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The Bessel Function, the Hankel Transform and an Application to Differential Equations

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THE BESSEL FUNCTION, THE HANKEL TRANSFORM AND AN APPLICATION TO DIFFERENTIAL EQUATIONS

by

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(Under the Direction of Yi Hu)

ABSTRACT

In this thesis we explore the properties of Bessel functions. Of interest is how they can be applied to partial differential equations using the Hankel transform. We use an example in two dimensions to demonstrate the properties at work as well as formulate thoughts on how to take the results further.

INDEX WORDS: Bessel function, Hankel transform, Schrödinger equation

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DEDICATION

This thesis is dedicated to my wonderful wife.

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TABLE OF CONTENTS

LIST OF SYMBOLS

$\mathbb{R}$	Real Numbers
$\mathbb{R}^+$	Nonnegative Real Numbers
$\mathbb{R}^n$	$\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, n-D space
$S^{n-1}(\mathbb{R})$	Unit Sphere of $\mathbb{R}^n$
$\mathbb{Z}$	Integers
$2\mathbb{Z}$	Even Integers
$2\mathbb{Z}+1$	Odd Integers
$\mathbb{N}$	Natural Numbers
$\mathbb{C}$	Complex Numbers
$L_p(\mathbb{F})$	p-integrable functions over $\mathbb{F}$
$\Gamma(x)$	Gamma function
$J_k(z)$	Bessel Function of order k
H_l	Hankel transform of order l
A_v	$-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{v^2}{r^2}$

CHAPTER 1

INTRODUCTION

The Schrödinger equation, known for its applications within the field of quantum mechanics, has many forms due to the potential of the original equation. We show here the general linear case

$$\begin{cases} i\partial_t u - \Delta_x u + V(x)u = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases}. \tag{1.1}$$

The equation $V(x)$ is known as the potential and is the source of variation for Schrödinger equations. If the potential is equal to zero, then it becomes the free-particle equation

$$\begin{cases} i\partial_t u - \Delta_x u = 0 \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = u_0(x) \end{cases}$$

which can be solved through separation of variables. In fact, the one dimensional solution, $u(x,t) = Ce^{i\lambda^2 t - i\lambda x}$, is fairly straight forward. Now we choose the inverse-square multiplied by a constant as our potential.

$$\begin{cases} i\partial_t u - \Delta_x u + \frac{a}{|x|^2}u = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.2}$$

where $a > \frac{-(n-2)^2}{4}$. The function $u_0(x)$ is the initial condition of (1.2), Δ_x is the Laplacian operator with respect to x, and $\partial_t u$ is the first derivative of u with respect to time. The third piece of the function, $\frac{a}{|x|^2}$, is the only portion that is open to change without leaving the structure of a Schrödinger equation. Figuring out solutions to various Schrödinger equations with different potentials, or determining restrictions on those solutions, is enlightening. In [2], Miao, Zheng and Zhang determined maximal solutions for Schrödinger equations in the form of

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} Y_{k,l}(\theta) \int_0^{\infty} (r\rho)^{-\frac{n-2}{2}} J_{\nu(k)}(r\rho) e^{it\rho^2} b_{k,l}^0(\rho) \rho^{n-1} d\rho. \tag{1.3}$$

We will use a basis constructed from [2] to attempt work towards a solution of the two dimensional Schrödinger equation with several different potentials. The different potentials include a constant potential a, $\frac{a}{|x|^{2+\varepsilon}}$, $\frac{a}{|x|^{2-\varepsilon}}$, where a is still a positive real number, ε is between 0 and 1 and m is an integer and finally a complex potential of the form $\frac{ae^{i\phi}}{|x|^2}$ where $a > 0$ and $-\pi < \phi < \pi$. To find their solutions, Miao, Zheng and Zhang used a coordinate transform, the Hankel transform and ordinary differential equation methods[2]. The coordinate transform relies on spherical harmonics and in essence takes x and replaces it with $r\theta$, where $r \in \mathbb{R}^+$ and $\theta \in S^{n-1}(\mathbb{R})$. This alteration opens the door, and we can find a series representation of the initial condition

$$u_0(x) = \sum_{k=0}^{\infty} \sum_{l=0}^{\delta_k} \alpha_{k,l}^0(r) Y_{k,l}(\theta), \tag{1.4}$$

where $\alpha_{k,l}^0(r)$ is a radial function, and $Y_{k,l}(\theta)$ is a spherical harmonic function that is orthogonal to any other $Y_{j,m}(\theta)$ as long as both $k \neq j$ and $l \neq m$. Equation (1.4), through separation of variables, becomes

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\delta_k} \alpha_{k,l}(r,t) Y_{k,l}(\theta). \tag{1.5}$$

In both equations, $\delta_k = \frac{2k+n-2}{k}\binom{n+k-3}{k-1}$ and it can also be observed that for $n = 2$, δ_k is a constant equal to 2 and thus independent of k.

CHAPTER 2

BESSEL FUNCTIONS AND THE HANKEL TRANSFORM

2.1 Properties of the Bessel functions

In order to discuss Bessel functions, we must first discuss the Gamma function. The Gamma function is defined as the following integral [6]

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1} dt \quad r > 0. \tag{2.1}$$

We can consider it to be related to the factorial function because it also has a property similar to factorials [6],

$$r\Gamma(r) = \Gamma(r+1). \tag{2.2}$$

The primary use for the Gamma function in this thesis will be as part of the Bessel functions of the first kind, which are solutions of the following partial differential equation

$$\frac{1}{z}\frac{\partial}{\partial z} z \frac{\partial u}{\partial z} + \left(1 - \frac{w^2}{z^2}\right) u = 0. \tag{2.3}$$

Proposition 2.1. *The solution to the Bessel equation*

$$\frac{1}{z}\frac{\partial}{\partial z} z \frac{\partial u}{\partial z} + \left(1 - \frac{w^2}{z^2}\right) u = 0$$

is

$$J_w(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(w+j+1)} \left(\frac{z}{2}\right)^{2j+w} \tag{2.4}$$

where w is a real number and is known as the order of the function [1].

Proof. We begin by expanding the equation into

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{z}\frac{\partial u}{\partial z} + \left(1 - \frac{w^2}{z^2}\right) u = 0, \tag{2.5}$$

and multiplying by z^2

$$z^2 \frac{\partial^2 u}{\partial z^2} + z\frac{\partial u}{\partial z} + \left(z^2 - w^2\right) u = 0. \tag{2.6}$$

We now proceed by using the method of Fröbenious and assume that the solution is of the form

$$u = z^w \sum_{k=0}^{\infty} c_k z^k. \tag{2.7}$$

We assume that $c_k \in \mathbb{R}$ and $c_0 \neq 0$. Now we determine the following using this series;

$$z\frac{\partial u}{\partial z} = wz^w \sum_{k=0}^{\infty} c_k z^k + z^w \sum_{k=0}^{\infty} k c_k z^k, \tag{2.8}$$

$$z^2\frac{\partial^2 u}{\partial z^2} = w(w-1)z^w \sum_{k=0}^{\infty} c_k z^k + 2wz^w \sum_{k=0}^{\infty} k c_k z^k + z^w \sum_{k=0}^{\infty} k(k-1) c_k z^k, \tag{2.9}$$

$$z^2 u = z^w \sum_{k=2}^{\infty} c_{k-2} z^k, \tag{2.10}$$

assuming that $c_{-2} = c_{-1} = 0$. Next we collect it all into (2.6),

$$\begin{aligned} 0 =& w(w-1)z^w \sum_{k=0}^{\infty} c_k z^k + 2wz^w \sum_{k=0}^{\infty} k c_k z^k + z^w \sum_{k=0}^{\infty} k(k-1) c_k z^k + wz^w \sum_{k=0}^{\infty} c_k z^k \\ &+ z^w \sum_{k=0}^{\infty} k c_k z^k + z^w \sum_{k=2}^{\infty} c_{k-2} z^k - w^2 z^w \sum_{k=0}^{\infty} c_k z^k \end{aligned} \tag{2.11}$$

$$= z^w \sum_{k=0}^{\infty} \left[w(w-1)c_k + 2wkc_k + k(k-1)c_k + wc_k + kc_k + c_{k-2} - w^2 c_k \right] z^k \tag{2.12}$$

$$= z^w \sum_{k=0}^{\infty} \left[w^2 c_k - wc_k + 2wkc_k + k^2 c_k - kc_k + wc_k + kc_k + c_{k-2} - w^2 c_k \right] z^k \tag{2.13}$$

$$= z^w \sum_{k=0}^{\infty} \left[2wkc_k + k^2 c_k + c_{k-2} \right] z^k \tag{2.14}$$

Then we compare coeficients and see,

$$0 = 2wkc_k + k^2 c_k + c_{k-2}. \tag{2.15}$$

Next, we can show that,

$$0 = -\frac{c_{k-2}}{2wk + k^2}, \tag{2.16}$$

for every k except when $k = 0$ and assuming that $-2w \notin \mathbb{N}$. We now recall that $c_{-1} = 0$ and see that thus through recursion $c_{2k-1} = 0$ for all k. Thus all that's left is to choose the value for c_0, which we will do later. We assume that $-w \notin \mathbb{N}$ to show that

$$c_{2k} = -\frac{c_{2k-2}}{4k(w+k)}, \tag{2.17}$$

and after applying some induction we find,

$$c_{2k} = \frac{c_0(-1)^k}{4^k k!(w+k)(w+k-1)\cdots(w+2)(w+1)}. \tag{2.18}$$

This allows us to rewrite Equation (2.7),

$$u = z^w \sum_{k=0}^{\infty} \frac{c_0(-1)^k}{4^k k!(w+k)(w+k-1)\cdots(w+2)(w+1)} z^{2k}. \tag{2.19}$$

Now we choose c_0 to be $\frac{1}{2^w w!}$. This leads to

$$u = \left(\frac{z}{2}\right)^w \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(w+k)!} \left(\frac{z}{2}\right)^{2k}, \tag{2.20}$$

or, if w is not a nonnegative integer, to

$$u = \left(\frac{z}{2}\right)^w \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(w+k+1)} \left(\frac{z}{2}\right)^{2k}. \tag{2.21}$$

□

Bessel functions of the first kind can also be described using an integral [6],

$$J_w(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin(\theta) - iw\theta} d\theta \tag{2.22}$$

Beyond this, there are a few properties that can be found for Bessel functions of the first kind. We will propose them as propositions and then prove their validity. The first can be discovered after careful manipulation of the series representation.

