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Precise Partitions Of Large Graphs

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PRECISE PARTITIONS OF LARGE GRAPHS

by

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(Under the Direction of Colton Magnant)

ABSTRACT

First by using an easy application of the Regularity Lemma, we extend some known results about cycles of many lengths to include a specified edge on the cycles. The results in this chapter will help us in rest of this thesis.

In 2000, Enomoto and Ota conjectured that if a graph G satisfies $\sigma_2(G) \geq n + k - 1$, then for any set of k vertices $v_1, \ldots, v_k$ and for any positive integers $n_1, \ldots, n_k$ with $\sum n_i = |G|$, there exists a partition of $V(G)$ into k paths $P_1, \ldots, P_k$ such that v_i is an end of P_i and $|P_i| = n_i$ for all i. We prove this conjecture when $|G|$ is large. Our proof uses the Regularity Lemma along with several extremal lemmas, concluding with an absorbing argument to retrieve misbehaving vertices.

Furthermore, sharp minimum degree and degree sum conditions are proven for the existence of a Hamiltonian cycle passing through specified vertices with prescribed distances between them in large graphs.

Finally, we prove a sharp connectivity and degree sum condition for the existence of a subdivision of a multigraph in which some of the vertices are specified and the distance between each pair of vertices in the subdivision is prescribed (within one).

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DEDICATION

This thesis is dedicated to Mr. Farshid Pahlavan.

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CHAPTER 1

INTRODUCTION

For all basic definitions and notation, see [1]. Let $\sigma_2(G)$ denote the minimum degree sum of a graph G.

The Traveling Salesman Problem involves finding the best (in some sense) cycle in a large graph that passes through each vertex exactly once. This corresponds to a most efficient route that a door-to-door salesman might take to visit each house in a particular city or each city in a particular region. As if this problem wasn't hard enough (it is known to be NP-Hard), what if the same salesman has a list of hotels he would like to visit on specific nights? The question of actually finding such a subgraph gets out of hand and it becomes natural to consider only existence results. One such result is the following, stated here in a simpler form from the original.

Theorem 1 (Faudree, Gould, Jacobson, Magnant [9]). *Given a constant k, the degree of each vertex in G is at least $\frac{|G|+k-1}{2}$, then for any selected set of k vertices (hotels), there exists a cycle passing through all the vertices of G such that the k selected vertices are approximately equally spaced. Furthermore, this degree condition is the best possible.*

More generally, one need not consider only a cycle. If we let H be any graph or even multigraph, we may consider the same problem in which we subdivide the edges of H a specified number of times, map the vertices of H into the vertices of a larger graph G and then try to find corresponding paths in G to represent those subdivided edges of H.

The result above is just one of many recent results involving placing specified vertices on graph substructures in which distances in the substructures are controlled. More examples can be found in [8, 10, 12, 15].

1.1 Preliminaries

Given two sets of vertices A and B, let $E(A, B)$ denote set of edges with one end in A and one end in B and let $e(A, B) = |E(A, B)|$. Define the *density between A and B* to be

$$d(A, B) = \frac{e(A, B)}{|A||B|}.$$

Definition 1. *Let $\epsilon > 0$. Given a graph G and two nonempty disjoint vertex sets $A, B \subset V$, we say that the pair (A, B) is ϵ-regular if for every $X \subset A$ and $Y \subset B$ satisfying*

$$|X| > \epsilon|A| \quad and \quad |Y| > \epsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \epsilon.$$

We will also use the following one-sided but stronger version of regularity.

Definition 2. *Let $\epsilon, d > 0$. Given a graph G and two nonempty disjoint vertex sets $A, B \subset V$, we say that the pair (A, B) is (ϵ, d)-super-regular if for every $X \subset A$ and $Y \subset B$ satisfying*

$$|X| > \epsilon|A| \quad and \quad |Y| > \epsilon|B|,$$

we have

$$e(X, Y) > d|X||Y|,$$

and furthermore $d_B(a) > d|B|$ for all $a \in A$ and $d_A(b) > d|A|$ for all $b \in B$.

The following is the famous Regularity Lemma of Szemerédi.

Lemma 1 (Regularity Lemma - Szemerédi [22]). *For every $\epsilon > 0$ and every positive integer m, there is an $M = M(\epsilon)$ such that if G is any graph and $d \in (0, 1)$ is any real number, then there is a partition of $V(G)$ into $r+1$ clusters $V_0, V_1, \ldots, V_r$, and there is a subgraph $G' \subseteq G$ with the following properties:*

(1) $m \leq r \leq M$,

(2) $|V_0| \leq \epsilon |V(G)|$,

(3) $|V_1| = \cdots = |V_r| = L \leq \epsilon |V(G)|$,

(4) $\deg_{G'}(v) > \deg_G(v) - (d + \epsilon)|V(G)|$ for all $v \in V(G)$,

(5) $e(G'[V_i]) = 0$ for all $i \geq 1$,

(6) for all $1 \leq i < j \leq r$ the graph $G'[V_i, V_j]$ is ϵ-regular and has density either 0 or greater than d.

Given a graph G and appropriate choices of ϵ and d, let G' be a spanning subgraph of G obtained from Lemma 1. The *reduced graph* $R = R(G, \epsilon, d)$ of G contains a vertex v_i for each cluster V_i in $G' \setminus V_0$ and has an edge between v_i and v_j if and only if $d(V_i, V_j) > d$. Hence, $V(R) = \{v_i \mid 1 \leq i \leq r\}$ and $E(R) = \{v_i v_j \mid 1 \leq i, j \leq r, \ d(V_i, V_j) > d\}$. Note that $r = |R|$.

This next lemma allows the creation of a super-regular pair from an ϵ-regular pair by simply removing some vertices.

Lemma 2 (Diestel [4], Lemma 7.5.1). *Let (A, B) be an ϵ-regular pair of density d and let $Y \subseteq B$ have size $|Y| \geq \epsilon |B|$. Then all but at most $\epsilon |A|$ of the vertices in A each have at least $(d - \epsilon)|Y|$ neighbors in Y.*

We use a simple corollary of this result.

Lemma 3. *Let (A, B) be an ϵ-regular pair of density d. There exist subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq (1 - \epsilon)|A|$ and $|B| \geq (1 - \epsilon)|B|$ such that the pair (A', B') is $(\epsilon, d - 2\epsilon)$-super-regular.*

Our next lemma follows trivially from the definition of super-regular pairs.

Lemma 4. *Given an (ϵ, d)-super-regular pair (A, B) and a pair of vertices $a \in A$ and $b \in B$, there exists a path of length at most 3 from a to b in (A, B).*

Lemma 5 (Sárközy and Selkow [21]). *For every $\delta > 0$, there are $\epsilon_0 = \epsilon_0(\delta), n_0 = n_0(\delta) > 0$ such that if $\epsilon \leq \epsilon_0$ and $n \geq n_0$, $G = (A, B)$ is an (ϵ, δ)-super-regular pair with $|A| = |B| = n$ and $x \in A$ and $y \in B$, then there is a Hamiltonian path of G from x to y.*

The following theorem gives us a degree sum condition on R based on our assumed degree sum condition on G.

Theorem 2 (Kühn, Osthus and Treglown [14]). *Given a constant c, if $\sigma_2(G) \geq cn$, then $\sigma_2(R) \geq (c - 2d - 4\epsilon)|R|$.*

We also use the following theorem of Ore.

Theorem 3 (Ore [19]). *If G is 2-connected, then G contains a cycle of length at least $\sigma_2(G)$.*

We use the following result of Williamson. Recall that a graph is called *panconnected* if, between every pair of vertices, there is a path of every possible length from 2 up to $n - 1$.

Theorem 4. *Let G be a graph of order n. If $\delta(G) \geq \frac{n+2}{2}$, then G is panconnected.*

CHAPTER 2

LENGTHS OF CYCLES CONTAINING A SPECIFIED EDGE

The search for cycles of specified lengths in graphs has been a blossoming area of Graph Theory ever since the classical works of Dirac [5] and Ore [19]. In particular, Dirac proved the following result which we will use in our proofs.

Theorem 5 ([5]). *Any graph G with $\delta(G) \geq 2$ contains a cycle of length at least $\delta(G) + 1$.*

As opposed to just a long cycle, one may also be interested in having cycles of all different lengths. A graph is called *pancyclic* if it contains a cycle of every length from 3 up to $n = |G|$. Recently, there has been a push to use weaker degree assumptions while making graphs *almost* pancyclic in some sense. More specifically, Gould, Haxell and Scott proved the following.

Theorem 6 ([20]). *For every real number $c > 0$, there exist constants n_0 and k such that the following holds. Let G be a graph with $n \geq n_0$ vertices and with minimum degree at least cn. Then G contains all even cycles from length 4 up to $ec(G) - k$ and all odd cycles from length k up to $oc(G) - k$, where $ec(G)$ and $oc(G)$ denote the length of longest even and odd cycles, respectively.*

Later, Nikiforov and Schelp showed the following. Here, the upper bounds on the cycle lengths are the notable differences between this and the previous result.

Theorem 7 ([18]). *For every integer $k \geq 2$, there exists n_0 such that the following holds. Let G be a graph with $n \geq n_0$ vertices and with minimum degree at least n/k. Then G contains all even cycles from length 4 up to $\delta(G) + 1$. Furthermore, if G is not bipartite, then G contains all odd cycles from length $2k - 1$ up to $\delta(G) + 1$, unless $k \geq 6$ and G belongs to a known exceptional class depending on k.*

Since the existence problem for cycles has been essentially solved, we consider placing selected vertices on these cycles of many different lengths. It turns out that we can get surprisingly similar results even when we place a specified pair of vertices on the cycles. Our main result is the following.

Theorem 8. *For all positive constants λ and c with $0 < \lambda < c < 1/2$, there exists n_0 such that the following holds. Let G be a graph with $n \geq n_0$ vertices and with minimum degree at least cn. Then for any pair of vertices $u, v \in V(G)$, there exists a cycle of length ℓ containing u and v for every integer $\ell \in [8/c, (2c-\lambda)n]$. Furthermore, if $uv = e \in E(G)$, the edge e can be used in each of the constructed cycles.*

The proof of Theorem 8 is presented in Section 2.1 and serves as an example of an easy application of the Regularity Lemma (Lemma 1 below).

