Summer 2017

Fiber Products in Commutative Algebra

Keller VandeBogert

FIBER PRODUCTS IN COMMUTATIVE ALGEBRA

by

KELLER VANDEBOGERT

(Under the Direction of Saeed Nasseh)

ABSTRACT

The purpose of this thesis is to introduce and illustrate some of the deep connections between commutative and homological algebra. We shall cover some of the fundamental definitions and introduce several important classes of commutative rings. The later chapters will consider a particular class of rings, the *fiber product*, and, among other results, show that any Gorenstein fiber product is precisely a one dimensional hypersurface. It will also be shown that any Noetherian local ring with a (nontrivially) decomposable maximal ideal satisfies the Auslander-Reiten conjecture. To conclude, generalizations of results by Takahashi [26] and Atkins-Vraciu [2] shall be presented.

INDEX WORDS: Auslander-Reiten conjecture, Cohen-Macaulay ring, Fiber products, Gorenstein ring, hypersurface, injective dimension, projective dimension, regular ring

2009 Mathematics Subject Classification: 13D05, 13D07, 13H10, 18A30

FIBER PRODUCTS IN COMMUTATIVE ALGEBRA

by

KELLER VANDEBOGERT

B.S., Georgia Southern University, 2016

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial

Fulfillment of the Requirements for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

FIBER PRODUCTS IN COMMUTATIVE ALGEBRA

by

KELLER VANDEBOGERT

Major Professor:	Saeed Nasseh
Committee:	Alina Iacob
	Jimmy Dillies

Electronic Version Approved:
July 2017

ACKNOWLEDGMENTS

I entered college thinking I knew everything, and by the end of my degree, I realized I knew next to nothing. The past 5 years have been a journey in which I slowly began to learn just how far one can go by truly pushing themselves. Admittedly, any drive I have is inspired by the seemingly superhuman abilities of my professors and the other excellent students I have encountered. That being said, I wish to acknowledge first David Stone, whose persistence on majoring in math eventually paid off when I finally joined the dark side 3 years ago. Also, without him, I would have never learned that no undergraduate is allowed to utter the word “clearly”.

I must also thank Andrew Sills. His confidence in my ability is truly humbling, and is the only reason I have been able to experience the beauty of advanced number theory. I could not be happier that I chose to take number theory as one of my first real math classes.

Jimmy Dillies allowed me a glimpse of the tough but rewarding work of independent research, and has been an invaluable source of help while planning my future. His lightning fast mind is both inspiring and intimidating.

Last but certainly not least, this thesis would not have been possible without the help of Saeed Nasseh. I cannot express my full gratitude for the time he has taken out of his schedule to watch me flounder helplessly amidst the walls of theorems and definitions he uses effortlessly. I can confidently say that I would not be *half* the student I am today had I not had the opportunity to work with him for this past year.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

In the past century, commutative ring theory and homological algebra have become nearly synonymous. The reasons for this can be traced to key results, for which either the technique of proof utilized theory of another branch, or the surprising correspondences stated by the conclusions of these results. Historically, homological algebra has its roots in some of the classical problems of differential geometry and algebraic topology, with regards to a more rigorous approach to the classification of surfaces. Even the modern language and terminology reflects this influence, with names such as "differentials" being used for the homomorphisms of a (co)chain complex. This is no accident; Poincaré's Theorem asserts that every exact form is closed. The measure for which the converse holds is precisely the notion of homology. During the mid 1800's, Riemann would be a pioneer of this search for invariance by defining the "connectedness numbers" of a surface S, which were later discovered to be related to the homology invariant $H_1(S,\mathbb{Z}/2\mathbb{Z})$ [28].

In the construction of connecting maps between certain sets, mathematicians would often notice that these maps obeyed a condition of nilpotence, that is, applying the same map twice would give 0. This condition has a natural interpretation in the case of the boundary operator in simplicial homology, for instance, where this nilpotence is the geometric statement that taking the boundary of a boundary should just give 0. However, there were many more exotic constructions that seemed to obey this condition, prompting the consideration of a more axiomatic approach. This led to the modern definitions of exact sequences and the more general (co)chain complexes.

Exact sequences, though modern, were hidden in results dating back to Euler. Euler's Theorem asserts that for a planar graph with F faces, E edges, and V vertices, $F-E+V=2$. We can write this in a way that is more indicative of the underlying structure by saying $(-1)^1+(-1)^2F+(-1)^3E+(-1)^4V+(-1)^5=0$. This is of course just an alternating sum, and the above is the result of the exactness guaranteed by a certain simplicial complex of any planar graph. In fact, for higher dimensions, the above alternating sum is an example of the more general *Euler-Poincaré* characteristic.

Fast-forwarding to the 20th century, the importance of commutative algebra becomes more apparent through the various results relating algebraic geometry and ring theory. The famous *Nullstellensatz* asserts a bijective correpondence between affine algebraic varieties and the radical ideals of polynomial rings, similar to the correspondence of subfields and subgroups in Galois theory. As one could imagine, this connection between commutative ring theory and geometry immediately implies a connection between homological algebra and commutative ring theory.

One of the first large results showing the power of homological methods in commutative algebra comes from a rather geometric question. Mathematicians are often very interested in singular points (and their resolutions) on curves, which may be interpreted as points where a line intersects itself or is otherwise "pointed". It turns out that these singularities can be defined algebraically, without the use of derivatives, by looking at the multiplicity of certain polynomials. The *Auslander-Buchsbaum-Serre* Theorem asserts that singular points are such that the residue field of the localization about these points must have infinite projective dimension.

Besides the nice characterization of singular points, this theorem more importantly solved an open problem of regular local rings. It was well known at the time that a regular local ring had finite global dimension. The Auslander-Buchsbaum-Serre Theorem showed that the converse was also true, thus giving a completely homological characterization of an object in commutative algebra. It also showed that the localization of a regular ring was again regular. Not only this, but the method of proof also used entirely homological methods, thus demonstrating the power of these techniques.

The purpose of this thesis is to introduce and illustrate some of the deep connections between commutative and homological algebra. The later chapters will consider a particular class of rings, the *fiber product*, and show that any Gorenstein fiber product is precisely a one dimensional hypersurface. It will also be shown that any Noetherian local ring with a (nontrivially) decomposable maximal ideal satisfies the Auslander-Reiten conjecture. To conclude, generalizations of results by Takahashi [26] and Atkins-Vraciu [2] shall be presented.

CHAPTER 2

HOMOLOGICAL ALGEBRA AND CLASSES OF COMMUTATIVE RINGS

Throughout this thesis, R will denote a commutative Noetherian ring. If R is local, then its maximal ideal and its residue field will be denoted $\mathfrak{m}$ and k, respectively. We denote by $\operatorname{Spec}(R)$ and $\operatorname{m-Spec}(R)$ the set of prime ideals and maximal ideals of R, respectively. Also, for the purposes of this thesis all R-modules will be assumed finitely generated. Keep in mind that many of the results in this chapter have more general statements and do not require the above assumptions.

2.1 Homological Algebra

2.1.1 The Basics

This section contains some fundamental results and definitions in homological algebra. The notation $\operatorname{mod} R$ will be used for the category of finitely generated R-modules.

Definition 2.1.1. For R-modules A and B, the pair of homomorphisms ϕ and ψ

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$

are called exact at B if $\operatorname{Ker}\phi = \operatorname{Im}\psi$. A sequence

$$\cdots \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_{n+1} \longrightarrow \cdots$$

is called exact if it is exact at each M_n.

The following is a trivial consequence of the above:

Proposition 2.1.2. *Let A, B, C be R-modules. Then:*

1. *The sequence* $0 \longrightarrow A \xrightarrow{\psi} B$ *is exact at A if and only if ψ is injective.*
2. *The sequence* $B \xrightarrow{\phi} C \longrightarrow 0$ *is exact at C if and only if ϕ is surjective.*

We then combine this to deduce the following.

Corollary 2.1.3. *The sequence*

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$$

of R-modules and R-module homomorphisms is exact if and only if ψ is injective, ϕ is surjective, and $\operatorname{Ker}\phi = \operatorname{Im}\psi$. *This type of sequence is called a short exact sequence.*

Remark 2.1.4. For any homomorphism of R-modules $\phi : B \to C$, the sequence

$$0 \longrightarrow \operatorname{Ker}\phi \longrightarrow B \xrightarrow{\phi} \operatorname{Im}\phi \longrightarrow 0$$

is a short exact sequence.

Example 2.1.5. An interesting example of exact sequences comes from Poincare's Lemma. If X is a smoothly contractible subset of $\mathbb{R}^n$, let $\Omega^p(X)$ denote the space of differential p-forms on X. This can be viewed at a module over the set of smooth functions, which is a commutative ring with addition and multiplication of functions. Then, the sequence

$$0 \xrightarrow{d} \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(X) \xrightarrow{d} \Omega^n(X) \xrightarrow{d} 0$$

is exact, where d denotes the exterior derivative.

Example 2.1.6. Consider a symplectic manifold (M, ω), where ω is a closed nondegenerate 2-form. Let $C^\infty(M)$ denote the set of smooth functions on M. A smooth vector field is defined as any smooth section of the tangent bundle TM. Define the symplectic gradient as the vector field drag$H$ for $H \in C^\infty(M)$ such that

$$i_{\mathrm{drag}H}\,\omega = -\mathrm{d}H$$

where i denotes the interior product of a differential form. Then a smooth vector field η is said to preserve ω if $L_\eta \omega = 0$, where L denotes the Lie Derivative. If the 1-form $i_\eta \omega$ is exact, then we call η a Hamiltonian vector field. Denote by $\mathrm{Ham}(M)$ the set of Hamiltonian vector fields on M. Viewed as modules (hence, vector spaces) over $\mathbb{R}$, we then have the following short exact sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \xrightarrow{\text{drag}} \mathrm{Ham}(M) \longrightarrow 0$$

Definition 2.1.7. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ a short exact sequence of R-modules and R-module homomorphisms. Then the sequence is said to be *split*, or *the sequence splits* if $B \cong A \oplus C$.

With this we have the following.

