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Survey of Results on the Schrodinger Operator with Inverse Square Potential

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SURVEY OF RESULTS ON THE SCHRÖDINGER OPERATOR WITH INVERSE SQUARE POTENTIAL

by

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(Under the Direction of Yi Hu)

ABSTRACT

In this paper we present a survey of results on the Schrödinger operator with Inverse Square potential, $\mathcal{L}_a = -\Delta + \frac{a}{|x|^2}, \quad a \geq -(\frac{d-2}{2})$. We briefly discuss the long-time behavior of solutions to the inter-critical focusing NLS with an inverse square potential (proof not provided). Later we present spectral multiplier theorems for the operator. For the case when $a \geq$, we use Hebisch [12] as a template for our attempt at a proof using estimates and results from [1], Sikora [3], [18] and [19]. The case when $0 > a \geq -(\frac{d-2}{2})$ was explored in [1], and their proof will be presented for completeness. No improvements on the sharpness of their proof as been obtained.

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CHAPTER 1

INTRODUCTION

1.1 THE OPERATOR

The operator

$$\mathcal{L}_a = -\Delta + \frac{a}{|x|^2} \quad with, \quad a \geq -\Big(\frac{d-2}{2}\Big)^2 \tag{1.1}$$

in dimensions $d \geq 3$. This operator was first introduced to us in [1] as defined below. The following related results were proved in [1]. $\mathcal{L}_a$ is the Friedrichs extension of the operator $\mathcal{L}_a^\circ$, where $\mathcal{L}_a^\circ$ denotes the natural action of $-\Delta + \frac{a}{|x|^2}$ on $\mathbb{C}_c^\infty(\mathbb{R}^d \backslash \{0\})$.

1.1.1 $\mathcal{L}_a^\circ$ IS A POSITIVE SEMI-DEFINITE SYMMETRIC OPERATOR

If we let

$$\sigma := \frac{d-2}{2} - \frac{1}{2}\sqrt{(d-2)^2 + 4a}$$

[1] shows that $\mathcal{L}_a^\circ$ can be seen to be positive via the factorization

$$\mathcal{L}_a^\circ = \Big(-\nabla + \sigma\frac{x}{|x|^2}\Big)\Big(\nabla + \sigma\frac{x}{|x|^2}\Big) = -\Delta + \sigma^2\frac{1}{|x|^2} = -\Delta + \sigma(d-2-\sigma)\frac{1}{|x|^2}.$$

If we pick $\theta \in \mathbb{C}_c^\infty(\mathbb{R}^d \backslash \{0\})$, then by functional calculus and the previous factorization of $\mathcal{L}_a^\circ$

$$\begin{aligned} \langle \theta, \mathcal{L}_a^\circ \theta \rangle &= \langle \theta, \Big(-\nabla + \sigma\frac{x}{|x|^2}\Big)\Big(\nabla + \sigma\frac{x}{|x|^2}\Big)\theta\rangle \\ &= \|\theta(x)\Big(\nabla + \sigma\frac{x}{|x|^2}\Big)\|^2 \\ &= \int_{\mathrm{R}^d} \Big|\nabla\theta(x) + \sigma\frac{x}{|x|^2}\theta(x)\Big|^2 \geq 0. \end{aligned}$$

Hence, $\mathcal{L}_a^\circ$ is positive semi-definite as needed.

1.1.2 General Theory of self-adjoint extensions

Below we present a version of Friedrich's Extension Theorem and Kato's Theorem from [8] (without proof). The Authors in in [1] used similar theorems to find a self-adjoint extension to the operator $\mathcal{L}_a^\circ$ (See [9, §X.3]).

Theorem 1.1. *Friedrich's Extension Theorem Let T_0 be a symmetric, semi-bounded Operator with domain $D(T_0)$ then, the quadratic form*

$$QT_0(\Phi,\Theta) := \langle \Phi, T_0\Theta\rangle, \Phi,\Theta \in D(T_0)$$

is closable.

Theorem 1.2. *Kato's Representation Theorem Let Q be a closed, semi-bounded quadratic form with domain D. Then it exists a unique, self-adjoint, semi-bounded operator T with domain $D(T) \subset D$ such that*

$$Q(\Phi,\Theta) = \langle \Phi,\Theta\rangle \quad \forall \Phi \in D, \forall \Theta \in D(T).$$

The Theorems mentioned above guarantee the existence of a unique self-adjoint extension $\mathcal{L}_a$ of $\mathcal{L}_a^\circ$, whose form domain $Q(\mathcal{L}_a) = D(\sqrt{\mathcal{L}_a}) \subseteq L^2(\mathcal{R}^d)$ is given by the completion of $C_c^\infty(\mathbb{R}^d \backslash \{0\})$ with respect to the norm

$$\|\Theta\|^2_{Q(\mathcal{L}_a)} = \int_{\mathrm{R}^d} \Big|\nabla\Theta\Big|^2 + \Big(1 + \frac{a}{|x^2|}\Big)\Big|\Theta\Big|^2 dx = \int_{\mathrm{R}^d} \Big|\nabla\Theta + \frac{\sigma x}{|x^2|}\Theta\Big|^2 + \Big|\Theta\Big|^2 dx.$$

Theorem 1.3. *(Equivalence of Sobolev norms) Suppose $d \geq 3, a \geq -\Big(\frac{d-2}{2}\Big)^2$, and $0 < s < 2$. If $1 < p < \infty$ satisfies $\frac{s+\sigma}{d} < \frac{1}{p} < min\Big\{1, \frac{d-\sigma}{\sigma}\Big\}$, then*

$$\|(-\Delta)^{\frac{s}{2}} f\|_{L^p} \lesssim_{d,p,s} \|\mathcal{L}_a^{\frac{s}{2}}\|, \forall f \in C_c^\infty(\mathbb{R}^d). \tag{1.2}$$

If $max\Big\{\frac{s}{d}, \frac{\sigma}{d}\Big\} < \frac{1}{p} < min\Big\{1, \frac{d-\sigma}{\sigma}\Big\}$, then

$$\|\mathcal{L}_a^{\frac{s}{2}} f\|_{L^p} \lesssim \|(-\Delta)^{\frac{s}{2}} f\|_{L^p}, \forall f \in C_c^\infty(\mathbb{R}^d) \tag{1.3}$$

1.1.3 HEAT AND RIESZ KERNELS

Theorem 1.4. *(The Heat Kernel Bounds) Assume* $d \geq 3$ *and* $a \geq \frac{-(d-2)}{2}$. *Then there exist positive constants* C_1, C_2 *and* c_1, c_2 *such that for all* $t > 0$ *and all* $x, y \in (\mathbb{R}^d \backslash \{0\})$,

$$C1\Big(1 \vee \frac{\sqrt{t}}{|x|}\Big)^{\sigma}\Big(1 \vee \frac{\sqrt{t}}{|y|}\Big)^{\sigma} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{c_1 t}} \leq e^{-t\mathcal{L}_a}(x, y) \\ \leq C2\Big(1 \vee \frac{\sqrt{t}}{|x|}\Big)^{\sigma}\Big(1 \vee \frac{\sqrt{t}}{|y|}\Big)^{\sigma} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{c_2 t}} \tag{1.4}$$

Theorem 1.5. *(Riesz Kernels) Let* $d \geq 3$ *and suppose* $0 < s < d$ *and* $d - s - 2\sigma$. *Then the Riesz potentials*

$$\mathcal{L}_a^{-\frac{s}{2}}(x, y) := \frac{1}{\Gamma(\frac{s}{2})} \int_0^{\infty} e^{-\mathcal{L}_a}(x, y) t^{\frac{s}{2}} \frac{dt}{t}$$

satisfy

$$\mathcal{L}_a^{-\frac{s}{2}}(x, y) \sim |x-y|^{s-d}\Big(\frac{|x|}{|x-y|} \wedge \frac{|y|}{|x-y|} \wedge 1\Big)^{-\sigma}. \tag{1.5}$$

