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Gallai-Ramsey Number of An 8-Cycle

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GALLAI-RAMSEY NUMBER OF AN 8-CYCLE

by

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(Under the Direction of Colton Magnant)

ABSTRACT

Given a graph G and a positive integer k, define the Gallai-Ramsey number to be the minimum number of vertices n such that any k-edge-coloring of K_n contains either a rainbow (all different colored) triangle or a monochromatic copy of G. In this work, we establish the Gallai-Ramsey number of an 8-cycle for all positive integers.

Key Words: Gallai-Ramsey numbers, Rainbow triangle, Monochromatic cycle.

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CHAPTER 1

INTRODUCTION

In this work, we consider only edge-colorings of graphs. A coloring of a graph is called *rainbow* if no two edges have the same color.

Edge-colorings of complete graphs which contain no rainbow triangle have very interesting and somewhat surprising structure. In 1967, Gallai [4] first examined this structure under the guise of transitive orientations (a translation of his paper is available in [6]). His result was restated in [5] in the terminology of graphs and can also be traced back to [1]. For the following statement, a *trivial partition* is a partition into only one part.

Theorem 1.1. *[4, 5, 6] In any coloring of a complete graph containing no rainbow triangle, there exists a non-trivial partition of the vertices (called a* Gallai partition*) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

In honor of this result, rainbow triangle-free colorings have been called *Gallai colorings*. Given a Gallai coloring of a complete graph and an associated Gallai partition, define the *reduced graph* of this partition to be the induced subgraph consisting of exactly one vertex from each part of the partition. Note that the reduced graph is a 2-colored complete graph.

When considering 2-colored complete graphs, a very natural problem to consider is the Ramsey problem of finding a monochromatic copy of some desired subgraph. Since we will be mostly considering cycles in this work, we define the classical Ramsey result for even cycles which will be used later in our proofs. Here, given a graph G, let $R_k(G)$ denote the k-color Ramsey number of G, namely the minimum number of vertices m such that any k coloring (using at most k colors) of K_m contains a monochromatic copy of G. The cycle of order m is denoted by C_m and let P_n be the path of order n.

Definition 1.2. *Given two graphs G and H, the k-colored* Gallai Ramsey number $gr_k(G : H)$ *is defined to be the minimum integer n such that every k-coloring (using all k colors) of the complete graph on n vertices contains either a rainbow copy of G or a monochromatic copy of H.*

A bipartite graph is a graph whose vertices can be divided into two disjoint sets, A and B, and such that every edge connects a vertex in A to one in B. Clearly we can see that C_8 is bipartite which means $gr_k(K_3 : C_8)$ is linear in k. With this result in mind, the orders of magnitude in the following general bounds for cycles should not be surprising.

Theorem 1.3 ([2]). *Let H be a fixed graph with no isolated vertices. Let k be an integer with $k \geq 1$. If H is not bipartite, then $gr_k(K_3 : H)$ is exponential in k. If H is bipartite, then $gr_k(K_3 : H)$ is linear in k.*

Theorem 1.4 ([10]). *Given integers $n \geq 2$ and $k \geq 1$,*

$$(n-1)k + n + 1 \leq gr_k(K_3 : C_2 n) \leq (n-1)k + 3n.$$

Theorem 1.5 ([10]). *Given integers $n \geq 2$ and $k \geq 1$,*

$$n2^k + 1 \leq gr_k(K_3 : C_{2n+1}) \leq (2^{k+3} - 3)n \log n.$$

For $gr_k(K_3 : C_n)$ with $3 \leq n \leq 6$, the exact numbers were shown below.

Theorem 1.6 ([8]). *For any positive integer k,*

$$gr_k(K_3 : C_3) = \begin{cases} 5^{k/2} + 1 & \text{for} \quad k \quad \text{even.} \\ 2 * 5^{(k-1)/2} + 1 & \text{otherwise.} \end{cases}$$

Theorem 1.7 ([8]). *For any positive integer $k \geq 2$,*

$$gr_k(K_3 : C_4) = k + 4.$$

Theorem 1.8 ([9]). *For any positive integer $k \geq 2$,*

$$gr_k(K_3 : C_5) = 2^{k+1} + 1.$$

Theorem 1.9 ([9]). *For any positive integer k,*

$$gr_k(K_3 : C_6) = 2k + 4.$$

Looking at these known results we can see that the result of even cycles is cleaner than the result of odd cycle. Note, there is no known sharp result for $gr_k(K_3 : C_7)$. From the bounds above, we can say that, $3k + 5 \leq gr_k(K_3 : C_8) \leq 3k + 12$ for $k \geq 1$. Except in the case when $C_n = K_3$, all of these exact results match the lower bounds in the above general results. With this in mind we prove the following.

Theorem 1.10. *For $k \geq 1$, $gr_k(K_3 : C_8) = 3k + 5$*

Our proof of Theorem 1.10 suggests that if the Gallai-Ramsey numbers were completely established for all paths, then we may be able to establish the numbers for all C_8. This is complementary to the results of [3] where the bounds for even cycles were used to establish bounds for paths.

We also show corresponding results for some subgraphs of C_8, completing the literature of Gallai-Ramsey numbers for all subgraphs of C_8. To obtain these subgraphs, we remove one of the vertices then we have the following.

Theorem 1.11. *For $k \geq 1$, $gr_k(K_3 : P_7) = 2k + 5$*

In Chapter 3, we prove Theorem 1.11 to strengthen our overall result. If we remove a second vertex then we have the following.

Theorem 1.12. *For $k \geq 1$, $gr_k(K_3 : 2P_3) = k + 5$*

In Chapter 4, we prove Theorem 1.12 to strengthen our overall result. Theorem 1.13 is also a result of removing a second vertex.

