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## Elements of Convergence Approach Theory

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# ELEMENTS OF CONVERGENCE APPROACH THEORY

by

**WILLIAM TROTT**

(Under the Direction of Frédéric Mynard)

## ABSTRACT

We introduce two generalizations to convergence approach spaces of classical results characterizing regularity of a convergence space in terms of continuous extensions of maps on one hand, and in terms of continuity of limits for the continuous convergence on the other. Characterizations are obtained for two alternative extensions of regularity to convergence-approach spaces: regularity and strong regularity. Along the way, we give a brief overview of the theory of convergence spaces and of convergence approach spaces.

*Key Words:* Regularity, Strong regularity, Convergence space, Convergence-approach space, Approach space, Strict subspace, Continuous extension, Contractive extension, Default of contraction, Continuous convergence, Diagonal axioms

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**ELEMENTS OF CONVERGENCE APPROACH THEORY**

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**ELEMENTS OF CONVERGENCE APPROACH THEORY**

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## LIST OF SYMBOLS

$\mathbb{R}$	Real Numbers
$\mathbb{N}$	Natural Numbers
$Y^X$	Morphisms from $X \rightarrow Y$
$C(X, Y)$	Continuous functions from $X$ to $Y$
$\mathbb{F}X$	Set of filters on $X$
$\mathcal{P}(X)$	Power set of $X$
$B(x, \epsilon)$	Ball of radius $\epsilon$ centered at $x$
$\mathcal{B}^\uparrow$	Closure of $\mathcal{B}$ under supersets
$\mathcal{F}\#\mathcal{G}$	The mesh of $\mathcal{F}$ and $\mathcal{G}$
$\mathcal{F}\vee\mathcal{G}$	Supremum of $\mathcal{F}$ and $\mathcal{G}$
$\mathcal{F}\wedge\mathcal{G}$	Infimum of $\mathcal{F}$ and $\mathcal{G}$
$\mathcal{F}\geq\mathcal{G}$	$\mathcal{F}$ finer than $\mathcal{G}$ ( $\mathcal{G}$ coarser than $\mathcal{F}$ )
$f[\mathcal{F}]$	The image filter of $\mathcal{F}$ under $f$
$[X, Y]$	The continuous convergence on $Y^X$
$h(S, f)$	Hull of extensionability
$\text{adh}A$	The adherence of $A$
$\text{adh}^\sharp\mathcal{F}$	$\{\text{adh}F : F \in \mathcal{F}\}^\uparrow$
$\mathcal{F}^\#$	Family of subsets that mesh with $\mathcal{F}$



# CHAPTER 1

## PRELIMINARIES

### 1.1 Categorical Problems of Topological Spaces

In order to work towards introducing convergence approach spaces, we must first start by introducing a generalization of topology known as *convergence theory*. In convergence theory, the structure of a space comes from how generalized sequences, known as *filters*, converge. Because convergence theory generalizes topology, it is possible to describe any topology using the language of convergence theory.

The study of convergence theory is motivated by problems that arise when we view topological spaces from a categorical perspective. Before we address these issues, we must introduce a few definitions from category theory.

**Definition 1.1.** [9] A category  $\mathcal{C}$ , is a class of objects,  $\text{ob}(\mathcal{C})$  and a class  $B^A$  of  $\mathcal{C}$ -morphisms from  $A$  to  $B$  for each pair  $A, B \in \text{ob}(\mathcal{C})$ , satisfying the following conditions:

1. For each  $A \in \text{ob}(\mathcal{C})$ , there is an identity,  $1_A \in A^A$ ,
2. For each  $A, B, C$ , there is a map  $B^A \times C^B \rightarrow C^A$  with  $(f, g) \mapsto g \circ f$  satisfying the following:
  - (a) If  $f \in B^A$ ,  $g \in C^B$ , and  $h \in D^C$ , then  $h \circ (g \circ f) = (h \circ g) \circ f$
  - (b) For every  $A, B, C \in \text{ob}(\mathcal{C})$ , every  $f \in B^A$  and every  $g \in A^C$ ,  $f \circ 1_A = f$  and  $1_A \circ g = g$
  - (c) The sets  $B^A$  are pairwise disjoint.

We will use the traditional method of denoting a category by writing the category in bold face, such as **Set** for the category of sets with functions between sets acting as

morphisms. Our focus for the time being will be on the category **Top**, of topological spaces with continuous maps as morphisms and we will introduce other categories as they appear.

**Definition 1.2.** [10] *A category  $\mathcal{C}$  is cartesian closed if it obeys the exponential law: for every  $A, B, C \in \text{ob}(\mathcal{C})$ , there is a bijection from  $C^{A \times B}$  to  $(C^B)^A$  which is written as  $C^{A \times B} \cong (C^B)^A$ .*

While this property holds for some categories, such as **Set**, it is well known that it does not hold in **Top**.

The problem comes when we take the objects  $X, Y, Z \in \text{ob}(\mathbf{Top})$  and consider the exponential condition  $C(X \times Y, Z) \cong C(X, C(Y, Z))$ , where  $C(X, Y)$  is the space of continuous maps between  $X$  and  $Y$ . In general, there is no topology  $C(Y, Z)$  that satisfies the exponential law. However, if we expand our focus to convergence spaces, we can define a convergence structure on  $C(Y, Z)$  for which the equality holds. Then we have that the category **Conv**, with convergence spaces (in particular topologies) as objects and continuous maps as morphisms, is cartesian closed.

## 1.2 Convergence Theory

### 1.2.1 Filters

Now that we have motivation for studying convergence theory, we need to build some machinery before we are able to define exactly what the objects of **Conv** are. In order to do this, we must first introduce the notion of a filter on a set.

**Definition 1.3.** *A (proper) filter  $\mathcal{F}$  on a set  $X$  is a family of subsets of  $X$  that satisfy the following conditions:*

1.  $\emptyset \notin \mathcal{F}$

2. For  $A \in \mathcal{F}$ , if  $A \subset B$ , then  $B \in \mathcal{F}$

3. If  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$

The space of all filters on a set  $X$  will be denoted as  $\mathbb{F}X$ . If a filter satisfies conditions 2 and 3, but not condition 1, it is called the *degenerate filter* on  $X$  and coincides with the powerset of  $X$ . Unless specified otherwise, filters are always assumed to be proper.

We say that a family  $\mathcal{B}$  of nonempty subsets of  $X$  is a *filter-base* if the family

$$\mathcal{B}^\uparrow := \{A \subset X : \exists B \in \mathcal{B} \text{ with } B \subseteq A\}$$

is a filter on  $X$ . If  $\mathcal{G} \subseteq \mathcal{P}(X)$  we will use the notation  $\mathcal{G}^\uparrow$  to denote the closure of  $\mathcal{G}$  with respect to supersets. If  $\mathcal{B}$  is a filter-base for some filter  $\mathcal{F}$ , then we say that  $\mathcal{B}$  *generates*  $\mathcal{F}$ .

**Example 1.4.** Let  $A \subset X$ , then the filter  $\{A\}^\uparrow := \{B \in X : A \subseteq B\}$  is called a principal filter.

**Example 1.5.** For a point  $x$  in a metric space  $X$ , we define  $\mathcal{B}(x)$  to be the family of balls centered at  $x$ . This family is a filter-base for the filter  $\mathcal{B}(x)^\uparrow$ , which is called the vicinity filter, and will be denoted  $\mathcal{V}(x)$ .

**Example 1.6.** A subset  $W$  of a topological space  $(X, \tau)$  is called a neighborhood of  $x \in X$  if there is an open set  $O \in \tau$  such that  $x \in O \subseteq W$ . Then the family of neighborhoods of  $x$ , denoted  $\mathcal{N}(x)$ , is a filter called the neighborhood filter of  $x$ .

**Example 1.7.** Let  $\{x_n\}_{n=1}^\infty$  be a sequence on  $X$ . The family of tails of  $\{x_n\}_{n=1}^\infty$  is the family

$$(x_n)_m := \{\{x_k : k \in \mathbb{N} \text{ and } k \geq n\} : m \in \mathbb{N}\}.$$

This family generates a filter,  $(x_n)_n^\uparrow$ , called a sequential filter.

Now that we have an idea of what sequences look like in  $\mathbb{F}X$ , a natural question is how can we interpret the convergence of a sequence in terms of the filter it generates? Recall that in a metric space, a sequence  $\{x_n\}$  converges to  $x$ , denoted  $x \in \lim_{n \rightarrow \infty} x_n$ , if for every  $\epsilon > 0$ , there is an  $N \in \mathbb{N}$  such that

$$B(x, \epsilon) \supset \{x_m : m > N\}.$$

To make a connection between these sets and the families that they are contained in, we need the following definition.

**Definition 1.8.** For  $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(X)$ , we say that  $\mathcal{F}$  is finer than  $\mathcal{G}$ , denoted  $\mathcal{F} \geq \mathcal{G}$ , if for every  $G \in \mathcal{G}$  there is an  $F \in \mathcal{F}$  such that  $G \supset F$ .

This relation is reflexive and transitive on  $\mathcal{P}(X)$ , and when it is restricted to  $\mathbb{F}X$  it becomes a partial order. With this new definition, we see that  $x \in \lim_{n \rightarrow \infty} x_n$  whenever  $(x_n)_n \geq \mathcal{B}(x)$ . This relation holds when both families are closed under super sets, so we have that  $x \in \lim_{n \rightarrow \infty} x_n$  if and only if

$$(x_n)^\uparrow \geq \mathcal{B}(x)^\uparrow = \mathcal{V}(x).$$

For this partial order, we can define the greatest lower bound of a family of filters  $(\mathcal{F}_i)_{i \in I}$ , by

$$\bigwedge_{i \in I} \mathcal{F}_i := \left\{ \bigcup_{i \in I} F_i : \forall i \in I, F_i \in \mathcal{F}_i \right\}^\uparrow.$$

The greatest lower bound of the family  $(\mathcal{F}_i)_{i \in I}$  is a filter on  $X$  called the *infimum of the filters*  $(\mathcal{F}_i)_{i \in I}$ .