Proposition 2.2. $J_w(z) = \frac{z}{2w}(J_{w-1}(z) + J_{w+1}(z))$

Proof. Beginning from the right hand side, we replace $J_{w-1}(z)$ with it's series representation

$$J_{w-1}(z)+J_{w+1}(z)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma(w+j)}\left(\frac{z}{2}\right)^{2j+w-1}+J_{w+1}(z). \tag{2.23}$$

Next we multiply, inside the sum, by $w+j$ in both the numerator and the denominator

$$J_{w-1}(z)+J_{w+1}(z)=\sum_{j=0}^{\infty}\frac{(w+j)(-1)^j}{j!\Gamma(w+j)(w+j)}\left(\frac{z}{2}\right)^{2j+w-1}+J_{w+1}(z). \tag{2.24}$$

By (2.2), $\Gamma(w+j)$ absorbs $w+j$ and becomes $\Gamma(w+j+1)$, so

$$J_{w-1}(z)+J_{w+1}(z)=\sum_{j=0}^{\infty}\frac{(w+j)(-1)^j}{j!\Gamma(w+j+1)}\left(\frac{z}{2}\right)^{2j+w-1}+J_{w+1}(z). \tag{2.25}$$

We expand the summation, splitting over the addition of $w+j$, we get

$$\begin{aligned}J_{w-1}(z)+J_{w+1}(z)=&\sum_{j=0}^{\infty}\frac{w(-1)^j}{j!\Gamma(w+j+1)}\left(\frac{z}{2}\right)^{2j+w-1}\\&+\sum_{j=0}^{\infty}\frac{j(-1)^j}{j!\Gamma(w+j+1)}\left(\frac{z}{2}\right)^{2j+w-1}+J_{w+1}(z).\end{aligned} \tag{2.26}$$

We extract $\frac{2w}{z}$ out of the first summation, so

$$\begin{aligned}J_{w-1}(z)+J_{w+1}(z)=&\frac{2w}{z}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma(w+j+1)}\left(\frac{z}{2}\right)^{2j+w}\\&+\sum_{j=1}^{\infty}\frac{j(-1)^j}{j!\Gamma(w+j+1)}\left(\frac{z}{2}\right)^{2j+w-1}+J_{w+1}(z).\end{aligned} \tag{2.27}$$

We replace the first term by a Bessel function of z with order w. In the second term, we set the stage to change the index of the summation as

$$J_{w-1}(z)+J_{w+1}(z)=\frac{2w}{z}J_w(z)-\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(j-1)!\Gamma(w+[j-1]+2)}\left(\frac{z}{2}\right)^{2[j-1]+w+1}+J_{w+1}(z) \tag{2.28}$$

Now we change the index of the middle term using $l = j - 1$ and get the following results

$$J_{w-1}(z) + J_{w+1}(z) = \frac{2w}{z} J_w(z) - \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma(w+l+2)} \left(\frac{z}{2}\right)^{2l+w+1} + J_{w+1}(z) \tag{2.29}$$

The middle term can now be replaced with a Bessel function of z with order $w+1$, so

$$J_{w-1}(z) + J_{w+1}(z) = \frac{2w}{z} J_w(z) - J_{w+1}(z) + J_{w+1}(z) = \frac{2w}{z} J_w(z) \tag{2.30}$$

Thus we have the following identity

$$J_w(z) = \frac{z}{2w}(J_{w-1}(z) + J_{w+1}(z)). \tag{2.31}$$

□

The next properties deal with the derivatives of the Bessel function. We begin by showing the function in series form.

Proposition 2.3.

1. $J_w'(z) = \frac{J_{w-1}(z) - J_{w+1}(z)}{2}$

2. $J_w'(z) = \frac{w}{z} J_w(z) - J_{w+1}(z)$

3. $J_w'(z) = J_{w-1}(z) - \frac{w}{z} J_w(z)$

Proof.

To prove (1), we know

$$J_w'(z) = \frac{d}{dz} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(w+j+1)} \left(\frac{z}{2}\right)^{2j+w}. \tag{2.32}$$

We then apply the derivative with respect to z,

$$J_w'(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(w+j+1)} \left(\frac{z}{2}\right)^{2j+w-1} \frac{2j+w}{2}. \tag{2.33}$$

Then the function is separated by addition, so

$$J_w'(z) = \sum_{j=0}^{\infty} \left[\frac{(-1)^j}{j!\Gamma(w+j+1)} \left(\frac{z}{2}\right)^{2j+w-1} \frac{j+w}{2} + \frac{(-1)^j}{j!\Gamma(w+j+1)} \left(\frac{z}{2}\right)^{2j+w-1} \frac{j}{2} \right]. \tag{2.34}$$

Using (2.2) on the first term and canceling a j from the factorial of the second, which in turn causes the sum of the second term to start from 1,

$$J_w'(z) = \frac{1}{2} \left[\sum_{j=0}^{\infty} \frac{(-1)^j}{j!(w+j)\Gamma(w+j)} \left(\frac{z}{2}\right)^{2j+w-1} (j+w) + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j-1)!\Gamma(w+j+1)} \left(\frac{z}{2}\right)^{2j+w-1} \right]. \tag{2.35}$$

We cancel the $j+w$ portions of the first term and take note of $w-1$. In the second term, we prepare to shift the sum from starting at one to starting at zero through factoring and adding zero where necessary, so

$$J_w'(z) = \frac{1}{2} \left[\sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma([w-1]+j+1)} \left(\frac{z}{2}\right)^{2j+[w-1]} - \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!\Gamma(w+[j-1]+2)} \left(\frac{z}{2}\right)^{2[j-1]+w+1} \right]. \tag{2.36}$$

We reconstitute the first term into a Bessel function of z with order $w-1$ and adjust the second term with a $l = j-1$ while also taking note of $w+1$ to get

$$J_w'(z) = \frac{1}{2} \left[J_{w-1}(z) - \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma([w+1]+l+1)} \left(\frac{z}{2}\right)^{2l+[w+1]} \right]. \tag{2.37}$$

Finally we reconstitute the second term into a Bessel function of z with order $w+1$,

$$J_w'(z) = \frac{J_{w-1}(z) - J_{w+1}(z)}{2}. \tag{2.38}$$

Now through (2.31) and what was just shown, we can achieve the two further forms for the first derivative. We know from (2.31) that

$$J_w(z) = \frac{z}{2w}(J_{w-1} + J_{w+1}). \tag{2.39}$$

We can then see that

$$J_{w-1}(z) = \frac{z}{2w}J_w - J_{w+1}. \tag{2.40}$$

and

$$J_{w+1}(z) = \frac{2w}{z}J_w - J_{w-1}, \tag{2.41}$$

Substituting (2.40) into (2.38), we get

$$J'_w(z) = \frac{1}{2}\left[\left(\frac{2w}{z}J_w - J_{w+1}\right) - J_{w+1}\right], \tag{2.42}$$

$$J'_w(z) = \frac{1}{2}\left[\frac{2w}{z}J_w - 2J_{w+1}\right], \tag{2.43}$$

$$J'_w(z) = \frac{w}{z}J_w - J_{w+1}. \tag{2.44}$$

Thus we have shown (2). Substituting (2.41) into (2.38), we get

$$J'_w(z) = \frac{1}{2}\left[J_{w-1} - \left(\frac{2w}{z}J_w - J_{w-1}\right)\right], \tag{2.45}$$

$$J'_w(z) = \frac{1}{2}\left[2J_{w-1} - \frac{2w}{z}J_w\right], \tag{2.46}$$

$$J'_w(z) = J_{w-1} - \frac{w}{z}J_w. \tag{2.47}$$

□

Normally the pattern of the first part of Proposition 2.3 is followed, as repeated applications easily build into the following form for higher Bessel derivatives.

Proposition 2.4. $J_w^{(i)}(z) = \frac{1}{2^i}\sum_{n=0}^{i}(-1)^n J_{w-i+2n}(z)\binom{i}{n}$ *where* $i \in \mathbb{N}$, *and* $\binom{i}{n}$ *is the nth term of the ith row of Pascal's triangle.*

Proof. The case $i = 1$ was already shown in Proposition 2.3. We shall assume this is true up to $i = k-1$. For $i = k$,

$$\left(\frac{d}{dz}\right)^k J_w(z) = \left(\frac{d}{dz}\right)^{k-1}\left(\frac{J_{w-1}(z) - J_{w+1}(z)}{2}\right) \tag{2.48}$$

$$= \left(\frac{d}{dz}\right)^{k-1}\left(\frac{J_{w-1}(z)}{2} - \frac{J_{w+1}(z)}{2}\right) \tag{2.49}$$

$$= \frac{1}{2}\left(\frac{d}{dz}\right)^{k-1} J_{w-1}(z) - \frac{1}{2}\left(\frac{d}{dz}\right)^{k-1} J_{w+1}(z) \tag{2.50}$$

$$= \frac{1}{2}\left(\frac{1}{2^{k-1}}\sum_{n=0}^{k-1}(-1)^n J_{w-k+2n}(z)\binom{k-1}{n} - \frac{1}{2^{k-1}}\sum_{n=0}^{k-1}(-1)^n J_{w-k+2n+2}(z)\binom{k-1}{n}\right) \tag{2.51}$$

$$= \frac{1}{2^k}\left(\sum_{n=0}^{k-1}(-1)^n J_{w-k+2n}(z)\binom{k-1}{n} - \sum_{n=0}^{k-1}(-1)^n J_{w-k+2(n+1)}(z)\binom{k-1}{n}\right) \tag{2.52}$$

$$= \frac{1}{2^k}\left(\sum_{n=0}^{k-1}(-1)^n J_{w-k+2n}(z)\binom{k-1}{n} + \sum_{n=1}^{k}(-1)^n J_{w-k+2n}(z)\binom{k-1}{n-1}\right) \tag{2.53}$$