1.1.4 HARDY INEQUALITY

Theorem 1.6. *(IV Hardy inequality for* $\mathcal{L}_a$*) Suppose* $d \geq 3, a < s < d, d - s - 2\sigma > 0$, *and* $1 < p < \infty$. *Then*

$$\||x|^{-s} f(x)\|_{L^p(\mathbb{R}^d)} \lesssim \|\mathcal{L}_a^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^d)} \tag{1.6}$$

holds, if and only if

$$s + \sigma < \frac{d}{p} < d - \sigma. \tag{1.7}$$

CHAPTER 2

TYPE-SETTING IN LATEX

2.1 Long-Time Behavior of Solutions to the Intercritical Focusing NLS with Inverse Square Potential

The results from this section originally appeared in [15], [16] and [17], which explored the long-time behavior of solutions to the intercritical NLS with inverse square potential:

$$i\partial_t u = \mathcal{L}_a u - |u|^p u, \tag{2.1}$$

where $u : \mathbb{R}_t^d x \mathbb{R}_x^d \to \mathbb{C}$, $\frac{4}{d} < p < \frac{4}{d-2}$ and $d \geq 3$.
For $a \in \left(-\left(\frac{d-2}{2}\right)^2, 0\right]$, equation (1) admits a global but non-scattering solution of the form $u(t) = e^{it} P_a$, where P_a (the *ground state*) solves the elliptic problem

$$-\mathcal{L}_a P_a - P_a + |P_a|^p P_a = 0. \tag{2.2}$$

2.1.1 Scattering / Blow-up Dichotomy

Theorem 2.1 (V)**.** *(Scattering/Blow-up Dichotomy) Suppose that* $d \geq 3$, $\frac{4}{d} < p < \frac{4}{d-2}$, *and* $a > -\left(\frac{d-2}{2}\right)^2$, *and let* $u_0 \in H^1(\mathbb{R}^d)$. *There exists a unique maximal-lifespan solution u to (1) with* $u|_{t=0} = u_0$. *If* u_0 *is below the ground state threshold, in the sense that*

$$M(u_0)^{\frac{4-p(d-2)}{dp-4}} E_a(u_0) < M(P_{a\wedge 0})^{\frac{4-p(d-2)}{dp-4}} E_{a\wedge 0}(P_{a\wedge 0}), \tag{2.3}$$

Then the following dichotomy holds: If

$$\|u_0\|_{L^2}^{\frac{4-p(d-2)}{dp-4}} \|u_0\|_{H_a^1} < \|P_{a\wedge 0}\|_{L^2}^{\frac{4-p(d-2)}{dp-4}} \|P_{a\wedge 0}\|_{H_a^1}, \tag{2.4}$$

Then u *is global in time and scatters in both time directions; that is, there exist solutions* $v_\pm$ *to the equation* $i\partial_t v_\pm = \mathcal{L}_{a\wedge 0} v_\pm$ *such that*

$$\lim_{t\to\pm\infty} \|u(t) - v_\pm(t)\|_{H^1} = 0.$$

Theorem 2.2 (VI cont'). *If*

$$\|u_0\|_{L^2}^{\frac{4-p(d-2)}{dp-4}}\|u_0\|_{H_a^1} > \|P_{a\wedge 0}\|_{L^2}^{\frac{4-p(d-2)}{dp-4}}\|P_{a\wedge 0}\|_{H_{a\wedge 0}^1},$$

and u_0 is radial or $xu_0 \in L^2$, then u blows up in finite time in both time directions.

2.1.2 Linear and Local Theory

Theorem 2.3 (VII Strichartz Estimates). *Let $a > -\left(\frac{d-2}{2}\right)^2$ and $d \geq 3$. Let (q,r) and $(\tilde{q},\tilde{r})$ be such that*

$$2 \leq q, \tilde{q} \leq \infty \quad \text{and} \quad \frac{2}{q} + \frac{d}{r} = \frac{2}{q} + \frac{d}{r} = \frac{d}{2},$$

with $(q,\tilde{q}) \neq (2,2)$. suppose $u : I \times \mathbb{R}^d \to \mathbb{C}$ solves

$$(i\partial_t - \mathcal{L}_a)u = F.$$

Then for any $t_0 \in I$, the following estimate holds:

$$\|u\|_{L_t^q L_x^r(Ix\mathbb{R}^d)} \lesssim \|u_0\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(Ix\mathbb{R}^d)}.$$

Theorem 2.4 (VIII Local Well-posedness). *Let $t_0 \in \mathbb{R}$, $u_0 \in H^1$,*

-There exist $T = T(\|u_0\|_{H_1}) > 0$ and a unique solution u to (1) on $(t_0 - T; t_0 + T)$ with $u(t_0) = u_0$. In particular, if u remains uniformly bounded in H^1 throughout its lifespan, then u extends to a global solution.

-Furthermore, there exists $\eta_0 > 0$ so that if

$$\|e^{-i(t-t_0)\mathcal{L}}u_0\|_{L_{t,x}^{q_0}((t_0,\infty)x\mathbb{R}^d)} < \eta.$$

The analogous statement holds backward in time and on all of $\mathbb{R}$.

-Finally, for any $\psi \in H^1$ there exists a solution to (1) that scatters to ψ as $t \to \infty$, and the analogous statement holds backwards in time.

Theorem 2.5 (IX Stability)**.** *Let $\tilde{u}$ solve*

$$i\partial_t \tilde{u} = \mathcal{L}_a \tilde{u} - |\tilde{u}|^p \tilde{u} + e$$

on an interval I for some function e. Suppose

$$\|u_0\|_{H^1} + \|\tilde{u}(t_0)\|_{H1} \leq E, \quad \|\tilde{u}\|_{L^{q_0}_{t,x}(Ix\mathbb{R}^d)} \leq L.$$

There exists $\varepsilon_0(E, L > 0)$ so that if $0 < \varepsilon < \varepsilon_0$ and

$$\|u_0 - \tilde{u}(t_0)\|_{H^1} + \||\nabla|^{s_c} e\|_{N(I)} < \varepsilon,$$

where $s_c = \frac{d}{2} - \frac{2}{p}$ and N is a sum of dual Strichartz spaces, the there exists a solution u to (1) with $u(t_0) = u_0$ satisfying

$$\|(\mathcal{L}_a)^{\frac{s_c}{2}}[u - \tilde{u}]\|_{S(I)} \lesssim \varepsilon, \quad \|(1 + \mathcal{L}_a)^{\frac{1}{2}} u\|_{S(I)} \lesssim_{E,L} 1$$

for any Strichartz space S.

2.1.3 Harmonic Analysis adapted to $\mathcal{L}_a$

The following set of tool-kits were developed in [1] and summarized in [15].

We present the Little-Paley projections defined via the heat kernel:

$$P_N^a := e^{-\mathcal{L}_a/N^2} - e^{-4\mathcal{L}_a/N^2} \quad for \quad N \in 2^{\mathbb{Z}}.$$

Let

$$\tilde{q} := \begin{cases} \infty & \text{if } a \geq 0, \\ \frac{d}{\sigma} & \text{if } -\left(\frac{d-2}{2}\right)^2 < a < 0. \end{cases}$$

We write $\tilde{q}'$ as the dual exponent to $\tilde{q}$. Using the previous definitions, we summarize the needed tools in the following:

Lemma 2.1.1 (Harmonic Analysis tools)**.** *For $\tilde{q}' < q \leq r < \tilde{q}$,*

$$f = \sum_{N \in 2^{\mathbb{Z}}} P_N^a f, \text{as elements of } L_x^r.$$

Furthermore, we have the following Bernstein estimates:

1. *The operators* P_N^a *are bounded on* L_x^r.

2. *The operators* P_N^a *map* L_x^q *to* L_x^r, *with the norm* $\mathcal{O}\Big(N^{\frac{d}{q}-\frac{d}{r}}\Big)$.

3. *For any* $s \in \mathbb{R}$,

$$N^s \|P_N^a f\|_{L_x^r} \sim \|\Big(\mathcal{L}_a\Big)^{\frac{s}{2}} P_N^a f\|_{L_x^r}.$$

Finally, for $0 \leq s < 2$, *we have the square function estimate*

$$\Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2s}|P_N^a f|^2\Big)^{\frac{1}{2}}\Big\|_{L_x^r} \sim \|(\mathcal{L}_a)^{\frac{s}{2}} f\|_{L_x^r}$$

2.2 Multiplier Theorem for the Case when $a \geq 0$

We present two multiplier theorems for the operator. We start with the case when $a \geq 0$. The theorem in part one was obtained from Hebicsh [12], we try to adapt the proof presented in the same paper to our operator. Some of the estimates used in the proof were obtained from [18] and [19]. For the purpose of completeness, we present a Mihklin-type multiplier theorem as presented in [1] for the case when $-(\frac{d-2}{2})^2 \leq a < 0$. We offer a brief restatement of the proof offered by [1].

Let E be the spectral measure of $\mathcal{L}_a$. If F is a bounded Borel measurable function we write

$$F(\mathcal{L}_a)f = \int F(\lambda)dE(\lambda)f.$$

Let

$$F_t(a) = F(tx).$$

By the spectral theorem $F(\mathcal{L}_a)$ is bounded on L^2.

Theorem 2.6. *(Hebisch[12]) If for some* $\epsilon > 0$, *a non-zero* $\phi \in C_c^\infty(R_+)$ *and constant C, we have*

$$\|\phi F_t\|_{H((d+1)/2+\epsilon)} \leq C, \tag{2.5}$$

then T is of weak type (1,1) and bounded on L^p for $1 < p < \infty$.