Proposition 2.1.8 (Splitting Lemma)**.** *Let R be a ring and $0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$ a short exact sequence of R-modules. Then the following are equivalent:*

1. *There exists a left inverse for ψ, that is, there exists $\psi' : B \to A$ such that $\psi' \circ \psi = Id_A$.*

2. *There exists a right inverse for ϕ, that is, there exists $\phi' : C \to B$ such that $\phi \circ \phi' = Id_C$.*

3. *The sequence splits, that is, $B \cong A \oplus C$.*

2.1.2 Free, Projective, Injective, and Flat modules

Definition 2.1.9. An R-module F is said to be *free* on the subset A of F if for every nonzero $x \in F$, x can be written *uniquely* as $x = r_1 a_1 + \ldots r_n a_n$ for $a_i \in A$, $r_i \in R$ and $n \in \mathbb{Z}^+$. A set with the above property will be called a basis for F.

Intuitively, a free module over a ring R is isomorphic to a direct sum of copies of the ring R. Trivially, this means that every ring R is free as an R-module.

Notation 2.1.10. The set $\mathrm{Hom}_R(M,N)$ will denote the set of R-module homomorphisms from M to N.

Consider a short exact sequence $0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$ of R-modules and R-module homomorphisms. Then it is often the case that there will exist a mapping f from another R-module M to the module B. Then we would have the following diagram:

$$\begin{tikzcd}
 & & M \arrow[dl, dashed] \arrow[d, "f"] \arrow[dr, dashed] & & \\
0 \arrow[r] & A \arrow[r, "\psi"] & B \arrow[r, "\phi"] & C \arrow[r] & 0
\end{tikzcd} \tag{2.1.1}$$

As the diagram suggests, one may be inclined to ask when this implies the existence of a mapping from M to each of the modules A and C making the diagram commute. Clearly we can just take $f' := \phi \circ f$ and this gives us a mapping for the right-hand side of the diagram. The left-hand side is not clear, however.

From a more functorial point of view, we can say

Proposition 2.1.11. *If for R-modules A, B, and M the sequence*

$$0 \longrightarrow A \xrightarrow{\psi} B$$

is exact, then the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(M,A) \xrightarrow{\psi'} \mathrm{Hom}_R(M,B)$$

is also exact with $\psi' = \psi \circ f$ for all $f : M \to A$.

We say that $\mathrm{Hom}_R(M,-)$ is a left-exact functor in this case. From here, if we have a map $\psi : A \to B$, then $\psi' : \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B)$ will be understood as the induced map. With this convention, we can say:

Theorem 2.1.12. *Let M, A, B, and C be R-modules. Then, if*

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$$

is exact, the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(M,A) \xrightarrow{\psi'} \mathrm{Hom}_R(M,B) \xrightarrow{\phi'} \mathrm{Hom}_R(M,C) \tag{2.1.2}$$

is also exact.

In the exact sequence 2.1.2, if ϕ' is surjective, then we say that $\mathrm{Hom}_R(M,-)$ is right exact. It is important to note that this does not happen in general. We hence give a characterization of the much less trivial case concerning the left-hand side of our original diagram (2.1.1):

Definition 2.1.13. If P is an R-module for which $\mathrm{Hom}_R(P,-)$ is right exact, then P is called a *projective module*.

Similarly, a module I will be called *injective* if the contravariant functor $\mathrm{Hom}(-,I)$ is exact, that is, both left and right exact.

Projective modules are intuitively modules for which if $\phi : M \to N$ and $f : P \to N$ are R-module homomorphisms (with ϕ surjective), then we can lift f to another mapping $F : P \to M$. Put more succinctly, the following diagram commutes, that is, $f = \phi \circ F$:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists F"'] \arrow[d, "f"] & \\ M \arrow[r, "\phi"] & N \arrow[r] & 0 \end{tikzcd}$$

Injective modules are dual to projective modules. Instead of asking for maps *from* the module I, we are asking about how maps *into* I lift to other maps into it. Put more precisely, if the sequence $0 \longrightarrow M \xrightarrow{\phi} N$ is exact, an injective module I is such that given and $f \in \mathrm{Hom}(M,I)$, there exists $F \in \mathrm{Hom}(N,I)$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\phi"] \arrow[d, "f"'] & N \arrow[dl, dashed, "\exists F"] \\ & I & \end{tikzcd}$$

Also, the induced maps are reversed. In this case, if we have $\phi \in \mathrm{Hom}(M,N)$, then $\phi' : \mathrm{Hom}(N,I) \to \mathrm{Hom}(M,I)$ (note the reversal due to contravariance) takes $f \mapsto f \circ \phi$.

In general, we have the following useful characterizations:

Proposition 2.1.14. *The following are equivalent:*

1. *An R-module P is projective.*

2. *For R-modules M, N, given a surjective map $\phi \in \mathrm{Hom}_R(M,N)$ and $f \in \mathrm{Hom}_R(P,N)$, there exists $F \in \mathrm{Hom}_R(P,M)$ making the following diagram commute:*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists F"'] \arrow[d, "f"] & \\ M \arrow[r, "\phi"] & N \arrow[r] & 0 \end{tikzcd}$$

3. *For R-modules M', N', any short exact sequence of the form*

$$0 \longrightarrow M' \longrightarrow N' \longrightarrow P \longrightarrow 0$$

splits.

4. *P is the direct summand of a free module.*

And similarly, the following are equivalent:

1. *An R-module I is injective.*

2. *For R-modules M, N, given $\phi \in \mathrm{Hom}(M,N)$ with $0 \longrightarrow M \xrightarrow{\phi} N$ exact and $f \in \mathrm{Hom}(M,I)$, there exists $F \in \mathrm{Hom}(N,I)$ making the following commute:*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\phi"] \arrow[d, "f"'] & N \arrow[dl, dashed, "F"] \\ & I & \end{tikzcd}$$

3. *For R modules M', N', any short exact sequence of the form*

$$0 \longrightarrow I \longrightarrow M' \longrightarrow N' \longrightarrow 0$$

splits.

To conclude, the following is fundamental and will be used in the later sections.

Theorem 2.1.15. *Let $M \in \operatorname{mod} R$. Then the following are equivalent:*

1. *M is projective.*

2. *M is locally free, that is, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ module for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

3. *$M_{\mathfrak{m}}$ is free for every $\mathfrak{m} \in \text{m-Spec}(R)$.*

In particular, if R is a local Noetherian ring, then $R_{\mathfrak{m}} = R$, and hence the properties of projectivity and free-ness coincide.

2.1.3 Resolutions and Ext

We now proceed to the construction of the so called *extension groups*. To do this, we want to give the following definition first:

Definition 2.1.16. Given an R-module A, a *projective resolution* of A is a sequence of modules M_n and homomorphisms d_n such that

$$\cdots \longrightarrow M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} M_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

is exact and each M_n a projective module. The minimal length of a projective resolution is called the *projective dimension* of M, denoted $\operatorname{pd}_R M$ or $\operatorname{pd} M$. Similarly, an *injective resolution* is such that

$$0 \longrightarrow A \xrightarrow{\varepsilon} M^0 \xrightarrow{d^0} \cdots \xrightarrow{d^n} M^{n+1} \xrightarrow{d^{n+1}} M^{n+2} \xrightarrow{d^{n+2}} \cdots$$

is an exact sequence and each M_i an injective module. The minimal length of an injective resolution is called the *injective dimension* of M, denoted $\mathrm{id}_R M$ or $\mathrm{id} M$.

It is important to consider the construction of such a resolution as given above, because it may not be obvious that such a resolution is always possible. Consider first the construction of a projective resolution. Given a module M, choose an ordered generating set $\{x_1, \ldots, x_n\}$. Then we have the obvious surjective map $\varepsilon : R^n := P_0 \to M$ by just sending $(r_1, \ldots, r_n) \mapsto r_1x_1 + \cdots + r_nx_n$. Now, do the same process to $\mathrm{Ker}\,\varepsilon$ in terms of finding a surjective map d_1 from some projective module P_1 onto $\mathrm{Ker}\,\varepsilon$. Proceeding inductively we will construct an exact sequence

$$\begin{tikzcd}
 & & & \mathrm{Ker}\,\varepsilon \arrow[dr] & & & \\
\cdots \arrow[r] & P_2 \arrow[rr, "d_2"] \arrow[dr] & & P_1 \arrow[rr, "d_1"] \arrow[ur] & & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
 & & \mathrm{Ker}\,d_1 \arrow[ur] & & & &
\end{tikzcd}$$

such that each P_i is projective. Hence, this is a projective resolution. Similarly, for the injective case, we can use the fact that every module sits inside of an *injective hull*, the "smallest" injective module containing M. By considering the inclusion of a module into its injective hull, we can again construct an exact sequence of injective modules satisfying the above definition. We then have the following:

Theorem 2.1.17. *Every module has a free (hence projective) and injective resolution.*

It is natural to consider the derived (co)homology groups after applying Hom to an appropriate resolution. We have the following:

Definition 2.1.18. Let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

be a projective resolution of an R-module A. For each R-module D, consider the induced cochain complex

$$0 \longrightarrow \mathrm{Hom}(A,D) \xrightarrow{\varepsilon} \mathrm{Hom}(P_0,D) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{Hom}(P_{n+1},D) \longrightarrow \mathrm{Hom}(P_{n+2},D) \longrightarrow \cdots$$

We define:

$$\mathrm{Ext}^n_R(A,D) = \mathrm{Ker}(d'_{n+1})/\,\mathrm{Im}(d'_n)$$

where the d'_i are the induced differentials by the Hom functor.

It can be seen that $\mathrm{Ext}^n_R(A,D)$ can also be derived using an injective resolution for D as follows. Let

$$0 \longrightarrow D \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} \cdots \xrightarrow{d^n} I^{n+1} \xrightarrow{d^{n+1}} I^{n+2} \xrightarrow{d^{n+2}} \cdots$$

be an injective resolution of A. Then, consider the induced cochain complex

$$0 \longrightarrow \mathrm{Hom}(A,D) \longrightarrow \mathrm{Hom}(A,I^0) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{Hom}(A,I^{n+1}) \longrightarrow \mathrm{Hom}(A,I^{n+2}) \longrightarrow \cdots$$

Then we have:

$$\mathrm{Ext}^n_R(A,D) = \mathrm{Ker}(d'^{n+1})/\,\mathrm{Im}(d'^n)$$

The following is a trivial consequence of the above

Proposition 2.1.19. *For any R-modules A and B, we have that* $\mathrm{Ext}^0_R(A,B) \cong \mathrm{Hom}_R(A,B)$.