$$= \frac{1}{2^k}\sum_{n=0}^{k}(-1)^n J_{w-k+2n}(z)\left(\binom{k-1}{n} + \binom{k-1}{n-1}\right) \tag{2.54}$$

$$= \frac{1}{2^k}\sum_{n=0}^{k}(-1)^n J_{w-k+2n}(z)\binom{k}{n}. \tag{2.55}$$

□

Instead of using Proposition 2.3, we construct an equivalent form of the second derivative of the Bessel function using Proposition 2.3 as appropriate to ensure that it requires only $J_w(z)$ and $J_{w+1}(z)$:

$$\begin{aligned}
J_w''(z) &= \left(\frac{w}{z}J_w(z) - J_{w+1}(z)\right)' \\
&= \frac{w}{z}J_w'(z) - \frac{w}{z^2}J_w(z) - J_{w+1}'(z) \\
&= \frac{w}{z}\left(\frac{w}{z}J_w(z) - J_{w+1}(z)\right) - \frac{w}{z^2}J_w(z) - \left(J_w(z) - \frac{w+1}{z}J_{w+1}(z)\right) \\
&= \frac{w^2}{z^2}J_w(z) - \frac{w}{z}J_{w+1}(z) - \frac{w}{z^2}J_w(z) - J_w(z) + \frac{w+1}{z}J_{w+1}(z) \\
&= \frac{w^2}{z^2}J_w(z) - \frac{w}{z^2}J_w(z) - J_w(z) + \frac{1}{z}J_{w+1}(z) \\
&= \left(\frac{w^2 - w - z^2}{z^2}\right)J_w(z) + \frac{1}{z}J_{w+1}(z). \qquad (2.56)
\end{aligned}$$

2.2 The Hankel Transform

The Hankel transform uses Bessel functions to transform from one coordinate system to another. It is defined in the following manner [2] for $f(x) : \mathbb{R}^n \to \mathbb{R}$, $|x| = r$, $\frac{x}{|x|} = \theta$, $|\xi| = \rho$, and $\frac{\xi}{|\xi|} = \phi$,

$$H_w[f(x)](\xi) = \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_w(r\rho) f(r\phi) r^{n-1} dr. \qquad (2.57)$$

The inverse transform of the Hankel transform is just the Hankel transform again but instead of starting from x, we use ξ:

$$H_w[F] = \int_0^\infty (r\rho)^{-\frac{n-2}{2}} J_w(r\rho) F(\rho\theta) \rho^{n-1} d\rho. \qquad (2.58)$$

According to Baruch, the real part of the order, w, must be greater than -1 for the inverse transform to exist [3]. When we apply the condition that $n = 2$ as well as assume that $f(x)$ is a radial function or a function that only depends on the magnitude of x, (2.57) and (2.58) become

$$H_w[f(r)](\rho) = \int_0^\infty J_w(r\rho) f(r) r dr = F(\rho), \qquad (2.59)$$

$$H_w[F(\rho)](r) = \int_0^\infty J_w(r\rho) F(\rho) \rho d\rho = f(r). \qquad (2.60)$$

An alternative definition of the Hankel transformation is as follows,

$$Ha_w[g(r)](\rho) = \int_0^\infty J_w(r\rho)g(r)(r\rho)^{\frac{1}{2}}dr = G(\rho), \tag{2.61}$$

$$Ha_w[G(\rho)](\xi) = \int_0^\infty J_w(r\rho)G(\rho)(r\rho)^{\frac{1}{2}}d\rho = g(r). \tag{2.62}$$

This alternative is the same if $f(r)r^{\frac{1}{2}} = g(r)$ and $F(\rho)\rho^{\frac{1}{2}} = G(\rho)$.

There is a property of the Hankel transform which involves (3.29), the polar form of the Laplacian. First we must assume that $f(x)$ is a function in $L^2(\mathbb{R}^+)$ and also a radial function. Then we proceed by defining an operator that is connected to (3.29)

$$A_\nu = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\nu^2}{r^2} \tag{2.63}$$

Proposition 2.5. $H_\nu A_\nu f(\xi) = |\xi|^2 \cdot H_\nu f(\xi)$

Proof.

$$[H_\nu A_\nu f](\xi) = \int_0^\infty (A_\nu f(r))J_\nu(r|\xi|)rdr$$

Replace A_ν,

$$[H_\nu A_\nu f](\xi) = \int_0^\infty J_\nu(r|\xi|)r\left(-f''(r) - \frac{f'(r)}{r} + \frac{\nu^2 f(r)}{r^2}\right)dr. \tag{2.64}$$

By integration by parts on $rf''(r)J_\nu(r|\xi|)$, we obtain

$$\begin{aligned}[H_\nu A_\nu f](\xi) &= \left[-\frac{df(r)}{dr}J_\nu(r|\xi|)r\right]_0^\infty + \int_0^\infty \Big(f'(r)\left(J_\nu(r|\xi|) + r|\xi|J_\nu'(r|\xi|)\right) \\ &\qquad - rJ_\nu(r|\xi|)\left(\frac{f'(r)}{r} + \frac{\nu^2 f(r)}{r^2}\right)\Big)dr \\ &= \left[-\frac{df(r)}{dr}J_\nu(r|\xi|)r\right]_0^\infty + \int_0^\infty \left(f'(r)r|\xi|J_\nu'(r|\xi|) + \frac{\nu^2 f(r)}{r}J_\nu(r|\xi|)\right)dr.\end{aligned} \tag{2.65}$$

Now we apply integration by parts on $rf'(r)J'_\nu(r|\xi|)$,

$$[H_\nu A_\nu f](\xi) = \left[r|\xi|f(r)J'_\nu(r|\xi|) - r\frac{df(r)}{dr}J_\nu(r|\xi|) \right]_0^\infty + \int_0^\infty \left(-|\xi|f(r)\left(J'_\nu(r|\xi|) + r|\xi|J''_\nu(r|\xi|)\right) + \frac{\nu^2 f(r)}{r}J_\nu(r|\xi|) \right) dr \tag{2.66}$$

$$= \left[r|\xi|f(r)J'_\nu(r|\xi|) - r\frac{df(r)}{dr}J_\nu(r|\xi|) \right]_0^\infty + \int_0^\infty f(r)r\left(-|\xi|^2 J''_\nu(r|\xi|) - \frac{|\xi|}{r}J'_\nu(r|\xi|) + \frac{\nu^2}{r^2}J_\nu(r|\xi|) \right) dr. \tag{2.67}$$

The first terms vanish and we are left with only the integral,

$$[H_\nu A_\nu f](\xi) = \int_0^\infty f(r)r\left(-|\xi|^2 J''_\nu(r|\xi|) - \frac{|\xi|}{r}J'_\nu(r|\xi|) + \frac{\nu^2}{r^2}J_\nu(r|\xi|) \right) dr. \tag{2.68}$$

According to our earlier work, we can replace the first and second Bessel function derivatives and arrive at

$$[H_\nu A_\nu f](\xi) = \int_0^\infty f(r)r\left(-|\xi|^2 \left(\frac{\nu^2 - \nu - (r|\xi|)^2}{(r|\xi|)^2}J_\nu(r|\xi|) + \frac{J_{\nu+1}(r|\xi|)}{r|\xi|} \right) - \frac{|\xi|}{r}\left(\frac{\nu}{r|\xi|}J_\nu(r|\xi|) - J_{\nu+1}(r|\xi|) \right) + \frac{\nu^2}{r^2}J_\nu(r|\xi|) \right) dr. \tag{2.69}$$

After multiplying everything through we get the following

$$[H_\nu A_\nu f](\xi) = \int_0^\infty f(r)r\left(\frac{-\nu^2}{r^2}J_\nu(r|\xi|) + \frac{\nu}{r^2}J_\nu(r|\xi|) + |\xi|^2 J_\nu(r|\xi|) - \frac{|\xi|}{r}J_{\nu+1}(r|\xi|) - \frac{\nu}{r^2}J_\nu(r|\xi|) + \frac{|\xi|}{r}J_{\nu+1}(r|\xi|) + \frac{\nu^2}{r^2}J_\nu(r|\xi|) \right) dr. \tag{2.70}$$

With simple cancellations we finally arrive at a final statement which can be restated as a Hankel transformation with a multiplier

$$[H_\nu A_\nu f](\xi) = \int_0^\infty f(r)r|\xi|^2 J_\nu(r|\xi|)dr = |\xi|^2 \cdot H_\nu f(\xi). \tag{2.71}$$

Thus we have shown the following property

$$H_\nu A_\nu f(\xi) = |\xi|^2 \cdot H_\nu f(\xi). \tag{2.72}$$

□

CHAPTER 3

FOURIER SERIES AND POLAR COORDINATES

3.1 ORTHOGONALITY OF SINE AND COSINE FUNCTIONS

Two functions, $f(x)$ and $g(x)$, are orthogonal on a interval (a,b) if $\int_a^b f(x)g(x)dx = 0$ [5]. The interval can be of any length and is not necessarily finite. There can also be a weight $w(x)$ is structured to cause $\int_a^b f^2(x)w(x)dx = 1$. A sequence of functions that are collectively orthogonal to each other can be arranged as $\{f_n\}_{n=0}^{\infty}$.

