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Exponentially Weighted Moving Average Charts for Monitoring the Process Generalized Variance

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EXPONENTIALLY WEIGHTED MOVING AVERAGE CHARTS FOR MONITORING THE PROCESS GENERALIZED VARIANCE

by

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(Under the Direction of Charles W. Champ)

ABSTRACT

The exponentially weighted moving average chart based on the sample generalized variance is studied under the independent multivariate normal model for the vector of quality measurements. The performance of the chart is based on an analysis of the chart's initial and steady-state run length distributions. The three methods that are commonly used to determinate run length distribution, simulation, the integral equation method, and the Markov chain approximation are discussed. The integral equation and Markov chain approaches are analytical methods that require a numerical method for determining the probability density and cumulative distribution functions describing the distribution of the sample generalized variance. Two methods for determining numerically these functions are discussed. The equivalence of the integral equation and Markov chain methods is shown resulting in a new method for obtaining a Markov chain approximation of the chart. Some examples of the implementation of these methods are given using MATLAB.

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CHAPTER 1

INTRODUCTION

The quality control chart was introduced in the early 1920's by Walter A. Shewhart (see [29]). It is based on a quality measurement(s) X whose distribution depends on one or more parameters that are to be monitored. Shewhart described two kinds of variability in a quality measurement X found in a process which he labeled as "natural" and "assignable." A process that is operating with only natural causes of variability present is said to be an "in-control" process. When at least one assignable cause is present, then the process is in an "out-of-control" state. The control chart is an aid to the practitioner in the attempt to discover assignable causes of variability in the process, as an aid in defining what is meant by a process being in an in-control state, and detecting when an assignable cause of variability has changed the process.

The actual distribution of the quality measurement(s) is not known. It is common practice to model this unknown distribution. The model we have selected for our study is the multivariate normal distribution for our vector of quality measurements. The set of parameters for this model are the mean vector and covariance matrix. In Chapter 2, we discuss this model along with our meaning of a process operating with only natural causes of variability and one that has at least one assignable cause of variability. Also, we discuss the method to be used in the collection of the outputted items from the process and the stochastic assumptions about the measurements on these items.

In chapter 3, we examine various statistics and their distributions that will be useful in making an inference about the quality of the process. We discuss in particular in this chapter two methods for determining the probability density and cumulative distribution function of the sample generalized variance and the natural logarithm of the sample generalized variance.

In general, a control chart is a plot of a statistic that summarizes the sample data at discrete points in time. Typically, a control chart has one and sometimes two control limits, although there might be more than two control limits for multi-dimensional charts. If the plotted statistic exceeds one of the control limits, this is taken as relatively strong evidence that the process is in an out-of-control state. Two lines in a control chart are chosen and called upper control limit (UCL) and lower control limit (LCL). When the plotted statistic falls at or below the LCL or at or above the UCL, the chart "signals" to the practitioner that the process is potentially in an out-of-control state; otherwise the process is said to be in control.

Monitoring for a change in the mean vector of distribution of a multivariate quality measurement is usually of primary interest to the practitioner. However, [7] showed various multivariate control charts for monitoring for a change in the mean vector are also effected by a change in the covariance matrix of the vector of quality measurements. [23] recommends in these situations that a chart be maintained for monitoring for a change in the covariance structure and that this chart should be examined first before examining a chart for monitoring the mean vector. Various control charts have been discussed in the literature for monitoring for a change in the covariance structure of a multivariate quality measurement.

Various authors have proposed/studied charts for monitoring for a change in the covariance matrix of the distribution of a multivariate quality measurement. These, include among others, [1], [31], [30], [24], [3], [12], [36], [33], [34] and [18]. A review of multivariate control charts for monitoring a covariance matrix is given by [35].

In this research, the Exponentially Weighted Moving Average (EWMA) charts based on sample generalized variance $|\mathbf{S}|$ and the natural logarithm $\ln(|\mathbf{S}|)$ of the sample generalize variance are examined. This chart is used for monitoring process variability measured by the process generalized variance $|\mathbf{\Sigma}|$. Changes in the process

variability can have a big impact on product quality. We discuss these charts in Chapter 4.

The length of time takes for a control chart to signal a potential "out-of-control" process is a random variable called the run length. The performance of a chart is often done by comparing properties of their run length distribution, in particular, the average run length (ARL). The commonly used methods, simulation, Markov chain approximation, and integral equations that are used to evaluate the run length distribution of control charts are discussed. A discuss of performance evaluation of the chart is presented in Chapter 5. We show that the Markov chain approximation and the integral equation methods are equivalent resulting in new ways to obtain a Markov chain approximation of a chart.

And, finally, some recommendations for the further research are given in our final chapter.

CHAPTER 2

MODEL AND SAMPLING METHODS

2.1 Introduction

In the design and evaluation of a control charting procedure, it is useful to have a data model. The most commonly used model to describe the distribution of a vector **X** of continuous quality measurements is the multivariate normal distribution. [29] discussed the concepts of a process being in a state of statistical in-control as well as being in a state of statistical out-of-control. In the designing a control charting procedure, one needs a definition of what is meant by the process being in the states of statistical in- and out-of-control. Our data model and these definitions are given in the next section.

Another assumption that is commonly made it that the samples are collected periodically from the output of the process and the **X** measurements on these items are random samples. Further, it is assumed that the random samples are independent. These assumptions about the data to be collected by the practitioner in an effort to bring a process into state of statistical in-control and then monitor for change in the process from one of being in-control to an out-of-control process are discussed in Section 3. Also, in this section, we discuss the common ways in which an in-control process is estimated based on our model.

2.2 Data Model

The most commonly used model for a $p \times 1$ vector **X** of quality measurements is the multivariate normal model. The joint probability density function of a multivariate normal distribution with p variables has the form

$$f_{\mathbf{X}}\left(\mathbf{x}\right) = \frac{1}{\left(2\pi\right)^{p/2}\left|\mathbf{\Sigma}\right|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^{\mathbf{T}}\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)},$$

where $\mathbf{x}$ is a $p \times 1$ vector of real numbers,

$$E\left(\mathbf{X}\right) = \mu_{\mathbf{X}} = \mu \text{ and } cov\left(\mathbf{X}\right) = \Sigma_{\mathbf{X}} = \Sigma.$$

Also, it is assumed that Σ is a positive definite matrix. It is convenient to write $\mathbf{X} \sim N_p\left(\mu, \Sigma\right)$ to state that the $p \times 1$ random vector $\mathbf{X}$ has a multivariate normal distribution with mean vector μ and covariance matrix Σ. We assume that the process is in a state of statistical in-control if $\mu = \mu_0$ and $\Sigma = \Sigma_0$, where μ_0 and Σ_0 are a fixed set of values that in general are unknown. The process is considered to be in an out-of-control state if $\mu \neq \mu_0$ or $\Sigma \neq \Sigma_0$.

The assumption that Σ is a positive definite matrix implies that the eigenvalue $\xi_1, \ldots, \xi_p$ of Σ are all positive real numbers. Associated with the p eigenvalues $\xi_1, \ldots, \xi_p$ is a set of corresponding orthonormal eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$. We can then express Σ as

$$\Sigma = \mathbf{VCV^T} = \mathbf{PP^T},$$

where $\mathbf{C} = Diagonal\left(\xi_1, \ldots, \xi_p\right)$, $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_p]$, and $\mathbf{P} = \mathbf{VC}^{1/2}$ with $\mathbf{C}^{1/2} = Diagonal\left(\xi_1^{1/2}, \ldots, \xi_p^{1/2}\right)$. Note that $\mathbf{V^T V} = \mathbf{I}$. This allows a "standardized" vector $\mathbf{Z}$ of the vector $\mathbf{X}$ to be defined by

$$\mathbf{Z} = \mathbf{P}^{-1}\left(\mathbf{X} - \mu\right) \text{ with } \mathbf{X} = \mu + \mathbf{PZ}.$$

The Jacobian of the transformation is $|\mathbf{P}|$. Since $\left|\mathbf{P^T}\right| = |\mathbf{P}|$, then we have

$$|\mathbf{P}| = |\Sigma|^{1/2}.$$

Hence, the joint distribution of the $p \times 1$ vector $\mathbf{Z}$ is

$$f_{\mathbf{Z}}\left(\mathbf{z}\right) = \frac{1}{\left(2\pi\right)^{p/2} |\mathbf{I}|^{1/2}} e^{-\frac{1}{2}\mathbf{z^T z}}.$$

We will refer to this distribution as the multivariate standard normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}$. Note that under the multivariate normal

model with positive definite covariance matrix, the components of the $p \times 1$ vector $\mathbf{Z}$ are stochastically independent each with mean 0.

The process generalized variance is the determinant $|\boldsymbol{\Sigma}|$ of the process covariance matrix $\boldsymbol{\Sigma}$. The process generalized variance can be used as an overall measure of the variability found in the process. [20] discussed the disadvantages of using the population/process generalized variance as a measure of variability. Observe that

$$|\boldsymbol{\Sigma}| = \left|\mathbf{VCV^T}\right| = \left|\mathbf{VV^T}\right||\mathbf{C}| = |\mathbf{C}| = \prod\nolimits_{i=1}^{p} \xi_i.$$

We also observe that when the process is in a state of statistical in-control, then we have

$$|\boldsymbol{\Sigma}_0| = |\mathbf{C}_0| = \prod\nolimits_{i=1}^{p} \xi_{i0},$$

where $\xi_{p0}, \ldots, \xi_{p0}$ are the eigenvalues of $\boldsymbol{\Sigma}_0$. As we will see later, it is convenient to define the parameter λ^2 as

$$\lambda^2 = \left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right| \text{ and } \theta = \ln\left(\left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right|\right).$$

It is easy to see that

$$\lambda^2 = \prod\nolimits_{i=1}^{p} \xi_{i0}^{-1}\xi_i \text{ and } \theta = \ln\left(\lambda^2\right) = \sum\nolimits_{i=1}^{p} \ln\left(\xi_{i0}^{-1}\xi_i\right).$$

Observe that if the process is in a state of statistical in-control, then

$$\lambda^2 = 1 \text{ and } \theta = 0.$$

If $\lambda^2 \neq 1$ or $\theta \neq 0$, then the process is out-of-control.

2.3   Data and Parameter Estimates

In order to make a determination of the state of the process, the practitioner will periodically take a sample of items from the output of the process and obtain on each

item the quality measurement $\mathbf{X}$. There are two phases of the process in which the practitioner collects data. In the first phase (Phase I), the practitioner works to bring the process into a state of statistical in-control. Data from a Phase I study of the process can be used to define what is meant by an in-control process. We assume here that in Phase I the practitioner has identified m independent random samples of the vector $\mathbf{X}$ of quality measurements that were taken on items produced when the process was in-control. We represent these mn data values by

$$\mathbf{X}_{i,1}, \ldots, \mathbf{X}_{i,n}$$

for $i = 1, \ldots, m$. These data can be used to estimate μ_0 and $\mathbf{\Sigma}_0$ when they are not known. The most commonly used estimators for these parameters are

$$\widehat{\mu}_0 = \overline{\mathbf{X}}_0 = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{X}_{i,j} \text{ and}$$
$$\widehat{\mathbf{\Sigma}}_0 = \overline{\mathbf{S}}_0 = \frac{1}{m(n-1)} \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\mathbf{X}_{i,j} - \overline{\mathbf{X}}_0\right) \left(\mathbf{X}_{i,j} - \overline{\mathbf{X}}_0\right)^{\mathbf{T}}.$$

It is easy to show that $\widehat{\mu}_0$ and $\widehat{\mathbf{\Sigma}}_0$ are unbiased estimators of μ_0 and $\mathbf{\Sigma}_0$, respectively.

In the second phase (known as Phase II or the monitoring phase), the practitioner periodically samples from the output of the process. We assume that the $\mathbf{X}$ measurements on these items are independent and identically distributed $N_p(\mu, \mathbf{\Sigma})$. These sets of measurements are represented by

$$\mathbf{X}_{t,1}, \ldots, \mathbf{X}_{t,n}$$

for $t = 1, 2, 3, \ldots$. Our estimates of μ and $\mathbf{\Sigma}$ at time t are, respectively,

$$\widehat{\mu}_t = \overline{\mathbf{X}}_t = \frac{1}{n} \sum_{j=1}^{n} \mathbf{X}_{t,j} \text{ and } \widehat{\mathbf{\Sigma}}_t = \overline{\mathbf{S}}_t = \frac{1}{n-1} \sum_{j=1}^{n} \left(\mathbf{X}_{t,j} - \overline{\mathbf{X}}_t\right) \left(\mathbf{X}_{t,j} - \overline{\mathbf{X}}_t\right)^{\mathbf{T}}.$$

One can show that $\widehat{\mu}_t$ and $\widehat{\mathbf{\Sigma}}_t$ are unbiased estimators of μ and $\mathbf{\Sigma}$, respectively.

2.4   Conclusion

We have discussed our model and data assumptions as well as a model for what is meant by a process being in a state of statistical in-control and a state of statistical out-of-control. These assumptions will be used in what follows in the design and evaluation of control charting procedures for monitoring for a change in the process generalized variance. There are other models for industrial processes in which the quality of an outputted item is measured by a vector of quality measurements. For processes in which it is more reasonable to assume that the vectors of quality measurements are autocorrelated, a multivariate time series model would be a better choice than our independent model.

CHAPTER 3

SOME DISTRIBUTIONAL RESULTS

3.1   Introduction

The distribution of a statistic is sometimes referred to as a sampling distribution. In the design and the performance analysis of a statistical method, the distribution of one or more statistics in a usable form is needed. For example, being able to analytically determine the distribution of a test statistic both under the null hypothesis and the alternative hypothesis is useful in determining the size of the test and the power function that can be used to answer the sample size question. In the performance analysis of a control charting procedure, it is of interest to examine the distribution of the run length both when the process is in a state of statistical in-control and when it is out-of-control The randomness found in the distribution of the run length comes from the randomness other statistics. In our study, we find that these other statistics are the sample covariance matrix and functions of the the sample covariance matrix such as the sample generalized variance and the natural logarithm of the sample generalized variance.

