Fall 2018

A Journey to the Adic World

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by

FAYADH KADHEM

(Under the Direction of Jimmy Dillies)

ABSTRACT

The first idea of this research was to study a topic that is related to both Algebra and Topology and explore a tool that connects them together. That was the entrance for me to the adic world. What was needed were some important concepts from Algebra and Topology, and so they are treated in the first two chapters. The reader is assumed to be familiar with Abstract Algebra and Topology, especially with Ring theory and basics of Point-set Topology. The thesis consists of a motivation and four chapters, the third and the fourth being the main ones. In the third chapter, we introduce the p-adic numbers and the f-adic rings, while adic spaces are considered in the last chapter. Adic spaces are relatively new topic in pure mathematics that have seen many developments in the last few years. Their significance enters from the bridging ability that they have to connect Algebra and Geometry, as we can see in the last chapter.

INDEX WORDS: Valuation, p-adic integer, f-adic ring, Huber pair, Presheaf, Adic space.

2009 Mathematics Subject Classification: 13F30, 13J, 14G22

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B.Sc., University of Bahrain, Bahrain, 2016

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial

Fulfillment of the Requirements for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

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Electronic Version Approved:
December 2018

DEDICATION

This thesis is dedicated to my lovely wife Huda Albasri, my father and my mother.

ACKNOWLEDGMENTS

First of all, I would like to stand and thank my great and kind supervisor Dr Jimmy Dillies for everything that he gave me; his effort, his time, his help and of course his knowledge. I had an unforgettable time under his supervision, I wish to him all the best in his life.

Also, I would like to express my appreciation to Dr Ahmed Matar, my undergraduate professor. He still helps me and gives me his useful comments and advice.

My family and my lovely wife Huda Albasri deserve my special thanks as well, they keep pushing me forward and give me everything that I need in order to succeed.

I also want to give special gratitude to my previous math professors and teachers that taught me in previous levels in my career, especially Dr Mohammed Larbi Labbi, Dr Kifah Alhami, T.Jafer Abdulhusain, T.Ahmed Hubail and T.Jafer Matar. I cannot forget their effect and their guidance to make correct choices at very important moments in my life.

TABLE OF CONTENTS

LIST OF SYMBOLS

$\mathbb{C}$	Complex Numbers
$\mathbb{R}$	Real Numbers
$\mathbb{R}^+$	Positive Real Numbers $\{x \in \mathbb{R} : x > 0\}$
$\mathbb{R}^-$	Negative Real Numbers $\{x \in \mathbb{R} : x < 0\}$
$\mathbb{Q}$	Rational Numbers
$\mathbb{Z}$	Integers
$\mathbb{N}$	Natural Numbers $\{0, 1, 2, 3, ...\}$

CHAPTER 1
VALUATIONS AND CATEGORIES

1.1 Motivation

Before we begin, let us try to motivate the ideas behind the concepts and notions that are defined later. Definitions are only useful if they provide a reasonable way to gain results.

We start our journey by defining a valuation to be a map $|\cdot|$ sending a ring A to a totally ordered monoid $\Gamma \cup \{0\}$, satisfying some axioms with the property that two valuations are equivalent if any $a, b \in A$ have the same total order as images of either valuations. Then we define the continuity of a valuation to be analogous to the continuity with respect to the ring topology on A. This guides us to define $\mathrm{Cont}(A)$ to be the set of equivalence classes of continuous valuations of A. We then consider the situation of an f-adic (or Huber) ring A, which is a topological ring with an open subring A_0 that is I-adic for a finitely generated ideal $I \subseteq A_0$, and a Huber (or affinoid) pair (A, A^+) where A is f-adic and A^+ is an integrally closed open subring. We then define the adic spectrum to be subspace $\mathrm{Spa}(A) := \{v \in \mathrm{Cont}(A) : \forall a \in A^+, v(a) \leq 1\}$ of $\mathrm{Cont}(A)$.

The idea behind that is motivated by the fact that the category of adic spaces (amazingly) provides a natural home for many important theories; in particular, this category contains the category of schemes, formal schemes, and their generic fibers (which are realizable as rigid analytic spaces and Berkovich spaces) as full subcategories. In fact, if one allows A to be any topological ring, one would lose the intuition for how adic spaces relate to formal geometry.

In this chapter we describe the notion of valuation ring. We begin by describing ordered groups and valuations and then move on to valuation ring. Valuations are a tool to measure the "size" of objects in a ring,.... Lastly, we give a brief overview to the language of Categories in order to describe more concisely the upcoming results. Our main reference for this chapter is the book by Samuel and Zariski [7].

1.2 Ordered Groups

Definition 1.2.1. A *totally ordered group* $(G, \cdot, \leq)$ is an abelian group with a binary operation $\cdot$ and a total order $\leq$ such that $a \leq b$ implies that $ac \leq bc$ for all $a, b, c \in G$. If there is no fear of ambiguity we may simply write G instead of $(G, \cdot, \leq)$.

Example 1.2.2. The set $(\mathbb{R}^+, \cdot, \leq)$ of positive real numbers with the usual multiplication and total order is a totally ordered group.

Example 1.2.3. Let G be a totally ordered group. Then every subgroup H of G is a totally ordered group. It is called a *totally ordered subgroup.*

Notation. Let G be a totally ordered group. If $g \in G$ and $H \subseteq G$ such that $g \leq h$ for all $h \in H$ then we denote this by writing $g \leq H$.

Definition 1.2.4. Let $G_1, G_2, ..., G_n$ be totally ordered groups, where $n \in \mathbb{Z}^+$. Let $(a_1, a_2, ..., a_n)$, $(b_1, b_2, ..., b_n) \in \prod_{i=1}^n G_i$ and $k = \min\{i \in \{1, ..., n\} : a_i \neq b_i\}$. Define $(a_1, a_2, ..., a_n) < (b_1, b_2, ..., b_n)$ if $a_k < b_k$. This ordering is called the *lexicographic order* (or the *dictionary order*).

Example 1.2.5. Let $G_1, G_2, ..., G_n$ be totally ordered groups, where $n \in \mathbb{Z}^+$. Then $\prod_{i=1}^n G_i$ is a totally ordered group under the lexicographic order.

Example 1.2.6. If $(a_1, a_2, ..., a_n)$, $(b_1, b_2, ..., b_n) \in \prod_{i=1}^n G_i$, then it is not a totally ordered group for the order defined as $(a_1, a_2, ..., a_n) < (b_1, b_2, ..., b_n)$ if and only if $a_i < b_i$ for all $i \in \{1, ..., n\}$.

Definition 1.2.7. Let G_1 and G_2 be two totally ordered groups. A *totally ordered group homomorphism* ϕ between G_1 and G_2 is a group homomorphism such that

$a \leq b$ in G_1 implies that $\phi(a) \leq \phi(b)$ in G_2. If ϕ is a group isomorphism with the same property then ϕ is said to be a *totally ordered group isomorphism.*

Remark. The inverse of a totally ordered group isomorphism is also totally ordered. In fact, for any $\phi(a) \leq \phi(b) \in G_2$ we have $\phi^{-1}[\phi(a)] = a \leq b = \phi^{-1}[\phi(b)]$.

Example 1.2.8. The map $e^x : (\mathbb{R}, +, \leq) \to (\mathbb{R}^+, \cdot, \leq)$ is an isomorphism of totally ordered groups.

Definition 1.2.9. Let G be a totally ordered group. We extend G to a totally ordered monoid $G \cup \{0\}$ by adding a *zero element* 0 and endowing it the following properties: for any $g \in G$, $0 < g$ and $0 \cdot g = g \cdot 0 := 0$.

Remark. If ϕ is a totally ordered group isomorphism between G_1 and G_2, then we may extend it to a *totally ordered monoid isomorphism* by defining $\phi(0) := 0$.

Example 1.2.10. The totally ordered groups $(\mathbb{R}, +, \leq)$ and $(\mathbb{R}^+, \cdot, \leq)$ can be extended to totally ordered monoids by adding $-\infty$ and 0 to be their zero elements respectively. Letting $e^{-\infty} = 0$ induces a totally ordered monoid isomorphism between $(\mathbb{R} \cup \{-\infty\}, +, \leq)$ and $(\mathbb{R}^+ \cup \{0\}, \cdot, \leq)$.

Proposition 1.2.11. *Let G be a totally ordered group with identity 1 and let $a, b \in G$. Then,*

1. $a < 1$ *if and only if* $a^{-1} > 1$.
2. $a, b \leq 1$ *implies that* $ab \leq 1$.
3. $a < 1, b \leq 1$ *implies that* $ab < 1$.
4. $a, b \geq 1$ *implies that* $ab \geq 1$.
5. $a > 1, b \geq 1$ *implies that* $ab > 1$.

6. *The only element of finite order is 1.*

Proof. We prove parts 1,2 and 6. The rest are similar to part 2. For part 1, note that $a < 1$ implies that $a \cdot a^{-1} < 1 \cdot a^{-1}$ by definition, which is equivalent to $1 < a^{-1}$. The converse is similar. For part 2, note that $a \leq 1$ implies that $ab \leq b$, but $b \leq 1$, therefore $ab \leq 1$. Now to prove part 6, without loss of generality assume that $a < 1$ then parts 3 and 5 imply that $a^n < 1$ or $a^n > 1$ for any nonzero integer n. □

Definition 1.2.12. A subgroup H of a totally ordered group G is said to be *convex* if for all $h_1, h_2 \in H$ and $g \in G$ with $h_1 \leq g \leq h_2$, we have $g \in H$.