2.2.1 PROOF ADAPTED TO $\mathcal{L}_a$

From (2.5), we get that $\|F\|L^\infty \leq C'C$, then

$$\|F(\mathcal{L}_a)\|_{L^2,L^2} \leq C'C. \tag{2.6}$$

By interpolation and duality argument,it is enough to prove that $F(\mathcal{L}_a)$ is of weak type $(1,1)$. Using the Trotter formula in [13] we obtain

$$0 \leq e^{-t\mathcal{L}_a}(x,y) \lesssim p_t(x,y), \tag{2.7}$$

where $p_t(x,y) = C\Big(1 \vee \frac{\sqrt{t}}{|x|}\Big)^\sigma \Big(1 \vee \frac{\sqrt{t}}{|y|}\Big)^\sigma t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{ct}}$. (2.7) implied the following

$$\int e^{-t\mathcal{L}_a} e^{s|x-y|} dx \lesssim Ce^{Cs^2t} \tag{2.8}$$

$$\int |e^{-t\mathcal{L}_a}(x,y)|^2 dx \lesssim Ct^{-\frac{d}{2}-\alpha} e^{2\lambda^2 t} \tag{2.9}$$

$$sup_{x,y}\Big|e^{-t\mathcal{L}_a}(x,y)\Big| \leq Ct^{-\frac{d}{2}-\alpha} e^{2\lambda^2 t} \tag{2.10}$$

for some constant C and all $s, t > 0, y \in \mathbb{R}^d$. We have

$$\|K\|_a = max\Big\{sup_x \int |K(x,y)|(1+|x-y|)^a dy, sup_y \int |K(x,y)|(1+|x-y|)^a dx\Big\}.$$

Lemma 2.2.1. *(see Hebisch [12] for proof) If $suppF \subset [1,4], \epsilon > 0, a \geq 0$, then*

$$\|F(\mathcal{L}_a)\|_a \leq C\|F\|_{H((d+1)/2+\epsilon+a)}$$

where C is independent of F and $\mathcal{L}_a$.

Proof. Set

$$K(\lambda) = F(-log(\lambda))\lambda^{-1}.$$

We have that

$$\|K\|_{H((d+1)/2+\epsilon+a)} \leq C_1 \|F\|_{h((d+1)/2+\epsilon+a)}, suppK \subset [e^{-4}, e].$$

Let $K(\lambda) = \sum \widehat{K}(n)e^{in\lambda}$, $en = e^{ine^{-\mathcal{L}_a}}e^{-\mathcal{L}_a}$, then

$$F(\mathcal{L}_a) = K(e^{-\mathcal{L}_a})e^{-\mathcal{L}_a} = \sum \widehat{K}(n)e_n.$$

2.8 and 2.9 allows us to use (3.1) from [14] to obtain

$$\|e_n\|_a \leq C_2(1+|n|)^{d/2+a}$$

so

$$\begin{aligned}
\|F\|_a &\leq C_2 \sum |\widehat{K}(n)|(1+|n|)^{d/2+a} \\
&\leq C_2\Big(\sum |\widehat{K}(n)|^2(1+|n|)^{d+2a+1+\epsilon}\Big)^{1/2}\Big(\sum(1+|n|)^{-1-\epsilon}\Big)^{1/2} \\
&\leq C_3\|K\|_{H((d+1+\epsilon)/2+a)} \leq C_4\|F\|_{H((d+1)/2+\epsilon+a)},
\end{aligned}$$

which ends the proof of the lemma.

□

Lemma 2.2.2. *(see Hebisch [12] for proof) For every $m \geq 0$ there exist $N, C > 0$ such that if $F \in H(N), suppF \subset [-1, 4]$, then*

$$|F(\mathcal{L}_a)(x,y)| \leq C\|F\|_{H(N)}(1+|x-y|)^{-m}$$

for all x, y and $\mathcal{L}_a$

Proof. Let $G(\lambda) = F(\lambda)e^{\lambda}$, $N = d/2 + m + 1$. Of course $\|G\|_{H(N)} \leq C_1\|F\|_{H(N)}$. By lemma 2.2.1, $\|G(\mathcal{L}_a)\|_m \leq C_2\|G\|_{H(N)}$ and by 2.7 and 2.10,

$$\begin{aligned}
|(1+|x-y|)^m F(\mathcal{L}_a)(x,y)| &= \Big|\int G(\mathcal{L}_a)(x,s)e^{-\mathcal{L}_a}(s,y)(1+|x-y|)^m ds\Big| \\
&\leq \int |G(\mathcal{L}_a)(x,s)|(1+|x-s|)^m e^{-\mathcal{L}_a}(s,y)(1+|s-y|)^m ds \\
&\leq \|G(\mathcal{L}_a)\|_m sup \quad p_1(x)(1+|x|)^m.
\end{aligned}$$

Then since, $G(\lambda) = F(\lambda)e^{\lambda}$

$$|F(\mathcal{L}_a)(x,y)| \leq C\|F\|_{H(N)}(1+|x-y|)^{-m}$$

$\square$

Let ϕ and ψ be in $C^{\infty}(\mathbb{R})$, where $supp\phi \subset [1/4, 2]$, $\sum = 1$ for every $x > 0$, and $supp\psi \subset [-1, 1]$, with $\psi(x) = 1$ for $x \in [0, 1/2]$. Let

$$F_k(\lambda) = \phi(2^{2k}\lambda)F(\lambda), \quad \psi_k(\lambda) = \psi(2^{2k}\lambda).$$

Choose $a < \epsilon$. There exists C such that

$$\|\psi_k F_k(\mathcal{L}_a)\|_{L^1,L^1} \leq C, \tag{2.11}$$

$$\int |F_k(\mathcal{L}_a)|(x,y)(1+2^{-k}|x-y|)^a dx \leq C, \tag{2.12}$$

$$|\psi_k(\mathcal{L}_a)|(x,y) \leq C2^{-kd}(1+2^{-k}|x-y|)^{-d-1}. \tag{2.13}$$

The proof for (2.11), (2.12) and (2.13) can be found in Hebisch [12], and has not been reproduced here.

Let f be an integrable function. We use Calderón-Zygmund decomposition on f at height λ with functions f_i and g and cubes Q_i such that

$$f = g + \sum f_i, \quad suppf_i \subset Q_i, \quad \int |f_i| \leq C\lambda|Q_i|,$$
$$|g| \leq C\lambda, \quad Q_i \cap Q_j = \emptyset \quad for \quad i \neq j, \quad \sum |Q_i| \leq C\|f\|_{L^1}/\lambda.$$

Let Q_i^* be the ball with the same center as Qi and radius $2diamQ_i$. We put $k_i = [log_2(diamQ_i)]$. Let h be an integrable function such that $supph \subset \{x : |x| \leq 1\} = B$. We have

$$\begin{aligned}\int_{|x|>2}|F_k(\mathcal{L}_a)h|(x)dx &\leq \|h\|_{L^1} sup_{y\in B}\int_{|x|>2}|F_k(\mathcal{L}_a)|(x,y)dx\\ &\leq \|h\|_{L^1} sup_y \int_{|x-y|>1}|F_k(\mathcal{L}_a)|(x,y)dx\\ &\leq 2^{ka}\|h\|_{L^1} sup_y \int |F_k(\mathcal{L}_a)|(x,y)(1+2^{-k}|x-y|)^a dx\\ &\leq C2^{ka}\|h\|_{L^1}\end{aligned}$$

and

$$\sum_{k\leq 0}\int_{|x|>2}|F_k(\mathcal{L}_a)h|(x)dx \leq C\sum_{k\leq 0}2^{ka}\|h\|_{L^1}\leq C_1\|h\|_{L^1}.$$

With the use of dilation we get

$$\sum_{j\leq k_i}\int_{(Q_i^*)^c}|F_j(\mathcal{L}_a)f_i|(x)dx \leq C\|f_i\|_{L^1}. \tag{2.14}$$

Lemma 2.2.3. *There exists C such that*

$$\Big\|\sum \psi_{k_i}(\mathcal{L}_a)f_i\Big\|_{L^2}^2 \leq C\lambda\|f\|_{L^1}.$$

Proof. First observe that there exists C_0 such that if $Q=\{x : max|x_i|\leq 1\}$ then for all x

$$\sup_{y\in Q}(1+|x-y|)^{-d-1}\leq C_0\inf_{y\in Q}(1+|x-y|)^{-d-1}.$$

As a result of this and using dilations we obtain for all i

$$\sup_{y\in Q_i}(1+2^{-k_i}|x-y|)^{-d-1}\leq C_0 \in_{y\in Q_i}(1+2^{-k_i}|x-y|)^{-d-1}. \tag{2.15}$$

Keeping i constant, let y_0 be the center of Q_i. By (2.15)

$$\begin{aligned}|\psi_{k_i}(\mathcal{L}_a)f_i|(x) &\leq \int 2^{-k_i d}(1+2^{-k_i}|x-y|)^{-d-1}|f_i|(y)dy\\ &\leq \lambda C_1|Q_i|2^{-k_i d}(1+2^{-k_i}|x-y_0|)^{-d-1}\\ &\leq \lambda C_2\int 2^{-k_i d}(1+2^{-k_i}|x-y|)^{-d-1}\mathcal{X}_{Q_i}(y)dy\\ &\leq \lambda C_3(2^{-k_i d}(1+2^{-k_i}|\cdot|)^{-d-1}*\mathcal{X}_{Q_i})(x).\end{aligned}$$