The distribution of the sample covariance is discussed in the next section. In Section 3, we discuss the distribution of the sample generalized variance, in particular, how the probability density function describing the distribution of the sample generalized variance can be expressed using the Meijer G function. This section is followed by a section about the distribution of the natural logarithm of the sample generalized variance. A closed form expression for the probability density function of the natural logarithm of the sample generalized variance is given.

3.2 Distribution of the Sample Covariance

Suppose that $\mathbf{X}^{p \times 1} \sim N_p(\mu, \mathbf{\Sigma})$. Let $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$ be a random samples of size n from the distribution of $\mathbf{X}$. The sample covariance matrix $\mathbf{S}$ is defined as

$$\mathbf{S} = \frac{1}{n-1} \sum_{i=1}^{n} \left(\mathbf{X}_i - \overline{\mathbf{X}} \right) \left(\mathbf{X}_i - \overline{\mathbf{X}} \right)^{\mathbf{T}},$$

where

$$\overline{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i.$$

It is convenient to define $\mathbf{A} = (n-1)\mathbf{S}$. Under the independent multivariate normal model with a positive definite covariance matrix, [14] shows that the sample covariance matrix $\mathbf{S}$ is positive definite with probability one. It follows that $\mathbf{A} = (n-1)\mathbf{S}$ is also positive definite with probability one. It is shown in [2] that $\mathbf{A}$ for $n > p$ can be expressed as

$$\mathbf{A} = \sum_{i=1}^{n-1} \mathbf{Z}_i \mathbf{Z}_i^{\mathbf{T}},$$

where $\mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}$ are independent and identically distributed $N_p(0, \mathbf{\Sigma})$. Further, he shows that the joint probability density function describing the distribution of the components of the positive definite matrix $\mathbf{A}_t$ is

$$f_{\mathbf{A}}(\mathbf{A}) = \frac{|\mathbf{A}|^{(n-p-2)/2} \, e^{-\frac{1}{2} tr\left(\mathbf{\Sigma}^{-1} \mathbf{A} \right)}}{2^{p(n-1)/2} \pi^{p(p-1)/4} \, |\mathbf{\Sigma}|^{(n-1)/2} \prod_{i=1}^{p} \Gamma\left(\frac{n-i}{2} \right)}.$$

Suppose that $\mathbf{\Sigma} = \mathbf{I}$, then

$$f_{\mathbf{A}}(\mathbf{A}) = \frac{|\mathbf{A}|^{(n-p-2)/2} \, e^{-\frac{1}{2} tr\left(\mathbf{I}^{-1} \mathbf{A} \right)}}{2^{p(n-1)/2} \pi^{p(p-1)/4} \, |\mathbf{I}|^{(n-1)/2} \prod_{i=1}^{p} \Gamma\left(\frac{n-i}{2} \right)}$$

$$= \frac{|\mathbf{A}|^{(n-p-2)/2} \, e^{-\frac{1}{2} tr(\mathbf{A})}}{2^{p(n-1)/2} \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(\frac{n-i}{2} \right)}.$$

It is not difficult to show that each component of the sample covariance matrix is an unbiased estimator of corresponding component of the population covariance

matrix. In this sense, we say that the sample covariance matrix is an unbiased estimator of the populations covariance matrix. We note here that

$$\mathbf{S}_t = S_t^2 \text{ and } (n-1)\, \mathbf{\Sigma}_0^{-1}\mathbf{S}_t = \frac{(n-1)\, S_t^2}{\sigma_0^2}.$$

3.3   Sample Generalized Variance

The population generalized variance is defined as the determinant $|\mathbf{\Sigma}|$ of the population (process) covariance $\mathbf{\Sigma}$. Its sample counterpart is the determinant $|\mathbf{S}|$ of the sample covariance matrix $\mathbf{S}$. [2] proves that under the independent multivariate normal model with $n > p$ that

$$|\mathbf{S}| \sim \frac{|\mathbf{\Sigma}|}{(n-1)^p} \prod_{i=1}^{p} \chi_{n-i}^2,$$

where $\chi_{n-1}^2, \ldots, \chi_{n-p}^2$ are stochastically independent Chi Square random variables with, respective, degrees of freedom $n-1, \ldots, n-p$. It follows that

$$W = \left|(n-1)\, \mathbf{\Sigma}^{-1}\mathbf{S}\right| \sim \prod_{i=1}^{p} \chi_{n-i}^2.$$

For the case in which $p = 2$, one can show from the results given in [2] that

$$W \sim \left(\chi_{2n-4}^2\right)^2 / 4.$$

[17] derived a closed form expression for the probability density function $f_{|\mathbf{S}|}(w)$ describing the distribution of $|\mathbf{S}|$ under the independent multivariate normal model. Their expression for $f_{|\mathbf{S}|}(w)$ is

$$
\begin{aligned}
f_{|\mathbf{S}|}(w) &= (n-1)^p \left|\mathbf{\Sigma}^{-1}\right| f_{|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|}\left((n-1)^p \left|\mathbf{\Sigma}^{-1}\right| w\right) \\
&= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{(n-1)^p \left|\mathbf{\Sigma}^{-1}\right| \left((n-1)^p \left|\mathbf{\Sigma}^{-1}\right| w\right)^{1/2} \prod_{i=1}^{p-1} w_i^{-1/2}}{\prod_{i=1}^{p} \Gamma\left(\frac{n-i}{2}\right) 2^{p(2n-p-1)/4}} \\
&\quad \times e^{-w_1/2} e^{-\left(\sum_{i=2}^{p-1}(w_i/w_{i-1}) + \left((n-1)^p \left|\mathbf{\Sigma}^{-1}\right| w/w_{p-1}\right)\right)/2} dw_1 \ldots dw_{p-1}.
\end{aligned}
$$

[26] show that the distribution of $|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}|$ can be expressed in terms of the Meijer G function. The Meijer G function is defined by

$$
G_{p,q}^{m\,r}\left(x\,\middle|\,
\begin{matrix}
a_1,\ldots,a_p \\
b_1,\ldots,b_q
\end{matrix}
\right)
= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j - s\right)\prod_{j=1}^{r}\Gamma\left(1 - a_j + s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1 - b_j + s\right)\prod_{j=r+1}^{p}\Gamma\left(a_j - s\right)}x^s ds,
$$

where the integral is along the complex contour L of a ratio of products of gamma functions. They ([26, p. 936]) express the pdf describing the distribution of $|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}|$ with some modification as follows

$$
f_{|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|}\left(w\right) = \frac{1}{2^p}\left(\prod_{j=1}^{p}\frac{1}{\Gamma\left(\frac{n-j}{2}\right)}\right)G_{0,p}^{p\,0}\left(\frac{w}{2^p}\,\middle|\,
\begin{matrix}
n-2 \\
2
\end{matrix},\ldots,\frac{n-1-p}{2}\right)I_{(0,\infty)}\left(w\right).
$$

The Meijer G function has been implemented in both MATLAB and Mathematica. The MATLAB code is

$$
G_{p,q}^{m\,r}\left(x\,\middle|\,
\begin{matrix}
a_1,\ldots,a_p \\
b_1,\ldots,b_q
\end{matrix}
\right)
$$
$$
= meijerG([[a_1,\ldots,a_r],[a_r+1,\ldots,a_p]],[[b_1,\ldots,b_m],[b_m+1,\ldots,b_q]],x).
$$

It follows that

$$
G_{0,p}^{p\,0}\left(x\,\middle|\,
\begin{matrix}
a_1,\ldots,a_p \\
b_1,\ldots,b_q
\end{matrix}
\right)
$$
$$
= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{p}\Gamma\left(b_j - s\right)\prod_{j=1}^{0}\Gamma\left(1 - a_j + s\right)}{\prod_{j=p+1}^{p}\Gamma\left(1 - b_j + s\right)\prod_{j=0+1}^{0}\Gamma\left(a_j - s\right)}x^s ds
$$
$$
= \frac{1}{2\pi i}\int_L \prod_{j=1}^{p}\Gamma\left(b_j - s\right)x^s ds
$$
$$
= meijerG([[a_1,\ldots,a_0],[a_0+1,\ldots,a_0]],[[b_1,\ldots,b_p],[b_p+1,\ldots,b_p]],x)
$$
$$
= meijerG([[],[]],[[b_1,\ldots,b_p],[]],x).
$$

It follows that

$$f_{|(n-1)\boldsymbol{\Sigma}^{-1}\mathbf{S}|}(w)$$

$$= \frac{1}{2^p} \left(\prod_{j=1}^{p} \frac{1}{\Gamma\left(\frac{n-j}{2}\right)} \right) meijerG([[], []], [[\frac{n-1}{2}, \ldots, \frac{n-p}{2}], []], w/2^p).$$

In Mathematica, the Meijer G function is implemented as

$$G_{p,q}^{m\,r} \left(x \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. \right)$$

$$= MeijerG[a1, ..., ar, a(r+1), ..., ap, b1, ..., bm, b(m+1), ..., bq, x].$$

We then have

$$f_{|(n-1)\boldsymbol{\Sigma}^{-1}\mathbf{S}|}(w)$$

$$= \frac{1}{2^p} \left(\prod_{j=1}^{p} \frac{1}{\Gamma\left(\frac{n-j}{2}\right)} \right) MeijerG([[], []], [[\frac{n-1}{2}, \ldots, \frac{n-p}{2}], []], w/2^p).$$

As we will see, the distributions of $\left|\overline{\mathbf{S}}_0\right|$ and $\left|m\left(n-1\right)\boldsymbol{\Sigma}_0^{-1}\overline{\mathbf{S}}_0\right|$ will be of interest. Using the results found in [2], the following theorem gives descriptions of their distributions.

Theorem 3.1. *If* $\mathbf{X}_{i,1}, \ldots, \mathbf{X}_{i,n}$ *for* $i = 1, \ldots, m$ *are* m *independent random samples each of size* n *from a common* $N_p\left(\mu_0, \boldsymbol{\Sigma}_0\right)$, *then for*

$$\overline{\mathbf{S}}_0 = \frac{1}{m\left(n-1\right)} \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\mathbf{X}_{i,j} - \overline{\mathbf{X}}_0\right) \left(\mathbf{X}_{i,j} - \overline{\mathbf{X}}_0\right)^{\mathbf{T}} \quad with$$

$$\overline{\mathbf{X}}_0 = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{X}_{i,j},$$

we have

$$\left|\overline{\mathbf{S}}_0\right| \sim \frac{|\boldsymbol{\Sigma}_0|}{m^p\left(n-1\right)^p} \prod_{i=1}^{p} \chi^2_{m(n-1)-(i-1)} \quad or$$

$$\left|m\left(n-1\right)\boldsymbol{\Sigma}_0^{-1}\overline{\mathbf{S}}_0\right| \sim \prod_{i=1}^{p} \chi^2_{m(n-1)-(i-1)},$$

where $\chi^2_{m(n-1)-(1-1)}, \ldots, \chi^2_{m(n-1)-(p-1)}$ are independent Chi Square random variables. In terms of the Meijer G function, we have

$$f_{\left|m(n-1)\Sigma_0^{-1}\overline{S}_0\right|}(w) = \frac{1}{2^p}\left(\prod_{j=1}^{p}\frac{1}{\Gamma\left(\frac{m(n-1)-(j-1)}{2}\right)}\right)$$
$$\times G_{0,p}^{p\,0}\left(\frac{w}{2^p}\left|\frac{m(n-1)-1}{2}, \ldots, \frac{m(n-1)-p}{2}\right.\right)I_{(0,\infty)}(w).$$

Proof. The proof of this theorem follows the proof given in [2] for the case in which $m = 1$ and using the results in [26]. $\qquad\square$

It is interesting to examine the means and variances of the distributions of the statistics $|\mathbf{S}|$, $\left|\overline{\mathbf{S}}_0\right|$, $\ln\left(|\mathbf{S}|\right)$, and $\ln\left(\left|\overline{\mathbf{S}}_0\right|\right)$. These parameters are determined under our independent multivariate normal model. The mean $\mu_{|\mathbf{S}|}$ is

$$\mu_{|\mathbf{S}|} = \frac{|\Sigma|}{(n-1)^p}\prod_{i=1}^{p}\mu_{\chi^2_{n-i}} = \frac{|\Sigma|}{(n-1)^p}\prod_{i=1}^{p}(n-i).$$

This suggest that an unbiased estimator of $|\Sigma|$ is

$$\frac{(n-1)^p|\mathbf{S}|}{\prod_{i=1}^{p}(n-i)}.$$

To determine the variance of the distribution of $|\mathbf{S}|$ under the independent multivariate normal model, we first determine $\mu_{|\mathbf{S}|^2}$. We see that

$$\mu_{|\mathbf{S}|^2} = \frac{|\Sigma|^2}{(n-1)^{2p}}\prod_{i=1}^{p}\mu_{\left(\chi^2_{n-i}\right)^2} = \frac{|\Sigma|^2}{(n-1)^{2p}}\prod_{i=1}^{p}\left(\sigma^2_{\chi^2_{n-i}} + \mu^2_{\chi^2_{n-i}}\right)$$
$$= \frac{|\Sigma|^2}{(n-1)^{2p}}\prod_{i=1}^{p}\left(2(n-i)+(n-i)^2\right)$$
$$= \frac{|\Sigma|^2}{(n-1)^{2p}}\left(\prod_{i=1}^{p}(n-i)\right)\left(\prod_{i=1}^{p}(2+n-i)\right).$$