Proposition 1.2.13. *Let G be a totally ordered group and H a totally ordered subgroup of G. The following are equivalent:*

1. *H is convex.*

2. *For all $h \in H$ and $g \in G$ with $h \leq g \leq 1$, we have $g \in H$.*

3. *If $g, h \leq 1$ and $gh \in H$, then $g, h \in H$.*

Proof. $(1) \Rightarrow (2)$: Any subgroup contains the identity 1. Hence, $g \in H$ by definition.
$(2) \Rightarrow (3)$: Note that $gh \leq g, h \leq 1$. Thus, $g, h \in H$ by (2).
$(3) \Rightarrow (1)$: Let $h_1, h_2 \in H$ and $g \in G$ with $h_1 \leq g \leq h_2$. Multiplying by g^{-1} gives $h_1 g^{-1} \leq 1$. Also, multiplying by ${h_2}^{-1}$ implies that $g{h_2}^{-1} \leq 1$. Now, $h_1 g^{-1}, g{h_2}^{-1} \leq 1$, with $h_1 g^{-1} \cdot g{h_2}^{-1} = h_1 {h_2}^{-1} \in H$. Thus, $g{h_2}^{-1} \in H$, and hence $g{h_2}^{-1} \cdot h_2 = g \in H$. □

Example 1.2.14.

1. Let G be a totally ordered group. Then, $\{1\}$ and G are convex.

2. Let $G_1, ..., G_n$ be totally ordered groups. Then $\prod_{i=k}^{n} G_i$ is a convex subgroup of $\prod_{i=1}^{n} G_i$ under the lexicographic order for any $k \in \{1, ..., n\}$.

Proposition 1.2.15. *Let G be a totally ordered group, H_1 and H_2 be convex subgroups. Then either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.*

Proof. Assume there exist $h_1 \in H_1 - H_2$ and $h_2 \in H_2 - H_1$. After possibly replacing these elements by their inverses we may assume that $h_1, h_2 < 1$. Also, after possibly swapping H_1 with H_2 we may assume that $h_1 < h_2$. But then $h_2 \in H_1$, because H_1 is convex. A contradiction to $h_2 \in H_2 - H_1$.

□

Proposition 1.2.16. *Let $\phi : G \to G'$ be a totally ordered group homomorphism. Then $ker\phi$ is a convex subgroup of G.*

Proof. Let $x, y \in ker\phi$ and $g \in G$ such that $x \leq g \leq y$. Since ϕ is a homomorphism, $\phi(x) \leq \phi(g) \leq \phi(y)$. But $x, y \in ker\phi$. Thus, $1 \leq \phi(g) \leq 1$. Hence, $\phi(g) = 1$ and $g \in ker\phi$. □

Definition 1.2.17. Let G be a totally ordered group. The number of convex subgroups different from $\{1\}$ and G is called the *height* of G and denoted by $ht(G)$.

Example 1.2.18. The only convex subgroups of $(\mathbb{R}^+, \cdot, \leq)$ are $\{1\}$ and $\mathbb{R}^+$. Therefore, the height of $(\mathbb{R}^+, \cdot, \leq)$ is 0.

1.3 Valuation

Definition 1.3.1. Let A be a ring. A *valuation* of A is a map $|\cdot| : A \to \Gamma \cup \{0\}$ where Γ is a totally ordered group, such that

1. $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in A$.

2. $|ab| = |a||b|$ for all $a, b \in A$.

3. $|0| = 0$ and $|1| = 1$.

The subgroup of Γ generated by $\mathrm{Im}(|\cdot|) - \{0\}$ is called the *value group* of $|\cdot|$. It is denoted by $\Gamma_{|\cdot|}$. The set $\mathrm{supp}(|\cdot|) := |\cdot|^{-1}(0)$ is called the *support* of $|\cdot|$.

The second and third conditions show that $|u| \neq 0$ for every $u \in A^{\times}$ and $|u^{-1}| = |u|^{-1}$. Moreover, $|-1||-1| = 1$ shows that $|-1| = 1$ because the only element of finite order in Γ is 1. Consequently, $|-x| = |x|$ for all $x \in A$.

Example 1.3.2. Let A be a ring and $\mathfrak{p}$ be a prime ideal of A. Then, $a \mapsto \begin{cases} 1, & \text{if } a \notin \mathfrak{p} \\ 0, & \text{if } a \in \mathfrak{p} \end{cases}$

is a valuation with value group $\{1\}$ and supp $= \mathfrak{p}$. Every valuation of A of this form is called a *trivial valuation*.

Example 1.3.3. Let $|\cdot|$ be a valuation on a field K and Γ be its value group. Let $A := K[x_1, ..., x_n]$ be the ring of n variables over K, and endow the group $(\mathbb{R}^+)^n \times \Gamma$ with the lexicographic order. Fix some real numbers $0 < s_1, ..., s_n < 1$. Then the map $|\cdot|' : A \to (\mathbb{R}^+)^n \times \Gamma \cup \{0\}$ defined by

$\sum_{\mathbf{i}=(i_1,\ldots,i_n)} a_{\mathbf{i}} {x_1}^{i_1} \cdots {x_n}^{i_n} \mapsto {s_1}^{j_1} \cdots {s_n}^{j_n} |a_{\mathbf{j}}|$, where $\mathbf{j} = \inf\{\mathbf{i} \in \mathbb{Z}^n : a_{\mathbf{i}} \neq 0\}$ is a valuation on A.

Remark. If $\varphi : B \to A$ is a ring homomorphism and $|\cdot|$ is a valuation of A. Then $|\cdot| \circ \varphi$ is a valuation of B.

Proposition 1.3.4. *Let A be a ring and $a, b \in A$ such that $|a| \neq |b|$. Then* $|a + b| = \max\{|a|, |b|\}$.

Proof. Without loss of generality, let $|a| < |b|$. Assume for contrary that $|a + b| < \max\{|a|, |b|\} = |b|$. Then, $|b| = |b + a - a| \leq \max\{|b + a|, |a|\} < |b|$. A contradiction. $\square$

Proposition 1.3.5. *Let $|\cdot|$ be a valuation of A. Then* $\mathrm{supp}(|\cdot|)$ *is a prime ideal of A.*

Proof. Let $ab \in \text{supp}(|\cdot|)$. Then $|a||b| = |ab| = 0$. Thus, we must have $|a| = 0$ or $|b| = 0$. Equivalently, $a \in \text{supp}(|\cdot|)$ or $b \in \text{supp}(|\cdot|)$. $\square$

Example 1.3.6. Consider the planar curve $y = x^2$, which can be represented by

$$C : \{(x, x^2) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2.$$

Define the valuation $|\cdot| : C \to \mathbb{R}$ by $|(x_1, x_1^2) + (x_2, x_2^2)| = \max\{|x_1|, |x_2|\}$. Then $\text{supp}(|\cdot|) = \{0\}$ which is a prime ideal.

Proposition 1.3.7. *Let $|\cdot| : A \to \Gamma \cup \{0\}$ be a valuation and K denote the field of fractions of $A/\text{supp}(|\cdot|)$. Then $|\cdot|' : K \to \Gamma \cup \{0\}$, defined by $\frac{\bar{a}}{\bar{b}} \mapsto |a||b|^{-1}$ is a valuation, where $\bar{a}$ and $\bar{b}$ are the images of a and b respectively in $A/supp(|\cdot|)$.*

The following illustrates the transformations between each algebraic structure:

$$A \to A/supp(|\cdot|) \to K \to \Gamma \cup \{0\}$$
$$a, b \mapsto \bar{a}, \bar{b} \mapsto \frac{\bar{a}}{\bar{b}} \mapsto |a||b|^{-1}$$

Proof. First, we show that $|\cdot|$ is well defined. Let $\frac{\bar{a}}{\bar{b}} = \frac{\bar{a'}}{\bar{b'}}$. If $\bar{a} = 0$, then $\bar{a'} = 0$ and $|a||b|^{-1} = |a'||b'|^{-1} = 0$. Otherwise, we have $\bar{a}\bar{b'} = \bar{a'}\bar{b}$. Which implies that $ab' = a'b + x$ where $x \in supp(|\cdot|)$. Thus, $|ab'| = |a'b + x|$. But, $|a'b + x| = |a'b|$ by Proposition1.2.4, since $|a'b| > 0 = |x|$. Therefore, $|a||b'| = |a'||b|$, or equivalently, $|a||b|^{-1} = |a'||b'|^{-1}$.

Now, let $\frac{\bar{a}}{\bar{b}}, \frac{\bar{c}}{\bar{d}} \in K$. In order to prove that, $|\frac{\bar{a}}{\bar{b}} + \frac{\bar{c}}{\bar{d}}|' \leq \max\{|\frac{\bar{a}}{\bar{b}}|', |\frac{\bar{c}}{\bar{d}}|'\}$, consider the following: $|\frac{\bar{a}}{\bar{b}} + \frac{\bar{c}}{\bar{d}}|' = |\frac{\bar{a}\bar{d} + \bar{c}\bar{b}}{\bar{b}\bar{d}}|' = |\frac{\overline{ad + cb}}{\overline{bd}}|' = |ad + cb||bd|^{-1} = |ad + cb||(bd)^{-1}| = |ad + cb||d^{-1}b^{-1}| = |add^{-1}b^{-1} + cbd^{-1}b^{-1}| = |ab^{-1} + cd^{-1}|$. Since $|\cdot|$ is a valuation, we get that $|\frac{\bar{a}}{\bar{b}} + \frac{\bar{c}}{\bar{d}}|' = |ab^{-1} + cd^{-1}| \leq \max\{|ab^{-1}|, |cd^{-1}|\} = \max\{|\frac{\bar{a}}{\bar{b}}|', |\frac{\bar{c}}{\bar{d}}|'\}$ by the definition of $|\cdot|'$.