If $h \in L^2$, then

$$\left|\left(h, 2^{-k_i d}(1+2^{-k_i}|\cdot|)\right)^{-d-1} * \mathcal{X}_{Q_i}\right| = \left|\left(2^{-k_i d}(1+2^{-k_i}|\cdot|)\right)^{-d-1}, h * \mathcal{X}_{Q_i}\right)\right| \leq C_4(Mh, \mathcal{X}_{Q_i})$$

where M is the Hardy-Littlewood maximal operator. Following is the Hardy-Littlewood maximal operator (Stein[11]). Since M is bounded on L^2,

$$\left|\left(h, \sum \psi_{k_i}(\mathcal{L}_a) f_i\right)\right| \leq C_5\left(Mh, \sum \lambda \mathcal{X}_{Q_i}\right) \leq C_6 \|h\|_{L^2} \left\| \sum \lambda \mathcal{X}_{Q_i} \right\|_{L^2}.$$

But $\| \sum \lambda \mathcal{X}_{Q_i} \|_{L^2}^2 = \sum \lambda^2 |Q_i| \leq C\lambda \|f\|_{L^1}$, which ends the proof.

Clearly, if $j < k$, then $\psi_k F_j = 0$ so $\psi_k(\mathcal{L}_a) F_j(\mathcal{L}_a) = 0$. Similarly, if $j > k$ then $\psi_k(\mathcal{L}_a) F_j(\mathcal{L}_a) = F_j(\mathcal{L}_a)$. Therefore

$$\begin{aligned}
F(\mathcal{L}_a) &= \sum_{i,j} F_j(\mathcal{L}_a) f_i + F(\mathcal{L}_a) g \\
&= \sum_i \Big(\sum_{j \leq k_i} F_j(\mathcal{L}_a) f_i + \sum_{j > k_i} F_j(\mathcal{L}_a) f_i \Big) + F(\mathcal{L}_a) g \\
&= \sum_i \sum_{j \leq k_i} F_j(\mathcal{L}_a) f_i + \sum_{i,j} F_j(\mathcal{L}_a) \psi_{k_i}(\mathcal{L}_a) f_i - \sum_i F_{k_i}(\mathcal{L}_a) \psi_{k_i}(\mathcal{L}_a) f_i + F(\mathcal{L}_a) g \\
&= \sum_i \sum_{j \leq k_i} F_j(\mathcal{L}_a) f_i + F(\mathcal{L}_a) \Big(\sum \psi_{k_i}(\mathcal{L}_a) f_i + g \Big) - \sum_i F_{k_i}(\mathcal{L}_a) \psi_{k_i}(\mathcal{L}_a) f_i.
\end{aligned}$$

Putting $S = \cup Q_i^*$, by (2.14) and the properties of the Calderón-Zygmund decomposition we have

$$\begin{aligned}
\left|\left\{ x : \Big| \sum_i \sum_{j \leq k_i} F_j(\mathcal{L}_a) f_i \Big| > \lambda/3 \right\}\right| &\leq |S| + (3/\lambda) \int_{S^c} \Big| \sum_i \sum_{j \leq k_i} F_j(\mathcal{L}_a) f_i \Big| \\
&\leq C\|f\|_{L^1}/\lambda + (C/\lambda) \sum \|f_i\|_{L^1} \\
&\leq C\|f\|_{L^1}/\lambda.
\end{aligned}$$

By lemma 2.23,

$$\Big\| \sum \psi_{k_i}(\mathcal{L}_a) f_i + g\|_{L^2}^2 \leq C\lambda \Big\| f\|_{L^1}.$$

and by (2.6)

$$\begin{aligned}
\Big|\Big\{x : \Big|F(\mathcal{L}_a)\Big(\sum \psi_{k_i}(\mathcal{L}_a) f_i + g\Big)\Big| > \lambda/3\Big\}\Big| & \\
&\leq (C/\lambda^2)\Big\| \sum \psi_{k_i}(\mathcal{L}_a) f_i + g\Big\|_{L^2}^2 \\
&\leq C'\lambda \|f\|_{L^1}/\lambda^2 = C\|f\|_{L^1}/\lambda.
\end{aligned}$$

By (2.8),

$$\begin{aligned}
\Big|\Big\{x : \Big|\Big(\sum F_{k_i}(\mathcal{L}_a)\psi_{k_i}(\mathcal{L}_a) f_i\Big)\Big| > \lambda/3\Big\}\Big| &\leq 3\Big\| \sum F_{k_i}(\mathcal{L}_a)\psi_{k_i}(\mathcal{L}_a) f_i\Big\|_{L^1}/\lambda \\
&\leq (C/\lambda)\sum \|f_i\|_{L^1} \leq C\|f\|_{L^1} \leq C\|f\|_{L^1}/\lambda,
\end{aligned}$$

This ends the proof of theorem 2.6. □

2.3 Mikhlin Multiplier Theorem for the case $-(\frac{d-2}{2})^2 \leq a < 0$

Below, we present a multiplier theorem, and summary of its proof for the case when $-(\frac{d-2}{2})^2 \leq a < 0$ Both the theorem and the major results of the proof were obtained from [1].

Theorem 2.7. (Mikhlin Multipliers) Fix $-\left(\frac{d-2}{2}\right)^2 \leq a < 0$ and suppose that $m : [0, \infty) \to \mathbb{C}$ satisfies

$$|\partial m(\lambda)| \lesssim \lambda^{-j} \quad \text{for all} \quad 0 \leq j \leq 3\left\lfloor \frac{d}{4} \right\rfloor + 3. \tag{2.16}$$

Then $m(\sqrt{\mathcal{L}_a})$ which we define via the L^2 functional calculus, extends uniquely from $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to a bounded operator on $L^p(\mathbb{R}^d)$ for all $r_0 < p < r_0' := \frac{d}{\sigma}$.

Proof. We present the major results of the proof provided by [1], a more complete proof can be found in said paper. By the Spectral theorem, the operator $T := m(\sqrt{\mathcal{L}_a})$ is bounded on L^2.) Thus using the Marcinkiewicz interpolation theorem and a duality argument, it suffices to show that T is of the weak-type (q,q)

$$|\{x : |Tf(x)| > h\}| \lesssim h^{-q}||f||^q_{L^q}(\mathbb{R}^d) \text{ for all } h > \mathit{0}.$$

The authors used Calderon-Zygmund decomposition to $|f|^q$ at height h^q to obtain a family of dyadic cubes $\{Q_k\}_k$, $Q_j \bigcap Q_k = \emptyset$, $\bigcup Q_j = \Omega$ if $j \neq k$ which allowed the original function f to be decomposed such that $f = g + b_k$, where $b = \sum_k b_k$ and $b_k = XQ_k f$ and $|g| \leq h$ almost everywhere. By construction,

$$\begin{aligned} h^q &< \frac{1}{|Q_k|}\int_{Q_k} |f(x)|^q dx \leq 2^n h^q \\ h^q|Q_k| &\leq \int_{Q_k} |f(x)|^q dx \leq 2^n |Q_k| h^q \\ |Q_k| &\leq \frac{1}{h^q}\int_{Q_k} |f(x)|^q dx \leq 2^n |Q_k| \end{aligned} \tag{2.17}$$

Multiplying (1.2) by h, we get

$$h|Q_k| \leq h^{1-q}\int_{Q_k} |f(x)|^q dx$$

By Holder's inequality and (2.17),

$$\int_{Q_k} |f(x)| dx \lesssim ||f||_{L^q(Q_k)} |Q_k|^{\frac{1}{q'}} \lesssim h|Q_k| \lesssim h^{1-q}\int_{Q_k} |f(x)|^q dx \tag{2.18}$$

We further decompose $b_k = g_k + \tilde{b}_k$ according to the definition below

$$\tilde{b}_k := (1 - e^{-r_k^2})^\mu b_k \quad \textit{and} \quad gk := [1 - (1 - e^{r_k^2 \mathcal{L}_a})^\mu] b_k$$

Using the Binomial Theorem we get that

$$\begin{aligned}(1+(-e^{-r_k^2}))^{\mu} &= \binom{\mu}{0}(-e^{-r_k^2\mathcal{L}_a})^0 + \binom{\mu}{1}(-e^{-r_k^2\mathcal{L}_a})^1 + \binom{\mu}{2}(-e^{-r_k^2\mathcal{L}_a})^2 + ... \\ &\quad + \binom{\mu}{\mu-1}(-e^{-r_k^2\mathcal{L}_a})^{\mu-1} + \binom{\mu}{\mu}(-e^{-r_k^2\mathcal{L}_a})^{\mu} \\ &= \sum_{\nu=0}^{\mu}\binom{\mu}{\nu}(-e^{-\nu r_k^2\mathcal{L}_a}) \\ &= \sum_{\nu=0}^{\mu}\frac{\mu!}{\nu!(\mu-\nu)!}(-e^{-\nu r_k^2\mathcal{L}_a}) \\ &= \sum_{\nu=1}^{\mu} c_\nu e^{-\nu r_k^2\mathcal{L}_a}\end{aligned}$$