It follows that

$$\sigma^2_{|\mathbf{S}|} = \frac{|\mathbf{\Sigma}|^2}{(n-1)^{2p}} \left(\prod_{i=1}^{p}(n-i)\right)\left(\prod_{i=1}^{p}(2+n-i)\right)$$

$$- \frac{|\mathbf{\Sigma}|^2}{(n-1)^{2p}} \left(\prod_{i=1}^{p}(n-i)\right)^2$$

$$= \frac{|\mathbf{\Sigma}|^2}{(n-1)^{2p}} \left(\prod_{i=1}^{p}(n-i)\right)$$

$$\times \left(\prod_{i=1}^{p}(2+n-i) - \prod_{i=1}^{p}(n-i)\right).$$

It follows that

$$\mu_{|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|} = \prod_{i=1}^{p}(n-i) \ \text{ and}$$

$$\sigma^2_{|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|} = \left(\prod_{i=1}^{p}(n-i)\right)\left(\prod_{i=1}^{p}(2+n-i) - \prod_{i=1}^{p}(n-i)\right).$$

Also, we have

$$V\left(\frac{(n-1)^p |\mathbf{S}|}{\prod_{i=1}^{p}(n-i)}\right) = \frac{(n-1)^{2p}}{(\prod_{i=1}^{p}(n-i))^2}\sigma^2_{|\mathbf{S}|}$$

$$= \frac{\prod_{i=1}^{p}(2+n-i) - \prod_{i=1}^{p}(n-i)}{(\prod_{i=1}^{p}(n-i))}|\mathbf{\Sigma}|^2.$$

Using the previous arguments, we have

$$\mu_{|\bar{\mathbf{S}}_0|} = \frac{|\mathbf{\Sigma}_0|}{m^p(n-1)^p}\prod_{i=1}^{p}\mu_{\chi^2_{m(n-1)-(i-1)}} = \frac{|\mathbf{\Sigma}_0|}{m^p(n-1)^p}\prod_{i=1}^{p}(m(n-1)-(i-1))$$

suggest that the statistic

$$\frac{m^p(n-1)^p|\bar{\mathbf{S}}_0|}{\prod_{i=1}^{p}(m(n-1)-(i-1))}$$

provides unbiased estimates of $|\mathbf{\Sigma}_0|$. The variance $\sigma^2_{|\bar{\mathbf{S}}_0|}$ is

$$\sigma^2_{|\bar{\mathbf{S}}_0|} = \frac{|\mathbf{\Sigma}|^2}{(n-1)^{2p}}\left(\prod_{i=1}^{p}(m(n-1)-(i-1))\right)$$

$$\times \left(\prod_{i=1}^{p}(2+m(n-1)-(i-1)) - \prod_{i=1}^{p}(m(n-1)-(i-1))\right).$$

Further, we have

$$\mu_{\left|m(n-1)\Sigma_0^{-1}\bar{\mathbf{S}}_0\right|} = \left(\prod_{i=1}^p \left(m\left(n-1\right) - \left(i-1\right)\right)\right) \text{ and}$$

$$\sigma_{\left|m(n-1)\Sigma_0^{-1}\bar{\mathbf{S}}_0\right|}^2 = \left(\prod_{i=1}^p \left(m\left(n-1\right) - \left(i-1\right)\right)\right)$$
$$\times \left(\prod_{i=1}^p \left(2 + m\left(n-1\right) - \left(i-1\right)\right) - \prod_{i=1}^p \left(m\left(n-1\right) - \left(i-1\right)\right)\right).$$

3.4   Natural Logarithm of the Sample Generalized Variance

We see from the results found in [2] that we can write

$$\ln\left(\left|\mathbf{S}\right|\right) \sim \ln\left(\frac{\left|\Sigma\right|}{\left(n-1\right)^p}\right) + \sum_{i=1}^p \ln\left(\chi_{n-i}^2\right) \text{ and}$$

$$U = \ln\left(W\right) = \ln\left(\left|\left(n-1\right)\Sigma^{-1}\mathbf{S}\right|\right) \sim \sum_{i=1}^p \ln\left(\chi_{n-i}^2\right).$$

For the case in which $p=2$, we have $W \sim \left(\chi_{2n-4}^2\right)^2/4$. Hence, it follows that

$$U = 2\left(\ln\left(\chi_{2n-4}^2\right) - \ln\left(2\right)\right).$$

The derivation of the distribution of

$$U = \sum_{i=1}^p \ln\left(\chi_{n-i}^2\right)$$

is given in [9]. We are interested as well in the distribution of

$$U_0 = \ln\left(\left|m\left(n-1\right)\Sigma_0^{-1}\bar{\mathbf{S}}_0\right|\right) = \sum_{i=1}^p \ln\left(\chi_{m(n-1)-(i-1)}^2\right).$$

Note that if $m=1$ and $\Sigma = \Sigma_0$, then U and U_0 have the same distribution. Expressions describing the distribution of U_0 are given in the following theorem.

The mean μ_U and variance σ_U^2 of the distribution of the random variable U can be expressed as

$$\mu_U = \sum_{i=1}^p \mu_{\ln\left(\chi_{n-i}^2\right)} \text{ and } \sigma_U^2 = \sum_{i=1}^p \sigma_{\ln\left(\chi_{n-i}^2\right)}^2.$$

The values $\mu_{\ln\left(\chi^2_{n-i}\right)}$ and $\sigma^2_{\ln\left(\chi^2_{n-i}\right)}$ must be obtained numerically for $i = 1, \ldots, p$. In general, we need to be able to determine the mean and variance of the distribution of the $\ln\left(\chi^2_\nu\right)$, where χ^2_ν is a random variable with a Chi Square distribution with ν degrees of freedom. It can be shown the probability density function describing the distribution of $\ln\left(\chi^2_\nu\right)$ has the form

$$f_{\ln(\chi^2_\nu)}(x) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right) 2^{\nu/2}} e^{-(e^x - \nu x)/2}$$

with support the reals. Closed form expressions are not available for the mean $\mu_{\ln(\chi^2_\nu)}$ and variance $\sigma^2_{\ln(\chi^2_\nu)}$ of the distribution of $\ln\left(\chi^2_\nu\right)$. Consequently, they must be determined numerically. For example, for $\nu = 5$, we find numerically that

$$\mu_{\ln\left(\chi^2_5\right)} = \int_{-\infty}^{\infty} x \frac{1}{\Gamma\left(\frac{5}{2}\right) 2^{5/2}} e^{-(e^x - 5x)/2} dx = 1.39630382\underline{1} \text{ and}$$

$$\sigma^2_{\ln\left(\chi^2_5\right)} = \int_{-\infty}^{\infty} x^2 \frac{1}{\Gamma\left(\frac{5}{2}\right) 2^{5/2}} e^{-(e^x - 5x)/2} dx$$

$$- \left(\int_{-\infty}^{\infty} x \frac{1}{\Gamma\left(\frac{5}{2}\right) 2^{5/2}} e^{-(e^x - 5x)/2} dx \right)^2$$

$$= 0.49035775\underline{61}.$$

The following theorem gives the distribution of

$$U_0 = \ln\left(\left| m\left(n - 1\right) \mathbf{\Sigma}_0^{-1} \overline{\mathbf{S}}_0 \right|\right).$$

Theorem 3.2. *If* $\chi^2_{m(n-1)}, \ldots, \chi^2_{m(n-1)-(p-1)}$ *are independent Chi Square random variables with degrees of freedom* $m\left(n - 1\right), \ldots, m\left(n - 1\right) - (p - 1)$, *respectively, with* $n > p$, *then the probability density function* $f_{U_0}(u)$ *describing the distribution of*

$$U_0 = \ln\left(\left| m\left(n - 1\right) \mathbf{\Sigma}_0^{-1} \overline{\mathbf{S}}_0 \right|\right) = \sum_{i=1}^{p} \ln\left(\chi^2_{m(n-1)-(i-1)}\right)$$

can be expressed as

$$f_{U_0}(u) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\ln\left(\chi^2_{m(n-1)-(1-1)}\right)}(u - x_2 - \ldots - x_p)$$

$$\times f_{\ln\left(\chi^2_{m(n-1)-(2-1)}\right)}(x_2) \cdots f_{\ln\left(\chi^2_{m(n-1)-(p-1)}\right)}(x_p)\, dx_2 \cdots dx_p;\ and$$

$$F_{U_0}(u) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F_{\ln\left(\chi^2_{m(n-1)-(1-1)}\right)}(u - x_2 - \ldots - x_p)$$

$$\times f_{\ln\left(\chi^2_{m(n-1)-(2-1)}\right)}(x_2) \cdots f_{\ln\left(\chi^2_{m(n-1)-(p-1)}\right)}(x_p)\, dx_2 \cdots dx_p.$$

Proof. The proof of this theorem follow the one in [9] for the distribution of U. $\quad\square$

A numerical method is given in [15] for evaluating multiple integrals of the form

$$I(f) = (2\pi)^{-(p-1)/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\mathbf{x}^{\mathbf{T}}\mathbf{x}/2} f(\mathbf{x})\, dx_2 \cdots dx_p.$$

For $n > p$, define

$$h_u(\mathbf{x}) = (2\pi)^{(p-1)/2} e^{\mathbf{x}^{\mathbf{T}}\mathbf{x}/2} f_{\ln\left(\chi^2_{m(n-1)-(1-1)}\right)}(u - x_2 - \ldots - x_p)$$

$$\times \cdots \times f_{\ln\left(\chi^2_{m(n-1)-(2-1)}\right)}(x_2) \cdots f_{\ln\left(\chi^2_{m(n-1)-(p-1)}\right)}(x_p),$$

where

$$\mathbf{x} = [x_2, \ldots, x_p]^{\mathbf{T}}.$$

The probability density function $f_{U_0}(u)$ describing the distribution of U_0 can be expressed as

$$f_{U_0}(u) = (2\pi)^{-(p-1)/2} \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} e^{-\mathbf{x}^{\mathbf{T}}\mathbf{x}/2} h_u(\mathbf{x})\, dx_2 \cdots dx_p.$$

It is easy to see that $f_{U_0}(u)$ is in the form of $I(f)$. The FORTRAN code to evaluate $I(f)$, available from Dr. Alan Genz, Department of Mathematics, Washington State University, Pullman, WA. can be used to evaluate $f_{U_0}(u)$.

Under the independent multivariate normal model, it is easy to see that

$$
\begin{aligned}
\mu_{\ln(|\bar{\mathbf{s}}_0|)} &= \ln\left(\frac{|\boldsymbol{\Sigma}_0|}{m^p (n-1)^p}\right) + \sum_{i=1}^p \mu_{\ln\left(\chi^2_{m(n)-(i-1)}\right)} \\
&= \ln\left(\frac{|\boldsymbol{\Sigma}_0|}{m^p (n-1)^p}\right) \\
&\quad + \sum_{i=1}^p \int_{-\infty}^\infty r \frac{1}{\Gamma\left(\frac{m(n)-(i-1)}{2}\right) 2^{(m(n)-(i-1))/2}} e^{-(e^r - (m(n)-(i-1))r)/2} dr \text{ and} \\
\sigma^2_{\ln(|\bar{\mathbf{s}}_0|)} &= \sum_{i=1}^p \sigma^2_{\ln\left(\chi^2_{m(n)-(i-1)}\right)} \\
&= \sum_{i=1}^p \left(\int_{-\infty}^\infty \frac{r^2 e^{-(e^r - (m(n)-(i-1))r)/2}}{\Gamma\left(\frac{m(n)-(i-1)}{2}\right) 2^{(m(n)-(i-1))/2}} dr - \mu^2_{\ln(|\bar{\mathbf{s}}_0|)} \right).
\end{aligned}
$$

It follows from these results that

$$
\begin{aligned}
\mu_{U_0} &= \sum_{i=1}^p \mu_{\ln\left(\chi^2_{m(n)-(i-1)}\right)} \text{ and} \\
\sigma^2_{U_0} &= \sum_{i=1}^p \sum_{i=1}^p \sigma^2_{\ln\left(\chi^2_{m(n)-(i-1)}\right)}.
\end{aligned}
$$

The distribution of U_0 can be expressed in terms of the distribution of W_0. Observe that the cumulative distribution function $F_{U_0}(u)$, where $U_0 = \ln(W_0)$ is given by

$$
F_{U_0}(u) = P(\ln(W_0) \leq u) = P(W_0 \leq e^u) = F_{W_0}(e^u).
$$

Hence, the probability density function $f_{U_0}(u)$ describing the distribution of U_0 can be expressed as

$$
f_{U_0}(u) = e^u f_{W_0}(e^u).
$$

Also, note that setting $m = 1$ yields the distribution of U. Using the results of [17], we have derived a closed form expression for the probability density function $f_{|\mathbf{S}|}(w)$ describing the distribution of $|\mathbf{S}|$ under the independent multivariate normal model.

