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Homogeneous Symplectic Manifolds of the Galilei Group

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HOMOGENEOUS SYMPLECTIC MANIFOLDS OF THE GALILEI GROUP

by

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(Under the Direction of François Ziegler)

ABSTRACT

In this Thesis we classify all symplectic manifolds admitting a transitive, 2-form preserving action of the Galilei group G. Using the moment map and a theorem of Kirillov-Kostant-Souriau, we reduce the problem to that of classifying the coadjoint orbits of a central extension of G discovered by Bargmann. We then develop a systematic inductive technique to construct a cross section of the coadjoint action. The resulting symplectic orbits are interpreted as the manifolds of classical motions of elementary particles with or without spin, mass, and color.

INDEX WORDS: Symplectic manifold, coadjoint orbit, differential 2-form, Galilei group.

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CHAPTER 1
INTRODUCTION

1.1 The Problem

Physicists have known for a long time that every elementary particle carries two characteristic numbers known as its *mass* and *spin*. This knowledge is the result of an analysis of *symmetry*, namely, the classification by E. Wigner in 1939 [17] of the "irreducible projective unitary representations of the Poincaré group". This group is the symmetry group of space-time according to special relativity, and its irreducible unitary representations are are regarded as all possible "quantum models" for elementary particles with Poincaré symmetry.

Nowadays particles carry are extra characteristic numbers (*isospin*, *color*, etc.) related to extra symmetry groups (SU(2), SU(3), etc.); still, all known particles fit in Wigner's scheme. Later Bargmann, Inönü and Wigner performed a similar analysis with the Galilei group in place of Poincaré's, with similar results [10, 3].

For a long time, it was believed that many of these systems—in particular those with zero mass or nonzero spin—had no classical model. But in the 1960's mathematicians realized that they actually do, provided the concept of "classical model" is given the proper generality. They found that the appropriate principles are [15]:

I. The space of all possible motions of a classical mechanical system is a *symplectic manifold* (X, σ).

II. If the system admits a Lie group G of symmetries, then there is a σ-*preserving action* of G on X.

III. If the system is elementary, then the action in II is *transitive* on X.

We will illustrate these notions by considering the simple example of a mass point in a constant gravity field, as on Earth. Newton's equations read:

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}, \qquad m\frac{d\boldsymbol{v}}{dt} = \mathbf{F} \tag{1.1}$$

with $\mathbf{F} = m\boldsymbol{a}$, $\boldsymbol{a} = constant$. Their solution is

$$\begin{cases} \boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{a}t^2 \\ \boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{a}t. \end{cases} \tag{1.2}$$

Each solution in (1.2) is a curve characterized by the pair $x = \left(\begin{smallmatrix}\boldsymbol{r}_0\\ \boldsymbol{v}_0\end{smallmatrix}\right)$. We can think of the space $\mathrm{X} = \{x = \left(\begin{smallmatrix}\boldsymbol{r}_0\\ \boldsymbol{v}_0\end{smallmatrix}\right) \mid \boldsymbol{r}_0, \boldsymbol{v}_0 \in \mathbb{R}^3\}$ of these pairs as the space of all possible motions of the system. (This is really a *chart* of it: intrinsically the space of motions in the set of all curves (1.2) in $\mathbb{R}^7 \ni (\boldsymbol{r}, \boldsymbol{v}, t)$.)

The symplectic structure of X comes from two observations of Lagrange:

(1) Newton's equations, viewed as a condition

$$\begin{cases} d\boldsymbol{r} - \boldsymbol{v}dt = 0 \\ md\boldsymbol{v} - \mathbf{F}dt = 0 \end{cases} \tag{1.3}$$

for the triple $dy = \begin{pmatrix} d\boldsymbol{r} \\ d\boldsymbol{v} \\ dt \end{pmatrix}$ to be tangent to a solution curve, is equivalent to the condition that

$$\langle md\boldsymbol{v} - \mathbf{F}dt, \delta\boldsymbol{r} - v\delta t\rangle - \langle m\delta\boldsymbol{v} - \mathbf{F}\delta t, d\boldsymbol{r} - \boldsymbol{v}dt\rangle = 0 \tag{1.4}$$

for all choices $\delta y = \begin{pmatrix} \delta\boldsymbol{r} \\ \delta\boldsymbol{v} \\ \delta t \end{pmatrix}$ of another vector in $\mathbb{R}^7$. This statement is essentially known as d'Alembert's "principle of virtual work".

(2) Writing $\sigma(dy, \delta y)$ for the quantity (1.4), Lagrange observed in addition that when we substitute the value of $y, dy, \delta y$ from (1.2), the result does not depend on $t, dt, \delta t$. To confirm this, we look at (1.2), which gives

$$\begin{cases} d\boldsymbol{r} = d\boldsymbol{r}_0 + td\boldsymbol{v}_0 + \boldsymbol{v}_0 dt + \boldsymbol{a}tdt \\ d\boldsymbol{v} = d\boldsymbol{v}_0 + \boldsymbol{a}dt, \end{cases} \tag{1.5}$$

hence

$$\begin{cases} d\boldsymbol{r} - \boldsymbol{v}dt = & d\boldsymbol{r}_0 + td\boldsymbol{v}_0 + \boldsymbol{v}_0 dt + \boldsymbol{a}tdt - (\boldsymbol{v}_0 + \boldsymbol{a}t)\, dt \\ = & d\boldsymbol{r}_0 + td\boldsymbol{v}_0 \\ md\boldsymbol{v} - \mathbf{F}dt = & m\,[d\boldsymbol{v}_0 + \boldsymbol{a}dt] - m\boldsymbol{a}dt \\ = & md\boldsymbol{v}_0. \end{cases} \tag{1.6}$$

We get comparable results for the δ's. Hence we get:

$$\begin{aligned} \sigma(dy, \delta y) = & \ \langle md\boldsymbol{v}_0, \delta\boldsymbol{r}_0 + t\delta\boldsymbol{v}_0\rangle - \langle m\delta\boldsymbol{v}_0, d\boldsymbol{r}_0 + td\boldsymbol{v}_0\rangle \\ = & \ \langle md\boldsymbol{v}_0, \delta\boldsymbol{r}_0\rangle - \langle m\delta\boldsymbol{v}_0, d\boldsymbol{r}_0\rangle + [m\langle d\boldsymbol{v}_0, \delta\boldsymbol{v}_0\rangle - m\,\langle\delta\boldsymbol{v}_0, d\boldsymbol{v}_0\rangle]\, t \\ = & \ \langle md\boldsymbol{v}_0, \delta\boldsymbol{r}_0\rangle - \langle m\delta\boldsymbol{v}_0, d\boldsymbol{r}_0\rangle \end{aligned} \tag{1.7}$$

which doesn't depend on $t, dt, \delta t$. The previous calculation was for constant gravity, but Lagrange's observation was that this elimination of $t, dt, \delta t$ always happens, for any system subject to any forces. The result is that we can regard (1.7) as a 2-form $\sigma(dx, \delta x)$ defined on the space of motions X. Moreover it is *nondegenerate* and *closed* ($d\sigma = 0$), thus giving X a symplectic structure. This is an example of Principle I from above and (X, σ) is our symplectic manifold.

To exemplify Principles II and III we specialize further to the case $\boldsymbol{a} = 0$ of a free mass point. We expect symmetry under the Galilei group, which is the group of space-time transformations made of:

rotations

$$g_{\mathrm{A}} : \begin{pmatrix} \boldsymbol{r} \\ t \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{A}\boldsymbol{r} \\ t \end{pmatrix}, \quad \mathrm{A} \in \mathrm{SO}(3), \tag{1.8}$$

boosts

$$g_{\boldsymbol{b}} : \begin{pmatrix} \boldsymbol{r} \\ t \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{r} + \boldsymbol{b}t \\ t \end{pmatrix}, \quad \boldsymbol{b} \in \mathbb{R}^3 \tag{1.9}$$

space translations

$$g_{\boldsymbol{c}} : \begin{pmatrix} \boldsymbol{r} \\ t \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{r} + \boldsymbol{c} \\ t \end{pmatrix}, \quad \boldsymbol{c} \in \mathbb{R}^3 \tag{1.10}$$

time translations

$$g_{e} : \begin{pmatrix} \boldsymbol{r} \\ t \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{r} \\ t + e \end{pmatrix}, \quad e \in \mathbb{R}. \tag{1.11}$$

Composing them in the order $g_e \circ g_{\boldsymbol{c}} \circ g_{\boldsymbol{b}} \circ g_{\mathrm{A}}$ we get a group of transformations conveniently expressed in matrix form as

$$\begin{pmatrix} \boldsymbol{r} \\ t \\ 1 \end{pmatrix} \mapsto \underbrace{\begin{pmatrix} \mathrm{A} & \boldsymbol{b} & \boldsymbol{c} \\ & 1 & e \\ & & 1 \end{pmatrix}}_{g} \begin{pmatrix} \boldsymbol{r} \\ t \\ 1 \end{pmatrix}. \tag{1.12}$$

This group of matrices, known as the *Galilei group*, acts naturally on the set of solution curves (1.2) (with $\boldsymbol{a} = 0$ now). Indeed we can think of each solution curve as a subset ("world-line")

$$\mathrm{W}_x = \left\{ \begin{pmatrix} \boldsymbol{r}_0 + \boldsymbol{v}_0 t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}. \tag{1.13}$$

of space-time, and the action (1.12) takes each of these to another. In more detail a simple computation gives $g(\mathrm{W}_x) = \mathrm{W}_{g(x)}$ where the action $g(x)$ on the right-hand side is given by:

rotations

$$g_{\mathrm{A}} : \begin{pmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{A}\boldsymbol{r}_0 \\ \mathrm{A}\boldsymbol{v}_0 \end{pmatrix} \tag{1.14}$$

boosts

$$g_{\boldsymbol{b}} : \begin{pmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 + \boldsymbol{b} \end{pmatrix} \tag{1.15}$$

space translations

$$g_{\boldsymbol{c}} : \begin{pmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{r}_0 + \boldsymbol{c} \\ \boldsymbol{v}_0 \end{pmatrix} \tag{1.16}$$

time translations

$$g_e : \begin{pmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{r}_0 - \boldsymbol{v}_0 e \\ \boldsymbol{v}_0 \end{pmatrix}. \tag{1.17}$$

Composing, we obtain an action of $g = g_e \circ g_{\boldsymbol{c}} \circ g_{\boldsymbol{b}} \circ g_{\mathrm{A}}$, or of the group of matrices (1.12), on (X, σ):

$$g \begin{pmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{pmatrix} = \begin{pmatrix} \mathrm{A}(\boldsymbol{r}_0 - \boldsymbol{v}_0 e) + \boldsymbol{c} - \boldsymbol{b}e \\ \mathrm{A}\boldsymbol{v}_0 + \boldsymbol{b} \end{pmatrix}. \tag{1.18}$$

One readily checks that this action is transitive, and that it preserves σ in the sense that

$$\sigma(d[g(x)], \delta[g(x)]) = \sigma(dx, \delta x) \qquad \forall\, g, x, dx, \delta x. \tag{1.19}$$

Thus $\mathrm{X} = \mathbb{R}^6$ with Lagrange's 2-form σ illustrates principles I–III and is our first example of a *homogeneous symplectic manifold of the Galilei group.* It is natural to ask if the Galilei group has any other homogeneous symplectic manifolds, and if so to classify them. Such is the purpose of this thesis.

1.2 The Solution

The solution to our problem is exposed in Theorems 4.1, 4.2 and 4.4 of Chapter 4. In short, we find 9 families of homogeneous symplectic manifolds with topologies as follows:

type	topology	parameters
(1)	{a point}	c
(2)	S^2	c, s
(3)	TS^2	c, s, n
(4)	$\mathrm{TS}^2 \times \mathbb{R}^2$	k, s
(4_a)	$\mathrm{TS}^2 \times \mathrm{TS}^1$	k, s
(5)	$\mathrm{TSO}(3) \times \mathbb{R}^2$	k, n
(5_a)	$\mathrm{TSO}(3) \times \mathrm{TS}^1$	k, n
(6)	$\mathbb{R}^6$	c, m
(7)	$\mathbb{R}^6 \times \mathrm{S}^2$	c, s, m

Each topological type carries a family of symplectic structures which we compute explicitly, with parameters c (interpreted as *rest energy*), s (*spin*), k (*color*), m (*mass*) and n (no interpretation).