Next, $|\frac{\bar{a}}{\bar{b}} \cdot \frac{\bar{c}}{\bar{d}}|' = |\frac{\bar{a}\bar{c}}{\bar{b}\bar{d}}|' = |\frac{\overline{ac}}{\overline{bd}}|' = |ac||bd|^{-1} = |acd^{-1}b^{-1}| = |ab^{-1}||cd^{-1}| = |\frac{\bar{a}}{\bar{b}}|'|\frac{\bar{c}}{\bar{d}}|'$.

Also, $|0|' = |\frac{\bar{0}}{\bar{b}}|' = |0||b|^{-1} = 0$ and $|1|' = |\frac{\bar{b}}{\bar{b}}|' = |b||b|^{-1} = 1$, completes the proof.

$\square$

Remark. In the previous Proposition, we have that $\Gamma_{|\cdot|} = \Gamma_{|\cdot|'}$.

Definition 1.3.8. Two valuations $|\cdot|_1$ and $|\cdot|_2$ on A are said to be *equivalent* if there is a totally ordered group isomorphism $\phi : \Gamma_{|\cdot|_1} \to \Gamma_{|\cdot|_2}$ such that $\phi \circ |\cdot|_1 = |\cdot|_2$.

Example 1.3.9. Let p be a prime number. The p-adic norm $|| \cdot ||_p$ (see definition 3.1.8) is a valuation on $\mathbb{Z}$. Fix an integer $a > 1$ such that $\gcd(a, p) = 1$. Let $n \in \mathbb{Z}$, define $|n|_{p,a}$ as follows:

$$|n|_{p,a} = \begin{cases} \dfrac{1}{a^{v(x)}}, & \text{if } n \neq 0 \\ 0, & \text{if } n = 0 \end{cases}.$$

Then $|| \cdot ||_p$ and $|\cdot|_{p,a}$ are equivalent valuations on $\mathbb{Z}$.

Remark. Often when we have a valuation on A and raise it to a certain positive power, we get equivalent valuations on A.

Proposition 1.3.10. *Let $|\cdot|_1$ and $|\cdot|_2$ be two valuations on A. Then the following are equivalent:*

1. *$|\cdot|_1$ and $|\cdot|_2$ are equivalent.*

2. *For all $a, b \in A$, $|a|_1 \leq |b|_1$ if and only if $|a|_2 \leq |b|_2$.*

Proof. $(\Rightarrow)$: Let $|a|_1 \leq |b|_1$. Then, $\phi(|a|_1) \leq \phi(|b|_1)$, since ϕ is a totally ordered group isomorphism. That is, $|a|_2 \leq |b|_2$ by definition. The converse follows, since ϕ is an isomorphism.

$(\Leftarrow)$: Define $f : \Gamma_{|\cdot|_1} \to \Gamma_{|\cdot|_2}$ by $f(|a|_1) = |a|_2$. We wish to show that f is an isomorphism between totally ordered groups.

- Surjectivity: Let $y \in \Gamma_{|\cdot|_2}$. Then, there exists an $x \in A$ such that $|x|_2 = y$. Thus, $f(|x|_1) = |x|_2 = y$.

- Injectivity: Let $|a|_1 = |b|_1 \in \Gamma_{|\cdot|_1}$ such that $f(|a|_1) = f(|b|_1)$. That is, $|a|_2 = |b|_2$. Hence, $|a|_1 = |b|_1$ by assumption.

- Homomorphism: Let $|a|_1, |b|_1 \in \Gamma_{|\cdot|_1}$. Then, Let $f(|a|_1|b|_1) = f(|ab|_1) = |ab|_2 = |a|_2|b|_2 = f(|a|_1) \cdot f(|b|_1)$. Also, $|a|_1 \leq |b|_1$ implies that $f(|a|_1) = |a|_2 \leq |b|_2 = f(|b|_1)$ by assumption.

□

Remark. The relation "equivalent" in valuations is an equivalence relation.

1.4 Valuation Rings

Definition 1.4.1. Let B be an integral domain and K be its field of fractions. Then B is called a *valuation ring* if for each $x \in K^\times$, either $x \in B$ or $x^{-1} \in B$ (or both).

Proposition 1.4.2. *Let B be a valuation ring and K be its field of fractions. Then,*

1. *B is a local ring.*

2. *If B' is a ring such that $B \subseteq B' \subseteq K$, then B' is a valuation ring of K.*

Proof. 1. Let $\mathfrak{m} := \{x \in B : x = 0 \text{ or } x^{-1} \notin B\}$, the set of non-units in B. To prove that B is local, it is enough to show that $\mathfrak{m}$ is an ideal. If $a \in B$ and $x \in \mathfrak{m}$ we have $ax \in \mathfrak{m}$, otherwise $(ax)^{-1} \in B$ and therefore $x^{-1} = a \cdot (ax)^{-1} \in B$ which contradicts the definition of $\mathfrak{m}$. Next let $x, y \in \mathfrak{m} - \{0\}$. Then either $xy^{-1} \in B$ or $x^{-1}y \in B$, since B is a valuation ring. If $xy^{-1} \in B$ then $x + y = (1 + xy^{-1})y \in B\mathfrak{m} \subseteq \mathfrak{m}$, and similarly if $x^{-1}y \in B$. Hence, $\mathfrak{m}$ is an ideal and therefore B is local.

2. Clear from the definitions.

□

Proposition 1.4.3. *Let B be an integral domain and K be its field of fractions. Then B is a valuation ring if and only if there exists a valuation $|\cdot|$ on K such that $B = \{x \in K : |x| \leq 1\}$.*

Proof. $(\Leftarrow)$: Given any $x \in K$, if $|x| \leq 1$ then $x \in B$ by definition. Otherwise $|x| > 1$, thus $|x^{-1}| < 1$ and $x^{-1} \in B$.

$(\Rightarrow)$: Define $|\cdot| : K \to K^{\times}/B^{\times} \cup \{0\}$, by

$$|x| = \begin{cases} 0, & \text{if } x = 0 \\ [x], & \text{if } x \in K^{\times} \end{cases}$$

where $[x] = x + B^{\times}$ and $[x] \leq [y]$ if $xy^{-1} \in B$. Then $|\cdot|$ is a valuation on K such that $B = \{x \in K : |x| \leq 1\}$. □

Example 1.4.4. Let F be any field. Then F is a valuation ring.

Example 1.4.5. Let $K = F(x)$ where F is a field and $F(x)$ is the field of fractions of $F[x]$. Let B be the set of all rational functions $f/g \in F(x)$ such that $\deg f(x) \leq \deg g(x)$, then B is a valuation ring.

1.5 Categories and Functors

Definition 1.5.1. A category $\mathcal{C}$ consists of a class of objects $\mathrm{ob}(\mathcal{C})$ and a class of *morphisms* (or *arrows*) between objects A and B denoted $\mathrm{Mor}(A, B)$ and a binary operation defined on compatible pairs called *composition* and denoted by $\circ$ in which

1. for any morphisms $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$ there exists a morphism $g \circ f \in \mathrm{Mor}(A, C)$,

2. if $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$ and $h \in \mathrm{Mor}(C, D)$ then $h \circ (g \circ f) = (h \circ g) \circ f$,

3. for any object $A \in \mathrm{ob}(\mathcal{C})$ there exists an identity morphism $id_A \in \mathrm{Mor}(A, A)$ such that $f \in \mathrm{Mor}(B, C)$ implies that $f = id_C \circ f = f \circ id_B$.

The key idea that we do not discuss elements here, but objects.

Example 1.5.2.

1. The class of sets as objects and functions as morphisms with the usual composition is a category of sets.

2. The class of groups as objects and group homomorphisms as morphisms with the usual composition is a category of groups.

3. The class of vector spaces as objects and linear transformations as morphisms with the usual composition is a category of vector spaces.

4. The class of topological spaces as objects and continuous maps as morphisms with the usual composition is a category of topological spaces.

Definition 1.5.3. Let $\mathcal{C}$ be a category such that $A, B, C \in \mathrm{ob}(\mathcal{C})$ and $f \in \mathrm{Mor}(A, B)$ then,

i) f is *monic* (or a *monomorphism*) if for all $g_1, g_2 \in \mathrm{Mor}(C, A)$ $f \circ g_1 = f \circ g_2$ implies that $g_1 = g_2$,

ii) f is *epic* (or an *epimorphism*) if for all $g_1, g_2 \in \mathrm{Mor}(B, C)$ $g_1 \circ f = g_2 \circ f$ implies that $g_1 = g_2$.

Definition 1.5.4. A *functor* F is a map between two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ that maps objects to objects and morphisms to morphisms such that

- if id_X is the identity morphism of the object X in $\mathcal{C}_1$ then $F(id_X) = id_{F(X)}$ is the identity morphism of the object $F(X)$ of $\mathcal{C}_2$,

- $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}_1$.

Example 1.5.5.
1. The map F_1 from the category of groups to the category of sets that maps each group to the set containing its elements and each group homomorphism to itself as a function is a functor.
2. The map F_2 that maps each topological space to its fundamental group and each continuous map to the induced map between the fundamental groups is a functor from the category of topological spaces to the category of groups.

Definition 1.5.6. Let $F : C \to D$ be a functor from C to D such that $\text{ob}(C), \text{Mor}(C), \text{ob}(D)$ and $\text{Mor}D$ are sets (not proper classes). Then F induces a function $F_{X,Y} : \text{Mor}_C(X, Y) \to \text{Mor}_D(F(X), F(Y))$ for every object X and Y in C. The functor F is called

i. *faithful* if $F_{X,Y}$ is injective.

ii. *full* if $F_{X,Y}$ is surjective.

iii. *fully faithful* if $F_{X,Y}$ is bijective.