Their expression for $f_{|\mathbf{S}|}(w)$ is

$$f_{U_0}(u) = e^u \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{m^p (n-1)^p |\mathbf{\Sigma}^{-1}| (m^p (n-1)^p |\mathbf{\Sigma}^{-1}| e^u)^{1/2} \prod_{i=1}^{p-1} w_i^{-1/2}}{\prod_{i=1}^p \Gamma\left(\frac{m(n-1)-(i-1)}{2}\right) 2^{p(2m(n-1)-p-3)/4}}$$

$$\times e^{-w_1/2} e^{-\left(\sum_{i=2}^{p-1}(w_i/w_{i-1}) + \left(m^p(n-1)^p|\mathbf{\Sigma}^{-1}|e^u/w_{p-1}\right)\right)/2} dw_1 \ldots dw_{p-1}.$$

Using the Meijer G function, we have

$$f_{U_0}(u) = \frac{e^u}{2^p} \left(\prod_{j=1}^p \frac{1}{\Gamma\left(\frac{m(n-1)-(j-1)}{2}\right)} \right)$$

$$\times G_{0,p}^{p\,0} \left(\frac{e^u}{2^p} \left| \frac{m(n-1)-1}{2}, \ldots, \frac{m(n-1)-p}{2} \right. \right) I_{(0,\infty)}(w).$$

We can write cumulative distribution function $F_{W_0}(w)$ in terms of the cumulative distribution function of U_0 as

$$F_{W_0}(w) = P\left(e^{U_0} \leq w\right) = F_{U_0}(\ln(w)).$$

Hence, their probability density functions are related by

$$f_{W_0}(w) = w^{-1} f_{U_0}(\ln(w)).$$

Using the results of [9], we have

$$f_{W_0}(w) = \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty w^{-1} f_{\ln\left(\chi^2_{m(n-1)-(1-1)}\right)} (\ln(w) - x_2 - \ldots - x_p)$$

$$\times f_{\ln\left(\chi^2_{m(n-1)-(2-1)}\right)} (x_2) \cdots f_{\ln\left(\chi^2_{m(n-1)-(p-1)}\right)} (x_p) \, dx_2 \cdots dx_p.$$

3.5 Conclusion

Some useful distributional results were given in this chapter for evaluating the run length distribution of the EWMA charts based on a various functions of the sample generalized variance. In particular, we discussed two methods for determining the distribution of $|\mathbf{S}|$, $\ln(|\mathbf{S}|)$, $|\overline{\mathbf{S}}_0|$, and $\ln\left(|\overline{\mathbf{S}}_0|\right)$ as well as functions of these statistics. The means and variances of these statistics were also given.

CHAPTER 4

EWMA $|\mathbf{S}|$ AND $\ln(|\mathbf{S}|)$ CHARTS

4.1 Introduction

[27]introduced the exponentially weighted moving average chart (EWMA) which he referred to as the geometric moving average chart. [11] introduced a family of control charting procedures of which the EWMA chart is a member. The EWMA chart was originally developed to monitor for a change in a process mean of a univariate quality measurement. The general form of an EWMA chart based on the statistic Y_t is a plot of the statistic E_t versus the sample size t, where

$$E_0 = \mu_{Y|\text{PIC}} \text{ and } E_t = (1 - r_t) E_{t-1} + r_t Y_t$$

with $0 < r_t \leq 1$ for $t = 1, 2, 3, \ldots$. Here, PIC stands for the "process is in-control." The value of Y_t is determined from the sample data $\mathbf{X}_{t,1}, \ldots, \mathbf{X}_{t,n}$. The chart signals a possible out-of-control process at time t if

$$E_t \leq h_t^- \text{ or } E_t \geq h_t^+.$$

The chart parameter r_t is called the smoothing parameter, h_t^- the lower control limit, and h_t^+ the upper control limit. Note that the smoothing parameter and control limits may depend on the sample number t as well as other parameters. If the value of $\mu_{Y|\text{PIC}}$ is not known, then an estimate Y_0 obtained from a Phase I analysis can be used.

If the chart parameters, the sample size, and/or the sampling interval are changed based on the information found in the sample data, the chart is referred to as an adaptive control chart. Several authors have discussed adaptive charts that change the sample size and sampling intervals. See [19] for a discuss of design issues for adaptive control charts. However, an adaptive chart that changes the value of the

smoothing parameter of an EWMA chart has not been studied. In what follows, we will only consider non-adaptive EWMA charts. Further, we only consider charts in which the smoothing parameter is fixed, that is, $r_t = r$.

One can show that expectation and variance of E_t are, respectively,

$$(1-r)^t \mu_{Y|\text{PIC}} + \sum\nolimits_{i=1}^{t} (1-r)^{t-i} r\mu_{Y_i} \text{ and } \sum\nolimits_{i=1}^{t} (1-r)^{2(t-i)} r^2 \sigma_{Y_i}^2.$$

If $\mu_{Y_i} = \mu_Y$ and $\sigma_{Y_i}^2 = \sigma_Y^2$ for all i, then the expectation and variance of E_t are, respectively,

$$(1-r)^t \mu_{Y|\text{PIC}} + \left(1 - (1-r)^t\right) \mu_Y \text{ and } \frac{1 - (1-r)^{2t}}{2 - r} \sigma_Y^2.$$

Further, it the process is in-control, then expectation of E_t is $\mu_{Y|\text{PIC}}$. Often the control limits for an EWMA chart are expressed in terms of the expectation and variance of E_t as

$$h_t^- = \mu_{Y|\text{PIC}} - k^- \sqrt{\frac{1 - (1-r)^{2t}}{2 - r} \sigma_Y^2} \text{ and}$$

$$h_t^+ = \mu_{Y|\text{PIC}} + k^+ \sqrt{\frac{1 - (1-r)^{2t}}{2 - r} \sigma_Y^2}.$$

It is common in the literature to use LCL_t and UCL_t for h_t^- and h_t^+, respectively. Note that the chart parameters k^- and k^+ do not depend on the value of t.

In the next section, we discuss EWMA charts in which $Y_t = |\mathbf{S}_t|$, the sample generalized variance. In Section 3, the EWMA charts for which $Y_t = \ln(|\mathbf{S}_t|)$.

4.2 EWMA |S| Chart

The EWMA $|\mathbf{S}|$ chart is an EWMA chart in which Y_t is selected to be the sample generalized variance $|\mathbf{S}_t|$ of the tth sample taken in Phase II. Under the independent multivariate normal model with $\mathbf{\Sigma}_0$ or $|\mathbf{\Sigma}_0|$ known, the chart is a plot of the statistic

E_t versus the sample number t, where

$$E_0 = \mu_{|\mathbf{S}||\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0} \text{ and } E_t = (1-r)E_{t-1} + r|\mathbf{S}_t|.$$

The control limits can be expressed as

$$h_t^- = \mu_{|\mathbf{S}||\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0} - k^- \sqrt{\frac{1-(1-r)^{2t}}{2-r}} \sigma_{|\mathbf{S}||\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0} \text{ and}$$

$$h_t^+ = \mu_{|\mathbf{S}||\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0} + k^+ \sqrt{\frac{1-(1-r)^{2t}}{2-r}} \sigma_{|\mathbf{S}||\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0},$$

where

$$\mu_{|\mathbf{S}||\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0} = \frac{|\boldsymbol{\Sigma}_0|}{(n-1)^p} \prod_{i=1}^{p}(n-i) \text{ and}$$

$$\sigma_{|\mathbf{S}||\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0}^2 = \frac{|\boldsymbol{\Sigma}_0|^2}{(n-1)^{2p}} \left(\prod_{i=1}^{p}(n-i) \right)$$
$$\times \left(\prod_{i=1}^{p}(2+n-i) - \prod_{i=1}^{p}(n-i) \right).$$

For the case in which $\boldsymbol{\Sigma}_0$ is estimated by $\overline{\mathbf{S}}_0$ obtained from the data collected in a Phase I study that is believed to be from an in-control process, we have obtain an estimate parameter version of the chart by substituting the unbiased estimator

$$\frac{m^p(n-1)^p |\overline{\mathbf{S}}_0|}{\prod_{i=1}^{p}(m(n-1)-(i-1))}$$

in place of the parameter $|\boldsymbol{\Sigma}_0|$. If follows that E_0 is

$$E_0 = \frac{m^p \prod_{i=1}^{p}(n-i)}{\prod_{i=1}^{p}(m(n-1)-(i-1))} |\overline{\mathbf{S}}_0|$$

and the control limits are

$$h_t^- = A - k^- B_t \text{ and } h_t^+ = A + k^+ B_t,$$

where

$$A = \frac{m^p |\overline{\mathbf{S}}_0|}{\prod_{i=1}^{p}(m(n-1)-(i-1))} \prod_{i=1}^{p}(n-i) \text{ and}$$

$$B_t = \sqrt{\frac{1-(1-r)^{2t}}{2-r}} \frac{m^p |\overline{\mathbf{S}}_0|}{\prod_{i=1}^{p}(m(n-1)-(i-1))} \sqrt{\prod_{i=1}^{p}(n-i)}$$
$$\times \sqrt{\prod_{i=1}^{p}(2+n-i) - \prod_{i=1}^{p}(n-i)}.$$

An equivalent form of the chart in the parameters known case can be obtained by defining the sequence of statistics E_t^* by

$$E_t^* = \left| (n-1) \, \mathbf{\Sigma}_0^{-1} \right| E_t.$$

We see that

$$E_t^* = \left| (n-1) \, \mathbf{\Sigma}_0^{-1} \right| \left((1-r) \, E_{t-1} + r \, |\mathbf{S}_t| \right)$$

$$= (1-r) \left| (n-1) \, \mathbf{\Sigma}_0^{-1} \right| E_{t-1} + r \left| (n-1) \, \mathbf{\Sigma}_0^{-1} \mathbf{S}_t \right|$$

$$= (1-r) \, E_{t-1}^* + r \left| (n-1) \, \mathbf{\Sigma}_0^{-1} \mathbf{S}_t \right|$$

with

$$E_0^* = \prod_{i=1}^{p} (n-i).$$

The control limits for this form of the chart are

$$h_t^{-*} = \left| (n-1) \, \mathbf{\Sigma}_0^{-1} \right| h^-$$

$$= \prod_{i=1}^{p} (n-i) - k^- \sqrt{\frac{1 - (1-r)^{2t}}{2-r}} \sqrt{\prod_{i=1}^{p} (n-i)}$$

$$\times \sqrt{\prod_{i=1}^{p} (2+n-i) - \prod_{i=1}^{p} (n-i)} \text{ and}$$

$$h_t^{+*} = \left| (n-1) \, \mathbf{\Sigma}_0^{-1} \right| h^+$$

$$= \prod_{i=1}^{p} (n-i) + k^+ \sqrt{\frac{1 - (1-r)^{2t}}{2-r}} \sqrt{\prod_{i=1}^{p} (n-i)}$$

$$\times \sqrt{\prod_{i=1}^{p} (2+n-i) - \prod_{i=1}^{p} (n-i)}.$$

Note that the chart parameters k^- and k^+ are the same for both forms of the chart. Substituting the unbiased estimator

$$\frac{\left| \overline{\mathbf{S}}_0 \right|}{\prod_{i=1}^{p} (m(n-1) - (i-1))}$$

in place of the parameter $|\mathbf{\Sigma}_0|$, we obtain the estimated parameters version of the chart. The sequence of EWMA statistics E_t^* and the control limits for this chart

become

$$E_t^* = (1 - r) E_{t-1}^* + r \frac{\prod_{i=1}^p \left(m \left(n - 1 \right) - \left(i - 1 \right) \right)}{m^p \left(n - 1 \right)^p} \left| (n - 1) \, \overline{\mathbf{S}}_0^{-1} \mathbf{S}_t \right|$$

with the same initial value E_0^* and control limits h_t^{-*} and h_t^{+*} as the known parameters version of the chart.

The control limits h_t^- and h_t^+ (h_t^{-*} and h_t^{+*}) are known as variable control limits. We observe that

$$h^- = \lim_{t \to \infty} h_t^- = \mu_{|\mathbf{S}| \mid \mathbf{\Sigma} = \mathbf{\Sigma}_0} - k^- \sqrt{\frac{1}{2 - r}} \sigma_{|\mathbf{S}| \mid \mathbf{\Sigma} = \mathbf{\Sigma}_0} \text{ and}$$

$$h^+ = \lim_{t \to \infty} h_t^+ = \mu_{|\mathbf{S}| \mid \mathbf{\Sigma} = \mathbf{\Sigma}_0} + k^+ \sqrt{\frac{1}{2 - r}} \sigma_{|\mathbf{S}| \mid \mathbf{\Sigma} = \mathbf{\Sigma}_0} .$$

Similarly, we have

$$h^{-*} = \lim_{t \to \infty} h_t^{-*} = \mu_{\left| (n-1) \mathbf{\Sigma}_0^{-1} \mathbf{S} \right| \mid \mathbf{\Sigma} = \mathbf{\Sigma}_0} - k^- \sqrt{\frac{1}{2 - r}} \sigma_{\left| (n-1) \mathbf{\Sigma}_0^{-1} \mathbf{S} \right| \mid \mathbf{\Sigma} = \mathbf{\Sigma}_0} \text{ and}$$

$$h^{+*} = \lim_{t \to \infty} h_t^{+*} = \mu_{\left| (n-1) \mathbf{\Sigma}_0^{-1} \mathbf{S} \right| \mid \mathbf{\Sigma} = \mathbf{\Sigma}_0} + k^+ \sqrt{\frac{1}{2 - r}} \sigma_{\left| (n-1) \mathbf{\Sigma}_0^{-1} \mathbf{S} \right| \mid \mathbf{\Sigma} = \mathbf{\Sigma}_0} .$$

The control limits h^- and h^+ (h^{-*} and h^{+*}) are known as asymptotic or fixed control limits. The fixed control limits formulas for the estimated parameters version of the chart can be easily determined by modifying the arguments for determining variable control limits.