We note that the literature contains some antecedents to our results. The earliest work is that of Souriau [15, §14], who classified homogeneous symplectic manifolds of the Poincaré group and derived galilean models in the limit where the speed of light goes to infinity. In that work only types (4), (6) and (7) are described explicitly. Later Bez [5] stated the results of a classification including types (2), (3), (4), (5), (6), (7). However his topologies for types (3) and (4) are wrong (TS^2 is confused with $\mathrm{S}^2 \times \mathbb{R}^2$) and the 2-forms are not computed. Finally Guillemin and Sternberg [9, pp. 437–441] give a rather brisk description of the classification, with remarkable mention on p. 440 of a "five-dimensional symplectic manifold" (!) but without computation of all the symplectic forms. We may describe our contribution as 1° the much more systematic and explicit calculation of a cross section for the coadjoint action (Theorem 4.1), 2° the complete calculation of all 2-forms (Theorem 4.2), and 3° the description of classes (4_a) and (5_a) (Theorem 4.4).

CHAPTER 2
ELEMENTS OF DIFFERENTIAL GEOMETRY

2.1 Differential Calculus

Differential calculus is the generalization of vector calculus to any number of dimensions. Its fundamental operation is the exterior derivative, d, which acts on *p-forms* and properly recasts and generalizes the familiar grad, div and curl.

A. Derivatives. By a ***numerical space***, we shall mean an open set X in some $\mathbb{R}^{\mathrm{M}}$. Its ***dimension*** is M. A map $\mathrm{F} : \mathrm{X} \to \mathrm{Y}$ between numerical spaces is called ***smooth*** if it has partial derivatives of all orders, and a ***diffeomorphism*** if it is a smooth bijection with smooth inverse.

— We may then form the ***derivative of* F *at* $\boldsymbol{x}$**, which is the linear map sending $\delta x \in \mathbb{R}^{\mathrm{M}}$ to

$$\begin{pmatrix} \delta y^1 \\ \vdots \\ \delta y^{\mathrm{N}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial y^1}{\partial x^1} & \cdots & \dfrac{\partial y^1}{\partial x^{\mathrm{M}}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial y^{\mathrm{N}}}{\partial x^1} & \cdots & \dfrac{\partial y^{\mathrm{N}}}{\partial x^{\mathrm{M}}} \end{pmatrix} \begin{pmatrix} \delta x^1 \\ \vdots \\ \delta x^{\mathrm{M}} \end{pmatrix} \in \mathbb{R}^{\mathrm{N}} \tag{2.1}$$

where $y = \mathrm{F}(x)$; here it is understood that the partial derivatives are evaluated at x. This linear map, or its matrix in (2.1), is classically denoted $\mathbf{DF}(\boldsymbol{x})$ or simply $\boldsymbol{\partial y/\partial x}$.

Thus, the expressions

$$\underset{\text{(Lagrange 1800)}}{\delta y,} \qquad \underset{\text{(Jacobi 1850)}}{\frac{\partial y}{\partial x}(\delta x),} \qquad \underset{\text{(Fréchet 1900)}}{\mathrm{DF}(x)(\delta x),} \tag{2.2}$$

all denote our vector (2.1), which can also be defined by

$$\mathrm{DF}(x)(\delta x) = \lim_{t \to 0} \frac{\mathrm{F}(x + t\delta x) - \mathrm{F}(x)}{t}. \tag{2.3}$$

When $\mathrm{M} = 1$ and we choose $\delta x = 1$, we obtain the ordinary derivative of the curve $y = \mathrm{F}(x)$, denoted

$$\frac{dy}{dx} = \mathrm{F}'(x) = \mathrm{DF}(x)(1). \tag{2.4}$$

— Now DF is a smooth map from X to the space $\mathbb{R}^{\mathrm{N \times M}}$ of all linear maps $\mathbb{R}^{\mathrm{M}} \to \mathbb{R}^{\mathrm{N}}$, which we can derive in its turn: we obtain the definition of the ***second derivative***

$\mathbf{D^2F} : \mathrm{X} \to \mathbb{R}^{\mathrm{N}\times\mathrm{M}\times\mathrm{M}}$, whose value at x sends δx to the linear map

$$\delta\Big[\frac{\partial y}{\partial x}\Big] = \frac{\partial^2 y}{\partial x^2}(\delta x) = \mathrm{D}^2\mathrm{F}(x)(\delta x); \quad \text{etc.} \tag{2.5}$$

The basic facts we shall use are the following:[1]

Proposition 2.1 (Properties of the derivative).

***Symmetry*:** $\mathrm{D}^2\mathrm{F}(x)(v)(w) = \mathrm{D}^2\mathrm{F}(x)(w)(v)$ (2.6)

***Leibniz Rule*:** $\delta[\Phi(a,..,z)] = \Phi(\delta a,..,z) + \cdots + \Phi(a,..,\delta z)$, (2.7)
if Φ is multilinear (i.e., linear in each variable)

***Chain Rule*:** $\mathrm{D}[\mathrm{G}\circ\mathrm{F}](x) = \mathrm{DG}(\mathrm{F}(x))\circ\mathrm{DF}(x)$ i.e. $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\circ\frac{\partial y}{\partial x}$ (2.8)

***Local Inversion*:** $\mathrm{DF}(x)$ is bijective $\Rightarrow$ the restriction of F to some open neighborhood of x is a diffeomorphism. (2.9)

— The converse of (2.9) is of course true, by (2.8): $\frac{\partial x}{\partial y}\circ\frac{\partial y}{\partial x} = \mathbb{1}$. Note that invertibility of the Jacobian matrix $\frac{\partial y}{\partial x}$ means that its *columns*

$$\frac{\partial y}{\partial x^1}, \ldots, \frac{\partial y}{\partial x^{\mathrm{M}}} \quad \text{make a basis of the column space } \mathbb{R}^{\mathrm{M}}, \tag{2.10}$$

while the relation $\frac{\partial x}{\partial y}\circ\frac{\partial y}{\partial x} = \mathbb{1}$ means that the *rows*

$$\frac{\partial x^1}{\partial y}, \ldots, \frac{\partial x^{\mathrm{M}}}{\partial y} \quad \text{make the dual basis of the dual row space } \overline{\mathbb{R}^{\mathrm{M}}}.^2 \tag{2.11}$$

In particular, applying this to the identity map $y = x$, we obtain the standard differential geometers' notation for the canonical bases of $\mathbb{R}^{\mathrm{M}}$ and its dual: $\frac{\partial x}{\partial x^i} = e_i$ and $\frac{\partial x^i}{\partial x} = e^i$ (the columns and rows of the identity matrix).

***Example* 1.** On the numerical space $\mathbf{GL}(\boldsymbol{n}) := \{g \in \mathbb{R}^{n\times n} \mid \det(g) \neq 0\}$, find the derivatives of $g \mapsto \overline{g}$ and $g \mapsto g^{-1}$.

Answer. (a): The transposition map $g \mapsto \overline{g}$ is linear (on $\mathbb{R}^{n\times n}$). So by the Leibniz rule (2.7), or directly (2.3), we obtain

$$\delta\overline{g} = \overline{\delta g}. \tag{2.12}$$

[1] See e.g. [1] or H. Cartan [7, Theorems 2.2.1, 2.4.3, 4.2.1, 5.1.1].

[2] Henceforth the bar $\overline{\cdot}$ shall denote transposition (of real matrices, row and column vectors).

(b): Deriving both sides of $\mathbb{1} = g \circ g^{-1}$ and using the Leibniz rule (2.7), we get $0 = \delta[g \circ g^{-1}]$ $= \delta g \circ g^{-1} + g \circ \delta[g^{-1}]$ and hence

$$\delta[g^{-1}] = -g^{-1} \circ \delta g \circ g^{-1}. \tag{2.13}$$

Note that when $n = 1$, this boils down to the familiar relation $\frac{d}{dt}[1/t] = -1/t^2$. Note also that this calculation would be rather more tedious if we stuck to the Fréchet notation (2.2).

***Example* 2.** If $x \in \mathbb{R}^{\mathrm{M}}$ is a smooth function of time and $y = \mathrm{F}(x)$, find the relation between the velocities dx/dt and dy/dt.

Answer. (2.4) tells us to evaluate both sides of $\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x} \circ \frac{\partial x}{\partial t}$ on the vector $1 \in \mathbb{R}$. This gives $\frac{dy}{dt} = \frac{\partial y}{\partial x}\left(\frac{dx}{dt}\right)$, or in other words

$$\frac{dy}{dt} = \mathrm{DF}(x)\left(\frac{dx}{dt}\right). \tag{2.14}$$

So the sought relation is $\mathrm{DF}(x)$. Note that this gives a method for computing $\mathrm{DF}(x)(\delta x)$: it is the initial velocity of the image, $t \mapsto (\mathrm{F} \circ \gamma)(t)$, of *any* smooth curve $t \mapsto \gamma(t)$ with initial velocity $\gamma'(0) = \delta x$. One advantage over (2.3), which uses the straight line $\gamma(t) = x + t\delta x$, is that (2.14) will still make sense on manifolds.

B. Vector fields and flows. Let $\mathrm{X} \subset \mathbb{R}^{\mathrm{M}}$ be a numerical space and V a ***vector field on* X**, that is, a smooth map $\mathrm{V} : \mathrm{X} \to \mathbb{R}^{\mathrm{M}}$. The next basic fact we need is:[3]

Proposition 2.2 (Existence, uniqueness, and smoothness for ODE). *For each $a \in \mathrm{X}$, the differential equation*

$$\frac{dx}{dt} = \mathrm{V}(x) \tag{2.15}$$

has a unique maximal solution, $x = \gamma_a(t)$, defined in an open interval about $t = 0$, with initial condition $\gamma_a(0) = a$. Moreover $\gamma_a(t)$ is a smooth function of $\binom{a}{t}$.

This allows us to define the ***exponential of* V** by $e^{\mathrm{V}}(a) = \gamma_a(1)$, with (possibly void!) $\mathrm{Domain}(e^{\mathrm{V}}) = \{a \in \mathrm{X} : 1 \in \mathrm{Domain}(\gamma_a)\}$; and the ***flow of* V** as the one-parameter family of maps

$$\{e^{t\mathrm{V}} : t \in \mathbb{R}\}. \tag{2.16}$$

Each is a smooth bijection with inverse $e^{-t\mathrm{V}}$, and we have $e^{t\mathrm{V}}(a) = \gamma_a(t)$. (Thus, computing the flow is the exact same thing as solving the differential equation, and

[3] See e.g. [1] or [7, Theorems 1.8.3, 3.7.1].

we can mostly drop the notation γ_a henceforth.) We note that the defining property (2.15) of the flow now takes the form

$$\frac{d}{dt} e^{t\mathrm{V}}(a) = \mathrm{V}(e^{t\mathrm{V}}(a)). \tag{2.17}$$

Moreover, uniqueness in (2.15) implies

$$e^{s\mathrm{V}} \circ e^{t\mathrm{V}} \subset e^{(s+t)\mathrm{V}} \tag{2.18}$$

i.e., these two maps coincide on the (possibly smaller) domain of the former. If (2.18) is an equality for all pairs $\binom{s}{t}$, or equivalently if Domain$(\gamma_a) = \mathbb{R}$ for all a, then V and its flow are called ***complete***.

Example* 1.** Verify that any ***linear vector field on $\mathbb{R}^{\mathrm{M}}$: $\mathrm{V}(x) = \mathrm{Z}x$ for some matrix $\mathrm{Z} \in \mathbb{R}^{\mathrm{M}\times\mathrm{M}}$, is complete.

Answer. Indeed its flow is given by the following formula, as one sees by deriving the series term by term:

$$e^{t\mathrm{Z}} = \lim_{n\to\infty} \left(\mathbb{1} + \frac{t\mathrm{Z}}{n}\right)^n = \mathbb{1} + t\mathrm{Z} + \frac{t^2\mathrm{Z}^2}{2!} + \frac{t^3\mathrm{Z}^3}{3!} + \cdots. \tag{2.19}$$

This shows that $e^{t\mathrm{Z}}$ is again linear, and that notation (2.16) is compatible with the preexisting notion of exponential for numbers or matrices. To compute (2.19) in practice, one can use the fact that Z satisfies some polynomial equation of degree at most M (e.g., $c(\mathrm{Z}) = 0$ where $c(\lambda) = \det(\lambda\mathbb{1} - \mathrm{Z})$) to rewrite the series as a linear combination of $\mathbb{1}, \mathrm{Z}, \ldots, \mathrm{Z}^{\mathrm{M}-1}$ with functions of t as coefficients.