Example 1.5.7.
1. F_1 in the previous example is faithful.
2. The inclusion functor from the category of abelian groups to the category of groups is fully faithful.

CHAPTER 2
SPECTRAL SPACES AND TOPOLOGICAL ALGEBRA

In this chapter we introduce the language of Spectral Space. Originally appearing in terms of Boolean Space, then in analysis (work of Gel'fand), they have formed the backbone of algebraic geometry since the second half of the twentieth century. We then describe the notion of topological algebra which will allow us to talk in the upcoming chapter of continuous valuations. The main references for this chapter are [4, 8].

2.1 Spectral Spaces

Definition 2.1.1. Let X be a non-empty topological space. Then X is called *irreducible* if it cannot be expressed as the union of two proper closed subsets. A nonempty subset Z of X is called *irreducible* if it is irreducible when we endow it with the subspace topology.

Remark. Equivalently, X is irreducible if every nonempty open subset of X is dense. Also, X is irreducible if and only if any two nonempty open subsets of X intersect.

Lemma 2.1.2. *Let X be a topological space. A subspace $Y \subseteq X$ is irreducible if and only if its closure $\overline{Y}$ is irreducible.*

Proof. A subset Z of X is irreducible if and only if for any two open subsets U and V of X with $Z \cap U \neq \emptyset$ and $Z \cap V \neq \emptyset$ we have $Z \cap (U \cap V) \neq \emptyset$. This implies the lemma because an open subset meets Y if and only if it meets $\overline{Y}$. $\square$

Remark. In particular, a subset of the form $\overline{\{x\}}$ for $x \in X$ is always irreducible.

Example 2.1.3. If X is a Hausdorff space, then the only irreducible subsets of X are the sets $\{x\}$ for $x \in X$.

Definition 2.1.4. Let X be a topological space.

i) We say that a point $\alpha \in X$ is a *generic point* if $\overline{\{\alpha\}} = X$.

ii) A topological space X is called *sobe*r if every irreducible closed subset of X has a unique generic point.

Example 2.1.5. 1. Let X be an indiscrete topological space. Then every point of X is a generic point.

2. Let Y be an infinite set endowed with the topology such that the closed sets different from Y are all finite subsets. Then Y is irreducible but has no generic point. In particular Y is not sober. If we add a single point α without changing the closed subsets different from $Y \cup \{\alpha\}$, then $Y \cup \{\alpha\}$ is sober.

Definition 2.1.6. A topological space X is called *spectral* if it is a sober compact space that has a basis consisting of compact open subsets which is closed under finite intersections. If X has an open covering by spectral spaces then it is called *locally spectral*.

Example 2.1.7. Let A be a ring and endow $\mathrm{Spec}A := \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal }\}$ with the usual topology (i.e. the closed sets of $\mathrm{Spec}A$ are of the form $V(\mathfrak{a}) := \{\mathfrak{p} : \mathfrak{a} \subseteq \mathfrak{p}\}$ for ideal $\mathfrak{a}$ of A). Then $\mathrm{Spec}A$ is spectral; for $f \in A$ set $D(f) := \{\mathfrak{p} : f \notin \mathfrak{p}\}$. Then the sets of the form $(D(f))_{f\in A}$ form a basis of open compact subsets of $\mathrm{Spec}A$ stable under finite intersections.

2.2 Topological Algebra

Definition 2.2.1. A topological space G that is also a group is called a *topological group* if the map $(x, y) \mapsto xy$ from $G \times G$ onto G is continuous and the map $x \mapsto x^{-1}$ from G onto G is also continuous.

Remark. Equivalently, a topological group is a group G equipped with a topology such that the map $(x, y) \mapsto xy^{-1} : G \times G \to G$ is continuous.

Theorem 2.2.2. *Let a be a fixed element of a topological group G. Then the left translation map $l_a : x \mapsto ax$, the right translation map $r_a : x \mapsto xa$, the inversion map $x \mapsto x^{-1}$ and the inner automorphism $x \mapsto axa^{-1}$ are all homeomorphisms of G.*

Proof.
1. Left translation l_a: It is clear that l_a is bijective. Let W be an open neighborhood of ax. Since G is a topological group, there exists an open set U such that $aU := \{au : a \in U\} \subseteq W$. This shows that l_a is continuous. Moreover, it is easy to see that the inverse of l_a is ${l_a}^{-1} : x \mapsto a^{-1}x$, which is continuous by the same argument as above.

2. Right translation r_a: It is a homeomorphism, by a similar argument to the previous case.

3. The inversion map: Let $f(x) = x^{-1}$. Clearly, f is a continuous bijective map. Also, $f^{-1}(x) = x^{-1}$. Hence, f is a homeomorphism.

4. The inner automorphism: It is a composition of two homeomorphisms, namely $x \mapsto xa^{-1}$ and $x \mapsto ax$. Hence, it is a homeomorphism.

□

Proposition 2.2.3. *Let G be a topological group and $H' \subseteq H \subseteq G$ be subgroups. If H' is open in G then H is open in G.*

Proof. Note that $H = \bigcup_{h \in H} hH'$ a union of open sets. Therefore H is open. □

Example 2.2.4. Let $G = \mathbb{R}$, the real line with addition as the group operation and the usual metric topology defined by $d(x, y) = |x - y|$. For each $\varepsilon > 0$, $|x| < \varepsilon/2$ and $|y| < \varepsilon/2$ imply that $|x + y| < \varepsilon$ and therefore addition is continuous. Similarly, one sees easily that the inversion $x \mapsto -x$ is continuous. Hence $\mathbb{R}$ is an abelian topological group.

Remark. The previous topological group can be extended to $\mathbb{R}^n$ by extending the addition and metric topology to $\mathbb{R}^n$.

Example 2.2.5. Let G be the quotient group $\mathbb{R}^n/\mathbb{Z}^n$ with the usual operation and the quotient topology of the additive group of $\mathbb{R}^n$. Then G is a compact abelian topological group.

Definition 2.2.6. A topological space R that is also a ring is said to be a *topological ring* if $(A, +)$ is a topological group and $(x, y) \mapsto xy$ is continuous from $R \times R$ to R. A topology τ on a topological ring is a *ring topology*.

Remark. Equivalently, a topological ring A is a ring equipped with a topology such that the mappings

$$(x, y) \mapsto x + y : A \times A \to A$$

$$(x, y) \mapsto x \cdot y : A \times A \to A$$

are continuous.

Note that the second axiom implies in particular that $y \mapsto -y$ is continuous (fix $x = -1$). Combined with the first, it shows that

$$(x, y) \mapsto x - y : A \times A \to A$$

is continuous and the additive group of A is a topological group.

Definition 2.2.7. A function N from a ring A to $[0, \infty)$ is a *norm* if the following conditions hold for all $x, y \in A$:

1. $N(x) = 0$ if and only if $x = 0$.

2. $N(x+y) \leq N(x) + N(y)$.

3. $N(-x) = N(x)$.

4. $N(xy) \leq N(x)N(y)$.

Remark. If N is a norm on a ring A, then d defined by $d(x,y) = N(x-y)$ for all $x, y \in A$, is a metric. Indeed, the first condition implies that $d(x,y) = 0$ if and only if $x = y$, the third implies that $d(x,y) = d(y,x)$ and the second yields the triangle inequality, since $d(x,z) = N(x-z) = N[(x-y)+(y-z)] \leq N(x-y) + N(y-z) = d(x,y) + d(y,z)$.

Theorem 2.2.8. *Let N be a norm on a ring A. The topology given by the metric d defined by N is a ring topology.*

Proof. Let $a, b \in A$. For all $x, y \in A$, $d(x+y, a+b) = N[(x+y)-(a+b)] = N[(x-a)+(y-b)] \leq N(x-a) + N(y-b) = d(x,a) + d(y,b)$. Thus, $(a,b) \mapsto a+b$ is continuous. For all $x \in A$, $d(-x,-a) = N(-x+a) = N(x-a) = d(x,a)$. Therefore, $a \mapsto -a$ is continuous as well. Finally, for all $x, y \in A$, $d(xy, ab) = N[(x-a)(y-b)+a(y-b)(x-a)b] \leq N(x-a)N(y-b)+N(a)N(y-b)+N(x-a)N(b)$. Hence, $(a,b) \mapsto ab$ is also continuous. $\square$

Example 2.2.9. On any ring A, the discrete topology is compatible with the ring structure. A topological ring whose topology is discrete is called a *discrete ring*.

Example 2.2.10. Let X be a set and $B(X)$ be the ring of all bounded real-valued (or complex valued) functions on X. Define $N(f) := \sup\{|f(x)| : x \in X\}$. Then N is a complete norm on $B(X)$, so $B(X)$ and each of its subrings is a topological ring for the topology defined by N.

Definition 2.2.11. Let F be a field. Then F is said to be a *topological field* if it is a topological ring such that $x \mapsto x^{-1}$ from $F^\times$ to $F^\times$ is continuous.

CHAPTER 3

P-ADIC INTEGERS AND F-ADIC RINGS

In this chapter, we finally move to the p-adic world. We start by a gentle introduction to p-adic integers and p-adic numbers. We then move on to the notion of f-adic or Huber rings, topological rings where the topology is generated by certain ideals. This material is based on [6, 9, 10].

3.1 p-adic Integers and p-adic Numbers

Definition 3.1.1. A *p-adic integer* is a formal series of the form $\alpha = a_0 + a_1p + a_2p^2 + a_3p^3 + ...$, where p is a prime integer and $a_i \in \{0, 1, 2, ..., p-1\}$ for all i. The set of all p-adic integers is denoted by $\mathbb{Z}_p$.