4.3 EWMA $\ln \left(|\mathbf{S}| \right)$ Charts

[3] studied the EWMA $\ln(|\mathbf{S}|)$ under the independent multivariate normal model with known covariance matrix $\mathbf{\Sigma}_0$. The EWMA $\ln(|\mathbf{S}|)$ is a plot of the EWMA statistic E_t, where

$$E_0 = \mu_{\ln(|\mathbf{S}|) \mid \mathbf{\Sigma} = \mathbf{\Sigma}_0} \text{ and } E_t = (1 - r) E_{t-1} + r \ln \left(|\mathbf{S}_t| \right) .$$

The control limits can be expressed as

$$h_t^- = \mu_{\ln(|\mathbf{S}|)|\mathbf{\Sigma}=\mathbf{\Sigma}_0} - k^- \sqrt{\frac{1-(1-r)^{2t}}{2-r}} \sigma_{\ln(|\mathbf{S}|)|\mathbf{\Sigma}=\mathbf{\Sigma}_0} \text{ and}$$

$$h_t^+ = \mu_{\ln(|\mathbf{S}|)|\mathbf{\Sigma}=\mathbf{\Sigma}_0} + k^+ \sqrt{\frac{1-(1-r)^{2t}}{2-r}} \sigma_{\ln(|\mathbf{S}|)|\mathbf{\Sigma}=\mathbf{\Sigma}_0} .$$

Recall that

$$\mu_{\ln(|\mathbf{S}|)|\mathbf{\Sigma}=\mathbf{\Sigma}_0} = \ln\left(\frac{|\mathbf{\Sigma}_0|}{(n-1)^p}\right) + \sum_{i=1}^{p} \mu_{\ln(\chi^2_{n-i})} \text{ and}$$

$$\sigma^2_{\ln(|\mathbf{S}|)|\mathbf{\Sigma}=\mathbf{\Sigma}_0} = \sigma^2_{\ln(|\mathbf{S}|)} = \sum_{i=1}^{p} \sigma^2_{\ln(\chi^2_{n-i})}.$$

The estimated parameters version of this chart is obtained from the parameter known version by replacing $|\mathbf{\Sigma}_0|$ with its unbiased estimator

$$\frac{m^p (n-1)^p |\overline{\mathbf{S}}_0|}{\prod_{i=1}^{p} (m(n-1)-(i-1))}.$$

It follows that initial value and control limits for the estimated parameters version are

$$E_0 = \ln\left(\frac{m^p |\overline{\mathbf{S}}_0|}{\prod_{i=1}^{p} (m(n-1)-(i-1))}\right) + \sum_{i=1}^{p} \mu_{\ln(\chi^2_{n-i})};$$

$$h_t^- = \ln\left(\frac{m^p |\overline{\mathbf{S}}_0|}{\prod_{i=1}^{p} (m(n-1)-(i-1))}\right) + \sum_{i=1}^{p} \mu_{\ln(\chi^2_{n-i})}$$
$$-k^- \sqrt{\frac{1-(1-r)^{2t}}{2-r}} \sqrt{\sum_{i=1}^{p} \sigma^2_{\ln(\chi^2_{n-i})}}; \text{ and}$$

$$h_t^- = \ln\left(\frac{m^p |\overline{\mathbf{S}}_0|}{\prod_{i=1}^{p} (m(n-1)-(i-1))}\right) + \sum_{i=1}^{p} \mu_{\ln(\chi^2_{n-i})}$$
$$+k^+ \sqrt{\frac{1-(1-r)^{2t}}{2-r}} \sqrt{\sum_{i=1}^{p} \sigma^2_{\ln(\chi^2_{n-i})}}.$$

For the case in which $p = 1$, the EWMA statistic becomes $\ln(S_t^2)$.

An equivalent form of the parameters known chart is an EWMA chart that plots the sequence of EWMA statistic E_t^* versus the sample number t, where

$$E_t^* = \ln\left(\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\right|\right) + E_t.$$

We see that

$$
\begin{aligned}
E_t^* &= \ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\right|\right) + (1-r)\,E_{t-1} + r\ln\left(|\mathbf{S}_t|\right) \\
&= (1-r)\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\right|\right) + (1-r)\,E_{t-1} \\
&\quad + r\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\right|\right) + r\ln\left(|\mathbf{S}_t|\right) \\
&= (1-r)\left(\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\right|\right) + E_{t-1}\right) \\
&\quad + r\left(\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\right|\right) + \ln\left(|\mathbf{S}_t|\right)\right) \\
&= (1-r)\,E_{t-1}^* + r\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}_t\right|\right).
\end{aligned}
$$

The control limits are

$$
h_t^{-*} = \mu_{\ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|\right)|\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0} - k^-\sqrt{\frac{1-(1-r)^{2t}}{2-r}}\,\sigma_{\ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|\right)|\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0} \quad \text{and}
$$

$$
h_t^{+*} = \mu_{\ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|\right)|\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0} + k^+\sqrt{\frac{1-(1-r)^{2t}}{2-r}}\,\sigma_{\ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|\right)|\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0}.
$$

Recall that

$$
\mu_U = \sum_{i=1}^p \mu_{\ln\left(\chi_{n-i}^2\right)} \quad \text{and} \quad \sigma_U^2 = \sum_{i=1}^p \sigma_{\ln\left(\chi_{n-i}^2\right)}^2.
$$

It follows that

$$
\mu_U = \mu_{\ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|\right)|\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0} \quad \text{and} \quad \sigma_{\ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|\right)|\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_0}^2 = \sigma_U^2.
$$

Hence our initial value E_0^* and control limits can be expressed as

$$
\begin{aligned}
E_0^* &= \ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\right|\right) + \ln\left(\frac{|\boldsymbol{\Sigma}_0|}{(n-1)^p}\right) + \sum_{i=1}^p \mu_{\ln\left(\chi_{n-i}^2\right)} \\
&= \sum_{i=1}^p \mu_{\ln\left(\chi_{n-i}^2\right)}; \\
h_t^{-*} &= \sum_{i=1}^p \mu_{\ln\left(\chi_{n-i}^2\right)} - k^-\sqrt{\frac{1-(1-r)^{2t}}{2-r}}\sqrt{\sum_{i=1}^p \sigma_{\ln\left(\chi_{n-i}^2\right)}^2}; \quad \text{and} \\
h_t^{+*} &= \sum_{i=1}^p \mu_{\ln\left(\chi_{n-i}^2\right)} + k^+\sqrt{\frac{1-(1-r)^{2t}}{2-r}}\sqrt{\sum_{i=1}^p \sigma_{\ln\left(\chi_{n-i}^2\right)}^2}.
\end{aligned}
$$

Note that the chart parameters k^- and k^+ are not effected by the transformation. We refer to this chart as the EWMA $\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|\right)$ chart. As with the EWMA

$\ln(|\mathbf{S}|)$, the estimated parameters version of the EWMA $\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|\right)$ chart is obtained from the parameter known version by replacing the parameter $|\boldsymbol{\Sigma}_0|$ with its unbiased estimator

$$\frac{m^p\,(n-1)^p\,\left|\overline{\mathbf{S}}_0\right|}{\prod_{i=1}^{p}\left(m\,(n-1)-(i-1)\right)}.$$

The initial value and the control limits for the estimated parameters version of the chart are of the same form as the parameters known case. That is, the initial values and control limits of this chart are E_0^*, h_t^{-*}, and h_t^{+*}. However, the plotted statistic E_t^* has the form

$$E_t^* = (1-r)\,E_{t-1}^* + r\ln\left(\frac{\prod_{i=1}^{p}\left(m\,(n-1)-(i-1)\right)}{m^p\,(n-1)^p}\left|(n-1)\,\overline{\mathbf{S}}_0^{-1}\mathbf{S}_t\right|\right).$$

For the case in which $p=1$, the EWMA statistic becomes $\ln\left((n-1)\,S_t^2/\sigma_0^2\right)$.

4.4   Conclusion

The parameters know and the estimated parameters versions of EWMA charts based on the sample generalized variance have been described.

CHAPTER 5

EVALUATING THE RUN LENGTH PERFORMANCE

5.1 Introduction

The run length of a chart is define to be the number T of the sample at which the chart first signals. The run length distribution is a discrete distribution whose support is the positive integers. In this section, we consider the case in which the control limits do not depend on t. It is convenient in the parameters known case to let

$$pr\,(t\,|z\,) = P\,(T = t\,|E_0 = z)\,,$$

where $h^- < z < h^-$. In the parameters estimated case, we will also condition on the value of $Y_0 = y_0$. In this case, we let

$$pr\,(t\,|z, y_0\,) = P\,(T = t\,|E_0 = z, Y_0 = y_0)\,.$$

When deriving expression describing various parameters of the run length distribution, we will use this expression since in general corresponding expressions for the parameters known case are special cases.

Using the results found in [32], one can show that the "tail" probabilities of the run length distribution can be approximated by a geometric distribution. There exist a value λ such that

$$pr\,(t^* + t\,|z, y_0\,) \approx \lambda^t pr\,(t^*\,|z, y_0\,)$$

with $0 < \lambda < 1$. An approximation to this approximation can be done by approximating λ by

$$\widehat{\lambda}_1 = \frac{pr\,(t^*\,|z, y_0\,)}{pr\,(t^* - 1\,|z, y_0\,)} \text{ and } \widehat{\lambda}_2 = \frac{1 - \sum_{t=1}^{t^*} pr\,(t\,|z, y_0\,)}{1 - \sum_{t=1}^{t^*-1} pr\,(t\,|z, y_0\,)}.$$

He suggested that $\widehat{\lambda}_2$ may in general provide a better estimate of λ than $\widehat{\lambda}_1$.

The parameters of the run length distribution that are commonly of interest to practitioners are the mean, standard deviation, and various percentiles. The mean is commonly referred to as the average run length (ARL). It can be expressed as

$$\mu_T = ARL = \sum_{t=1}^{\infty} t \cdot pr\left(t \,|z, y_0\right).$$

Using the approximation given in [32], we have

$$
\begin{aligned}
\mu_T &= \sum_{t=1}^{t^*} t \cdot pr\left(t \,|z, y_0\right) + \sum_{t=t^*+1}^{\infty} t \cdot pr\left(t \,|z, y_0\right) \\
&= \sum_{t=1}^{t^*} t \cdot pr\left(t \,|z, y_0\right) + \sum_{t=1}^{\infty} \left(t^* + t\right) \cdot pr\left(t^* + t \,|z, y_0\right) \\
&\approx \sum_{t=1}^{t^*} t \cdot pr\left(t \,|z, y_0\right) + \sum_{t=1}^{\infty} \left(t^* + t\right) \cdot \lambda^t pr\left(t^* \,|z, y_0\right) \\
&= \sum_{t=1}^{t^*} t \cdot pr\left(t \,|z, y_0\right) + pr\left(t^* \,|z, y_0\right) \left(t^* \sum_{t=1}^{\infty} \lambda^t + \sum_{t=1}^{\infty} t \cdot \lambda^t\right).
\end{aligned}
$$

It is easy to show that

$$\sum_{t=1}^{\infty} \lambda^t = \frac{\lambda}{1 - \lambda} \text{ and } \sum_{t=1}^{\infty} t \cdot \lambda^t = \frac{\lambda}{(1 - \lambda)^2}.$$

Thus, we have

$$
\begin{aligned}
\mu_T &\approx \sum_{t=1}^{t^*} t \cdot pr\left(t \,|z, y_0\right) + \lambda pr\left(t^* \,|z, y_0\right) \left(\frac{t^*}{1 - \lambda} + \frac{1}{(1 - \lambda)^2}\right) \\
&= \sum_{t=1}^{t^*} t \cdot pr\left(t \,|z, y_0\right) + pr\left(t^* + 1 \,|z, y_0\right) \left(\frac{t^*}{1 - \lambda} + \frac{1}{(1 - \lambda)^2}\right).
\end{aligned}
$$

Similarly, one can show that

$$\mu_{T^2} \approx \sum_{t=1}^{t^*} t \cdot pr\left(t \,|z, y_0\right) + pr\left(t^* + 1 \,|z, y_0\right) \left(\frac{(t^*)^2}{1 - \lambda} + \frac{2t^* - 1}{(1 - \lambda)^2} + \frac{2}{(1 - \lambda)^3}\right).$$

These are the results given in [32]. The variance σ_T^2 of the run length distribution can be approximated by replacing the approximations of μ_T and μ_{T^2} in the following expression.

$$\sigma_T^2 = \mu_{T^2} - \mu_T^2.$$

The standard deviation σ_T will be referred to as $SDRL$.

The $100\left(1-\alpha\right)$th percentile T_α is the solution to the system of inequalities

$$P\left(T \le T_\alpha\right) \ge 1 - \alpha \text{ and } P\left(T < T_\alpha\right) < 1 - \alpha.$$

If $T_\alpha \le t^*$, it is easy to obtain its value by a simple search. If $T_\alpha > t^*$, then using the approximation in [32] an approximation for T_α is

$$T_\alpha \approx t^* - 1 + \frac{\ln\left(\left(1 - \lambda\right)\left(\sum_{t=1}^{t^*} pr\left(t \,|\, z, y_0\right) - \left(1 - \alpha\right)\right) / pr\left(t^* \,|\, z, y_0\right) + \lambda\right)}{\ln\left(\lambda\right)}.$$

This expression is similar to the one given in [32].

The most commonly used methods for evaluating the run length distribution are simulation, integral equations, and Markov chains. In the next section, we discuss the simulation methods. This is followed by a section on in which exact expressions are given for the run length distribution using integral equations. In the third section, we discuss how the chart can be approximated by a Markov chain. In Section 5, we show that the integral equation and Markov chain methods are equivalent.