***Example* 2.** Compute the flow for the Newton equations

(a) $\frac{d}{dt}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -q \\ p \end{pmatrix}$ (harmonic oscillator); (2.20)

(b) $\frac{d}{dt}\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{a} \end{pmatrix}$ (free fall in constant gravity $\boldsymbol{a} =$ downward 32 ft/s^2). (2.21)

Answer. (a): This is linear with matrix $\mathrm{J} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $\mathrm{J}^2 = -\mathbb{1}$, the series for $e^{t\mathrm{J}}$ splits into $\mathbb{1}\cos t + \mathrm{J}\sin t$. So the flow consists of rotations in the $\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)$-plane,

$$e^{t\mathrm{J}}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}. \tag{2.22}$$

(b): The vector field here is affine: $\mathrm{V}\left(\begin{smallmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 \\ \boldsymbol{a} \end{smallmatrix}\right)$. This allows us to rewrite the equation as linear in one more dimension:

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} & 0 \\ & 0 & \boldsymbol{a} \\ & & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \\ 1 \end{pmatrix}. \tag{2.23}$$

Because this matrix Z satisfies $\mathrm{Z}^3 = 0$, the series (2.19) boils down to $\mathbb{1} + t\mathrm{Z} + \frac{1}{2}t^2\mathrm{Z}^2$ and we find

$$e^{t\mathrm{V}}\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{r} + \boldsymbol{v}t + \frac{1}{2}\boldsymbol{a}t^2 \\ \boldsymbol{v} + \boldsymbol{a}t \end{pmatrix}. \tag{2.24}$$

C. The adjoint action. There are two basic operations on vector fields, defined as follows. Given a diffeomorphism $\mathrm{F} : \mathrm{X} \to \mathrm{Y}$ between numerical spaces, and vector fields U and V on X, we obtain new vector fields $\mathrm{Ad}_\mathrm{F}\mathrm{V}$ and $[\mathrm{U}, \mathrm{V}]$ on Y (resp. X) by

Definition 2.3 (Operations on vector fields).

Push-forward:
$$\mathrm{Ad}_\mathrm{F}\mathrm{V}(y) := \frac{d}{dt}\left[\mathrm{F} \circ e^{t\mathrm{V}} \circ \mathrm{F}^{-1}\right](y)\Big|_{t=0} \tag{2.25}$$

Lie bracket:[4]
$$[\mathrm{U}, \mathrm{V}](x) := \frac{d}{ds}\mathrm{Ad}_{e^{s\mathrm{U}}}\mathrm{V}(x)\Big|_{s=0}. \tag{2.26}$$

One also calls $\mathrm{Ad}_\mathrm{F}\mathrm{V}$ the ***image*** of V by F. Note that we might as well define it by the property—verified using (2.18)—that its flow is given by the 'change of variables formula'

$$e^{t\mathrm{Ad}_\mathrm{F}\mathrm{V}} = \mathrm{F} \circ e^{t\mathrm{V}} \circ \mathrm{F}^{-1}. \tag{2.27}$$

Both operations can be computed without explicit knowledge of the flows $e^{s\mathrm{U}}$, $e^{t\mathrm{V}}$. In fact, for $\mathrm{Ad}_\mathrm{F}\mathrm{V}$, an application of (2.14) and (2.17) gives the formula

$$\mathrm{Ad}_\mathrm{F}\mathrm{V}(y) = \mathrm{DF}(x)(\mathrm{V}(x)), \qquad x = \mathrm{F}^{-1}(y). \tag{2.28}$$

For the bracket, we have:

Proposition 2.4 (Properties of the Lie bracket).

Bracket Formula:
$$[\mathrm{U}, \mathrm{V}](x) = \mathrm{DU}(x)(\mathrm{V}(x)) - \mathrm{DV}(x)(\mathrm{U}(x)) \tag{2.29}$$

Naturality:
$$\mathrm{Ad}_\mathrm{F}[\mathrm{U}, \mathrm{V}] = [\mathrm{Ad}_\mathrm{F}\mathrm{U}, \mathrm{Ad}_\mathrm{F}\mathrm{V}] \tag{2.30}$$

Jacobi Identity:
$$[\mathrm{T}, [\mathrm{U}, \mathrm{V}]] + [\mathrm{U}, [\mathrm{V}, \mathrm{T}]] + [\mathrm{V}, [\mathrm{T}, \mathrm{U}]] = 0. \tag{2.31}$$

Remark. Jacobi's identity (2.31), plus the bilinearity and antisymmetry apparent on (2.29), are expressed by saying that Lie bracket turns $\mathrm{Vect}(\mathrm{X}) = \{\text{vector fields on X}\}$ into a ***Lie algebra***. Naturality (2.30) says that the adjoint action Ad preserves this structure.

[4] We use the definition of Arnol'd [2], which differs in sign from that of Lie and other authors.

Proof of (2.29). Put $\mathrm{R}\left(\begin{smallmatrix} s\\ t\\ u\end{smallmatrix}\right) := e^{s\mathrm{U}}\circ e^{t\mathrm{V}}\circ e^{u\mathrm{U}}(x)$. We have

$$
\begin{aligned}
[\mathrm{U},\mathrm{V}](x) &= \frac{d}{ds}\frac{d}{dt}\left[\mathrm{R}\left(\begin{smallmatrix} s\\ t\\ -s\end{smallmatrix}\right)\right]_{s=t=0} && \text{by (2.25) and (2.26)}\\
&= \frac{d}{dt}\frac{d}{ds}\left[\mathrm{R}\left(\begin{smallmatrix} s\\ t\\ -s\end{smallmatrix}\right)\right]_{s=t=0} && \text{by (2.6)}\\
&= \frac{d}{dt}\left[\mathrm{DR}\left(\begin{smallmatrix} 0\\ t\\ 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1\\ 0\\ -1\end{smallmatrix}\right)\right]_{t=0} && \text{by (2.14)}\\
&= \frac{d}{dt}\left[\mathrm{DR}\left(\begin{smallmatrix} 0\\ t\\ 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1\\ 0\\ 0\end{smallmatrix}\right) - \mathrm{DR}\left(\begin{smallmatrix} 0\\ t\\ 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 0\\ 0\\ 1\end{smallmatrix}\right)\right]_{t=0} && \text{by linearity of } \mathrm{DR}\left(\begin{smallmatrix} 0\\ t\\ 0\end{smallmatrix}\right)\\
&= \frac{d}{dt}\frac{d}{ds}\left[\mathrm{R}\left(\begin{smallmatrix} s\\ t\\ 0\end{smallmatrix}\right) - \mathrm{R}\left(\begin{smallmatrix} 0\\ t\\ s\end{smallmatrix}\right)\right]_{s=t=0} && \text{by (2.14)}\\
&= \frac{d}{dt}\frac{d}{ds}\left[e^{s\mathrm{U}}\circ e^{t\mathrm{V}}(x) - e^{t\mathrm{V}}\circ e^{s\mathrm{U}}(x)\right]_{s=t=0} && \text{by definition of R}\\
&= \frac{d}{dt}\Big[\mathrm{U}(e^{t\mathrm{V}}(x))\Big]_{t=0} - \frac{d}{ds}\Big[\mathrm{V}(e^{s\mathrm{U}}(x))\Big]_{s=0} && \text{by (2.6) and (2.17)}\\
&= \mathrm{DU}(x)(\mathrm{V}(x)) - \mathrm{DV}(x)(\mathrm{U}(x)) && \text{by (2.14) and (2.17).} \qquad \square
\end{aligned}
$$

Proof of (2.30). Write $x = \mathrm{F}^{-1}(y)$. We have

$$
\begin{aligned}
\mathrm{Ad_F}[\mathrm{U},\mathrm{V}](y) &= \mathrm{DF}(x)([\mathrm{U},\mathrm{V}](x)) && \text{by (2.28)}\\
&= \mathrm{DF}(x)\Big(\frac{d}{ds}\mathrm{Ad}_{e^{s\mathrm{U}}}\mathrm{V}(x)\Big|_{s=0}\Big) && \text{by (2.26)}\\
&= \frac{d}{ds}\mathrm{DF}(x)(\mathrm{Ad}_{e^{s\mathrm{U}}}\mathrm{V}(x))\Big|_{s=0} && \text{by linearity of } \mathrm{DF}(x)\\
&= \frac{d}{ds}\mathrm{DF}(x)\Big(\frac{d}{dt}\Big[e^{s\mathrm{U}}\circ e^{t\mathrm{V}}\circ e^{-s\mathrm{U}}(x)\Big]_{t=0}\Big)\Big|_{s=0} && \text{by (2.25)}\\
&= \frac{d}{ds}\frac{d}{dt}\left[\mathrm{F}\circ e^{s\mathrm{U}}\circ e^{t\mathrm{V}}\circ e^{-s\mathrm{U}}\circ \mathrm{F}^{-1}(y)\right]_{s=t=0} && \text{by (2.14)}\\
&= \frac{d}{ds}\frac{d}{dt}\left[e^{s\mathrm{Ad_F}\mathrm{U}}\circ e^{t\mathrm{Ad_F}\mathrm{V}}\circ e^{-s\mathrm{Ad_F}\mathrm{U}}(y)\right]_{s=t=0} && \text{by (2.27)}\\
&= [\,\mathrm{Ad_F}\mathrm{U},\ \mathrm{Ad_F}\mathrm{V}\,](y) && \text{by (2.25, 2.26).} \qquad \square
\end{aligned}
$$

Proof of (2.31). Define $\mathrm{U}_s = \mathrm{Ad}_{e^{s\mathrm{T}}}\mathrm{U}$ and $\mathrm{U}'_s = \frac{d}{ds}\mathrm{U}_s$ so that $\mathrm{U}_0 = \mathrm{U}$ and $\mathrm{U}'_0 = [\mathrm{T},\mathrm{U}]$; and likewise for V. We compute

$$
\begin{aligned}
[\mathrm{T},[\mathrm{U},\mathrm{V}]](x) &= \frac{d}{ds}\,\mathrm{Ad}_{e^{s\mathrm{T}}}[\mathrm{U},\mathrm{V}](x)\Big|_{s=0} && \text{by (2.26)}\\
&= \frac{d}{ds}\,[\mathrm{U}_s,\ \mathrm{V}_s](x)\Big|_{s=0} && \text{by (2.30)}\\
&= \frac{d}{ds}\Big\{\mathrm{DU}_s(x)(\mathrm{V}_s(x)) - \mathrm{DV}_s(x)(\mathrm{U}_s(x))\Big\}_{s=0} && \text{by (2.29)}
\end{aligned}
$$

$$= \mathrm{DU}_0'(x)(\mathrm{V}_0(x)) + \mathrm{DU}_0(x)(\mathrm{V}_0'(x)) - \mathrm{DV}_0'(x)(\mathrm{U}_0(x)) - \mathrm{DV}_0(x)(\mathrm{U}_0'(x))$$

by (2.7) and (2.6)

$$= [\mathrm{U}_0', \mathrm{V}_0](x) + [\mathrm{U}_0, \mathrm{V}_0'](x) \qquad \text{by (2.29)}$$

$$= [[\mathrm{T},\mathrm{U}],\mathrm{V}](x) + [\mathrm{U},[\mathrm{T},\mathrm{V}]](x). \qquad \square$$

***Example* 1.** Let $\mathrm{V}(x) = \mathrm{Z}x$ be a linear vector field. Compute:

(a) its image $\mathrm{Ad}_g\mathrm{Z}$ by any linear bijection, $g \in \mathrm{GL}(\mathrm{M})$,

(b) its bracket $[\mathrm{Z}, \mathrm{Z}']$ with any other linear vector field.

Answer. Since any linear map is its own derivative, (2.28) and (2.29) give

$$\mathrm{Ad}_g\mathrm{Z} = g\mathrm{Z}g^{-1} \qquad \text{and} \qquad [\mathrm{Z}, \mathrm{Z}'] = \mathrm{Z}\mathrm{Z}' - \mathrm{Z}'\mathrm{Z}. \tag{2.32}$$

Thus we see that linear vector fields make a *Lie subalgebra* $\mathfrak{gl}(n)$ of $\mathrm{Vect}(\mathbb{R}^n)$, which moreover is invariant under the adjoint action of G.

***Example* 2.** On $\mathrm{X} = \mathbb{R} \setminus \{0\}$, compute (a) the image of the vector field $\mathrm{U}(x) = x^2$ by the diffeomorphism $\mathrm{F} : x \mapsto 1/x$; (b) the Lie bracket of $\mathrm{V}(x) = x^p$ and $\mathrm{W}(x) = x^q$.

Answer. (a): $\mathrm{Ad}_\mathrm{F}\mathrm{U}(y) = -1$.

(b): $[x^p, x^q] = (p-q)x^{p+q-1}$.

