$, we have

$$\begin{aligned}
\underline{d}(\pi(f)) &= \underline{d}(f + f^G) \\
&= d(f + f^G) + d(f + f^G)^G \\
&= d(f) + d(f^G) + d(f)^G + d(f^G)^G \\
&= d(f) + d(f^G) + d(f)^G + d(f^G) \\
&= d(f) + d(f)^G \\
&= \pi(d(f)),
\end{aligned}$$

where $d(f^G)^G = d(f^G)$ because no term in $d(f^G)$ lies in $\text{LT}(I)$. Therefore we have a short exact sequence of chain complexes over K:

$$0 \longrightarrow (S_I, d) \hookrightarrow (S, d) \xrightarrow{\pi} (I, \underline{d}) \longrightarrow 0,$$

which induces, for each $i > 0$, the following short exact sequences:

$$0 = H_i(S) \longrightarrow H_i(I) \xrightarrow{d} H_{i-1}(S_I) \longrightarrow H_{i-1}(S) = 0.$$

where d is obtained from the connecting map. In more detail, d maps the element $[f] \in H_i(I)$ to the element $[d(f)] \in H_{i-1}(S_I)$. □

Decomposing $H_i(S_I)$

Let g be a homogeneous polynomial of degree j and let G' be the reduced Gröbner basis for $\langle I, g\rangle$ with respect to our fixed monomial ordering. In Commutative Algebra, we learn about the following short exact sequence of graded S-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (S/(I:g))(-j) & \xrightarrow{\;g\;} & S/I & \longrightarrow & S/\langle I,g\rangle & \longrightarrow & 0. \\ & & \overline{f} & \longmapsto & \overline{fg} & & & & \end{array}$$

We want to use this short exact sequence to our advantage. First, using the isomorphisms $S_{I:g} \cong S/(I:g)$, $S_I \cong S/I$, and $S_{\langle I,g\rangle} \cong S/\langle I, g\rangle$, we get, for each i, a short exact sequence of K-vector spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (S_{I:g})_{j-i} & \xrightarrow{\;\cdot g\;} & (S_I)_i & \xrightarrow{\;\_^{G'}\;} & (S_{\langle I,g\rangle})_i & \longrightarrow & 0, \\ & & f & \longmapsto & (fg)^G & & & & \\ & & & & f & \longmapsto & f^{G'} & & \end{array}$$

or in other words, a short exact sequence of graded K-vector spaces

$$0 \longrightarrow (S_{I:g})(-j) \xrightarrow{\;\cdot g\;} S_I \xrightarrow{\;\_^{G'}\;} S_{\langle I,g\rangle} \longrightarrow 0.$$

We want to know under what conditions this becomes a short exact sequence of chain complexes over K, that is, when does the following diagram commute?

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & (S_{I:g})_{j-i} \arrow[r, "\cdot g"] \arrow[d, "d"'] & (S_I)_i \arrow[r, "_^{G'}"] \arrow[d, "d"'] & (S_{\langle I,g\rangle})_i \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & (S_{I:g})_{j-i-1} \arrow[r, "\cdot g"] \arrow[d] & (S_I)_{i-1} \arrow[r, "_^{G'}"] \arrow[d] & (S_{\langle I,g\rangle})_{i-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The conditions which need to be satisfied are the following:

$$(gd(m))^G = d((gm)^G) \text{ for all monomials } m \text{ which are not in } \mathrm{LT}(I:g) \tag{3.1}$$

$$d(m)^{G'} = d(m^{G'}) \text{ for all monomials } m \text{ which are not in } \mathrm{LT}(I) \tag{3.2}$$

For the moment, let's assume that these conditions are satisfied so that we have a short exact sequence of chain complexes. Then by the usual argument, the short exact sequence of chain complexes gives rise to a long exact sequence in homology:

$$\begin{tikzcd}
& \cdots \arrow[r] & H_{i+1}(S_{\langle I,g\rangle}) \arrow[dll, rounded corners, "\lambda"] \\
H_{i-j}(S_{I:g}) \arrow[r, "\cdot g"] & H_i(S_I) \arrow[r, "-^{G'}"] & H_i(S_{\langle I,g\rangle}) \arrow[dll, rounded corners, "\lambda"] \\
H_{i-j-1}(S_{I:g}) \arrow[r, "\cdot g"] & H_{i-1}(S_I) \arrow[r, "-^{G'}"] & \cdots
\end{tikzcd}$$

It is easy to see that the connecting maps λ all induce the zero map. Thus, we get for each i, the short exact sequence of K-vector spaces:

$$0 \longrightarrow H_{i-j}(S_{I:g}) \xrightarrow{\cdot g} H_i(S_I) \xrightarrow{-^{G'}} H_i(S_{\langle I,g\rangle}) \longrightarrow 0,$$

and since the inclusion map $S_{\langle I,g\rangle} \hookrightarrow S_I$ splits the map $-^{G'}$, we obtain the following isomorphism

$$H_{i-j}(S_{I:g}) \oplus H_i(S_{\langle I,g\rangle}) \cong H_i(S_I) \tag{3.3}$$

where we map the representative (f_1, f_2) in $H_{i-j}(S_{I:g}) \oplus H_i(S_{\langle I,g\rangle})$ to the representative $gf_1 + f_2$ in $H_i(S_I)$. We summarize our findings in the form of a theorem.