And, much less obvious is the following important result:

Theorem 2.1.20. *For R-modules A and B, the modules* $\operatorname{Ext}^i_R(A,B)$ *do not depend on the choice of projective resolution for A or injective resolution for B.*

Combining 2.1.19 with the long exact cohomology sequence:

Theorem 2.1.21. *Suppose*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of R-modules and R-module homomorphisms. Then, we have the induced long exact sequence:

$$0 \longrightarrow \operatorname{Hom}_R(C,D) \longrightarrow \operatorname{Hom}_R(B,D) \longrightarrow \operatorname{Hom}_R(A,D) \longrightarrow$$

$$\operatorname{Ext}^1_R(C,D) \longrightarrow \operatorname{Ext}^1_R(B,D) \longrightarrow \operatorname{Ext}^1_R(A,D) \longrightarrow \cdots$$

Similarly, we have the induced long exact sequence:

$$0 \longrightarrow \operatorname{Hom}_R(D,A) \longrightarrow \operatorname{Hom}_R(D,B) \longrightarrow \operatorname{Hom}_R(D,C) \longrightarrow$$

$$\operatorname{Ext}^1_R(D,A) \longrightarrow \operatorname{Ext}^1_R(D,B) \longrightarrow \operatorname{Ext}^1_R(D,C) \longrightarrow \cdots$$

Noting that for an injective R-module Q, $0 \longrightarrow Q \longrightarrow Q \longrightarrow 0$ is an injective resolution (and similarly in the projective case), we get the following:

Proposition 2.1.22. *The following are equivalent:*

1. *An R-module P is projective.*
2. $\operatorname{Ext}^1_R(P,A) = 0$ *for all R-modules A.*
3. $\operatorname{Ext}^n_R(P,A) = 0$ *for all R-modules A and all* $n \geqslant 1$.

Similarly, the following are also equivalent:

1. *An R-module Q is injective.*

2. $\operatorname{Ext}^1_R(A,Q) = 0$ *for all R-modules A.*

3. $\operatorname{Ext}^n_R(A,Q) = 0$ *for all R-modules A and all* $n \geqslant 1$.

Remark 2.1.23. The notation Ext comes from extension. Extension refers to the fact that any short exact sequence can be thought of as an extension of the first term A by the last term C. For a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there is always a trivial extension with $B = A \oplus C$. The Ext functor, or more precisely, $\operatorname{Ext}^1_R(C,A)$, is a measure of how many inequivalent extensions of A by C there are. Equivalence is taken in terms of certain commutative diagrams between short exact sequences. This description of Ext was discovered by Yoneda, see the paper by Weibel [28].

2.2 Ring Theory

With the language of homological algebra, we will soon see that many unsightly ring theoretic definitions have very elegant characterizations in terms of Ext.

2.2.1 Some Dimension Theory

Definition 2.2.1. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the *height* of $\mathfrak{p}$ is defined to be the maximal length r of any strictly descending chain of prime ideals $\mathfrak{p} = \mathfrak{p}_r \supset \mathfrak{p}_{r-1} \supset \cdots \supset \mathfrak{p}_0$. This quantity is denoted $\operatorname{ht}\mathfrak{p}$.

The *coheight* of $\mathfrak{p} \in \operatorname{Spec}(R)$ is the maximal length of ascending chains of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$. This quantity is denoted $\operatorname{coht}\mathfrak{p}$.

Now, consider taking the supremum over heights for all $\mathfrak{p} \in \operatorname{Spec}(R)$. This is called the *Krull Dimension* of R, and is denoted simply $\dim R$.

Definition 2.2.2. Let $I \subset R$ be an ideal. Then, define

$$V(I) := \{\mathfrak{p} \mid \ \mathfrak{p} \supset I, \ \mathfrak{p} \in \operatorname{Spec}(R)\}$$

Then, we define a topology on $\operatorname{Spec}(R)$ in which $V(I)$ is closed for all ideals $I \subset R$. This topology is known as the *Zariski topology*. For the rest of the thesis, $\operatorname{Spec}(R)$ will always be considered with the Zariski topology.

Example 2.2.3. In a topological space X, a closed set is irreducible if it cannot be expressed as the union of two nonempty (disjoint) closed sets. The combinatorial dimension is taken as the maximal length of all strictly increasing or decreasing chains of *irreducible* closed subsets of X. It can be shown that the combinatorial dimension of $\operatorname{Spec}(R)$ is precisely the Krull dimension of R [16].

Proposition 2.2.4. *Let A be a ring with $\mathfrak{p} \in \operatorname{Spec}(A)$. We have the following:*

$$\operatorname{ht}\mathfrak{p} = \dim A_{\mathfrak{p}}$$

$$\operatorname{coht}\mathfrak{p} = \dim A/\mathfrak{p}$$

$$\operatorname{ht}\mathfrak{p} + \operatorname{coht}\mathfrak{p} \leqslant \dim A$$

Example 2.2.5. Every principal ideal domain (that is not a field) has Krull dimension 1.

Example 2.2.6. Suppose that R is Artinian, and note that since $R/\mathfrak{p}$ is an Artinian integral domain for all $\mathfrak{p} \in \operatorname{Spec}(R)$, every prime ideal is a maximal ideal. However, this immediately implies that there exist no nontrivial chains of decreasing ideals in R, and hence R has Krull dimension 0.

This in fact gives us a converse for a theorem of Akizuki (see [16, Theorem 3.2])

Theorem 2.2.7. *A ring R is Artinian if and only if it is Noetherian of dimension* 0.

Definition 2.2.8. We can define the height of any ideal $I \subset A$ as

$$\operatorname{ht} I := \inf\{\operatorname{ht}\mathfrak{p} : \ \mathfrak{p} \in V(I)\}$$

We then have the analog to the final inequality of 2.2.4: $\operatorname{ht} I + \operatorname{ht} A/I \leqslant \dim A$. We want to now extend this idea of dimension to modules.

Definition 2.2.9. For any A-module M, define

$$\dim M := \dim \big(A/\operatorname{Ann}(M)\big)$$

Where $\operatorname{Ann}(M)$ denotes the annihilator of M, that is, the set of all $r \in R$ such that $rM = 0$.

Recalling the definition of the support of a module, it is easy to see that $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$. By definition of the Zariski topology, this is a closed subset of $\operatorname{Spec}(A)$, and if M is finitely generated we have that $\dim M$ is precisely the combinatorial dimension of $\operatorname{Supp}(M) \subset \operatorname{Spec}(A)$.

Using a prime avoidance argument, the following can be proved:

Theorem 2.2.10. *Let I be a proper ideal of R of height n. Then there exist $x_1, \ldots, x_n \in I$ such that* $\operatorname{ht}(x_1, \ldots, x_i) = i$ *for all $i = 1, \ldots, n$.*

And we have a useful characterization in the local case:

Theorem 2.2.11. *For the local ring $(R, \mathfrak{m})$, the following are equivalent:*

1. $\dim R = n$
2. $\operatorname{ht}\mathfrak{m} = n$
3. *n is precisely the infimum over all m such that there exist $x_1, \ldots, x_m \in \mathfrak{m}$ with $\sqrt{(x_1, \ldots, x_m)} = \mathfrak{m}$.*

4. *n is precisely the infimum over all m for which there exist $x_1, \dots, x_m \in \mathfrak{m}$ such that $R/(x_1, \dots, x_m)$ is Artinian.*

Remark 2.2.12. The property of the x_i given in part (3) of the above theorem is known as being a *system of parameters* of A. Indeed, every local Noetherian ring admits a system of parameters. This definition will be used again once we begin classifying rings.

This definition can be extended to $M \in \operatorname{mod} R$. If $\dim M = n$, then $x_1, \dots, x_n \in \mathfrak{m}$ is a system of parameters for M if $M/(x_1, \dots, x_n)M$ is Artinian.

Proposition 2.2.13. *Let $(R, \mathfrak{m})$ be a local ring with $M \in \operatorname{mod} R$ and $x_1, \dots, x_r \in \mathfrak{m}$. Then*

$$\dim(M/(x_1, \dots, x_r)M) \geqslant \dim M - r$$

Equality holds if and only if $x_1, \dots, x_r$ is part of a system of parameters of M.

2.2.2 Associated Primes, Regular Sequences, Grade, and Depth

We begin with the following

Definition 2.2.14. An *associated prime* $\mathfrak{p} \in \operatorname{Spec}(R)$ of an R-module M is an ideal such that $\mathfrak{p} = \operatorname{Ann}(z)$ for some $z \in M$.

The set of associated primes is denoted $\operatorname{Ass} M$.

It would be nice to have an explicit realization of an associated prime. This is not too hard to construct, however. Consider the family of annihilators

$$F := \{\operatorname{Ann}(x) \mid 0 \neq x \in M\}$$

Since R is Noetherian, we can choose some maximal element (with respect to inclusion) $\operatorname{Ann}(x) := m \in F$. Suppose $ab \in m$ for $a,\ b \in R$. By definition, $abx = 0$, and by symmetry there is no loss of generality in assuming $b \notin m$. Then we see that $a \in \operatorname{Ann}(bx) \subset m$, by maximality. Hence $a \in m$, and by definition m is prime. We have just proved

Theorem 2.2.15. *Let M be a nonzero R-module.*

1. *Associated primes are precisely the maximal elements of the family of ideals*

$$F = \{\mathrm{Ann}(x) \mid 0 \neq x \in M\}$$

2. *The set of zero divisors is the union of all $\mathfrak{p} \in \mathrm{Ass}\, M$.*

Remark 2.2.16. For an R-module M and $\mathfrak{p} \in \mathrm{Ass}\, M$ with $\mathfrak{p} = \mathrm{Ann}(z)$, $z \in M$, let $I \subset \mathfrak{p}$ be an ideal. Consider the homomorphism $\overline{\phi} : R/\mathfrak{p} \to M$ induced by sending $1 + \mathfrak{p} \mapsto z$. The kernel is precisely $\mathfrak{p}$, so this is an injection. Restricting, this gives us a non-zero homomorphism $\phi : R/I \to M$. We will use this seemingly inconsequential observation soon.

Definition 2.2.17. An element $x \in R$ is M-regular (or just regular) if $xm = 0$ for $m \in M$ implies $m = 0$.

A regular M-sequence (or just M-sequence) is a sequence $x_1, \ldots, x_n$ such that each x_i is a regular element of $M/(x_1, \ldots x_{i-1})M$, and $M/\underline{x}M \neq 0$ (where $\underline{x}$ denotes either the ideal $(x_1, \ldots, x_n)$ or just $x_1, \ldots x_n$. The context should make this meaning clear).