If  $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(X)$ , we say that  $\mathcal{F}$  and  $\mathcal{G}$  *mesh*, denoted  $\mathcal{F} \# \mathcal{G}$ , if  $F \cap G \neq \emptyset$  for every  $F \in \mathcal{F}$  and  $G \in \mathcal{G}$ . When one of the filters is a principal filter,  $\{A\}^\uparrow$ , we denote the mesh of  $\{A\}^\uparrow$  and  $\mathcal{F}$  by  $A \# \mathcal{F}$  or  $A \in \mathcal{F}^\#$ , where

$$\mathcal{F}^\# := \{A \subset X : A \# \mathcal{F}\}.$$

If  $\mathcal{F}, \mathcal{G} \in \mathbb{F}X$ , then we can get a least upper bound for  $\mathcal{F}$  and  $\mathcal{G}$ , whenever  $\mathcal{F} \# \mathcal{G}$ , which we denote by

$$\mathcal{F} \vee \mathcal{G} := \{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}.$$

In order to generalize the least upper bound to a family of filters, we have to generalize the notion of two filters meshing.

**Definition 1.9.** *A family  $\mathcal{F}$  of subsets of  $X$  has the finite intersection property if  $\bigcap_{A \in \mathcal{B}} A \neq \emptyset$  for any finite subset  $\mathcal{B}$  of  $\mathcal{F}$ .*

A family of filters  $(\mathcal{F}_i)_{i \in I}$  admits a least upper bound in  $\mathbb{F}X$  whenever  $\bigcup_{i \in I} \mathcal{F}_i$  has the finite intersection property. The *least upper bound of the family  $(\mathcal{F}_i)_{i \in I}$*  is given by

$$\bigvee_{i \in I} \mathcal{F}_i := \left\{ \bigcap_{A \in \mathcal{B}} A : \mathcal{B} \subset \bigcup_{i \in I} \mathcal{F}_i, \text{card} \mathcal{B} < \infty \right\}^\uparrow.$$

From time to time, we can simplify results by looking at a special class of filters.

**Definition 1.10.** *A proper filter  $\mathcal{U}$  is called an ultrafilter on  $X$ , denoted  $\mathcal{U} \in \mathbb{U}X$ , if it satisfies the following equivalent conditions:*

1. *If  $A \cup B \in \mathcal{U}$ , then either  $A \in \mathcal{U}$  or  $B \in \mathcal{U}$ .*
2. *For  $A \subseteq X$ , either  $A \in \mathcal{U}$  or  $A^C \in \mathcal{U}$ .*
3. *If  $\mathcal{G} \in \mathbb{F}X$  and  $\mathcal{G} \geq \mathcal{U}$ , then  $\mathcal{G} = \mathcal{U}$ .*
4.  *$\mathcal{U}^\# = \mathcal{U}$ .*

**Example 1.11.** *The principal filter  $\{x\}^\uparrow$  is an ultrafilter since if  $A \cup B \in \{x\}^\uparrow$ , then  $x \in A$  or  $x \in B$ .*

Assuming the axiom of choice, we have the following proposition.

**Proposition 1.12.** *Every family of subsets of  $X$  with the finite intersection property (in particular every filter) is contained in an ultrafilter.*

**Definition 1.13.** If  $f : X \rightarrow Y$  and  $\mathcal{F} \in \mathbb{F}X$ , then the image filter is defined as

$$f[\mathcal{F}] = \{f(F) : F \in \mathcal{F}\}^\uparrow = \{A \subseteq Y : f^{-1}(A) \in \mathcal{F}\}.$$

## 1.2.2 Convergences

In this section we define the basics of convergence theory, for the most part without proofs. The interested reader should consult [3] or [4] for details.

Given a nonempty set  $X$ , the family of filters  $\mathbb{F}X$ , and a relation  $\xi$  between  $X$  and  $\mathbb{F}X$ , we say that  $\mathcal{F} \in \mathbb{F}X$  *converges* to  $x \in X$ , denoted  $x \in \lim_\xi \mathcal{F}$ , whenever  $(x, \mathcal{F}) \in \xi$ .

**Definition 1.14.** A relation  $\xi$  from  $X$  to  $\mathbb{F}X$  is called a *convergence* if it satisfies the following properties:

1. For  $\mathcal{F}, \mathcal{G} \in \mathbb{F}X$ ,  $\mathcal{F} \leq \mathcal{G} \implies \lim_\xi \mathcal{F} \subseteq \lim_\xi \mathcal{G}$ ,
2. For every  $\mathcal{F}, \mathcal{G} \in \mathbb{F}X$ ,  $\lim_\xi \mathcal{F} \cap \lim_\xi \mathcal{G} \subseteq \lim_\xi (\mathcal{F} \wedge \mathcal{G})$ ,
3. For  $x \in X$ ,  $x \in \lim_\xi \{x\}^\uparrow$ .

The pair  $(X, \xi)$  is called a *convergence space*, and these are the objects of the category **Conv** that was mentioned earlier. If the relation  $\xi$  only satisfies 1 and 2, then we call  $\xi$  a *preconvergence* and the pair  $(X, \xi)$  is likewise called a *preconvergence space*.

**Definition 1.15.** A function  $f : (X, \xi) \rightarrow (Y, \tau)$  is *continuous* if for every  $\mathcal{F} \in \mathbb{F}X$  and  $x \in X$

$$x \in \lim_\xi \mathcal{F} \implies f(x) \in \lim_\tau f[\mathcal{F}].$$

An equivalent formulation of the continuity of  $f$  is that  $f$  is continuous if

$$f(\lim_\xi \mathcal{F}) \subseteq \lim_\tau f[\mathcal{F}].$$

**Example 1.16.** We define the usual notion of convergence on  $\mathbb{R}$ , denoted  $\nu$ , by saying that for  $x \in \mathbb{R}$ ,  $\mathcal{F} \in \mathbb{F}\mathbb{R}$

$$x \in \lim_{\nu} \mathcal{F} \iff \mathcal{F} \geq \left\{ \left( x - \frac{1}{n}, x + \frac{1}{n} \right) : n \in \mathbb{N} \right\}^{\uparrow}.$$

**Example 1.17.** A map  $\mathcal{V}(\cdot) : X \rightarrow \mathbb{F}X$  with  $\{x\}^{\uparrow} \geq \mathcal{V}(x)$  for every  $x$  determines a convergence  $\xi$  on  $X$  by

$$x \in \lim_{\xi} \mathcal{F} \iff \mathcal{F} \geq \mathcal{V}(x),$$

then  $\xi$  is called a pretopology.

An equivalent method of classifying a convergence as a pretopology is that  $\xi$  is a pretopology if for any family of filters  $\{\mathcal{F}_i\}_{i \in I}$  we have the following

$$\lim_{\xi} \left( \bigwedge_{i \in I} \mathcal{F}_i \right) = \bigcap_{i \in I} \lim_{\xi} \mathcal{F}_i.$$

We say that  $O \subset X$  is  $\xi$ -open if whenever  $O$  contains limit points of  $\mathcal{F} \in \mathbb{F}X$  we have that  $O \in \mathcal{F}$ , or written symbolically,

$$\lim_{\xi} \mathcal{F} \cap O \neq \emptyset \implies O \in \mathcal{F}.$$

If we let  $\mathcal{O}_{\xi}$  denote the collection of  $\xi$ -open sets on  $X$ , then it is easy to check that  $\mathcal{O}_{\xi}$  satisfies the following conditions

1.  $\emptyset, X \in \mathcal{O}_{\xi}$
2. For any  $\mathcal{B} \subseteq \mathcal{O}_{\xi}$ ,  $\bigcup_{O \in \mathcal{B}} O \in \mathcal{O}_{\xi}$
3. For any  $\mathcal{B} \subseteq \mathcal{O}_{\xi}$  with  $\text{card} \mathcal{B} < \infty$ ,  $\bigcap_{O \in \mathcal{B}} O \in \mathcal{O}_{\xi}$ .

In other words,  $\mathcal{O}_{\xi}$  defines a topology on  $X$ . By Ex. 1.6, we can define a filter at each  $x \in X$ , called the *neighborhood filter*, by saying that a set  $A$  is a *neighborhood* of  $x$  if there is an  $O \in \mathcal{O}_{\xi}$  such that  $x \in O \subset A$  and considering the family  $\mathcal{N}_{\xi}(x)$  of neighborhoods of  $x$ .

**Example 1.18.** A convergence space  $(X, \xi)$  is a topological space if and only if  $x \in \lim_{\xi} \mathcal{N}_{\xi}(x)$  for every  $x \in X$ . Equivalently, for  $\mathcal{F} \in \mathbb{F}X$ ,  $x \in \lim_{\xi} \mathcal{F}$  if and only if  $\mathcal{F} \geq \mathcal{N}_{\xi}(x)$ .

Obviously, each topology is a pretopology, but the converse is not true in general.

**Example 1.19.** Let  $X = \{x_{\infty}\} \cup \{x_n : n < \infty\} \cup \{x_{n,k} : n, k < \infty\}$  with every element distinct. We can define a convergence  $\xi$  on  $X$  by  $x_{n,k} \in \lim_{\xi} \mathcal{F}$  if  $\mathcal{F} = \{x_{n,k}\}^{\uparrow}$ ,  $x_n \in \lim_{\xi} \mathcal{F}$  if  $\{x_n\}^{\uparrow} \wedge \{x_{n,k}\}_k^{\uparrow} \leq \mathcal{F}$ , and  $x_{\infty} \in \lim_{\xi} \mathcal{F}$  if  $\{x_{\infty}\}^{\uparrow} \wedge \{x_n\}_n^{\uparrow}$ .