Then,

$$g_k = \sum_{\nu=1}^{\mu} c_\nu e^{-\nu r_k^2\mathcal{L}_a} b_k$$

Where r_k denotes the radius of Q_k and $\mu := \lfloor \frac{d}{4} \rfloor + 1$. Therefore,

$$\begin{aligned} f &= g + b \\ &= g + \sum_k b_k \\ &= g + \sum_k g_k + \sum_k \tilde{b}_k \end{aligned}$$

Applying the operator T to the above quantity, we get

$$Tf = Tg + \sum_k Tg_k + \sum_k T\tilde{b}_k.$$

By the Marcinkiewicz Interpolation Theorem

$$|Tf| \leq |Tg| + |\sum_k Tg_k| + |\sum_k T\tilde{b}_k|.$$

Then

$$\{|Tf|>h\}\subset\left\{|Tg|>\frac{1}{3}h\right\}\cup\left\{|T\sum_k g_k|>\frac{1}{3}h\right\}\cup\left\{|T\sum_k\tilde{b}_k|>\frac{1}{3}h\right\}$$

By Chebyshev's inequality, and the boundedness of T in L^2, and (2.17)

$$\left|\left\{|Tg|>\frac{1}{3}h\right\}\right|\lesssim h^{-2}||Tg||_{L^2}^2\lesssim h^{-2}||g||_{L^2}^2\lesssim h^{-q}||g||_{L^q}^q\lesssim h^{-q}||f||_{L^q}^q$$

Using an argument similar to what was used above we obtain that

$$\left|\left\{|T\sum_k g_k|>\frac{1}{3}h\right\}\right|\lesssim h^{-2}||T\sum_k g_k||_{L^2}^2\lesssim h^{-2}||\sum_k g_k||_{L^2}^2 \tag{2.19}$$

To control g_k

$$\begin{aligned}
\Big\|\sum_k g_k\Big\|_{L^2}^2 &= \int\Big|\sum_k g_k\Big|^2 \qquad (2.20)\\
&= \int\sum_k g_k\sum_l g_l\\
&= \int\sum_k\sum_\nu c_\nu e^{-\nu r_k^2\mathcal{L}_a}b_k\sum_l\sum_{\nu'}c'_\nu e^{-\nu' r_l^2\mathcal{L}_a}b_l\\
&= \int\sum_{\nu,\nu'}c_\nu c_{\nu'}\sum_k e^{-\nu r_k^2\mathcal{L}_a}b_k\sum_l e^{-\nu' r_l^2\mathcal{L}_a}b_l\\
&= \sum_{\nu,\nu'}c_\nu c_{\nu'}\sum_{k,l}\int b_k e^{-(\nu r_k^2+\nu' r_l^2)\mathcal{L}_a}b_l\\
&= \sum_{\nu,\nu'}c_\nu c_{\nu'}\sum_{k,l}\left\langle b_k, e^{-(\nu r_k^2+\nu' r_l^2)\mathcal{L}_a}b_l\right\rangle\\
&\lesssim \sum_{k,l}\left\langle b_k, e^{-(\nu r_k^2+\nu' r_l^2)\mathcal{L}_a}b_l\right\rangle \qquad (2.21)
\end{aligned}$$

Using the heat kernel in theorem 1.4 we obtain

$$\begin{aligned}
\|\sum_k gk\|_{L^2}^2 &= \sum_{\nu,\nu'}c_\nu c_{\nu'}\sum_{k,l}\langle b_k, e^{-(\nu r_k^2+\nu' r_l^2)}\rangle \qquad (2.22)\\
&\lesssim \sum_{r_k\geq r_l}r_k^{-d}\int_{Q_l}\int_{Q_k}\Big(\frac{r_k}{|x|}\vee 1\Big)^\sigma|b_k(x)|e^{-\frac{|x-y|^2}{cr_k^2}}\Big(\frac{r_k}{|y|}\vee 1\Big)^\sigma|b_l(y)|dxdy
\end{aligned}$$

Now, all that is needed is to show that the quantity on the far right is bounded. Integrating over Q_k and Q_l, we get

$$\sum_{l:r_k \geq r_l} \int_{Q_l} \int_{Q_k} r_k^{-d} \Big(\frac{r_k}{|x|} \vee 1 \Big)^{\sigma} |b_k(x)| e^{-\frac{|x-y|^2}{cr_k^2}} \Big(\frac{r_k}{|y|} \vee 1 \Big)^{\sigma} |b_l(y)| dx dy \tag{2.23}$$

From here, we freeze k, and $xc \in Q_k$ so we can focus on

$$\begin{aligned} \sum_{l:r_l \leq r_k} \int_{Q_l} e^{-\frac{|x-y|^2}{cr_k^2}} \Big(\frac{r_k}{|y|} \vee 1 \Big)^{\sigma} |b_l(y)| dy \ &\lesssim \ \sum_{l:r_l \leq r_k} \int_{Q_l} e^{-\frac{|x-y|^2}{cr_k^2}} |b_l(y)| dy & (2.24) \\ &+ \ \sum_{l:Q_l \subset B(0,2r_k)} \int_{Q_l} \Big(\frac{r_k}{|y|} \Big)^{\sigma} |b_l(y)| dy & (2.25) \end{aligned}$$

We are assuming that $Q_l \cap B(0, 2r_k) \neq \emptyset$ implies $Q_l \subseteq B(0, 2r_k)$ because $r_l \leq r_k$. r_l is the radius of Q_l, and $r_l \leq r_k$, then $dima(Q_l) \leq 2r_k$. x has been fixed in Q_k. Pick a point y in Q_l, then $|x - y| \leq 2r_k$

$$\begin{aligned} |x - y| - 2r_k &\leq 0 \\ (|x - y| - 2r_k)^2 &= |x - y|^2 - 2r_k |x - y| + 4r_k^2 \geq 0 \\ |x - y|^2 &\geq 2r_k |x - y| - 4r_k^2 \end{aligned}$$

We find some $y' \in Q_l$ such that $|x - y'|^2 \leq 2r_k |x - y|$. This is from the fact that $|x - y| \leq 2r_k$ for any $y \in Q_l$, then

$$|x - y|^2 \geq \frac{1}{2} |x - y'|^2 - 4r_k^2$$

for all $y, y' \in Q_l$. Then

$$|b_l(y)| = ||b_l(y)||_{L^1} \lesssim h|Q_l|$$

And

$$\begin{aligned}\sum_{l:r_l\leq r_k}\int_{Q_l} e^{-\frac{|x-y|^2}{cr_k^2}}|b_l(y)|dy &\lesssim \sum_{l:r_l\leq r_k}\int_{Q_l} e^{-\frac{|x-y'|^2}{2cr_k^2}}|b_l(y)|dy \\ &\lesssim \sum_{l:r_l\leq r_k}|b_l(y)|\int_{Q_l} e^{-\frac{|x-y'|^2}{2cr_k^2}}dy \\ &\lesssim \sum_{l:r_l\leq r_k}||b_l(y)||_{L^1}\frac{1}{|Q_l|}\int_{Q_l} e^{-\frac{|x-y'|^2}{2cr_k^2}}dy \\ &\lesssim \sum_{l:r_l\leq r_k} h\int_{Q_l} e^{-\frac{|x-y'|^2}{2cr_k^2}}dy \\ &\lesssim h\sum_{l:r_l\leq r_k}\int_{Q_l} e^{-\frac{|x-y'|^2}{2cr_k^2}}dy \\ &\lesssim hr_k^d\end{aligned}$$

On the other hand

$$\begin{aligned}&\lesssim \Big[\sum_{l:Q_l\subset B(0,2r_k)}\int_{Q_l}\Big(\frac{r_k}{|y|}\Big)^{\sigma q'}\Big]^{\frac{1}{q'}}\Big[\sum_{l:Q_l\subset B(0,2r_k)}\int_{Q_l}|b_l(y)|^q\Big]^{\frac{1}{q}} \\ &\lesssim \Big[\sum_{B(0,2r_k)}\int_{Q_l}\Big(\frac{r_k}{|y|}\Big)^{\sigma q'}\Big]^{\frac{1}{q'}}\Big[\sum_{l:Q_l\subset B(0,2r_k)} h^q|Q_l|\Big]^{\frac{1}{q}} \\ &\lesssim \Big[\sum_{B(0,2r_k)} r^{\sigma q'}\frac{y}{(1-\sigma q')|y|^{\sigma q'}}\Big|_{B(0,2r_k)}\Big]^{\frac{1}{q'}}\Big[\sum_{l:Q_l\subset B(0,2r_k)} h^q|Q_l|\Big]^{\frac{1}{q}} \\ &\lesssim \Big[\sum_{B(0,2r_k)} r^{\sigma q'}\Big]^{\frac{1}{q'}}\Big[\sum_{l:Q_l\subset B(0,2r_k)} h^q r_k^d\Big]^{\frac{1}{q}} \\ &\lesssim hr_k^{\frac{d}{q'}}r_k^{\frac{d}{q}} = hr_k^{d(\frac{1}{q}+\frac{1}{q'})} = hr_k^d\end{aligned}$$