5.2   Simulation

We recall that the parameters known version of the EWMA $\left|\left(n - 1\right)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|$ is define by the sequence

$$E_0^* = \prod_{i=1}^{p}\left(n - i\right) \text{ and } E_t^* = \left(1 - r\right)E_{t-1}^* + r\left|\left(n - 1\right)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}_t\right|$$

and the control limits

$$h_t^{-*} = \prod_{i=1}^{p} (n - i) - k^- \sqrt{\frac{1 - (1 - r)^{2t}}{2 - r}} \sqrt{\prod_{i=1}^{p} (n - i)}$$

$$\times \sqrt{\prod_{i=1}^{p} (2 + n - i) - \prod_{i=1}^{p} (n - i)} \text{ and }$$

$$h_t^{+*} = \prod_{i=1}^{p} (n - i) + k^+ \sqrt{\frac{1 - (1 - r)^{2t}}{2 - r}} \sqrt{\prod_{i=1}^{p} (n - i)}$$

$$\times \sqrt{\prod_{i=1}^{p} (2 + n - i) - \prod_{i=1}^{p} (n - i)}.$$

Note that E_0^*, h_t^{-*}, and h_t^{+*} do not depend on the values of $|\boldsymbol{\Sigma}_0|$. Further, recall that

$$\begin{aligned}\left|(n - 1) \boldsymbol{\Sigma}_0^{-1} \mathbf{S}_t\right| &= \left|\boldsymbol{\Sigma}_0^{-1} \boldsymbol{\Sigma}\right| \left|(n - 1) \boldsymbol{\Sigma}^{-1} \mathbf{S}_t\right| \\ &= \lambda^2 W_t = \lambda^2 \prod_{i=1}^{p} \left(\chi_{n-i}^2\right)_t,\end{aligned}$$

where the Chi Square random variables $\left(\chi_{n-1}^2\right)_t, \ldots, \left(\chi_{n-p}^2\right)_t$ are stochastically independent. We now see that

$$E_t^* = (1 - r) E_{t-1}^* + r \left(\lambda^2 \prod_{i=1}^{p} \left(\chi_{n-i}^2\right)_t\right).$$

To simulate the sequence of EWMA statistics E_t^* one only needs to simulate the p independent Chi Square random variates $\left(\chi_{n-1}^2\right)_t, \ldots, \left(\chi_{n-p}^2\right)_t$ at each time t. The first time T the chart signals is a simulated run length.

The value of Y_t for the estimated parameters version of this chart is

$$Y_t = \left(\frac{\prod_{i=1}^{p} (m (n - 1) - (i - 1))}{m^p (n - 1)^p}\right) \left|(n - 1) \overline{\mathbf{S}}_0^{-1} \mathbf{S}_t\right|.$$

This can be expressed as

$$\begin{aligned}Y_t &= \left(\frac{\prod_{i=1}^{p} (m (n - 1) - (i - 1))}{m^p (n - 1)^p}\right) \left|\boldsymbol{\Sigma}_0^{-1} \boldsymbol{\Sigma}\right| \left|m (n - 1) \boldsymbol{\Sigma}_0^{-1} \overline{\mathbf{S}}_0\right|^{-1} \left|(n - 1) \boldsymbol{\Sigma}^{-1} \mathbf{S}_t\right| \\ &= \left(\frac{\prod_{i=1}^{p} (m (n - 1) - (i - 1))}{m^p (n - 1)^p}\right) \lambda^2 W_0^{-1} W_t.\end{aligned}$$

Recall that

$$W_0 = \prod_{i=1}^{p} \left(\chi_{m(n-1)-(i-1)}^2\right)_0.$$

To simulate this charting procedure under the independent multivariate normal model, first the value of W_0 is obtained as the product of the simulated values of the independent Chi Square random variables $\left(\chi^2_{m(n-1)-(1-1)}\right)_0, \ldots, \left(\chi^2_{m(n-1)-(p-1)}\right)_0$. Then at each time t the value of W_t is the product of the simulated values of p independent Chi Square random variables $\left(\chi^2_{n-1}\right)_t, \ldots, \left(\chi^2_{n-p}\right)_t$. The EWMA statistic is then up-dated as

$$E_t^* = (1-r) E_{t-1}^* + r \left(\frac{\prod_{i=1}^p (m(n-1) - (i-1))}{m^p (n-1)^p} \right) \lambda^2 W_0^{-1} W_t.$$

The first time T the chart signals is a simulated run length.

In the known parameters versions of the EWMA $\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|\right)$ chart, we can express the sequence of EWMA statistics E_t^* as

$$E_0^* = \sum_{i=1}^p \mu_{\ln\left(\chi^2_{n-i}\right)} \text{ and } E_t^* = (1-r) E_{t-1}^* + r \left(\theta + \sum_{i=1}^p \ln\left(\chi^2_{n-i}\right)_t \right),$$

where $\theta = \ln\left(\lambda^2\right)$ and $\left(\chi^2_{n-1}\right)_t, \ldots, \left(\chi^2_{n-9}\right)_t$ are stochastically independent.

5.3 Integral Equation Method

[10] show how integral equations can be derived that are useful in evaluating the run length distribution of the chart. First, we consider the EWMA chart in which the support of the distribution of Y_t is the reals. It is useful to consider the EWMA chart in which the initial value is some arbitrary value z between the fixed control limits h^- and h^+. Thus, the sequence of EWMA statistics are defined by

$$E_0 = z \text{ and } E_t = (1-r) E_{t-1} + rY_t$$

with the chart signalling a potential out-of-control process if

$$E_t \leq h^- \text{ or } E_t \geq h^+.$$

For convenience, we let

$$pr\left(t\,|z\right) = P\left(T = t\,|E_0 = z\right).$$

For the case in which $t = 1$, we have

$$
\begin{aligned}
pr\left(1\,|z\right) &= P\left(T = 1\,|E_0 = z\right) \\
&= P\left((1-r)\,z + rY_1 \le h^-\right) + P\left((1-r)\,z + rY_1 \ge h^+\right) \\
&= F_Y\left(\frac{h^- - (1-r)\,z}{r}\right) + 1 - F_Y\left(\frac{h^+ - (1-r)\,z}{r}\right).
\end{aligned}
$$

For the case in which $t > 1$,

$$pr\left(t\,|z\right) = P\left(T = t\,|E_0 = z\right) = P\left(T = t, h^- < E_1 < h^+\,|E_0 = z\right).$$

This holds since $t > 1$ which implies that the chart did not signal at time $t = 1$.
Thus, the event

$$\left\{-\infty < E_1 < \infty\right\} = \left\{h^- < E_1 < h^+\right\}.$$

Hence,

$$
\begin{aligned}
P\left(T = t\,|E_0 = z\right) &= P\left(T = t, -\infty < E_1 < \infty\,|E_0 = z\right) \\
&= P\left(T = t, h^- < E_1 < h^+\,|E_0 = z\right).
\end{aligned}
$$

We can now write

$$
\begin{aligned}
pr\left(t\,|z\right) &= P\left(T = t\,|E_0 = z, h^- < E_1 < h^+\right) \\
&\quad \times P\left(h^- < E_1 < h^+\,|E_0 = z\right) \\
&= \int_{h^-}^{h^+} P\left(T = t\,|E_0 = z, E_1 = e_1\right) f_{E_1|E_0}\left(e_1\,|z\right) de_1.
\end{aligned}
$$

Observe that

$$P\left(T = t\,|E_0 = z, E_1 = e_1\right) = P\left(T - 1 = t - 1\,|E_0 = z, E_1 = e_1\right).$$

Note that the remaining run length $T - 1$ given $E_1 = e_1$ has the same distribution as T given $E_0 = e_1$. Hence, we have

$$P\left(T = t \,|E_0 = z, E_1 = e_1\right) = P\left(T = t - 1 \,|E_0 = e_1\right) = pr\left(t - 1 \,|e_1\right).$$

We now have

$$pr\left(t \,|z\right) = \int_{h^-}^{h^+} pr\left(t - 1 \,|e_1\right) f_{E_1|E_0}\left(e_1 \,|z\right) de_1.$$

The cumulative distribution function $F_{E_1|E_0}\left(e_1 \,|z\right)$ of E_1 given $E_0 = z$ is

$$
\begin{aligned}
F_{E_1|E_0}\left(e_1 \,|z\right) &= P\left(E_1 \le e_1 \,|E_0 = z\right) \\
&= P\left((1 - r) z + r Y_1 \le e_1 \,|E_0 = z\right) \\
&= F_Y\left(\frac{e_1 - (1 - r) z}{r}\right).
\end{aligned}
$$

Thus, the probability density function $f_{E_1|E_0}\left(e_1 \,|z\right)$ can be expressed in terms of the probability density function of Y as

$$f_{E_1|E_0}\left(e_1 \,|z\right) = r^{-1} f_Y\left(\frac{e_1 - (1 - r) z}{r}\right).$$

It then follows that

$$pr\left(t \,|z\right) = \int_{h^-}^{h^+} pr\left(t - 1 \,|e_1\right) r^{-1} f_Y\left(\frac{e_1 - (1 - r) z}{r}\right) de_1.$$

While the aforementioned sequence of integral equations have the run length distribution as their exact solution, an exact numerical solution cannot be obtained exactly. However, an approximate numerical solution can be obtained. This done by using what is commonly referred to as numerical integration.

Making the transformation

$$e_1 = h^- + \frac{h^+ - h^-}{2}\left(v + 1\right) \text{ with } de_1 = \frac{h^+ - h^-}{2} dv,$$

we have

$$pr\left(t \,|z\right) = \int_{-1}^{1} pr\left(t - 1 \,|e_1\right) r^{-1} f_Y\left(\frac{e_1 - (1 - r) z}{r}\right) \frac{h^+ - h^-}{2} dv.$$

Note that we did not substitute the expression for e_1 in terms of v. The nodes $v_1, \ldots, v_\eta$ and associated weights $\omega_1, \ldots, \omega_\eta$ of a Gaussian quadrature using Legurre polynomials can be used to approximate the integral in the previous equation. We now have

$$pr\left(t \,|z\right) \approx \sum_{j=1}^{\eta} pr\left(t - 1 \,|e_{1j}\right) r^{-1} f_Y \left(\frac{e_{1j} - (1 - r)\,z}{r}\right) \frac{h^+ - h^-}{2} \omega_j,$$

where

$$e_{1j} = h^- + \frac{h^+ - h^-}{2}\left(v_j + 1\right).$$

For the $t > 1$, we have

$$\mathbf{p}_t = \begin{bmatrix} pr\left(t \,|e_{11}\right) \\ pr\left(t \,|e_{12}\right) \\ \vdots \\ pr\left(t \,|e_{1\eta}\right) \end{bmatrix} \approx \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1\eta} \\ q_{21} & q_{22} & \cdots & q_{2\eta} \\ \vdots & \vdots & \ddots & \vdots \\ q_{\eta 1} & q_{\eta 2} & \cdots & q_{\eta\eta} \end{bmatrix} \begin{bmatrix} pr\left(t - 1 \,|e_{11}\right) \\ pr\left(t - 1 \,|e_{12}\right) \\ \vdots \\ pr\left(t - 1 \,|e_{1\eta}\right) \end{bmatrix}$$

$$= \mathbf{Q}\mathbf{p}_{t-1},$$

where

$$q_{ij} = r^{-1} f_Y \left(\frac{e_{1j} - (1 - r)\,e_{1i}}{r}\right) \frac{h^+ - h^-}{2} \omega_j \text{ and}$$

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1\eta} \\ q_{21} & q_{22} & \cdots & q_{2\eta} \\ \vdots & \vdots & \ddots & \vdots \\ q_{\eta 1} & q_{\eta 2} & \cdots & q_{\eta\eta} \end{bmatrix}.$$

for $i, j = 1, \ldots, \eta$. For the case $t = 1$,

$$
\mathbf{p}_1 = \begin{bmatrix} pr\left(1 \,|e_{11}\right) \\ pr\left(1 \,|e_{12}\right) \\ \vdots \\ pr\left(1 \,|e_{1\eta}\right) \end{bmatrix} = \begin{bmatrix} F_Y\left(\frac{h^- - (1-r)e_{11}}{r}\right) + 1 - F_Y\left(\frac{h^+ - (1-r)e_{11}}{r}\right) \\ F_Y\left(\frac{h^- - (1-r)e_{12}}{r}\right) + 1 - F_Y\left(\frac{h^+ - (1-r)e_{12}}{r}\right) \\ \vdots \\ F_Y\left(\frac{h^- - (1-r)e_{1\eta}}{r}\right) + 1 - F_Y\left(\frac{h^+ - (1-r)e_{1\eta}}{r}\right) \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} F_Y\left(\frac{h^- - (1-r)e_{11}}{r}\right) - F_Y\left(\frac{h^+ - (1-r)e_{11}}{r}\right) \\ F_Y\left(\frac{h^- - (1-r)e_{12}}{r}\right) - F_Y\left(\frac{h^+ - (1-r)e_{12}}{r}\right) \\ \vdots \\ F_Y\left(\frac{h^- - (1-r)e_{1\eta}}{r}\right) - F_Y\left(\frac{h^+ - (1-r)e_{1\eta}}{r}\right) \end{bmatrix}.
$$

Observe that

$$
\begin{bmatrix} F_Y\left(\frac{h^- - (1-r)e_{11}}{r}\right) + 1 - F_Y\left(\frac{h^+ - (1-r)e_{11}}{r}\right) \\ F_Y\left(\frac{h^- - (1-r)e_{12}}{r}\right) + 1 - F_Y\left(\frac{h^+ - (1-r)e_{12}}{r}\right) \\ \vdots \\ F_Y\left(\frac{h^- - (1-r)e_{1\eta}}{r}\right) + 1 - F_Y\left(\frac{h^+ - (1-r)e_{1\eta}}{r}\right) \end{bmatrix} = \begin{bmatrix} 1 - \int_{h^-}^{h^+} r^{-1} f_Y\left(\frac{e_1 - (1-r)e_{11}}{r}\right) de_1 \\ 1 - \int_{h^-}^{h^+} r^{-1} f_Y\left(\frac{e_1 - (1-r)e_{12}}{r}\right) de_1 \\ \vdots \\ 1 - \int_{h^-}^{h^+} r^{-1} f_Y\left(\frac{e_1 - (1-r)e_{1\eta}}{r}\right) de_1 \end{bmatrix}
$$