It is easy to verify that  $\xi$  is a pretopology. To see that  $\xi$  is not a topology, let  $O$  be an open set containing  $x_{\infty}$ . Then there is are  $n_0, k_0 \in \mathbb{N}$  such that  $x_n \in O$  and  $x_{n,k} \in O$  for  $n \geq n_0$  and  $k \geq k_0$ . So the neighborhood filter of  $x_{\infty}$  is generated by sets like

$$\{x_{\infty}\} \cup \{x_n : n > n_0\} \cup \{x_{n,k} : k > k_0, n > n_0\}$$

and thus does not converge to  $x_{\infty}$  with respect to  $\xi$ .

However, this definition of a topology isn't always useful, and we instead find ourselves using the equivalent view of our convergence being *diagonal*. Before we do that, we have to define a *selection* map, which takes each point  $x \in X$  to a filter  $\mathcal{S}(x) \in \mathbb{F}X$  that converges to  $x$ . When we let this map act on a filter on  $X$ , we get what is called the *contour filter*  $\mathcal{S}(\mathcal{F})$  which is defined as

$$\mathcal{S}(\mathcal{F}) := \bigvee_{F \in \mathcal{F}} \bigwedge_{x \in F} \mathcal{S}(x).$$

**Definition 1.20.** A convergence  $\xi$  is diagonal if for every selection  $\mathcal{S} : X \rightarrow \mathbb{F}X$  and every filter  $\mathcal{F}$  converging to  $x \in X$ , the contour filter converges to  $x$ :

$$x \in \lim_{\xi} \mathcal{S}(\mathcal{F}).$$

With diagonality, we obtain a more useful characterization of topologies as convergences.



**Proposition 1.21.** *A convergence  $\xi$  is a topology if and only if  $\xi$  is a pretopology and is diagonal.*

**Theorem 1.22.** *A convergence  $\xi$  is a topology if and only if for every set  $A$ , every map  $l : A \rightarrow X$  and  $\mathcal{S} : X \rightarrow \mathbb{F}X$  such that  $l(a) \in \lim_{\xi} \mathcal{S}(a)$  for each  $a \in A$ ,*

$$\lim_{\xi} l[\mathcal{F}] \subseteq \lim_{\xi} \mathcal{S}(\mathcal{F})$$

*for every  $\mathcal{F} \in \mathbb{F}A$ .*

Since we have a way of talking about a topology on a space by considering the convergence structure of the space and have seen how to interpret continuous maps in this language, a natural question is how we can talk about other topological ideas in terms of convergences. While we could give convergence generalizations of all of the ideas of topology, this is outside of the scope of this paper, and we will instead only look at the notions that we will need to state the theorems of interest.

The first notion that we will look to generalize is that of the topological closure. Recall that the topological closure of  $A \subseteq X$ , denoted  $\text{cl}(A)$ , is the collection of all points  $x \in X$  such that every neighborhood of  $x$  contains a point of  $A$ . Note that this means that  $A \subseteq \text{cl}A$ . So in the case of filters, when the convergence  $\xi$  is a topology, we want our generalization of closure to have  $A \# \mathcal{N}_{\xi}(x)$  for every  $x$  in the generalized closure.

**Definition 1.23.** *If  $A$  is a subset of a convergence space  $X$ , then the adherence of  $A$  is defined by*

$$\text{adh}_{\xi} A := \bigcup_{A \in \mathcal{F}^{\#}} \lim_{\xi} \mathcal{F}$$

**Proposition 1.24.** *For any subset  $A$  of a convergence space*

$$\text{adh}_{\xi} A = \bigcup_{A \in \mathcal{F}^{\#}} \lim_{\xi} \mathcal{F} = \bigcup_{A \in \mathcal{G}} \lim_{\xi} \mathcal{G} = \bigcup_{A \in \mathcal{U} \in \mathbb{U}X} \lim_{\xi} \mathcal{U}$$

**Claim 1.25.** *If  $\xi$  is a topology, then  $\text{cl}(A) = \text{adh}_\xi A$  for every  $A \subset X$*

*Proof.* First, let  $x \in \text{cl}(A)$ . Then we have that  $A \in \mathcal{N}_\xi(x)^\#$ , and since  $\xi$  is a topology, then we have that  $x \in \lim_\xi \mathcal{N}_\xi(x)$ , so  $x \in \text{adh}_\xi A$ .

Conversely, let  $x \in \text{adh}_\xi A$ . Then there is a  $\mathcal{F} \in \mathbb{F}X$  with  $A \in \mathcal{F}^\#$  such that  $x \in \lim_\xi \mathcal{F}$ . Since  $\xi$  is a topology,  $x \in \lim_\xi \mathcal{F}$  if and only if  $\mathcal{F} \geq \mathcal{N}_\xi(x)$ . Then we have that  $A \in \mathcal{N}_\xi(x)^\#$ , so we conclude that  $x \in \text{cl}(A)$  since  $A \in \mathcal{N}_\xi(x)^\#$ .  $\square$

Since we can apply  $\text{adh}_\xi$  to any subset of  $X$ , we can also apply it to any family of subsets of  $X$ . In particular, if  $\mathcal{F} \in \mathbb{F}X$ , we consider

$$\text{adh}_\xi^\natural \mathcal{F} := \{\text{adh}_\xi F : F \in \mathcal{F}\}^\uparrow.$$

Since several of our main results deal with regularity, we need to generalize the idea into convergence spaces. Recall that a topological space  $(X, \tau)$  is said to be *regular* if for every closed  $A \subset X$  and every  $x \notin A$ , there are disjoint  $U, V \in \tau$  such that  $x \in U$  and  $A \subset V$ .

**Proposition 1.26.** *A topological space  $(X, \xi)$  is regular if and only if for every  $x \in X$ ,  $\mathcal{N}_\xi(x) = \text{adh}_\xi^\natural(\mathcal{N}_\xi(x))$ .*

This leads to the the following definition.

**Definition 1.27.** *A convergence  $\xi$  is regular if for every filter  $\mathcal{F}$ ,*

$$\lim_\xi \mathcal{F} \subseteq \lim_\xi \text{adh}_\xi^\natural \mathcal{F}.$$

We can obtain a characterization of regularity dual to that of topologies given in Theorem 1.22.

**Theorem 1.28.** *A convergence space  $(X, \xi)$  is regular if and only if for every set  $A$ , every map  $l : A \rightarrow X$ , and  $\mathcal{S} : A \rightarrow \mathbb{F}X$  with  $l(a) \in \lim_\xi \mathcal{S}(a)$  for each  $a \in A$ ,*

$$\lim_\xi \mathcal{S}(\mathcal{F}) \subseteq \lim_\xi l[\mathcal{F}].$$

The first of our main theorems is a generalization of Thm. 2.6 in [12]. Adapting this theorem to the convergence space setting requires us to revisit the problem that first motivated our interest in convergence spaces: for  $X, Y, Z \in \text{ob}(\mathbf{Conv})$ , what is the structure required on  $C(Y, Z)$  so that  $C(X \times Y, Z) \equiv C(X, C(Y, Z))$ ?

To answer this question, we begin by defining the relation  $[X, Y]$  on the space of all functions from  $X$  to  $Y$ . To do this, we say that for  $f \in Y^X$  and  $\mathcal{F} \in \mathbb{F}(Y^X)$ ,

$$f \in \lim_{[X, Y]} \mathcal{F} \Leftrightarrow \forall x \in X, \forall \mathcal{G} \in \mathbb{F}X, (x \in \lim_{\mathcal{G}} \Rightarrow f(x) \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle).$$

where

$$\langle \mathcal{G}, \mathcal{F} \rangle := \{ \langle G, F \rangle : G \in \mathcal{G}, F \in \mathcal{F} \}^\dagger$$

and

$$\langle G, F \rangle := \{ h(g) : g \in G, h \in F \}.$$

It turns out that  $[X, Y]$  only satisfies properties 1 and 2 in 1.14, and that  $f \in \lim_{[X, Y]} \{f\}^\dagger$  only when  $f \in C(X, Y)$ . So  $[X, Y]$  defines a convergence on  $C(X, Y)$  called the *continuous convergence*, and it is exactly this structure that is required to make  $C(X \times Y, Z) \equiv C(X, [Y, Z])$  in  $\mathbf{Conv}$ .

**Theorem 1.29.** [12, Thm. 2.6] *A convergence space  $(Y, \tau)$  is regular if and only if for every topological space  $X$ , every  $f \in Y^X$  and every  $\mathcal{F} \in \mathbb{F}(Y^X)$ ,  $f \in \lim_{[X, Y]} \mathcal{F}$  implies that  $f \in C(X, Y)$ .*

This theorem generalizes the fact that while the pointwise limit of a sequence of continuous functions need not be continuous, the uniform limit of a sequence of continuous functions is continuous.

The second theorem that we are interested in generalizing is due to Frič and Kent and deals with extending a function from what they call a *strict subspace* to a larger subspace.

**Definition 1.30.** Let  $(X, \xi)$  be a convergence space and  $S \subseteq X$ . Then  $S$  is a strict subspace if for every  $x \in \text{adh}_\xi S$  and  $\mathcal{F} \in \mathbb{F}(\text{adh}_\xi S)$  with  $x \in \lim_\xi \mathcal{F}$ , there is a  $\mathcal{G} \in \mathbb{F}S$  such that  $\text{adh}_\xi^h \mathcal{G} \leq \mathcal{F}$  and  $x \in \lim_\xi \mathcal{G}$ .