Theorem 1.13. *For $k \geq 1$, $gr_k(K_3 : P_4 \cup P_2) = 2k + 4$*

In Chapter 5, we prove Theorem 1.13 to strengthen our overall result. A third case of removing two vertices would be a P_5 and a single vertex. This result is already known [3].

In our arguments, we occasionally use classical Ramsey numbers. The following case will be helpful.

Theorem 1.14. $R_2(C_8, C_8) = 11$

At times, we consider a G-partition as a 2-coloring of a reduced graph by choosing one vertex from each part. For the sake of notation, we define a t-blowup of a colored graph G to be the graph created by replacing each vertex of G with t vertices and each edge of color i in G with all edges of color i between the corresponding sets.

More generally than the Gallai-Ramsey numbers, define $gr_k(G : H_1, H_2, \ldots, H_k)$ to be the minimum integer N such that every coloring of K_n for $n \geq N$ using at most k colors contains either a rainbow copy of G or a monochromatic copy of H_i in color i for some i.

We will commonly use the following definition of a colored complete graph in our construction of sharpen examples. Define a lexical k-coloring of K_n, say $L(n_1, n_2, \ldots, n_k)$ with $\sum n_i = n$ to be; start with K_{n_1}, in red, call this G_1 and for each $i > 1$, add n_i vertices to G_{i-1} with all incident edges color i. Then $L(n_1, \ldots, n_k) := G_k$. One of the main properties of a lexical coloring that we will be using is that it contains no rainbow triangles.

CHAPTER 2

PROOF OF THEOREM 10

In order to prove Theorem 1.10, we actually prove the following slightly stronger result. For the precise statement, let $G_3 = C_8$, $G_2 = P_7$, $G_1 = P_5$, and $G_0 = P_3$. Note that all of these graphs are subgraphs of C_8 and represent the results of removing vertices from C_8. Theorem 1.10 follows from Theorem 2.1 by setting $i_j = 3$ for all j.

Theorem 2.1. *For $k \geq 1$, and for $0 \leq i_j \leq 3$ for all $1 \leq j \leq k$,*

$$gr_k(K_3 : G_{i_1}, G_{i_2}, \ldots, G_{i_k}) = \sum_j = 1^k i_j + 5.$$

Proof. Let $\Sigma = \sum i_j$. The proof is by induction on Σ. If $\Sigma = 0$, the result is trivial since each color is only looking for either P_2 or P_3 and it is easy to see that $gr_k(K_3 : P_3) = 3$. Thus, suppose $\Sigma \geq 1$ so $n \geq \Sigma + 5 \geq 6$. Let G be a coloring of K_n with no rainbow triangle and no monochromatic G_i for any i.

Let T be a largest set of vertices in G with the properties that

1. each vertex in T has one color on all its edges to $G \setminus T$, and

2. $|G \setminus T| \geq 4$.

Note that $T = \emptyset$ is possible. Let $T_1, T_2, \ldots, T_k$ denote the sets of vertices in T such that each vertex in T_j has all edges in color j to the vertices in $G \setminus T$. If $|T_j| > i_j$, then $T_j \cup (G \setminus T)$ contains the desired monochromatic copy of a graph in G_{i_j} in color j. Thus, $|T_j| \leq i_j$ for all j. More generally, if $T \neq \emptyset$, say with $|T_j| = a$ for some $1 \leq a \leq 3$ and for some j, then by induction on Σ applied to $G \setminus T$, we have the desired result. Thus, we may assume that $T = \emptyset$.

Consider a G-partition of G and let A be a largest part of this partition. Note that if $|A| \geq 4$, we can let $T = G \setminus A$ and apply induction as above so we may assume $|A| \leq 3$. By the choice of A, this means that every part of the G-partition has order at most 3. We now prove some helpful claims.

Claim 1. *If three parts have order at least* 3 *and at least one additional part has order at least* 2*, then there is a monochromatic* C_8.

Note that the reduced graph R of the four sets is a 2-colored K_4.

Proof. Any 2-coloring of K_4 contains either a monochromatic K_3 or a monochromatic P_4 in some color.

For the first case, suppose that we have a blue K_3. Let A, B and C be the three corresponding sets. Let $a_1, a_2, a_3 \in A, \quad b_1, b_2, b_3 \in B$ and $c_1, c_2 \in C$. Note that we may have $|C| \geq 3$. Since A, B and C form a blue K_3 in the reduced graph all edges between the sets A, B and C are blue. Then $a_1 - b_1 - a_2 - b_2 - c_1 - a_3 - b_3 - c_2 - a_1$ induces the desired monochromatic C_8.

For the second case, suppose that we have a blue P_4. Suppose the corresponding sets of the P_4 are A, B, C and D and, by symmetry that $|B| \geq 3$. Let $a_1, a_2 \in A, \quad b_1, b_2, b_3 \in B, \quad c_1, c_2 \in C$ and $d_1 \in D$. Since A, B, C and D form a blue P_4 in the reduced graph, all edges between consecutive sets are blue. Then $a_1 - b_1 - c_1 - d_1 - c_2 - b_2 - a_2 - b_3 - a_1$ induces the desired monochromatic C_8 □

For the sake of our next result, we need an extra definition. Given sets of graphs $\mathscr{G}$ and $\mathscr{H}$, define $R(\mathscr{G}, \mathscr{H})$ to be the minimum integer N such that any 2-coloring of K_n for $n \geq N$ contains either a copy of a graph in $\mathscr{G}$ in red or a copy of a graph in $\mathscr{H}$ in blue. This is a simple generalization of Ramsey numbers that has been studied for several specific classes of graphs (see [3] for example).