Theorem 3.12. *Let I be a homogeneous ideal and let g be a homogeneous polynomial of degree j. Let G be the reduced Gröbner basis for I and G' be the reduced Gröbner basis for $\langle I, g\rangle$ with respect to our fixed monomial ordering. Suppose that the following conditions are satisfied:*

1. $(gd(m))^G = d((gm)^G)$ *for all monomials* m *which are not in LT*$(I : g)$.

2. $d(m)^{G'} = d(m^{G'})$ *for all monomials* m *which are not in LT*(I).

Then we have an isomorphism

$$H_{i-j}(S_{I:g}) \oplus H_i(S_{\langle I,g\rangle}) \cong H_i(S_I)$$

given by mapping the representative (f_1, f_2) *in* $H_{i-j}(S_{I:g}) \oplus H_i(S_{\langle I,g\rangle})$ *to the representative* $gf_1 + f_2$ *in* $H_i(S_I)$.

Decomposing $H_i(S_I)$ in a Special Case

We will now discuss a special case of when the conditions in Theorem (3.12) are satisfied. Consider the case where I is a monomial ideal and g is a monomial of degree j which is not in I. Then condition (3.1) is satisfied since if m is not in $I : g$, then gm is not in I, and so $(gm)^G = gm$ which implies $(gd(m))^G = gd(m)$.

For condition (3.2) first assume that m is not in $\langle I, g\rangle$. Then then $m^{G'} = m$, which implies $d(m)^{G'} = d(m) = d(m^{G'})$. Thus condition (3.2) is satisfied in this case. Now assume that $m = g$. Then $m^{G'} = 0$, which implies $d(m^{G'}) = 0$. Thus, we must have $d(g) = 0$ in order for condition (3.2) to be satisfied in this case. So assume $d(g) = 0$ and consider the final case where $m = m_1 g$. Since $d(g) = 0$, we obtain $d(m)^{G'} = (d(m_1)g)^{G'} = 0$, and thus (3.2) is satisfied in this case as well.

In the next example, we show how we can apply Theorem (3.12) recursively. In what follows, we frequently use the notation I, g to mean $I + \langle g\rangle$ and $I : g$ to mean $I : \langle g\rangle$. For example, $I, g_1 : g_2 = \langle I, g_1\rangle : \langle g_2\rangle$, and $I : g_1, g_2 = \langle (I : g_1), \langle g_2\rangle\rangle$, and so on.

Example 3.13. *Consider* $S = K[x, y, z]$ *and* $I = \langle x^3y, yz^3\rangle$. *Then* $d(x^2) = d(z^2) = 0$, *and so*

$$\begin{aligned} H_i(S_I) &= x^2 H_{i-2}(S_{I:x^2}) \oplus H_i(S_{I,x^2}) \\ &= x^2(z^2 H_{i-4}(S_{I:x^2z^2}) \oplus H_{i-2}(S_{I:x^2,z^2})) \oplus z^2 H_{i-2}(S_{I,x^2:z^2}) \oplus H_i(S_{I,x^2,z^2}) \\ &= x^2 z^2 H_{i-4}(S_{I:x^2z^2}) \oplus x^2 H_{i-2}(S_{I:x^2,z^2}) \oplus z^2 H_{i-2}(S_{I,x^2:z^2}) \oplus H_i(S_{I,x^2,z^2}) \end{aligned}$$

We calculate

$$\begin{aligned} I : x^2 : z^2 &= \langle xy, yz \rangle \\ I, x^2 : z^2 &= \langle x^2, yz \rangle \\ I : x^2, z^2 &= \langle xy, z^2 \rangle \\ I, x^2, z^2 &= \langle x^2, z^2 \rangle \end{aligned}$$

The only part which has nontrivial homology is $S_{I:x^2:z^2}$. Thus, $H_5(S_I) = [d(x^3yz^2)]K$ and $H_i(S_I) = 0$ for all $i \neq 5$.