Consider  $S \subseteq X$  and a continuous function  $f : S \rightarrow Y$ . If  $x \in \text{adh}S$ , then for  $f$  to extend to  $S \cup \{x\}$  continuously, it is necessary (and sufficient) that  $f(x) \in \bigcap_{\mathcal{F} \in \mathbb{F}S, x \in \lim \mathcal{F}} \lim_\tau f[\mathcal{F}]$ .

**Definition 1.31.** Let  $(X, \xi)$  and  $(Y, \tau)$  be two convergence spaces,  $S \subseteq X$ , and  $f : (S, \xi|_S) \rightarrow (Y, \tau)$  be continuous. The hull of extensionability of  $S$  for  $f$  is

$$h(S, f) := \{x \in \text{adh}_\xi S : \bigcap_{\mathcal{F} \in \mathbb{F}S, x \in \lim_\xi \mathcal{F}} \lim_\tau f[\mathcal{F}] \neq \emptyset\}.$$

**Theorem 1.32.** A convergence space  $(Y, \tau)$  is regular if and only if whenever  $S$  is a strict subspace of a convergence space  $(X, \xi)$  and  $f : (S, \xi|_S) \rightarrow (Y, \tau)$  is a continuous map, there is a continuous map  $\hat{f} : (h(S, f), \xi|_{h(S, f)}) \rightarrow (Y, \tau)$  such that  $\hat{f}|_S = f$ .

Theorem 3.10 generalizes this result to **Cap**.

## CHAPTER 2

### APPROACH SPACES

In [8], R. Lowen showed that there is a category that contains the categories **Top** (topological spaces with continuous maps) and **Met** (metric spaces with contractive maps) as full subcategories. That is, from a categorical point of view, that topological spaces and metric spaces can be considered as special cases of a common type of object. One of the insights that sparked this result was that certain topological notions have metric counterparts that have similar characterizations.

For example, in topological spaces, compactness of a topological space is similar to the concept of total boundedness of a metric space. Recall that a topological space  $(X, \tau)$  is a *compact space* if for any open covering of the space, there is a finite subcover. A metric space  $(X, d)$  is a *totally bounded space* if and only if for every  $\epsilon > 0$ , there is a finite collection of open balls of radius  $\epsilon$  that covers  $X$ . It turns out that both notions are instances of *measure of compactness 0* in approach spaces.

This insight led to the introduction of the category **Ap**, with *approach spaces* as objects and *contractions* as morphisms. Since **Ap** contains both **Top** and **Met**, we are able to combine the notions that make topological spaces nice to use without having to give up the ability to obtain the quantifications that make metric spaces nice. In fact, in **Ap** we are able to measure how much structure a space has, such as how close a topological space is to being compact.

However, it turns out that **Ap** and **Top** share the same categorical problems. Luckily **Ap** can be embedded in a larger but better behaved category, in the same way that we were able to embed **Top** into **Conv**. This generalization, **Cap** (convergence approach spaces with contractions as morphisms) contains **Conv** and **Met** as full subcategories and is free of the categorical problems present in **Ap**.

This chapter will be focused on providing the basics of the theory of convergence

approach spaces, and, as in the first chapter, many results will be stated without proof. The interested reader is directed to [9] for a reference on approach spaces.

## 2.1 Convergence Approach Spaces

In our treatment of convergence approach spaces, we will define the spaces by focusing on functions  $\lambda : \mathbb{F}X \rightarrow [0, \infty]^X$  known as *limits*, that we compare pointwise.

**Definition 2.1.** *A function  $\lambda : \mathbb{F}X \rightarrow [0, \infty]^X$  is called a (convergence-approach) limit if it satisfies the following properties:*

1. For any  $x \in X$ ,  $\lambda(\{x\}^\dagger)(x) = 0$ ,
2. For  $\mathcal{F}, \mathcal{G} \in \mathbb{F}X$ , if  $\mathcal{F} \leq \mathcal{G}$ , then  $\lambda(\mathcal{G}) \leq \lambda(\mathcal{F})$ ,
3. For all  $\mathcal{F}, \mathcal{G} \in \mathbb{F}X$ ,  $\lambda(\mathcal{F} \wedge \mathcal{G}) = \lambda(\mathcal{F}) \vee \lambda(\mathcal{G})$ ,

where  $\lambda(\mathcal{F}) \vee \lambda(\mathcal{G}) = \sup\{\lambda(\mathcal{F}), \lambda(\mathcal{G})\}$  in  $[0, \infty]^X$  ordered pointwise.

A limit can be thought of as a map that measures how close a filter is to converging to a point  $x \in X$ . So, the first condition can be interpreted as saying that the principal filter of a point fully converges to the point. Similarly, the second condition states the finer the filter, the better it converges. The pair  $(X, \lambda)$  is called a *convergence approach space*, and these are the objects of **Cap**.

Now, we turn our attention to defining the morphisms of **Cap**.

**Definition 2.2.** *For two convergence approach spaces  $(X, \lambda_X)$  and  $(Y, \lambda_Y)$  a map  $f : (X, \lambda_X) \rightarrow (Y, \lambda_Y)$  is called a contraction if for every  $x \in X$*

$$\lambda_Y(f[\mathcal{F}])(f(x)) \leq \lambda_X(\mathcal{F})(x)$$

<b>Cap</b>	$\lambda(\{x\}^\uparrow)(x) = 0$ $\mathcal{F} \geq \mathcal{G} \implies \lambda(\mathcal{G}) \leq \lambda(\mathcal{F})$ $\lambda(\mathcal{F} \wedge \mathcal{G}) = \lambda(\mathcal{F}) \vee \lambda(\mathcal{G})$	<b>Conv</b>	$x \in \lim_\xi \{x\}^\uparrow$ $\mathcal{F} \geq \mathcal{G} \implies \lim_\xi \mathcal{G} \subseteq \lim_\xi \mathcal{F}$ $\lim_\xi(\mathcal{F} \wedge \mathcal{G}) = \lim_\xi(\mathcal{F}) \cap \lim_\xi(\mathcal{G})$
<b>PrAp</b>	$\lambda(\bigwedge_{i \in I} \mathcal{F}_i) = \bigvee_{i \in I} \lambda(\mathcal{F}_i)$	<b>PrTop</b>	$\lim_\xi(\bigwedge_{i \in I} \mathcal{F}_i) = \bigcap_{i \in I} \lim_\xi \mathcal{F}_i$
<b>Ap</b>	<b>PrAp</b> + $\lambda(\mathcal{S}(\mathcal{F})) = \lambda(\mathcal{F}) + \bigvee_{x \in X} \lambda(\mathcal{S}(x))(x)$	<b>Top</b>	<b>PrTop</b> + diagonal

Table 2.1: The relationship between **Cap** and **Conv**

We can consider any convergence space  $(X, \xi)$  as a convergence approach space by defining its limit in the following way

$$\lambda_\xi(\mathcal{F})(x) = \begin{cases} 0 & \text{if } x \in \lim_\xi \mathcal{F} \\ \infty & \text{otherwise.} \end{cases}$$

Also, a map from  $f : (X, \xi) \rightarrow (Y, \tau)$  is continuous if and only if it is a contraction from  $(X, \lambda_\xi)$  to  $(Y, \lambda_\tau)$ .

Table 2.1 demonstrates the relationship between convergence approach spaces and convergence spaces, as well as their associated subcategories. There are several other ways of defining an approach space, such as using distance functions, but each of these can be shown to be equivalent [9], so we will focus on limit functions.

Comparing the convergence side of the table and the convergence approach side, we see that the condition that

$$\lambda(\mathcal{S}(\mathcal{F})) = \lambda(\mathcal{F}) + \bigvee_{x \in X} \lambda(\mathcal{S}(x))(x)$$

is the convergence approach generalization of diagonality.

The generalization of adherence to **Cap** is defined by letting  $A \subseteq X$  and  $\epsilon \geq 0$  and considering the set

$$A^{(\epsilon)} := \{x \in X : \exists \mathcal{U} \in \mathbb{U}X, A \in \mathcal{U}, \lambda(\mathcal{U})(x) \leq \epsilon\},$$

and extending it to  $\mathcal{G} \subseteq \mathcal{P}(X)$  by

$$\mathcal{G}^{(\epsilon)} := \{G^{(\epsilon)} : G \in \mathcal{G}\}.$$

With this generalization of adherence to convergence approach spaces, we can now generalize regularity of a space to convergence approach spaces.

**Definition 2.3.** A convergence approach space  $(X, \lambda)$  is regular if for every  $\mathcal{F} \in \mathbb{F}X$ ,  $\epsilon \geq 0$ , and  $x \in X$

$$\lambda(\mathcal{F}^{(\epsilon)})(x) \leq \lambda(\mathcal{F})(x) + \epsilon.$$

**Definition 2.4.** A convergence approach space  $(X, \lambda)$  is strongly regular if for every  $\mathcal{F} \in \mathbb{F}X$ ,  $\epsilon \geq 0$ , and  $x \in X$

$$\lambda(\mathcal{F}^{(\epsilon)})(x) \leq \lambda(\mathcal{F})(x) \vee \epsilon.$$

Let  $\oplus : [0, \infty] \rightarrow [0, \infty]$  be a commutative and associate binary operation that satisfies the following two conditions

$$0 \oplus r = r \tag{2.1}$$

$$r \oplus \bigwedge_{a \in A} a = \bigwedge_{a \in A} (r \oplus a) \tag{2.2}$$

for every  $r \in [0, \infty]$  and  $A \subset [0, \infty]$ . This is the same as saying that  $[0, \infty]$  with reverse order is a unital quantale in the sense of [11]. In the case of the non-negative reals, the two main examples of a unital quantale are standard addition  $+$  and pairwise maximum  $\vee$ . This tensor preserves order, that is

$$a \leq b \text{ and } c \leq d \implies a \oplus c \leq b \oplus d, \tag{2.3}$$

and it also respects limits,

$$(a + \epsilon) \oplus (b + \epsilon) \rightarrow a \oplus b, \text{ as } \epsilon \rightarrow 0. \tag{2.4}$$

Combining (2.1) and (2.3), it is easy to see that  $\vee \leq \oplus$ .