2.1 Proof of Theorem 8

Choose constants such that

$$0 < \epsilon \ll d \ll \lambda < c < 1$$

and let $n \geq n(\epsilon)$.

First some helpful observations. For the first lemma, after applying Lemma 1, we consider a regular pair in R with density at least d and, using Lemma 3 , remove very few vertices from each of the clusters to make the pair of clusters balanced and (ϵ, δ)-super-regular, where $\delta = d - \epsilon$.

Lemma 6. *Let X and Y be two sufficiently large clusters forming a balanced (ϵ, δ)-super-regular pair with $|X| = |Y| = L$. Then for every pair of vertices $x \in X$ and $y \in Y$, there exist paths of all odd lengths ℓ between x and y satisfying*

(a) $3 \leq \ell \leq \delta L$ *and*

(b) $2(1 - \delta)L + 1 \leq \ell \leq 2L - 1$.

Proof. In order to prove this lemma, we first state an easy fact which follows from the definition of super-regularity.

Fact 1. *Between any two vertices in opposite sets of a super-regular pair, there is a path of length* 3.

If fewer than $\delta L/2$ vertices are removed from each of the clusters, then they are clearly still super-regular (for an appropriately smaller choice of ϵ and δ). Applying Fact 1 repeatedly on the clusters after the removal of all but the two center vertices of a path allows for the extension of a path. This yields all short paths and proves part (a). Applying Lemma 5 repeatedly on the clusters after the removal of some unused vertices allows for the creation of almost-spanning paths in $X \cup Y$. This yields all long paths and proves part (b). $\qquad\square$

Our next observation is a simple application of Lemma 19. Let $\mathcal{C}$ be a cycle in R (and corresponding sets of clusters in G). To this cycle, we apply Lemma 3 to alternating pairs and remove very few vertices to leave alternating pairs balanced and super-regular (leaving out one cluster in the cycle from our pairs if $|\mathcal{C}|$ is odd). The resulting cycle of clusters in G is called *alternating*.

Corollary 9. *Let $\mathcal{C}$ be an alternating cycle of clusters using $t \geq (c - \epsilon - d)|R|$ clusters and let L denote the minimum order of a cluster in $\mathcal{C}$. Then, between any given two vertices in the clusters of $\mathcal{C}$, there is a path of length ℓ in G for all $\ell \in [t + 3, (t-2)L]$ of the appropriate parity. Furthermore, there is also a cycle containing the two selected vertices of all even lengths $\ell \in [2t + 6, (t - 2)L]$.*

Proof. Since $(c - \epsilon - d) \gg \epsilon$, we see that $(c - \epsilon - d)|R|\delta L \gg \epsilon n > L$ so combinations of applications of Lemma 19(a) and Lemma 19(b) easily yield the desired result. $\square$

Recall the statement of our main result, that for every sufficiently large 2-connected non-bipartite graph G with minimum degree at least cn and for every pair of vertices u, v,there is a cycle of all possibly lengths from $8/c$ up to $(2c - \lambda)n$ containing u and v. Furthermore, for any edge $e \in E(G)$, e is contained in cycles of all possible lengths from $8/c$ up to $(2c - \lambda)n$. We are now able to prove Theorem 8.

Proof. Since, by Lemma 1, we have $\delta(R) \geq (c - d - \epsilon)|R|$, the following fact is an easy exercise.

Fact 2. *The shortest cycle in the reduced graph has order at most* 4.

Although the graph G is 2-connected, we cannot even assume that R is connected, so the longest cycle we can hope for in R comes from Theorem 5.

Fact 3. *The longest cycle in the reduced graph has order at least* $(c - d - \epsilon)|R|$.

By considering a shortest odd cycle and observing that such a cycle must have no chords, it is an easy exercise to obtain the following fact.

Fact 4. *There is an odd cycle of order at most* $\frac{2}{c} + 1$ *in* G.

Let O be a shortest odd cycle in G obtained from Fact 4. Also let $\mathcal{C}$ be a cycle in R (and, for convenience, also let $\mathcal{C}$ denote the corresponding cycle of clusters in G). Since G is 2-connected, there are two disjoint paths, each with length at most $\frac{1}{c}$, between vertices of O and vertices of $\mathcal{C}$. Let H denote the subgraph consisting of $\mathcal{C}$, O and these two short paths.

First choose $\mathcal{C}$ to have length 4 (such a cycle exists by Fact 2). For any two vertices u and v in G, there exist two paths of length at most $\frac{1}{c}$ from each of u and v

to H (or possibly one path or edge between u and v and then one path each to H). By Lemma 19, since we use two super-regular pairs, we can build all of the desired short cycles (of both parities) containing u and v of length ℓ for all $\ell \in [8/c, 2(d-\epsilon)L]$.

Next choose $\mathcal{C}$ to be a longest cycle in R with length $t \geq (c-d-\epsilon)|R|$ (such a cycle exists by Fact 3). By Corollary 9, if we make $\mathcal{C}$ alternating and if L is the minimum order of a cluster in $\mathcal{C}$, then there is a cycle containing (or path between) any pair of vertices in H of all possible lengths ℓ (of both parities) for $\ell \in [4/c + 2t, (t-2)L]$. Since $2(d-\epsilon)L \gg 3|R| > 4/c + 2t$ for n sufficiently large, we have therefore produced cycles containing u and v of all lengths $\ell \in [8/c, (t-2)L]$.

If R contains a cycle of length at least $(2c - d - \epsilon)|R|$, then choosing this cycle to be $\mathcal{C}$ above produces the desired long paths and cycles to complete the proof, so suppose not. Then R contains at least two cycles $\mathcal{C}$ and $\mathcal{D}$ each of length at least $(c - d - \epsilon)|R|$ by Theorem 5. Again, let H be the subgraph defined above, so there must be two disjoint paths from $\mathcal{D}$ to H. Let H' denote this new subgraph induced on $H \cup \mathcal{D}$ along with the two disjoint paths in between. Then the same argument as above yields paths and cycles between every pair of vertices in H' and therefore paths and cycles between any pair of vertices in G of all lengths from $4L$ up to $2(c - d - \epsilon)|R|L \geq 2(c - d - \epsilon)(1 - \epsilon)n$ so if $\epsilon, d \ll \lambda$, the desired result holds. $\qquad\square$

CHAPTER 3

ENOMOTO-OTA'S CONJECTURE HOLDS FOR LARGE GRAPHS

In 2000, Enomoto and Ota conjectured the following and proved several cases.

Conjecture 1 (Enomoto and Ota [7]). *Given an integer $k \geq 3$, let G be a graph of order n and let $n_1, n_2, \ldots, n_k$ be a set of k positive integers with $\sum n_i = n$. If $\sigma_2(G) \geq n + k - 1$, then for any k distinct vertices $x_1, x_2, \ldots, x_k$ in G, there exists a set of vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that $|P_i| = n_i$ and P_i starts at x_i for all i with $1 \leq i \leq k$.*

A partial solution was provided by Magnant and Martin in the sense that the path lengths could only be prescribed within a small fraction of n.

Theorem 10 (Magnant and Martin [15]). *Given an integer $k \geq 3$, for every set of k positive real numbers $\eta_1, \ldots \eta_k$ with $\sum_{i=1}^{k} \eta_i = 1$, and for every $\epsilon > 0$, there exists n_0 such that for every graph G of order $n \geq n_0$ with $\sigma_2(G) \geq n + k - 1$ and for every choice of k vertices $S = \{x_1, \ldots, x_k\} \subseteq V(G)$, there exists a set of vertex disjoint paths $P_1, \ldots, P_k$ which span $V(G)$ with P_i beginning at the vertex x_i and $(\eta_i - \epsilon)n < |P_i| < (\eta_i + \epsilon)n$. Also the condition on $\sigma_2(G)$ is sharp.*

When n is sufficiently large relative to k, we prove that Conjecture 1 holds.

Theorem 11. *Given an integer $k \geq 3$, let G be a graph of sufficiently large order n and let $n_1, n_2, \ldots, n_k$ be a set of k positive integers with $\sum n_i = n$. If $\sigma_2(G) \geq n + k - 1$, then for any k distinct vertices $x_1, x_2, \ldots, x_k$ in G, there exists a set of vertex disjoint paths $P_1, P_2, \ldots, P_k$ such that $|P_i| = n_i$ and P_i starts at x_i for all i with $1 \leq i \leq k$.*

Our proof utilizes several extremal lemmas based on the structure of the reduced graph provided by the Regularity Lemma. Our lemmas deal with the cases where

the minimum degree is small, the reduced graph has a large independent set and the connectivity of the reduced graph is small.

3.1 Proof Outline

Given an integer $k \geq 3$ and desired path orders $n_1, n_2, \ldots, n_k$, we choose constants ϵ and d as follows:

$$0 < \epsilon \ll d \ll \frac{1}{k},$$

where $a \ll b$ is used to indicate that a is chosen to be sufficiently small relative to b. Let n be sufficiently large to apply Lemma 1 with constant ϵ to get large clusters and let R be the corresponding reduced graph. Note that, when applying Lemma 1, there are at least $\frac{1-\epsilon}{\epsilon}$ clusters so $|R| \geq \frac{1-\epsilon}{\epsilon}$.

We use a sequence of lemmas to eliminate extremal cases of the proof. Without loss of generality, we assume $n_1 \leq n_2 \leq \cdots \leq n_k$. Our first lemma establishes the case when $\delta(G)$ is small.

Lemma 7. *Conjecture 1 holds when* $\delta(G) \leq \frac{n_k}{8}$.

Lemma 7 is proven in Section 4.3. By Lemma 7, we may assume $\delta(G) \geq \frac{n_k}{8} \geq \frac{n}{8k}$. Our next lemma establishes the case when $\kappa(R) \leq 1$.

Lemma 8. *Given a positive integer* k, *let* $\epsilon = \epsilon_k, d = d_k > 0$, *and let* G *be a graph of order* $n \geq n(\epsilon, d, k)$ *with* $\sigma_2(G) \geq n + k - 1$ *and* $\delta(G) \geq \frac{n_k}{8}$. *If* $\kappa(R) \leq 1$, *then the conclusion of Conjecture 1 holds.*

Lemma 8 is proven in Section 3.4. Our final lemma establishes the case where G contains a large independent set.