Using this new information, we obtain

$$\begin{aligned}
\Big\|\sum_k g_k\Big\|_{L^2}^2 &\lesssim h\sum_k \int_{Q_k} \Big(\frac{r_k}{|x|}\vee 1\Big)^{\sigma} |b_k(x)|dx \\
&\lesssim h\Big[\sum_k \int_{Q_k} \Big(\frac{r_k}{|x|}\vee 1\Big)^{\sigma q'} dx\Big]^{\frac{1}{q'}} h\Big[\sum_k \int_{Q_k} |b_k(x)|^q dx\Big]^{\frac{1}{q}} \\
&\lesssim h\Big[\sum_k \int_{Q_k} \Big(\frac{r_k}{|x|}\vee 1\Big)^{\sigma q'} dx\Big]^{\frac{1}{q'}} h\Big[\int_{Q_k} \sum_k |b_k(x)|^q dx\Big]^{\frac{1}{q}} \\
&\lesssim h\Big[\sum_k \int_{Q_k} (1)^{\sigma q'} dx\Big]^{\frac{1}{q'}} h\Big[\int_{Q_k} |f|^q dx\Big]^{\frac{1}{q}} \\
&\lesssim h\Big[\sum_k |Q_k|\Big]^{\frac{1}{q'}} ||f||_{L_q} dx \\
&\lesssim h|Q_k|^{\frac{1}{q'}} ||f||_{L_q} dx \\
&\lesssim h^{2-q} \int_{Q_k} |f(x)|^q dx \\
&\lesssim h^{2-q} ||f||_{L^q}^q
\end{aligned}$$

At this point all that is required is to estimate $\{|T\sum_k \tilde{b}_k| > \frac{1}{3}h\}$. Define Q_k^* as the $2\sqrt{d}$ dilate of Q_k. As

$$\Big|\Big\{\Big|T\sum_k \tilde{b}_k| > \frac{1}{3}h\Big|\Big\}\Big| \subset \cup_j Q^* \cup \Big\{x \in R^d \setminus \cup_j Q_j^* : \Big|T\sum_k \tilde{b}_k\Big| > \frac{1}{3}h\Big\}.$$

Using Chebyshev's inequality

$$\begin{aligned}
\Big|\Big\{\Big|T\sum_k \tilde{b}_k\Big| > \frac{1}{3}h\Big\}\Big| &\lesssim \sum_j |Q_j^*| + h^{-1}\sum_k ||T\tilde{b}_k||_{L^1(R^d\setminus Q_k^*)} \\
&\lesssim h^{-q}||f||_{L^q}^q + h^{-1}\sum_k ||T\tilde{b}_k||_{L^1(\mathbb{R}^d\setminus Q_k^*)}
\end{aligned}$$

In order to complete the proof, we need to show

$$\|T\tilde{b}_k\|_{L^1(R^d\setminus Q_k^*)} \lesssim h^{1-q}||b_k||_{L^q}^q \tag{2.26}$$

To do this, we divide the region $\mathbb{R}^d \setminus Q_k^*$ into dyadic annuli of the form $R < dist\{x, Q_k\} \leq 2R$ for $r_k \leq R \in 2^{\mathbb{Z}}$. The following will be proved:

$$\|T\tilde{b}_k\|_{L^2(dist\{x,Q_k\}>R)} \lesssim \Big(\frac{r_k}{R}\Big)^{2\mu} R^{-d(\frac{1}{2}-\frac{1}{q'})}\|b_k\|_{L^q}, \tag{2.27}$$

Claim (2.26) follows

$$\begin{aligned}
\|T\tilde{b}_k\|_{L^1(R^d\setminus Q_k^*)} &= \sum_{R\geq r_k} \|T\tilde{b}_k\|_{L^1(R<dist\{x,Q_k\})\leq 2R} \\
&\lesssim \sum_{R\geq r_k} R^{\frac{d}{2}} \|T\tilde{b}_k\|_{L^2(dist\{x,Q_k\})>R} \\
&\lesssim \sum_{R\geq r_k} R^{\frac{d}{2}} \left(\frac{r_k}{R}\right)^{2\mu} R^{-d(\frac{1}{2}-\frac{1}{q'})} \|b_k\|_{L^q}^q \\
&\lesssim r_k^{\frac{d}{q'}} \|b_k\|_{L^q} \lesssim h^{1-q} \|b_k\|_{L^q}.
\end{aligned}$$

In order for the sum above to converge, we need $\frac{d}{q'} < 2\mu$, which is guaranteed under the hypothesis presented

To proved (2.27), we write

$$(T\tilde{b}_k)(x) = \int_{Q_k} \left[m(\sqrt{\mathcal{L}_a})(1-e^{-r_k^2\mathcal{L}_a})^{\mu}\right](x,y)b_k(y)dy \tag{2.28}$$

The function defined below is extended to all of $\mathbb{R}$ as an even function.

$$a(\lambda) := m(\lambda)(1-e^{-r_k^2\lambda^2})^{\mu} \tag{2.29}$$

We need to show that

$$|\partial^j a(\lambda)| \lesssim |\lambda|^{-j}\Big(1 \wedge r_k|\lambda|\Big)^{2\mu}. \tag{2.30}$$

To start the proof, we need to state the following lemmas.

2.3.1 First Lemma

Lemma 2.3.1. *For* $s = 1, 2, 3, 4....$

$$\partial_\lambda^s(e^{-r_k^2\lambda^2}) = \lambda^{-s} P_{2,s}(r\lambda) e^{-r_k^2\lambda^2}$$

Where $P_{2,s}$ is a polynomial of degree s.

$$P_k(\alpha) = a_k x^k + a_{k-1} x^{k-1} + ... + a_1 x + a_0.$$

Proof. Induction If $b = 0$,

$$\partial_\lambda^0(e^{-r_k^2\lambda^2}) = e^{-r_k^2\lambda^2} = a_0 e^{-r_k^2\lambda^2},\ a_0 = 1\ LHS = RHS$$

Now suppose

$$\partial^{s-1}(e^{-r_k^2\lambda^2}) = \lambda^{-(s-1)} P_{2(s-1)}(r\lambda) e^{-r_k^2\lambda^2}$$

Then,

$$\begin{aligned}
\partial^s(e^{-r_k^2\lambda^2}) &= \partial^1\partial^{s-1}(e^{-r_k^2\lambda^2}) \\
&= \partial^1\left[\lambda^{-(s-1)} * P_{2(s-1)}(r\lambda) * e^{-r_k^2\lambda^2}\right] \\
&= -(s-1)\lambda^s * P_{2(s-1)}(r\lambda)e^{-r_k^2\lambda^2} + \lambda^{-(s-1)} * r * P_{2(s-1)-1}(r\lambda) * e^{-r_k^2\lambda^2} \\
&+ \lambda^{-(s-1)} * P_{2(s-1)-1}(r\lambda) * e^{-r_k^2\lambda^2}(-r^2 2\lambda) \\
&= \lambda^{-s} P_{2s}(r\lambda) e^{-r_k^2\lambda^2}
\end{aligned}$$

□

2.3.2 SECOND LEMMA (LEIBNIZ RULE)

Lemma 2.3.2. *(Leibniz rule)*

$$\begin{aligned}
\partial^s(U * V) &= \sum_{k=0}^{s} \binom{s}{k} \partial^k U * \partial^{s-k} V \\
&= U * \partial^s V + s * \partial^1 U * \partial^{s-1} V + ... + \partial^s U * V
\end{aligned}$$

2.3.3 THIRD LEMMA

Lemma 2.3.3.

$$\partial^s\left[\left(1-e^{-r_k^2\lambda^2}\right)^{\mu}\right] \lesssim |\lambda|^{-s}\left(1\wedge r_k|\lambda|\right)^{2\mu}$$

Recall

$$a(\lambda) = m(\lambda)\left(1-e^{-r_k^2\lambda^2}\right)^{\mu}$$

$$\begin{aligned}
\partial^j &= \sum_{l=0}^{j}\binom{j}{l}\partial^l m(\lambda)\partial^{j-l}\left[\left(1-e^{-r_k^2\lambda^2}\right)^{\mu}\right] \\
&\leq \sum_{l=0}^{j}\binom{j}{l}|\lambda|^{-l}|\lambda|^{-(j-l)}\left(1\wedge r_k|\lambda|\right)^{2\mu} \\
&\lesssim |\lambda|^{-j}\left(1\wedge r_k|\lambda|\right)^{2\mu}
\end{aligned}$$