Using this tensor, we are able to generalize both cases of regularity that were just introduced with the following definition.



**Definition 2.5.** A convergence approach space  $(X, \lambda)$  is  $\oplus$ -regular if for every  $\mathcal{F} \in \mathbb{F}X$ ,  $\epsilon \geq 0$ , and  $x \in X$

$$\lambda(\mathcal{F}^{(\epsilon)})(x) \leq \lambda(\mathcal{F})(x) \oplus \epsilon.$$

Of course, in this definition, if we take  $\oplus$  to be  $+$  or  $\vee$  then we obtain the definitions for regularity or strong regularity respectively. If for every  $\mathcal{F} \in \mathbb{F}X$  2.5 holds for some  $x \in X$ , then we call  $x$  an  $\oplus$ -regularity point.

We are also able to generalize diagonality using the tensor.

**Definition 2.6.** A convergence approach space  $(X, \lambda)$  is  $\oplus$ -diagonal if for every map  $\mathcal{S} : X \rightarrow \mathbb{F}X$  and  $\mathcal{F} \in \mathbb{F}X$  we have that

$$\lambda(\mathcal{S}(\mathcal{F}))(\cdot) = \lambda(\mathcal{F})(\cdot) \oplus \bigvee_{x \in X} \lambda(\mathcal{S}(x))(x)$$

Similar to the case in convergence spaces, we have an alternative way of characterizing regularity using maps from a non-empty subset and selection maps.

**Proposition 2.7.** A convergence approach space is  $\oplus$ -regular if and only if for every  $A \neq \emptyset$ ,  $l : A \rightarrow X$ ,  $\mathcal{F} \in \mathbb{F}A$ , and  $\mathcal{S} : X \rightarrow \mathbb{F}X$ ,

$$\lambda(l[\mathcal{F}])(\cdot) \leq \lambda(\mathcal{S}(\mathcal{F}))(\cdot) \oplus \bigvee_{a \in A} \lambda(\mathcal{S}(a))(l(a)).$$

The final thing that we need to generalize is the continuous convergence  $[X, Y]$  on  $Y^X$ . For two convergence approach spaces  $(X, \lambda_X)$  and  $(Y, \lambda_Y)$ , the limit on the space  $C(X, Y)$  of contractions from  $X$  to  $Y$ , is defined by

$$\lambda_{[X, Y]}(\mathcal{F})(f) := \inf\{\alpha \in [0, \infty] : \forall \mathcal{G} \in \mathbb{F}X, \mathcal{S} \lambda_Y \langle \mathcal{G}, \mathcal{F} \rangle (f(\cdot)) \leq \lambda_X(\mathcal{G})(\cdot) \vee \alpha\},$$

where  $\langle \mathcal{G}, \mathcal{F} \rangle$  is defined as it was in the case of convergence spaces.

**CHAPTER 3**  
**REGULARITY IN Cap**

**3.1 Regularity and continuous convergence**

We define the *default of contraction* of a function  $f \in Y^X$ , denoted  $m_+(f)$ , in the following way:

$$m_+(f) := \inf\{\alpha \in [0, \infty] : \forall \mathcal{G} \in \mathbb{F}X, \lambda_Y(f[\mathcal{G}]) \circ f \leq \lambda_X(\mathcal{G}) + \alpha\}.$$

The default of contraction measures how far away the function is from being a contraction, so it is clear that  $f$  is a contraction if and only if  $m_+(f) = 0$ . We generalize the default of contraction using the tensor  $\oplus$  that was defined in the previous chapter by

$$m_{\oplus}(f) := \inf\{\alpha \in [0, \infty] : \forall \mathcal{G} \in \mathbb{F}X, \lambda_Y(f[\mathcal{G}]) \circ f \leq \lambda_X(\mathcal{G}) \oplus \alpha\}.$$

For our two examples of this tensor,  $+$  and  $\vee$ , it is easy to see that for every  $f$ ,

$$m_+(f) \leq m_{\vee}(f)$$

because  $a \vee b \leq a + b$  for all  $a, b \in [0, \infty]$ .

Theorem 1.29 states that if the codomain is regular, then  $[X, Y]$ -limits are automatically continuous. In the case of convergence approach spaces, we will see that the level of convergence in  $[X, Y]$  controls the default of contraction:

**Theorem 3.1.** [1] *If  $Y$  is a  $\oplus$ -regular convergence-approach space,  $X$  is a convergence-approach space, and  $f \in Y^X$  then*

$$m_{\oplus}(f) \leq \left( \bigwedge_{\mathcal{F} \in \mathbb{F}(Y^X)} \lambda_{[X, Y]}(\mathcal{F})(f) \right) \oplus \left( \bigwedge_{\mathcal{F} \in \mathbb{F}(Y^X)} \lambda_{[X, Y]}(\mathcal{F})(f) \right).$$

*Conversely, if  $Y$  is not  $\oplus$ -regular, there is a topological space  $X$  and  $f \in Y^X$  with*

$$m_{\oplus}(f) > \left( \bigwedge_{\mathcal{F} \in \mathbb{F}(Y^X)} \lambda_{[X, Y]}(\mathcal{F})(f) \right).$$

In particular, considering convergence spaces as convergence-approach spaces, we get as an immediate corollary:

**Corollary 3.2.** *Let  $Y$  be a convergence space. The following are equivalent:*

1.  $Y$  is regular;

2.

$$f \in \lim_{[X,Y]} \mathcal{F} \implies f \in C(X, Y)$$

for every convergence space  $X$ , every  $f \in Y^X$  and every  $\mathcal{F} \in \mathbb{F}(Y^X)$ ;

3.

$$f \in \lim_{[X,Y]} \mathcal{F} \implies f \in C(X, Y)$$

for every topological space  $X$ , every  $f \in Y^X$  and every  $\mathcal{F} \in \mathbb{F}(Y^X)$ .

In particular, this result generalizes [12, Theorem 2.6] of Wolk, which establishes the equivalence between (1) and (3), under the assumption that  $Y$  be topological.

**Corollary 3.3.** *If a convergence-approach space  $Y$  is regular then for every convergence-approach space  $X$  and  $f \in Y^X$ ,*

$$m_+(f) \leq 2 \bigwedge_{\mathcal{F} \in \mathbb{F}(Y^X)} \lambda_{[X,Y]}(\mathcal{F})(f).$$

*If  $Y$  is not regular, there is a topological space  $X$  and  $f \in Y^X$  with*

$$\bigwedge_{\mathcal{F} \in \mathbb{F}(Y^X)} \lambda_{[X,Y]}(\mathcal{F})(f) < m_+(f).$$

**Corollary 3.4.** *A convergence-approach space  $Y$  is strongly regular if and only if for every convergence approach (equivalently, topological) space  $X$  and  $f \in Y^X$ ,*

$$m_\vee(f) \leq \bigwedge_{\mathcal{F} \in \mathbb{F}(Y^X)} \lambda_{[X,Y]}(\mathcal{F})(f).$$

We will need the following observation to prove Theorem 3.1.

**Lemma 3.5.** *If  $\alpha \in [0, \infty]$ ,  $\mathcal{G} \in \mathbb{F}X$ ,  $\mathcal{F} \in \mathbb{F}(Y^X)$  and  $f \in Y^X$  satisfy*

$$\lambda_Y(\langle x, \mathcal{F} \rangle)(f(x)) \leq \alpha,$$

for every  $x \in X$ , then

$$f[\mathcal{G}] \geq \langle \mathcal{G}, \mathcal{F} \rangle^{(\alpha)}.$$

Note that the case  $\alpha = 0$  states that if  $\mathcal{F}$  converges pointwise to  $f \in Y^X$  then for any  $\mathcal{G} \in \mathbb{F}X$ ,  $f[\mathcal{G}] \geq \text{adh}_{c(Y)}^{\natural} \langle \mathcal{G}, \mathcal{F} \rangle$ , where  $c(Y)$  is the convergence defined by  $x \in \lim_{c(Y)} \mathcal{F}$  if and only if  $\lambda_Y(\mathcal{F})(x) = 0$ . <sup>(1)</sup>

*Proof.* Let  $x \in G$  for some  $G \in \mathcal{G}$ . We consider the filter  $\langle \{x\}^\uparrow, \mathcal{F} \rangle$  on  $\langle G, F \rangle$  for  $F \in \mathcal{F}$ . Then by the assumption,

$$\lambda_Y(\langle \{x\}^\uparrow, \mathcal{F} \rangle)(f(x)) \leq \alpha,$$

so  $f(x) \in \langle G, F \rangle^{(\alpha)}$ . Thus  $f(G) \subseteq \langle G, F \rangle^{(\alpha)}$  for any  $G \in \mathcal{G}$  and  $F \in \mathcal{F}$ , so  $f[\mathcal{G}] \geq \langle \mathcal{G}, \mathcal{F} \rangle^{(\alpha)}$ .  $\square$

*Proof of Theorem 3.1.* Let  $Y$  be a  $\oplus$ -regular convergence-approach space and let

$$c := \bigwedge_{\mathcal{F} \in \mathbb{F}(Y^X)} \lambda_{[X, Y]}(\mathcal{F})(f).$$

For  $\epsilon > 0$ , there is an  $\mathcal{F}_\epsilon \in \mathbb{F}(Y^X)$  such that  $\lambda_{[X, Y]}(\mathcal{F}_\epsilon)(f) < c + \epsilon$ , and, by definition of  $\lambda_{[X, Y]}$ , there is  $\alpha_\epsilon < \lambda_{[X, Y]}(\mathcal{F}_\epsilon)(f) + \epsilon < c + 2\epsilon$  such that  $\lambda \langle \mathcal{G}, \mathcal{F}_\epsilon \rangle \circ f \leq \lambda_X(\mathcal{G}) \vee \alpha_\epsilon$  for every  $\mathcal{G} \in \mathbb{F}X$ . In particular,

$$\lambda \langle \{x\}^\uparrow, \mathcal{F}_\epsilon \rangle (f(x)) \leq \lambda_X(\{x\}^\uparrow)(x) \vee \alpha_\epsilon = \alpha_\epsilon$$

so that  $f[\mathcal{G}] \geq \langle \mathcal{G}, \mathcal{F}_\epsilon \rangle^{(\alpha_\epsilon)}$  by Lemma 3.5.