Lemma 9. *Given a positive integer k, let $\epsilon = \epsilon_k > 0$ be small, and let G be a graph of order $n \geq n(\epsilon)$. If $\sigma_2(G) \geq n + k - 1$ and $\alpha(G) \geq \left(\frac{1}{2} - \epsilon\right) n$, then G satisfies Conjecture 1.*

Lemma 9 is proven in Section 3.5.

With all these lemmas in place, we use Ore's Theorem (Theorem 3) to construct a long cycle in the reduced graph of G. Alternating edges of this cycle are then made into super-regular pairs of G. This structure is then used to construct the desired paths. The complete proof of our main result, assuming the above lemmas, is presented in the following section.

3.2   Proof of Theorem 11

By Lemma 8, we may assume R is 2-connected. By Theorem 2, we know that $\sigma_2(R) \geq (1 - 2d - 4\epsilon)|R|$. Thus, we may apply Theorem 3 to obtain a cycle C of length at least $(1 - 2d - 4\epsilon)|R|$ in R. Define a "garbage set" to include V_0 and those clusters not used in C.

Color the edges of C with red and blue such that no two red edges are adjacent and at most one consecutive pair of edges is blue (if $|C|$ is odd). Apply Lemma 3 on the pairs of clusters in G corresponding to the red edges of R to obtain super-regular pairs where the two sets of each super-regular pair have the same order. All vertices discarded in this process are added to the garbage set and define the clusters C_i to be the original clusters without the removed vertices. Note that we have added a total of at most ϵn vertices to the garbage set.

If C is odd, then let c_0 be the vertex in R with two blue edges, let C_0 be the corresponding cluster in G, and let C_0^+ and C_0^- be the neighboring clusters in G. Since the pairs (C_0^-, C_0) and (C_0, C_0^+) are both large and ϵ-regular, there exists a set

of at least k vertices $T_0 \subseteq C_0$ with a matching to each of C_0^- and C_0^+. We use these vertices as transportation and move all of $C_0 \setminus T_0$ to the garbage set.

Let G_C denote the graph induced on the set of vertices remaining in clusters associated with C that have not been moved to the garbage set, and let D denote the garbage set. Then $V(G) = \bigcup_{i=1}^{|C|} C_i \cup C_0 \cup D$ (if C_0 exists), with $|D| \leq (2d + 7\epsilon)n$.

By Lemma 7, we may assume $\delta(G) \geq \frac{n_k}{8}$. In particular, the vertices in D each have at least $\frac{n_k}{8} - (|D| - 1) \gg \epsilon n$ edges to G_C.

A path is said to *balance* the super-regular pairs in G_C if, for every super-regular pair the path visits, it uses an equal number of vertices from each set in the pair. Note that the removal of a balancing path preserves the fact that each super-regular pair of clusters is balanced. Let (A, B) be a super-regular pair of clusters in G_C. A balancing path starting in A and ending in B which contains a vertex $v \in D$ is called *v-absorbing*.

Claim 1. *Avoiding any selected set of at most ϵr clusters and any set of at most $\frac{16(2d+7\epsilon)n}{\epsilon r}$ vertices in each of the remaining clusters, there exists a v-absorbing path of order at most 17. Otherwise the desired path partition already exists.*

Proof. Absorbing paths are constructed iteratively, one for each vertex of D, in an arbitrary order. Suppose some number of such absorbing paths have been created. If we have created one for each vertex of D within the restrictions of the claim, the proof is complete so suppose we have constructed at most $|D| - 1$ absorbing paths. Vertices that have already been used and clusters that have lost at least $\frac{16(2d+7\epsilon)n}{\epsilon r}$ vertices are removed from consideration in following iterations.

Let L' be the order of the smallest cluster in C.

Fact 5. *If we have created at most $|D| - 1$ such paths, at most ϵr clusters would have order at most $L' - \frac{16(2d+7\epsilon)n}{\epsilon r}$.*

Proof. Since each absorbing path constructed in this claim has order at most 16 (other than the vertex v), we lose at most 16 vertices from G_C for each vertex of D. The result follows. □

Let $v \in D$ such that there is no absorbing path for v of order at most 17. Since $d(v) \geq \frac{n_k}{8}$, v must have edges to at least $\frac{r}{8k}$ clusters. Let A and B be two clusters which are not already ignored to which v has at least two edges to vertices that are not already in a path or an absorbing path. For convenience, we call two clusters X and Y a *couple* or *spouses* if X and Y are consecutive on C and the pair (X, Y) is super-regular.

The following facts are easily proven using the structure we have provided and the lemmas proven before.

Fact 6. *A and B are not a couple.*

Otherwise it would be trivial to produce a v-absorbing path.

Let A' and B' denote the spouses of A and B, respectively, let $a', b' \in R$ correspond to A' and B', respectively, and define the following sets of clusters:

- $X_A := \{$couples PQ of clusters such that pa' and qa' are edges in $R\}$,

- $X_B := \{$couples PQ of clusters such that pb' and qb' are edges in $R\}$, and

- $X_{AB} := \{$all couples of clusters such that one spouse has an edge to both A' and B' in $R\}$. In particular, let X'_{AB} denote the clusters in X_{AB} that are not the neighbors of A' and B'.

  Since we are considering two neighbors of v in A (and two neighbors of v in B), say v_1 and v_2, if X_A (or similarly X_B) contains even a single couple (Q, R), then we can absorb v using a path of the form $v_1vv_2 - A' - Q - R - A'$. Thus, we may actually assume $X_A = X_B = \emptyset$.

Our next fact follows from the fact that $\sigma_2(R) \geq (1 - 2d - 4\epsilon)|R|$.

Fact 7. *There are at most $(2d - 4\epsilon)|R|$ clusters in C which are not in X_{AB}.*

If there is an edge xy between two (not already used) vertices in clusters in X'_{AB}, then there is a v-absorbing path of the form $v_1 v v_2 - A' - (X_{AB} \setminus X'_{AB}) - xy - (X_{AB} \setminus X'_{AB}) - A'$. Thus, the graph induced on the vertices in clusters in X'_{AB} contains no edges. By Lemma 9, we have the desired set of paths. This completes the proof of Claim 7. $\qquad\qquad\square$

For each chosen vertex x_i, if $x_i \notin G_C$, use Menger's Theorem [17] to construct a shortest path to a vertex, say x'_i, in G_C. Using an edge of a super-regular pair first, construct a balancing path from x'_i through every cluster of G_C. Note that, since the pairs are either ϵ-regular or (ϵ, d)-super-regular, using Lemma 4, this path can be constructed to use at most 2 vertices from each cluster.

First suppose the path starting at x_i already has order at least $n_i - 17$. In this case, we add at most 76 vertices using a super-regular pair (and Lemma 4) or discard any excess vertices to obtain the desired path. If a coupled pair of clusters in G_C is left unbalanced by this process, we simply remove a vertex from the larger cluster to D. Note that repeating this for each short path adds at most $k - 1$ vertices to D.

By Claim 7, since $|D| \leq (2d + 7\epsilon)n$, we can construct an absorbing path for each vertex $v \in D$ where these paths are all disjoint. Let P^v be an absorbing path for v with ends of P^v in clusters C_i and C_{i+1}. Suppose uw is the edge of P_k from C_i to C_{i+1}. Then using Lemma 4, we can replace the edge uw with the path P^v with the addition of at most 2 extra vertices at either end. Note that absorbing a vertex $v \in D$ into a path P_i using the absorbing path will always change the parity of the length of P_i.

For each path P_i that is not already completed and not the correct parity, absorb a single vertex from D into P_i. This will correct the parity of the path.

Recall the assumption (without loss of generality) that $n_1 \leq \cdots \leq n_k$. By the same process, all remaining vertices of D can be absorbed into P_k. This makes $|P_k|$ larger but since $|D| \leq (2d + 7\epsilon)n$ and each absorbing path P^v for $v \in D$ has order at most 17 with at most two extra at either end, we get $|P_k| \leq 3|C| + 17(2d + 7\epsilon)n < n_k$.

The following lemma, stated in [16], is an easy exercise using the definition of (ϵ, d)-super-regular pairs and the Blow-Up Lemma [13].

Lemma 10 (Magnant and Salehi Nowbandegani [16]). *Let U and V be two clusters forming a balanced (ϵ, d)-super-regular pair with $|U| = |V| = L$. Then for every pair of vertices $u \in U$ and $v \in V$, there exist paths of all odd lengths ℓ between u and v satisfying*

rm (a) $3 \leq \ell \leq dL$ and

rm (b) $(1 - d)L \leq \ell \leq L$.

For each i with n_i small, absorb a few pairs of vertices from each super-regular pair, using Lemma 19, until P_i has the desired order. For each remaining index i, using Lemma 19 absorb entire super-regular pairs at a time (along with possibly a few vertices from other super-regular pairs) until each path P_i has the desired order to complete the proof. □

3.3 Proof of Lemma 7

Recall that Lemma 7 claims Conjecture 1 holds when $\delta(G) \leq \frac{n_k}{8}$.

Proof. Let $a \in V(G)$ with $|N(a)| = \delta(G) \leq \frac{n_k}{8}$, and partition $V(G)$ as follows:

$$B = G \setminus (a \cup N(a))$$

$$A = \left\{ v \in a \cup N(a) : |N(v) \cap V(B)| < \frac{1}{8}(n + k - \delta(G) - 1) \right\}$$

$$C = \left\{ v \in a \cup N(a) : |N(v) \cap V(B)| \geq \frac{1}{8}(n + k - \delta(G) - 1) \right\}$$

Note that, since $\sigma_2(G) \geq n + k - 1$, the set A induces a complete graph. Furthermore, the set B has order $n - 1 - \delta(G)$, and A is nonempty since $a \in A$. Since $\sigma_2(G) \geq n + k - 1$ and a has no edges to B, each vertex in B has degree at least $n + k - 1 - \delta(G)$ which means $\delta(G[B]) \geq n + k - 1 - 2\delta(G)$. Note that G is also $(k + 1)$-connected. First, a claim about subsets of B.