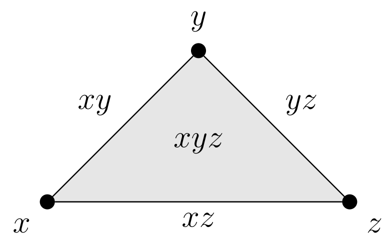
3.4 Topological Interpretation of $H(S_I)$

In this section, we will give a topological interpretation of $H(S_I)$ in the case where I is a squarefree monomial ideal. More specifically, we will show that $H(S_I)$ is isomorphic to the simplicial homology of a corresponding simplicial complex.

Reinterpreting Simplicial Complexes

We want to reinterpret the theory simplicial complexes using the language of monomials. There is a bijection between the set of subsets of $\{x_1, \ldots, x_n\}$ and the set of squarefree monomials in the variables $x_1, \ldots, x_n$. Indeed, if m is a squarefree monomial, then the corresponding subset of $\{x_1, \ldots, x_n\}$ is supp$(m) := \{x_\lambda \mid x_\lambda$ divides $m\}$. Moreover, if m and m' are squarefree monomials, then m divides m' if and only if supp$(m) \subseteq$ supp(m').

Example 3.14. *Here's how we think of squarefree monomials in x, y, z and how they sit on their corresponding faces.*



STANLEY-REISNER RINGS

Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$. We denote by I_Δ to be the ideal of nonfaces of Δ, that is, I_Δ is generated by the squarefree monomials m in S which are not in Δ. We define the **Stanley-Reisner ring** $K[\Delta]$ of the simplicial complex Δ to be the K-algebra $K[\Delta] := S/I_\Delta$. We will also denote by I_Δ^{sq} to mean $I_\Delta^{\text{sq}} := I_\Delta + \langle x_1^2, \ldots, x_n^2 \rangle$.

Conversely, if I is a squarefree monomial ideal, then we denote by Δ_I the simplicial complex on $\{x_1, \ldots, x_n\}$ whose ideal of nonfaces is I. Thus, Δ_I consists of all squarefree monomials which do not belong to I.

Lemma 3.4.1. *Suppose I is a monomial ideal and let $\mathcal{M} = \{m_1, \ldots, m_r\}$ be the unique minimal basis of I. For each $\lambda = 1, \ldots, n$, let k_λ be a nonnegative even integer such that $x_\lambda^{k_\lambda}$ does not divide any monomial in $\mathcal{M}$. Then*

$$H(S_I) \cong H(S_{I + \langle x_1^{k_1}, \ldots, x_n^{k_n} \rangle}).$$

Proof. We prove by induction on $\lambda = 1, \ldots, n$. The base case $\lambda = 1$ will follow if $H(S_{I:x_1^{k_1}}) \cong 0$, since

$$H(S_I) \cong H(S_{I:x_1^{k_1}}) \oplus H(S_{\langle I, x_1^{k_1} \rangle}).$$

by Theorem (3.12). Since $x_1^{k_1}$ does not divide any monomial in $\mathcal{M}$, a basis for $I : x_1^{k_1}$ is given by $\mathcal{M}' = \{m_1', \ldots, m_r'\}$, where if $m_\mu = x_1^{\alpha_{\mu 1}} x_2^{\alpha_{\mu 2}} \cdots x_n^{\alpha_{\mu n}}$, then $m_\mu' = x_2^{\alpha_{\mu 2}} \cdots x_n^{\alpha_{\mu n}}$

for all $\mu = 1, \ldots, r$. In particular, if $f \in S_{I:x_1^{k_1}}$ represents a cycle in $H(S_{I:x_1^{k_1}})$, then $x_1 f \in S_{I:x_1^{k_1}}$ represents a boundary of f in $H(S_{I:x_1^{k_1}})$. Thus $H(S_{I:x_1^{k_1}}) = 0$, and our claim is proved.

For the induction step, assume that

$$H(S_I) \cong H(S_{\langle I, x_1^{k_1}, \ldots, x_\lambda^{k_\lambda}\rangle})$$

for some $1 \leq \lambda < n$. By the same argument as in the base case (with $I + \langle x_1^{k_1}, \ldots, x_\lambda^{k_\lambda}\rangle$ replaced with I), we have $H(S_{(I+\langle x_1^{k_1}, \ldots, x_\lambda^{k_\lambda}\rangle):x_{\lambda+1}^{k_{\lambda+1}}}) \cong 0$. Thus

$$\begin{aligned}
H(S_I) &\cong H(S_{I+\langle x_1^{k_1}, \ldots, x_\lambda^{k_\lambda}\rangle}) \\
&\cong H(S_{(I+\langle x_1^{k_1}, \ldots, x_\lambda^{k_\lambda}\rangle):x_{\lambda+1}^{k_{\lambda+1}}}) \oplus H(S_{I+\langle x_1^{k_1}, \ldots, x_\lambda^{k_\lambda}, x_{\lambda+1}^{k_{\lambda+1}}\rangle}) \\
&\cong H(S_{I+\langle x_1^{k_1}, \ldots, x_\lambda^{k_\lambda}, x_{\lambda+1}^{k_{\lambda+1}}\rangle}).
\end{aligned}$$

This proves the induction step, and hence the lemma.