Proof. Case 1:

$$r_k|\lambda| < 1$$

We need to show

$$\partial^s\left[\left(1-e^{-r_k^2\lambda^2}\right)\right] \lesssim |\lambda|^{-s}(r_k|\lambda|)^{2\mu}$$

When s=0,

$$\left(1-e^{-r_k^2\lambda^2}\right)^{2\mu} \lesssim \left(r_k|\lambda|\right)^{2\mu}$$

Suppose

$$\partial^{s-1}\left[\left(1-e^{-r_k^2\lambda^2}\right)^{2\mu}\right] \lesssim |\lambda|^{-(s-1)}\left(r_k|\lambda|\right)^{2\mu}.$$

Then,

$$\begin{aligned}
\partial^s &= \partial^{s-1}\partial^1\Big[\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu}\Big] \\
&= \partial^{s-1}\Big[\mu\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big(+e^{-r_k^2\lambda^2}r_k^2 2\lambda\Big)\Big] \\
&= 2\mu r_k^2\partial^{s-1}\Big[\lambda e^{-r_k^2\lambda^2}\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big] \\
&= 2\mu r_k^2\partial^{s-1}\Big[\lambda\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}-\lambda\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu}\Big] \\
&= 2\mu r_k^2\Big(\partial^{s-1}\Big[\lambda\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big]-\partial^{s-1}\Big[\lambda\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu}\Big]\Big)
\end{aligned}$$

The first quantity in the RHS is then bounded by

$$\begin{aligned}
\partial^{s-1}\Big[\lambda\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big] &= \lambda\partial^{s-1}\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}+(s-1)\partial^{s-2}(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1} \\
&\lesssim \lambda|\lambda|^{-(s-1)}\Big(r_k|\lambda|\Big)^{2(\mu-1)}+|\lambda|^{-(s-2)}\Big(r_k|\lambda|\Big)^{2(\mu-1)} \\
&\lesssim |\lambda|^{-s+2}\Big(r_k|\lambda|\Big)^{2(\mu-1)}
\end{aligned}$$

The second quantity is bounded by

$$\partial^{s-1}\Big[\lambda\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big]\lesssim|\lambda|^{-s+2}\Big(r_k|\lambda|\Big)^{2\mu}$$

So,

$$\begin{aligned}
\partial^s\Big[\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu}\Big] &\lesssim 2\mu r_k^2|\lambda|^{-s+2}\Big(r_k|\lambda|\Big)^{2(\mu-1)} \\
&\lesssim |\lambda|^{-s}\Big(r_k|\lambda|\Big)^2\Big(r_k|\lambda|\Big)^{2(\mu-1)} \\
&\lesssim |\lambda|^{-s}\Big(r_k|\lambda|\Big)^{2\mu}
\end{aligned}$$

Case 2: $r_k|\lambda|\geqslant 1$.

We need to show

$$\partial^s\Big[\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu}\Big]\lesssim|\lambda|^{-s}.$$

When $s=0$,

$$\partial^s\Big[\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu}\Big]\lesssim 1^{\mu}\lesssim|\lambda|^0=1.$$

Then,

$$\begin{aligned}\partial^s\Big[\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu}\Big] &= \partial^{s-1}\Big[\mu\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big(+e^{-r_k^2\lambda^2}r_k^2 2\lambda\Big)\Big]\\ &= 2\mu r_k^2\partial^{s-1}\Big[\lambda e^{-r_k^2\lambda^2}\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big]\\ &= 2\mu r_k^2\Big\{\lambda\partial^{s-1}\Big[e^{-r_k^2\lambda^2}\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big]+(s-1)\partial^{s-2}\Big[e^{-r_k^2\lambda^2}\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big]\Big\}\end{aligned}$$

To bound the first half of the quantity on the RHS we see that

$$\begin{aligned}\partial^{s-1}\Big[e^{-r_k^2\lambda^2}\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big] &= \sum_{l=0}^{s-1}\binom{s-1}{l}\partial^l\Big(e^{-r_k^2\lambda^2}\Big)\partial^{s-1-l}\Big[\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big]\\ &\lesssim \sum_{l=0}^{s-1}\binom{s-1}{l}|\lambda|^{-l}P_{2l}(r_k\lambda)e^{-r_k^2\lambda^2}|\lambda|^{-(s-1-l)}\\ &\lesssim |\lambda|^{-(s-1)}P_{2(s-2)}(r_k\lambda)e^{-r_k^2\lambda^2}\end{aligned}$$

Similarly, the second quantity on the RHS can be bounded by

$$\partial^{s-2}\Big[e^{-r_k^2\lambda^2}\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu-1}\Big]\lesssim|\lambda|^{-(s-2)}P_{2(s-2)}(r_k\lambda)e^{-r_k^2\lambda^2}$$

Then the whole thing can be bounded. And we have

$$\begin{aligned}\partial^s\Big[\Big(1-e^{-r_k^2\lambda^2}\Big)^{\mu}\Big] &\lesssim r_k^2|\lambda|^{-s+2}P_{2(s-1)}(r_k\lambda)e^{-r_k^2\lambda^2}\\ &\lesssim |\lambda|^{-s}(r_k|\lambda|)^2P_{2(s-1)}(r_k\lambda)e^{-r_k^2\lambda^2}\\ &\lesssim |\lambda|^{-s}P_{2s}(r_k|\lambda|)e^{-r_k^2\lambda^2}\\ &\lesssim |\lambda|^{-s}\end{aligned}$$

□

Define φ to be a smooth, positive, even function supported on $[-\frac{1}{2},\frac{1}{2}]$, and such that $\varphi(\tau)=1$ for $|\tau|<\frac{1}{4}$. Then the Fourier transform of φ is

$$\hat{\varphi}(\lambda)=\int e^{-i\lambda\tau}\varphi(\tau)d\tau$$

and,

$$\check{\varphi}(\lambda) = \frac{1}{2\pi}\int e^{i\lambda\tau}\varphi(\tau)d\tau$$

Now, let R be a number such that $\Big[-\frac{1}{2},\frac{1}{2}\Big] \subseteq \Big[-\frac{R}{2},\frac{R}{2}\Big]$,

$$\begin{aligned}
\check{\varphi}_R(\lambda) &= R\check{\varphi}(R\lambda) \\
&= \frac{R}{2\pi}\int e^{i\lambda R\tau}\varphi(\frac{R\tau}{R})\frac{d\tau}{R} \\
\textit{Letting } \tau = R\tau & \\
\check{\varphi}_R(\lambda) &= \frac{1}{2\pi}\int e^{i\lambda\tau}\varphi\Big(\frac{\tau}{R}\Big)d\tau,
\end{aligned}$$

Both a and φ are even by definition, then convolution

$$\begin{aligned}
a_1(\lambda) &:= (a * \check{\varphi}_R)(\lambda) = \int_{-\infty}^{\infty} a(\tau)\check{\varphi}_R(\lambda-\tau)d\tau \\
&= \int_{-\infty}^{\infty} a(\tau)\check{\varphi}_R(\lambda-\tau)d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\tau)\check{\varphi}_R(\lambda-\tau)d\tau e^{-i\lambda\tilde{\tau}}d\lambda \\
&= \int_{-\infty}^{\infty} a(\tau)\int_{-\infty}^{\infty}\check{\varphi}_R(\tilde{\tau})d\tau e^{-i(\lambda+\tilde{\lambda})\tilde{\tau}}d\tilde{\lambda} \\
&= \int_{-\infty}^{\infty} a(\tau)e^{-i\lambda\tilde{\tau}}d\tau\int_{-\infty}^{\infty}\check{\varphi}_R(\tilde{\tau})d\tau e^{-i\tilde{\lambda}\tilde{\tau}}d\tilde{\lambda} \\
&= \int_{-\infty}^{\infty} a(\tau)e^{-i\lambda\tilde{\tau}}d\tau\varphi\Big(\frac{\tau}{R}\Big)
\end{aligned}$$

Now applying an inverse Fourier transform we get

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{a}(\tau)e^{i\lambda\tilde{\tau}}d\tau\varphi\Big(\frac{\tau}{R}\Big) \tag{2.31}$$

$$= \frac{1}{\pi}\int_{0}^{\infty}cos(\lambda\tau)\hat{a}(\tau)\varphi\Big(\frac{\tau}{R}\Big)d\tau \tag{2.32}$$

$$\tag{2.33}$$

since the function is even, and letting $\tau = \tilde{\tau}$.