Proposition 3.1. *$f_k(\theta) = \sin(k\theta)$ and $g_k(\theta) = \cos(k\theta)$ are orthogonal to each other and to any other f_m or g_m.*

Proof. We start by assuming that $m \neq k$ and both are nonzero integers as the cases when m, k or both are zero are trivial. Choosing to work through cosine first, We begin with a product-to-sum trigonometric identity

$$\int_0^{2\pi} \cos(mx)\cos(kx)dx = \int_0^{2\pi} \frac{1}{2}\left[\cos((m-k)x) + \cos((m+k)x)\right]dx. \tag{3.1}$$

Then integrate

$$\begin{aligned}\int_0^{2\pi} \cos(mx)\cos(kx)dx &= \frac{1}{2}\int_0^{2\pi} (\cos((m-k)x) + \cos((m+k)x))dx \\ &= \frac{1}{2}\left[\frac{\sin((m-k)x)}{m-k} + \frac{\sin((m+k)x)}{m+k}\right]_0^{2\pi} = 0.\end{aligned} \tag{3.2}$$

As both m and k are nonzero and not equivalent, no denominator is zero also because the sine of an even multiple of π is always zero the result is zero. If we decide that $m = k$ and still nonzero, we get the following after starting from the end of (3.1).

$$\frac{1}{2}\int_0^{2\pi} (\cos((m-m)x) + \cos((m+m)x))dx = \frac{1}{2}\int_0^{2\pi} 1 + \cos(2mx)dx = \frac{1}{2}\left[x + \frac{1}{2m}\sin(2mx)\right]_0^{2\pi} = \pi. \tag{3.3}$$

All together we see that

$$\int_0^{2\pi} \cos(kx)\cos(mx)dx = \begin{cases} 2\pi, & k = m = 0, \\ \pi, & k = m \neq 0, \\ 0, & k \neq m. \end{cases} \tag{3.4}$$

The results for the orthogonality of the sine function are similar:

$$\int_0^{2\pi} \sin(mx)\sin(kx)dx = \begin{cases} \pi, & m = k \neq 0, \\ 0, & m \neq k \text{ or } m = k = 0. \end{cases} \tag{3.5}$$

Finally, we check the result of multiplying cosine and sine together while still assuming that m and k are not equal integers. Beginning with a product-to-sum identity,

$$\begin{aligned} \int_0^{2\pi} \cos(mx)\sin(kx)dx &= \frac{1}{2}\int_0^{2\pi} (\sin((k+m)x) + \sin((k-m)x))dx \qquad (3.6) \\ &= \frac{1}{2}\left[-\frac{\cos((k+m)x)}{k+m} - \frac{\cos((k-m)x)}{k-m}\right]_0^{2\pi} \\ &= \frac{1}{2}\left[-\frac{\cos((k+m)2\pi)}{k+m} - \frac{\cos((k-m)2\pi)}{k-m}\right] - \frac{1}{2}\left[-\frac{\cos(0)}{k+m} - \frac{\cos(0)}{k-m}\right] \\ &= \frac{1}{2}\left[-\frac{1}{k+m} - \frac{1}{k-m}\right] + \frac{1}{2}\left[\frac{1}{k+m} + \frac{1}{k-m}\right] = 0. \qquad (3.7) \end{aligned}$$

Returning to (3.6), we proceed again but this time with the assumption that $m = k \neq 0$:

$$\begin{aligned} \int_0^{2\pi} \cos(mx)\sin(mx)dx &= \frac{1}{2}\int_0^{2\pi} (\sin((m+m)x) + \sin((m-m)x))dx \qquad (3.8) \\ &= \frac{1}{2}\int_0^{2\pi} \sin(2mx)dx \qquad (3.9) \\ &= \frac{1}{2}\left[\frac{-\cos(2mx)}{2m}\right]_0^{2\pi} = \frac{1}{2}\left[\frac{-1}{2m} + \frac{1}{2m}\right] = 0 \qquad (3.10) \end{aligned}$$

Now we see that no matter what index, sine and cosine are always orthogonal to each other in the same interval. In total the result

$$\int_0^{2\pi} \cos(kx)\cos(mx)dx = \begin{cases} 2\pi, & k = m = 0, \\ \pi, & k = m \neq 0, \\ 0, & k \neq m, \end{cases} \tag{3.11}$$

$$\int_0^{2\pi} \sin(mx)\sin(kx)dx = \begin{cases} \pi, & m = k \neq 0, \\ 0, & m \neq k \text{ or } m = k = 0, \end{cases} \tag{3.12}$$

$$\int_0^{2\pi} \cos(kx)\sin(mx)dx = 0. \tag{3.13}$$

□

Therefore, we define a group of functions $\{f_n(z)\}_{n=0}^{\infty}$ such that

$$f_n(z) = \begin{cases} \cos(\frac{n}{2}z), & n \in 2\mathbb{Z}, \\ \sin(\frac{n-1}{2}z), & n \in 2\mathbb{Z}+1. \end{cases} \tag{3.14}$$

This group is an orthogonal group and can also be defined by adding a second index, m, that is either 1 or 2. Now,

$$f_{n,m}(z) = \begin{cases} \cos(nz), & m = 1, \\ \sin(nz), & m = 2. \end{cases} \tag{3.15}$$

3.2 Polar Transformation

Shifting from the Cartesian coordinate construction into the polar coordinate construction is fairly straight forward. The radius, r, is simply the 2-norm of any vector and the angle, θ, is a value ranging from $[0, 2\pi)$ which corresponds to a specific unit vector. With these ideas, it is a simple task to setup each substitution. Note that the expression for θ is conditional based on the sign of both y and x:

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad r = \sqrt{x^2+y^2},$$
$$\theta = \begin{cases} \arctan\left(\frac{y}{x}\right), & y>0, x>0, \\ \pi + \arctan\left(\frac{y}{x}\right), & x<0, \\ 2\pi + \arctan\left(\frac{y}{x}\right), & x>0, y<0. \end{cases} \tag{3.16}$$

Using the equations listed in (3.16) we construct the derivatives of r and θ with respect to x first:

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos(\theta)}{r} = \cos(\theta), \\ \frac{\partial \theta}{\partial x} &= \frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right) = \frac{y}{x^2+y^2} = -\frac{\sin(\theta)}{r}. \end{aligned} \tag{3.17}$$

Similar to (3.17), we find the same pair of derivatives with respect to y this time:

$$\begin{aligned} \frac{\partial r}{\partial y} &= \frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin(\theta)}{r} = \sin(\theta), \\ \frac{\partial \theta}{\partial y} &= \frac{\partial}{\partial y}\tan^{-1}\left(\frac{y}{x}\right) = \frac{x}{x^2+y^2} = \frac{\cos(\theta)}{r}. \end{aligned} \tag{3.18}$$

Next we start from the conclusions of (3.17) and find the second derivatives with respect to x as

$$\begin{aligned} \frac{\partial^2 r}{\partial x^2} &= \frac{\partial}{\partial x}\cos(\theta) = -\sin(\theta)\frac{\partial \theta}{\partial x} = \frac{\sin^2(\theta)}{r}, \\ \frac{\partial^2 \theta}{\partial x^2} &= \frac{\partial}{\partial x}\frac{-\sin(\theta)}{r} = \frac{\cos(\theta)\frac{\partial \theta}{\partial x}r + \sin(\theta)\frac{\partial r}{\partial x}}{r^2} = 2\frac{\cos(\theta)\sin(\theta)}{r^2}. \end{aligned} \tag{3.19}$$

The last step of setting up begins from (3.18) and we find the second derivatives with respect to y:

$$\begin{aligned} \frac{\partial^2 r}{\partial y^2} &= \frac{\partial}{\partial y}\sin(\theta) = \cos(\theta)\frac{\partial \theta}{\partial y} = \frac{\cos^2(\theta)}{r}, \\ \frac{\partial^2 \theta}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\cos(\theta)}{r} = \frac{-\sin(\theta)\frac{\partial \theta}{\partial y}r - \cos(\theta)\frac{\partial r}{\partial y}}{r^2} = -2\frac{\cos(\theta)\sin(\theta)}{r^2}. \end{aligned} \tag{3.20}$$

Because of (3.16) we know that we can change an equation from being expressed in the Cartesian coordinate system to the polar coordinate system. We shall indicate this as follows

$$u(x,y) = v(r,\theta). \tag{3.21}$$

3.3 Laplacian in Polar Coordinates

The Laplacian of a function, also know as the divergence of the gradient, is easily represented in Cartesian coordinates as the sum of the second derivatives with respect to each coordinate. We use our new representation of $u(x,y)$ that we showed in (3.21) then we can replace it in the expanded Cartesian Laplacian:

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}. \tag{3.22}$$

Obviously derivatives with respect to x and y on a function of r and θ is not the best way of expressing the Laplacian. Thus we use (3.16) and treat $v(r,\theta)$ as $v(r(x,y),\theta(x,y))$ and then use chain rule as

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}\right). \tag{3.23}$$

Expanding according to the product rule of derivatives,

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = {} & \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial \theta}\right)\frac{\partial \theta}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) \\ & + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial r}\right)\frac{\partial r}{\partial y} + \frac{\partial v}{\partial r}\frac{\partial}{\partial y}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial \theta}\right)\frac{\partial \theta}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial y}\right).\end{aligned} \tag{3.24}$$

Note that $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$ will only apply to the expression in parentheses follows it. Upon applying those partial derivatives, we get

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)=&\left(\frac{\partial^2 v}{\partial r^2}\frac{\partial r}{\partial x}+\frac{\partial^2 v}{\partial\theta\partial r}\frac{\partial\theta}{\partial x}\right)\frac{\partial r}{\partial x}+\frac{\partial v}{\partial r}\frac{\partial^2 r}{\partial x^2}+\left(\frac{\partial^2 v}{\partial r\partial\theta}\frac{\partial r}{\partial x}+\frac{\partial^2 v}{\partial\theta^2}\frac{\partial\theta}{\partial x}\right)\frac{\partial\theta}{\partial x}\\&+\frac{\partial v}{\partial\theta}\frac{\partial^2\theta}{\partial x^2}+\left(\frac{\partial^2 v}{\partial^2 r}\frac{\partial r}{\partial y}+\frac{\partial^2 v}{\partial\theta\partial r}\frac{\partial\theta}{\partial y}\right)\frac{\partial r}{\partial y}+\frac{\partial v}{\partial r}\frac{\partial^2 r}{\partial y^2}\\&+\left(\frac{\partial^2 v}{\partial r\partial\theta}\frac{\partial r}{\partial y}+\frac{\partial^2 v}{\partial\theta^2}\frac{\partial\theta}{\partial y}\right)\frac{\partial\theta}{\partial y}+\frac{\partial v}{\partial\theta}\frac{\partial^2\theta}{\partial y^2}.\end{aligned}\tag{3.25}$$