□

Theorem 3.15. *Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$. Then*

$$H_i(S_{I_\Delta}) \cong H_i(S_{I_\Delta^{sq}}) \cong \widetilde{H}_{i-1}(\Delta; K)$$

for all $i \in \mathbb{Z}$.

Proof. The first isomorphism $H_i(S_{I_\Delta}) \cong H_i(S_{I_\Delta^{\mathrm{sq}}})$ follows from Lemma (3.4.1). Indeed, I_Δ is a squarefree monomial ideal, and hence x_λ^2 does not divide any monomial in the minimal monomial basis of I_Δ for all $\lambda = 1, \ldots, n$.

Now we will show that $H_i(S_{I_\Delta^{\mathrm{sq}}}) \cong \widetilde{H}_{i-1}(\Delta; K)$. The map $\varphi : S_{I_\Delta^{\mathrm{sq}}} \to S(\Delta)$, given by $\varphi(m) = \mathrm{supp}(m)$ for all monomials $m \in S_{I_\Delta^{\mathrm{sq}}}$, is a graded isomorphism of degree -1. Moreover, it is easy to check that $\varphi d = \partial\varphi$. Thus φ induces an isomorphism $H_i(S_{I_\Delta^{\mathrm{sq}}}) \cong \widetilde{H}_{i-1}(\Delta; K)$.

□

Corollary 3.16. *Suppose I is a squarefree monomial ideal. Then*

$$H_i(S_I) \cong \widetilde{H}_{i-1}(\Delta_I; K)$$

CHAPTER 4

HOMOLOGICAL CONSTRUCTIONS OVER A RING OF CHARACTERISTIC 2

Throughout this chapter, let R be a ring of characteristic 2. The purpose of this chapter is to generalize some of our results from chapter 3. In particular, we will determine the structure of differential graded R-algebras which are finitely-generated and commutative. To avoid repetition, all differential graded R-algebras mentioned in this section are assumed to be finitely generated and commutative.

4.1 Constructing Differential Graded R-Algebras

Theorem 4.1. *Let S_w denote the weighted polynomial ring $R[x_1, \ldots, x_n]_{(w_1,\ldots,w_n)}$. Define the map*

$$d := \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda},$$

where f_λ is a nonzero homogeneous polynomial in S_w of weighted degree $w_\lambda - 1$ for all $\lambda = 1, \ldots, n$. Then

1. *d is a graded endomorphism $d\colon S_w \to S_w$ of degree -1 which satisfies Leibniz law.*

2. *Moreover, let $I \subset S_w$ be any d-stable homogeneous ideal such that $d(f_\lambda) \in I$ for all $\lambda = 1, \ldots, n$ and such that $I \subset \langle x_1, \ldots, x_n \rangle$. Then d induces a map $\overline{d}\colon S_w/I \to S_w/I$, given by $\overline{d}(\overline{f}) = \overline{d(f)}$ for all $\overline{f} \in S_w/I$, and $(S_w/I, \overline{d})$ is a differential graded R-algebra.*

Proof. We first show that d is a graded endomorphism $d\colon S_w \to S_w$ of degree -1 which satisfies Leibniz law:

- R-linearity: We have

$$\begin{aligned}
d(r_1g_1 + r_2g_2) &= \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(r_1g_1 + r_2g_2) \\
&= \sum_{\lambda=1}^{n} f_\lambda (r_1\partial_{x_\lambda}(g_1) + r_2\partial_{x_\lambda}(g_2)) \\
&= r_1 \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_1) + r_2 \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_2)) \\
&= r_1 d(g_1) + r_2 d(g_2),
\end{aligned}$$

  for all $r_1, r_2 \in R$ and $g_1, g_2 \in S_w$.

- Leibniz law: We have

$$\begin{aligned}
d(g_1g_2) &= \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_1g_2) \\
&= \sum_{\lambda=1}^{n} f_\lambda (\partial_{x_\lambda}(g_1)g_2 + g_1\partial_{x_\lambda}(g_2)) \\
&= \left(\sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_1)\right) g_2 + g_1 \left(\sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_2))\right) \\
&= d(g_1)g_2 + g_1 d(g_2),
\end{aligned}$$

  for all $g_1, g_2 \in S_w$.