Claim 2. *Every subset of B of order at least $\frac{3n_k}{8}$ is panconnected.*

Proof. With $|B| = n - \delta(G) - 1$ and $\delta(G[B]) \geq n + k - 1 - 2\delta(G)$, we see that $\delta(G[B]) \geq |B| - \delta(G) \geq |B| - \frac{n_k}{8}$. Therefore, for any subset $B' \subseteq B$ with $|B'| \geq \frac{3n_k}{8}$, we have $\delta(G[B']) \geq |B'| - \frac{n_k}{8} > \frac{|B'| + 2}{2}$. By Theorem 4, we see that B' is panconnected. $\square$

Consider k selected vertices $X = \{x_1, \ldots, x_k\} \subseteq V(G)$. Let X_A denote the (possibly empty) set $X \cap A$ and let X'_A denote $X_A \cup v$ where $v \in A \setminus X_A$ if such a vertex v exists. If no such vertex v exists, then let $X'_A = X_A$. The vertices of X'_A will serve as start vertices for paths that will be used to cover all of A. By Menger's Theorem and since $\kappa(G) \geq k + 1$, there exists a set of disjoint paths $\mathscr{P}_A$ starting at the vertices of X'_A and ending in B and avoiding all other vertices of X. Choose $\mathscr{P}_A$ so that each path is as short as possible, contains only one vertex in B and, by construction, has order at most 4. If any of the paths in $\mathscr{P}_A$ begins at a selected vertex x_i and has order at least n_i, we call this desired path completed and remove the first n_i vertices of the path from the graph and continue the construction process. If $A \setminus V(\mathscr{P}_A) \neq \emptyset$, let P_v be a path using all remaining vertices and ending at v. This path P_v together with the path of $\mathscr{P}_A$ corresponding to v provides a single path that cleans up the remaining vertices of A and ends in B. The ending vertices of

these paths, the vertices of B, will serve as proxy vertices for the start vertices (v or $x_i \in X \cap A$). Thus far, we have constructed paths that cover all of A, start at vertices of $X \cap A$ (when such vertices exist) and end in B.

As vertices of B are selected and used on various paths, we continuously call the set of vertices in B that have not already been prescribed or otherwise mentioned the *remaining* vertices in B. For example, so far $B \setminus (X \cup V(\mathscr{P}_A))$ is the set of remaining vertices of B. Our goal is to maintain at least $\frac{3n_k}{8} + 1$ remaining vertices to be able to apply Claim 8 as needed within these remaining vertices.

Since $|C| \leq \delta(G) \leq \frac{n_k}{8}$ and $d_B(u) \geq \frac{1}{8}(n + k - \delta(G) - 1)$ for all $u \in C$, there exists a set of two distinct neighbors in $B \setminus (X \cup V(\mathscr{P}_A))$ for each vertex in C. For each vertex $x_i \in X \cap C$, select one such vertex to serve as a proxy for x_i and leave the other aforementioned neighbor in the remaining vertices of B. By Claim 8, there exists a path through the remaining vertices of B with at most one intermediate vertex from one neighbor of a vertex of C to a neighbor of another vertex of C. Since $|C| \leq \frac{n_k}{8}$, such paths can be built and strung together into a single path P_C starting and ending in B, containing all vertices of $C \setminus X$ with $|P_C| < 4|C| \leq \frac{n_k}{2}$.

We may now construct what is left of the desired paths within B. The paths $P_1, P_2, \ldots, P_{k-1}$ can be constructed in any order starting at corresponding proxy vertices and ending at arbitrary remaining vertices of B using Claim 8 in the remaining vertices of B. Finally, there are at least

$$|B| - \left|B \cap \left(\cup_{i=1}^{k-1} V(P_i)\right)\right| - |B \cap V(\mathscr{P}_A)| - |B \cap V(P_C)|$$
$$\geq (n - 1 - \delta(G)) - (k + 1) - (3|C|)$$
$$> \frac{3n_k}{8} + 1$$

remaining vertices in B. With these and Claim 8, we construct a path with at most one internal vertex from an end of P_C to the proxy of v (if such a vertex exists) and

a path containing all remaining vertices of B from x_k (or its proxy) to the other end of P_C. This completes the construction of the desired paths and thereby completes the proof of Lemma 7. □

3.4   Proof of Lemma 8

Assume $\sigma_2(G) \geq n + k - 1$. We begin with a result ensuring that low connectivity in the reduced graph R results in at most two components after removal of a minimum cut set.

Lemma 11. *Let $\epsilon, d > 0$ be small real numbers and k be a positive integer. If G is a graph with $\sigma_2(G) \geq n + k - 1$ and reduced graph R with connectivity at most 1, then R consists of only two components after removal of a minimum cut set.*

Proof. Applying Lemma 1 to G, let $G'' = G'[V(G) \setminus V_0]$. Since $d_{G''}(v) > d_G(v) - (d + 2\epsilon)n$, it immediately follows that $\sigma_2(R) > (1 - 2(d + 2\epsilon))|R|$. Let D be a minimum cutset of R (if one exists) so $|D| \in \{0, 1\}$. Suppose R (or $R \setminus D$) contains at least three components, three of which being A, B, and C. Let $a \in A$, $b \in B$ and $c \in C$. Then $d(a) + d(b) > (1 - 2(d + 2\epsilon))|R|$, which implies $|A| + |B| > (1 - 2(\delta + 2\epsilon))|R| - 2|D|$. Similarly, the same is true for $|B| + |C|$ and $|A| + |C|$. Finally $2(|R| - |D|) = 2(|A| + |B| + |C|) > 3(1 - 2(d + 2\epsilon))|R| - 6|D|$, or $|D| > \frac{1}{4}(1 - 2(d + 2\epsilon))|R|$, a contradiction. □

Remark 1. Given small real numbers $\epsilon, d > 0$ and a positive integer k, let G be a graph of order $n = \sum_{i=1}^{k} n_i \geq n(\epsilon, d, k)$ with $\sigma_2(G) \geq n + k - 1$ and $\delta(G) \geq \frac{n_k}{8}$. Let G' be the subgraph of G from Lemma 1 and let E' be the set of edges that were removed from G to obtain G'. We replace the smallest matching M possible (from E' back into G') to recover the condition that $\kappa(G') \geq k + 1$. Since the reduced R graph of G' is assumed to have connectivity at most 1, let $D \subset V(G')$ be the

cluster corresponding to a cut vertex of R. (If R contains no cut vertices, then $D = \emptyset$.) Let V_0 be the garbage cluster of G' resulting from Lemma 1, and let C be a minimum cutset of G'. By Lemma 1, each vertex of R corresponds to a cluster in G' of order $L \leq \epsilon n$. Since there is a cutset with $C \subseteq D \cup V_0 \cup M$, we have $k + 1 \leq |C| \leq |D| + |V_0| + (k+1) \leq 2\epsilon n$. By Corollary 18, we may define A and B to be the two components of $G' \setminus C$ and write $G' = A \cup C \cup B$. It immediately follows from $\sigma_2(G) \geq n + k - 1$ that $\sigma_2(G') \geq n + k - 1 - 2\epsilon n$ and

$$\delta(G'[A]) > |A| - |C| - 2\epsilon n \geq |A| - 4\epsilon n,$$
$$\delta(G'[B]) > |B| - |C| - 2\epsilon n \geq |B| - 4\epsilon n. \tag{3.1}$$

From the condition $\delta(G) \geq \frac{n_k}{8} \geq \frac{n}{2k}$ (Lemma 7), we know $|A|, |B| \geq \frac{n_k}{8} - |C| - 2\epsilon n \geq \left(\frac{1}{8k} - 4\epsilon\right) n > \frac{n}{8(k+1)}$.

While panconnected sets give paths of arbitrary length, only the endpoints are specified. Hence, to create disjoint paths of arbitrary length, we must create sets using vertices that are not part of an already existing desired path. Fortunately, even small subsets of A and B induce panconnected graphs.

Lemma 12. *Let ϵ, d, k, and $G' = A \cup C \cup B$ be defined as in Remark 2. Then the induced graph on any subgraph of A or B of order at least $8\epsilon n$ is panconnected.*

Proof. We see from (4.4) that $\delta(G'[A]) > |A| - 4\epsilon n$. Then for all $U \subset A$ of order at least $8\epsilon n$, we have

$$\delta(G[U]) \geq |U| - 4\epsilon n + 1$$
$$\geq \frac{|U| + 2}{2}.$$

By Theorem 4, the graph $G'[U]$ is panconnected. A symmetric argument shows that if $U \subset B$ has order at least $8\epsilon n$, then $G'[U]$ is panconnected. $\square$

With this information, we prove the following lemma which is completes the proof of Lemma 8.

Lemma 13. *Given small real numbers $\epsilon, d > 0$ and a positive integer k, let G be a graph of order $n = \sum_{i=1}^{k} n_i \geq n(\epsilon, d, k)$ with $\sigma_2(G) \geq n + k - 1$ and $\delta(G) \geq \frac{n_k}{8}$. If $\kappa(R) \leq 1$, then the conclusion of Theorem 11 holds.*

Proof. Suppose $\kappa(R) \leq 1$, and let $G' = A \cup C \cup B$ as in Remark 2. As noted before (4.4), we know $k + 1 \leq |C| \leq 2\epsilon n$. As noted after (4.4), we know $|A|, |B| > \frac{n}{8(k+1)}$. Since C is a minimum cut set, for each vertex $c \in C$, we may reserve 2 unique neighbors $a_c \in A \setminus X$ and $b_c \in B \setminus X$. Call $A_C = \{a_c \in A \setminus X \mid c \in C\}$ (symmetrically $B_C = \{b_c \in B \setminus X \mid c \in C\}$) the set of *proxy* vertices in A (symmetrically B). Then we have

$$|C| = |A_C| = |B_C|.$$

Given a vertex x, let an *x-path* be a path containing x as an endpoint. Namely, each desired path P_i in G' is an x_i-path.

Our strategy is as follows, first we suppose that $G'[A]$ and $G'[B]$ are complete and create "shadows" of our desired paths with some simple properties. Then we use Lemma 20 to create the desired paths, based on the shadows, in $G'[A]$ and $G'[B]$.