Claim 2. $R(\{C_4, P_5\}, \{C_4, P_5\}) = 5$

Proof. If we consider the unique 2-coloring of a K_5 with no monochromatic triangles, then there is a C_5 in each color. Thus, we also have the desired P_5 in both colors. We may therefore assume that all other 2-colorings of K_5 have a monochromatic triangle. Let $a_1, a_2, a_3 \in A$ be a monochromatic K_3 in red and $b_1, b_2 \in B$ be the two remaining vertices of the K_5. If all the edges from A to B are in one color, then there exists a monochromatic C_4 in that color. Without loss of generality, let e be a red edge a_1b_1. To avoid a C_4 in red, we let edges a_2b_1 and a_3b_1 be blue. To avoid getting a P_5 in red we let edges a_2b_2 and a_3b_2 be blue. Now we can clearly see that our blue edges make a C_4, $b_1 - a_2 - b_2 - a_3 - b_1$. $\square$

By Claim 2, if there are at least five parts of order at least 2, then there is a monochromatic C_8 since the 2-blow-up of a C_4 or a P_5 each contains a C_8. Thus, by Claims 1 and 2 and Theorem 1.14, if $n \geq 17$, there is already a monochromatic C_8. This means that, we may assume $n \leq 16$ in addition to the assumption that $T = \emptyset$ and so $|A| \leq 3$. We know that $R(C_8, C_8) = 11$ from Theorem 1.14. Since the (2-colored) reduced graph is a subgraph of A by choosing one vertex from each set, there must be at most 10 sets in the G-partition.

Claim 3. *If there are two sets of order 3 and at least five more vertices, then there exist a monochromatic C_8.*

Proof. Let A and B be the sets that contains 3 vertices each and have red edges between them. Let $i \leq 5$ and let v_i be the other vertices. If four vertices have red edges to both sets this induces a $K_{6,4}$ which contains a C_8 in red. Similarly, if four vertices have blue edges to both sets then we have our desired C_8 in blue. Therefore, let v_1 have red edges to both sets. This means that the other four vertices have blue edges to both sets induces a $K_{6,4}$ which contains a C_8 with blue edges. Therefore, let v_1 and v_2 have red edges to both sets. This gives us a C_8 in red, $A - v_1 - B - A - v_2 - B - A - B - A$, thus v_1 and v_2 can only have red edges to one set, say A. If any other vertex outside has red edges to be B then we can find a red C_8, which means all vertices outside must have blue edges to B. Similarly we can say the same thing about v_3 and v_4 such that at least four vertices will have red edges to A. The last vertex must have can have either color to A. We do not need it for this proof so we will ignore the last vertex. Now we need to look at the edges between these 4 vertices with colored edges to both sets. Specifically we want to look a the color of the edges in a path of these four vertices. With three edges and two colors we know by the pigeon hole principle at least two edges will have the same color. Since at least two edges have the same color then we have our desire C_8 in that color, blue C_8, $v_1 - B - v_2 - v_3 - B - v_4 - v_5 - B - B$, and red C_8, $v_1 - v_2 - v_3 - A - B - A - B - A - v_1$. $\square$

Claim 4. *If there is one set of order 3, two set of order 2 and at least four other vertices, then there exists a monochromatic C_8.*

Proof. Let A be the set of order 3 and let B be one of the sets of order 2 and let C be the other set of order 2. Let v_i be the singletons such that $1 \leq i \leq 4$. Without loss of generality, suppose that red appears on most of the edges between A, B and C. Suppose first that A, B and C all have red edges between them so that $c(A, B) = c(B, C) = c(C, A)$. If one of the singletons has a red edge to any of the three sets then we can find a C_8 in red. Therefore, all of the singletons will have blue edges to A, B and C which induces a $K_{7,4}$ which contains our desired C_8.

Now suppose that B and C have blue edges between them such that $c(A, B) = c(A, C)$. If any of the singletons have a red edge to either B or C then we have a C_8 in red. Therefore, all singletons must have blue edges to B and C which induces a $K_{4,4}$ which contains a C_8 in blue.

Finally suppose that A and C have blue edges between them such that $c(A, B) = c(B, C)$. To avoid a C_8 in red, at most one singleton can have red edges to A. Thus, at least three singletons have blue edges to A. None of these three singletons can have a blue edge to C, so they must all have red to C. This is induces a red C_8, $v_1 - A - B - C - v_2 - C - B - A - v_1$. Therefore all singletons must have blue to A and red to C. To avoid a C_8 in blue we can have at most one singleton with blue edges to B thus at least three red edges to B, which then we can find our desired C_8 in red, $v_1 - B - A - B - v_2 - C - v_3 - C - v_1$. $\square$

Claim 5. *If there is one set of order 3, one set of order 2 and at least six singletons, then there exist a monochromatic C_8.*

Proof. Let A be the set of order 3 and let B be the set of order 2. Let v_i be the singletons such that $1 \leq i \leq 6$. Let C be the largest set of singletons with blue edges to A. Suppose A and B have red edges between them. If at least three singletons have red edges to A and B then we can find a red C_8, $v_1 - A - B - v_2 - A - B - v_3 - A - v_1$. To avoid a C_8 in red, let four singletons have blue edges to A. In this case, $|C| = 4$. At most one vertex in C can have a blue edge to B, else we have a C_8 in blue, $v_1 - A - v_2 - A - v_3 - C - v_4 - C - v_1$. The remaining singletons, v_5 and v_6, must have red edges to A by definition of C. If v_5 or v_6 have at least two blue edges to the vertices in C then we have a blue C_8, $C - A - C - A - C - v_4 - C - A - C$. Therefore, v_5 and v_6 must have at least two red edges to the vertices in C. This induces our desired C_8 with red edges, $C - B - A - v_5 - A - B - C - v_6 - C$.

Now, let $|C| = 5$. Again, C can have at most one vertex with blue edges to B. By definition of C, v_6 must have red edges to A. If v_6 has at least two blue edge to the vertices in C then we can find a blue C_8, $C - A - C - v_6 - C - A - C - A - C$, therefore v_6 can have at most one blue edge to C. If there is at least two blue edges between the vertices in C then we can find a blue C_8, $C - C - C - A - C - A - C - A - C$, which means there can be at most one blue edge between the vertices in C. Therefore, there must be at least two red edges between them, which induces our desired C_8, $C - C - C - B - C - B - A - v_6 - C$.