- Graded of degree -1: By R-linearity, we only need to check this on monomials. Let $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of weighted degree i. A term in $d(x_1^{\alpha_1} \cdots x_n^{\alpha_n})$ has the form $\alpha_\lambda f_\lambda x_1^{\alpha_1} \cdots x_\lambda^{\alpha_\lambda - 1} \cdots x_n^{\alpha_n}$ where $\alpha_\lambda \equiv 1$ mod 2, and

$$\begin{aligned}
\deg_w \left(\alpha_\lambda f_\lambda x_1^{\alpha_1} \cdots x_\lambda^{\alpha_\lambda - 1} \cdots x_n^{\alpha_n}\right) &= \deg_w \left(f_\lambda x_1^{\alpha_1} \cdots x_\lambda^{\alpha_\lambda - 1} \cdots x_n^{\alpha_n}\right) \\
&= \deg_w (f_\lambda) + \deg_w \left(x_1^{\alpha_1} \cdots x_\lambda^{\alpha_\lambda - 1} \cdots x_n^{\alpha_n}\right) \\
&= w_\lambda - 1 + w_1\alpha_1 + \cdots + w_\lambda(\alpha_\lambda - 1) + \cdots + w_n\alpha_n \\
&= -1 + w_1\alpha_1 + \cdots + w_\lambda\alpha_\lambda + \cdots + w_n\alpha_n \\
&= -1 + i.
\end{aligned}$$

So every term in $d(x_1^{\alpha_1}\cdots x_n^{\alpha_n})$ has weighted degree $-1+i$. This implies that d is graded of degree -1.

Now we will show that $(S_w/I, \overline{d})$ is a differential graded R-algebra. Since I is d-stable, the map $\overline{d}$ is well-defined. The map $\overline{d}$ inherits the properties of being a graded endomorphism of degree -1 which satisfies Leibniz law from d, thus we just need to show that $\overline{d}^2 = 0$, or in other words, that $d^2(g) \in I$ for all $g \in S_w$. So let $g \in S_w$. Then

$$\begin{aligned} d^2(g) &= d\left(\sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g)\right) \\ &= \sum_{\lambda=1}^{n} d(f_\lambda \partial_{x_\lambda}(g)) \\ &= \sum_{\lambda=1}^{n} d(f_\lambda)\partial_{x_\lambda}(g) + f_\lambda d(\partial_{x_\lambda}(g)) \\ &= \sum_{\lambda=1}^{n} d(f_\lambda)\partial_{x_\lambda}(g) \in I, \end{aligned}$$

where we used the fact that $\partial_{x_\lambda}^2 = 0$ and $\partial_{x_\mu}\partial_{x_\lambda} = \partial_{x_\lambda}\partial_{x_\mu}$ to conclude that

$$\begin{aligned} \sum_{\lambda=1}^{n} f_\lambda d(\partial_{x_\lambda}(g)) &= \sum_{\lambda=1}^{n} f_\lambda \sum_{\mu=1}^{n} f_\mu \partial_{x_\mu}(\partial_{x_\lambda}(g)) \\ &= 0. \end{aligned}$$

$\square$

Remark 4.2.

1. *We often denote this differential graded R-algebra as $(S_w/I, f_1, \ldots f_n)$ instead of $(S_w/I, \overline{d})$.*

2. *When we write "let $(S_w/I, f_1, \ldots f_n)$ be a differential graded R-algebra", it is understood that the conditions in Theorem (4.1) are satisfied.*

Proposition 4.3. *Let (S_w, d) be a differential graded R-algebra and let I be a homogeneous ideal in S. Then I is d-stable if and only if for some generating set $\{g_1, \ldots, g_r\}$ of I, we have $d(g_\lambda) \in I$ for all $\lambda = 1, \ldots, r$.*

Proof. One direction is trivial, so let's prove the other direction. Let $\{g_1, \ldots, g_r\}$ be a generating set for I such that $d(g_\lambda) \in I$ for all $\lambda = 1, \ldots, r$ and let $g \in I$. Since $\{g_1, \ldots, g_r\}$ generates I, we can write $g = \sum_{\lambda=1}^r q_\lambda g_\lambda$ for some $q_1, \ldots, q_r \in S$. Thus, by Leibniz law, we have

$$\begin{aligned} d(g) &= d\left(\sum_{\lambda=1}^r q_\lambda g_\lambda\right) \\ &= \sum_{\lambda=1}^r d(q_\lambda g_\lambda) \\ &= \sum_{\lambda=1}^r (d(q_\lambda) g_\lambda + q_\lambda d(g_\lambda)) \in I. \end{aligned}$$

Thus, I is d-stable. □

Proposition 4.4. *Let $(S_w/I, f_1, \ldots, f_n)$ be a differential graded R-algebra and let g be a homogeneous polynomial in S of degree j such that $d(g)$ is in I. Then $(S_w/\langle I, g\rangle, f_1, \ldots, f_n)$ and $(S/(I : g), f_1, \ldots, f_n)$ are differential graded R-algebras.*