First suppose that $G'[A]$ and $G'[B]$ are complete. If $|C|$ is even, build paths $P_1, P_2, \ldots P_k$ such that each time a path visits a vertex in C, the path passes from $A \setminus A_C$, to A_C, to C, to B_C and then to $B \setminus B_C$ (or the opposite direction) and furthermore, all except at most one path segment of P_i in $G'[A]$ and one path segment of P_i in $G'[B]$ have length 2 for all $1 \leq i \leq k$. If $|C|$ is odd, we first move one vertex of $C \setminus X \neq \emptyset$ to either A or B (this vertex must have many edges to at least one of A or B) and then create the paths as above. Let P_i^A and P_i^B denote the segments $P_i \cap G'[A]$ and $P_i \cap G'[B]$ respectively.

Arrange the set of path segments $\{P_1^A, \ldots P_t^A\}$ of the shadows in nondecreasing order, $|P_1'| \leq \cdots \leq |P_t'|$, and suppose P_i' is a path from v_i to v_i' where $v_i, v_i' \in A$. By

construction we have $2 \leq P'_i$ for all $1 \leq i \leq t$.

Back in the original graph G', our goal is to construct path segments with same lengths and end vertices as P'_i for all $1 \geq i \geq t$. Since $|A| > \frac{n}{8(k+1)} > 8\epsilon n$ by Lemma 20 we can build a path from v_1 to v'_1 of order $|P'_1|$. We inductively construct the remaining paths in $G'[A]$ with the following claim. Here we let A^* denote the vertices in $A \setminus A_C$ that have not already been used on a path.

Claim 3. *After constructing $P'_1, \ldots P'_{t-1}$, there are at least $8\epsilon n$ vertices remaining in A^* to apply Lemma 20 and create P'_t.*

Proof. By construction, at most k paths of P'_i have length greater than 2. Thus, we get $|P'_t| \geq \frac{|A|-2(t-k)}{k}$. Also, $t \leq |C| + k$ because each path segment in A connects two vertices in $X \cup A_C$. Therefore, $|P'_t| \geq \frac{|A|-2(|C|)}{k}$ and since $A \geq \frac{n}{8(k+1)}$, we have $\frac{|A|-2(|C|)}{k} \geq \frac{n}{8k(k+1)} - \frac{2(|C|)}{k} > 8\epsilon n$ as desired. $\square$

By Claim 9, we can create all desired paths in $G'[A]$ based on their shadows. Similarly, we can create the desired paths in $G'[B]$. Using corresponding vertices of C to connect the constructed paths yields the desired final paths. $\square$

3.5   Proof of Lemma 9

Recall that Lemma 9 says for a positive integer k, a small $\epsilon = \epsilon_k > 0$, and a graph G of order $n \geq n(\epsilon)$, if $\sigma_2(G) \geq n + k - 1$ and $\alpha(G) \geq \left(\frac{1}{2} - \epsilon\right)n$, then G satisfies Conjecture 1.

Proof. Let A be a maximum independent set of G, and let $B = V(G) \setminus A$. By the assumption on $\sigma_2(G)$, we have $|B| \geq \frac{1}{2}(n + k - 1)$. This implies

$$\left(\frac{1}{2} - \epsilon\right)n \leq |A| \leq \frac{1}{2}(n - k + 1)$$

and

$$\frac{1}{2}(n + k - 1) \leq |B| \leq \left(\frac{1}{2} + \epsilon\right) n.$$

Claim 4. *Let B' be the set of vertices in B each with at least $\left(\frac{1}{2} - \frac{1}{16k}\right) n$ edges to A. Then $|B'| \geq \frac{n}{2}(1 - 16k\epsilon)$.*

Proof. Each vertex in A (except possibly one) has at least $\frac{1}{2}(n + k - 1)$ neighbors in B, which means there are at least $(|A| - 1) \cdot \frac{1}{2}(n + k - 1)$ edges between A and B. On the other hand, there are fewer than $|B'| \left(\frac{1}{2} - \epsilon\right) n + (|B \setminus B'|) \left(\frac{1}{2} - \frac{1}{16k}\right) n$ edges out of B. Since $|A| \geq \left(\frac{1}{2} - \epsilon\right) n$ and $|B| \leq \left(\frac{1}{2} + \epsilon\right) n$, we get

$$|B'| \left(\frac{1}{2} - \epsilon\right) n + \left(\left(\frac{1}{2} + \epsilon\right) n - |B'|\right) \left(\frac{1}{2} - \frac{1}{16k}\right) n \geq \left(\left(\frac{1}{2} - \epsilon\right) n - 1\right) \frac{1}{2}(n + k - 1)$$

which implies that $|B'| \geq \frac{n}{2}(1 - 16k\epsilon)$ as desired. $\qquad\square$

A bipartite graph $U \cup V$ is *bipanconnected* if for every pair of vertices $x, y \in U \cup V$, there exist (x, y)-paths of all possible lengths at least 2 of appropriate parity in $U \cup V$. That is, for every pair of vertices $x \in U$ and $y \in V$, there exist (x, y)-paths of every possible odd length except 1, and for every pair of vertices $x, y \in U$ (and V), there exist (x, y)-paths of every even length. Note that we must exclude the value 1 from our definition in order to allow graphs $U \cup V$ that are not complete bipartite. Also observe that the sets U and V need not be balanced, so the longest possible length may be only $2 \min\{|U|, |V|\}$.

Lemma 14 (Coll, Halperin and Magnant [2]). *If $G[U \cup V]$ is a balanced bipartite graph of order $2m$ with $\delta(G[U \cup V]) \geq \frac{3m}{4}$, then $G[U \cup V]$ is bipanconnected.*

Our next claim shows that any reasonably large subsets of B' induce bipanconnected subgraphs when paired with any corresponding subset of A.

Claim 5. *For all $m \geq \frac{n}{4k}$, if $U \subseteq A$ and $V \subseteq B'$ with $|U|, |V| \geq m$, then $U \cup V$ induces a bipanconnected subgraph of G.*

Proof. For $m \geq \frac{n}{4k}$, let U and V be subsets of A and B' respectively with $|U|, |V| \geq m$. Each vertex in U has at least $|V| - 2\epsilon n > \frac{3m}{4}$ neighbors in V. Each vertex in B' misses at most $\frac{1}{16k}n$ vertices in A so each vertex in V has degree at least $|U| - \frac{1}{16k}n \geq \frac{3m}{4}$ into U. It follows from Lemma 14 that $G[U \cup V]$ is bipanconnected. $\qquad\square$

Let $D = B \setminus B'$ so

$$
\begin{aligned}
|D| \; &\leq \; \max\{|B|\} - \min\{|B'|\} & (3.2)\\
&= \; \left(\frac{1}{2} + \epsilon\right) n - \frac{n}{2}(1 - 16k\epsilon) & (3.3)\\
&= \; (16k + 1)\epsilon n. & (3.4)
\end{aligned}
$$

Let M be a maximum matching between D and A and let D' be the vertices of D that are not covered by edges of M. In particular, vertices in D' must have many edges to B, and therefore behave as if they are in A. Let τ be the minimum number of edges in $B \setminus D'$ that must be used in order to ensure all paths can be constructed as desired. In particular, if we let $o(S)$ denote the number of odd ordered desired paths that must start at a vertex in the set S, we get

$$
\tau = |B \setminus D'| - |A \cup D'| - o(B \setminus D') + o(A \cup D').
$$

By definition, there are no edges from D' to $A \setminus V(M)$. This means that if we pick two vertices u and v in $A \setminus V(M)$, we get $d(u) + d(v) \leq 2|B \setminus D'|$ but since $\sigma_2(G) \geq n + k - 1$, this means

$$
|B| - |A| - k + 1 \geq 2|D'|,
$$

which implies that $\tau \geq -1$.

We first assume $\tau \geq 0$ and build the desired paths. We will address the other case in Claim 6 later.

By the definition of $D \setminus D'$, each vertex in this set has many neighbors in $B \setminus D$. Let D^* denote those vertices in $D \setminus D'$ that can be strung together on a path (or a set of at most k paths vertices of X are involved) that starts in A and ends in B' and uses an equal number of vertices from $B \setminus D'$ and $A \cup D'$. Let D_0 be the remaining vertices in $D \setminus (D' \cup D^*)$. If we choose a vertex $u \in A$ with no neighbors to $D_0 \cup D'$ (note that such a vertex must exist by the definitions of these sets) and $v \in D_0$, we get

$$n + k - 1 \leq d(u) + d(v) \leq |B \setminus (D' \cup D_0)| + |B \setminus D'| - 1$$

which reduces down to $|D_0| \leq \tau$. Thus, we may assume τ is rather large and we need only show that we are always able to use τ edges within B in constructing the desired path system.

By definition, each vertex of $D \setminus D'$ can use either 1 or 2 edges within B in constructions. If τ larger than $|D \setminus D'|$, the degree sum condition, applied to vertices in B, provides many edges within B, easily enough to find τ edges that can be used. Finally, applying Claim 5, we may construct each desired path starting at the corresponding selected vertex, with the prescribed length. We construct these paths in order from shortest to longest so that, when constructing the final path, there will certainly be enough vertices remaining to apply Claim 5. This completes the proof in the case when $\tau \geq 0$. We now show that this assumption is justified.

Claim 6. *We have $\tau \geq 0$.*

Proof. For a contradiction, suppose $\tau = -1$. In order for this to occur, all desired paths must be odd and all selected vertices must be in $B \setminus D'$. The set $A \cup D'$ must contain no edges, since otherwise such an edge could be used in the construction of the paths as above. Thus, by the choice of A with $|A| = \alpha(G)$, we must have $D' = \emptyset$. This gives us $|B| = |A| + k - 1$ which means $A \cup B$ induces a complete bipartite graph.

If k is even, this means n is even (since $n = \sum n_i$) but $n = |A| + |B| = 2|A| + k - 1$ which is odd, a contradiction. If k is odd, this means n is odd but $n = 2|A| + k - 1$ which is even, again a contradiction. This completes the proof of Claim 6 $\qquad\square$

This also completes the proof of Lemma 9. $\qquad\square$

CHAPTER 4

PLACING SPECIFIED VERTICES AT PRECISE LOCATIONS ON A HAMILTONIAN CYCLE

Recently, there have been several results concerning the placement of specified vertices on a Hamiltonian cycle. Kaneko and Yoshimoto proved the following result.