If we have that $M/\underline{x}M = 0$ but the other condition for the sequence $\underline{x}$ is satisfied, then $\underline{x}$ is called a *weak* M-sequence. In the local case, these definitions coincide when $\underline{x} \subset \mathfrak{m}$ since if $M/\underline{x}M = 0$, Nakayama's Lemma gives that $M = 0$.

Consideration of the length of an M-regular sequence will lead us into the first natural question: what is the maximal length of any M-sequence? However, even more nuanced is the consideration of a *maximal* M-sequence, which is any such M-sequence with the property that it cannot be extended to any strictly larger M-sequence. It turns out that this length is extremely important for classification of rings. We can also add the additional restriction that each x_i in our desired regular sequence must be contained in some ideal I. It is not obvious that the length of any maximal M-sequence contained in I is invariant.

Now from Remark 2.2.16, if any ideal I consists entirely of zero divisors, there exists some nonzero homomorphism. Similarly, consider trying to find a homomorphism between two R modules M, N. If there is some M-regular element $z \in \operatorname{Ann}(N)$, then for any $\phi \in \operatorname{Hom}_R(N, M)$, $n \in N$:

$$\phi(zn) = 0 = z\phi(n)$$

Since z is M-regular, we conclude that $\phi(n) = 0$ for all $n \in N$. We have just proved the first part of the following:

Proposition 2.2.18. *Let $M, N \in \operatorname{mod} R$. Set $I := \operatorname{Ann}(N)$.*

1. *If I contains any M-regular element, then $\operatorname{Hom}_R(N, M) = 0$.*
2. *Conversely, if R is Noetherian and $M, N \in \operatorname{mod} R$, $\operatorname{Hom}_R(N, M) = 0$ implies that I contains some M-regular element.*

We now have the following fundamental theorem due to Rees:

Theorem 2.2.19. *Let $M \in \operatorname{mod} R$ and I be an ideal such that $M \neq IM$. Then all maximal M-sequences have the same length given by*

$$n = \min\{i : \operatorname{Ext}^i_R(R/I, M) \neq 0\}$$

Definition 2.2.20. Let $M \in \operatorname{mod} R$ and I be an ideal such that $M \neq IM$. Then the common length of all maximal M-sequences in I is called the *grade* of I on M, and is denoted $\operatorname{grade}(I, M)$. By the previous theorem,

$$\operatorname{grade}(I, M) = \min\{i : \operatorname{Ext}^i_R(R/I, M) \neq 0\}$$

Of course, as is usual, we can consider the case when $(R,\mathfrak{m})$ is a local ring and $M \in \operatorname{mod} R$ satisfy the above hypotheses. We have a special notation in this case:

$$\operatorname{grade}(\mathfrak{m},M) := \operatorname{depth} M$$

and we call this the *depth* of M. The following proposition consists of a collection of very useful formulas for grade.

Proposition 2.2.21. *Let I, J be ideals of R, and $M \in \operatorname{mod} R$. Then:*

1. $\operatorname{grade}(I,M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in V(I)\}$

2. $\operatorname{grade}(I,M) = \operatorname{grade}(\sqrt{I},M)$

3. $\operatorname{grade}(I \cap J,M) = \min\{\operatorname{grade}(I,M), \operatorname{grade}(J,M)\}$

4. *If $\underline{x}$ is an M-sequence in I, then*

$$\operatorname{grade}(I/(\underline{x}), M/\underline{x}M) = \operatorname{grade}(I, M/\underline{x}M) = \operatorname{grade}(I,M) - n$$

5. *If $N \in \operatorname{mod} R$ with $\operatorname{Supp} N = V(I)$, then*

$$\operatorname{grade}(I,M) = \inf\{i : \operatorname{Ext}^i_R(N,M) \neq 0\}$$

2.2.3 Relationships between Depth, Krull Dimension, and Projective Dimension

It turns out that the depth of a module and the dimension of a module are intimately connected. The next inequality is fundamental, and the case in which equality occurs is used to define a very special class of rings.

Proposition 2.2.22. *Let $(R,\mathfrak{m})$ be a local ring and $0 \neq M \in \operatorname{mod} R$. Then every M-sequence is part of a system of parameters of M, and we have the inequality:*

$$\operatorname{depth} M \leqslant \dim R/\mathfrak{p}$$

for all $\mathfrak{p} \in \operatorname{Ass} M$. *In particular,*

$$\operatorname{depth} M \leqslant \dim M$$

Definition 2.2.23. Let $(R,\mathfrak{m},k)$ be a local ring. For $M \in \operatorname{mod} R$ with $\operatorname{depth} M = n$, we define the *type* of M, denoted γ_M, as the dimension of the vector space $\operatorname{Ext}^n_R(k,M)$. More precisely:

$$\gamma_M = \dim_k \operatorname{Ext}^n_R(k,M)$$

The following elegant and useful formula relates depth and projective dimension:

Theorem 2.2.24 (Auslander-Buchsbaum Formula)**.** *Let* $(R,\mathfrak{m})$ *be a local ring with* $0 \neq M \in \operatorname{mod} R$. *If* $\operatorname{pd}_R M < \infty$, *then*

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} R$$

2.3 Classification of Rings

This section will introduce the fundamental classes of rings with examples of each type. Similar to how in elementary algebra, one learns that field $\implies$ Euclidean Domain $\implies$ Principal Ideal Domain $\implies$ Unique Factorization Domain, there is a similar string of implications for the classes we introduce here.

2.3.1 Regular Rings

We have already introduced the notion of a system of parameters $x_1,\ldots,x_n$ for a local ring $(R,\mathfrak{m})$, and moreover mentioned that any Noetherian local ring admits a system of parameters. One may be tempted to ask what happens if we can strengthen the condition that $\sqrt{(x_1,\dots,x_n)} = \mathfrak{m}$ to simple $(x_1,\dots,x_n) = \mathfrak{m}$. We have:

Definition 2.3.1. A Noetherian local ring $(R,\mathfrak{m})$ is *regular* if it has a system of parameters generating $\mathfrak{m}$. In this case, the system of parameters is called a *regular system of parameters*.

In the case that R is not local, if the localization $R_{\mathfrak{m}}$ of a ring R at each $\mathfrak{m} \in \text{m-Spec}\, R$ is a regular local ring, then R is called regular.

Proposition 2.3.2. *A Noetherian local ring $(R,\mathfrak{m})$ is regular if and only if its $\mathfrak{m}$-adic completion $\widehat{R}$ is regular.*

Remark 2.3.3. Denoting by $\mu(\mathfrak{m})$ the minimal number of generators of $\mathfrak{m}$, the above definition is equivalent to stating that $\mu(\mathfrak{m}) = \dim R$. Krull's Height Theorem gives that any ideal I generated by n elements has $\operatorname{ht} I \leqslant n$. In the local case, $\dim R = \operatorname{ht} \mathfrak{m} \leqslant \mu(\mathfrak{m})$. Then, we see that the case of equality is precisely the condition that the ring R be a regular local ring.

Proposition 2.3.4 (Auslander-Buschsbaum-Nagata)**.** *Let $(R,\mathfrak{m})$ be a regular local ring. Then R is an integral domain.*

Theorem 2.3.5 (Auslander-Buchsbaum-Serre Theorem)**.** *Let $(R,\mathfrak{m},k)$ be a local ring. Then the following are equivalent:*

1. *R is regular.*

2. $\operatorname{pd} M < \infty$ *for every* $M \in \operatorname{mod} R$.

3. $\operatorname{pd} k < \infty$.

Given an affine variety V, the property of being regular corresponds to the geometric property of being a nonsingular point of V. The following theorem shows that regular local rings are in fact a smaller class than the more standard class of UFD's.

Theorem 2.3.6 (Auslander-Buschsbaum-Nagata)**.** *A regular local ring R is a Unique Factorization Domain.*

Theorem 2.3.7. *A Noetherian ring R is regular if and only if* $R[X_1,\ldots,X_n]$ *is regular. Similarly, a Noetherian ring R is regular if and only if* $R[[X_1,\ldots,X_n]]$ *is regular.*

2.3.2 Complete Intersection Rings and Hypersurfaces

We now expand the class of regular rings by considering a new definition. As mentioned above, a regular local ring is the result of the localization at a smooth point on an algebraic variety. This motivates the terminology of calling a Noetherian local ring *singular* if it is not regular. If we do not have regularity, we can hope for a weaker condition by considering even smaller neighborhoods of a point on an algebraic variety. Of course, more precisely this means taking the $\mathfrak{m}$-adic completion $\widehat{R}$.

Upon taking the completion, we know by 2.3.2 that this must also be singular, however we can hope that $\widehat{R}$ is in fact the quotient of some regular local ring, since we may be able to pull back the elements of $\widehat{R}$ to a ring with nicer properties. In this case, we have a definition

Definition 2.3.8. A local ring $(R,\mathfrak{m})$ is a *complete intersection (ring)* if its $\mathfrak{m}$-adic completion $\widehat{R}$ is a residue class ring of a regular local ring S with respect to an ideal generated by an S-sequence. More precisely,

$$\widehat{R} = S/(x_1,\ldots,x_n)$$

where $(x_1,\ldots,x_n)$ is S-regular.

It should be obvious that a regular ring R is a complete intersection ring, since $\widehat{R}$ is regular by 2.3.2 and is the quotient of itself with the ideal (0), which is generated by $\varnothing$ (and this is vacuously an R-sequence). The converse does not hold, however:

Example 2.3.9. Consider the ring $k[x]/(x^2)$ where k is some field. Then the maximal ideal is $(x)/(x^2)$ and taking the completion yields $k[[x]]/(x^2)$. This is clearly the quotient of a regular ring by some regular element, and hence is a complete intersection. However, this is not an integral domain (since $x^2 = 0$), so our ring is certainly not regular.

Finally, we give another definition that will be used in the upcoming chapter:

Definition 2.3.10. A ring R is called a *hypersurface* if its $\mathfrak{m}$-adic completion $\widehat{R}$ is the quotient of a regular local ring by some principal ideal.

2.3.3 Gorenstein Rings

In this section we shall see that consideration of the length of injective resolutions for a ring will lead us to the definition of another new class of rings.