After factoring out the partial derivatives of v, there is

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)=&\frac{\partial^2 v}{\partial r^2}\left(\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right)+\frac{\partial v}{\partial r}\left(\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}\right)\\&+2\frac{\partial^2 v}{\partial\theta\partial r}\left(\frac{\partial\theta}{\partial x}\frac{\partial r}{\partial x}+\frac{\partial\theta}{\partial y}\frac{\partial r}{\partial y}\right)+\frac{\partial v}{\partial\theta}\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right)\\&+\frac{\partial^2 v}{\partial\theta^2}\left(\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2\right).\end{aligned}\tag{3.26}$$

Substituting (3.17), (3.18), (3.19) and (3.20) in where they show up, (3.26) becomes

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)=&\frac{\partial^2 v}{\partial r^2}\left(\cos^2(\theta)+\sin^2(\theta)\right)+\frac{\partial v}{\partial r}\left(\frac{\sin^2(\theta)}{r}+\frac{\cos^2(\theta)}{r}\right)\\&+2\frac{\partial^2 v}{\partial\theta\partial r}\left(\frac{-\sin(\theta)\cos(\theta)}{r}+\frac{\sin(\theta)\cos(\theta)}{r}\right)\\&+\frac{\partial v}{\partial\theta}\left(\frac{2\cos(\theta)\sin(\theta)}{r^2}-\frac{2\sin(\theta)\cos(\theta)}{r^2}\right)\\&+\frac{\partial^2 v}{\partial\theta^2}\left(\left(\frac{\sin(\theta)}{r}\right)^2+\left(\frac{\cos(\theta)}{r}\right)^2\right).\end{aligned}\tag{3.27}$$

After adding it all up and factoring out the inverse powers of r, we get

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)=&\frac{\partial^2 v}{\partial r^2}\left(\cos^2(\theta)+\sin^2(\theta)\right)+\frac{1}{r}\frac{\partial v}{\partial r}\left(\sin^2(\theta)r+\cos^2(\theta)\right)\\&+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}\left(\sin^2(\theta)+\cos^2(\theta)\right).\end{aligned}\tag{3.28}$$

As $\sin^2(\theta)+\cos^2(\theta)=1$ this reduces to

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}.\tag{3.29}$$

Thus we now have the expression of the polar coordinate type Laplacian.

CHAPTER 4

APPLICATION

4.1 Originating Idea

In [2], they were searching for a maximal solution estimate to a specific Schrödinger potential equation. However, in the process they demonstrated how to find a general solution, a process we will follow and change with the additional detail of $n = 2$.

Proposition 4.1. *The solution to*

$$\begin{cases} i\partial_t u - \Delta u + \frac{a}{|x|^2} u = 0, \quad a > 0 \\ u(x,0) = f(x), \end{cases} \tag{4.1}$$

is

$$v(r,\theta,t) = H_\nu \left[e^{it\rho^2} H_\nu a_{0,1}^0 \right] + \sum_{k=1}^{\infty} \left(H_\nu \left[e^{it\rho^2} H_\nu a_{k,1}^0 \right] \cos(k\theta) + H_\nu \left[e^{it\rho^2} H_\nu a_{k,2}^0 \right] \sin(k\theta) \right), \tag{4.2}$$

where $\nu = \sqrt{k^2 + a}$*.*

Proof. Miao, Zhang and Zheng used spherical harmonics to write

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \sum_{l=1}^{2} a_{k,l}^0(r) Y_{k,l}(\theta) \\ &= a_{0,1}^0(r) + \sum_{k=1}^{\infty} \left[a_{k,1}^0(r)\cos(k\theta) + a_{k,2}^0(r)\sin(k\theta) \right] \end{aligned} \tag{4.3}$$

where the second line is our adaption through Fourier series since we are working in 2 dimensions. They also set $a > \frac{(n-2)^2}{4}$Then we run (4.1) through (3.16), the coordinate transform into polar coordinates,

$$\begin{cases} i\partial_t v - \partial_{rr} v - \frac{1}{r}\partial_r v - \frac{1}{r^2}\partial_{\theta\theta} v + \frac{a}{r^2} v = 0, \\ v(r,\theta,0) = g(r,\theta). \end{cases} \tag{4.4}$$

As it follows from (4.3)

$$
\begin{aligned}
g(r,\theta) &= \sum_{k=0}^{\infty}\sum_{l=1}^{2} a_{k,l}^{0}(r)Y_{k,l}(\theta) \\
&= a_{0.1}^{0}(r) + \sum_{k=1}^{\infty}\left(a_{k,1}^{0}(r)\cos(k\theta) + a_{k,2}^{0}(r)\sin(k\theta)\right).
\end{aligned}
\tag{4.5}
$$

Through the orthogonality of the sine and cosine, we find

$$
\begin{aligned}
a_{k,l}^{0}(r) &= C_{k,l}\int_{0}^{2\pi} g(r,\theta)Y_{k,l}(\theta)d\theta \\
&= \begin{cases}
\frac{1}{2\pi}\int_{0}^{2\pi} g(r,\theta)d\theta, & l=1, k=0, \\
\frac{1}{\pi}\int_{0}^{2\pi} g(r,\theta)\cos(k\theta)d\theta, & l=1, k>0, \\
\frac{1}{\pi}\int_{0}^{2\pi} g(r,\theta)\sin(k\theta)d\theta, & l=2.
\end{cases}
\end{aligned}
\tag{4.6}
$$

Though separation of variables, we construct the following from (4.5)

$$
\begin{aligned}
v(r,\theta,t) &= \sum_{k=0}^{\infty}\sum_{l=1}^{2} v_{k,l}(r,t)Y_{k,l}(\theta) \\
&= v_{0.1}(r,t) + \sum_{k=1}^{\infty}\left(v_{k,1}(r,t)\cos(k\theta) + v_{k,2}(r,t)\sin(k\theta)\right),
\end{aligned}
\tag{4.7}
$$

where $v_{k,l}(r,t)$ is given by

$$
\begin{cases}
i\partial_t v_{k,l} - \partial_{rr}v_{k,l} - \frac{1}{r}\partial_r v_{k,l} + \frac{k^2+a}{r^2}v_{k,l} = 0, \\
v_{k,l}(r,0) = a_{k,l}^{0}(r),
\end{cases}
\tag{4.8}
$$

and was found by utilizing the orthogonality of sine and cosine. Let $\nu = \sqrt{k^2+a}$. Then we see

$$
\begin{cases}
i\partial_t v_{k,l} + A_\nu v_{k,l} = 0, \\
v_{k,l}(r,0) = a_{k,l}^{0}(r).
\end{cases}
\tag{4.9}
$$

Normally the Hankel transform changes both r and θ however because $v_{k,l}$ is a radial equation, we can drop the angular portion of the transform and only focus on the radial distance, ρ. Let $\hat{v}_{k,l} = H_\nu v_{k,l}(\rho,t)$ and apply H_ν on (4.9)

$$
\begin{cases}
i\partial_t \hat{v}_{k,l} + H_\nu\left[A_\nu v_{k,l}\right] = 0, \\
\hat{v}_{k,l}(\rho,0) = H_\nu a_{k,l}^{0}(\rho).
\end{cases}
\tag{4.10}
$$

We see that now would be a good time to use Proposition 2.5, however we first assume that $v_{k,l}(r,t) \in L^2$ and then apply Proposition 2.5 to get

$$\begin{cases} i\partial_t \hat{v}_{k,l} + \rho^2 \hat{v}_{k,l} = 0, \\ \hat{v}_{k,l}(\xi,t) = H_\nu a^0_{k,l}(\rho). \end{cases} \tag{4.11}$$

Solving the ODE presented in (4.11),

$$i\partial_t \hat{v}_{k,l}(\rho,t) = -\rho^2 \hat{v}_{k,l}(\rho,t) \tag{4.12}$$

$$\partial_t \hat{v}_{k,l}(\rho,t) = i\rho^2 \hat{v}_{k,l}(\rho,t) \tag{4.13}$$

$$\int \partial_t \hat{v}_{k,l}(\rho,t)dt = \int i\rho^2 \hat{v}_{k,l}(\rho,t)dt \tag{4.14}$$

$$\hat{v}_{k,l}(\rho,t) = \int i\rho^2 \hat{v}_{k,l}(\rho,t)dt \tag{4.15}$$

$$\hat{v}_{k,l}(\rho,t) = B_{k,l}(\rho)e^{i\rho^2 t}. \tag{4.16}$$

Once we use the initial condition from (4.11) we have our solution for $\hat{v}_{k,l}(\rho,t)$

$$\hat{v}_{k,l}(\xi,t) = e^{it\rho^2}\left[H_\nu a^0_{k,l}\right](\rho). \tag{4.17}$$

We then reverse the transformation to find $v_{k,l}(r,t)$

$$\begin{aligned} v_{k,l}(r,t) &= H_\nu\left[e^{it\rho^2} H_\nu a^0_{k,l}\right] \\ &= \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left[H_\nu a^0_{k,l}\right](\rho)d\rho \\ &= \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left(\int_0^\infty rJ_\nu(r\rho)a^0_{k,l}(r)dr\right)d\rho \\ &= \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left(\int_0^\infty rJ_\nu(r\rho)\left(C_{k,l}\int_0^{2\pi} g(r,\theta)Y_{k,l}(\theta)d\theta\right)dr\right)d\rho \\ &= \begin{cases} \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left(\int_0^\infty rJ_\nu(r\rho)\left(\frac{1}{2\pi}\int_0^{2\pi} g(r,\theta)d\theta\right)dr\right)d\rho, & l=1, k=0, \\ \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left(\int_0^\infty rJ_\nu(r\rho)\left(\frac{1}{\pi}\int_0^{2\pi} g(r,\theta)\cos(k\theta)d\theta\right)dr\right)d\rho, & l=1, k>0, \\ \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left(\int_0^\infty rJ_\nu(r\rho)\left(\frac{1}{\pi}\int_0^{2\pi} g(r,\theta)\sin(k\theta)d\theta\right)dr\right)d\rho, & l=2. \end{cases} \end{aligned} \tag{4.18}$$

Thus,

$$v(r,\theta,t) = H_\nu\left[e^{it\rho^2}H_\nu a^0_{0,1}\right] + \sum_{k=1}^{\infty}\left(H_\nu\left[e^{it\rho^2}H_\nu a^0_{k,1}\right]\cos(k\theta) + H_\nu\left[e^{it\rho^2}H_\nu a^0_{k,2}\right]\sin(k\theta)\right) \tag{4.19}$$

is the solution. □

4.2 Constant Potential

Before any unique potentials, we first observe the constant potential. There are two notable changes to the solution of Proposition 4.1. The first is the multipication of e^{iat} on everything. The second is that the order of the Hankel transforms is now k instead of $\sqrt{k^2+a}$.