Now, let $|C| = 6$. Again, C can have at most one vertex with blue edges to B. Of the vertices in C that have red edges to B, there can be at most one blue edge between them. Therefore, at least three of the vertices have red edges such that they do not form a triangle. of the vertices in C with red edge to A, when can see that of the two vertices with blue edges between and red edges to A and the rest of C acts the claim, if there is one set of order 3, two set of order 2 and at least four more vertices, then there exist a monochromatic C_8 which follows from Claim 4.

Finally, suppose v_1, v_2, v_3 have red edges to A. To avoid a C_8 in red, v_1, v_2, v_3 must blue edges to B. If v_4, v_5, v_6 have a red edge to B, then we will have a C_8 in red. Therefore, all six singletons must have blue edges to B. Of the singletons that have red edges to A, v_1, v_2, v_3, no two vertices can have a red edge to a singleton that does not have a red edge to A, v_4, v_5, v_6. Furthermore, those two vertices must have blue edges to a singleton that does not have a red edge to A. This will give us at least a P_5 in blue, combined with all the singletons having blue edges to B, we can find our desired C_8, $v_1 - v_4 - v_2 - v_5 - v_3 - B - v_6 - B - v_1$. □

Claim 6. *If there is one set of order at least 3 and at least nine more vertices, then there exist a monochromatic* C_8.

Proof. Let A be our set order 3. We define B to be the set of vertices with red edges to A and we define C to be the set of vertices with blue edges to A. By the pigeon hole principle at least five edges will have the same color edges to A. Therefore let us say $|B| = 5$ which induces a $K_{3,5}$ in red. Therefore, $|C| = 4$ and induces $K_{3,4}$ in blue. To avoid a rainbow a triangle, one vertex from either set, say B, must have red or blue edges, say red, to the other set and this induces a C_8, $B - A - B - A - B - A - B - C - B$. □

Claim 7. *If there are four sets of order at least 2 and at least three more vertices, then there exist a monochromatic* C_8.

Proof. Let A, B, C and D each be sets of order 2. The trivial case is $ABCD$ all have red edges between them. Therefore, suppose we have a P_4 in the reduced graph, $ABCD$, with red edges and all other edges edges in the reduced graph must be blue which is also a P_4, $CADB$. If any of the vertices outside have red edges to A or D then we can find a C_8 with red edges. Therefore, all the vertices outside must have blue edges to A and D, this induces our desired C_8 with blue edges on $v_1 - A - C - A - v_2 - D - B - D - v_1$. □

Claim 8. *If there are three sets of order at least 2 and at least five more vertices, then there exist a monochromatic C_8.*

Proof. Let A, B and C each be sets of order 2. Let A, B and C have red edges between them. If two of the vertices outside have red edges to at least two of the sets, say A and B, we can find a red C_8, $v_1 - A - C - B - v_2 - B - C - A - v_1$. Therefore, we can have at most one vertex from outside with red edges to the sets which means at least 4 vertices outside have blue edges to the sets. This induces a $K_{6,4}$ which contains C_8 with blue edges. Now suppose that we have red edges between A and B and also between B and C, and therefore blue edges between A and C. If at least 4 outside vertices have blue edges to both set A and C then this induces a $K_{4,4}$ which contains a blue C_8. So we can only have at most three blue edges to A and C which means at least 2 vertices have red edges to A and C. This induces a C_8 in red, $v_1 - A - B - C - v_2 - A - B - C - v_1$. Therefore all five outside vertices have red edges to A and blue edges to C. If at least 3 vertices have red edges to B then have a C_8 in red, $v_1 - A - v_2 - B - C - B - v_3 - A - v_1$. So we can have at most 2 vertices outside with red edges to B, which means at least 3 of the vertices must have blue edge to B, this also induces a blue C_8, $v_1 - C - A - C - v_2 - B - v_3 - B - v_1$. □

Claim 9. *If there are two sets of order at least 2 and at least 8 more vertices, then there exist a monochromatic C_8.*

Proof. Let A and B each be a set of order 2. If at least 4 vertices have red edges to both A and B this induces a $K_{4,4}$ which contains a C_8. Therefore, let at most 3 vertices have red edges to A and B. This means that least 5 vertices have blue edges to A and B which induces a $K_{4,5}$ which contains a C_8 in blue. Thus, let all of the outside vertices have red edges to A and blue edges to B. We know that we have a P_3 in red,$v_1 - A - v_2$, and a P_3 in blue, $v_1 - B - v_2$. We need to find a P_6 in red or blue in the outside vertices. From [3] we know that $R(P_6, P_6) = 8$, which means can find a P_6 in, say red, to connect to our red

P_3 to give us our desired C_8 with red edges. □

Lemma 2.2. *Let $Q_i = P_5$ for all i. Then $gr_k(K_3 : Q_1, Q_2, \ldots, Q_t, P_3, \ldots, P_3) = t + 5$.*

Proof. The proof is by induction on t. If $t = 0$, the result is trivial since each color is looking for a P_3 and it is easy to see that $gr_k(K_3 : P_3) = 5$ for all $k \geq 3$. If $t = 1$, $gr_k(K_3 : Q_1, P_3) = 6$, we are looking for a P_5 in the first color and a P_3 in the second color. Let H be the biggest partition. If H has $3 \leq |H| \leq n - 3$, then at least two vertices outside have the same color on edges to all of H. Then we have as induced $K_{3,2}$ in that color, which contains our desired P_5. Now we want to assume that the set H is small. If $|H| = 2$, then we can have at most two vertices outside H with edges in blue to H (to avoid a P_5). Therefore the other 2 vertices must outside of H must have edges in red H. To avoid a rainbow triangle, we know that the edge between a vertex outside of H with a red edge and a vertex outside of H with blue edge, must be either red or blue. Either one will give of our desired P_5. If $|H| = 1$, which means we just have all singletons making this a 2-coloring Ramsey number, thus $R(P_5, P_5) = 6$

Thus, suppose $t \geq 2$ so $n \geq t + 5 \geq 7$. Let H be the biggest set in the partition. If H has $2 \leq |H| \leq n - 2$, then at least two vertices outside have the same color on edges to all of H. Then we have as induced $K_{2,3}$ in that color, which contains our desired P_5. Now we want to assume that the set H is small. If $|H| = 1$, which means we just have all singletons making this a 2-coloring Ramsey number, thus $R(P_5, P_5) = 6$.