***Example* 3.** It is traditional to write $\partial/\partial x^i$ for the vector fields $y \mapsto \partial y/\partial x^i$ attached to a diffeomorphism $\mathrm{F} : \mathrm{X} \to \mathrm{Y}$ (2.10). Find (a) their flows, and (b) their Lie brackets.

Answer. (a): $e^{t\partial/\partial x^i}(\mathrm{F}(x)) = \mathrm{F}(x + te_i)$.

(b): $\left[\partial/\partial x^i, \partial/\partial x^j\right] = 0$.

D. Differential forms. An ***exterior p-form*** on $\mathbb{R}^\mathrm{M}$ is an alternating multilinear map $\omega : \mathbb{R}^\mathrm{M} \times \cdots \times \mathbb{R}^\mathrm{M} \to \mathbb{R}$. Here there are p factors, and alternating means that $\omega(v_1, .., v_p)$ changes sign whenever two v_i are interchanged. We denote the space of exterior p-forms by $\boldsymbol{\bigwedge^p(\overline{\mathbb{R}^\mathrm{M}})}$, and we agree that $\bigwedge^0(\overline{\mathbb{R}^\mathrm{M}}) = \mathbb{R}$. Thus,

– 0-forms are just scalars $\omega \in \mathbb{R}$;

– 1-forms are rows $(\omega_1 \ldots \omega_\mathrm{M})$ in the dual space $\overline{\mathbb{R}^\mathrm{M}}$ (2.11);

– 2-forms can be regarded as antisymmetric matrices (ω_{ij}), with

$$\omega(u,v) = \overline{u} \begin{pmatrix} \omega_{11} & \cdots & \omega_{1\mathrm{M}} \\ \cdots & \cdots & \cdots \\ \omega_{\mathrm{M}1} & \cdots & \omega_{\mathrm{MM}} \end{pmatrix} v\,; \tag{2.33}$$

– ...

– M-forms are all scalar multiples of the standard volume form **vol**, defined by $\mathrm{vol}(v_1, .., v_\mathrm{M})$ = determinant of the matrix with columns $v_1, .., v_\mathrm{M}$;

and in general:

Proposition 2.5 (Dimension formula).
The dimension of $\bigwedge^p(\overline{\mathbb{R}^\mathrm{M}})$ *is the binomial coefficient* $\binom{\mathrm{M}}{p}$, *i.e.,*

$$\begin{array}{rccccccccccc} \mathrm{M}=0: & & & & & & 1 \\ \mathrm{M}=1: & & & & & 1 & & 1 \\ \mathrm{M}=2: & & & & 1 & & 2 & & 1 \\ \mathrm{M}=3: & & & 1 & & 3 & & 3 & & 1 \\ \cdots\cdots & & 1 & & 4 & & 6 & & 4 & & 1 \\ \mathrm{M}: & \binom{\mathrm{M}}{0} & & \binom{\mathrm{M}}{1} & & \cdots & & \binom{\mathrm{M}}{p} & & \cdots & & \binom{\mathrm{M}}{\mathrm{M}}. \end{array} \tag{2.34}$$

Proof. We can map p-forms into $\mathbb{R}^{\binom{\mathrm{M}}{p}}$ by sending each ω to the system of its ***components***

$$\omega_{kl..r} := \omega(e_k, e_l, .., e_r) \tag{2.35}$$

where $\{k < \ldots < r\}$ runs over all $\binom{\mathrm{M}}{p}$ choices of an increasing p-tuple in $\{1, .., \mathrm{M}\}$. This map is one-to-one because if all components are zero, so is ω by multilinearity and antisymmetry. It is onto because for each p-tuple $\{a < \ldots < h\}$, the formula

$$\mathrm{Minor}^{a\ldots h}(v_1, .., v_p) = \begin{vmatrix} v_1^a & \cdots & v_p^a \\ \cdots & \cdots & \cdots \\ v_1^h & \cdots & v_p^h \end{vmatrix} \tag{2.36}$$

defines a p-form $\omega = \mathrm{Minor}^{a\ldots h}$ whose components (2.35) are all zero except $\omega_{a\ldots h} = 1$. □

— Let $\mathrm{X} \subset \mathbb{R}^\mathrm{M}$ be a numerical space. A ***differential p-form*** on X is a smooth map $x \mapsto \omega$ sending each x to an exterior p-form on $\mathbb{R}^\mathrm{M}$, or in other words, a smooth map $\mathrm{X} \to \bigwedge^p(\overline{\mathbb{R}^\mathrm{M}})$. It is traditional to have a symbol not for the map itself, but for the 'variable form' ω that results. When we need to emphasize that the form is evaluated at some point y other than x, however, we may denote that value by ω_y.

E. Operations on differential forms. There are five basic operations on differential forms. We define the first three by the following formulas, where the v_i are elements of $\mathbb{R}^\mathrm{M}$:

Definition 2.6 (Pointwise operations on p-forms).

***Pull-back*:** $$\mathrm{F}^*\omega(v_1,..,v_p) := \omega_{\mathrm{F}(x)}(\mathrm{DF}(x)(v_1),..,\mathrm{DF}(x)(v_p)) \tag{2.37}$$

***Interior product*:** $$\mathrm{i_V}\omega(v_2,..,v_p) := \omega(\mathrm{V}(x),v_2,..,v_p) \tag{2.38}$$

***Exterior product*:** $$\theta\wedge\omega(v_0,..,v_p) := \sum_{i=0}^{p}(-1)^i\theta(v_i)\omega(v_0,..,\widehat{v_i},..,v_p) \tag{2.39}$$

where the hat $\widehat{\cdot}$ indicates a term to be omitted.

In more detail:

— In (2.37), ω is a p-form on the target space of a smooth map $\mathrm{F}:\mathrm{X}\to\mathrm{Y}$. Then $\mathrm{F}^*\omega$ is a p-form on X, obtained by declaring its value at x on given vectors in $\mathbb{R}^{\mathrm{M}}$ to be the value of ω at $\mathrm{F}(x)$ on their images by $\mathrm{DF}(x)$. Note that the chain rule (2.8) implies

$$(\mathrm{G}\circ\mathrm{F})^* = \mathrm{F}^*\mathrm{G}^*. \tag{2.40}$$

If $\boxed{\mathrm{F}^*\omega=\omega}$, we say that F ***preserves*** ω, or leaves ω ***invariant***, or is a ***symmetry*** of ω. ("If the object looks the same after being transformed, then the transformation is a symmetry.") More generally we also say this—e.g. after (2.45)—if $\mathrm{X}\subset\mathrm{Y}$ and $\mathrm{F}^*\omega=\omega$ on X.

— In (2.38), ω is a p-form and V is a vector field on X. Then $\mathrm{i_V}\omega$ is the $(p-1)$-form on X obtained by inserting $\mathrm{V}(x)$ as the first argument of ω; when $p=0$, we agree that $\mathrm{i_V}\omega=0$.

— In (2.39), ω is a p-form and θ is a 1-form on X. Then $\theta\wedge\omega$ is the $(p+1)$-form on X obtained by *antisymmetrizing* $\theta(v_0)\omega(v_1,..,v_p)$ to make it alternating in all its arguments. Its components (2.35) are

$$[\theta\wedge\omega]_{kl\ldots r} = \theta_k\omega_{\hat{k}l\ldots r} - \theta_l\omega_{k\hat{l}\ldots r} + \ldots + (-1)^p\theta_r\omega_{kl\ldots\hat{r}}. \tag{2.41}$$

These three operations are 'pointwise' in the sense that the result doesn't require knowledge of ω, θ, or V at points other than x (or $\mathrm{F}(x)$). So we can extend the notation by also writing $\mathrm{i}_v\omega$ or $\theta\wedge\omega$ for a single vector $v\in\mathbb{R}^{\mathrm{M}}$ or covector $\theta\in\overline{\mathbb{R}^{\mathrm{M}}}$. In particular, we can define at each point x the *kernel*

$$\ker(\omega) := \{v\in\mathbb{R}^{\mathrm{M}} : \mathrm{i}_v\omega=0\}. \tag{2.42}$$

The remaining two operations, in contrast, involve deriving ω:

Definition 2.7 (Differential operations on *p*-forms).

***Lie derivative*:**

$$\mathrm{L}_{\mathrm{V}}\omega(v_1,..,v_p) := \frac{d}{dt}\, e^{t\mathrm{V}*}\omega(v_1,..,v_p)\Big|_{t=0} \tag{2.43}$$

***Exterior derivative*:**[5]

$$\mathrm{d}\omega(v_0,..,v_p) := \sum_{i=0}^{p} (-1)^i \frac{\partial\omega}{\partial x}(v_i)(v_0,..,\widehat{v_i},..,v_p) \tag{2.44}$$

where the hat $\widehat{\cdot}$ indicates a term to be omitted.

In more detail:

— In (2.43), V and ω are a vector field and a p-form on X. Then $\mathrm{L}_{\mathrm{V}}\omega$ is again a p-form on X (whose definition makes sense because every $x \in \mathrm{X}$ is in the domain of $e^{t\mathrm{V}}$ for t small enough). Using (2.18) and (2.40) we see that (2.43) implies more generally

$$\frac{d}{dt}\, e^{t\mathrm{V}*}\omega = e^{t\mathrm{V}*}\mathrm{L}_{\mathrm{V}}\omega \qquad \forall\, t. \tag{2.45}$$

Thus we have $\mathrm{L}_{\mathrm{V}}\omega = 0$ iff the flow of V consists of symmetries of ω. (Hence the interest of Cartan's formula (2.48) below, which will compute $\mathrm{L}_{\mathrm{V}}\omega$ without knowledge of the entire flow.)

— In (2.44), dω is the $(p+1)$-form on X defined by taking the ordinary derivative of ω and then antisymmetrizing, much as in (2.39). Its components (2.35) are

$$[\mathrm{d}\omega]_{kl\ldots r} = \partial_k\omega_{\hat{k}l\ldots r} - \partial_l\omega_{k\hat{l}\ldots r} + \ldots + (-1)^p\partial_r\omega_{kl\ldots\hat{r}} \tag{2.46}$$

where ∂_i is short for $\partial/\partial x^i$. Comparing this to (2.41), we see that the definition of d can be formally written $\mathrm{d}\omega = \partial \wedge \omega$ where ∂ is a 'fake 1-form' analogous to the fake vector ∇ of vector calculus. In particular, we get for

$p = 0:$	$[\mathrm{d}\omega]_i = \partial_i\omega$	(cf. 'gradient')
$p = 1:$	$[\mathrm{d}\omega]_{ij} = \partial_i\omega_j - \partial_j\omega_i$	(cf. 'curl')
$p = 2:$	$[\mathrm{d}\omega]_{ijk} = \partial_i\omega_{jk} + \partial_j\omega_{ki} + \partial_k\omega_{ij}$	(cf. 'divergence').

Proposition 2.8 (Properties of the exterior and Lie derivatives).

***Naturality*:**

$$\mathrm{d}[\mathrm{F}^*\omega] = \mathrm{F}^*[\mathrm{d}\omega]. \tag{2.47}$$

[5]Note the upright d, not to be confused with d.