Proposition 2.3.11. *Let M be an R-module. The following are equivalent:*

1. $\operatorname{id} M \leqslant n$
2. $\operatorname{Ext}_R^{n+1}(N,M) = 0$ *for all R-modules N.*
3. $\operatorname{Ext}_R^{n+1}(R/J,M) = 0$ *for all ideals J of R*

The above proposition is obvious from the material of the previous section on homological algebra. Similar to the characterization of projective dimension in terms of Ext, we see:

Proposition 2.3.12. *Let $(R,\mathfrak{m},k)$ be a local ring and $M \in \operatorname{mod} R$. Then*

$$\operatorname{id} M = \sup\{i : \operatorname{Ext}_R^i(k,M) \neq 0\}$$

Now we can introduce our next class of rings:

Definition 2.3.13. A local ring R is called a *Gorenstein Ring* if $\mathrm{id}_R R < \infty$. A not necessarily local ring is called *Gorenstein* if its localization at every maximal ideal is Gorenstein.

We then have the following Proposition:

Proposition 2.3.14.

1. *Suppose R is Gorenstein. Then, $R_\mathfrak{p}$ is Gorenstein for every $\mathfrak{p} \in \mathrm{Spec}(R)$ (indeed this holds for any multiplicative subset).*
2. *Suppose that $\underline{x}$ is an R-regular sequence. If R is Gorenstein, then so is $R/(\underline{x})$. If R is local, the converse also holds.*
3. *Suppose that R is local. Then R is Gorenstein if and only if its $\mathfrak{m}$-adic completion is Gorenstein.*

Now suppose that the local ring $(R, \mathfrak{m})$ is a complete intersection. Then, $\widehat{R} = S/(\underline{x})$ and hence by (2) in the above proposition, it suffices to show that S is Gorenstein. However, since $(S, \mathfrak{n}, \ell)$ is a regular local ring, it has finite projective dimension which in turn implies that $\mathrm{Ext}^i_R(\ell, S) = 0$ for $i \gg 0$.

To see this, we can find a finite projective resolution of the residue field ℓ by 2.3.5. Applying the contravariant functor $\mathrm{Hom}_R(-, R)$, we find that $\mathrm{Ext}^i_R(k, R)$ must eventually vanish. Then, using 2.3.12, we conclude that injective dimension is finite and hence S is Gorenstein. Then, R is Gorenstein as well. We have proved:

Proposition 2.3.15. *Any complete intersection is Gorenstein.*

Example 2.3.16. Consider the ring of formal power series over a field k,

$$k[[X, Y, Z]]/(X^2 - Y^2, Y^2 - Z^2, XY, YZ, ZX)$$

this is Gorenstein. However, it can be shown that this is not a Complete Intersection [16].

Remark 2.3.17. There is another characterization of Gorenstein rings that will be discussed in the following subsection that requires another definition. This characterization allows one to construct entire classes of rings that are Gorenstein but not complete intersections (see [7]).

To conclude, we state the following which will be used in the next subsection.

Theorem 2.3.18. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let $M \in \operatorname{mod} R$ be of finite injective dimension. Then,*

$$\dim M \leqslant \operatorname{id} M = \operatorname{depth} R$$

2.3.4 Cohen-Macaulay Rings

Recalling 2.2.22, we can now define another class of rings:

Definition 2.3.19. Let R be a local ring. $0 \neq M \in \operatorname{mod} R$ is called a *Cohen-Macaulay module* if $\operatorname{depth} M = \dim M$. If R is a Cohen-Macaulay module over itself, then it is called a *Cohen-Macaulay Ring*.

A Cohen-Macaulay module is called *maximal* if M is Cohen-Macaulay and $\dim M = \dim R$. In the non-local case, a module is called Cohen-Macaulay if the localization $M_{\mathfrak{m}}$ at every maximal ideal $\mathfrak{m}$ is Cohen-Macaulay.

Note that if R is Gorenstein, then 2.3.18 combined with 2.2.22 shows that $\dim R = \operatorname{depth} R$. Hence:

Proposition 2.3.20. *Any Gorenstein ring is Cohen-Macaulay.*

Theorem 2.3.21. *Let $(R, \mathfrak{m})$ be a local ring and $M \neq 0$ a Cohen-Macaulay module. Then*

1. $\dim R/\mathfrak{p} = \operatorname{depth} M$ *for all* $\mathfrak{p} \in \operatorname{Ass} M$.

2. $\operatorname{grade}(I,M) = \dim M - \dim M/IM$ *for all ideals* $I \subset \mathfrak{m}$.

3. $\underline{x} = x_1, \ldots, x_n$ *is an M sequence if and only if* $\dim M/\underline{x}M = \dim M - n$.

4. $\underline{x}$ *is an* M*-sequence if and only if it is part of a system of parameters of* M.

Recalling the definition of type, we have the following:

Theorem 2.3.22. *Let* $(R, \mathfrak{m}, k)$ *be a local ring. The the following are equivalent:*

1. *R is Gorenstein.*

2. *R is Cohen-Macaulay of type* 1.

Definition 2.3.23. Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring. A maximal Cohen-Macaulay module C with type 1 and finite injective dimension is called a *canonical module* of R.

With the above definition, we see that 2.3.22 tells us that, up to isomorphism, a Gorenstein ring is its own canonical module.

Example 2.3.24. Consider the ring of formal power series $k[[t^3, t^5, t^7]]$. It can be shown that this ring is Cohen-Macaulay but not Gorenstein.

Remark 2.3.25. To conclude this subsection, we shall leave the reader with the following chain of inclusions:

$$\text{Regular} \subset \text{Complete Intersection} \subset \text{Gorenstein} \subset \text{Cohen-Macaulay} \tag{2.3.1}$$

2.3.5 Poincaré and Bass series

Definition 2.3.26. Let X be a generating set for an R-module M. If no proper subset of X generates M, then X is called a *minimal basis* or *minimal generating set*.

Consider the following construction: given an R-module M over a local ring $(R, \mathfrak{m}, k)$, take the quotient $M/\mathfrak{m}M \cong k \otimes M$. As a module over a field, this is actually a vector space. Choose a basis $\{\bar{x}_1, \ldots, \bar{x}_n\}$ of this vector space and consider the preimage $x_i \in M$ of each $\bar{x}_i$ with respect to the canonical projection. Then it is obvious that any proper subset of this set of generators cannot generate M. Also, the set $X = \{x_1, \ldots, x_n\}$ generates all of M since for any $x \in M$ its image in $M/\mathfrak{m}M$ is in the span of our $\{\bar{x}_i\}$. Taking the preimage of this linear combination, we find that $x = r_1x_1 + \cdots + r_nx_n$ for some $r_i \in R$.

Set $\beta_0 := \dim_k \operatorname{Hom}_R(M,k)$ and construct a homomorphism $\phi_0 : R^{\beta_0} \to M$ with $\phi_0(e_i) = x_i$, where $x_1, \ldots, x_{\beta_0}$ is a minimal basis for M. The kernel of this map is called the *first syzygy*. Now, set $\beta_1 = \mu(\operatorname{Ker}\phi_0)$, where $\mu(-)$ denotes the minimal number of generators for the given module, and construct the same type of map $\phi_1 : R^{\beta_1} \to \operatorname{Ker}\phi_0$. Continuing inductively, we construct what is called a *minimal free resolution* for M, since we are constructing a resolution by successively finding minimal bases of the ith syzygy. The number $\beta_i(M)$ is called the ith *Betti number*. We have the following characterization:

Proposition 2.3.27. *Let $(R, \mathfrak{m}, k)$ be a local ring and $M \in \operatorname{mod} R$. Then, $\beta_i(M) = \dim_k \operatorname{Ext}^i_R(M,k)$ for all i, and,*

$$\operatorname{pd}_R M = \sup\{i : \operatorname{Ext}^i_R(M,k) \neq 0\} = \sup_i\{i : \beta_i(M) neq 0\}$$

We have a similar dual to the Betti numbers:

Definition 2.3.28. The *Bass numbers* $\mu_i(M)$ of a module M over a local ring $(R, \mathfrak{m})$ are defined as the vector space dimension of $\operatorname{Ext}^i_R(k,M)$.

Definition 2.3.29. Let $M \in \operatorname{mod} R$. The *Poincaré series* and the *Bass series* of M, denoted

$P_R^M(t)$ and $I_M^R(t)$, respectively, are the formal Laurent series defined as follows:

$$P_R^M(t) := \sum_{i \geqslant 0} \beta_i(M)t^i$$

$$I_M^R(t) := \sum_{i \geqslant 0} \mu_i(M)t^i.$$

We simply denote $I_R^R(t)$ by $I_R(t)$.

The coefficient of $t^{\operatorname{depth} R}$ in $I_R(t)$ is precisely the *type* of R introduced in Definition 2.2.23. Note that $\gamma_R \neq 0$ and all the coefficients of t^i in $I_R(t)$ for $i < \operatorname{depth} R$ are zero by the characterization of depth given in Theorem 2.2.19 (with $I = \mathfrak{m}$ of course).

Also, note that the constant term in $P_R^k(t)$ is 1. This is because we have that the coefficient of t^0 is $\operatorname{Ext}_R^0(k,k) \cong \operatorname{Hom}_R(k,k) \cong \operatorname{Hom}_k(k,k) \cong k$, where the last isomorphism is the natural map $f \mapsto f(1)$.

For simplicity we will denote $P_R^M(t)$ and $I_M^R(t)$ by P_R^M and I_M^R, where it is understood that these are functions of t. These series will be used to prove results in the next chapter.

2.3.6 The Auslander-Reiten Conjecture

To conclude this chapter, we introduce a conjecture put forth originally as a generalization of Nakayama's Conjecture. As will be seen, this thesis will show that another class of rings shall satisfy the Auslander-Reiten conjecture. This conjecture claims: *over a local ring R, if M is a finitely generated R-module such that* $\operatorname{Ext}_R^i(M, M \oplus R) = 0$ *for* $i > 0$*, then M is projective.*

Definition 2.3.30. A ring R is said to satisfy the *Auslander Condition* if for every finitely generated R-module M there exists a nonnegative integer b_M such that for every finitely generated R-module N one has that $\operatorname{Ext}_R^i(M,N) = 0$ for $i > b_M$ whenever $\operatorname{Ext}_R^i(M,N) = 0$ for $i \gg 0$.