Proof. First note that $d(f_\lambda) \in I$ implies $d(f_\lambda) \in \langle I, g\rangle$ and $d(f_\lambda) \in I : g$ for all $\lambda = 1, \ldots, n$. So we just need to show that $\langle I, g\rangle$ and $I : g$ are d-stable. Since $d(g)$ is in I, Proposition (4.3) implies that $\langle I, g\rangle$ is d-stable. To prove that $I : g$ is d-stable, let $f \in I : g$. Then since $fg \in I$ and I is d-stable, it follows that $d(fg) = d(f)g + fd(g) \in I$, which implies $d(f)g \in I$, since $d(g) \in I$. Therefore $d(f) \in I : g$, which implies that $I : g$ is d-stable. □

4.2 Classifying Differential Graded R-Algebras

Theorem 4.5. *Every differential graded R-algebra is isomorphic to one described in Theorem (4.1).*

Proof. Let (A, d_A) be a differential graded R-algebra with generators $a_1, \ldots, a_n$. Then for each $\lambda = 1, \ldots n$, we have $a_\lambda \in A_{w_\lambda}$, where $w_\lambda \in \mathbb{Z}_{\geq 0}$. Let S_w denote the weighted polynomial ring $R[x_1, \ldots, x_n]_{(w_1,\ldots,w_n)}$, and let $\varphi\colon S_w \to A$ be the unique morphism of graded R-algebras such that $\varphi(x_\lambda) = a_\lambda$ for all $\lambda = 1, \ldots, n$. Then A is isomorphic to $S_w/\text{Ker}(\varphi)$ as graded R-algebras. Choose $f_\lambda \in S_w$ such that $\varphi(f_\lambda) = d_A(a_\lambda)$ and define the map $d\colon S_w \to S_w$ by

$$d := \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}.$$

Then d is a graded endomorphism of degree -1 which satisfies Leibniz law, by Theorem (4.1). We claim that Ker(φ) is d-stable and that $d(f_\lambda) \in \text{Ker}(\varphi)$ for all $\lambda = 1, \ldots, n$. We do this in two steps:

Step 1: We will show that $\varphi d = d_A \varphi$. It suffices to show that for all monomials m, we have $\varphi(d(m)) = d_A(\varphi(m))$. We prove this by induction on deg(m). For the base case deg$(m) = 1$, we have $m = x_\lambda$ for some $\lambda \in \{1, \ldots, n\}$. Then

$$\begin{aligned}\varphi(d(x_\lambda)) &= \varphi(f_\lambda)\\ &= d_A(a_\lambda)\\ &= d_A(\varphi(x_\lambda)).\end{aligned}$$

Now suppose that $\varphi(d(m)) = d_A(\varphi(m))$ for all monomials m in S of degree less than i, where $i > 1$. Let $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S whose degree is $i + 1$. We may assume that $\alpha_1, \alpha_\lambda \geq 1$ for some $\lambda \in \{1, \ldots, n\}$. Then using Leibniz law together with induction, we obtain

$$\begin{aligned}\varphi(d(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})) &= \varphi(d(x_1^{\alpha_1})x_2^{\alpha_2}\cdots x_n^{\alpha_n} + x_1^{\alpha_1}d(x_2^{\alpha_2}\cdots x_n^{\alpha_n}))\\ &= \varphi(d(x_1^{\alpha_1}))\varphi(x_2^{\alpha_2}\cdots x_n^{\alpha_n}) + \varphi(x_1^{\alpha_1})\varphi(d(x_2^{\alpha_2}\cdots x_n^{\alpha_n}))\\ &= \varphi(d(x_1^{\alpha_1}))a_2^{\alpha_2}\cdots a_n^{\alpha_n} + a_1^{\alpha_1}\varphi(d(x_2^{\alpha_2}\cdots x_n^{\alpha_n}))\\ &= d_A(a_1^{\alpha_1})a_2^{\alpha_2}\cdots a_n^{\alpha_n} + a_1^{\alpha_1}d_A(a_2^{\alpha_2}\cdots a_n^{\alpha_n})\\ &= d_A(a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_n^{\alpha_n})\\ &= d_A(\varphi(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})).\end{aligned}$$

This establishes Step 1.

Step 2: We show that Ker(φ) is d-stable and that $d(f_\lambda) \in$ Ker(φ) for all $\lambda = 1, \ldots, n$. Let $g \in$ Ker(φ). Then by Step 1, we have

$$\begin{aligned}\varphi(d(g)) &= d_A(\varphi(g))\\ &= d_A(0)\\ &= 0.\end{aligned}$$

Thus $d(g) \in$ Ker(φ), which implies Ker(φ) is d-stable. Step 1 also implies

$$\begin{aligned}\varphi(d(f_\lambda)) &= d_A(\varphi(f_\lambda))\\ &= d_A(d_A(a_\lambda))\\ &= 0,\end{aligned}$$

for all $\lambda = 1, \ldots, n$.