É. Cartan's Formula: $\mathrm{L_V}\omega = \mathrm{d}\mathrm{i_V}\omega + \mathrm{i_V}\mathrm{d}\omega.$ (2.48)

H. Cartan's Formula: $\mathrm{i}_{[\mathrm{U},\mathrm{V}]}\omega = \mathrm{i_V}\mathrm{L_U}\omega - \mathrm{L_U}\mathrm{i_V}\omega.$ (2.49)

Lie's Formula: $\mathrm{L}_{[\mathrm{U},\mathrm{V}]}\omega = \mathrm{L_V}\mathrm{L_U}\omega - \mathrm{L_U}\mathrm{L_V}\omega.$ (2.50)

Poincaré's Theorem: $\mathrm{Image(d)} \subset \mathrm{Kernel(d)}$, *i.e.* $\mathrm{d}^2 = 0.$ (2.51)

Poincaré's Lemma: $\mathrm{Image(d)} = \mathrm{Kernel(d)}$, *if* X *is star-shaped.* (2.52)

Proof of (2.47). We do this for a 3-form, as it will suffice to make the general case clear. Put $y = \mathrm{F}(x)$, $\mathrm{M} = \mathrm{DF}(x)$ and $\mathrm{N} = \frac{\partial \mathrm{M}}{\partial x} = \mathrm{D}^2\mathrm{F}(x)$. By definition $[\mathrm{F}^*\omega]_x = \omega_y \circ (\mathrm{M}\times\mathrm{M}\times\mathrm{M})$, so by Leibniz (2.7) we have

$$\begin{aligned}\frac{\partial \mathrm{F}^*\omega}{\partial x}(u_0) = \frac{\partial \omega}{\partial y}(\mathrm{M}(u_0)) \circ (\mathrm{M}\times\mathrm{M}\times\mathrm{M}) &+ \omega\circ(\mathrm{N}(u_0)\times\mathrm{M}\times\mathrm{M})\\ &+ \omega\circ(\mathrm{M}\times\mathrm{N}(u_0)\times\mathrm{M})\\ &+ \omega\circ(\mathrm{M}\times\mathrm{M}\times\mathrm{N}(u_0)).\end{aligned}$$

Evaluating this on vectors (u_1, u_2, u_3), we get the first term $\frac{\partial \mathrm{F}^*\omega}{\partial x}(u_0)(u_1, u_2, u_3)$ in (2.44) as

$$\frac{\partial \omega}{\partial y}(v_0)(v_1, v_2, v_3) + \omega(w_{01}, v_2, v_3) + \omega(v_1, w_{02}, v_3) + \omega(v_1, v_2, w_{03})$$

where $v_i = \mathrm{M}(u_i)$ and $w_{ij} = \mathrm{N}(u_i)(u_j)$. Antisymmetrizing as in (2.44), we obtain therefore

$$\begin{aligned}&\mathrm{d}[\mathrm{F}^*\omega](u_0, u_1, u_2, u_3)\\ &= \frac{\partial \omega}{\partial y}(v_0)(v_1, v_2, v_3) + \omega(w_{01}, v_2, v_3) + \omega(v_1, w_{02}, v_3) + \omega(v_1, v_2, w_{03})\\ &- \frac{\partial \omega}{\partial y}(v_1)(v_0, v_2, v_3) - \omega(w_{10}, v_2, v_3) - \omega(v_0, w_{12}, v_3) - \omega(v_0, v_2, w_{13})\\ &+ \frac{\partial \omega}{\partial y}(v_2)(v_0, v_1, v_3) + \omega(w_{20}, v_1, v_3) + \omega(v_0, w_{21}, v_3) + \omega(v_0, v_1, w_{23})\\ &- \frac{\partial \omega}{\partial y}(v_3)(v_0, v_1, v_2) - \omega(w_{30}, v_1, v_2) - \omega(v_0, w_{31}, v_2) - \omega(v_0, v_1, w_{32}).\end{aligned}$$

Now, using the symmetry $w_{ij} = w_{ji}$ (2.6) and the antisymmetry of ω, we see that all terms in the last three columns cancel in pairs, in the pattern

	01	02	03
	10	12	13
	20	21	23
	30	31	32

.

Meanwhile the first column adds up to $[\mathrm{d}\omega]_y(v_0, v_1, v_2, v_3)$, which by definition, is precisely $\mathrm{F}^*[\mathrm{d}\omega](u_0, u_1, u_2, u_3)$. □

Proof of (2.48). Again we do it for the case of a 3-form. First we expand Definition (2.43) of the Lie derivative:

$$\begin{aligned}
\mathrm{L_V}\omega(u,v,w) &= \tfrac{d}{dt}\omega_{e^{t\mathrm{V}}(x)}(\mathrm{D}e^{t\mathrm{V}}(x)(u), \mathrm{D}e^{t\mathrm{V}}(x)(v), \mathrm{D}e^{t\mathrm{V}}(x)(w))\big|_{t=0} && \text{by (2.37)}\\
&= \tfrac{d}{dt}\omega_{e^{t\mathrm{V}}(x)}\big|_{t=0}(u,v,w) + \omega\left(\tfrac{d}{dt}\mathrm{D}e^{t\mathrm{V}}(x)(u)\big|_{t=0}, v, w\right)\\
&\qquad + \omega\left(u, \tfrac{d}{dt}\mathrm{D}e^{t\mathrm{V}}(x)(v)\big|_{t=0}, w\right) && \text{by (2.7)}\\
&\qquad + \omega\left(u, v, \tfrac{d}{dt}\mathrm{D}e^{t\mathrm{V}}(x)(w)\big|_{t=0}\right)\\
&= \frac{\partial\omega}{\partial x}(\mathrm{V}(x))(u,v,w) + \omega(\mathrm{DV}(x)(u), v, w)\\
&\qquad + \omega(u, \mathrm{DV}(x)(v), w) \qquad \text{by (2.6) and (2.17)} && \text{(A)}\\
&\qquad + \omega(u, v, \mathrm{DV}(x)(w)).
\end{aligned}$$

Secondly we have

$$\begin{aligned}
\mathrm{i_V}\mathrm{d}\omega(u,v,w) &= \mathrm{d}\omega(\mathrm{V}(x), u, v, w) && \text{by (2.38)}\\
&= \frac{\partial\omega}{\partial x}(\mathrm{V}(x))(u,v,w) - \frac{\partial\omega}{\partial x}(u)(\mathrm{V}(x), v, w)\\
&\qquad + \frac{\partial\omega}{\partial x}(v)(\mathrm{V}(x), u, w) \qquad \text{by (2.44)} && \text{(B)}\\
&\qquad - \frac{\partial\omega}{\partial x}(w)(\mathrm{V}(x), u, v).
\end{aligned}$$

Third we observe that $\mathrm{i_V}\omega = \omega(\mathrm{V}(x), \cdot, \cdot)$ is a bilinear function of the pair $(\omega, \mathrm{V}(x))$. So the Leibniz rule (2.7) applies and gives

$$\frac{\partial \mathrm{i_V}\omega}{\partial x}(u) = \frac{\partial\omega}{\partial x}(u)(\mathrm{V}(x), \cdot, \cdot) + \omega(\mathrm{DV}(x)(u), \cdot, \cdot). \tag{2.53}$$

Therefore we have

$$\begin{aligned}
\mathrm{d}\mathrm{i_V}\omega(u,v,w) &= \frac{\partial \mathrm{i_V}\omega}{\partial x}(u)(v,w) - \frac{\partial \mathrm{i_V}\omega}{\partial x}(v)(u,w) + \frac{\partial \mathrm{i_V}\omega}{\partial x}(w)(u,v) && \text{by (2.44)}\\
&= \frac{\partial\omega}{\partial x}(u)(\mathrm{V}(x), v, w) + \omega(\mathrm{DV}(x)(u), v, w) && \text{by (2.53)}\\
&\quad - \frac{\partial\omega}{\partial x}(v)(\mathrm{V}(x), u, w) - \omega(\mathrm{DV}(x)(v), u, w) && \text{(C)}\\
&\quad + \frac{\partial\omega}{\partial x}(w)(\mathrm{V}(x), u, v) + \omega(\mathrm{DV}(x)(w), u, v).
\end{aligned}$$

The desired relation (A) = (B) + (C) now follows, because the last column of (B) cancels the first of (C), and the last of (C) equals the last of (A) (by antisymmetry of ω). □

Proof of (2.49). We apply formula (A) established during the proof of (2.48). This gives

$$[\mathrm{i_V}\mathrm{L_U}\omega](v_2, .., v_p) = \frac{\partial\omega}{\partial x}(\mathrm{U}(x))(\mathrm{V}(x), v_2, .., v_p) + \omega(\mathrm{DU}(x)(\mathrm{V}(x)), v_2, .., v_p)$$

$$\begin{aligned}&+ \omega(\mathrm{V}(x), \mathrm{DU}(x)(v_2), .., v_p)\\ &+ \dots\\ &+ \omega(\mathrm{V}(x), v_2, .., \mathrm{DU}(x)(v_p))\end{aligned} \tag{D}$$

and, by the same token,

$$\begin{aligned}[\mathrm{L_U i_V}\omega](v_2, .., v_p) &= \frac{\partial \mathrm{i_V}\omega}{\partial x}(\mathrm{U}(x))(v_2, .., v_p) + \mathrm{i_V}\omega(\mathrm{DU}(x)(v_2), .., v_p)\\ &\qquad + \dots\\ &\qquad + \mathrm{i_V}\omega(v_2, .., \mathrm{DU}(x)(v_p))\\ &= \frac{\partial\omega}{\partial x}(\mathrm{U}(x))(\mathrm{V}(x), v_2, .., v_p) + \omega(\mathrm{DV}(x)(\mathrm{U}(x)), v_2, .., v_p)\\ &\qquad + \omega(\mathrm{V}(x), \mathrm{DU}(x)(v_2), .., v_p)\\ &\qquad + \dots\\ &\qquad + \omega(\mathrm{V}(x), v_2, .., \mathrm{DU}(x)(v_p))\end{aligned} \tag{E}$$

where we have used formula (2.53) for $\frac{\partial \mathrm{i_V}\omega}{\partial x}$. Subtracting (E) from (D), we see that everything cancels except $\omega(\mathrm{DU}(x)(\mathrm{V}(x)), v_2, .., v_p) - \omega(\mathrm{DV}(x)(\mathrm{U}(x)), v_2, .., v_p)$, which by (2.29) is just $\mathrm{i}_{[\mathrm{U,V}]}\omega(v_2, .., v_p)$. □

Proof of (2.50). First we observe that naturality (2.47) has an infinitesimal version

$$\mathrm{L_V d}\omega = \mathrm{dL_V}\omega, \tag{2.54}$$

obtained from (2.47) by taking $\mathrm{F} = e^{t\mathrm{V}}$ and deriving both sides at $t = 0$. Now we have

$$\begin{aligned}\mathrm{L}_{[\mathrm{U,V}]} &= \mathrm{di}_{[\mathrm{U,V}]} + \mathrm{i}_{[\mathrm{U,V}]}\mathrm{d} && \text{by (2.48)}\\ &= \mathrm{d}\,(\mathrm{i_V L_U} - \mathrm{L_U i_V}) + (\mathrm{i_V L_U} - \mathrm{L_U i_V})\,\mathrm{d} && \text{by (2.49)}\\ &= (\mathrm{di_V} + \mathrm{i_V d})\,\mathrm{L_U} - \mathrm{L_U}\,(\mathrm{di_V} + \mathrm{i_V d}) && \text{by (2.54)}\\ &= \mathrm{L_V L_U} - \mathrm{L_U L_V} && \text{by (2.48).}\end{aligned}$$

□

Proof of (2.51). Since any tangent vector is a value of some (e.g. constant) vector field V, it is enough to show that $\mathrm{i_V d^2}\omega = 0$ for all V. To this end we substitute in (2.54) the value of $\mathrm{L_V}$ drawn from Cartan's formula (2.48). We obtain: $\mathrm{i_V d^2}\omega = \mathrm{d^2 i_V}\omega$. Now this relation inductively reduces the matter to forms of lower degree; and for a 0-form ω it gives the desired conclusion (since $\mathrm{i_V}\omega = 0$). □

Proof of (2.52). Let $\omega \in \mathrm{Kernel(d)}$, and assume that X is star-shaped. By naturality (2.47) applied to a translation F, we may as well assume that X is star-shaped about the origin. In other words, the flow $e^{t\mathrm{E}}(x) = e^t x$ of the *Euler* vector field $\mathrm{E}(x) = x$ stays in X for all $t \leqslant 0$. Since $\mathrm{d}\omega = 0$, formulas (2.45, 2.48, 2.47) and $e^{t\mathrm{E}*}\mathrm{i_E} = \mathrm{i_E} e^{t\mathrm{E}*}$ imply

$$\frac{d}{dt}\, e^{t\mathrm{E}*}\omega = \mathrm{di_E} e^{t\mathrm{E}*}\omega \qquad\qquad \forall\, t \in (-\infty, 0].$$

Now we have $e^{t\mathrm{E}*}\omega = \omega_{e^t x} \circ (e^t \mathbb{1} \times \cdots \times e^t \mathbb{1}) = e^{pt}\omega_{e^t x}$. So the previous equation becomes

$$\frac{d}{dt}\, e^{pt}\omega_{e^t x} = \mathrm{d}\mathrm{i}_{\mathrm{E}} e^{pt}\omega_{e^t x} \qquad\qquad \forall\, t \in (-\infty, 0].$$

Integrating both sides, we obtain

$$\omega = \int_{-\infty}^{0} \frac{d}{dt}\, e^{pt}\omega_{e^t x}\, dt = \mathrm{d}\left[\int_{-\infty}^{0} \mathrm{i}_{\mathrm{E}} e^{pt}\omega_{e^t x}\, dt\right]$$

which shows that $\omega \in \mathrm{Image}(\mathrm{d})$. (We note that the interchange of d (2.44) and $\int_{-\infty}^{0}$ needs no justification if we use the *Denjoy integral*: see e.g. [4, 12.8–12.13, p. 291]. If an improper Riemann or Lebesgue integral is used instead, one readily justifies the interchange after reducing to an integral over $[0, 1]$ by means of the substitution $s = e^t$.) □

F. Vector-valued differential forms. The theory of differential forms generalizes *mutatis mutandis* if we replace $\bigwedge^p(\overline{\mathbb{R}^{\mathrm{M}}})$ by the space $\bigwedge^p(\overline{\mathbb{R}^{\mathrm{M}}}, \mathrm{E})$ of E-valued alternating p-linear maps $\omega : \mathbb{R}^{\mathrm{M}} \times \cdots \times \mathbb{R}^{\mathrm{M}} \to \mathrm{E}$, where E is a finite-dimensional vector space over $\mathbb{R}$. Just about the only changes are that the dimensions (2.34) get multiplied by $\dim(\mathrm{E})$, and that θ in (2.39) should be scalar-valued.