---

<sup>1</sup> $c(Y)$  is known as the **Conv-coreflection** of  $Y$ .

Since  $Y$  is  $\oplus$ -regular,

$$\begin{aligned} \lambda_Y(\langle \mathcal{G}, \mathcal{F}_\epsilon \rangle^{(\alpha_\epsilon)}) \circ f &\leq \lambda \langle \mathcal{G}, \mathcal{F}_\epsilon \rangle \circ f \oplus \alpha_\epsilon \leq (\lambda_X(\mathcal{G}) \vee \alpha_\epsilon) \oplus \alpha_\epsilon, \\ &\leq \lambda_X(\mathcal{G}) \oplus (\alpha_\epsilon \oplus \alpha_\epsilon), \end{aligned}$$

using (2.3) and the fact that  $\vee \leq \oplus$ . Thus  $\lambda_Y(f[\mathcal{G}]) \circ f \leq \lambda_X(\mathcal{G}) \oplus (\alpha_\epsilon \oplus \alpha_\epsilon)$  because  $f[\mathcal{G}] \geq \langle \mathcal{G}, \mathcal{F}_\epsilon \rangle^{(\alpha_\epsilon)}$ . However,  $\alpha_\epsilon \oplus \alpha_\epsilon < (c + 2\epsilon) \oplus (c + 2\epsilon)$ , and since  $\epsilon$  is arbitrary, the inequality becomes  $\lambda_Y(f[\mathcal{G}]) \circ f \leq \lambda_X(\mathcal{G}) \oplus (c \oplus c)$  by (2.4), and we conclude that  $m_\oplus(f) \leq c \oplus c$ .

For the converse, assume that  $Y$  is not  $\oplus$ -regular. Then in view of Proposition 2.7, there exists  $A \neq \emptyset$ ,  $l : A \rightarrow Y$ ,  $\mathcal{S} : A \rightarrow \mathbb{F}Y$ ,  $\mathcal{H} \in \mathbb{F}A$ , and  $y_0 \in Y$  such that

$$\lambda_Y(l[\mathcal{H}])(y_0) > \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \oplus \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a)). \quad (3.1)$$

From this, we build a topological approach space  $X$ , a filter  $\mathcal{F}_0$  on  $Y^X$ , and a function  $f \in Y^X$  with  $m_\oplus(f) > \lambda_{[X,Y]}(\mathcal{F}_0)(f)$ .

### The space $X$ and function $f$

Let  $X := (Y \times A) \cup A \cup \{x_\infty\}$  where  $x_\infty \notin A$ . Define  $p_Y : Y \times A \rightarrow Y$  by  $p_Y(y, a) = y$  for all  $(y, a) \in Y \times A$ , and let  $f : X \rightarrow Y$  be defined by  $f|_A = l$ ,  $f|_{Y \times A} = p_Y$ , and  $f(x_\infty) = y_0$ . Let

$$\mathcal{N} := \bigcup_{H \in \mathcal{H}} \bigcap_{a \in H} (\mathcal{S}(a) \times \{a\}^\dagger) \wedge \{a\}^\dagger.$$

Now we define  $\lambda_X$  by, for all  $a \in A$  and  $y \in Y$ :

$$\lambda_X(\mathcal{G})((y, a)) := \begin{cases} 0 & \text{if } \mathcal{G} = \{(y, a)\}^\uparrow \\ \infty & \text{otherwise} \end{cases}$$

$$\lambda_X(\mathcal{G})(a) := \begin{cases} 0 & \text{if } \mathcal{G} \geq (\mathcal{S}(a) \times \{a\}^\uparrow) \wedge \{a\}^\uparrow \\ \infty & \text{otherwise} \end{cases}$$

$$\lambda_X(\mathcal{G})(x_\infty) := \begin{cases} 0 & \text{if } \mathcal{G} \geq \mathcal{N} \wedge \{x_\infty\}^\uparrow \\ \infty & \text{otherwise.} \end{cases}$$

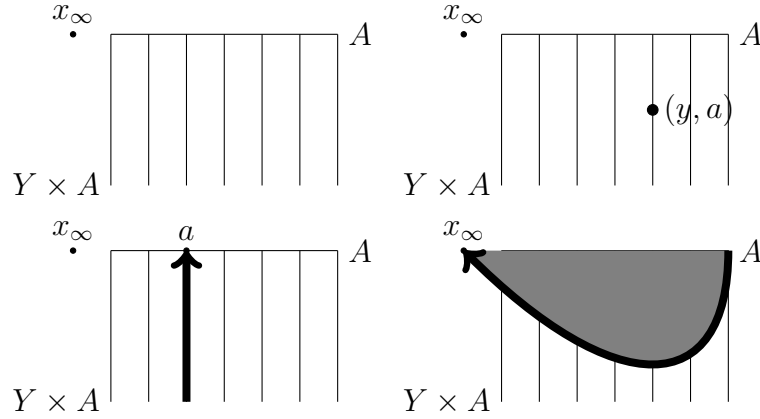


Figure 3.1: The space  $X$  and the approach structure defined on  $X$

Note that  $X$  is then a topological CAP space, and that

$$m_\oplus(f) > \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \oplus \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a)),$$

because of (3.1) and  $f[\mathcal{H}] = l[\mathcal{H}]$ .

### The filter $\mathcal{F}_0$

Let  $P := \{h \in Y^X : h|_{Y \times A} = p_Y \text{ and } h(x_\infty) = y_0\}$ , and for each  $a \in A$ , let

$$\hat{a} : Y^X \rightarrow Y$$

$$h \mapsto h(a).$$

Let

$$\mathcal{A} := \bigcup_{a \in A} \{\hat{a}^{-1}(S) \cap P : a \in A, S \in \mathcal{S}(a)\}$$

$$\mathcal{B} := \bigcup_{H \in \mathcal{H}} \left\{ \bigcap_{a \in H} \hat{a}^{-1}(S_a^H) \cap P : (S_a^H)_{a \in H} \in \prod_{a \in H} \mathcal{S}(a) \right\}.$$

Then  $\mathcal{A} \cup \mathcal{B}$  has the finite intersection property, in the sense of Def 1.9, and thus generates a filter  $\mathcal{F}_0$  on  $Y^X$ .

Controlling  $\lambda_{[X,Y]}(\mathcal{F}_0)(f)$

It suffices to show that

$$\lambda_{[X,Y]}(\mathcal{F}_0)(f) \leq \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \vee \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a))$$

because then

$$\lambda_{[X,Y]}(\mathcal{F}_0)(f) \leq \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \oplus \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a)) < m_{\oplus}(f).$$

To this end, by definition of  $[X, Y]$ , we only need to show that

$$\lambda_Y(\langle \mathcal{G}, \mathcal{F}_0 \rangle)(f(x)) \leq \lambda_Y(\mathcal{S}(\mathcal{H}))(f(x)) \vee \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a)) \vee \lambda_X(\mathcal{G})(x)$$

for every  $\mathcal{G} \in \mathbb{F}X$  and  $x \in X$ .

If  $\mathcal{G} \geq \mathcal{N} \wedge \{x_{\infty}\}^{\uparrow}$  then  $\langle \mathcal{G}, \mathcal{F}_0 \rangle \geq \mathcal{S}(\mathcal{H}) \wedge \{y_0\}^{\uparrow}$  since for each  $B \in \mathcal{S}(\mathcal{H})$  there is  $H_B \in \mathcal{H}$  and for each  $a \in H_B$ , there is  $S_a \in \mathcal{S}(a)$  such that  $\bigcup_{a \in H_B} S_a \subseteq B$  and

$$\left\langle \bigcup_{a \in H_B} ((S_a \times \{a\}) \cup \{a\}), \bigcap_{a \in H_B} \hat{a}^{-1}(S_a) \cap P \right\rangle \subseteq \bigcup_{a \in H_B} S_a.$$

Thus

$$\lambda_Y(\langle \mathcal{G}, \mathcal{F}_0 \rangle)(y_0) \leq \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) = \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \vee \lambda_X(\mathcal{G})(x_{\infty}).$$

If  $\mathcal{G} \geq (\mathcal{S}(a) \times \{a\})^{\uparrow} \wedge \{a\}^{\uparrow}$  for some  $a \in A$ , then

$$\langle \mathcal{G}, \mathcal{F}_0 \rangle \geq \mathcal{S}(a).$$

Indeed, by definition of  $P$ ,  $\langle S \times \{a\}, \hat{a}^{-1}(S) \cap P \rangle \subseteq S$  for any  $S \in \mathcal{S}(a)$ . Moreover,  $\langle a, \hat{a}^{-1}[\mathcal{S}(a)] \vee P \rangle \geq \mathcal{S}(a)$ , so that  $\langle a, \mathcal{F}_0 \rangle \geq \mathcal{S}(a)$ .