Now Theorem (4.1) implies that $(S_w/\text{Ker}(\varphi), \overline{d})$ is a differential graded R-algebra. Moreover, Step 1 implies $\varphi : (S_w/\text{Ker}(\varphi), \overline{d}) \to (A, d_A)$ is an isomorphism of differential graded R-algebras. $\square$

DIFFERENTIAL GRADED R-ALGEBRAS OF THE FORM $(S/I, r_1, \dots, r_n)$

We now want to consider some special cases of Theorem (4.1). In particular, we want to consider the case where the weighted vector is $w = (1, \dots, 1)$. As usual, we will write S to denote the polynomial ring $R[x_1, \dots, x_n]$. Let $r_1, \dots, r_n$ be nonzero elements in R, and define $d\colon S \to S$ by

$$d := \sum_{\lambda=1}^{n} r_\lambda \partial_{x_\lambda}.$$

Since $d(r_\lambda) = 0$ for all $\lambda = 1, \dots, n$, it follows from Theorem (4.1) that $(S, r_1, \dots, r_n)$ is a differential graded R-algebra. Moreover, if I is a d-stable ideal, then $(S/I, r_1, \dots, r_n)$ is a differential graded R-algebra.

Koszul Complex

Recall from Example (2.19) that the Koszul complex $\mathcal{K}(r_1, \dots, r_n)$ is a differential graded R-algebra. Indeed, $\mathcal{K}(r_1, \dots, r_n)$ is isomorphic to the differential graded R-algebra $(S/I, r_1, \dots, r_n)$, where I is generated by $\{x_1^2, \dots, x_n^2\}$. Clearly I is d-stable since $d(x_\lambda^2) = 0$ for all $\lambda = 1, \dots, n$.

Example 4.6. *Let $R = \mathbb{F}_2[x,y]/\langle xy \rangle$ and let $r_1 = x$ and $r_2 = y$. Then $S = R[u,v]$ has a differential graded R-algebra structure with the differential d given by*

$$d := x\partial_u + y\partial_v.$$

Using graded lexicographical ordering on the monomials, we can explicitly write S as a chain complex over R using matrices as the linear maps:

$$\cdots \longrightarrow R^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & y & 0 \\ 0 & x & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow 0$$

Now let I be the homogeneous ideal in S generated by $\{x^2, y^2\}$. Then $(S/I, r_1, r_2)$ is isomorphic to the Koszul complex $\mathcal{K}(r_1, r_2)$. Using graded lexicographical ordering on the monomials, we can explicitly write S/I as a chain complex over R using matrices as the linear maps:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow 0$$

Blowup algebras

Proposition 4.7. *Let Q be a finitely generated ideal in R with generating set $\{a_1, \ldots, a_n\}$. Then the blowup algebra $B_Q(R)$ can be given the structure of differential graded R-algebra.*

Proof. Let $\varphi : S \to B_Q(R)$ be the unique graded R-algebra homomorphism such that $\varphi(x_\lambda) = ta_\lambda$ for all $\lambda = 1, \ldots, n$ and let $d := \sum_{\lambda=1}^n a_\lambda \partial_\lambda$. We claim that Ker$(\varphi)$ is d-stable. Indeed, let $f \in$ Ker(φ). Since Ker(φ) is homogeneous, we may assume that f is homogeneous. Write f and $d(f)$ in terms of the monomial basis:

$$f = \sum_{\lambda=1}^{r} b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_n^{\alpha_{n\lambda}} \qquad \text{and} \qquad d(f) = \sum_{\substack{1 \le \mu \le n \\ 1 \le \lambda \le r}} \alpha_{\mu\lambda} a_\mu b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_\mu^{\alpha_{\mu\lambda}-1} \cdots x_n^{\alpha_{n\lambda}}.$$

where $b_\lambda \in R$ and $\alpha_{\mu\lambda} \in \mathbb{Z}_{\geq 0}$ for all $\lambda = 1, \ldots, r$ and $\mu = 1, \ldots n$. Observe that

$$\begin{aligned}
0 &= \varphi(f) \\
&= \varphi\left(\sum_{\lambda=1}^{r} b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_n^{\alpha_{n\lambda}}\right) \\
&= \sum_{\lambda=1}^{r} b_\lambda \varphi(x_1)^{\alpha_{1\lambda}} \cdots \varphi(x_n)^{\alpha_{n\lambda}} \\
&= t^i \left(\sum_{\lambda=1}^{r} b_\lambda a_1^{\alpha_{1\lambda}} \cdots {a_n}^{\alpha_{n\lambda}}\right)
\end{aligned}$$