Example. The ***Maurer-Cartan 1-form*** on $\mathrm{GL}(n)$ is the $\mathfrak{gl}(n)$-valued 1-form Θ defined by $\Theta(\delta g) = g^{-1}\delta g$. Compute its exterior derivative $\mathrm{d}\Theta$.

Answer. We find

$$\mathrm{d}\Theta(\delta g, \delta' g) = \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g \qquad \text{by (2.44)} \qquad (2.55)$$

$$= [g^{-1}\delta' g, g^{-1}\delta g]. \qquad \text{by (2.13, 2.32).} \qquad (2.56)$$

2.2 Manifolds and Lie Groups

A manifold is a space obtained by patching numerical spaces together smoothly. One can then define its tangent bundle, and differential calculus on it, by patching together their tangent and p-form bundles. Examples are all matrix groups, and all orbits under the smooth action of a matrix group.

A. Manifolds. Let X be a set. An ***atlas*** on X is a set A of bijections $a : \mathrm{U}_a \to \mathrm{X}_a$ called ***charts***, such that (i) their domains U_a are numerical spaces, (ii) the union of their ranges X_a is X, and (iii) for any two charts a and b the ***chart changer*** $a^{-1} \circ b : b^{-1}(\mathrm{X}_a \cap \mathrm{X}_b) \to a^{-1}(\mathrm{X}_a \cap \mathrm{X}_b)$ is a diffeomorphism between numerical spaces. Two atlases on X are called ***equivalent*** if their union is an atlas.

— A ***manifold*** is a set X with an equivalence class of atlases on it. The charts of any atlas in the class are then called ***charts of* X**—or, in the older literature, 'systems of curvilinear coordinates'; a chart whose range contains x is called a ***chart at x***. A map $\mathrm{F} : \mathrm{X} \to \mathrm{Y}$ between manifolds is called ***smooth*** if for any charts a and c of X and Y, the composed map

$$c^{-1} \circ \mathrm{F} \circ a \tag{2.57}$$

is smooth as a map of numerical spaces. A ***diffeomorphism*** is a smooth bijection with smooth inverse.

— Every manifold comes equipped with the ***manifold topology***, defined by declaring a subset ***open*** if its preimage under every chart of an atlas is open in numerical space. (This only depends on the atlas's equivalence class.) So the usual terminology applies: a manifold may or may not be compact, connected, etc. To be on the safe side, we assume in this work that all manifolds are ***Hausdorff*** and ***second countable***. Smooth maps are continuous in the manifold topology.

B. The tangent bundle and tangent maps. Let X be a manifold and let $x \in \mathrm{X}$. We consider pairs $\left(\begin{smallmatrix}\delta u\\ a\end{smallmatrix}\right)$ where a is a chart at x and δu is an element of the space $\mathbb{R}^{\mathrm{M}}$ containing the domain of a. We call two such pairs $\left(\begin{smallmatrix}\delta u\\ a\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\delta v\\ b\end{smallmatrix}\right)$ ***equivalent*** if the derivative of $a^{-1} \circ b$ at $b^{-1}(x)$ sends δv to δu, i.e.,

$$\delta u = \frac{\partial u}{\partial v}(\delta v) \tag{2.58}$$

where $\frac{\partial u}{\partial v} = \mathrm{D}[a^{-1} \circ b](b^{-1}(x))$ (obviously an equivalence relation by the chain rule (2.8)). An equivalence class $\left[\begin{smallmatrix}\delta u\\ a\end{smallmatrix}\right]_x$ of such pairs is called a ***tangent vector*** to X at x. The set of such tangent vectors is called the ***tangent space*** to X at x and is denoted $\mathbf{T}_{\boldsymbol{x}}\mathbf{X}$. The map $\theta_{a,x} : \mathbb{R}^{\mathrm{M}} \to \mathrm{T}_x\mathrm{X}$ defined by $\theta_{a,x}(\delta u) = \left[\begin{smallmatrix}\delta u\\ a\end{smallmatrix}\right]_x$ is a bijection; by declaring it to be linear, we endow $\mathrm{T}_x\mathrm{X}$ with a vector space structure which, in view of the linearity of (2.58), does not depend on the chosen chart a. Its dimension is called the ***dimension*** of X at x. If $\mathrm{X} \subset \mathbb{R}^{\mathrm{M}}$ is a numerical space (with the obvious atlas consisting of its identity map alone), then each $\mathrm{T}_x\mathrm{X}$ identifies canonically with the ambient $\mathbb{R}^{\mathrm{M}}$.

— Let $\mathrm{F} : \mathrm{X} \to \mathrm{Y}$ be a smooth map between manifolds, a a chart at x and c a chart at $y = \mathrm{F}(x)$. The linear map $\mathbf{DF}(\boldsymbol{x})$ defined by the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{T}_x\mathrm{X} & \xrightarrow{\ \mathrm{DF}(x)\ } & \mathrm{T}_y\mathrm{Y} \\ \uparrow{\scriptstyle \theta_{a,x}} & & \uparrow{\scriptstyle \theta_{c,y}} \\ \mathbb{R}^{\mathrm{M}} & \xrightarrow{\ \mathrm{D}[c^{-1} \circ \mathrm{F} \circ a](a^{-1}(x))\ } & \mathbb{R}^{\mathrm{N}} \end{array} \tag{2.59}$$

does not depend on the choice of a and c; it is called the ***derivative of* F *at* x** and is also denoted $\boldsymbol{\partial y/\partial x}$. It clearly satisfies the chain rule (2.8) as well as properties (2.9) and (2.14). Also, in this notation the map $\theta_{a,x}$ is nothing but $\mathrm{D}a(a^{-1}(x))$.

— We define the ***tangent bundle* TX** of X as the disjoint union of the $\mathrm{T}_x\mathrm{X}$, or in other words, the set of all pairs $\left(\begin{smallmatrix}\delta x\\ x\end{smallmatrix}\right)$ where $x \in \mathrm{X}$ and $\delta x \in \mathrm{T}_x\mathrm{X}$. If $\mathrm{F} : \mathrm{X} \to \mathrm{Y}$ is a smooth map between manifolds, its ***tangent map*** is the map $\mathbf{F}_* : \mathrm{TX} \to \mathrm{TY}$ defined by

$$\mathrm{F}_* \begin{pmatrix}\delta x\\ x\end{pmatrix} = \begin{pmatrix}\mathrm{DF}(x)(\delta x)\\ \mathrm{F}(x)\end{pmatrix}. \tag{2.60}$$

Whenever the base point x of a tangent vector is clear from the context (as it is when we use the notation δx for a tangent vector at x), we drop it from the notation so that (2.60) may be written

$$\mathrm{F}_*(\delta x) = \mathrm{DF}(x)(\delta x). \tag{2.61}$$

If A is an atlas of X, then the maps $\{a_*\}_{a\in\mathrm{A}}$ make an atlas of TX, which is thus canonically a manifold. (Axiom (iii) holds because $(a_*^{-1} \circ b_*)(\delta v) = \mathrm{D}[a^{-1} \circ b](v)(\delta v)$ is clearly a smooth function of the pair $\left(\begin{smallmatrix}\delta v\\ v\end{smallmatrix}\right) \in \mathbb{R}^{\mathrm{M}} \times \mathrm{U}_b$.)

We emphasize that on a manifold there is no notion of second derivative $\mathrm{D}^2\mathrm{F}$, because DF takes its values in the *variable* space $\mathrm{Hom}(\mathrm{T}_x\mathrm{X}, \mathrm{T}_y\mathrm{Y})$. There is however a second tangent map $\mathrm{F}_{**} : \mathrm{TTX} \to \mathrm{TTY}$.

C. Vector fields and flows. A ***vector field*** on the manifold X is a section of its tangent bundle, i.e. a smooth map $\mathrm{V} : \mathrm{X} \to \mathrm{TX}$ such that $\mathrm{V}(x) \in \mathrm{T}_x\mathrm{X}$. As X is Hausdorff, Proposition 2.2 carries over verbatim[6] so that we can define the flow $e^{t\mathrm{V}}$ with properties (2.17, 2.18), as well as the push-forward and bracket (2.25, 2.26) with properties (2.27, 2.28, 2.30, 2.31).

We note that (2.26) contains a slight abuse of notation: since $\mathrm{Ad}_{e^{s\mathrm{U}}}\mathrm{V}(x)$ is a curve in $\mathrm{T}_x\mathrm{X}$, its derivative at 0 is really a member of $\mathrm{T}_{\mathrm{V}(x)}\mathrm{T}_x\mathrm{X}$ rather than $\mathrm{T}_x\mathrm{X}$. The issue is resolved by observing that these two spaces can be canonically identified. More seriously, formula (2.29) makes no sense on manifolds, since there is no way to subtract $\mathrm{DV}(x)(\mathrm{U}(x)) \in \mathrm{T}_{\mathrm{V}(x)}\mathrm{TX}$ from $\mathrm{DU}(x)(\mathrm{V}(x)) \in \mathrm{T}_{\mathrm{U}(x)}\mathrm{TX}$. Fortunately, we can still prove Jacobi's identity (2.31) by using (2.30) to transport the question into the domain of a chart, where the formula is already established.

D. *p*-form bundles and differential forms. Let X be a manifold with atlas A. We denote by $\boldsymbol{\bigwedge^p(\mathrm{T}_x^*\mathrm{X})}$ the space of exterior p-forms at x, i.e., alternating p-linear maps $\mathrm{T}_x\mathrm{X} \times \cdots \times \mathrm{T}_x\mathrm{X} \to \mathbb{R}$ (p factors). The ***p-form bundle*** $\boldsymbol{\bigwedge^p(\mathrm{T}^*\mathrm{X})}$ of X is the disjoint union of these spaces, or in other words, the set of all pairs $\left(\begin{smallmatrix}\omega\\ x\end{smallmatrix}\right)$ where

[6] See e.g. [1, Proposition 4.1.17] or [16, Theorem 1.48].

$x \in \mathrm{X}$ and ω is an exterior p-form at x. Whenever the base point x is clear from the context (such as when we evaluate ω on vectors $\delta_1 x, \dots, \delta_p x \in \mathrm{T}_x\mathrm{X}$), we drop it from the notation and write simply ω for an element of the p-form bundle. For each chart $a : u \mapsto x$ of X with domain $\mathrm{U}_a \subset \mathbb{R}^{\mathrm{M}}$, we define a chart $a_\natural : \bigwedge^p(\overline{\mathbb{R}^{\mathrm{M}}}) \times \mathrm{U}_a \to \bigwedge^p(\mathrm{T}^*\mathrm{X})$ of the p-form bundle by

$$a_\natural \begin{pmatrix} \alpha \\ u \end{pmatrix} = \begin{pmatrix} \alpha \circ \left(\frac{\partial u}{\partial x} \times \cdots \times \frac{\partial u}{\partial x}\right) \\ a(u) \end{pmatrix} \tag{2.62}$$

where $\frac{\partial u}{\partial x} = \mathrm{D}a(u)^{-1}$. Then the charts $\{a_\natural\}_{a\in\mathrm{A}}$ make an atlas of the p-form bundle, which is thus canonically a manifold. (Axiom (i) holds because we can regard $\bigwedge^p(\overline{\mathbb{R}^{\mathrm{M}}})$ as $\mathbb{R}^{\binom{\mathrm{M}}{p}}$ (2.35), and Axiom (iii) holds because

$$(a_\natural^{-1} \circ b_\natural) \begin{pmatrix} \beta \\ v \end{pmatrix} = \begin{pmatrix} \beta \circ \left(\frac{\partial v}{\partial u} \times \cdots \times \frac{\partial v}{\partial u}\right) \\ (a^{-1} \circ b)(v) \end{pmatrix}, \tag{2.63}$$

where $\frac{\partial v}{\partial u} = \mathrm{D}[a^{-1} \circ b](v)^{-1}$, is clearly a smooth function of the pair $\left(\begin{smallmatrix} \beta \\ v \end{smallmatrix}\right)$.) The 1-form bundle is also called the ***cotangent bundle*** and denoted $\mathbf{T^*X}$.