Thus, taking into account that  $f(a) = l(a)$ ,

$$\lambda_Y(\langle \mathcal{G}, \mathcal{F}_0 \rangle)(f(a)) \leq \lambda_Y(\mathcal{S}(a))(l(a)) = \lambda_Y(\mathcal{S}(a))(l(a)) \vee \lambda_X(\mathcal{G})(a).$$

Finally, if  $\mathcal{G}$  is a principal ultrafilter  $\{t\}^\uparrow$  then  $\langle \{t\}^\uparrow, \mathcal{F}_0 \rangle = \{f(t)\}^\uparrow$  if  $t \in (Y \times A) \cup \{x_\infty\}$ , by definition of  $f$  and  $\mathcal{F}_0$ . If  $t = a \in A$ , however, we have  $\langle a, \mathcal{F}_0 \rangle \geq \mathcal{S}(a)$ , so that  $\lambda_Y \langle a, \mathcal{F}_0 \rangle (f(a)) \leq \lambda_Y(\mathcal{S}(a))(l(a))$ .

Thus

$$\lambda_{[X,Y]}(\mathcal{F}_0)(f) \leq \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \vee \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a)).$$

□

### 3.2 Regularity and contractive Extensions

In this section, we investigate the conditions under which a contractive map  $f : S \rightarrow Y$ , where  $S \subset X$  and  $X, Y$  are CAP spaces, can be extended to a contraction defined on a larger subset of  $X$ . In particular, we will provide a generalization of Theorem 1.32. First, we need a convergence approach analogue of the hull of extensionability of Def 1.31.

We proceed following the terminology used in [6]. Given two CAP spaces  $X$  and  $Y$ ,  $x \in X$ ,  $S \subset X$ ,  $f : S \rightarrow Y$  and  $\alpha, \epsilon \in [0, \infty]$ , define

$$\begin{aligned} H_S^\epsilon(x) &:= \{ \mathcal{F} \in \mathbb{F}S : \lambda_X(\mathcal{F})(x) \leq \epsilon \} \\ F_S^\epsilon(x) &:= \{ y \in Y : \forall \mathcal{F} \in H_S^\epsilon(x), \lambda_Y(f[\mathcal{F}])(y) \leq \epsilon \} \\ h(S, f, \alpha) &:= \left\{ x \in S^{(\alpha)} : \bigcap_{\epsilon \in [0, \infty]} F_S^\epsilon(x) \neq \emptyset \right\} \\ h(S, f) &:= h(S, f, 0). \end{aligned}$$



Note that  $F_S^\epsilon(x) = Y$  if  $H_S^\epsilon(x) = \emptyset$ , that  $S \subseteq h(S, f) \subseteq h(S, f, \alpha)$  for each  $\alpha$ , and that if  $X$  and  $Y$  are convergence spaces (considered as CAP spaces) then

$$h(S, f) = \left\{ x \in \text{adh}S : \bigcap_{\mathcal{F} \in \mathbb{F}S, x \in \lim_X \mathcal{F}} \lim_Y f[\mathcal{F}] \neq \emptyset \right\},$$

is the hull of extensionability as in definition 1.31.

**Definition 3.6.** *Given a contraction  $f : S \rightarrow Y$  where  $S \subseteq X$ , and  $\alpha \in [0, \infty]$ , we call a function  $g : h(S, f, \alpha) \rightarrow Y$  with  $g|_S = f$  and  $g(x) \in \bigcap_{\epsilon \in [0, \infty]} F_S^\epsilon(x)$  for each  $x \in h(S, f, \alpha)$  an admissible extension of  $f$ . If each  $g(x)$  is also a  $\oplus$ -regularity point, then we call  $g$  a  $\oplus$ -regular extension of  $f$ .*

Note that we can adopt a similar terminology in **Conv** <sup>(2)</sup>.

**Definition 3.7.** *Let  $X$  be a CAP space and  $S \subseteq X$  and  $\alpha \in [0, \infty]$ . Then  $S$  is called an  $\alpha$ - $\oplus$ -strict subspace if for every  $x \in S^{(\alpha)}$  and every  $\mathcal{F} \in \mathbb{F}S^{(\alpha)}$  there is  $\mathcal{G} \in \mathbb{F}S$  such that  $\mathcal{G}^{(\alpha)} \leq \mathcal{F}$  and*

$$\lambda(\mathcal{G})(x) \leq \lambda(\mathcal{F})(x) \oplus \alpha. \quad (3.2)$$

$S$  is called  $\oplus$ -strict if it is  $\alpha$ - $\oplus$ -strict for every  $\alpha \in [0, \infty]$ .

**Definition 3.8.**  *$S$  is called a uniformly  $\alpha$ - $\oplus$ -strict subspace if for every  $\mathcal{F} \in \mathbb{F}S^{(\alpha)}$  there is  $\mathcal{G} \in \mathbb{F}S$  such that  $\mathcal{G}^{(\alpha)} \leq \mathcal{F}$  and*

$$\lambda(\mathcal{G}) \leq \lambda(\mathcal{F}) \oplus \alpha. \quad (3.3)$$

on  $S^{(\alpha)}$ .

$S$  is called *uniformly  $\oplus$ -strict* if it is uniformly  $\alpha$ - $\oplus$ -strict for every  $\alpha \in [0, \infty]$ .

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<sup>2</sup>Namely if  $f : S \rightarrow Y$  is continuous for  $S \subset X$ , we call a function  $g : h(S, f) \rightarrow Y$  with  $g|_S = f$  and  $g(x) \in \bigcap_{x \in \lim_X \mathcal{F}; S \in \mathcal{F}} \lim_Y f[\mathcal{F}]$  for each  $x \in h(S, f)$  an *admissible extension* of  $f$ . If moreover each  $g(x)$  is a regularity point,  $g$  is a *regular extension* of  $f$ .

Thus a subspace of a convergence space is strict, in the sense of definition 1.30, if and only if it is  $\oplus$ -strict, equivalently  $\epsilon$ - $\oplus$ -strict for some  $\epsilon < \infty$ , when considered as a CAP space.

**Proposition 3.9.** *If  $X$  is a  $\oplus$ -diagonal CAP space, then every subspace is uniformly  $\oplus$ -strict.*

*Proof.* Let  $S \subseteq X$ , let  $\alpha \in [0, \infty]$  and take a filter on  $S^{(\alpha)}$  and call  $\mathcal{F}$  is the filter generated on  $X$ . For each  $x \in S^{(\alpha)}$ , take  $\mathcal{S}(x) \in \mathbb{F}S$  such that  $\lambda(\mathcal{S}(x))(x) \leq \alpha$  and for  $x \notin S^{(\alpha)}$  let  $\mathcal{S}(x) = \{x\}^\uparrow$ . Since  $X$  is  $\oplus$ -diagonal, for  $\mathcal{G} = \mathcal{S}(\mathcal{F})$  we have

$$\lambda(\mathcal{G}) = \lambda(\mathcal{S}(\mathcal{F})) \leq \lambda(\mathcal{F}) \oplus \bigvee_{x \in X} \lambda(\mathcal{S}(x))(x) \leq \lambda(\mathcal{F}) \oplus \alpha$$

on  $X$ . Clearly  $S \in \bigcap_{x \in S^{(\alpha)}} \mathcal{S}(x)$  and since  $S^{(\alpha)}$  belongs to  $\mathcal{F}$  we have  $S \in \mathcal{G}$ . We finally check that  $\mathcal{G}^{(\alpha)} \leq \mathcal{F}$ . Let  $Z \in \bigcap_{x \in F \cap S^{(\alpha)}} \mathcal{S}(x)$  for some  $F \in \mathcal{F}$ . With  $u \in F \cap S^{(\alpha)}$  the filter  $\mathcal{S}(u)$  contains  $Z$  and  $\lambda \mathcal{S}(u)(u) \leq \alpha$ . So  $u \in Z^{(\alpha)}$ . It follows that  $F \cap S^{(\alpha)} \subseteq Z^{(\alpha)}$ .  $\square$

We are now ready to generalize Theorem 1.32 to convergence approach spaces.

**Theorem 3.10.** *Let  $\alpha \in [0, \infty]$  and let  $Y$  be a CAP space. If  $S$  is an  $\alpha$ - $\oplus$ -strict subspace of a convergence approach space  $X$  and  $f : S \rightarrow Y$  is a contraction, then every  $\oplus$ -regular extension  $g : h(S, f, \alpha) \rightarrow Y$  of  $f$  satisfies  $m_\oplus(g) \leq \alpha \oplus \alpha$ .*

*Proof.* We may assume  $\alpha < \infty$ . Let  $g$  be an  $\oplus$ -regular extension  $g : h(S, f, \alpha) \rightarrow Y$ . Let  $\mathcal{F} \in \mathbb{F}(h(S, f, \alpha))$  and  $x_0 \in h(S, f, \alpha)$ . Since  $S$  is an  $\alpha$ - $\oplus$ -strict subspace of  $X$  there is a  $\mathcal{G} \in \mathbb{F}S$  such that  $\mathcal{G}^{(\alpha)} \leq \mathcal{F}$  and

$$\lambda_X(\mathcal{G})(x_0) \leq \lambda_X(\mathcal{F})(x_0) \oplus \alpha.$$

Since  $\mathcal{G}^{(\alpha)} \vee h(S, f, \alpha) \leq \mathcal{F}$  we have

$$(f[\mathcal{G}])^\alpha \leq g[\mathcal{G}^{(\alpha)} \vee h(S, f, \alpha)] \leq g[\mathcal{F}],$$

where the first inequality follows from the assertion  $g(G^\alpha \cap h(S, f, \alpha)) \subseteq (f(G))^\alpha$ . Indeed for  $x \in G^{(\alpha)} \cap h(S, f, \alpha)$  there is an ultrafilter  $\mathcal{U}$  on  $G$  with  $\lambda(\mathcal{U})(x) \leq \alpha$ , that is with  $\mathcal{U} \in H_S^\alpha(x)$ . Since  $g$  is an admissible extension of  $f$ ,  $g(x) \in \bigcap_{\beta \in [0, \infty]} F_S^\beta(x)$  so that in particular  $g(x) \in F_S^\alpha(x)$  and  $\lambda_Y(f[\mathcal{U}])(g(x)) \leq \alpha$ . Thus  $g(x) \in (f(G))^{(\alpha)}$ .