If we truncate $\alpha \in \mathbb{Z}_p$ at its kth term, we get $\alpha_k = a_0 + a_1p + a_2p^2 + a_3p^3 + ... + a_{k-1}p^{k-1}$, which represents an element in $\mathbb{Z}/p^k\mathbb{Z}$, which yields a map $\mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}$.

A sequence $\{\alpha_k\}$ in which $\alpha_k \equiv \alpha_{k'} \mod p^{k'}$ for all $k' < k$ defines a unique p-adic integer $\alpha \in \mathbb{Z}_p$. Start with $k = 1$, $\alpha_1 = a_0$, then for $k = 2$, $\alpha_2 = a_0 + a_1p$ and so on, then $\alpha = \lim_{k\to\infty} \alpha_k$. A formal way to discuss this guides to the notion of inverse limits.

Definition 3.1.2. Let $\{A_i\}_{i\in I}$ be a family of groups and $\mu_{ij} : A_j \to A_i$ be a family of homomorphisms for all $i \leq j$ such that:

i. μ_{ii} is the identity map of A_i.

ii. $\mu_{ij} \circ \mu_{jk} = \mu_{ik}$ for all $i \leq j \leq k$.

Define the *inverse limit* of the system $\{A_i\}$ as follows:

$$\lim_{\leftarrow} A_i := \{(a_i) \in \prod_{i\in I} A_i : a_i = \mu_{ij}(a_j)\}.$$

Remark. In the discussion before this definition, letting $A_k := \mathbb{Z}/p^k\mathbb{Z}$ introduces a bijection between

$$\mathbb{Z}_p \to \lim_{\leftarrow} \mathbb{Z}/p^k\mathbb{Z}.$$

Moreover, since each element in $\mathbb{Z}/p^k\mathbb{Z}$ can be expressed as $\alpha_k \in \mathbb{Z}_p$ and each element element in $\mathbb{Z}_p$ can be written uniquely as a sequence $\{\alpha_k\}$ in which $\alpha_k \in \mathbb{Z}/p^k\mathbb{Z}$ for all k, we may redefine $\mathbb{Z}_p$ as:

$$\mathbb{Z}_p := \lim_{\leftarrow} \mathbb{Z}/p^k\mathbb{Z} = \{(\alpha_i) \in \prod_{i\in\mathbb{N}} \mathbb{Z}/p^i\mathbb{Z} : \alpha_i = \mu_{ij}(\alpha_j)\}.$$

Remark. The sum of a geometric series $1 + r + r^2 + r^3 + ... = \dfrac{1}{1-r}$ if $|r| < 1$. The denominator still makes since if $r \neq 1$, so we may extend the convergence radius by allowing r to be anything different from 1. For instance, $1 + 2 + 2^2 + 2^3 + ... = \dfrac{1}{1-2} = -1$. The sum of course does not converge in the usual sense. It does however converge in the sense of *Abel*.

Theorem 3.1.3. *Any integer can be uniquely written as a p-adic integer for any prime p.*

Proof. Let $n \in \mathbb{Z}$ and p be any prime. We wish to write n as $n = a_0 + a_1p + a_2p^2 + ...$ with $a_i \in \{0, 1, 2, ..., p-1\}$.

i. $n \geq 0$. Without loss of generality assume that $n > p$ and $p \nmid n$. Let $a_0 = n \mod p \in \{0, 1, 2, ..., p-1\}$. Then $n - a_0 > 0$ is divisible by p. Let $n_1 = \dfrac{n-a_0}{p}$, consider the following:

   $n = a_0 + a_1p + a_2p^2 + ...$

   $\Rightarrow n - a_0 = a_1p + a_2p^2 + a_3p^3 ...$

   $\Rightarrow n_1 = \dfrac{n-a_0}{p} = a_1 + a_2p + a_3p^2 + ...$

   Now, let $a_1 = n_1 \mod p$ and repeat the same process until getting $p \leq n_i < 2p$. Then $n_i - a_i = p$ and $n_{i+1} = 1$ so $a_{i+1} = 1$ and $a_k = 0$ for all $k \geq i+2$. The uniqueness follows from the fact that each $a_i \in \{0, 1, 2, ..., p-1\}$.

ii. $n < 0$. We repeat the same procedure of case i, but here we will have an infinite series, since the sum of finite positive numbers cannot be negative.

□

Corollary 3.1.4. *For any prime p, $\mathbb{Z} \subseteq \mathbb{Z}_p$.*

Example 3.1.5. In the previous remark we introduced the idea of geometric series with $r > 1$ and found that $\sum 2^n = -1$. Here we start reversely using the method of the proof of the previous theorem, with $n = -1$ and $p = 2$. Let $-1 = a_0 + a_1 2 + a_2 2^2 + ...$, then $a_0 = -1 \mod 2 = 1$. Thus, $-2 = a_1 2 + a_2 2^2 + a_3 2^3 + ...$, so $-1 = a_1 + a_2 2 + a_3 2^2 + ...$ and again $a_1 = 1$. Continue in this way, we find that $-1 = 1 + 2 + 2^2 + 2^3 + ...$.

Proposition 3.1.6. *For any prime p, $\mathbb{Z}_p$ is an integral domain.*

Proof. 1. $(\mathbb{Z}_p,+)$ is an abelian group:

- Let $a_0 + a_1 p + a_2 p^2 + ...$, $b_0 + b_1 p + b_2 p^2 + ... \in \mathbb{Z}_p$. Then $(a_0 + a_1 p + a_2 p^2 + ...) + (b_0 + b_1 p + b_2 p^2 + ...) = (a_0 + b_0) + (a_1 + b_1)p + (a_2 + b_2)p^2 + ... \in \mathbb{Z}_p$. (Addition mod p).
- 0 is the additive identity in $\mathbb{Z}_p$.
- The existence of inverses follows from the existence of them in $\dfrac{\mathbb{Z}}{p\mathbb{Z}}$.
- The associativity follows from that one in $\dfrac{\mathbb{Z}}{p\mathbb{Z}}$.
- The commutativity of $\mathbb{Z}_p$ follows from the commutativity of usual addition in $\mathbb{Z}/p\mathbb{Z}$.

2. $(\mathbb{Z}_p,\cdot)$ is a semi-group:

- Let $a_0 + a_1 p + a_2 p^2 + ...$, $b_0 + b_1 p + b_2 p^2 + ... \in \mathbb{Z}_p$. Then $(a_0 + a_1 p + a_2 p^2 + ...) \cdot (b_0 + b_1 p + b_2 p^2 + ...) = (a_0 b_0) + (a_0 b_1 + a_1 b_0)p + (a_0 b_2 + a_1 b_1 + a_0 b_2)p^2 + ... \in \mathbb{Z}_p$. (Multiplication mod p).

- The associativity of the multiplication of $\mathbb{Z}_p$ follows from the associativity of usual multiplication and the associativity of $\mathbb{Z}/p\mathbb{Z}$.

3. 1 is the identity of $\mathbb{Z}_p$.

4. The commutativity of $\mathbb{Z}_p$ follows easily from the commutativity of $\mathbb{R}$.

5. Assume $a_0 + a_1p + a_2p^2 + \ldots$, $b_0 + b_1p + b_2p^2 + \ldots \in \mathbb{Z}_p$ and their product $(a_0b_0) + (a_0b_1 + a_1b_0)p + (a_0b_2 + a_1b_1 + a_0b_2)p^2 + \ldots = 0$. Assume for contrary that there exist $i, j \in \mathbb{N}$ such that $a_i, b_j \neq 0$. Without loss of generality suppose the rest of a_i's and b_j's are all zero. Then $a_ib_j \equiv 0 \mod p$. But $p \nmid a_i$ and $p \nmid b_j$ implies $p \nmid a_ib_j$ a contradiction. Hence $\mathbb{Z}_p$ has no zero divisors.

$\square$

Definition 3.1.7. Let $a = \sum_{i\in\mathbb{N}} a_ip^i$ be a p-adic integer. If $a \neq 0$ there is a first index $v = v(a) \geq 0$ such that $a_v \neq 0$. This index is the *p-adic order* of a, $v = v(a) = ord_p(a)$.

Definition 3.1.8. Let $x, y \in \mathbb{Z}_p$. Define the *p-adic norm* by

$$||x||_p = \begin{cases} \dfrac{1}{p^{v(x)}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases},$$

and the *p-adic metric* by

$$d_p(x, y) = ||x - y||_p.$$

Remark. $||\cdot||_p$ is a norm and ($\mathbb{Z}_p$,d_p) is a metric space.

Remark. With this metric, multiplication by p in $\mathbb{Z}_p$ is a contracting map

$$d(px, py) = \frac{1}{p}d(x, y)$$

and hence is continuous.

Theorem 3.1.9. *For any prime p, $\mathbb{Z}_p$ is a topological ring.*

Proof. We need to prove that $\mathbb{Z}_p$ is a topological group and continuous under multiplication.

i. $\mathbb{Z}_p$ is a topological group.

   We have indeed, $x \in a + p^n\mathbb{Z}_p$ and $y \in b + p^n\mathbb{Z}_p$ implies $x - y \in a - b + p^n\mathbb{Z}_p$ for all $n \in \mathbb{N}$. In other words, using the p-adic metric, we have

   $$||x-a||_p \leq ||p^n||_p = p^{-n}, ||y-b||_p \leq ||p^n||_p = p^{-n} \Rightarrow ||(x-y)-(a-b)||_p \leq p^{-n},$$

   proving the continuity of the map $(x, y) \mapsto x - y$ at any point (a, b).

ii. Multiplication is continuous.