Theorem 12 (Kaneko and Yoshimoto [12]). *Let G be a graph of order n, $d \leq \frac{n}{4}$ a positive integer and A a set of at most $\frac{n}{2d}$ vertices. If $\delta(G) \geq \frac{n}{2}$ then there exists a hamiltonian cycle in G with the distance, along the cycle, between any pair of vertices of A at least d.*

Sárközy and Selkow managed to produce specified distances between almost all the consecutive pairs but did not put the specified vertices in order.

Theorem 13 (Sárközy and Selkow [21]). *There are $\omega, n_0 > 0$ such that if G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq n/2$, d is an arbitrary integer with $3 \leq d \leq \omega n/2$ and S is an arbitrary set of vertices in G with $2 \leq |S| = k \leq \omega n/d$, then for every sequence d_i of integers with $3 \leq d_i \leq d$, $1 \leq i \leq k-1$, there is a Hamiltonian cycle C of G and an ordering of the vertices of S, $a_1, a_2, \ldots, a_k$, such that the vertices of S are visited in this order on C and we have $|dist_C(a_i, a_{i+1}) - d_i| \leq 1$ for all but one $1 \leq i \leq k-1$.*

More recently, Faudree et al. placed specified vertices in order but were unable to prescribe the distances exactly.

Theorem 14 (Faudree et al. [9]). *Let $t \geq 3$ be an integer and $\epsilon, \gamma_1, \gamma_2, \ldots, \gamma_t$ positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < \min\{\frac{\gamma_i^2}{2}\}$. For $n \geq \frac{7t^{12} \times 10^{10}}{\epsilon^6}$, let G be a graph of order n having $\delta(G) \geq \frac{n+t-1}{2}$ or $\delta(G) \geq \frac{n}{2}$ and $\kappa(G) \geq \frac{3t}{2}$. For every $X = \{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, there exists a hamiltonian cycle H containing*

the vertices of X in order such that $(\gamma_i - \epsilon)n \leq dist_H(x_i, x_{i+1}) \leq (\gamma_i + \epsilon)n$ for all $1 \leq i \leq t$. Furthermore, the minimum degree and connectivity conditions are sharp.

In the case where there are only two vertices specified, the following conjecture has been attributed to Enomoto.

Conjecture 2 ([6]). *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2 + 1$, then for any $x, y \in V(G)$, there is a Hamiltonian cycle C of G such that $d_C(x, y) = flrn/2$.*

This conjecture was generalized by Faudree and Li.

Conjecture 3 (Faudree and Li [11]). *If G is a graph of order n with $\delta(G) \geq n/2 + 1$, then for any integer $2 \leq m \leq n/2$ and for any $x, y \in V(G)$, there is a Hamiltonian cycle C of G such that $d_C(x, y) = m$.*

The case of Conjecture 3 where $n \geq 6m$ was proven in [10]. Faudree and Gould recently provided a sharp minimum degree condition for the placement of specified vertices at precise locations relative to each other on a Hamiltonian cycle.

Theorem 15 (Faudree and Gould [8]). *Let $n_1, \ldots, n_{k-1}$ be a set of $k-1$ integers each at least 2 and $\{x_1, \ldots, x_k\}$ be a fixed set of k ordered vertices in a graph G of order n. If $\delta(G) \geq (n + 2k - 2)/2$, then there is $N = N(k, n_1, \ldots, n_{k-1})$ such that if $n \geq N$, there is a Hamiltonian cycle C of G such that $d_C(x_i, x_{i+1}) = n_i$ for all $1 \leq i \leq k - 1$.*

Our first result is very closely related to Theorem 15. Our degree assumption is lower but since our choices of the lengths n_i must be large, we are not bound by the sharpness example noted in [8].

Theorem 16. *Given an integer $k \geq 3$, let G be a graph of sufficiently large order n. Then there exists n_0 such that if $n_1, n_2, \ldots, n_k$ are a set of k positive integers*

with $n_i \geq n_0$ for all i, $\sum n_i = n$, and $\delta(G) \geq \frac{n+k}{2}$, then for any k distinct vertices $x_1, x_2, \ldots, x_k$ in G, there exists a Hamiltonian cycle such that the length of the path between x_i to X_{i+1} on the Hamiltonian cycle is n_i.

Theorem 16 includes the best possible bound on $\delta(G)$ by the following example. Suppose n is large and satisfies appropriate divisibility constraints with respect to k. Consider two complete graphs A and B each of order $\frac{n-(k+1)}{2}$. Let C be the remaining $k+1$ vertices. If we let $+$ denote the standard graph join of inserting all edges between two disjoint sets, then let $G = A+C+B$ where this notation means $(A+C)\cup(C+B)$ where the copies of vertices of C are identified. If all of the vertices $x_1, \ldots, x_k$ are chosen from A and each length n_i is chosen to be $\frac{n}{k}$, there is no Hamiltonian cycle with consecutive specified vertices at distance precisely n_i apart. Furthermore, this graph has $\delta(G) = |A| + |C| - 1 = \frac{n-k-1}{2} + k = \frac{n+k-1}{2}$.

Our next result provides a degree sum condition for the same placement of specified vertices on a Hamiltonian cycle.

Theorem 17. *Given an integer $k \geq 3$, let G be a graph of sufficiently large order n. Then there exists n_0 such that if $n_1, n_2, \ldots, n_k$ are a set of k positive integers with $n_i \geq n_0$ for all i, $\sum n_i = n$, and $\sigma_2(G) \geq n + 2k - 2$, then for any k distinct vertices $x_1, x_2, \ldots, x_k$ in G, there exists a Hamiltonian cycle such that the length of the path between x_i to X_{i+1} on the Hamiltonian cycle is n_i.*

Theorem 17 includes the best possible bound on $\sigma_2(G)$ by the following example. Let A be a complete graph on k vertices and B be a complete graph on $n - (3k - 2)$ vertices. Let C be the remaining $2k - 2$ vertices. If we let $G = A + C + B$, choose all vertices $x_i \in A$, and choose lengths of paths n_i to be at least 3 each, there is no Hamiltonian cycle with these specified vertices at distance n_i apart. Furthermore, this graph has $\sigma_2(G) = (|A| - 1) + |C| + (|B| - 1) = n + 2k - 3$.

Theorems 16 and 17 are proven in Section 4.2.

Our proof follows the same outline as the proof in [3]. It utilizes several extremal lemmas based on the structure of the reduced graph provided by the Regularity Lemma. The lemmas deal with the cases where the minimum degree is small, the reduced graph has a large independent set and when the connectivity of the reduced graph is small.

4.1 Proof Outline

Given an integer $k \geq 3$ and desired path segment lengths $n_1, n_2, \ldots, n_k$, we choose constants ϵ and d as follows:

$$0 < \epsilon \ll d \ll \frac{1}{k},$$

where $a \ll b$ is used to indicate that a is chosen to be sufficiently small relative to b. Let n be sufficiently large to apply the Regularity Lemma 1 with constant ϵ to get large clusters and let R be the corresponding reduced graph. Note that, when applying Lemma 1, there are at least $\frac{1}{\epsilon}$ clusters so $|R| \geq \frac{1}{\epsilon}$.

We use a sequence of lemmas to eliminate extremal cases of the proof. Without loss of generality, we assume $n_k \geq n_i$ for all i. Our first lemma establishes the case of Theorem 17 when $\delta(G)$ is small.

Lemma 15. *Theorem 17 holds when $\delta(G) \leq \frac{n_k}{8}$.*

Lemma 15 is proven in Section 4.3. By Lemma 15, we may assume $\delta(G) \geq \frac{n_k}{8} \geq \frac{n}{8k}$ in the proof of Theorem 17. Our next lemmas establish the case where G contains a large independent set. Although the statements are almost identical, the different degree assumptions require slightly different proofs.

Lemma 16. *Let $\epsilon > 0$ be a sufficiently small real number. If $\alpha(G) \geq \left(\frac{1}{2} - \epsilon\right) n$, then Theorem 16 and Theorem 17 hold.*

The proofs of Lemma 16 is almost identical to the proof of Lemma 9.

Our final lemmas establish the case when $\kappa(R) \leq 1$. Again the statements look identical but the different assumptions of the theorems require slightly different proofs.

Lemma 17. *If $\kappa(R) \leq 1$, then Theorem 16 holds.*

Lemma 18. *If $\kappa(R) \leq 1$, then Theorem 17 holds.*

Lemmas 17 and 18 are proven in Section 4.4.

Once all these lemmas are in place, we use Ore's Theorem (Theorem 3) to construct a long cycle in the reduced graph. Alternating edges of this cycle are made into super-regular pairs of the graph. This structure is then used to construct the desired paths. The complete proof of our main results are presented in the following section. Since the non-extremal case, the case when all of the above lemmas are assumed, looks generally the same for both of our main results and the conclusions are the same, we are able to combine both proofs into one.

4.2   Proof of Theorems 16 and 17

Since the conclusions of both results are identical and the proofs are the same aside from applications of different lemmas, we provide a single proof for both theorems.

Proof. By Lemma 17 or 18, we may assume R is 2-connected. By Theorem 2, we know that $\sigma_2(R) \geq (1 - 2d - 4\epsilon)|R|$. Thus, we may apply Theorem 3 to obtain a cycle C of length at least $(1 - 2d - 4\epsilon)|R|$ in R. Define a "garbage set" to include V_0 and those clusters not used in C.

Color the edges of C with red and blue such that no two red edges are adjacent and, as few blue edges as possible are adjacent. Note that if C is even, the colors will

alternate and if C is odd, there will be only one consecutive pair of blue edges while all others are alternating. Apply Lemma 3 on the pairs of clusters in G corresponding to the red edges of R to obtain super-regular pairs where the two sets of each super-regular pair have the same order. All vertices discarded in this process are added to the garbage set and redefine the clusters C_i to be the clusters without the removed vertices. Note that we have added at most ϵn vertices to the garbage set.

If C is odd, let c_0 be the vertex with two blue edges, let C_0 be the corresponding cluster and let C_0^+ and C_0^- be the neighboring clusters. Since the pairs (C_0^-, C_0) and (C_0, C_0^+) are both large and ϵ-regular, there exists a set of k vertices $T_0 \subseteq C_0$ with a matching to each of C_0^- and C_0^+. We will use these vertices as transportation and move all of $C_0 \setminus T_0$ to the garbage set.