This means we may assume that $|H| \geq n - 2$ and the vertices outside H each have only one color to H. We will assume $|H| = n - 2$ since the case $|H| = n - 1$ follows similarly. Suppose colors t and $t - 1$. By induction on t, H has a monochromatic P_5 in color i, where $1 \leq i \leq t - 2$, or a monochromatic P_3 in color j where $t - 1 \leq j \leq k$. Any of these would complete the proof except P_3 in color $i = t$ or $t - 1$. Suppose we have a vertex, w, outside of H and we have a $P_3 = v_1, v_2, v_3$ inside of H in color i. Let w have color i edges to H. Therefore, we must have a vertex inside of H called u that was an edge in color i from w. Thus if we have such a P_3, along with the corresponding vertex outside H, makes a P_5 in color i to complete the proof. $\square$

To complete the proof, we consider cases based on small values of Σ.

Case 1. $\Sigma = 1$.

With loss of generality, $G_1 = P_5$ and $G_i = P_3$ for $i \geq 2$. Therefore, we have $G = K_6$ we want to show $gr_k(K_3 : P_5, P_3, P_3, ..., P_3) = 6$. Since red is the only color allowed to contain adjacent edges, each other color induces only a matching. In fact, to avoid a rainbow triangle, the edges induced on all colors other than red together must induce a matching. The compliment of this matching contains a P_5 in red to complete the proof in this case.

Case 2. $\Sigma = 2$.

Subcase 2.1. $gr_k(K_3 : P_7, P_3, \ldots, P_3) = 7$

In this case, all colors other than red together induce a matching M. In $K_7 \setminus M$, it is easy to find a P_7.

Subcase 2.2. $gr_k(K_3 : P_5, P_5, P_3, \ldots, P_3) = 7$.

This result follows from Lemma 16.

Case 3. $\Sigma = 3$.

Subcase 3.1. $gr_k(K_3 : C_8, P_3, P_3, \ldots, P_3) = 8$.

In this case, all colors other than red together induce a matching M. In $K_8 \setminus M$, it is easy to find a C_8.

Subcase 3.2. $gr_k(K_3 : P_7, P_5, P_3, \ldots, P_3) = 8^{\cdot}$

Since $R_2(P_7, P_5) = 8$, we may assume that there are at most 7 parts in the partition. Thus, there must exist a part of the partition of order at least 2. Other than colors red and blue, all other colors together induce a matching so if we choose our G-partition to have the most possible parts, we may assume all parts have order at most 2.

Let A be a part of order 2. At most two of the vertices outside can have blue to A (to avoid a P_5). Therefore at least 4 vertices outside all have red to A. This induces a $K_{2,4}$ in red. Each of blue vertices can have at most one red edge to the red set and actually only total. All other edges are of blue. This gives us our desired result of a P_5 in blue.

Next suppose 2 sets have size 2 called A and B. If blue appears between A and B then all other edges will be red to the 2 sets. This gives us a $K_{4,4}$ which contains a P_7. Therefore the edges between A and B must be red. If there are at least 2 vertices outside with red to A and one vertex to B then there is a P_7 in red. On the other hand if there are 2 vertices outside with blue to A, then we might as well have blue in between the 2 sets. Therefore we have found our desired P_7 in one color and P_5 in the other color.

Subcase 3.3. $gr_k(K_3 : P_5, P_5, P_5, P_3, \ldots P_3) = 8.$

This result follows from Lemma 16.

The cases $\Sigma = 4 - 7$ follow similarly and tediously. $\square$

CHAPTER 3

PROOF OF THEOREM 11

In order to prove Theorem 1.11, we actually prove the following slightly stronger result. Theorem 1.11 then follows from this result in the case when $t = k$.

Theorem 3.1. *Given* $1 \leq t \leq k$, *let* $G_1, G_2, \ldots, G_t = P_7$ *and* $G_{t+1}, \ldots, G_k = P_5$. *Then* $gr_k(K_3 : G_1, G_2, \ldots, G_k) = k + t + 5$.

Proof. For the lower bound, the graph $L(k, 6, 2, \ldots, 2, 1, \ldots, 1)$ where 2 occurs $t-1$ times, has no rainbow triangles, no monochromatic P_7 in any of the first t colors, no monochromatic P_5 in any of the remaining colors and has order $k + t + 4$.

For the upper bound, suppose $n = k + t + 5$ and G is a k-coloring of K_n with no rainbow triangle. If $k \geq 2$, the result is trivial or follows from the classical Ramsey number. Thus suppose $k \geq 3$, so $n \geq 9$.

Consider a G-partition of G. Let A be a largest part of this partition.

Claim 10. *If* $3 \leq |A| \leq n - 5$, *then there exists the desired monochromatic* P_7.