   Fix $a, b \in \mathbb{Z}_p$ and consider $x = a + h$ and $y = b + k$ in $\mathbb{Z}_p$. Then,

   $||xy - ab||_p = ||(a + h)(b + k) - ab||_p = ||ak + hb + hk||_p$

   $\leq \max(||a||_p, ||b||_p)(||h||_p + ||k||_p) + ||h||_p||k||_p \to 0$ as $||h||_p, ||k||_p \to 0$.

   This proves the continuity of multiplication at any point $(a, b) \in \mathbb{Z}_p \times \mathbb{Z}_p$.

□

Proposition 3.1.10. *Let p and q be prime numbers with* $\gcd(p, q) = 1$*, then* $\dfrac{1}{q} \in \mathbb{Z}_p$.

Proof. Let $\dfrac{1}{q} = a_0+a_1p+a_2p^2+a_3p^3+...$, where $a_i \in \{0, 1, ..., p-1\}$ for all $i \in \mathbb{N}$. Then, $1 = qa_0+qa_1p+qa_2p^2+qa_3p^3+...$ and so $1 \equiv qa_0 \mod p$, which has a nonzero solution since $\gcd(p, q) = 1$. Now, let $n := 1 - qa_0 < 0$. Then $n = qa_1p + qa_2p^2 + qa_3p^3 + ...$ and all a_i's can be found using the method in the proof theorem 3.1.3 with the fact that $\gcd(p, q) = 1$. Hence, $1/q \in \mathbb{Z}_p$. □

Corollary 3.1.11. *Let $a, b \in \mathbb{Z}$ and p be prime with* $\gcd(p, b) = 1$*. Then* $\dfrac{a}{b} \in \mathbb{Z}_p$.

Proof. Knowing that $\mathbb{Z}_p$ is an integral domain with $a \in \mathbb{Z}_p$ and using the fact that b can be written uniquely as a product of distinct primes that are relatively prime to p completes the proof. □

Example 3.1.12. **p-adic Long Division**: Let $n_1 = 7, n_2 = 4$ and $p = 5$. Now, $\frac{4}{7} = a_0 + a_1 5 + a_2 5^2 + ...,$ thus $4 = 7a_0 + 7a_1 \cdot 5 + 7a_2 \cdot 5^2 + 7a_3 \cdot 5^3 + ...$. So, $7a_0 \equiv 4 \mod 5$ and $a_0 = 2$. Therefore, $-10 = 7a_1 \cdot 5 + 7a_2 \cdot 5^2 + 7a_3 \cdot 5^3 + ...$. So, $-2 = 7a_1 + 7a_2 \cdot 5 + 7a_3 \cdot 5^2 + ...$ and $7a_1 \equiv -2 \mod 5$. This gives that $a_1 = 4$. Continue in this way and find all such a_i's.

Remark. $\mathbb{Q}$ is not a subset of $\mathbb{Z}_p$ because $\frac{1}{p} \notin \mathbb{Z}_p$. For example, let $p = 2$. Assume that $\frac{1}{2} = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + ...,$ which implies that $1 + 0 \cdot 2 + 0 \cdot 2^2 + 0 \cdot 2^3 + ... = 1 = a_0 \cdot 2 + a_1 \cdot 2^2 + a_2 \cdot 2^3 + ...$. Thus $1 \equiv 0 \mod 2$ by the uniqueness of a_i's. A contradiction, hence $\frac{1}{2} \notin \mathbb{Z}_p$.

Remark. $\mathbb{Z}_p$ is not a field. In fact, the element $a = a_0 + a_1 p + a_2 p^2 + ...$ is invertible in $\mathbb{Z}_p$ if and only if a_0 is invertible in $\frac{\mathbb{Z}}{p\mathbb{Z}}$ (if and only if $a_0 \neq 0$). In other words

$$\mathbb{Z}_p{}^{\times} = \left\{ a = \sum_{i \in \mathbb{N}} a_i \in \mathbb{Z}_p : a_0 \neq 0 \right\}.$$

Definition 3.1.13. The field of fractions of $\mathbb{Z}_p$ is denoted by $\mathbb{Q}_p$ and called the field of *p-adic numbers*.

Remark. For any prime p, $\mathbb{Q} \subseteq \mathbb{Q}_p$.

3.2 f-adic Rings

Definition 3.2.1. A topological ring A is called *adic* if there exists an ideal I of A such that $\{I^n : n \in \mathbb{N}\}$ is a basis of neighborhoods of 0 in A. Such an ideal I is called an *ideal of definition* and such a topology is called an *I-adic topology*.

Example 3.2.2. $I = (p)$ is an ideal of definition of $\mathbb{Z}_p$.

Definition 3.2.3. A topological ring A is called an *f-adic ring* or a *Huber ring* if there exists an open adic subring $A_0 \subseteq A$ with finitely generated ideal of definition I. Such a subring is called a *ring of definition* of A.

Example 3.2.4. Let $A = \mathbb{Q}_p$ (or $A = \mathbb{Z}_p$) and $A_0 = \mathbb{Z}_p$. Then A is an f-adic ring by the previous example.

Example 3.2.5. Any ring A can be given the discrete topology is f-adic with $A_0 = A$ and $I = \{0\}$.

Definition 3.2.6. A subset S of a topological ring A is *bounded* if for every open neighborhood U of 0 there exists an open neighborhood V of 0 such that $VS = \{vs : v \in Vs \in S\} \subseteq U$. An element $a \in A$ is *power bounded* if $\{a^n : n \in \mathbb{Z}^+\}$ is bounded. If $\lim a^n = 0$, then a is *topologically nilpotent*. Set

$$A^\circ := \{a \in A : a \text{ is power bounded}\},$$

$$A^{\circ\circ} := \{a \in A : a \text{ is topologically nilpotent}\}.$$

Remark. A° is a subring of A, and $A^{\circ\circ}$ is an ideal of A.

Definition 3.2.7. A topological ring A is called *Tate* if A is f-adic and has a topologically nilpotent unit.

Example 3.2.8. $\mathbb{Q}_p$ is Tate with a topological nilpotent unit p.

Proposition 3.2.9. *Let A be a Tate ring and let B be a ring of definition of A. Then B contains a topologically nilpotent unit u of A. For any such u, $A = B_u$ and uB is an ideal of definition of B.*

Proof. Since B is an open neighborhood of 0, there exists for every topologically nilpotent element t an $n \in \mathbb{N}$ such that $u = t^n \in B$. This shows the first assertion.

For every $a \in A$ there exists $n \in \mathbb{N}$ such that $au^n \in B$, hence $A = B_u$. Multiplication with u^n is a homeomorphism $A \to A$. This shows that $u^n B$ is open. Moreover, as B is bounded, for every neighborhood V of 0 there exists $n \in \mathbb{N}$ such that $u^n B \subseteq V$. $\square$

3.3 Adic Homomorphisms

Definition 3.3.1. Let A and B be f-adic rings. A ring homomorphism $\varphi : A \to B$ is called *adic* if there exist rings of definitions A_0 of A and B_0 of B and an ideal of definition I of A_0 such that $\varphi(I)B_0$ is an ideal of definition of B_0.

Remark. Any adic ring homomorphism is continuous. Conversely, for every continuous homomorphism $\varphi : A \to B$ there exist always rings of definitions A_0 of A and B_0 of B and finitely generated ideals of definition I of A_0 and J of B_0 such that $\varphi(A_0) \subseteq B_0$ and such that $\varphi(I) \subseteq J$. But in general $\varphi(I)B_0$ is not an ideal of definition of B_0.

Example 3.3.2. Let A be a discrete ring. Then A is adic with ideal of definition $I = \{0\}$. Any homomorphism $\varphi : A \to B$ to an f-adic ring B is continuous. It is adic if and only if B also carries the discrete topology.

Proposition 3.3.3. *Let $\varphi : A \to B$ be a continuous ring homomorphism between f-adic rings. Assume that A is Tate. Then B is Tate, φ is adic and for every ring of definition B_0 of B we have $\varphi(A) \cdot B_0 = B$.*

Proof. Let A_0 and B_0 be rings of definition of A and B respectively such that $\varphi(A_0) \subseteq B_0$. Let $s \in A$ be a topologically nilpotent unit of A. Then $\varphi(s)$ is a topologically nilpotent unit of B and hence B is a Tate ring. Replacing s by some power, we may assume that $s \in A_0$. Then sA_0 is an ideal of definition of A_0 and $\varphi(s)B_0$ is an ideal of definition of B_0 (proposition 3.2.9). This shows that φ is adic. Let $B_0 \subseteq B$ be an arbitrary ring of definition. Replacing s by some power we may assuume that $\varphi(s) \in B_0$. Moreover one has $A = (A_0)_s$ and $B = (B_0)\varphi(s)$ by proposition 3.2.9 again and hence $\varphi(A) \cdot B_0 = B$. $\square$

Remark. Let $\varphi : A \to B$ and $\psi : B \to C$ be continuous ring homomorphisms of f-adic rings.

1. If φ and ψ are adic, then $\psi \circ \varphi$ is adic.

2. If $\psi \circ \varphi$ is adic, then ψ is adic.

3.4 Huber Pairs and Adic Spectrum

Definition 3.4.1. 1. Let A be an f-adic ring and let A° be the subring of power bounded elements. A subring B of A is called a *ring of integral elements* if B is open and integrally closed in A and if $B \subseteq A^\circ$.

2. A *Huber pair* (or an *affinoid pair*) is a pair (A, A^+), where A is an f-adic ring and A^+ is a ring of integral elements. If it is not ambiguous we may simply write A instead of (A, A^+).

3. A Huber pair (A, A^+) is called *complete* (respectively *adic*, respectively *Tate*, etc.) if A has this property.

4. A *morphism* of Huber pairs $(A, A^+) \to (B, B^+)$ is a ring homomorphism $\phi : A \to B$ such that $\phi(A^+) \subseteq B^+$. It is called *continuous* (respectively *adic*) if $\phi : A \to B$ is continuous (respectively adic).