— A ***differential p-form*** on X is a section of its p-form bundle, i.e. a smooth map $\mathrm{X} \to \bigwedge^p(\mathrm{T}^*\mathrm{X})$ which sends each x to an exterior p-form ω at x. As before, it is traditional to have a symbol not for the map $x \mapsto \omega$ itself but for the 'variable form' ω that results; when we need to emphasize that the form is evaluated at some point y other than x, we may denote that value by ω_y.

— Formulas (2.37, 2.38, 2.39, 2.43) can be taken over verbatim to define the ***pull-back***, ***interior product***, ***exterior product*** and ***Lie derivative*** on manifolds, with properties (2.40) and (2.45). The exterior derivative, on the other hand, cannot be defined by formula (2.44) because $\frac{\partial\omega}{\partial x}(v_i)$ lies in $\mathrm{T}_\omega \bigwedge^p(\mathrm{T}^*\mathrm{X})$, not $\bigwedge^p(\mathrm{T}^*_x\mathrm{X})$. One circumvents this difficulty by making property (2.47), when F is a chart a, into the definition of the ***exterior derivative*** $\mathrm{d}\omega$ on manifolds. The consistency of this is ensured by property (2.47) itself, as already established in numerical spaces; and properties (2.48–2.51) follow as well.

E. Submanifolds and quotient manifolds. A map $i : \mathrm{Y} \to \mathrm{X}$ between manifolds is called an ***immersion*** if it is smooth and the derivative $\mathrm{D}i(y)$ is injective for each $y \in \mathrm{Y}$. A subset Y of a manifold X is called an ***initial submanifold*** if it admits a manifold structure such that (i) the inclusion $i : \mathrm{Y} \hookrightarrow \mathrm{X}$ is an immersion, and (ii) for any manifold Z, an arbitrary map $\mathrm{F} : \mathrm{Z} \to \mathrm{Y}$ is smooth iff $i \circ \mathrm{F} : \mathrm{Z} \to \mathrm{X}$ is smooth. That manifold structure is then unique. (Indeed let Y' be another and F the identity $\mathrm{Y}' \to \mathrm{Y}$; then by hypothesis $i \circ \mathrm{F}$ and $i \circ \mathrm{F}^{-1}$ are smooth (immersions), so F and F^{-1} are smooth and hence diffeomorphisms.)

— A map $p : \mathrm{X} \to \mathrm{Y}$ between manifolds is called an ***submersion*** if it is smooth and the derivative $\mathrm{D}p(x)$ is surjective for each $x \in \mathrm{X}$. The quotient Y of a manifold

X by an equivalence relation is called a ***quotient manifold*** if it admits a manifold structure such that (i) the projection $p : \mathrm{X} \to \mathrm{Y}$ is a submersion, and (ii) for any manifold Z, an arbitrary map $\mathrm{F} : \mathrm{Y} \to \mathrm{Z}$ is smooth iff $\mathrm{F} \circ p : \mathrm{X} \to \mathrm{Z}$ is smooth. That manifold structure is then unique. (Indeed let Y' be another and F the identity $\mathrm{Y}' \to \mathrm{Y}$; then by hypothesis $\mathrm{F} \circ p$ and $\mathrm{F}^{-1} \circ p$ are smooth (submersions), so F and F^{-1} are smooth and hence diffeomorphisms.)

F. Lie groups and group actions. A ***Lie group*** is a group G with a manifold structure such that the product $(g, h) \mapsto gh$ and the inversion $g \mapsto g^{-1}$ are smooth maps from $\mathrm{G} \times \mathrm{G}$ (resp. G) to G. Its ***Lie algebra*** is the tangent space $\mathfrak{g} = \mathrm{T}_e\mathrm{G}$ at the identity element. A basic example is the numerical space $\mathrm{GL}(n)$ (2.13), whose Lie algebra $\mathfrak{gl}(n)$ identifies canonically with the space $\mathbb{R}^{n\times n}$ of all real $n \times n$ matrices.

— A ***smooth action*** of G on a manifold X is a group morphism $\rho : \mathrm{G} \to \mathrm{Diff}(\mathrm{X})$ of G into diffeomorphisms of X, such that $(g, x) \mapsto \rho(g)(x)$ is a smooth map $\mathrm{G} \times \mathrm{X} \to \mathrm{X}$. It is customary to drop ρ from the notation and write $g(x)$ or $g_\mathrm{X}(x)$ for $\rho(g)(x)$. The ***orbit*** of $x \in \mathrm{X}$ is

$$\mathrm{G}(x) = \{g(x) : g \in \mathrm{G}\} \tag{2.64}$$

and is in natural set-theoretic bijection, $g(x) \mapsto g\mathrm{G}_x$, with the quotient G/G_x of G by the ***stabilizer*** $\mathrm{G}_x = \{g \in \mathrm{G} : g(x) = x\}$. The orbits form a partition of X. If this partition is trivial (i.e. there is just one orbit), the action is called ***transitive*** and X is called a ***homogeneous space*** of G.

— We shall depend on the following key facts, which provide us with many examples of manifolds, Lie groups, and homogeneous spaces:[7]

Proposition 2.9 (Matrix Lie groups and their homogeneous spaces).

(1) *Every subgroup* G *of* $\mathrm{GL}(n)$ *is an initial submanifold, and as such is a Lie group with Lie algebra*

$$\mathfrak{g} = \left\{\mathrm{Z} \in \mathfrak{gl}(n) : e^{t\mathrm{Z}} \in \mathrm{G} \quad \forall t \in \mathbb{R}\right\}. \tag{2.65}$$

(2) *Every quotient* G/H *of a Lie group* G *by a closed subgroup* H *is a quotient manifold, on which the natural transitive action* $g(g'\mathrm{H}) = gg'\mathrm{H}$ *is smooth.*

(3) *Every orbit* $\mathrm{G}(x)$ *of a smooth action of* G *on a manifold* X *is an initial submanifold, and as such is diffeomorphic to the quotient manifold* G/G_x.

[7] For (1) see [6, Chap. III, §§4.5 (Prop. 9), 6.4 (Cor. 2)] or [8, §6.14 (Thm 11)] or [14, pp. 48 (Cor. 7), 55 (Prop. 1), 134 (Ex. 2)]. For (2) see [8, §4.9 (Thm 4 and Corollaire)]. For (3) see [13, Prop. 2.3.12].

— It is clear from (2.25, 2.26, 2.32) that the Lie algebra (2.65) of a matrix group is not only a vector subspace of $\mathfrak{gl}(n)$, but is also invariant under the adjoint action Ad_g ($g \in \mathrm{G}$) as well as under the bracket $[\cdot,\cdot]$. Thus $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n)$, and each smooth action ρ of G on a manifold X induces an ***infinitesimal action*** homomorphism $\rho_* : \mathfrak{g} \to \mathrm{Vect}(\mathrm{X})$ by

$$\rho_*(\mathrm{Z})(x) = \frac{d}{dt} e^{t\mathrm{Z}}(x)\Big|_{t=0}. \tag{2.66}$$

As with ρ one usually drops ρ_* from the notation so that (2.66) writes $\mathrm{Z}(x)$ or $\mathrm{Z}_{\mathrm{X}}(x)$. If the action is transitive, then (3) above ensures that the tangent space to an orbit (2.64) at x is

$$\mathrm{T}_x\mathrm{G}(x) = \{\mathrm{Z}(x) : \mathrm{Z} \in \mathfrak{g}\} \simeq \mathfrak{g}/\mathfrak{g}_x \tag{2.67}$$

where $\mathfrak{g}_x = \{\mathrm{Z} \in \mathfrak{g} : \mathrm{Z}(x) = 0\}$. We note also that every matrix group comes equipped with its ***Maurer-Cartan 1-form***, induced by that of the ambient $\mathrm{GL}(n)$ and whose exterior derivative is computed by (2.56), thanks to naturality (2.47).

2.3 Symplectic Geometry

Symplectic geometry is the geometry of a manifold X with a nondegenerate, closed 2-form on it. A key construction is the symplectic gradient, drag, which attaches to every function on X a vector field whose flow preserves the 2-form. When a group acts by such flows, this gives rise to the concept of *moment map* by means of which one can classify all homogeneous symplectic manifolds.

A ***symplectic manifold*** is a manifold X with a 2-form σ which is closed ($\mathrm{d}\sigma = 0$) and nondegenerate, i.e. its kernel (2.42) is everywhere zero, or in other words, the antisymmetric matrix (2.33) of its coefficients is everywhere invertible. It follows that the dimension of X must be even.

A. Hamiltonian vector fields. Let (X, σ) be a connected symplectic manifold. A vector field η on X is called ***symplectic*** if its flow preserves the 2-form: $\mathrm{L}_\eta\sigma = 0$. By Cartan's formula (2.48), this holds iff the 1-form $\mathrm{i}_\eta\sigma$ is closed. When this 1-form is actually exact, so that there is a smooth function $x \mapsto \mathrm{H}$ on X with

$$\mathrm{i}_\eta\sigma = -\mathrm{dH}, \tag{2.68}$$

we say that η is ***hamiltonian***, $\eta \in \mathrm{Ham}(\mathrm{X})$, and we write $\eta = \mathrm{drag}\,\mathrm{H}$ ('symplectic gradient'). We define the ***Poisson bracket*** of two functions H, H' by

$$\{\mathrm{H}, \mathrm{H}'\} = \sigma(\mathrm{drag}\,\mathrm{H}', \mathrm{drag}\,\mathrm{H}). \tag{2.69}$$

Proposition 2.10 (The Poisson Lie algebra of a symplectic manifold). *Poisson bracket makes the space* $\mathrm{C}^\infty(\mathrm{X})$ *of smooth functions on* X *into a Lie algebra, i.e. it satisfies the Jacobi identity*

$$\{\{\mathrm{H},\mathrm{H}'\},\mathrm{H}''\}+\{\{\mathrm{H}',\mathrm{H}''\},\mathrm{H}\}+\{\{\mathrm{H}'',\mathrm{H}\},\mathrm{H}'\}=0. \tag{2.70}$$

Moreover one has the exact sequence

$$0\longrightarrow\mathbb{R}\longrightarrow\mathrm{C}^\infty(\mathrm{X})\xrightarrow{\text{drag}}\mathrm{Ham}(\mathrm{X})\longrightarrow 0. \tag{2.71}$$

Proof. We recall that *exact sequence* means that the arrows are Lie algebra homomorphisms and the image of each is the kernel of the next. For instance $\mathbb{R}$ injects in $\mathrm{C}^\infty(\mathrm{X})$ as the constant functions, on which the bracket (2.69) is zero. To prove the proposition, let $\eta=\operatorname{drag}\mathrm{H}$ and $\eta'=\operatorname{drag}\mathrm{H}'$ be two hamiltonian vector fields. We have

$$\begin{aligned}
\mathrm{i}_{[\eta,\eta']}\sigma &= -\mathrm{L}_\eta\mathrm{i}_{\eta'}\sigma && \text{by (2.49) since } \mathrm{L}_\eta\sigma=0\\
&= -\left(\mathrm{d}\mathrm{i}_\eta+\mathrm{i}_\eta\mathrm{d}\right)\mathrm{i}_{\eta'}\sigma && \text{by (2.48)}\\
&= -\mathrm{d}\mathrm{i}_\eta\mathrm{i}_{\eta'}\sigma && \text{since } \mathrm{i}_{\eta'}\sigma \text{ is closed}\\
&= -\mathrm{d}\{\mathrm{H},\mathrm{H}'\}. && \text{by (2.69).}
\end{aligned}$$

This shows that

$$[\operatorname{drag}\mathrm{H},\operatorname{drag}\mathrm{H}']=\operatorname{drag}\{\mathrm{H},\mathrm{H}'\}. \tag{2.72}$$

Thus drag sends Poisson brackets to Lie brackets (2.26), and in particular Ham(X) is closed under Lie bracket. To prove Jacobi's identity, we note that (2.69) can be rewritten

$$\{\mathrm{H},\mathrm{H}'\}=\mathrm{i}_{\operatorname{drag}\mathrm{H}}\mathrm{i}_{\operatorname{drag}\mathrm{H}'}\sigma=-\mathrm{i}_{\operatorname{drag}\mathrm{H}}\mathrm{d}\mathrm{H}'=-\mathrm{L}_{\operatorname{drag}\mathrm{H}}\mathrm{H}' \tag{2.73}$$

where we have used (2.68) and (2.48). Therefore we have

$$\begin{aligned}
\{\{\mathrm{H},\mathrm{H}'\},\mathrm{H}''\} &= -\mathrm{L}_{\operatorname{drag}\{\mathrm{H},\mathrm{H}'\}}\mathrm{H}'' && \text{by (2.73)}\\
&= -\mathrm{L}_{[\eta,\eta']}\mathrm{H}'' && \text{by (2.72)}\\
&= \mathrm{L}_\eta\mathrm{L}_{\eta'}\mathrm{H}''-\mathrm{L}_{\eta'}\mathrm{L}_\eta\mathrm{H}'' && \text{by (2.50)}\\
&= \{\mathrm{H},\{\mathrm{H}',\mathrm{H}''\}\}-\{\mathrm{H}',\{\mathrm{H},\mathrm{H}''\}\} && \text{by (2.73).}
\end{aligned}$$

Finally it is clear from (2.68) that $\operatorname{drag}\mathrm{H}$ vanishes iff $\mathrm{dH}=0$, or in other words, that the kernel of drag consists of the constant functions; so the sequence (2.71) is exact. □

B. Hamiltonian group actions. Let a Lie group G act on X in σ-preserving fashion: $g^*\sigma=\sigma$, so that the infinitesimal action (2.66) is by symplectic vector fields.