Proposition 4.2. *The solution to*

$$\begin{cases} i\partial_t u - \Delta u + au = 0, \\ u(x,0) = f(x), \end{cases} \tag{4.20}$$

is

$$v(r,\theta,t) = e^{ait}H_k\left[e^{i\rho^2 t}H_k a^0_{0,1}\right] + e^{ait}\sum_{k=1}^{\infty}\left(H_k\left[e^{it\rho^2}H_k a^0_{k,1}\right]\cos(k\theta) + H_k\left[e^{it\rho^2}H_k a^0_{k,2}\right]\sin(k\theta)\right). \tag{4.21}$$

Proof. The proof is similar to the proof for Proposition 4.1. □

4.3 Inverse Power Potential

4.3.1 $r^{-2-\varepsilon}$

We set up an equation similar to before with one difference

$$\begin{cases} i\partial_t u - \Delta u + \frac{a}{r^{2+\varepsilon}}u, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+, a > 0, 1 > \varepsilon > 0, \\ u(x,0) = f(x), \end{cases} \tag{4.22}$$

and that difference is the potential. Instead of a simple inverse-square potential, there is something more. Because of this and the fact that ε is not an integer, we simply call it an inverse-power potential. We then use the same beginning steps that we used to prove Proposition 4.1 and arrive at

$$\begin{cases} i\partial_t v_{k,l} - \partial_{rr} v_{k,l} - \frac{1}{r}\partial_r v_{k,l} + \frac{k^2}{r^2} v_{k,l} + \frac{a}{r^{2+\varepsilon}} v_{k,l} = 0, \quad (r,t) \in \mathbb{R}^+ \times \mathbb{R}^+, a > 0, 1 > \varepsilon > 0, \\ v_{k,l}(r,0) = a^0_{k,l}(r). \end{cases} \tag{4.23}$$

From here we tried a few different ways that would all use the Hankel transform.

Attempt: A_ν

For the first attempt we chose to try and use A_ν like we did for (4.9). However to do so, requires that we add zero

$$\begin{cases} i\partial_t v_{k,l} - \partial_{rr} v_{k,l} - \frac{1}{r}\partial_r v_{k,l} + \frac{k^2}{r^2} v_{k,l} + \frac{a}{r^{2+\varepsilon}} v_{k,l} + \frac{a}{r^2} v_{k,l} - \frac{a}{r^2} v_{k,l} = 0, \\ v_{k,l}(r,0) = a^0_{k,l}(r). \end{cases} \tag{4.24}$$

Now we can replace the requisite parts with A_ν

$$\begin{cases} i\partial_t v_{k,l} + A_\nu v_{k,l} + \frac{a}{r^{2+\varepsilon}} v_{k,l} - \frac{a}{r^2} v_{k,l} = 0, \\ v_{k,l}(r,0) = a^0_{k,l}(r), \end{cases} \tag{4.25}$$

and then rearranging the last two terms into one term

$$\begin{cases} i\partial_t v_{k,l} + A_\nu v_{k,l} + \frac{a(1-r^\varepsilon)}{r^{2+\varepsilon}} v_{k,l} = 0, \\ v_{k,l}(r,0) = a^0_{k,l}(r). \end{cases} \tag{4.26}$$

Now we apply the Hankel transform use (2.5). Call $\hat{v}_{k,l} = H_\nu v_{k,l}$ then

$$\begin{cases} i\partial_t \hat{v}_{k,l} + \rho^2 \hat{v}_{k,l} + H_\nu \left[\frac{a(1-r^\varepsilon)}{r^{2+\varepsilon}} v_{k,l}\right] = 0, \\ \hat{v}_{k,l}(\rho,0) = H_\nu a^0_{k,l}(\rho). \end{cases} \tag{4.27}$$

All that we need is some substitution that will easily replace $H_\nu\left[\frac{a(1-r^\varepsilon)}{r^{2+\varepsilon}}v_{k,l}\right]$ with some function with $\hat{v}_{k,l}$ in it rather than the mess. So, let us focus on that piece. First we note that the constant a can be extracted:

$$H_\nu\left[\frac{a(1-r^\varepsilon)}{r^{2+\varepsilon}}v_{k,l}\right] = aH_\nu\left[\frac{1-r^\varepsilon}{r^{2+\varepsilon}}v_{k,l}\right]. \tag{4.28}$$

With the constant out of the way, we can begin dissecting the expression

$$H_\nu\left[\frac{1-r^\varepsilon}{r^{2+\varepsilon}}v_{k,l}\right] = \int_0^\infty J_\nu(r\rho)\frac{1-r^\varepsilon}{r^{2+\varepsilon}}v_{k,l}(r,t)rdr, \tag{4.29}$$

using integration by parts, with $u = J_\nu(r\rho)v_{k,l}(r,t)$,

$$\begin{aligned} H_\nu\left[\frac{1-r^\varepsilon}{r^{2+\varepsilon}}v_{k,l}\right] = &\left[\left(\frac{r^\varepsilon-1}{\varepsilon r^\varepsilon}+\ln\left(\frac{1}{x}\right)\right)v_{k,l}(r,t)J_\nu(r\rho)\right]_0^\infty \\ &-\int_0^\infty\left(\frac{r^\varepsilon-1}{\varepsilon r^\varepsilon}+\ln\left(\frac{1}{x}\right)\right)\left(\frac{\partial v_{k,l}(r,t)}{\partial r}J_\nu(r\rho)+v_{k,l}(\rho,t)\frac{\partial J_\nu(r\rho)}{\partial r}\right)dr. \end{aligned} \tag{4.30}$$

This shows us that this result will not get simpler as we try to go further. This is before we consider that the first portion in brackets will not simplify to zero or a constant.

Attempt: H_k

This time we try with $\nu = k$ instead of $\nu = \sqrt{k^2+a}$. Thus (4.23) becomes

$$\begin{cases} i\partial_t v_{k,l} + A_k v_{k,l} + \frac{a}{r^{2+\varepsilon}}v_{k,l} = 0, \\ v_{k,l}(r,0) = a^0_{k,l}(r). \end{cases} \tag{4.31}$$

Then based on Proposition 2.5, when we apply a Hankel transform of order k to 4.31 we get

$$\begin{cases} i\partial_t \widetilde{v}_{k,l} + \rho^2\widetilde{v}_{k,l} + H_k\left[\frac{a}{r^{2+\varepsilon}}v_{k,l}\right] = 0, \\ \widetilde{v}_{k,l}(\rho,0) = H_k a^0_{k,l}(\rho), \end{cases} \tag{4.32}$$

where $\widetilde{v}_{k,l} = H_k v_{k,l}$. Focusing on the relatively unchanged portion of 4.32, we again can extract the constant:

$$H_k\left(\frac{av_{k,l}(r,t)}{r^{2+\varepsilon}}\right)(\xi) = aH_k\left(\frac{v_{k,l}(r,t)}{r^{2+\varepsilon}}\right)(\xi). \tag{4.33}$$

We can begin to break it down by definition,

$$H_k\left(\frac{v_{k,l}(r,t)}{r^{2+\varepsilon}}\right)(\xi) = \int_0^\infty \frac{v_{k,l}(r,t)}{r^{2+\varepsilon}} J_k(r|\xi|)rdr = \int_0^\infty \frac{v_{k,l}(r,t)J_k(r|\xi|)}{r^{1+\varepsilon}}dr, \tag{4.34}$$

using integration by parts, with $dv = \frac{dr}{r^{1+\varepsilon}}$,

$$H_k\left(\frac{v_{k,l}(r,t)}{r^{2+\varepsilon}}\right)(\xi) = \left[\frac{-1}{\varepsilon r^\varepsilon} J_k(r\rho)v_{k,l}(r,t)\right]_0^\infty + \int_0^\infty \frac{1}{\varepsilon r^\varepsilon}\left(\rho J_k'(r\rho)v_{k,l}(r,t) + J_k(r\rho)\frac{\partial v_{k,l}}{\partial r}(r,t)\right)dr, \tag{4.35}$$

and again using integration by parts, with $dv = \frac{dr}{\varepsilon r^\varepsilon}$

$$\begin{aligned} H_k\left(\frac{v_{k,l}(r,t)}{r^{2+\varepsilon}}\right)(\xi) = & \left[\frac{-1}{\varepsilon r^\varepsilon} J_k(r\rho)v_{k,l}(r,t) + \frac{r^{1-\varepsilon}}{\varepsilon(1-\varepsilon)}(\rho J_k'(r\rho)v_{k,l}(r,t) + J_k(r\rho)\frac{\partial v_{k,l}}{\partial r}(r,t))\right]_0^\infty \\ & - \int_0^\infty \frac{r^{1-\varepsilon}}{\varepsilon(1-\varepsilon)}(\rho^2 J_k''(r\rho)v_{k,l}(r,t) + 2\rho J_k'(r\rho)\frac{\partial v_{k,l}}{\partial r}(r,t) + J_k(r\rho)\frac{\partial^2 v_{k,l}}{\partial r^2}(r,t))dr. \end{aligned} \tag{4.36}$$

The second half of the portion to be evaluated contains r in the numerator as well as the L^2 function $v_{k,l}$, thus it cancels out. We are left with

$$\begin{aligned} = & \left[\frac{-1}{\varepsilon r^\varepsilon} J_k(r\rho)v_{k,l}(r,t)\right]_0^\infty - \int_0^\infty \frac{r^{1-\varepsilon}}{\varepsilon(1-\varepsilon)}\Bigg(\rho^2 J_k''(r\rho)v_{k,l}(r,t) \\ & + 2\rho J_k'(r\rho)\frac{\partial v_{k,l}}{\partial r}(r,t) + J_k(r\rho)\frac{\partial^2 v_{k,l}}{\partial r^2}(r,t)\Bigg)dr \end{aligned} \tag{4.37}$$

and again we see that the answer cannot be found directly.