Let G_C denote the set of vertices remaining in clusters associated with C that have not been moved to the garbage set and let D denote the garbage set. Note that $|D| \leq (2d + 7\epsilon)n$.

By Lemma 15, we may assume $\delta(G) \geq \frac{n_k}{8}$ for both proofs. In particular, the vertices in D each have at least $\frac{n_k}{8} - (|D| - 1) \gg \epsilon n$ edges to G_C, even in the proof of Theorem 17.

A path is said to *balance* the super-regular pairs in G_C if for every super-regular pair the path visits, it uses an equal number of vertices from each set in the pair. Note that the removal of a balancing path preserves the fact that if a pair of clusters is super-regular, then the two clusters have the same order.

For each chosen vertex x_i, if $x_i \notin G_C$, use Menger's Theorem to construct two shortest paths to vertices, say x_i' and x_i^{pp}, in G_C. By Lemma 15, these paths are actually just single edges.

For each index i with $1 \leq i \leq k$, construct a path P_i from x_i^{pp} through every cluster of G_C by going all the way around the cycle C and continuing on to x_{i+1}',

where indices are taken modulo k. This path is constructed to be balancing except possibly at one end (since both x_i^{pp} and x_{i+1}' might be in the later spouse of their respective couples for a particular orientation of C). If the path is not balancing, we remove a single vertex from the unbalanced pair to make it balanced again, and place that vertex in D. This process adds at most $k \ll |D|$ vertices to D. Using Lemma 4, this path can be constructed to use at most 2 vertices from each cluster for each time the path passes through. Since each path goes around the cycle at most twice, each path has length at most $4|R|$. Together these paths form a cycle with the selected vertices x_i appearing in order. The goal of the rest of the proof is to extend these paths to their desired lengths in a controlled way.

Let (A, B) be a super-regular pair of clusters on C. A balancing path starting in A and ending in B which contains a vertex $v \in D$ is called v-absorbing. The following claim was proven in [3]

Claim 7 (Coll et al. [3]). *Avoiding any selected set of at most $\epsilon\ell$ clusters and any set of at most $\frac{16(2d+7\epsilon)n}{\epsilon\ell}$ vertices in each of the remaining clusters, there exists a v-absorbing path of order at most* 17. *Otherwise the desired Hamiltonian cycle already exists.*

By Claim 7, since $|D| \leq (2d + 7\epsilon)n$, we can construct an absorbing path for each vertex $v \in D$ where these paths are all disjoint. Let P^v be an absorbing path for v with ends of P^v in clusters C_i and C_{i+1}. Suppose uw is the edge of P_k from C_i to C_{i+1}. Then using Lemma 4, we can replace the edge uw with the path P^v with the addition of at most 4 extra vertices at either end. Note that absorbing a vertex $v \in D$ into a path P_i using the absorbing path will always change the parity of the length of P_i.

For each path P_i that is not already completed and not the correct parity, absorb

a single vertex from D into P_i. This will change the parity of the path. At this point, every path has the correct parity and has length at most $4|R| + 17 \leq 4M + 17$ where M is a function of ϵ provided by Lemma 1.

By the same process, all remaining vertices of D can be absorbed into P_k. This makes $|P_k|$ larger but since $|D| \leq (2d + 7\epsilon)n$ and each absorbing path P^v for $v \in D$ has order at most 17, we get $|P_k| \leq 3|C| + 17(2d + 7\epsilon)n < n_k$.

The following lemma, stated in [16], is an easy exercise using the definition of (ϵ, δ)-super-regular pairs and Lemma 5.

Lemma 19 ([16]). *Let U and V be two clusters forming a balanced (ϵ, δ)-super-regular pair with $|U| = |V| = L$. Then for every pair of vertices $u \in U$ and $v \in V$, there exist paths of all odd lengths ℓ between u and v satisfying*

(a) $3 \leq \ell \leq \delta L$ *and*

(b) $(1 - \delta)L \leq \ell \leq L$.

For each i with n_i small, absorb a few pairs of vertices from each super-regular pair (using Lemma 19) until P_i has the desired order. For each remaining index i, using Lemma 19 absorb entire super-regular pairs at a time (along with possibly a few vertices from other super-regular pairs) until each path P_i has the desired order to complete the proof. $\square$

4.3 Proof of Lemma 15

Recall that Lemma 15 claims Theorem 17 holds when $\delta(G) \leq \frac{n_k}{8}$.

Proof. Let $a \in V(G)$ with $|N(a)| = \delta(G) \leq \frac{n_k}{8}$, and partition $V(G)$ as follows:

$$B = G \setminus (a \cup N(a))$$

$$A = \left\{ v \in a \cup N(a) : |N(v) \cap V(B)| < \frac{1}{8}(n + k - \delta(G) - 1) \right\}$$

$$C = \left\{ v \in a \cup N(a) : |N(v) \cap V(B)| \geq \frac{1}{8}(n + k - \delta(G) - 1) \right\}$$

Note that, since $\sigma_2(G) \geq n+2k-2$, the set A induces a complete graph. Furthermore, the set B has order $n - 1 - \delta(G)$, and A is nonempty since $a \in A$. Since $\sigma_2(G) \geq n+2k-2$ and a has no edges to B, each vertex in B has degree at least $n+2k-2-\delta(G)$ which means $\delta(G[B]) \geq n+2k-3-2\delta(G)$. Also note that $\kappa(G) \geq 2k$. First, a claim about subsets of B.

Claim 8. *Every subset of B of order at least $\frac{3n_k}{8}$ is panconnected.*

Proof. With $|B| = n - \delta(G) - 1$ and $\delta(G[B]) \geq n + 2k - 2 - 2\delta(G)$, we see that $\delta(G[B]) \geq |B|-\delta(G) \geq |B|-\frac{n_k}{8}$. Therefore, for any subset $B' \subseteq B$ with $|B'| \geq \frac{3n_k}{8}$, we have $\delta(G[B']) \geq |B'|-\frac{n_k}{8} > \frac{|B'|+2}{2}$. By Theorem 4, we see that B' is panconnected. $\square$

Consider k selected vertices $X = \{x_1, \ldots, x_k\} \subseteq V(G)$. Let X_A denote the (possibly empty) set $X \cap A$. First assume that $|X_A| = t \geq 1$. Since $\kappa(G) \geq 2k \geq 2t$, by Menger's theorem there exists a set of $2t$ disjoint paths $\mathscr{P}_A$ starting at the vertices of X_A, each vertex of X_A having two paths, and ending in B, avoiding all other vertices of X. If there are any vertices remaining in $A \setminus X_A$ that have not been used on the $2t$ paths, we redefine one of the paths to include these vertices using Menger's theorem and the fact that A is complete. If $t = 0$, then simply use Menger's Theorem to find two paths from A to B and use all of A to patch these two together into a single path including all of A.

As vertices of B are selected and used on various paths, we continuously call the set of vertices in B that have not already been prescribed or otherwise mentioned the *remaining* vertices in B. For example, so far, the set $B \setminus (X \cup V(\mathscr{P}_A))$ is all the

remaining vertices of B. Our goal is to maintain at least $\frac{3n_k}{8} + 1$ remaining vertices to be able to apply Claim 8 as needed within these remaining vertices.

Since $|C| \leq \delta(G) \leq \frac{n_k}{8}$ and $d_B(u) \geq \frac{1}{8}(n + k - \delta(G) - 1)$ for each $u \in C$, there exists a set of two distinct neighbors in $B \setminus (X \cup V(\mathscr{P}_A))$ for each vertex in C. For each vertex $x_i \in X \cap C$, select these two vertices to serve as proxies for x_i. By Claim 8, there exists a path, through the remaining vertices of B, with at most one intermediate vertex from one neighbor of a vertex of C to a neighbor of another vertex of C. Since $|C| \leq \frac{n_k}{8}$, such paths can be built and strung together into a single path P_C starting and ending in B and containing all vertices of $C \setminus X$ with $|P_C| < 4|C| \leq \frac{n_k}{2}$.

We may now construct paths within B to build the desired Hamiltonian cycle. Each path P_i for $1 \leq i \leq k-1$ can be constructed can be constructed in any order by starting at x_i (or a corresponding proxy vertex) and ending at x_{i+1} (or a corresponding proxy vertex) using Claim 8 within the remaining vertices of B so that the path P_i has precisely the desired length n_i. Finally, there are at least

$$|B| - \left|B \cap \left(\cup_{i=1}^{k-1} V(P_i)\right)\right| - |B \cap V(\mathscr{P}_A)| - |B \cap V(P_C)|$$
$$\geq (n - 1 - \delta(G)) - (k + 1) - (3|C|)$$
$$\geq \frac{3n_k}{8} + 2$$

remaining vertices in B. With these and Claim 8, we construct a short path with at most one internal vertex from an end of P_C to x_k (or a corresponding proxy vertex) and a path containing all remaining vertices of B from x_1 (or a corresponding proxy vertex) to the other end of P_C. This completes the construction of the desired Hamiltonian cycle and thereby completes the proof of Lemma 15. $\qquad\square$

4.4   Proof of Lemmas 17 and 18

Assume $\sigma_2(G) \geq n + 2k - 2$. We begin with a result ensuring that low connectivity in the reduced graph R results in at most two components after removal of a minimum cut set, which is an easy corollary of the corresponding lemma in [3].