implies that $\sum_{\lambda=1}^{r} b_\lambda a_1^{\alpha_{1\lambda}} \cdots {a_n}^{\alpha_{n\lambda}} = 0$. Therefore

$$\begin{aligned}
\varphi(d(f)) &= \varphi\left(\sum_{\substack{1\le\mu\le n\\ 1\le\lambda\le r}} \alpha_{\mu\lambda} a_\mu b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_\mu^{\alpha_{\mu\lambda}-1} \cdots x_n^{\alpha_{n\lambda}}\right)\\
&= \sum_{\substack{1\le\mu\le n\\ 1\le\lambda\le r}} \alpha_{\mu\lambda} a_\mu b_\lambda \varphi(x_1)^{\alpha_{1\lambda}} \cdots \varphi(x_\mu)^{\alpha_{\mu\lambda}-1} \cdots \varphi(x_n)^{\alpha_{n\lambda}}\\
&= t^{i-1}\left(\sum_{\substack{1\le\mu\le n\\ 1\le\lambda\le r}} \alpha_{\mu\lambda} a_\mu b_\lambda a_1^{\alpha_{1\lambda}} \cdots a_\mu^{\alpha_{\mu\lambda}-1} \cdots a_n^{\alpha_{n\lambda}}\right)\\
&= t^{i-1}\left(\left(\sum_{\mu=1}^{n} \alpha_{\mu\lambda}\right)\left(\sum_{\lambda=1}^{r} b_\lambda a_1^{\alpha_{1\lambda}} \cdots {a_n}^{\alpha_{n\lambda}}\right)\right)\\
&= 0.
\end{aligned}$$

Therefore $(S/\text{Ker}(\varphi), a_1, \ldots, a_n)$ is a differential graded R-algebra where $S/\text{Ker}(\varphi) \cong B_Q(R)$. $\square$

Remark 4.8. *It isn't too difficult to show that this differential graded R-algebra is $(B_Q(R), \partial_t)$, where ∂_t is defined in the obvious way.*

Example 4.9. *Let $R = \mathbb{F}_2[x,y]/\langle y^2 + x^3 + x^2\rangle$, $\mathfrak{m}$ be the maximal ideal in R generated by $\{\overline{x}, \overline{y}\}$, S denote the polynomial ring $R[u,v]$, and $d = \overline{x}\partial_u + \overline{y}\partial_v$. There is a surjective R-algebra homomorphism from S to the blowup algebra at $\mathfrak{m}$ given by*

$$\varphi : S := \mathbb{F}_2[x,y,u,v]/\langle y^2 + x^3 + x^2\rangle \to B_{\mathfrak{m}}(R),$$

where φ is induced by $\varphi(u) = t\overline{x}$ and $v \mapsto t\overline{y}$. Using Singular, we find that the kernel of φ is an ideal which is homogeneous in the variables u, v, and is generated by the set $\{f_1, f_2, f_3\}$, where

$$\begin{aligned}
f_1 &= \overline{x}v + \overline{y}u\\
f_2 &= \overline{x}u^2 + u^2 + v^2\\
f_3 &= \overline{x}^2u + \overline{x}u + \overline{y}v
\end{aligned}$$

Note that $d(f_1) = d(f_2) = d(f_3) \in$ Ker(φ). It follows from Proposition (4.3) that Ker(φ) is d-stable, which we already knew from Proposition (4.7).

4.3 Homology Computations

Proposition 4.10. *Let $(S/I, r_1, \ldots, r_n)$ be a differential graded R-algebra. Suppose that there are $t_1, \ldots, t_n \in R$ such*

$$\sum_{\lambda=1}^{n} t_\lambda r_\lambda = 1. \tag{4.1}$$

Then $H(S/I, r_1, \ldots, r_n) = 0$.

Proof. First note that $\sum_{\lambda=1}^{n} t_\lambda x_\lambda \notin I$, otherwise $d\left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) = 1 \notin I$ would imply that I is not d-stable. Let f be a homogeneous polynomial of degree i such $d(f) \in I$; so f represents a cycle of $(S/I, \overline{d})$. Then

$$\begin{aligned} d\left(\left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) f\right) &= d\left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) f + \left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) d(f) \\ &= \left(\sum_{\lambda=1}^{n} t_\lambda r_\lambda\right) f + \left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) d(f) \\ &= f + \left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) d(f) \\ &\equiv f \bmod I. \end{aligned}$$

thus Ker$(\overline{d}) =$ Im$(\overline{d})$, which proves the claim. □

Remark 4.11. *The condition (4.1) is equivalent to saying that $\{r_1, \ldots, r_n\}$ generates the unit ideal.*

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