[3] shows that the wave equation with inverse-square potential $u_{tt} + \mathcal{L}_a u = 0$ obeys finite speed of propagation. Noting that $\phi(\frac{\tau}{R})$ is supported on the set $\{\tau : |\tau| \leq \frac{R}{2}\}$, the following is obtained

$$supp\Big(a_1\big(\sqrt{\mathcal{L}_a}\big)\delta_y\Big) \subseteq \bigcup_{\tau\leq\frac{R}{2}} supp\Big(cos\big(\tau\sqrt{\mathcal{L}_a}\big)\delta_y\Big) \subseteq B\Big(y, \frac{1}{2}R\Big).$$

Thus, this part of the multiplier a does not contribute to (2.27).

The remaining part of a is shown to be bounded. Define

$$a_2(\lambda) := a_1(\lambda) - a(\lambda) = \int[a(\theta) - a(\lambda)]\check{\varphi}_R(\lambda-\theta)d\theta$$

When $|\lambda| \leq R^{-1}$

$$|a_2(\lambda)| \lesssim \Big(1 \wedge r_k|\lambda|\Big)^{2\mu}\Big(|\lambda|R\Big)^{-2\mu} \tag{2.34}$$

and when $|\lambda| \geq R^{-1}$

$$|a_2(\lambda)| \lesssim \int\Big|\varepsilon(\theta)\Big|\Big|\check{\varphi}_R(\lambda-\theta)\Big|d\theta \lesssim \Big(1 \wedge r_k|\lambda|\Big)^{2\mu}\Big(|\lambda|R\Big)^{-j}. \tag{2.35}$$

Combining (2.34) and (2.35) with the assumption that $R \geq r_k$

$$|a_2(\lambda)| \lesssim \Big(1\wedge r_k|\lambda|\Big)^{2\mu}\Big((|\lambda|R)^{-2\mu}+(|\lambda|R)^{-j}\Big) = \Big(\frac{1\wedge r_k|\lambda|}{|\lambda|R}\Big)^{2\mu}\Big(1+R^2\lambda^2\Big)^{\frac{2\mu-j}{2}} \tag{2.36}$$

The first part of the quantity on the far right can be controlled by $\Big(\frac{r_k}{R}\Big)^{2\mu}$, and the remaining can be decomposed into the following

$$\left(1+R^2\lambda^2\right)^{\frac{2\mu-j}{2}} \approx \int_0^\infty \left(\frac{t}{R^2}\right)^{\frac{j-2\mu}{2}} e^{\frac{-t(1+R^2\lambda^2)}{R^2}} \frac{dt}{t}$$

Combining the two gives equation (2.37)

$$|a_2(\lambda)| \lesssim \left(\frac{1\wedge r_k|\lambda|}{|\lambda|R}\right)^{2\mu}\left(1+R^2\lambda^2\right)^{\frac{2\mu-j}{2}} \lesssim \left(\frac{r_k}{R}\right)^{2\mu}\int_0^\infty \left(\frac{t}{R^2}\right)^{\frac{j-2\mu}{2}} e^{\frac{-t(1+R^2\lambda^2)}{R^2}} \frac{dt}{t} \quad (2.37)$$

By the spectral theorem (Appendix A.2) and the triangle inequality, we obtain the next result.

$$\|a_2(\sqrt{\mathcal{L}_a})\|_{L^2(\mathbb{R})} \lesssim \left(\frac{r_k}{R}\right)^{2\mu}\int_0^\infty \left(\frac{t}{R^2}\right)^{\frac{j-2\mu}{2}} e^{-\frac{t}{R^2}} \left\|e^{-t\mathcal{L}_a}b_k\right\|_{L^2}\frac{dt}{t} \quad (2.38)$$

We state the following quantity without proof. The proof can be found in [1]:

$$\left\|e^{-t\mathcal{L}_a}b_k\right\|_{L^2} \lesssim t^{-\frac{d}{4}}(t+r_k^2)^{\frac{d}{2q'}}\|b_k\|_{L^q} \quad (2.39)$$

Which leads to

$$\begin{aligned} \|a_2(\sqrt{\mathcal{L}_a})b_k\|_{L^2(\mathbb{R}^d)} &\lesssim \left(\frac{r_k}{R}\right)^{2\mu}R^{-d(\frac{1}{2}-\frac{1}{q'})}\|b_k\|_{L^q}\int_0^\infty \left(\frac{t}{R^2}\right)^{\frac{j-2\mu}{2}-\frac{d}{4}}\left(1+\frac{t}{R^2}\right)^{\frac{d}{2q'}} e^{-\frac{t}{R^2}}\frac{dt}{t} \\ &\lesssim (\frac{r_k}{R})^{2\mu}R^{-d(\frac{1}{2}-\frac{1}{q'})}\|b_k\|L^q, \end{aligned}$$

for any $R \geq r_k$. This completes the proof of theorem 2.7.

□

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APPENDIX A

NOTABLE THEOREMS

Here we present a short selection of harmonic analysis theorems that were useful in our work, either explicitly or implicitly.

Theorem A.1. (Calderon-Zygmund). Let $f \in L^1(\mathbb{R}^n)$, and let $h > 0$. There exists a countable collection of cubes with sides parallel to the axes, Q_j with disjoint interiors, such that, for each j,

$$h < \frac{1}{|Q_j|} \int_{Q_j} |f| dx \leq 2^n h.$$

Consider $\Omega = \bigcup Q_j$ and $F = \mathbb{R} \setminus \Omega$. Then,

$$|\Omega| \leq h^{-1} ||f||_{L^1(\mathbb{R}^n)}.$$

Moreover,

$$|f(x)| \leq h$$

holds almost everywhere for $x \in F$. There exist a decomposition

$$f(x) = g(x) + b(x)$$

such that $|g(x)| \leq 2^n h$ almost everywhere, moreover, for $1 < p < \infty$,

$$||g||_{L^p(R^n)} \leq h^{\frac{p-1}{p}} (1 + 2^{np})^{\frac{1}{p}} ||f||_{L^1(\mathbb{R}^n)}^{\frac{1}{p}}$$

Theorem A.2. (Chebyshev Theorem) Let $(X, \sum, \mu)$ be measurable space, and let f be an extended real-valued measurable function defined on X. Then for any real number $h > 0$ and $0 < q < \infty$,

$$\mu\{x \in X : |f(x)| \geq h\} \leq \frac{1}{t^q} \int_{|f| \geq h} |f|^q d\mu.$$

Theorem A.3. (Spectral Theorem) Suppose that $\mathcal{L}_a$ is a self-adjoint positive definite operator acting on $\mathrm{L}^2\Big(TX, \mu\Big)$. Such an operator admits a spectral decomposition $\mathrm{E}_L(\lambda)$ and for any bounded Borel function $F : [0, \infty) \to \mathbf{C}$, we define the operator $F(\mathcal{L}_a) : L^2(TX) \to L^2(TX)$ by the formula

$$F(\mathcal{L}_a) = \int_0^\infty F(\lambda) dE_{\mathcal{L}_a}(\lambda.)$$

Suppose that S is a bounded operator from $L^p(TX)$ to $L^q(TX)$. We write $\|S\|_{L^p(TX) \to L^q(TX)}$ for the usual operator norm of S. If S is of the weak type $(1, 1)$, i.e., if

$$\mu(x \in X : |Sf(x)| > \lambda) \leq C \frac{\|f\|_{L^1(TX)}}{\lambda} \quad \forall \lambda \in R^+ \quad \forall f \in L^1(TX),$$

where the least possible of C is $\|S\|_{L^1 \to L^{1,\infty}}$.

Theorem A.4. (Marcinkiewicz interpolation Theorem, Stein 21) Suppose that $1 \leq r \leq \infty$. If T is a sub-additive mapping from $L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$ to the space of measurable functions on $(\mathbb{R}^n)$ which is simultaneously of weak type $(1, 1)$ and weak type (r, r), then T is also of type (p, p), for all p such that $1 < p < r$. More explicitly: Suppose that for all $f, g \in L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$

(i) $|T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|$

(ii) $m\{: |Tf(x)| > h\} \leq \frac{A_1}{h} ||f||_1, \ f \in L^1(\mathbb{R}^n)$

(iii) $m\{x : |Tf(x)| > h\} \leq \left(\frac{A_r}{h} ||f||_r\right)^r, \ f \in L^r(\mathbb{R}^n)$

Then

$$||Tf(x)||_p \leq A_p ||f||_p, \ f \in L^p(\mathbb{R}^n)$$

for all $1 < p < r$, where A_p depends only on $A_1, A_2, \ pand \, r$.

Theorem A.5. (Holder's Inequality) Let (S, σ, μ) be a measure space and let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$.Then, for all measurable real, or complex-valued functions f and g on S

$$||fg||_1 \leq ||f||_p ||g||_q.$$

If in addition $p, q \in (1, \infty)$ and $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then Holder's inequality becomes an equality iff $|f|^p$ and $|g|^q$ are linearly dependent in $L^1(\mu)$, meaning that there exist real numbers, $\alpha, \beta \geq 0$, not both of them zero, such that $\alpha|f|^p = \beta|g|^q$ on μ almost everywhere.