4.3.2 $r^{-2+\varepsilon}$

We decide to move along to finding a solution for the second inverse power potential,

$$\begin{cases} i\partial_t u - \Delta u + \frac{a}{r^{2-\varepsilon}}u, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+, a > 0, 1 > \varepsilon > 0, \\ u(x,0) = f(x). \end{cases} \tag{4.38}$$

Instead of adding epsilon, we now subtract it. We then use the same beginning steps that we used to show Equation 4.23 and arrive at

$$\begin{cases} i\partial_t v_{k,l} - \partial_{rr} v_{k,l} - \frac{1}{r}\partial_r v_{k,l} + \frac{k^2}{r^2} v_{k,l} + \frac{a}{r^{2-\varepsilon}} v_{k,l} = 0, \quad (r,t) \in \mathbb{R}^+ \times \mathbb{R}^+, a > 0, 1 > \varepsilon > 0, \\ v_{k,l}(r,0) = a^0_{k,l}(r). \end{cases} \tag{4.39}$$

Because of how it played earlier, we shall use H_k, rather than H_ν, and reveal, by Proposition 2.5,

$$\begin{cases} i\partial_t \widetilde{v}_{k,l} + \rho^2 \widetilde{v}_{k,l} + aH_k\left[\frac{1}{r^{2-\varepsilon}} v_{k,l}\right] = 0, \\ \widetilde{v}_{k,l}(\rho,0) = H_k a^0_{k,l}(\rho), \end{cases} \tag{4.40}$$

where $\widetilde{v}_{k,l} = H_k v_{k,l}$. As before we again focus on finding a substitution for $H_k\left[\frac{1}{r^{2-\varepsilon}} v_{k,l}\right]$.

$$H_k\left[\frac{1}{r^{2-\varepsilon}} v_{k,l}\right] = \int_0^\infty \frac{1}{r^{1-\varepsilon}} v_{k,l}(r,t) J_k(r\rho) dr. \tag{4.41}$$

Using integration by parts, with $u = v_{k,l}(r,t)J_k(r\rho)$,

$$H_k\left[\frac{1}{r^{2-\varepsilon}} v_{k,l}\right] = \left[\frac{r^\varepsilon}{\varepsilon} v_{k,l}(r,t) J_k(r\rho)\right]_0^\infty - \int_0^\infty \frac{r^\varepsilon}{\varepsilon}\left(\frac{\partial v_{k,l}}{\partial r}(r,t) J_k(r\rho) + \rho v_{k,l}(r,t) J'_k(r\rho)\right) dr. \tag{4.42}$$

Unlike before, the first term will evaluate to zero,

$$H_k\left[\frac{1}{r^{2-\varepsilon}} v_{k,l}\right] = -\int_0^\infty \frac{r^\varepsilon}{\varepsilon}\left(\frac{\partial v_{k,l}}{\partial r}(r,t) J_k(r\rho) + \rho v_{k,l}(r,t) J'_k(r\rho)\right) dr, \tag{4.43}$$

so we try another round of integration by parts with $dv = \frac{r^\varepsilon}{\varepsilon} dr$:

$$\begin{aligned} H_k\left[\frac{1}{r^{2-\varepsilon}} v_{k,l}\right] = &-\left[\frac{r^\varepsilon}{\varepsilon}\left(\frac{\partial v_{k,l}}{\partial r}(r,t) J_k(r\rho) + \rho v_{k,l}(r,t) J'_k(r\rho)\right)\right] \\ &+ \int_0^\infty \frac{r^{\varepsilon+1}}{\varepsilon^2+\varepsilon}\left(\frac{\partial^2 v_{k,l}}{\partial r^2}(r,t) J_k(r\rho) + 2\rho\frac{\partial v_{k,l}}{\partial r}(r,t) J'_k(r\rho) + \rho^2 v_{k,l}(r,t) J''_k(r\rho)\right) dr. \end{aligned} \tag{4.44}$$

Again the first part cancels to zero

$$H_k\left[\frac{1}{r^{2-\varepsilon}}v_{k,l}\right]=\int_0^\infty \frac{r^{\varepsilon+1}}{\varepsilon^2+\varepsilon}\left(\frac{\partial^2 v_{k,l}}{\partial r^2}(r,t)J_k(r\rho)+2\rho\frac{\partial v_{k,l}}{\partial r}(r,t)J_k'(r\rho)+\rho^2 v_{k,l}(r,t)J_k''(r\rho)\right)dr, \tag{4.45}$$

even if the rest will not simplify.

4.3.3 Possible Replacements

As mentioned in Chapter 2, there are two forms of the Hankel transform. The tables of integral transforms collected in the Bateman Manuscript Project use the form shown in Equations (2.61) and (2.62) when covering Hankel transforms. In order to accurately utilize the information of that source, we must either change which Hankel transform pattern we have been using or alter the transformations to align with our work. Since altering the transformations means we can keep the same notation as before, that is the path we shall tread. The listing that is going to be used is the eighth general formula and has a condition that must be met first. The transform begins from

$$f(r)r^{-\mu}, \tag{4.46}$$

and with the condition that

$$\mathrm{Re}\nu+1>\mathrm{Re}\mu>0,$$

or that the real part of ν, the order of the transformation, plus one is greater than the real part of μ. Applying Ha or the alternate Hankel Transform the result, as seen in the Bateman integral tables, is

$$2^{1-\mu}[\Gamma(\mu)]^{-1}\rho^{\frac{1}{2}-\nu}\times\int_0^\rho \eta^{\nu-\mu+\frac{1}{2}}(\rho^2-\eta^2)^{\mu-1}g(\eta;\nu-\mu)d\eta, \tag{4.47}$$

where $g(\rho;\nu)=Ha_\nu[f(r)]$ [8]. As we have shown, for the potentials $\frac{a}{r^{-2\pm\varepsilon}}$, finding solutions is a little tricky because of the method we have chosen to attempt to follow. However we now have a possible way of evaluating the third term that would not simplify or reduce earlier. To begin we must first make sure that we do the appropriate substitutions. First we need

$$f(r,t) = v_{k,l}(r,t)r^{\frac{1}{2}}, \tag{4.48}$$

along with

$$g(\rho,t;k) = Ha_k[f(r,t)](\rho,t) = \rho^{\frac{1}{2}}H_k[v_{k,l}(r,t)](\rho,t). \tag{4.49}$$

Finally we need the correct setup

$$\rho^{\frac{1}{2}}H_k\left[\frac{v_{k,l}(r,t)}{r^{2\mp\varepsilon}}\right] = Ha_k\left[r^{\frac{1}{2}}\frac{v_{k,l}(r,t)}{r^{2\mp\varepsilon}}\right], \tag{4.50}$$

or else we would not be approaching the structure properly. Now, we see that,

$$f(r,t)r^{-\mu} = v_{k,l}(r,t)r^{\frac{1}{2}}r^{-\mu} = v_{k,l}r^{\frac{1}{2}}r^{-2\mp\varepsilon} \tag{4.51}$$

Thus we see that $\mu = 2\pm\varepsilon$ and due to the condition of the formula, the following cases occur:

k	$2+\varepsilon$	$2-\varepsilon$
0	Condition Failed	Condition Failed
1	Condition Failed	Condition Met
$2 \geq$	Condition Met	Condition Met

As can be seen, there is no representation for $k = 0$ either way. Even so, it does give a representation for all of the k orders of 2 and beyond. Now we use Equation (4.47),

$$Ha_k\left[r^{\frac{1}{2}}\frac{v_{k,l}(r,t)}{r^{2\mp\varepsilon}}\right] = 2^{1-(2\pm\varepsilon)}[\Gamma(2\pm\varepsilon)]^{-1}\rho^{\frac{1}{2}-k} \times \int_0^{\rho} \eta^{k-(2\pm\varepsilon)+\frac{1}{2}}(\rho^2-\eta^2)^{2\pm\varepsilon-1}g(\eta,t;k-(2\pm\varepsilon))\,d\eta,$$

and multiply the negative signs throughout,

$$Ha_k\left[r^{\frac{1}{2}}\frac{v_{k,l}(r,t)}{r^{2\mp\varepsilon}}\right] = 2^{-1\mp\varepsilon}[\Gamma(2\pm\varepsilon)]^{-1}\rho^{\frac{1}{2}-k} \times \int_0^{\rho} \eta^{k-\frac{3}{2}\mp\varepsilon}(\rho^2-\eta^2)^{1\pm\varepsilon}g(\eta,t;k-2\mp\varepsilon)\,d\eta.$$

Next we replace g with the alternate Hankel transform,

$$Ha_k\left[r^{\frac{1}{2}}\frac{v_{k,l}(r,t)}{r^{2\mp\varepsilon}}\right]=2^{-1\mp\varepsilon}[\Gamma(2\pm\varepsilon)]^{-1}\rho^{\frac{1}{2}-k}\times\int_0^\rho \eta^{k-\frac{3}{2}\mp\varepsilon}(\rho^2-\eta^2)^{1\pm\varepsilon}Ha_{k-2\mp\varepsilon}\left[v_{k,l}r^{\frac{1}{2}}\right](\eta,t)d\eta.$$