Example 3.4.2. 1. Let A be an adic ring with finitely generated ideal of definition I. Then A is f-adic with $A = A_0$ and ideal of definition I. Every subset of A is bounded and $A^\circ = A$. Hence (A, A) is a Huber pair.
A special case is a ring A endowed with the discrete topology, i.e. $I = \{0\}$. Then for every subring A^+ of A that is integrally closed in A, (A, A^+) is a Huber pair.

2. Let k be a non-archimedean field (i.e., k is a topological field whose topology is given by a nontrivial non-archimedean norm $|\cdot| : k \to [0, \infty)$). Then $\mathcal{O}_k :=$

$\{a \in k : |a| \leq 1\} = k^\circ$ and $(k, \mathcal{O}_k)$ is a Huber pair. Every element $\bar{\omega} \in k - \{0\}$ with $|\bar{\omega}| < 1$ is a topologically nilpotent unit.

3. More generally, let $(A, ||\cdot||)$ be any k-Banach algebra with power-multiplicative norm. Then A is a Tate ring with topologically nilpotent unit $\bar{\omega}$. As a ring of definition one can take $A_0 = A^\circ = \{a \in A : ||a|| \leq 1\}$ which is an $\mathcal{O}_k$-algebra.

4. Let R be any ring with its discrete topology; then the power series ring $A = R[[T_1, ..., T_n]]$ is a Huber ring with ring of definition $A_0 = A$ and ideal of definition $(T_1, ..., T_n)$. Note that R is not Tate.

5. Let $A = \mathbb{Q}_p[[T]]$. It is tempting to say that A is a Huber ring with ring of definition $A_0 = \mathbb{Z}_p[[T]]$ and ideal of definition (p, T). But in fact one cannot put a topology on A which makes this work. Indeed, in such a topology $T^n \to 0$, and since multiplication by p^{-1} is continuous, $p^{-1}T^n \to 0$ as well. But this sequence never enters A_0, and therefore $A_0 \subseteq A$ is not open. (It is fine to say that $\mathbb{Q}_p[[T]]$ is a Huber ring with ring of definition $\mathbb{Q}_p[[T]]$ and ideal of definition (T), but then you are artificially suppressing the topology of $\mathbb{Q}_p$, so that the sequence p^n does not approach 0.)

Definition 3.4.3. Let A be a topological ring. Let v be a valuation on A and Γ be its valuation group. Then v is called *continuous* if for every $\gamma \in \Gamma$, $\{a \in A : v(a) < \gamma\}$ is open in A.

Notation. We denote the set of equivalence classes of continuous valuations of A by $\mathrm{Cont}(A)$.

Definition 3.4.4. Let $A = (A, A^+)$ be Huber. The *adic spectrum* of A is the subspace $\mathrm{Spa}(A) := \{v \in \mathrm{Cont}(A) : \forall a \in A^+, v(a) \leq 1\}$ of $\mathrm{Cont}(A)$.

Notation. If $x \in \mathrm{Spa}(A)$ and $f \in A$, then $x(f)$ is denoted by $|f(x)|$.

Remark. We endow $\mathrm{Spa}(A)$ with the topology generated by the subsets $\{x \in \mathrm{Spa}(A) : |f(x)| \leq |g(x)| \neq 0\}$ with $f, g \in A$.

Definition 3.4.5. Let A be a topological ring. A point $x \in \mathrm{Cont}(A)$ is called *analytic* if $\mathrm{supp}(x)$ is not open in A.

Notation. If $A = (A, A^+)$ is Huber, then the subset of analytic points in $\mathrm{Spa}A$ is denoted by $(\mathrm{Spa}A)_a$. Its complement in $\mathrm{Spa}A$ is denoted by $(\mathrm{Spa}A)_{na}$.

Remark. Let $A = (A, A^+)$ be Huber. For $x \in \mathrm{Cont}(A)$ the following assertions are equivalent.

- x has non-open support.
- There exists $a \in A^{\circ\circ}$ such that $x(a) \neq 0$.

Lemma 3.4.6. *Let $\varphi : A \to B$ be a continuous homomorphism between Huber pairs and let $f = \mathrm{Spa}(\varphi) : X := \mathrm{Spa}B \to Y := \mathrm{Spa}A$ be the attached continuous map.*

1. *$f(X_{na}) \subseteq Y_{na}$, and if φ is adic, then $f(X_a) \subseteq Y_a$.*
2. *If φ is adic, then for every rational subset V of Y the preimage $f^{-1}(V)$ is rational.*

   In particular f is spectral.

Proof. 1. The first assertion is clear and the second follows from the previous remark.

2. Let $s \in A$ and $T \subseteq A$ be a finite subset such that $T \cdot A$ is open in A. If φ is adic, then $\varphi(T) \cdot B$ is open in B. Hence $f^{-1}(R(\frac{T}{s}))$ is the rational subset $R(\frac{\varphi(T)}{s})$ of $\mathrm{Spa}(B)$.

□

CHAPTER 4
ADIC SPACES

In this chapter we describe Adic spaces, spaces described locally by Huber pairs. We show that they define a general framework englobing rigid geometry, schemes, etc. We conclude by working out the case of scheme and highlighting how to see them as 'trivial' adic spaces. This chapter is based on [9, 10].

4.1 The Presheaf

Definition 4.1.1. Let (A, A^+) be a Huber pair and $X := \mathrm{Spa}(A, A^+)$. Let $s_1, ..., s_n \in A$ and $T_1, ..., T_n \subseteq A$ be nonempty finite subsets such that T_iA is an open ideal of A for all i. We define a subset

$$U\left(\left\{\frac{T_i}{s_i}\right\}\right) = U\left(\frac{T_1}{s_1}, ..., \frac{T_n}{s_n}\right) := \{x \in X : \forall t_i \in T_i,\ |t_i(x)| \leq |s_i(x)| \neq 0\}.$$

This is open because it is an intersection of a finite collection of the sort of opens which generate the topology on X. Subsets of this form are called *rational subsets*.

Remark. Note that a finite intersection of rational subsets is again rational, just by concatenating the data that define the individual rational subsets.

Theorem 4.1.2. *Let $U \subseteq \mathrm{Spa}(A, A^+)$ be a rational subset. Then there exists a complete Huber pair $(\mathcal{O}_X(U), \mathcal{O}^+{}_X(U))$ and a morphism $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}^+{}_X(U))$ such that the map $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}^+{}_X(U)) \to \mathrm{Spa}(A, A^+)$ factors over U, and is universal for such maps. Moreover this map is a homeomorphism onto U. In particular U is quasi-compact.*

Proof. Choose s_i and T_i such that $U = U(\{\frac{T_i}{s_i}\})$. Choose $A_0 \subseteq A$ a ring of definition and $I \subseteq A_0$ a finitely generated ideal of definition. Take $(A, A^+) \to (B, B^+)$ such that $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors over U. Then,

1. The s_i's are invertible in B, so that we get a map $A[\{\frac{1}{s_i}\}] \to B$.
2. All t_i/s_i are of $|\cdot| \leq 1$ everywhere on $\mathrm{Spa}(B, B^+)$, so that $t_i/s_i \in B^+ \subseteq B^\circ$.
3. Since B° is the inductive limit of the rings of definition B_0, we can choose a B_0 which contains all t_i/s_i. We get a map $A_0[t_i/s_i] \to B_0$, where $i = 1, ..., n$ and $t_i \in T_i$ for all i. Endow $A_0[\{t_i/s_i\}]$ with the $IA_0[\{t_i/s_i\}]$-adic topology.

Remark. This defines a ring topology on $A[\{1/s_i\}]$ making $A_0[\{t_i/s_i\}]$ an open subring.

Lemma 4.1.3. *If $T \subseteq A$ is a subset such that $TA \subseteq A$ is open, then TA_0 is open.*

Proof. After possibly replacing I with some power we may assume that $I \subseteq TA$. Write $I = (f_1, ..., f_k)$. There exists a finite set R such that $f_1, ..., f_k \in TR$. Since I is topologically nilpotent, there exists n such that $RI_n \subseteq A_0$. Then for all $i = 1, ..., k$, $f_i I^n \subseteq TRI^n \subseteq TA_0$. Sum this over all i and conclude that $I^{n+1} \subseteq TA_0$. □

Back to the proof of the theorem. We have $A[\{1/s_i\}]$, a (non-complete) Huber ring. Let $A[\{1/s_i\}]^+$ be the integral closure of the image of $A^+[\{t/s_i\}]$ in $A[\{1/s_i\}]$. Let $(A\langle\{T_i/s_i\}\rangle, A\langle\{T_i/s_i\}\rangle^+)$ be its completion, a Huber pair. This has the desired universal property.

For the claim that Spa of this pair is homeomorphic to U: Use that Spa does not change under completion. (Also that the operation of taking the integral closure does not change much, either.) □

Definition 4.1.4. Define a *presheaf* $\mathcal{O}_X$ of topological rings on $\mathrm{Spa}(A, A^+)$: If $U \subseteq X$ is rational, $\mathcal{O}_X(U)$ is as in the theorem. On a general open $W \subseteq X$, we put

$$\mathcal{O}_X(W) = \varprojlim_{U \subseteq W \text{ is rational}} \mathcal{O}_X(U).$$

One defines $\mathcal{O}_X^+$ similarly. If $\mathcal{O}_X$ is a sheaf, we call (A, A^+) a *sheafy* Huber pair.