It is well known (see [27]) that any ring R satisfying the above Auslander condition must also satisfy the Auslander-Reiten conjecture. In the case of commutative Noetherian rings, it has been shown that any Gorenstein local ring of codimension $\leqslant 4$ and any fiber product of commutative Noetherian local rings must satisfy the Auslander-Reiten conjecture (see [23] and [18], respectively).

Other notable classes for which the Auslander-Reiten conjecture holds are Artinian local rings $(R.\mathfrak{m})$ with $\mathfrak{m}^3 = 0$ (see [13]), Gorenstein normal Rings (see [3]), and Complete Intersections (see [6]).

In [27, Theorem 1.2], a string of implications is introduced for different conditions of commutative Noetherian local rings, showing that different classes are forced to obey the Auslander-Reiten conjecture.

CHAPTER 3

COHEN-MACAULAY AND GORENSTEIN FIBER PRODUCTS

In this chapter, $(S,\mathfrak{m}_S,k)$ *and* $(T,\mathfrak{m}_T,k)$ *will denote local rings with common residue field* k *such that* $\mathfrak{m}_S \neq 0 \neq \mathfrak{m}_T$.

3.1 Fiber Products, Poincaré Series and Bass Series

Definition 3.1.1. The fiber product of S and T, denoted $S\times_k T$, is the pull-back in the following commutative diagram

$$\begin{tikzcd} S\times_k T \arrow[r] \arrow[d] & T \arrow[d, "\pi_T"] \\ S \arrow[r, "\pi_S"] & k \end{tikzcd}$$

in which π_S and π_T are the natural surjections onto the residue field k. In fact,

$$S\times_k T = \{(s,t)\in S\times T \mid \pi_S(s)=\pi_T(t)\}.$$

Remark 3.1.2. The ring $S\times_k T$ is a non-trivial noetherian local ring with maximal ideal $\mathfrak{m}_{S\times_k T}=\mathfrak{m}_S\oplus\mathfrak{m}_T$, and there are ring isomorphisms $(S\times_k T)/\mathfrak{m}_T\cong S$ and $(S\times_k T)/\mathfrak{m}_S\cong T$.

We also have the following equalities:

$$\dim S\times_k T = \max\{\dim S,\dim T\} \tag{3.1.1}$$

$$\operatorname{depth} S\times_k T = \min\{\operatorname{depth} S,\operatorname{depth} T,1\} \tag{3.1.2}$$

The first equality follows immediately by noting that the prime ideals of $S\times_k T$ are of the form $\mathfrak{p}\times_k T$ and $S\times_k\mathfrak{q}$ for $\mathfrak{p}\in\operatorname{Spec}(S)$ and $\mathfrak{q}\in\operatorname{Spec}(T)$.

The second equality is a consequence of the formulas presented in [8, Remark 3.1].

For the rest of this section, we collect some well-known results that we will need in the future. Recall that the definitions of the Poincaré series and Bass series are given in Definition 2.3.29.

Remark 3.1.3. The following equality is due to Kostrikin and Šafarevič [14] (see also Christensen, Striuli, and Vraciu [8])

$$\frac{1}{P^k_{S\times_k T}} = \frac{1}{P^k_S} + \frac{1}{P^k_T} - 1 \tag{3.1.3}$$

which relates the Poincaré series of the residue field k over the fiber product and over each of the rings defining the fiber product.

The following are due to Lescot [15, Theorem 3.1].

Case 1: If S and T are singular (that is, are not regular rings), then we have the following equality:

$$\frac{I_{S\times_k T}}{P^k_{S\times_k T}} = t + \frac{I_S}{P^k_S} + \frac{I_T}{P^k_T}. \tag{3.1.4}$$

Case 2: If S is singular and T is regular with $\dim T = n$, then we have

$$\frac{I_{S\times_k T}}{P^k_{S\times_k T}} = t + \frac{I_S}{P^k_S} - \frac{t^{n+1}}{(1+t)^n}. \tag{3.1.5}$$

Case 3: And finally, if S and T are both regular with $\dim S = m$ and $\dim T = n$, then we have

$$\frac{I_{S\times_k T}}{P^k_{S\times_k T}} = t - \frac{t^{m+1}}{(1+t)^m} - \frac{t^{n+1}}{(1+t)^n}. \tag{3.1.6}$$

Using (3.1.3) we can find the following relations, found through basic manipulations. First, when S and T are singular:

$$I_A\left(P^k_T + P^k_S - P^k_T P^k_S\right) = tP^k_T P^k_S + I_S P^k_T + I_T P^k_S. \tag{3.1.7}$$

When S is singular and T is regular with $\dim T = n$, we obtain:

$$I_A\left(P^k_T + P^k_S - P^k_T P^k_S\right)(1+t)^n = \left(t(1+t)^n - t^{n+1}\right)P^k_T P^k_S + (1+t)^n I_S P^k_T. \tag{3.1.8}$$

3.2 Cohen-Macaulay Fiber Products and Hypersurfaces of Dimension 1

Using (3.1.1) and (3.1.2), we have the following proposition, which we shall employ in the proof of some results in the future.

Theorem 3.2.1 ([19]). *The fiber product $S \times_k T$ is Cohen-Macaulay if and only if* $\dim S \times_k T \leqslant 1$ *and both S and T are Cohen-Macaulay satisfying the following equality*

$$\dim S = \dim T = \dim S \times_k T.$$

Proof. Suppose first the $A := S \times_k T$ is Cohen Macaulay. Hence, $\dim A = \operatorname{depth} A$, and we automatically find that $\dim A \leqslant 1$, since $\operatorname{depth} A \leqslant 1$ by (3.1.2). Without loss of generality, we can assume $\dim A = \dim S$. Then, $\dim S \geqslant \dim T$, and we have 3 cases. Assume first that $\dim S = \operatorname{depth} S$. Then, S is Cohen-Macaulay, and we have the following string of inequalities:

$$\dim S = \operatorname{depth} S \leqslant \operatorname{depth} T \leqslant \dim T \leqslant \dim S$$

So that all inequalities in the above are actually equality. Hence, $\dim T = \operatorname{depth} T$ and T is Cohen-Macaulay as well.

If $\dim S = \operatorname{depth} T$, we have two similar strings of inequalities:

$$\dim S = \operatorname{depth} T \leqslant \operatorname{depth} S \leqslant \dim S$$

$$\dim S = \operatorname{depth} T \leqslant \dim T \leqslant \dim S$$

So the above is again actually equality, and we conclude that $\dim T = \operatorname{depth} T$ and $\dim S = \operatorname{depth} S$, so S and T are again Cohen-Macaulay.

Finally, for $\dim S = \operatorname{depth} A = 1$, this means that $\operatorname{depth} S \geqslant 1$ and $\operatorname{depth} T \geqslant 1$. Then we have:

$$1 \leqslant \operatorname{depth} S \leqslant \dim S = 1$$

$$1 \leqslant \operatorname{depth} T \leqslant \dim T \leqslant \dim S = 1$$

And we conclude that $\operatorname{depth} S = \dim S = 1$, and likewise for T, so that S and T are Cohen-Macaulay.

Conversely, if $\dim S \times_k T \leqslant 1$ and S, T are Cohen-Macaulay with $\dim S = \dim T = \dim S \times_k T$, then we see that

$$\operatorname{depth} A = \min\{\dim A, 1\}$$

But since $\dim A \leqslant 1$, we have $\operatorname{depth} A = \dim A$, so A is Cohen-Macaulay. $\square$

Next result will be used in the proof of a result in the next section.

Proposition 3.2.2 ([19]). *Assume that R is a hypersurface of dimension 1 that is complete, and let I be a non-zero ideal of R such that R/I is a regular ring of dimension 1. Then, $R/I \cong Q/(f)$, where Q is a regular local ring and $f \in Q$ is a prime element.*

Proof. By definition of hypersurface (given by 2.3.10), we have $R \cong Q/(g)$ with Q a regular local ring, and g is a regular element in Q. With this, we see that $1 = \dim R = \dim Q/(g) = \dim Q - 1$, so that $\dim Q = 2$. Thus, (g) is not the maximal ideal of Q.

Now, since any regular local ring is an integral domain, we see that I is a prime ideal so there corresponds $\mathfrak{q} \in V\big((g)\big) \cap \operatorname{Spec}(Q)$ such that I corresponds to $\mathfrak{q}/(g)$.

By the Auslander-Buchsbaum-Nagata Theorem 2.3.6, Q is a UFD and we can factorize $g = g_1 g_2 \dots g_n$ into primes. Note that $\mathfrak{q}$ is a prime ideal, so by definition at least one prime $g_i \in \mathfrak{q}$. Define $f := g_i$.

We claim that $(f) = \mathfrak{q}$. To see this, since by assumption $\dim R/I = 1$ and R is Cohen-Macaulay, we have that

$$1 = \dim R/I = \dim R - \operatorname{ht}_R I$$

implying that $\mathrm{ht}_R\, I = 0$. This implies that $\mathrm{ht}_Q \mathfrak{q} = \mathrm{ht}_Q(f)$. Since $f \in Q$ is prime, the ideal (f) of Q is a prime ideal and therefore, $(f) = \mathfrak{q}$, as claimed.

From here, we see:

$$R/I \cong \frac{Q/(g)}{\mathfrak{q}/(g)} = \frac{Q/(g)}{(f)/(g)} \cong Q/(f) \tag{3.2.1}$$

as asserted. □

Finally, we record the following result without proof, as it will be used later.

Proposition 3.2.3 ([8])**.** *If S and T are regular rings of dimension* 1*, then $S \times_k T$ is a hypersurface.*

3.3 Gorenstein Fiber Products

This section contains a result that characterizes the Gorenstein fiber products. In fact, it says that the class of Gorenstein fiber products coincides with that of hypersurface fiber products and hence, with that of complete intersection fiber products; see the diagram in (2.3.1).

Theorem 3.3.1 ([19])**.** *The fiber product $A := S \times_k T$ is Gorenstein if and only if it is a hypersurface with* $\dim A = 1$*. We also have S and T are regular rings with* $\dim S = 1 = \dim T$.

Moreover, under the above assumptions, if A is complete, then

$$A \cong Q/(p) \times_k Q/(q)$$

where Q, $Q/(p)$, and $Q/(q)$ are regular rings with residue field k and p and q are prime elements in Q.

Proof. First note that if A is a hypersurface, then by (2.3.1) it is Gorenstein.