Then replace the alternate transform with regular transform,

$$Ha_k\left[r^{\frac{1}{2}}\frac{v_{k,l}(r,t)}{r^{2\mp\varepsilon}}\right]=2^{-1\mp\varepsilon}[\Gamma(2\pm\varepsilon)]^{-1}\rho^{\frac{1}{2}-k}\times\int_0^\rho \eta^{k-\frac{3}{2}\mp\varepsilon}(\rho^2-\eta^2)^{1\pm\varepsilon}H_{k-2\mp\varepsilon}\left[v_{k,l}\right](\eta,t)\eta^{\frac{1}{2}}d\eta,$$

and combine the powers of η,

$$Ha_k\left[r^{\frac{1}{2}}\frac{v_{k,l}(r,t)}{r^{2\mp\varepsilon}}\right]=2^{-1\mp\varepsilon}[\Gamma(2\pm\varepsilon)]^{-1}\rho^{\frac{1}{2}-k}\times\int_0^\rho \eta^{k-1\mp\varepsilon}(\rho^2-\eta^2)^{1\pm\varepsilon}H_{k-2\mp\varepsilon}\left[v_{k,l}\right](\eta,t)d\eta,$$

and finally remove one $\rho^{\frac{1}{2}}$ to find the result for the transform to get

$$H_k\left[\frac{v_{k,l}(r,t)}{r^{2\mp\varepsilon}}\right]=\frac{2^{-1\mp\varepsilon}}{\rho^k\Gamma(2\pm\varepsilon)}\times\int_0^\rho \eta^{k-1\mp\varepsilon}(\rho^2-\eta^2)^{1\pm\varepsilon}H_{k-2\mp\varepsilon}\left[v_{k,l}\right](\eta,t)d\eta. \tag{4.52}$$

Now we have a replacement for $H_k\left[\frac{v_{k,l}(r,t)}{r^{2\mp\varepsilon}}\right]$. However, we were trying to solve

$$i\partial_t H_k\left[v_{k,l}\right]+\rho^2 H_k\left[v_{k,l}\right]+aH_k\left[\frac{v_{k,l}(r,t)}{r^{2\mp\varepsilon}}\right]=0 \tag{4.53}$$

through replacing the final term. We can replace that term now, however the replacement makes it apparent that using the Hankel transform to find a solution of an inverse power potential is not quite possible at the current time.

4.4 Complex Potential

This is the last potential that was examined.

$$\begin{cases} i\partial_t u-\Delta u+\frac{ae^{i\phi}}{r^2}u=0, \quad a>0,-\pi<\phi<\pi, \\ u(x,0)=f(x). \end{cases} \tag{4.54}$$

The restrictions on a and ϕ are because a and ϕ are later used to define ν and these restrictions will keep the real part of ν greater than 0. The reason we choose to keep the real part of the order greater than zero is because it follows along with the paper by Miao, Zhang and Zheng in which they keep $a > 0$ for simplicity even though they could have said to let $a > -\frac{1}{2}$ [2]. As ν will be a complex number, thus leading to a complex order Hankel transformation, we can refer to Baruch's paper and see that we are also have $-\frac{1}{2}$ as our boundary, but only for the real portion of the order. A second restriction that we will hold to is to only use the primary root of a complex number, this will also keep the real part of ν greater than zero. We proceed like when the potential was just $\frac{a}{r^2}$.

Proposition 4.3. *The solution to*

$$\begin{cases} i\partial_t u - \Delta u + \frac{ae^{i\phi}}{|x|^2} u = 0, \quad a > 0, -\pi < \phi < \pi, \\ u(x,0) = f(x), \end{cases} \tag{4.55}$$

is

$$v(r,\theta,t) = H_\nu \left[e^{it\rho^2} H_\nu a^0_{0,1} \right] + \sum_{k=1}^{\infty} \left(H_\nu \left[e^{it\rho^2} H_\nu a^0_{k,1} \right] \cos(k\theta) + H_\nu \left[e^{it\rho^2} H_\nu a^0_{k,2} \right] \sin(k\theta) \right), \tag{4.56}$$

where $\nu = \sqrt{k^2 + ae^{i\phi}}$

Proof. As we mentioned, we start with;

$$f(x) = a^0_{0,1}(r) + \sum_{k=1}^{\infty} \left[a^0_{k,1}(r) \cos(k\theta) + a^0_{k,2}(r) \sin(k\theta) \right]. \tag{4.57}$$

Then we run (4.55) through (3.16), the coordinate transform into polar coordinates,

$$\begin{cases} i\partial_t v - \partial_{rr} v - \frac{1}{r}\partial_r v - \frac{1}{r^2}\partial_{\theta\theta} v + \frac{ae^{i\phi}}{r^2} v = 0, \\ v(r,\theta,0) = g(r,\theta). \end{cases} \tag{4.58}$$

As it follows from (4.57)

$$g(r,\theta) = a^0_{0.1}(r) + \sum_{k=1}^{\infty} \left(a^0_{k,1}(r) \cos(k\theta) + a^0_{k,2}(r) \sin(k\theta) \right). \tag{4.59}$$

Through the orthogonality of the sine and cosine, we find

$$
\begin{aligned}
a^0_{k,l}(r) &= C_{k,l}\int_0^{2\pi} g(r,\theta)Y_{k,l}(\theta)d\theta \\
&= \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} g(r,\theta)d\theta, & l=1, k=0, \\ \frac{1}{\pi}\int_0^{2\pi} g(r,\theta)\cos(k\theta)d\theta, & l=1, k>0, \\ \frac{1}{\pi}\int_0^{2\pi} g(r,\theta)\sin(k\theta)d\theta, & l=2. \end{cases}
\end{aligned} \tag{4.60}
$$

Though separation of variables, we construct the following from (4.59)

$$
\begin{aligned}
v(r,\theta,t) &= \sum_{k=0}^{\infty}\sum_{l=1}^{2} v_{k,l}(r,t)Y_{k,l}(\theta) \\
&= v_{0.1}(r,t) + \sum_{k=1}^{\infty}\left(v_{k,1}(r,t)\cos(k\theta) + v_{k,2}(r,t)\sin(k\theta)\right),
\end{aligned} \tag{4.61}
$$

where $v_{k,l}(r,t)$ is given by

$$
\begin{cases} i\partial_t v_{k,l} - \partial_{rr} v_{k,l} - \frac{1}{r}\partial_r v_{k,l} + \frac{k^2+ae^{i\phi}}{r^2} v_{k,l} = 0, \\ v_{k,l}(r,0) = a^0_{k,l}(r), \end{cases} \tag{4.62}
$$

and was found by utilizing the orthogonality of sine and cosine. Let $\nu = \sqrt{k^2+ae^{i\phi}}$. Then we see

$$
\begin{cases} i\partial_t v_{k,l} + A_\nu v_{k,l} = 0, \\ v_{k,l}(r,0) = a^0_{k,l}(r). \end{cases} \tag{4.63}
$$

Normally the Hankel transform changes both r and θ however because $v_{k,l}$ is a radial equation, we can drop the angular portion of the transform and only focus on the radial distance, ρ. Let $\hat{v}_{k,l} = H_\nu v_{k,l}(\rho,t)$ and apply H_ν on (4.63)

$$
\begin{cases} i\partial_t \hat{v}_{k,l} + H_\nu\left[A_\nu v_{k,l}\right] = 0, \\ \hat{v}_{k,l}(\rho,0) = H_\nu a^0_{k,l}(\rho). \end{cases} \tag{4.64}
$$

We see that now would be a good time to use Proposition 2.5, however we first assume that

$v_{k,l}(r,t) \in L^2$ and then apply Proposition 2.5 to get

$$\begin{cases} i\partial_t \hat{v}_{k,l} + \rho^2 \hat{v}_{k,l} = 0, \\ \hat{v}_{k,l}(\xi,t) = H_\nu a^0_{k,l}(\rho). \end{cases} \tag{4.65}$$

Solving the ODE presented in (4.65),

$$i\partial_t \hat{v}_{k,l}(\rho,t) = -\rho^2 \hat{v}_{k,l}(\rho,t) \tag{4.66}$$

$$\partial_t \hat{v}_{k,l}(\rho,t) = i\rho^2 \hat{v}_{k,l}(\rho,t) \tag{4.67}$$

$$\int \partial_t \hat{v}_{k,l}(\rho,t)dt = \int i\rho^2 \hat{v}_{k,l}(\rho,t)dt \tag{4.68}$$

$$\hat{v}_{k,l}(\rho,t) = \int i\rho^2 \hat{v}_{k,l}(\rho,t)dt \tag{4.69}$$

$$\hat{v}_{k,l}(\rho,t) = B_{k,l}(\rho)e^{i\rho^2 t}. \tag{4.70}$$

Once we use the initial condition from (4.65) we have our solution for $\hat{v}_{k,l}(\rho,t)$

$$\hat{v}_{k,l}(\xi,t) = e^{it\rho^2}\left[H_\nu a^0_{k,l}\right](\rho). \tag{4.71}$$

We then reverse the transformation to find $v_{k,l}(r,t)$

$$\begin{aligned} v_{k,l}(r,t) &= H_\nu\left[e^{it\rho^2} H_\nu a^0_{k,l}\right] \\ &= \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left[H_\nu a^0_{k,l}\right](\rho)d\rho \\ &= \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left(\int_0^\infty rJ_\nu(r\rho)a^0_{k,l}(r)dr\right)d\rho \\ &= \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left(\int_0^\infty rJ_\nu(r\rho)\left(C_{k,l}\int_0^{2\pi} g(r,\theta)Y_{k,l}(\theta)d\theta\right)dr\right)d\rho \\ &= \begin{cases} \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left(\int_0^\infty rJ_\nu(r\rho)\left(\frac{1}{2\pi}\int_0^{2\pi} g(r,\theta)d\theta\right)dr\right)d\rho, & l=1, k=0, \\ \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left(\int_0^\infty rJ_\nu(r\rho)\left(\frac{1}{\pi}\int_0^{2\pi} g(r,\theta)\cos(k\theta)d\theta\right)dr\right)d\rho, & l=1, k>0, \\ \int_0^\infty \rho J_\nu(r\rho)e^{it\rho^2}\left(\int_0^\infty rJ_\nu(r\rho)\left(\frac{1}{\pi}\int_0^{2\pi} g(r,\theta)\sin(k\theta)d\theta\right)dr\right)d\rho, & l=2. \end{cases} \end{aligned} \tag{4.72}$$

Thus,

$$v(r,\theta,t) = H_\nu\left[e^{it\rho^2}H_\nu a^0_{0,1}\right] + \sum_{k=1}^{\infty}\left(H_\nu\left[e^{it\rho^2}H_\nu a^0_{k,1}\right]\cos(k\theta) + H_\nu\left[e^{it\rho^2}H_\nu a^0_{k,2}\right]\sin(k\theta)\right) \tag{4.73}$$

is the solution. □

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