Therefore

$$\lambda_Y(g[\mathcal{F}])(g(x_0)) \leq \lambda_Y(f[\mathcal{G}]^{(\alpha)})(g(x_0)) \leq \lambda_Y((f[\mathcal{G}])(g(x_0))) \oplus \alpha,$$

since  $g(x_0)$  is a regularity point of  $Y$ . With  $\lambda_X(\mathcal{G})(x_0) = \gamma$ , using the fact that  $g(x_0) \in F_S^\gamma(x_0)$  we obtain

$$\lambda_Y(f[\mathcal{G}])(g(x_0)) \leq \gamma = \lambda_X(\mathcal{G})(x_0).$$

Finally we obtain

$$\lambda_Y(g[\mathcal{F}])(g(x_0)) \leq \lambda \mathcal{G}(x_0) \oplus \alpha \leq \lambda \mathcal{F}(x_0) \oplus \alpha \oplus \alpha$$

□

**Corollary 3.11.** *If  $S$  is a  $\oplus$ -strict subspace of a CAP space  $X$ , and  $Y$  is a  $\oplus$ -regular CAP space, then every admissible extension  $g : h(S, f) \rightarrow Y$  of a contraction  $f : S \rightarrow Y$  is a contraction.*

The restriction of Theorem 3.10 to **Conv** is essentially (in fact, it is slightly more general than) the direct part of [5, Theorem 1.1]:

**Corollary 3.12.** *If  $S$  is a strict subspace of a convergence space  $X$  and  $Y$  is a convergence space, then every regular extension  $g : h(S, f) \rightarrow Y$  of a continuous map  $f : S \rightarrow Y$  is continuous. In particular, if  $Y$  is regular, every admissible extension  $g : h(S, f) \rightarrow Y$  of a continuous map  $f : S \rightarrow Y$  is continuous.*

Using a construction similar to that of the proof of Theorem 3.1, we obtain a partial converse:

**Theorem 3.13.** *If  $Y$  is not  $\oplus$ -regular, then there is a  $\oplus$ -approach space  $X$ , a (uniformly  $\oplus$ -strict) subspace  $S$ , a contraction  $f : S \rightarrow Y$ ,  $\alpha \in [0, \infty)$ , and an admissible extension  $g : h(S, f, \alpha) \rightarrow Y$  that is not contractive (that is,  $m_{\oplus}(g) > 0$ ).*

*Proof.* Since  $Y$  is not  $\oplus$ -regular, there exists  $A \neq \emptyset$ ,  $l : A \rightarrow Y$ ,  $\mathcal{S} : A \rightarrow \mathbb{F}Y$ ,  $\mathcal{H} \in \mathbb{F}A$ , and  $y_0 \in Y$  such that

$$\lambda_Y(l[\mathcal{H}])(y_0) > \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \oplus \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a)). \quad (3.4)$$

Let  $X := (Y \times A) \cup A \cup \{x_{\infty}\}$ ,  $S := Y \times A$ , and  $f : S \rightarrow Y$  be  $f(y, a) = y$ . Let

$$\mathcal{N} := \bigcup_{H \in \mathcal{H}} \bigcap_{a \in H} (\mathcal{S}(a) \times \{a\}^{\uparrow}) \wedge \{a\}^{\uparrow}.$$

On  $X$ , we define the following CAP structure:

$$\begin{aligned} \lambda_X(\mathcal{G})((y, a)) &:= \begin{cases} 0 & \text{if } \mathcal{G} = \{(y, a)\}^{\uparrow} \\ \infty & \text{otherwise} \end{cases} \\ \lambda_X(\mathcal{G})(a) &:= \begin{cases} 0 & \text{if } \mathcal{G} = \{a\}^{\uparrow} \\ \lambda_Y(\mathcal{S}(a))(l(a)) & \text{if } \mathcal{G} \geq (\mathcal{S}(a) \times \{a\}^{\uparrow}) \wedge \{a\}^{\uparrow} \text{ and } \mathcal{G} \neq \{a\}^{\uparrow} \\ \infty & \text{otherwise} \end{cases} \\ \lambda_X(\mathcal{G})(x_{\infty}) &:= \begin{cases} 0 & \text{if } \mathcal{G} = \{x_{\infty}\}^{\uparrow} \\ \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) & \text{if } \mathcal{G} \geq \mathcal{N} \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Note that  $f$  is a contraction and that  $X$  is a  $\oplus$ -approach space. Thus, in view of Proposition 3.9,  $S$  is a  $\oplus$ -strict subspace.

We claim that  $h(S, f, \alpha) = X$  for  $\alpha := \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \vee \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a))$ , which is finite by (3.4). Indeed,  $A \subseteq h(S, f, \alpha)$  because if  $\mathcal{G} \geq (\mathcal{S}(a) \times \{a\}^{\uparrow}) \wedge \{a\}^{\uparrow}$ ,

$\lambda_X(\mathcal{G})(a) = \lambda_Y(\mathcal{S}(a))(l(a)) \leq \alpha$ . Also,  $l(a) \in \bigcap_{\epsilon \in [0, \infty]} F_S^\epsilon(a)$ , since  $H_S^\epsilon(a) = \{\mathcal{G} \in \mathbb{F}S : \mathcal{G} \geq \mathcal{S}(a) \times \{a\}^\uparrow\}$  and  $f[\mathcal{G}] = \mathcal{S}(a)$  imply that  $\lambda_Y(f[\mathcal{G}])(l(a)) = \lambda_Y(\mathcal{S}(a))(l(a))$ .

Moreover  $x_\infty \in S^{(\alpha)}$  because  $\lambda_X(\mathcal{N} \vee S)(x_\infty) = \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \leq \alpha$ , and  $x_\infty \in h(S, f, \alpha)$  because  $y_0 \in \bigcap_{\epsilon \in [0, \infty]} F_S^\epsilon(x_\infty)$ . Indeed, if  $\epsilon < \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0)$  then  $H_S^\epsilon(x_\infty) = \emptyset$ , so that  $F_S^\epsilon(x_\infty) = Y$ . If  $\lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \leq \epsilon < \infty$ , then

$$H_S^\epsilon(x_\infty) = \{\mathcal{G} \in \mathbb{F}S : \lambda_X(\mathcal{G})(x_\infty) \leq \epsilon\} = \{\mathcal{G} \in \mathbb{F}S : \mathcal{G} \geq \mathcal{N} \vee S\}.$$

Thus if  $\mathcal{G} \in H_S^\epsilon(x_\infty)$  then  $f[\mathcal{G}] \geq \mathcal{S}(\mathcal{H})$ , and  $\lambda_Y(f[\mathcal{G}])(y_0) \leq \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) = \lambda_X(\mathcal{G})(x_\infty) \leq \epsilon$ . Thus  $y_0 \in F_S^\epsilon(x_\infty)$ .

Consider the admissible extension  $g : h(S, f, \alpha) \rightarrow Y$  of  $f$  defined by  $g|_S = f$ ,  $g|_A = l$  and  $g(x_\infty) = y_0$ . Then

$$m_\oplus(g) > \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a)) \geq 0$$

because

$$\begin{aligned} \lambda_Y(g[\mathcal{H}])(y_0) = \lambda_Y(l[\mathcal{H}])(y_0) &> \lambda_Y(\mathcal{S}(\mathcal{H}))(y_0) \oplus \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a)) \\ &> \lambda_X(\mathcal{H})(x_\infty) \oplus \bigvee_{a \in A} \lambda_Y(\mathcal{S}(a))(l(a)). \end{aligned}$$

□

Note that if, in the proof above,  $Y$  is a convergence space (considered as a CAP space), then we can assume  $\lambda_Y(\mathcal{S}(a))(l(a))$  to be 0 for each  $a \in A$ , and  $\lambda_Y(\mathcal{S}(\mathcal{H}))(y_0)$  to be 0. Thus,  $X$  is then a *topological* CAP space. Therefore, we recover:

**Corollary 3.14.** [5, Theorem 1.1] *A convergence space  $Y$  is regular if and only if, whenever  $S$  is a strict subspace of a convergence (equivalently, topological) space  $X$  and  $f : S \rightarrow Y$  is a continuous map there exists a continuous map  $\bar{f} : h(S, f) \rightarrow (Y, \tau)$  with  $\bar{f}|_S = f$ .*

Since every subspace of a diagonal convergence space is strict, we also recover:

**Corollary 3.15.** [2] *A Hausdorff convergence space  $Y$  is regular if and only if for every diagonal convergence space  $X$ , every subspace  $S$  of  $X$ , and every continuous map  $f : S \rightarrow Y$  there exists a (unique) continuous map  $\bar{f} : h(S, f) \rightarrow Y$  with  $\bar{f}|_S = f$ .*

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