Assume now that A is Gorenstein. Note that by Proposition 2.3.15, A is Cohen-Macaulay. Hence, by Theorem 3.2.1 we have $\dim A$ is either 0 or 1. Note also that we have 3 possible cases for the rings S and T, as given by the conditions on (3.1.4), (3.1.5), and (3.1.6). Thus, in total we have 6 possible cases. We shall show that 5 of these cases are impossible, as we describe below.

Case 1: Suppose $\dim A = 0$ and S and T are both singular. Since A is Gorenstein, it is Cohen-Macaulay and the first nonzero coefficient of I_A is the coefficient γ_A of 1. Since A is isomorphic to its canonical module by definition of Gorenstein, we deduce immediately that $\gamma_A = 1$. Using this, we want to compare coefficients of (3.1.4). On the left side of this equality, the constant term of $P_T^k + P_S^k - P_T^k P_S^k$ is $1+1-1 = 1$, so the constant term of $I_A\left(P_T^k + P_S^k - P_T^k P_S^k\right)$ is just $\gamma_A = 1$. On the right, note that by Proposition 3.2.1, S and T also have 0 depth so that the constant term is merely $\gamma_S + \gamma_T$, since $tP_T^k P_S^k$ has no constant term. Thus, comparing the left and right side, we see that $1 = \gamma_S + \gamma_T$. But this is clearly impossible, since the right hand side it the sum of two strictly positive integers. Hence, this case cannot occur.

Case 2: Suppose $\dim A = 1$ and S and T are both singular. Again, examining the left of (3.1.4), there will be no constant term since I_A starts at the term $t^{\operatorname{depth} A} = t$, with coefficient $\gamma_A = 1$. Examining the right, we want to find the coefficients of t. Again, S and T are Cohen Macaulay and also have depth 1 by Proposition 3.2.1. Also, the term $tP_T^k P_S^k$ now contributes a 1, and comparing both sides, we see that $\gamma_A = 1 = 1 + \gamma_S + \gamma_T$. Subtracting, this means $\gamma_S + \gamma_T = 0$, which is again impossible.

Case 3: Suppose $\dim A = 0$, S singular, and T regular. Since T is regular of dimension 0, it is equal to its residue field k, which is a contradiction.

Case 4: Suppose $\dim A = 1$, S singular, and T regular. We want to compare coefficients of the t terms in (3.1.8). Then, almost identically as in the second case, we see the the left of (3.1.8) has constant term $\gamma_A = 1$. On the right, first notice that $\left(t(1+t)^n - t^{n+1}\right) P_T^k P_S^k$ has coefficient 1 of t. Similarly, we know that S is Cohen Macaulay with $\operatorname{depth} S = 1$, so the term $(1+t)^n I_S P_T^k$ has coefficient γ_S on t. Comparing coefficients, we find $\gamma_A = 1 = 1 + \gamma_S$, implying $\gamma_S = 0$. Again, this is impossible.

Case 5: Suppose $\dim A = 0$, and S and T are regular. Then, since S and T are regular rings of dimension 0, they are in fact equal to the residue field k, which is another contradiction. Hence, the only possible case is the next case.

Case 6: Suppose $\dim A = 1$, and S and T are regular. Then, by using Proposition 3.2.3 we see that A is a hypersurface of dimension 1.

Now assume further that A is complete. Then, $A/\mathfrak{m}_S \cong T$ and $A/\mathfrak{m}_T \cong S$ are both regular of dimension 1. We can now employ the result of Proposition 3.2.2 to conclude that $S \cong Q/(p)$ and $T \cong Q/(q)$ for prime elements p and q in Q, where Q, $Q/(p)$ and $Q/(q)$ are regular rings. Hence,

$$A \cong Q/(p) \times_k Q/(q).$$

This completes the proof. □

As mentioned before, the above theorem has shown that a Gorenstein fiber Product is indeed a hypersurface. The natural question is whether or not it is regular as well. However, we shall now show that this is the best we can get:

Proposition 3.3.2 ([19])**.** *The fiber product $A := S \times_k T$ is not regular.*

Proof. Suppose that A is regular. Note that A is not a field because $\mathfrak{m}_S \neq 0 \neq \mathfrak{m}_T$. Note also that A is Cohen-Macaulay. Thus, by Theorem 3.2.1 we have $\dim A = 1$.

Since A is regular we have $\operatorname{pd}_A(A/\mathfrak{m}_T) < \infty$ by Theorem 2.3.5. By the Auslander-Buchsbaum formula 2.2.24, we also have

$$\operatorname{depth} A/\mathfrak{m}_T + \operatorname{pd}_A(A/\mathfrak{m}_T) = \operatorname{depth} A = 1.$$

Therefore, $\mathrm{pd}_A(A/\mathfrak{m}_T) \leqslant 1$. However, using the short exact sequence

$$0 \to \mathfrak{m}_T \to A \to A/\mathfrak{m}_T \to 0$$

this then implies that $\mathrm{pd}_A \mathfrak{m}_T = 0$. Thus, we find that $\mathfrak{m}_T$ is isomorphic to a free A-module. However, $\mathfrak{m}_T\mathfrak{m}_S = 0$, implying that $\mathfrak{m}_S = 0$, which is a contradiction. Thus, A is not regular, as desired. $\square$

CHAPTER 4
APPLICATIONS

The notation used in this chapter comes from Chapters 2 and 3.

4.1 A Characterization of Fiber Products and Vanishing of Ext

In this chapter we shall apply the results of Chapter 3 to generalize results of Takahashi [26] and Atkins-Vraciu [2]. The following is a result of Ogoma that provides a useful characterization of the fiber products.

Proposition 4.1.1 ([20]). *A local ring $(R,\mathfrak{m})$ is a fiber product of the form $S \times_k T$ if and only if its maximal ideal is decomposable.*

Proof. Suppose first that we have a fiber product $S \times_k T$. Then, as we mentioned in Remark 3.1.2 its maximal ideal is decomposable.

Conversely, suppose $\mathfrak{m} \cong I \oplus J$. Then consider the natural homomorphism

$$\Theta \colon R \to R/I \times_{R/\mathfrak{m}} R/J$$

sending $r \mapsto (r+I, r+J)$. We show that Θ is bijective.

Suppose $\Theta(r) = (I,J)$. Then, $r \in I \cap J = (0)$, so $r = 0$ and Θ is injective.

Now let $(r_1+I, r_2+J) \in R/I \times_{R/\mathfrak{m}} R/J$. Following Definition 3.1.1 consider the diagram of ring homomorphism

$$\begin{tikzcd} R/I \times_k R/J \arrow[r] \arrow[d] & R/J \arrow[d, "\pi_{R/J}"] \\ R/I \arrow[r, "\pi_{R/I}"] & R/\mathfrak{m} \end{tikzcd}$$

Then, by definition of $\pi_{R/I}$ and $\pi_{R/J}$, we see that $r_1 + \mathfrak{m} = r_2 + \mathfrak{m}$, which implies that $r_1 - r_2 \in \mathfrak{m}$. Since $\mathfrak{m} \cong I \oplus J$, we see that $r_1 - r_2 = i + j$ for $i \in I$ and $j \in J$. Set $r := r_1 - i = r_2 + j$. Then, $r + I = r_1 - i + I = r_1 + I$, and likewise $r + J = r_2 + j + J = r_2 + J$. Hence, $\Theta(r) = (r_1 + I, r_2 + J)$, and this implies that Θ is a surjective map. $\square$

Corollary 4.1.2 ([19]). *If R has a decomposable maximal ideal, then R is not regular.*

Proof. By Proposition 4.1.1 we have that R is a fiber product of the form $S \times_k T$. Now Proposition 3.3.2 implies that R is not regular. $\square$

We now proceed to examine vanishing properties of Ext over rings with decomposable maximal ideals. Since 4.1.1 tells us that decomposable maximal ideals force the structure of a fiber product, we can now apply results of fiber products to deduce deeper properties. First, we will use the following (see [18, Corollary 4.2]):

Proposition 4.1.3 ([18]). *Let R be a fiber product and let $M,N \in \operatorname{mod} R$. If $\operatorname{Ext}^i_R(M,N) = 0$ for $i \gg 0$, then $\operatorname{pd}_R(M) \leqslant 1$ or $\operatorname{id}_R(N) \leqslant 1$.*

Corollary 4.1.4 ([19]). *Let the maximal ideal $\mathfrak{m}$ of the local ring R be decomposable, and let $M,N \in \operatorname{mod} R$. If $\operatorname{Ext}^i_R(M,N) = 0$ for $i \gg 0$, then $\operatorname{pd}_R(M) \leqslant 1$ or $\operatorname{id}_R(N) \leqslant 1$.*

Proof. By Proposition 4.1.1, R is isomorphic to a fiber product. By Proposition 4.1.3, we see that if R is a fiber product and $\operatorname{Ext}^i_R(M,N) = 0$ for $i \gg 0$, then $\operatorname{pd}_R(M) \leqslant 1$ or $\operatorname{id}_R(N) \leqslant 1$. $\square$

We state the following without proof:

Theorem 4.1.5 ([18]). *Let $A := S \times_k T$ and assume that $M \in \operatorname{mod} A$.*

(a) *If $\operatorname{Ext}^i_A(M, M \oplus A) = 0$ for $i \gg 0$, then $\operatorname{pd}_A(M) \leqslant 1$.*

(b) *If $\operatorname{Ext}^i_A(M, M \oplus A) = 0$ for $i \geqslant 1$, then M is A free.*

The following is an immediate corollary of the above results (see Section 2.3.6 of Chapter 2).

Corollary 4.1.6 ([19]). *Let the maximal ideal $\mathfrak{m}$ of the local ring R be decomposable. Then R satisfies the Auslander-Reiten Conjecture.*

Proof. Again, since $\mathfrak{m}$ is decomposable, R is a fiber product. Using Theorem 4.1.5, the conclusion follows immediately. $\square$

4.2 Generalization of a Result of Takahashi

In this section we will give a brief excursion into the notion of totally reflexive modules and Gorenstein dimension.

Definition 4.2.1. An R-module M will be called *totally reflexive* if

$$\mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R) \cong M$$

and $\mathrm{Ext}^i_R(M,R) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M,R),R)$ for all $i > 0$.