Substituting

$$
\sum_{j=1}^{\eta} r^{-1} f_Y\left(\frac{e_{1j} - (1-r)\,e_{1i}}{r}\right) \omega_j \text{ for } \int_{h^-}^{h^+} r^{-1} f_Y\left(\frac{e_1 - (1-r)\,e_{1j}}{r}\right) de_1,
$$

we have

$$
\mathbf{p}_1 \approx \begin{bmatrix} 1 - \sum_{j=1}^{\eta} r^{-1} f_Y\left(\frac{e_{1j} - (1-r)e_{11}}{r}\right) \omega_j \\ 1 - \sum_{j=1}^{\eta} r^{-1} f_Y\left(\frac{e_{1j} - (1-r)e_{12}}{r}\right) \omega_j \\ \vdots \\ 1 - \sum_{j=1}^{\eta} r^{-1} f_Y\left(\frac{e_{1j} - (1-r)e_{1\eta}}{r}\right) \omega_j \end{bmatrix}
$$

$$
= (\mathbf{I} - \mathbf{Q})\,\mathbf{1}.
$$

Hence, the run length distribution can be expressed (approximately) as

$$
\mathbf{p}_1 \approx (\mathbf{I} - \mathbf{Q})\,\mathbf{1} \text{ and } \mathbf{p}_t \approx \mathbf{Q}^{t-1}(\mathbf{I} - \mathbf{Q})\,\mathbf{1}.
$$

$$\mathbf{p}_1 \approx (\mathbf{I} - \mathbf{Q}) \, \mathbf{1} \text{ and } \mathbf{p}_t \approx \mathbf{Q}^{t-1} (\mathbf{I} - \mathbf{Q}) \, \mathbf{1}.$$

For the EWMA chart based on the statistic

$$Y_t = \ln\left(\left|(n-1) \, \mathbf{\Sigma}_0^{-1} \mathbf{S}_t\right|\right),$$

then

$$f_Y\left(\frac{e_1 - (1-r) \, z}{r}\right) = f_U\left(\frac{e_1 - (1-r) \, z - r\theta}{r}\right).$$

If EWMA statistic Y_t is

$$Y_t = \ln\left(\left|(n-1) \, \overline{\mathbf{S}}_0^{-1} \mathbf{S}_t\right|\right),$$

then we must consider the conditional distribution of Y_t given $U_0 = u_0$. In this case,

we are interested in replacing

$$f_Y\left(\frac{e_1 - (1-r) \, z}{r}\right)$$

with

$$f_{Y|U_0}\left(\frac{e_1 - (1-r) \, z}{r} \,\middle|\, u_0\right) = f_U\left(\frac{e_1 - (1-r) \, z - r\left(\theta + \ln\left(m^p \, (n-1)^p\right) - u_0\right)}{r}\right).$$

It follows that $pr\left(t \,|\, z\right)$ is a function $pr\left(t \,|\, z, u_0\right)$ of u_0. We then have

$$
\begin{aligned}
pr\left(t \,|\, z, u_0\right) &= \int_{h^-}^{h^+} pr\left(t - 1 \,|\, e_1, u_0\right) r^{-1} \\
&\quad \times f_U\left(\frac{e_1 - (1-r) \, z - r\left(\theta + \ln\left(m^p \, (n-1)^p\right) - u_0\right)}{r}\right) de_1.
\end{aligned}
$$

Further, we see that in our numerical approximation the matrix $\mathbf{Q}$ is a function of

u_0. Hence, $\mathbf{p}_t$ is a function $\mathbf{p}_t\left(u_0\right)$ of u_0. The unconditional run length distribution

is then obtained as

$$\mathbf{p}_t = \int_{-\infty}^{\infty} \mathbf{p}_t\left(u_0\right) f_{U_0}\left(u_0\right) du_0.$$

5.4   Markov Chain Approximation

[4] developed a Markov chain approximation for the one-sided cumulative sum (CUSUM) chart introduced by [25]. [21] introduced a fast-initial response feature to the CUSUM chart. This method was used by [22] to evaluate the run length performance of the EWMA chart with and without a fast-initial response used to monitor the mean of a univariate quality measurement. We discuss the Markov chain approximation for a more general version of the EWMA chart in this section.

Recall that the sequence of EWMA statistics based on the statistic Y_t are defined by

$$E_0 = \widehat{\mu}_{Y|\text{PIC}} \text{ and } E_t = (1 - r) E_{t-1} + rY_t$$

and the chart signals at time t if

$$E_t \leq h^- \text{ or } E_t \geq h^+.$$

To obtain a general Markov chain approximation for this EWMA chart, we first divide the interval (h^-, h^+) into the η subintervals

$$(b_0, b_1], (b_1, b_2], \ldots, (b_{\eta-2}, b_{\eta-1}], (b_{\eta-1}, b_\eta)$$

with $b_0 = h^- < b_1 < b_2 < \ldots < b_{\eta-1} < b_\eta = h^+$. We see that

$$\left(h^-, h^+\right) = (b_0, b_1] \cup (b_1, b_2] \cup \ldots \cup (b_{\eta-2}, b_{\eta-1}] \cup (b_{\eta-1}, b_\eta)$$

Next we select the points $a_1, a_2, \ldots, a_\eta$ such that

$$b_0 < a_1 < b_1 < a_2 < b_2 < \ldots < b_{\eta-1} < a_\eta < b_\eta.$$

The non-absorbing states of our Markov chain are the values $a_1, a_2, \ldots, a_\eta$. The transition probabilities from the non-absorbing state $E_t = a_i$ to the non-absorbing

state $E_{t+1} = a_j$ is

$$p_{ij} = P\left(b_{j-1} < E_{t+1} < b_j \,|\, E_t = a_i\right).$$

If we let A be the absorbing state, then $p_{Aj} = 0$ for $j = 1, \ldots, \eta$ and $p_{AA} = 1$. Also, we have

$$p_{iA} = 1 - \sum_{j=1}^{\eta} p_{ij}.$$

We then define the matrix $\mathbf{P}$ by

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1\eta} & p_{1A} \\ p_{21} & p_{2} & \cdots & p_{2\eta} & p_{2A} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{\eta 1} & p_{\eta 2} & \cdots & p_{\eta\eta} & p_{\eta A} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^{(\eta+1)\times(\eta+1)}.$$

Note that the (i, j)th coordinate of $\mathbf{P}^t$ is

$$p_{ij}^{(t)} = \begin{cases} P\left(b_{j-1} < E_t < b_j \,|\, E_0 = a_i\right), & \text{for } i, j = 1, 2, \ldots, \eta \text{ and } j = A; \\ p_{Aj}, & \text{for } j = 1, 2, \ldots, \eta, A. \end{cases}$$

It is useful to define the matrix $\mathbf{Q}$ by

$$\mathbf{Q} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1\eta} \\ p_{21} & p_{2} & \cdots & p_{2\eta} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\eta 1} & p_{\eta 2} & \cdots & p_{\eta\eta} \end{bmatrix}^{\eta\times\eta}.$$

The matrix $\mathbf{Q}$ is the submatrix of $\mathbf{P}$ obtained by removing the last row and column of $\mathbf{P}$. For the Markov chain to transition from the non-absorbing state i to the absorbing state A for the first time at time t, then the EWMA chart must signal for the first time at time t. This happens at time $t = 1$ with probability p_{iA}. The values $p_{1A}, \ldots, p_{\eta A}$

can be expressed in matrix form as

$$
\begin{bmatrix} p_{1A} \\ p_{2A} \\ \vdots \\ p_{\eta A} \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1\eta} \\ p_{21} & p_{2} & \cdots & p_{2\eta} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\eta 1} & p_{\eta 2} & \cdots & p_{\eta\eta} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}
$$

$$
= \ (\mathbf{I} - \mathbf{Q})\,\mathbf{1},
$$

where $\mathbf{1}$ is an $\eta \times 1$ vector of ones. For the Markov chain to enter the absorbing state for the first time at time t, then it must transition from a non-absorbing state to a non-absorbing state for the first $t-1$ transitions and then transition to the absorbing state. Thus, we have in matrix form

$$
\mathbf{p}_t \ = \ \begin{bmatrix} p_{1A}^{(t)} \\ p_{2A}^{(t)} \\ \vdots \\ p_{\eta A}^{(t)} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1\eta} \\ p_{21} & p_{2} & \cdots & p_{2\eta} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\eta 1} & p_{\eta 2} & \cdots & p_{\eta\eta} \end{bmatrix}^{t-1} \begin{bmatrix} p_{1A} \\ p_{2A} \\ \vdots \\ p_{\eta A} \end{bmatrix}
$$

$$
= \ \mathbf{Q}^{t-1}\,(\mathbf{I} - \mathbf{Q})\,\mathbf{1}.
$$

It follows that

$$
\begin{bmatrix} P\left(T = t \,|\, E_0 = a_1\right) \\ P\left(T = t \,|\, E_0 = a_2\right) \\ \vdots \\ P\left(T = t \,|\, E_0 = a_\eta\right) \end{bmatrix} = \mathbf{Q}^{t-1}\,(\mathbf{I} - \mathbf{Q})\,\mathbf{1}.
$$

If $\widehat{\mu}_{Y|\mathrm{PIC}} = a_k$, then

$$
P\left(T = t \,\big|\, E_0 = \widehat{\mu}_{Y|\mathrm{PIC}}\right) = P\left(T = t \,|\, E_0 = a_k\right).
$$

If $a_{k-1} < \widehat{\mu}_{Y|\mathrm{PIC}} < a_k$, then we can interpolate to obtain an approximations for

$P\left(T=t \,\middle|\, E_0=\widehat{\mu}_{Y|\text{PIC}}\right)$. Using linearly interpolation, we have

$$
\begin{aligned}
P\left(T=t \,\middle|\, E_0=\widehat{\mu}_{Y|\text{PIC}}\right) \;=\;& P\left(T=t \,\middle|\, E_0=a_{k-1}\right)+\frac{\widehat{\mu}_{Y|\text{PIC}}-a_{k-1}}{a_k-a_{k-1}} \\
& \times\left(P\left(T=t \,\middle|\, E_0=a_k\right)-P\left(T=t \,\middle|\, E_0=a_{k-1}\right)\right).
\end{aligned}
$$

The vector of average run lengths can now be determined by

$$
\mu_{T|\mathbf{a}}=\begin{bmatrix} \mu_{T|E_0=a_1} \\ \mu_{T|E_0=a_2} \\ \vdots \\ \mu_{T|E_0=a_\eta} \end{bmatrix}=\sum_{t=1}^{\infty} t\mathbf{Q}^{t-1}\left(\mathbf{I}-\mathbf{Q}\right)\mathbf{1}=\left(\mathbf{I}-\mathbf{Q}\right)^{-1}\mathbf{1}.
$$

Further, we have

$$
\begin{aligned}
\mu_{T^2|\mathbf{a}}\;=\;& \begin{bmatrix} \mu_{T^2|E_0=a_1} \\ \mu_{T^2|E_0=a_2} \\ \vdots \\ \mu_{T^2|E_0=a_\eta} \end{bmatrix}=\sum_{t=1}^{\infty} t^2\mathbf{Q}^{t-1}\left(\mathbf{I}-\mathbf{Q}\right)\mathbf{1} \\
\;=\;& \left(\mathbf{I}+2\mathbf{Q}\left(\mathbf{I}-\mathbf{Q}\right)^{-1}\right)\left(\mathbf{I}-\mathbf{Q}\right)^{-2}\left(\mathbf{I}-\mathbf{Q}\right)\mathbf{1} \\
\;=\;& \left(\mathbf{I}+2\mathbf{Q}\left(\mathbf{I}-\mathbf{Q}\right)^{-1}\right)\left(\mathbf{I}-\mathbf{Q}\right)^{-1}\mathbf{1}.
\end{aligned}
$$

It follows that

$$
\sigma_{T|\mathbf{a}}^2=\begin{bmatrix} \sigma_{T|E_0=a_1}^2 \\ \sigma_{T|E_0=a_2}^2 \\ \vdots \\ \sigma_{T|E_0=a_\eta}^2 \end{bmatrix}=\begin{bmatrix} \mu_{T^2|E_0=a_1} \\ \mu_{T^2|E_0=a_2} \\ \vdots \\ \mu_{T^2|E_0=a_\eta} \end{bmatrix}-\begin{bmatrix} \mu_{T|E_0=a_1}^2 \\ \mu_{T|E_0=a_2}^2 \\ \vdots \\ \mu_{T|E_0=a_\eta}^2 \end{bmatrix}.
$$

We can use the method given in [32] to obtain geometric tail approximations to the run length distribution. This allows us to use the Markov chain approximation to approximate the average run length (ARL), standard deviation of the run length $(SDRL)$, and percentage points of the run length distribution.

This method of obtaining the Markov chain approximation to the EWMA chart is a generalization of the one used by [22]. They state that the properties of an EWMA control scheme can be approximated using a procedure similar to that described by [4]. Although they suggested discretizing the control statistic and then evaluating the exact properties of the discretized statistic, we evaluate the properties of the continuous state Markov chain by discretizing the infinite-state transition probability matrix. Their method divides the interval between the lower and upper control into $2m + 1$ equal width intervals of length 2δ. It follows that

$$\delta = \frac{(h^+ - h^-)/2}{2m + 1}.$$

Setting $\eta = 2m + 1$,

$$b_j = h^- + 2j\delta \text{ and } a_j = h^- + (2j - 1)\delta$$

for $j = 1, \ldots, 2m + 1$. Note that for $j = 0$, $b_0 = h^-$.