Corollary 18. *Let $\epsilon, d > 0$ be small real numbers and k be a positive integer. If G is a graph with $\sigma_2(G) \geq n + 2k - 2$ and reduced graph R with connectivity at most 1, then R consists of only two components after removal of a minimum cut set.*

Remark 2. Given small real numbers $\epsilon, d > 0$ and a positive integer k, let G be a graph of order $n = \sum_{i=1}^{k} n_i \geq n(\epsilon, d, k)$ with $\sigma_2(G) \geq n + 2k - 2$ and $\delta(G) \geq \frac{n_k}{8}$. Let G' be the subgraph of G from Lemma 1 and let E' be the set of edges that were removed from G to obtain G'. We replace the smallest matching M possible (from E' back into G') to recover the condition that $\kappa(G') \geq 2k$. Since the reduced R graph of G' is assumed to have connectivity at most 1, let $D \subset V(G')$ be the cluster corresponding to a cut vertex of R. (If R contains no cut vertices, then $D = \emptyset$.) Let V_0 be the garbage cluster of G' resulting from Lemma 1, and let C be a minimum cutset of G'. By Lemma 1, each vertex of R corresponds to a cluster in G' of order $L \leq \epsilon n$. Since there is a cutset with $C \subseteq D \cup V_0 \cup M$, we have $2k \leq |C| \leq |D| + |V_0| + 2k \leq 2\epsilon n$. By Corollary 18, we may define A and B to be the two components of $G' \setminus C$ and write $G' = A \cup C \cup B$. It immediately follows from $\sigma_2(G) \geq n + 2k - 2$ that $\sigma_2(G') \geq n + 2k - 2 - 2\epsilon n$ and

$$
\begin{aligned}
\delta(G'[A]) &> |A| - |C| - 2\epsilon n \geq |A| - 4\epsilon n, \\
\delta(G'[B]) &> |B| - |C| - 2\epsilon n \geq |B| - 4\epsilon n.
\end{aligned}
\tag{4.1}
$$

From the condition $\delta(G) \geq \frac{n_k}{8} \geq \frac{n}{2k}$ (Lemma 15), we know $|A|, |B| \geq \frac{n_k}{8} - |C| - 2\epsilon n \geq \left(\frac{1}{8k} - 4\epsilon\right) n > \frac{n}{8(k+1)}$.

While panconnected sets give paths of arbitrary length, only the endpoints are specified. Hence, to create disjoint paths of arbitrary length, we must create sets using vertices that are not part of an already existing desired path. Fortunately, even small subsets of A and B induce panconnected graphs.

Lemma 20. *Let ϵ, d, k, and $G' = A \cup C \cup B$ be defined as in Remark 2. Then the induced graph on any subgraph of A or B of order at least $8\epsilon n$ is panconnected.*

Proof. We see from (4.4) that $\delta(G'[A]) > |A| - 4\epsilon n$. Then for all $U \subset A$ of order at least $8\epsilon n$, we have

$$\delta(G[U]) \geq |U| - 4\epsilon n + 1$$
$$\geq \frac{|U| + 2}{2}.$$

By Theorem 4, the graph $G'[U]$ is panconnected. A symmetric argument shows that if $U \subset B$ has order at least $8\epsilon n$, then $G'[U]$ is panconnected. $\square$

With this information, we prove the following lemma which is completes the proof of Lemma 18.

Lemma 21. *Given small real numbers $\epsilon, d > 0$ and a positive integer k, let G be a graph of order $n = \sum_{i=1}^{k} n_i \geq n(\epsilon, d, k)$ with $\sigma_2(G) \geq n + 2k - 2$ and $\delta(G) \geq \frac{n_k}{8}$. If $\kappa(R) \leq 1$, then the conclusion of Theorem 17 holds.*

Proof. Suppose $\kappa(R) \leq 1$, and let $G' = A \cup C \cup B$ as in Remark 2. As noted before (4.4), we know $2k \leq |C| \leq 2\epsilon n$. As noted after (4.4), we know $|A|, |B| > \frac{n}{8(k+1)}$. Since C is a minimum cut set, for each vertex $c \in C$, we may reserve 2 unique neighbors $a_c \in A \setminus X$ and $b_c \in B \setminus X$. Call $A_C = \{a_c \in A \setminus X \mid c \in C\}$ (symmetrically $B_C = \{b_c \in B \setminus X \mid c \in C\}$) the set of *proxy* vertices in A (symmetrically B). Then we have

$$|C| = |A_C| = |B_C|.$$

Given vertices x and y, let an *x-y-path* be a path containing x and y as endpoints. Namely, each desired path P_i in G' is an x_i-x_{i+1}-path.

Our strategy is as follows, first we suppose that $G'[A]$ and $G'[B]$ are complete and create "shadows" of our desired paths with some simple properties. Then we use Lemma 20 to create the desired paths, based on the shadows, in $G'[A]$ and $G'[B]$.

First suppose that $G'[A]$ and $G'[B]$ are complete. If $|C|$ is even, build paths $P_1, P_2, \ldots P_k$ (of the Hamiltonian cycle) such that each time a path visits a vertex in C, the path passes from $A \setminus A_C$, to A_C, to C, to B_C and then to $B \setminus B_C$ (or the opposite direction) and furthermore, all except at most one path segment of P_i in $G'[A]$ and one path segment of P_i in $G'[B]$ have length 2 for all $1 \leq i \leq k$. If $|C|$ is odd, we first move one vertex of $C \setminus X \neq \emptyset$ to either A or B (this vertex must have many edges to at least one of A or B) and then create the Hamiltonian cycle as above. Let P_i^A and P_i^B denote the segments $P_i \cap G'[A]$ and $P_i \cap G'[B]$ respectively.

Arrange the set of path segments $\{P_1^A, \ldots P_t^A\}$ of the shadows in nondecreasing order, $|P_1'| \leq \cdots \leq |P_t'|$, and suppose P_i' is a (v_i, v_i')-path where $v_i, v_i' \in A$. By construction we have $2 \leq P_i'$ for all $1 \leq i \leq t$.

Back in the original graph G', our goal is to construct path segments with same lengths and end vertices as P_i' for all $1 \geq i \geq t$. Since $|A| > \frac{n}{8(k+1)} > 8\epsilon n$ by Lemma 20 we can build a v_1-v_1'-path of order $|P_1'|$. We inductively construct the remaining paths in $G'[A]$ with the following claim. Here we let A^* denote the vertices in $A \setminus A_C$ that have not already been used on a path.

Claim 9. *After constructing $P_1', \ldots P_{t-1}'$, there are at least $8\epsilon n$ vertices remaining in A^* to apply Lemma 20 and create P_t'.*

Proof. By construction, at most k paths of P_i' have length greater than 2. Thus, we get $|P_t'| \geq \frac{|A| - 2(t-k)}{k}$. Also, $t \leq |C| + k$ because each path segment in A connects

two vertices in $X \cup A_C$. Therefore, $|P'_t| \geq \frac{|A|-2(|C|)}{k}$ and since $A \geq \frac{n}{8(k+1)}$, we have $\frac{|A|-2(|C|)}{k} \geq \frac{n}{8k(k+1)} - \frac{2(|C|)}{k} > 8\epsilon n$ as desired. $\qquad \square$

By Claim 9, we can create all desired paths in $G'[A]$ based on their shadows. Similarly, we can create the desired paths in $G'[B]$. Using corresponding vertices of C to connect the constructed paths yields the desired Hamiltonian cycle. $\qquad \square$

By same argument as proof of Lemma 18 we can prove Lemma 17. The only additional case occurs when $\kappa(G) < 2k$. In this case, the sets A and B are almost complete and we can create the desired Hamiltonian cycle trivially using Lemma 20 and an analogy of Lemma 21 within A and B.

CHAPTER 5

SEMI-LINKAGE WITH ALMOST PRESCRIBED LENGTHS IN LARGE GRAPHS

If we let H be any graph or even multigraph, we may consider the same problem in which we subdivide the edges of H a specified number of times, map the vertices of H into the vertices of a larger graph G and then try to find corresponding paths in G to represent those subdivided edges of H. The following is our main results

Theorem 19. *Let H be a multigraph with e edges, $S \subseteq V(H)$, $s = v_0(S) + \sum_{v \in S} d_H(v)$ and $c > 0$. There exists ℓ_0 and n_0 such that if G is a graph of order $n \geq n_0$ with $\kappa(G) \geq s$ and $\sigma_2(G) \geq cn$ and $\mathscr{L} = \{\ell_1, \ell_2, \ldots, \ell_e\}$ is a set of lengths with $\ell_0 \leq \ell_i \leq \frac{cn}{3k}$ for all i, then G is $(H, S, \mathscr{L}, 1)$-semi-linked.*

5.1   Proof of Theorem 19

Given $0 < c < 1$, we choose constants ϵ and d so that

$$0 < \epsilon \ll d \ll c < 1$$

where $a \ll b$ is used to indicate that a is chosen to be sufficiently small relative to b. Apply Lemma 1 on G to obtain the reduced graph R.

By Theorems 2 and 3, there is a cycle C in R of length at least $\frac{1}{2}(c - 2d - 4\epsilon)|R|$. For convenience, we will also use C to denote the corresponding set of clusters in G. Color the edges of C with red and blue such that no two red edges are adjacent and at most two consecutive blue edges are adjacent. Note that if C is even, then the colors will alternate. Apply Lemma 3 on the pairs of clusters in G corresponding to the red edges of C to obtain super-regular pairs where the two clusters of each super-regular pair have the same order. Note that we lose at most ϵn vertices in total

from the clusters in C. Let L' be the smallest order of a cluster in C. Any vertices not removed from the clusters of C and not already selected (as the image of a vertex in S) or used on a path will be called *remaining*. If we let C be the set of all vertices in G remaining in the clusters of C, then $|C| \geq tL' - |H| \geq \frac{1}{2}(c - 2d - 6\epsilon)n$.

By Menger's Theorem, there is a set of paths starting at those images of vertices in S that fall outside C and ending in C such that the paths are disjoint except at the starting vertices and each image of a vertex $v \in S$ has $d(v)$ paths to C. We will let the ends of these paths in C serve as proxy vertices for the images of the vertices of S. Note that, since $\sigma_2(G) \geq cn$ and vertices at distance 3 on these paths share no neighbors (except possibly some vertices on other paths), these paths can be chosen to have length less than $\frac{12}{c}$ each.

For each edge $uv \in E(H)$, we construct a preliminary path from the image of u to the image of v (or their corresponding proxies) using at least one remaining vertex of each cluster in C. By Lemma 19, such paths can be constructed to use at most four vertices of each cluster, going around the cycle C once and then continuing on to the destination. At this point, each preliminary path has length at most $\frac{24}{c} + 4|R|$ so this defines ℓ_0.

For each i with ℓ_i small (say less than $2L'$), absorb a few pairs of vertices from each super-regular pair into the preliminary path, using Lemma 19, until the path is within one of the desired order. For each remaining index i, using Lemma 19, absorb entire super-regular pairs at a time (along with possibly a few vertices from some other super-regular pairs) until each path is within one of the desired order to complete the proof. $\qquad\square$

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