Proof. Since $|A| \leq n - 5$, there are at least $n - (n - 5) = 5$ vertices in $G \setminus A$. Let $a_1, a_2, a_3, a_4 \in A$ and let $b_1, b_2, b_3, b_4, b_5 \in G \setminus A$. Since A is a part of the G-partition, for each $b_i, c(b_i a_j) = c(b_i a_\ell)$ for all j, ℓ. With at least 5 vertices in $G \setminus A$, by the pigeon hole principle three of them must have the same color on all edges to A, let b_1 b_2 b_3 have all red edges to A. Then $\{a_1, a_2, a_3, a_4\} \cup \{b_1, b_2, b_3\}$ induces a monochromatic $K_{4,3}$, which contains the desired monochromatic P_7. Now let A only have three vertices, a_1, a_2, a_3. Since n is at least 9 then $G \setminus A$ has at least 6. Since A is a part of the G-partition, for each $b_i, c(b_i a_j) = c(b_i a_\ell)$ for all j, ℓ. With at least 6 vertices in $G \setminus A$, by the pigeon hole principle three of them must have the same color on all edges to A, let b_1 b_2 b_3 have all red edges to A. Then $\{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$ induces a monochromatic $K_{3,3}$ in red and in blue. To avoid a rainbow triangle the edge between b_3 and b_4 must be either red or blue, which contains the desired monochromatic P_7. $\square$

We break the remainder of the proof into two cases based on $|A|$.

Case 1. *Suppose* $|A| \geq n - 4$.

If there exist 3 vertices outside of A with all edges to A in a single color, then G contains a monochromatic P_7 since this induces a monochromatic $K_{3,|A|}$ with $|A| \geq 4$. Thus by structure of the G-partition, suppose there are at most 2 vertices in $G \setminus A$, each with its own color on all edges to A. We will assume there are actually two vertices u and v with all one color on the edges to A, since the proof is similar if there was only one. Suppose $c(uA) = i$ and $c(vA) = j$. If $G_i = P_7$ or $G_j = P_7$, the proof is complete so suppose $G_i = G_j = P_5$. By induction on $\sum_{l=1}^{k} G_l$, we have

$$gr_k(G_1, G_2, \cdots, G_i - 1, \cdots, G_j - 1, \cdots, G_k) = (k-1) + (t-1) + 5,$$

so we can find either G_l in color l or P_5 in color i or j in $G \setminus \{u, v\}$. Suppose without loss of generality, that we find a P_5 in color i in $G \setminus \{u, v\}$. Then this P_5, along with another P_3 in color i centered at u, completes the proof. The base of this induction is when $t = 1, k = 1$ and here the result follows $gr_1 4(K_3 : P_7) = 7$.

Case 2. *All sets in the partition have order at most* 2.

Subcase 2.1. *There exist at least four sets with order* 2.

Let A_1, A_2, A_3, A_4 be these sets of order 2. Let $A_i = \{a_{i1}, a_{i2}\}$ for $1 \geq i \geq 4$. Since $k \geq 3$, $n \geq 9$ and we know there is a vertex in $G \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)$. First suppose the edges between three pairs of the sets have a single color, say red, to make a $K_{2,2,2}$. Say $c(A_1, A_2) = c(A_2, A_3) = c(A_3, A_1)$. Then all we need the vertex in $G \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)$ to have red edges to any of the three sets A_1, A_2, A_3 then we have our desired P_7. Else we can find a P_7 in the opposite color. Now suppose there is no monochromatic $K_{2,2,2}$. Then there must exists a permutation of the sets $A_1, \ldots, A_4$ so that $c(A_1, A_2) = c(A_2, A_3) = c(A_3, A_4)$. Then we can find our desired P_7, $A_1 - A_2 - A_3 - A_4 - A_3 - A_2 - A_1$.

Subcase 2.2. *There exists three sets with order 2.*

Let A_1, A_2, A_3 be the three sets of order 2. Without loss of generality, suppose $c(A_1, A_2) = c(A_2, A_3)$ is red. Since $k \geq 3, n \geq 9$ so there at at least 3 vertices in $G \setminus (A_1 \cup A_2 \cup A_3)$. Then all we need is a vertex from $G \setminus (A_1 \cup A_2 \cup A_3)$ to either set A_1 or A_3 with a red edge to get our desired P_7. If none of the 3 vertices in $G \setminus (A_1 \cup A_2 \cup A_3)$ have a red edge to A_1 or A_3 then it must have a blue edge. That induces a $K_{4,5}$ and we can easily find a P_7 from this.

Subcase 2.3. *There exists two sets with order 2.*

Let A_1, A_2 be the two sets of order 2. Since $k \geq 3, n \geq 9$ there are 5 vertices in $G \setminus (A_1 \cup A_2)$. If at least 3 vertices in $G \setminus (A_1 \cup A_2)$ in both sets induces a $K_{4,3}$ which contains a P_7.

Subcase 2.4. *There exists at most one set A with order 2.*

Let A be the set of order 2, if one exists. Since $k \geq 3, n \geq 9$ so there are at least 7 vertices in $G \setminus A$. Choosing one vertex from A along with 7 of the singletons induces a 2-colored K_8, which contains the desired monochromatic P_7.

Subcase 2.5. *All singletons.*

Since $R_2(P_7) = 9$ [7], this case is trivial. $\square$

CHAPTER 4

PROOF OF THEOREM 12

In order to prove Theorem 1.12, we actually prove the following slightly stronger result. Theorem 1.12 then follows from this result in the case when $t = k$.

Theorem 4.1. *Given* $0 \leq t \leq k$*, let* $m_1 = \cdots = m_t = 2$ *and* $m_{t+1} = \cdots = m_k = 1$*. Then* $gr_k(K_3 : m_1P_3, \ldots, m_tP_3, m_{t+1}P_3, \ldots, m_kP_3) = t + 5$.

Proof. For the lower bound, $L(t, 5, 1, \ldots, 1)$ has no rainbow triangles, no mono-chromatic $2P_3$ and has order $t + 5$.

For the upper bound, suppose G is a k-coloring of K_n with no rainbow triangle. Consider a G-partition of G. Let A be largest part of this partition.

Claim 11. *If* $3 \leq |A| \leq n - 5$*, then there exists the desired monochromatic* $2P_3$.