5.5   Some Numerical Results and Comparisons

In the known parameters case, suppose a practitioner is interested in using the EWMA chart that plots the statistic E_t versus t with smoothing parameter r and control limits h_t^- and h_t^+, where

$$E_0 = \sum_{i=1}^{p} \mu_{\ln\left(\chi_{n-i}^2\right)} \text{ and } E_t = (1 - r)E_{t-1} + r\ln\left(\left|(n-1)\mathbf{\Sigma}_0^{-1}\mathbf{S}_t\right|\right)$$

and

$$h_t^- = \sum_{i=1}^{p} \mu_{\ln\left(\chi_{n-i}^2\right)} - k^- \sqrt{\frac{1 - (1 - r)^{2t}}{2 - r}}\sqrt{\sum_{i=1}^{p} \sigma_{\ln\left(\chi_{n-i}^2\right)}^2}$$

and

$$h_t^+ = \sum_{i=1}^{p} \mu_{\ln\left(\chi_{n-i}^2\right)} + k^+ \sqrt{\frac{1 - (1 - r)^{2t}}{2 - r}}\sqrt{\sum_{i=1}^{p} \sigma_{\ln\left(\chi_{n-i}^2\right)}^2}.$$

Recall that

$$\ln\left(\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}_t\right|\right) \sim \theta + \sum_{i=1}^{p} \ln\left(\chi_{n-i}^2\right),$$

where

$$\lambda^2 = \left|\mathbf{\Sigma}_0^{-1}\mathbf{\Sigma}\right| \ \text{and} \ \theta = \ln\left(\lambda^2\right).$$

In order to use the EWMA chart, the practitioner first selects the values of the chart parameters r, k^-, and k^+. Using simulation, values of $k = k^- = k^+$ were determined for various values of r so that the chart has an in-control average run length of 100. These results are presented in table 5.1.

p = 2, n = 10, $\theta = 0$		p = 3, n = 10, $\theta = 0$	
r	k	r	k
0.1	2.20	0.1	2.20
0.2	2.35	0.2	2.40
0.3	2.50	0.3	2.45
0.4	2.55	0.4	2.50
0.5	2.55	0.5	2.55
0.6	2.60	0.6	2.60
0.7	2.60	0.7	2.60
0.8	2.65	0.8	2.60
0.9	2.65	0.9	2.60

Table 5.1: A chosen (r,k) combination that yields ARL=100.

Similar results could be obtained using the integral equation and Markov chain methods.

It is our interest to compare the three methods, simulation, integral equation, and Markov chain methods in determining the run length distribution. In table 5.2, some average run lengths (*ARLs*) were determined using the three methods. The

ARLs using simulation, integral equation, and Markov chain methods are denoted, respectively, by ARL_{SIM}, ARL_{IntEq}, and ARL_{MC}.

p = 2, n = 10, r = 0.5, k = 2.6				
λ^2	θ	ARL_{SIM}	$ARL_{INT.EQ}$	ARL_{MC}
0.6	-0.5	16.72	17.04	17.05
0.8	-0.22	48.18	48.06	48.01
1	0	109.29	107.24	106.9
1.2	0.18	92.24	94.47	94.12
1.4	0.34	45.25	46.11	46.03
p = 3, n = 10, r = 0.5, k = 2.55				
λ^2	θ	ARL_{SIM}	$ARL_{INT.EQ}$	ARL_{MC}
0.6	-0.5	23.31	23.38	23.29
0.8	-0.22	54.78	55.63	55.10
1	0	99.14	98.8	96.25
1.2	0.18	90.19	91.7	88.09
1.4	0.34	57.15	56.36	54.43

Table 5.2: Comparison of the methods in determining the run length distribution.

We see that the three methods given very similar results.

5.6   Equivalence of the Integral Equation and Markov Chain Methods

[8] showed that the integral equation and Markov chain methods for evaluating the run length distribution for a cumulative sum (CUSUM) and EWMA charts are equivalent provided the support of the statistic whose CUSUM/EWMA values being plotted is the reals. This equivalence is based on the Markov chain approximation of the CUSUM chart by [4] and the Markov chain approximation of the EWMA chart by

[22].

As we have seen, the run length distribution of the EWMA chart can be expressed at the exact solution to a sequence of iteratively obtained integral equations. Under a given model, it is not possible to obtain a numerical solution. However, using numerical methods one can obtain an approximate solution. On the other hand, the Markov chain method begins with a Markov chain approximation of the chart. The run length distribution of this approximate charting procedure can be obtained numerically and this becomes the approximate run length distribution of the EWMA chart. For each method of obtaining a numerical solution to the integral equation representation of the run length there is a Markov chain approximation that yields the same approximate run length distribution. In what follows, we will show that the integral equation method when a particular numerical approximation is used will result in the same approximate run length distribution of the EWMA chart when the [22] method based on [4] method is used to approximate the run length distribution. Further, we show how one can select a Markov chain approximation to the chart that will provide a more accurate approximation to the run length distribution. We consider only the case in which the support of the statistic whose EWMA values are being plotted is the reals.

As we have seen

$$
\begin{aligned}
pr\left(t \,|z\right) \;\; &= \;\; \int_{h^-}^{h^+} pr\left(t-1 \,|e_1\right) r^{-1} f_Y\left(\frac{e_1 - (1-r)\,z}{r}\right) de_1 \\
&= \;\; \sum_{j=1}^{\eta} \int_{b_{j-1}}^{b_j} pr\left(t-1 \,|e_1\right) r^{-1} f_Y\left(\frac{e_1 - (1-r)\,z}{r}\right) de_1.
\end{aligned}
$$

By the Weighted Mean Value Theorem for Integrals ([5]), there exists a real number

α_j such that

$$\int_{b_{j-1}}^{b_j} pr\,(t-1\,|e_1)\,r^{-1} f_Y\left(\frac{e_1-(1-r)\,z}{r}\right) de_1$$

$$= pr\,(t-1\,|\alpha_j) \int_{b_{j-1}}^{b_j} r^{-1} f_Y\left(\frac{e_1-(1-r)\,z}{r}\right) de_1.$$

We have then that

$$pr\,(t\,|z) = \sum_{j=1}^{\eta} pr\,(t-1\,|\alpha_j) \int_{b_{j-1}}^{b_j} r^{-1} f_Y\left(\frac{e_1-(1-r)\,z}{r}\right) de_1$$

$$= \sum_{j=1}^{\eta} pr\,(t-1\,|\alpha_j) \left(F_Y\left(\frac{b_j-(1-r)\,z}{r}\right) - F_Y\left(\frac{b_{j-1}-(1-r)\,z}{r}\right)\right).$$

Approximating α_j with a_j, we have

$$pr\,(t\,|z) \approx \sum_{j=1}^{\eta} pr\,(t-1\,|a_j) \left(F_Y\left(\frac{b_j-(1-r)\,z}{r}\right) - F_Y\left(\frac{b_{j-1}-(1-r)\,z}{r}\right)\right).$$

In particular, we have

$$pr\,(t\,|a_i) \approx \sum_{j=1}^{\eta} pr\,(t-1\,|a_j) \left(F_Y\left(\frac{b_j-(1-r)\,a_i}{r}\right) - F_Y\left(\frac{b_{j-1}-(1-r)\,a_i}{r}\right)\right).$$

This yields the same results as the Markov chain method for approximating the run length distribution.

In a previous section on the integral equation method, the integral

$$\int_{h^-}^{h^+} pr\,(t-1\,|e_1)\,r^{-1} f_Y\left(\frac{e_1-(1-r)\,z}{r}\right) de_1$$

was approximated using a Gaussian quadrature method. This resulted a set of quadrature points and a matrix $\mathbf{Q}$. The Markov chain in which this set of quadrature points are the states and the matrix $\mathbf{Q}$ the transition matrix can be used as a Markov chain approximation of the chart. Clearly, this Markov chain approximation and the associated integral equation method lead to the same approximation of the run length distribution. Hence, the two methods are equivalent.

5.7 Steady-State Run Length Distribution

The run length distribution that as been presented is known as the initial state run length distribution since it depends on the initial value of the EWMA chart. If the initial is random, then the run length distribution is known as the steady-state run length distribution. [13] suggested a technique for approximating the steady-state distribution with a cyclic steady-state distribution.[6] applied his method to obtain the steady-state run length distribution of a Shewhart chart supplemented with runs rules.

The method suggested by [13] for determining an approximate cyclic steady-state probability vector, $\mathbf{p}_{ss}$, is to replace the transition matrix $\mathbf{P}$ with the matrix $\mathbf{P}^*$ given as

$$\mathbf{P}^* = \begin{pmatrix} \mathbf{Q} & (\mathbf{I} - \mathbf{Q})\,\mathbf{1} \\ 1, 0, \ldots, 0 & 0 \end{pmatrix}$$

with the transition probabilities determined under the assumption the process is in-control. The vector $\mathbf{p}_{ss}$ is the probability distribution of the initial-state of the Markov chain. The matrix $\mathbf{P}^*$ is referred to as the ergodic transition matrix. [22] method of determining the value of $\mathbf{p}_{ss}$ is a two stage procedure. First a vector $\mathbf{p}$ is determined by solving the equation $\mathbf{p} = \mathbf{P}^{*\mathbf{T}}\mathbf{p}$ subject to $\mathbf{1}^{\mathbf{T}}\mathbf{p} = 1$. Then the steady-state probability vector is found by

$$\mathbf{p}_{ss} = \left(\mathbf{1}^{\mathbf{T}}\mathbf{q}\right)^{-1}\mathbf{q},$$

where $\mathbf{q}$ is determined from $\mathbf{p}$ by eliminating the component associated with the absorbing state. [6] showed that $\mathbf{q}$ could be simply determined by

$$\mathbf{q} = \left(\mathbf{G} - \mathbf{Q}^{\mathbf{T}}\right)^{-1}\mathbf{u},$$

where

$$\mathbf{G} = \begin{bmatrix} 2 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \text{ and } \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The steady-state probability distribution $\mathbf{p}_{ss,t}$ of the run length is

$$p_{ss,t} = \mathbf{p}_{ss}^{\mathbf{T}} \mathbf{p}_t$$

for $t = 1, 2, 3, \dots$. The steady-state average run length of the chart is then given by

$$\mu_{T|\mathbf{a}} = \mathbf{p}_{ss}^{\mathbf{T}} \mu_{T|\mathbf{a}}.$$

Other parameters of the steady-state run length distribution can be determined in a similar way.

5.8   Conclusion

Methods for evaluating the run length distribution of the various EWMA charts based on the sample generalized variance chart were examined. This included both fixed and steady-state run length distributions. Although simulation, integral equation, and a Markov approximation are the most commonly used methods for studying properties of the run length distribution, these are not the only methods. [28] look at the sequence of EWMA statistics E_t as an autoregressive process of order 1. From this view, an Edgewood expansion was used to approximate the probability density and cumulative density function of the process. By summing an approximated survival function and truncating the sum at a finite state to obtain the average run length of the chart. Their method does not have the easy of use as the integral equation method nor does it provide more accurate results.

CHAPTER 6

CONCLUSION

6.1   General Conclusions

The exponentially weighted moving average (EWMA) chart based on the sample generalized variance was studied under the independent multivariate normal model for the vectors of quality measurements $\mathbf{X}$. In Chapter 2, the model and sampling methods for this study were discussed. In Chapter 3 some distributional results were presented. In particular, two methods for determining numerically probability density and cumulative density functions of the sample generalized variance were examined. Two EWMA charts used to monitor for a change in the process generalized variances were outlined. One is based on the sample generalized variance and the other on the natural logrithm of the sample generalized variance. The estimated parameters verison of these two charting procedures were introduced.

Evaluating the performance of a chart is typically done by examining the run length distribution. Three methods, simulation, integral equations, and Markov chain approximation, used to determine the run length distribution of the chart were studied. The equivalence of the integral equation and Markov chain methods was shown. Finally, some examples of the implementation of these methods using MATLAB were given.

6.2   Areas for Further Research

Various control charting procedures for monitoring for a change in the process generalized variance have been proposed in the literature. We are interested in comparing the EWMA chart based on the sample generalized variance with these charts both in the parameters known and estimated cases.

As previously mentioned, it maybe more reasonable to assume that the vectors of quality measurements are autocorrelated. One question that arises is how well does an EWMA based on the sample generalized variance designed under the assumption the vectors of quality measurements are stochastically independent perform when in fact the vectors of quality measurements are autocorrelated. Autocorrelated models for multivariate measurement of the Box-Jenkin type could be used for this study.

The EWMA charts discussed in this thesis are based on the sample generalized variance. We are interested in studying a multivariate version of the EWMA chart that plots the statistic $|\mathbf{E}_t|$ versus t, where the matrix $\mathbf{E}_t$ is defined by

$$\mathbf{E}_0 = \boldsymbol{\Sigma}_0 \text{ and } \mathbf{E}_t = (1 - r)\,\mathbf{E}_{t-1} + r\mathbf{S}_t$$

when $\boldsymbol{\Sigma}_0$ is known and $\mathbf{E}_t$ is defined by

$$\mathbf{E}_0 = \overline{\mathbf{S}}_0 \text{ and } \mathbf{E}_t = (1 - r)\,\mathbf{E}_{t-1} + r\mathbf{S}_t$$

in the parameters estimated version of the chart. The chart signals a potential out-of-control process with respect to the process covariance matrix $\boldsymbol{\Sigma}$ if $|\mathbf{E}_t| \leq h^-$ or $|\mathbf{E}_t| \geq h^+$.

We are also interested in the robustness of the charts. We plan a study that includes examining actual process data to determine how robust are model is to these data.

It was suggested that the charts be based on the Frobenius norm of the sample covariance matrix instead of its determinant. The Frobenius norm of the sample covariance matrix $\mathbf{S}$ is

$$||\mathbf{S}||_F = \sqrt{\sum_{i=1}^{p} \sum_{i=1}^{p} S_{ij}^2},$$

where S_{ij} is the (i, j)th component of $\mathbf{S}$. See [16, p. 55]. We are interested in comparing the EWMA charts based on this sample measure with those base on the sample generalized variance.

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