Formally, a reflexive module is one for which $\mathrm{Hom}_R(-,R)$ is idempotent when applied to M, and total reflexivity is an analog of projectivity but with the additional vanishing of $\mathrm{Ext}^i_R(\mathrm{Hom}_R(M,R),R)$. We use this to define another example of a homological dimension:

Definition 4.2.2. Let M be an R-module. The *Gorenstein Dimension* of M is the smallest integer n for which there exists a resolution of M by totally reflexive modules of length n. This quantity will be denoted $\operatorname{G-dim}_R M$ or just $\operatorname{G-dim} M$ when R is clear.

The above immediately implies that $\operatorname{G-dim} M = 0$ for a totally reflexive module. Gorenstein dimension also obeys the following:

Theorem 4.2.3 (Compare 2.3.5)**.** *Let $(R,\mathfrak{m},k)$ be a local ring. Then the following are equivalent:*

1. *R is Gorenstein.*

2. $\operatorname{G-dim} M < \infty$ *for every* $M \in \operatorname{mod} R$.

3. $\operatorname{G-dim} k < \infty$.

Theorem 4.2.4 (Compare 2.2.24)**.** *Let $(R,\mathfrak{m})$ be a local ring with $0 \neq M \in \operatorname{mod} R$. If $\operatorname{G-dim}_R M < \infty$, then*

$$\operatorname{G-dim}_R M + \operatorname{depth} M = \operatorname{depth} R$$

The above immediately gives that when both $\operatorname{pd} M$ and $\operatorname{G-dim} M$ are finite, they are equal, however in general we have that $\operatorname{G-dim} M \leqslant \operatorname{pd} M$.

We give a result of Takahashi [26, Theorem A]:

Theorem 4.2.5 ([26])**.** *Let $(R, \mathfrak{m})$ be a complete local ring. Then the following are equivalent:*

1. *There is an R-module M with $\operatorname{G-dim}_R M < \infty = \operatorname{pd}_R M$, and $\mathfrak{m}$ is decomposable.*

2. *R is Gorenstein, and $\mathfrak{m}$ is decomposable.*

3. *There are a complete regular local ring S of dimension 2 and a regular system of parameters x, y of S such that $R \cong S/(xy)$.*

To give a generalization of this result, we will need the following result of Foxby.

Lemma 4.2.6 ([12])**.** *If there exists a non-zero R-module with finite projective and injective dimension, then R is Gorenstein.*

Here is the main result of this subsection, generalizing Takahashi's result.

Theorem 4.2.7 ([19])**.** *If the maximal ideal $\mathfrak{m}$ of the complete local ring R is decomposable, then the following are equivalent.*

(i) *There is $E \in \operatorname{mod} R$ with $\operatorname{id}_R(E) < \infty$ such that $\operatorname{Ext}^i_R(E, R) = 0$ for all $i \gg 0$;*

(ii) *R is Gorenstein;*

(iii) *R is a hypersurface of dimension 1. In this case, R is isomorphic to a fiber product $Q/(p) \times_k Q/(q)$, where Q, $Q/(p)$, and $Q/(q)$ are regular local rings with residue field k and $p, q \in Q$ are prime elements;*

(iv) *There is $M \in \operatorname{mod} R$ with $\operatorname{pd}_R(M) = \infty$ such that $\operatorname{Ext}^i_R(M, R) = 0$ for all $i \gg 0$.*

Proof. (i) $\implies$ (ii): By 4.1.4 we know that $\mathrm{pd}_R(E) \leqslant 1$ or $\mathrm{id}_R(R) \leqslant 1$. If $\mathrm{id}_R(R) \leqslant 1$, then by definition R is Gorenstein. If $\mathrm{pd}_R(E) \leqslant 1$, then as E already has finite injective dimension, by using 4.2.6 we conclude that R is Gorenstein as well.

(ii) $\implies$ (iii): This holds obviously by noting that R is a fiber product and hence, by Theorem 3.3.1, the statement of (iii) holds.

(iii) $\implies$ (iv): By Corollary 4.1.2 we know that R is not regular. Hence, by Theorem 2.3.5 we have $\mathrm{pd}_R(k) = \infty$. Also, since R is Gorenstein we have $\mathrm{id}_R(R) < \infty$. Therefore, $\mathrm{Ext}^i_R(k,R) = 0$ for all $i \gg 0$.

(iv) $\implies$ (i): Let $M \in \mathrm{mod}\, R$ with $\mathrm{pd}_R(M) = \infty$ such that $\mathrm{Ext}^i_R(M,R) = 0$ for all $i \gg 0$. Then by Corollary 4.1.4 we conclude that $\mathrm{id}_R(R) < \infty$. Hence, letting $E := R$ we obviously have $\mathrm{Ext}^i_R(R,R) = 0$ for all $i > 0$ because R is free. $\square$

4.3 Generalization of a Result of Atkins-Vraciu

Recall the definition of a totally reflexive module as given in the previous section. We state the result of Atkins-Vraciu here for reference:

Theorem 4.3.1 ([2]). *Let $(R,\mathfrak{m})$ be an Artinian local ring with $\mathfrak{m}^3 = 0$. Assume that $\mathfrak{m}$ is decomposable, and $\mu(\mathfrak{m}) \geqslant 3$. Then M has no non-free totally reflexive modules.*

The following is a generalization of this theorem.

Theorem 4.3.2. *Assume that the maximal ideal $\mathfrak{m}$ of the local ring R is decomposable. If R is Artinian, then R has no non-free finitely generated modules M such that $\mathrm{Ext}^i_R(M,R) = 0$ for all $i \gg 0$. (Hence, R has no non-free totally reflexive modules.)*

Proof. Assume on the contrary that there exists non-free $M \in \mathrm{mod}\, R$ with $\mathrm{Ext}^i_R(M,R) = 0$ for all $i \gg 0$. Then, by Corollary 4.1.4 we have $\mathrm{pd}_R(M) \leqslant 1$ or $\mathrm{id}_R(R) \leqslant 1$. If $\mathrm{pd}_R(M) \leqslant 1$, then by Auslander-Buchsbaum we have

$$\mathrm{pd}_R(M) = \mathrm{depth}\, R - \mathrm{depth}_R M = 0$$

and hence, M is free over R by Theorem 2.1.15, which is a contradiction. This means that we must have $\mathrm{id}_R(R) \leqslant 1$, and therefore R is Gorenstein. Hence, by Theorem 4.2.7 we must have $\dim R = 1$, which is again a contradiction.

These contradictions show that there is no non-free finitely generated R-module M such that $\mathrm{Ext}^i_R(M,R) = 0$ for all $i \gg 0$ over an Artinian ring R. $\square$

REFERENCES

[1] H. Ananthnarayan, L. Avramov, and W. F. Moore, *Connected sums of Gorenstein local rings*, J. Reine Angew. Math., **667** (2012), 149–176.

[2] J. C. Atkins and A. Vraciu, *On the existence of non-free totally reflexive modules*, preprint (2016), `arXiv:1602.08385v2`.

[3] T. Araya, *The Auslander-Reiten Conjecture for Gorenstein Rings*, `arXiv:0805.0210`

[4] M. Auslander and D. A. Buchsbaum, *Codimension and multiplicity*, Ann. of Math. (2) **68** (1958), 625–657.

[5] M. Auslander and I. Reiten, *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. **52** (1975), 69–74.

[6] M. Auslander, S. Ding, O. Solberg, *Liftings and Weak Liftings of Modules*, Journal of Algebra, Volume 156, Issue 2, 15 April 1993, Pages 273-317

[7] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Camb. Studies in Adv. Math. **39**

[8] L. W. Christensen, J. Striuli, and O. Veliche, *Growth in the minimal injective resolution of a local ring*, J. Lond. Math. Soc. (2) **81** (2010), no. 1, 24–44.

[9] M. D'Anna, *A construction of Gorenstein rings*, J. Alg. **306** (2006), 507–519.

[10] A. Dress and H. Krämer, *Bettireihen von Faserprodukten lokaler Ringe*, Math. Ann. **215** (1975), 79–82.

[11] D. S. Dummit and R. M. Foote, *Abstract Algebra*, Wiley (2006), ed. 3

[12] H.-B. Foxby, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), 5–19.

[13] C. Huneke, L.M. Sega, A.N. Vraciu, *Vanishing of Ext and Tor over Cohen-Macaulay Local Rings*, `arXiv:1409.1141`

[14] A. I. Kostrikin and I. R. Šafarevič, *Groups of homologies of nilpotent algebras*, Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 1066–1069.

[15] J. Lescot, *La série de Bass d'un produit fibré d'anneaux locaux*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), no. 12, 569–571.

[16] H. Matsumura *Commutative Ring Theory*, Camb. Studies in Adv. Math. **8**

[17] W. F. Moore, *Cohomology over fiber products of local rings*, J. Algebra **321** (2009), no. 3, 758–773.

[18] S. Nasseh and S. Sather-Wagstaff, *Vanishing of Ext and Tor over fiber products*, Proc. Amer. Math. Soc., to appear.

[19] S. Nasseh, R. Takahashi, and K. VandeBogert, *On Gorenstein fiber products and applications*, preprint 2017.

[20] T. Ogoma, *Existence of dualizing complexes*, J. Math. Kyoto Univ., **24** (1984), no. 1, 27–48.

[21] T. Ogoma, *Fiber products of Noetherian rings*, Commutative algebra and combinatorics (Kyoto, 1985), 173–182, Adv. Stud. Pure Math., 11, North-Holland, Amsterdam, 1987.

[22] T. Ogoma, *Fiber products of Noetherian rings and their applications*, Math. Proc. Camb. Phil. Soc., **97** (1985), 231–241.

[23] L.M. Sega, *Vanishing of Cohomology over Gorenstein Rings of Small Codimension*, Proceedings of the AMS, Volume 131, Number 8, Pages 23132323

[24] J.-P. Serre, *Sur la dimension homologique des anneaux et des modules noethériens*, Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955 (Tokyo), Science Council of Japan, 1956, pp. 175–189.

[25] J. Shapiro, *On a construction of Gorenstein rings proposed by M. D'Anna*, J. Alg. **323** (2010), 1155–1158.

[26] R. Takahashi, *Direct summands of syzygy modules of the residue class field*, Nagoya Math. J., **189** (2008), 1–25.

[27] R. Takahashi, *Auslander-Reiten Conjecture and Auslander-Reiten Duality*, `arXiv:1112.6201`

[28] C. A. Weibel, *History of Homological Algebra*, Preprint, December 24, 1997, K-theory Preprint Archives, `http://www.math.uiuc.edu/K-theory/0245/`