Proof. Since $|A| \leq n - 5$, there are at least $n - (n - 5) = 5$ vertices in $G \setminus A$. Let $a_1, a_2, a_3 \in A$ and let $b_1, b_2, b_3, b_4, b_5 \in G \setminus A$. Since A is a part of the G-partition, for each $b_i, c(b_ia_j) = c(b_ia_\ell)$ for all j, ℓ. With at least 5 vertices in $G \setminus A$, by the pigeon hole principle three of them must have the same color on all edges to A, let b_1 b_2 b_3 have all red edges to A. Then $\{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$ induces a monochromatic $K_{3,3}$, which contains the desired monochromatic $2P_3$. □

We break the remainder of the proof into two cases based on $|A|$.

Case 1. *Suppose* $|A| \geq n - 4$.

If there exist 2 vertices outside of A with all edges to A in a single color, then G contains a monochromatic $2P_3$ since this induces a monochromatic $K_{2,|A|}$ with $|A| \geq 4$. Thus by the structure of the G-partition, there are at most 2 vertices in $G \setminus A$, each with its own color on all edges to A. We will assume there are actually two vertices u and v with all one color each on edges to A, since the proof is similar if there was only one.

Suppose $c(uA) = i$ and $c(vA) = j$. If $m_i = 1$ or $m_j = 1$, the proof is complete so suppose $m_i = m_j = 2$. By induction on $\sum_{\ell=1}^{k} m_\ell$, we have

$$gr_k(m_1P_3, m_2P_3, \cdots, (m_i - 1)P_3, \cdots, (m_j - 1)P_3, \cdots, m_kP_3) = (t - 2) + 5,$$

so we can find either $m_\ell P_3$ in color ℓ or P_3 in color i or j in $G \setminus \{u, v\}$. Suppose, without loss of generality, that we find a P_3 in color i in $G \setminus \{u, v\}$. Then this P_3, along with another P_3 in color i centered at u, completes the proof. The base of this induction is when $t = 0$ and here the result follows from the trivial observation that $gr_k(K_3 : P_3) = 3$ for all k.

Case 2. *All sets in the partition have order at most* 2.

Subcase 2.1. *There exist at least three sets with order* 2.

Let A_1, A_2, A_3 be three sets of order 2. Then the edges between two pairs of sets must have a single color. Say $c(A_1, A_2) = c(A_2, A_3)$. Then $A_2 \cup (A_1 \cup A_3)$ induces a $K_{2,4}$ in this color, containing the desired $2P_3$.

Subcase 2.2. *There exist two sets with order* 2.

Let A_1 and A_2 be the two sets of order 2. Without loss of generality, suppose $c(A_1, A_2)$ is red. Since $k \geq 3$, we get $n \geq 8$, so there are at least 4 vertices in $G \setminus (A_1 \cup A_2)$. To avoid creating a red $2P_3$, at most one vertex $v \in G \setminus (A_1 \cup A_2)$ may have red edges to $A_1 \cup A_2$. This means that three of the vertices in $G \setminus (A_1 \cup A_2)$ and must have all the other color, say blue, on all edges to $A_1 \cup A_2$. This induces a blue $K_{3,4}$ which contains the desired monochromatic $2P_3$.

Subcase 2.3. *There exists at most one set A with order* 2.

Let A be the set of order 2, if one exists. Since $k \geq 3$, $n \geq 8$ so there are at least 6 singletons in $G \setminus A$. Choosing one vertex from A along with six of the singletons induces a 2-colored K_7, which contains the desired monochromatic $2P_3$.

Subcase 2.4. *All singletons.*

Since $R_2(2P_3) = 7$ [3], this case is trivial. $\square$

CHAPTER 5

PROOF OF THEOREM 13

In order to prove Theorem 1.13, we actually prove the following stronger result. Theorem 1.13 then follows from this result in the case when $t = k$.

Theorem 5.1. *Given $1 \leq t \leq k$, let $G_1, G_2, \ldots, G_t = P_4 \cup P_2$ and $G_{t+1}, \ldots, G_k = 2P_2$. Then $gr_k(K_3 : G_1, \ldots, G_t, G_{t+1}, \ldots, G_k) = k + t + 4$.*

Proof. For the lower bound, $L(k, t, 2, \cdots, 2, 1, \cdots, 1)$ where 2 occurs t times, has no rainbow triangles, no monochromatic $P_4 \cup P_2$ in any of the first t colors, no monochromatic $2P_2$ in any of the remaining colors, and has order $k + t + 4$.

For the upper bound, suppose G is a k-coloring of K_n with no rainbow triangle. Consider a G-partition of G. Let A be a largest part of this partition.

Claim 12. *If $3 \leq |A| \leq n - 5$, then there exists the desired monochromatic $P_4 \cup P_2$.*

Proof. Since $|A| \leq n - 5$, there are at least $n - (n - 5) = 5$ vertices in $G \setminus A$. Let $a_1, a_2, a_3 \in A$ and let $b_1, b_2, b_3, b_4, b_5 \in G \setminus A$. Since A is a part of the G-partition, for each $b_i, c(b_i a_j) = c(b_i a_\ell)$ for all j, ℓ. With at least 5 vertices in $G \setminus A$, by the pigeon hole principle three of them must have the same color on all edges to A, let b_1 b_2 b_3 have all red edges to A. Then $\{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$ induces a monochromatic $K_{3,3}$, which contains the desired monochromatic $P_4 \cup P_2$. $\square$

We break for the remainder of the proof into two cases based on $|A|$.

Case 1. $|A| \geq n - 4$.