Proposition 4.1.5. *For all $U \subseteq X = \mathrm{Spa}(A, A^+)$,*

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : |f(x)| \leq 1, \text{ for all } x \in U\}.$$

In particular $\mathcal{O}_X^+(U)$ is a sheaf if $\mathcal{O}_X(U)$ is. If (A, A^+) is complete, then $\mathcal{O}_X(X) = A$ and $\mathcal{O}_X^+(X) = A^+$.

4.2 Adic Spaces

Definition 4.2.1. An *adic space* is an object of V that is locally isomorphic to $\mathrm{Spa}(A, A^+)$ for some sheafy Huber pair (A, A^+). The category of adic spaces is the full subcategory of V whose objects are the adic spaces. An adic space is called *affinoid*, if it is isomorphic to $\mathrm{Spa}(A, A^+)$ for some sheafy Huber pair (A, A^+).

We obtain a functor $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ from the category of sheafy Huber pairs to the category of adic spaces. The canonical morphism of adic spaces $\mathrm{Spa}(\hat{A}, \hat{A}^+) \to \mathrm{Spa}(A, A^+)$ is an isomorphism of adic spaces. The functor $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ from the category of sheafy complete Huber pairs to the category of adic spaces is fully faithful. More precisely one has for every adic space Y and every sheafy Huber pair (A, A^+) a bijection

$$\mathrm{Hom}(Y, \mathrm{Spa}(A, A^+)) \xrightarrow{\sim} \mathrm{Hom}((A, A^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X))),$$

where the right hand side denotes continuous ring homomorphisms $\phi : A \to \mathcal{O}_X(X)$ such that $\phi(A^+) \subseteq \mathcal{O}_X^+(X)$.

Example 4.2.2. i) Let k be a non-archimedean field. Then $\mathrm{Spa}(k, k^\circ)$ consists of a single point x, the equivalence class of the valuation $|\cdot| : k \to [0, \infty]$ defining the topology of k. One has $\kappa(x) = \hat{k}$.

ii) Let A be a valuation ring of height 1. Then $\mathrm{Spa}(A, A)$ consists of an open point η and a closed point s with $\kappa(\eta) = \mathrm{Frac}(A) =: k$ and $\kappa(s) = \frac{A}{\mathfrak{m}_A}$. The canonical morphism $S^0 = \mathrm{Spa}(k, A) \to S$ is an open immersion onto the open point.

Example 4.2.3. Endow $\mathbb{Z}$ and $\mathbb{Z}[t]$ with the discrete topology. Then $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$ is the final object in the category of adic spaces and for every adic space X we find

$$\mathrm{Hom}(X, \mathrm{Spa}(\mathbb{Z}[t], \mathbb{Z})) = \mathcal{O}_X(X),$$

$$\mathrm{Hom}(X, \mathrm{Spa}(\mathbb{Z}[t], \mathbb{Z}[t])) = \mathcal{O}_X^+(X).$$

Endow $\mathbb{Z}[[t]]$ with the t-adic topology and set $D^0 := \mathrm{Spa}(\mathbb{Z}[[t]], \mathbb{Z}[[t]])$. Then for every affinoid adic space $X = \mathrm{Spa} A$ we have

$$\mathrm{Hom}(X, D^0) = \mathcal{O}_X(X)^{\circ\circ} = \hat{A}^{\circ\circ}.$$

Indeed, first note that every integrally closed open subring of $\hat{A}$ contains $\hat{A}^{\circ\circ}$. In particular $(\hat{A}^+)^{\circ\circ} = \hat{A}^{\circ\circ}$. Every continuous homomorphism $\varphi : \mathbb{Z}[[t]] \to \mathcal{O}_X^+(X) = \hat{A}^+$ is determined by the image a of t. As t is topologically nilpotent, a is topologically nilpotent. Conversely, let $a \in \mathcal{O}_X^+(X)$ be topologically nilpotent. As $\mathcal{O}_X^+(X)$ is complete and 0 has a fundamental system of neighborhoods consisting of additive subgroups, a series $\sum_n \lambda_n a^n$ converges if and only if $\lim_n {}_n a^n = 0$. But this is the case if ${}_n \in \varphi(\mathbb{Z})$ because $\varphi(\mathbb{Z})$ is automatically bounded (as $\varphi(\mathbb{Z})$ is contained in every ring of definition of A) and a is topologically nilpotent. We view D^0 as the formal open unit disc.

Definition 4.2.4. Let X be an adic space. A point $x \in X$ is called *analytic* if there exists an open neighborhood U of x such that $\mathcal{O}(U)$ contains a topologically nilpotent unit.

Proposition 4.2.5. *Let X be an adic space and $x \in X$. Then the following are equivalent:*

1. *x is analytic.*

2. *For every open affinoid neighborhood $U = \mathrm{Spa}A$ of x, the point* $\mathrm{supp}x \subseteq A$ *is not open in A.*

We set $X_a := \{x \in X : x \text{ is analytic}\}$ *and* $X_{na} := X - X_a$.

Proof. We may assume that $X = \mathrm{Spa}A$, A is a complete affinoid ring.
$(\Leftarrow)$: Let $x \in \mathrm{Spa}A$ such that $\mathrm{supp}x$ is not open in A. By the remark after definition 3.4.5 there exists a topologically nilpotent element s of A with $x(s) \neq 0$. Then $U := \{y \in \mathrm{Spa}A : y(s) \neq 0\}$ is an open neighborhood of x in $\mathrm{Spa}A$. As the restriction $A = \mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is a continuous ring homomorphism, the image of s in $\mathcal{O}_X(U)$ is again a topologically nilpotent unit.
$(\Rightarrow)$: Let $x \in \mathrm{Spa}A$ such that $\mathrm{supp}x$ is open in A, and let U be an open neighborhood of x. We have to show that $\mathcal{O}_X(U)$ has no topologically nilpotent unit.
Let V be a rational subset of $\mathrm{Spa}A$ with $x \in V \subseteq U$ and set $p := \{f \in \mathcal{O}_X(V) : v_x(f) = 0\}$. Then p is a prime ideal of $\mathcal{O}_X(V)$ with $p \cap A = \mathrm{supp}x$. As $\mathrm{supp}x$ contains an ideal of definition of (a ring of definition of) A, p contains an ideal of definition of $\mathcal{O}_X(V)$ by definition of $\mathcal{O}_X(V)$. Thus p is an open prime ideal of $\mathcal{O}_X(V)$ and contains therefore all topologically nilpotent elements of $\mathcal{O}_X(V)$. As p contains no units, $\mathcal{O}_X(V)$ contains no topologically nilpotent units. Hence $\mathcal{O}_X(U)$ does not contain a topologically nilpotent unit. $\square$

Remark. Let X be an adic space. Then for every open subspace U of X we have $U_a = X_a \cap U$ and $U_{na} = X_{na} \cap U$.

Definition 4.2.6. A morphism $f : X \to Y$ of adic spaces is called *adic* if for every $x \in X$ there exist open affinoid neighborhoods U of x in X and V of $f(x)$ in Y with $f(U) \subseteq V$ such that the ring homomorphism of f-adic rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ induced by f is adic.

Proposition 4.2.7. *Let $f : X \to Y$ be a morphism of adic spaces.*

1. *If f is adic then $f(X_a) \subseteq Y_a$.*

2. *$f(X_{na}) \subseteq Y_{na}$.*

Proof. It follows from the previous remark that we may assume that X and Y are affinoid. But in this case we have already shown the all results in Lemma 3.4.6. $\square$

4.3 Formal Schemes as Adic Spaces

Definition 4.3.1. A ringed space $(X, \mathcal{O}_X)$ is a pair of a topological space X with a sheaf of rings $\mathcal{O}_X$ on X. The sheaf $\mathcal{O}_X$ is called the *structure sheaf* of X.

Remark. For every complete noetherian adic ring A let $\mathrm{Spf}(A)$ denote its formal spectrum. Then the functor $\mathrm{Spf}(A) \to \mathrm{Spa}(A, A)$ from noetherian affine formal schemes to the category of adic spaces can be globalized to a fully faithful functor $t : X \to X^{\mathrm{ad}}$ from the category of locally noetherian formal schemes to the category of adic spaces. More precisely for every locally noetherian formal scheme X there exists an adic space X^{ad} and a morphism of locally and topologically ringed spaces $\pi : (X^{\mathrm{ad}}, \mathcal{O}^+_{X^{\mathrm{ad}}}) \to (X, \mathcal{O}_X)$ satisfying the following universal property. For every adic space Z and for every morphism $f : (Z, \mathcal{O}^+_Z) \to (X, \mathcal{O}_X)$ of locally and topologically ringed spaces there exists a unique morphism of adic spaces $g : Z \to X^{\mathrm{ad}}$ making the following diagram commutative

$$\begin{tikzcd}
(Z, \mathcal{O}^+_Z) \arrow[r, "f"] \arrow[dr, dashed, "g^+"'] & (X, \mathcal{O}_X) \\
& (X^{\mathrm{ad}}, \mathcal{O}^+_{X^{\mathrm{ad}}}) \arrow[u, "\pi"]
\end{tikzcd}$$

For $X = \mathrm{Spf}(A)$ for a complete noetherian adic ring A, the underlying continuous map of π is given by $X^{\mathrm{ad}} = \mathrm{Spa}(A, A) \ni x \mapsto \{f \in A : |f(x)| < 1\}$ which is an open prime ideal of A.

CHAPTER 5
CONCLUSION

In this thesis we built the notion of adic spaces starting from the concept of ordered group and progressively adding the geometric and algebraic features necessary. We have shown how a relatively simple axiomatic construction is able to englobe many geometric models and can be used to work with objects coming from the p-adic world.

Huber (or f-adic) spaces are a relatively new construction in geometry but their usefulness is undeniable. This thesis can for example serve as an introduction to the incredible work of Scholze and the latest developments in the area.

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