If there exist 3 vertices outside of A with all edges to A in a single color, then G contains a monochromatic $P_4 \cup P_2$ since this induces a monochromatic $K_{3,|A|}$ with $|A| \geq 4$. Thus by the structure of the G-partition, assume there are at most 2 vertices in $G \setminus A$, each with its own color on all edges to A. We will assume there are actually two vertices u and

v with all one color each on edges to A, since the proof is similar if there was only one. Suppose $c(uA) = i$ and $c(vA) = j$. If $G_i = P_4 \cup P_2$ or $G_j = P_4 \cup P_2$, the proof is complete so suppose $G_i = G_j = 2P_2$. By induction on $\sum_{\ell=1}^{k} G_\ell$, we have

$$gr_k(G_1, G_2, \cdots, G_i - 1 \cdots, G_j - 1, \cdots, G_k) = (k-1) + (t-1) + 4$$

so we can find either G_ℓ in color ℓ or $2P_2$ in color i or j in $G \setminus \{u, v\}$. Suppose without lose of generality we find $2P_2$ in color i in $G \setminus \{u, v\}$. Then this $2P_2$ along with another P_3 is color i centered at u this completes the proof. The base of this induction is when $t = 1, k = 1$ and here the result follows from $gr_k(K_3 : 2P_2) = 4$.

Case 2. *All sets in the partition have order at most* 2.

Subcase 2.1. *There exist at least four sets with order* 2.

Let A_1, A_2, A_3, A_4 be these sets of order 2. Let $A_i = \{a_{i1}, a_{i2}\}$ for $1 \leq i \leq 4$. First suppose the edges between three pairs of sets have a single color to make a $K_{2,2,2}$. Say $c(A_1, A_2) = c(A_2, A_3) = c(A_3, A_1)$. Then $a_{11}a_{21}a_{31}a_{12}$ and $a_{22}a_{32}$ form a monochromatic $P_4 \cup P_2$. Now suppose there is no monochromatic $K_{2,2,2}$. Then there exists a permutation of the sets $A_1, \cdots, A_4$ so that $c(A_1, A_2) = c(A_2.A_3) = c(A_3, A_4)$. Then $a_{11}a_{21}a_{31}a_{41}$ and $a_{22}a_{32}$ form a monochromatic $P_4 \cup P_2$.

Subcase 2.2. *Suppose there exist three sets with order* 2.

Let A_1, A_2, A_3 be the three sets of order 2. Without loss of generality, Suppose $c(A_1, A_2) = c(A_2, A_3)$ is red. Since $k \geq 3$, $n \geq 10$ so there are at least 4 vertices in $G \setminus (A_1 \cup A_2 \cup A_3)$. We define these 4 vertices to be s_i for $1 \leq i \leq 4$. Let all vertices in $G \setminus (A_1 \cup A_2 \cup A_3)$ have all the other color, say blue, on all edges to $A_1 \cup A_3$ and $c(A_1, A_3)$ is blue. This induces a blue K_6 which contains the desired monochromatic $P_4 \cup P_2$.

Subcase 2.3. *Suppose there exists two sets with order* 2.

If there are two sets, let A_1, A_2 be the two sets of order 2. Since $k \geq 3$, $n \geq 10$ so there are at least 6 singletons in $G \setminus (A_1 \cup A_2)$. Choosing one vertex from each set of order 2 (even if there are fewer than 2 such sets) along with all of the singletons induces a 2-colored K_8

Subcase 2.4. *Suppose there exists at most one set A with order 2.*

Let A be the set of order 2, if one exists. Since $k \geq 3, n \geq 10$ so there are at least 8 singletons in $G \setminus A$. Choosing one vertx from A along with eight of the vertices induces a 2-colored K_9, which induces a monochromatic $P_4 \cup P_2$.

Subcase 2.5. *All singletons.*

Since $R_2(P_4 \cup P_2) = 8$ [3], this case is trivial. $\square$

CHAPTER 6

CONCLUSION

In this work, we have proven Theorem 1.10 the Gallai-Ramsey number for an 8-cycle.

$$gr_k(K_3 : C_8) = 3k + 5$$

We also show corresponding results for some subgraphs of C_8, completing the literature of Gallai-Ramsey numbers of all subgraphs of C_8, proving Theorem 1.11, Theorem 1.12 and Theorem 1.13. All other subgraphs have already been proven and cited in this work.

REFERENCES

[1] K. Cameron and J. Edmonds. Lambda composition. *J. Graph Theory*, 26(1):9–16, 1997.

[2] A. Gyárfás, G. Sárközy, A. Sebő, and S. Selkow. Ramsey-type results for gallai colorings. *J. Graph Theory*, 64(3):233–243, 2010.

[3] S. Fujita, C. Magnant, and K. Ozeki. Rainbow generalizations of ramsey theory: a survey (upated). *http://math.georgiasouthern.edu/∼cmagnant/*, 2011.

[4] T. Gallai. Transitiv orientierbare Graphen. *Acta Math. Acad. Sci. Hungar*, 18:25–66, 1967.

[5] A. Gyárfás and G. Simonyi. Edge colorings of complete graphs without tricolored triangles. *J. Graph Theory*, 46(3):211–216, 2004.

[6] J. L. Ramírez Alfonsín and B. A. Reed, editors. *Perfect graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Ltd., Chichester, 2001.

[7] L. Gerencśer and A. Gyárfás. On Ramsey-Type Problems. *Annales Universitatis Scientiarum Budapestinensis*, Eotvos Sect. Math., 10 (1967) 167-170.

[8] Faudree, R. J. and Gould, R. and Jacobson, M. and Magnant, C., Ramsey numbers in rainbow triangle free colorings. *Australas. J. Combin.*, 46 (2010) 269-284.

[9] Fujita, S. and Magnant, C., Gallai-ramsey numbers for cycles. *Discrete Math.*, 311 (2011) 1247-1254.

[10] Hall, Mi. and Magnant, C. and Ozeki, K. and Tsugaki, M., Improved upper bounds for Gallai-Ramsey numbers of paths and cycles. *Journal of Graph Theory*, 75 (2014) 59-74