We say that a ***moment map*** exists if these infinitesimal generators are actually hamiltonian, so that a map $\Phi : \mathrm{X} \to \mathfrak{g}^*$ exists with

$$\mathrm{i}_{\mathrm{Z_X}}\sigma = -\mathrm{dH_Z} \qquad \text{where} \qquad \mathrm{H_Z} = \langle \Phi(x), \mathrm{Z}\rangle. \tag{2.74}$$

It is of interest to find if the moment map Φ is ***equivariant***, i.e. $\Phi(g(x)) = g(\Phi(x))$ where G and $\mathfrak{g}$ act on $\mathfrak{g}^*$ by the ***coadjoint action***:

$$\langle g(y), \mathrm{Z}\rangle = \langle y, \mathrm{Ad}_{g^{-1}}\mathrm{Z}\rangle, \qquad \langle \mathrm{Z}(y), \mathrm{Z}'\rangle = \langle y, [\mathrm{Z}', \mathrm{Z}]\rangle. \tag{2.75}$$

For this we have the following criterion:

Proposition 2.11 (Equivariance of the moment map). *Assume that* G *is connected. Then the moment map* (2.74) *is* G*-equivariant if and only if it satisfies*

$$\{\mathrm{H_Z}, \mathrm{H_{Z'}}\} = \mathrm{H}_{[\mathrm{Z},\mathrm{Z}']}. \tag{2.76}$$

Proof. We have

$$\begin{aligned}
\{\mathrm{H_{Z'}}, \mathrm{H_Z}\} - \mathrm{H}_{[\mathrm{Z}',\mathrm{Z}]} &= \sigma(\mathrm{Z}(x), \mathrm{Z}'(x)) - \langle \Phi(x), [\mathrm{Z}', \mathrm{Z}]\rangle && \text{by (2.69, 2.74)} \\
&= \langle \mathrm{D}\Phi(x)(\mathrm{Z}(x)) - \mathrm{Z}(\Phi(x)), \mathrm{Z}'\rangle && \text{by (2.74, 2.75)} \qquad \text{(A)} \\
&= \tfrac{d}{dt} \left\langle \Phi(e^{t\mathrm{Z}}(x)) - e^{t\mathrm{Z}}(\Phi(x)), \mathrm{Z}'\right\rangle\Big|_{t=0}.
\end{aligned}$$

Thus we see that if Φ is G-equivariant then $\{\mathrm{H_{Z'}}, \mathrm{H_Z}\} - \mathrm{H}_{[\mathrm{Z}',\mathrm{Z}]} = 0$. Conversely, assume that the expression (A) is identically zero. Fix x and Z and put $\mathrm{F}(t) = \Phi(e^{t\mathrm{Z}}(x)) - e^{t\mathrm{Z}}(\Phi(x))$. Then we have

$$\begin{aligned}
\tfrac{d}{dt}\mathrm{F}(t) &= \tfrac{d}{ds}\, \Phi(e^{s\mathrm{Z}}(e^{t\mathrm{Z}}(x))) - e^{s\mathrm{Z}}(e^{t\mathrm{Z}}(\Phi(x)))\Big|_{s=0} \\
&= \mathrm{D}\Phi(e^{t\mathrm{Z}}(x))(\mathrm{Z}(e^{t\mathrm{Z}}(x))) - \mathrm{Z}(e^{t\mathrm{Z}}(\Phi(x))) \\
&= \mathrm{Z}(\Phi(e^{t\mathrm{Z}}(x))) - \mathrm{Z}(e^{t\mathrm{Z}}(\Phi(x))) && \text{since (A)} \equiv 0 \\
&= \mathrm{Z}(\mathrm{F}(t)).
\end{aligned}$$

Thus $\mathrm{F}(t)$ is a solution of the ordinary differential equation $\frac{d}{dt}\mathrm{F}(t) = \mathrm{Z}(\mathrm{F}(t))$ with initial condition $\mathrm{F}(0) = 0$. By uniqueness of such a solution, $\mathrm{F}(t)$ must be identically zero. This shows the equivariance $\Phi(g(x)) = g(\Phi(x))$ for all g in one-parameter subgroups of G. Since one-parameter subgroups generate G, it follows that Φ is G-equivariant. ☐

Proposition 2.12 (The Kirillov-Kostant-Souriau Theorem [11, 12, 15]).

(1) *Every coadjoint orbit of a Lie group is a homogeneous symplectic manifold when endowed with the* **KKS 2-*form***

$$\sigma(\mathrm{Z}(x), \mathrm{Z}'(x)) = \langle x, [\mathrm{Z}', \mathrm{Z}] \rangle. \tag{2.77}$$

(2) *Conversely, every homogeneous symplectic manifold of a connected Lie group* G *is, up to a possible covering, a coadjoint orbit of some central extension of* G.

Proof. We must first show that (2.77) is well-defined, i.e. the right-hand side only depends on Z and Z′ via $\mathrm{Z}(x)$ and $\mathrm{Z}'(x)$. But this is clear by (2.75) which shows that the definition may be rewritten

$$\sigma(\mathrm{Z}(x), \mathrm{Z}'(x)) = \langle \mathrm{Z}(x), \mathrm{Z}' \rangle = -\langle \mathrm{Z}'(x), \mathrm{Z} \rangle. \tag{2.78}$$

That same relation shows that σ is nondegenerate; for if $\mathrm{Z}(x)$ is such that (2.78) vanishes for all Z′ then clearly $\mathrm{Z}(x) = 0$. To see that σ is closed, we evaluate its pull-back by the orbit map $\pi : \mathrm{G} \to \mathrm{Y}$ sending g to $\pi(g) = g(x_0)$ on two tangent vectors $\delta g = \mathrm{Z}g$, $\delta' g = \mathrm{Z}'g$ (we assume without loss of generality that G is a matrix group):

$$\begin{aligned}
(\pi^*\sigma)(\delta g, \delta' g) &= \sigma(\pi_*(\mathrm{Z}g), \pi_*(\mathrm{Z}'g)) \\
&= \sigma(\mathrm{Z}(g(x_0)), \mathrm{Z}'(g(x_0))) && \text{by (2.66)} \\
&= \langle g(x_0), [\mathrm{Z}', \mathrm{Z}] \rangle && \text{by (2.77)} \qquad (2.79) \\
&= \langle x_0, g^{-1}[\mathrm{Z}', \mathrm{Z}]g \rangle && \text{by (2.75)} \\
&= \langle x_0, [g^{-1}\mathrm{Z}'g, g^{-1}\mathrm{Z}g] \rangle \\
&= \langle x_0, [g^{-1}\delta' g, g^{-1}\delta g] \rangle \\
&= \langle x_0, \mathrm{d}\Theta(\delta g, \delta' g) \rangle && (2.80)
\end{aligned}$$

where $\mathrm{d}\Theta$ is the exterior derivative (2.56) of the Maurer-Cartan 1-form. This shows that $\pi^*\sigma = \mathrm{d}\langle x_0, \Theta \rangle$, so $\pi^*\sigma$ is not only closed but exact. By (2.47) it follows that $\pi^*\mathrm{d}\sigma = 0$, whence $\mathrm{d}\sigma = 0$ since π is a submersion. To see that σ is G-invariant, we observe that $g_*(\mathrm{Z}(x)) = (\mathrm{Ad}_g\mathrm{Z})(g(x))$ and therefore

$$\begin{aligned}
(g^*\sigma)(\mathrm{Z}(x), \mathrm{Z}'(x)) &= \sigma((\mathrm{Ad}_g\mathrm{Z})(g(x)), (\mathrm{Ad}_g\mathrm{Z}')(g(x))) \\
&= \langle g(x), \mathrm{Ad}_g[\mathrm{Z}', \mathrm{Z}] \rangle \\
&= \langle x, [\mathrm{Z}', \mathrm{Z}] \rangle \\
&= \sigma(\mathrm{Z}(x), \mathrm{Z}'(x)).
\end{aligned}$$

Whence (1). Conversely, let (X, σ) be a symplectic manifold with a transitive, σ-preserving action of G. Then the infinitesimal action is by symplectic vector fields, i.e. the 1-forms

$\mathrm{i}_{\mathrm{Z_X}}\sigma$ are all closed. Replacing X and G by coverings if necessary, we can assume that these 1-forms are exact. As a result there is a moment map but it is not necessarily equivariant. To circumvent this obstruction, let $\bar{\mathfrak{g}}$ denote the set of pairs $\overline{\mathrm{Z}} = \binom{h}{\mathrm{Z}}$ in $\mathrm{C}^\infty(\mathrm{X}) \times \mathfrak{g}$ such that $\mathrm{Z_X} = \operatorname{drag} h$. As $\mathrm{Z_X}$ then determines h up to an additive constant (2.71), we obtain a central extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \bar{\mathfrak{g}} \xrightarrow{\mathrm{pr}_2} \mathfrak{g} \longrightarrow 0$$

where $\bar{\mathfrak{g}}$ has Lie bracket $\left[\binom{h}{\mathrm{Z}}, \binom{h'}{\mathrm{Z'}}\right] = \binom{\{h,h'\}}{[\mathrm{Z},\mathrm{Z'}]}$. Now the corresponding simply connected Lie group $\overline{\mathrm{G}}$ acts on X via G, with moment map Φ given by $\mathrm{H}_{\overline{\mathrm{Z}}} = h$ (2.74). By construction this satisfies the equivariance condition (2.76): so Φ maps X onto a coadjoint orbit Y of $\overline{\mathrm{G}}$, and since (2.76) also writes

$$\sigma(\overline{\mathrm{Z}}(x), \overline{\mathrm{Z}}'(x)) = \langle y, [\overline{\mathrm{Z}}', \overline{\mathrm{Z}}]\rangle \qquad \text{where } y = \Phi(x),$$

we see that $\sigma = \Phi^*\sigma_{\mathrm{KKS}}$ where σ_{KKS} is the 2-form (2.77). Finally the relation $\langle \mathrm{D}\Phi(x)(\delta x), \mathfrak{g}\rangle = \sigma(\delta x, \mathfrak{g}(x))$, already observed during the proof of (2.76), shows that the kernel of $\mathrm{D}\Phi(x)$ is σ-orthogonal to $\mathfrak{g}(x) = \mathrm{T}_x\mathrm{X}$ (by transitivity), hence is zero by nondegeneracy. Therefore we conclude that $\Phi : \mathrm{G}/\mathrm{G}_x \to \mathrm{G}/\mathrm{G}_y$ is a covering map of homogeneous spaces. □

In general, using formula (2.77) to evaluate σ on given tangent vectors $\delta x, \delta' x$ requires us to express them in the form $\mathrm{Z}(x)$—that is, we must find solutions $\mathrm{Z} \in \mathfrak{g}$ of the equation $\delta x = \mathrm{Z}(x)$. While this is always possible (2.67), we will be able to avoid much of the tedium by using another version:

Corollary 2.13 (of proof). *Suppose that* G *is a matrix group, and fix* $x_0 \in \mathfrak{g}^*$. *Then the 2-form* (2.77) *of the coadjoint orbit* $\mathrm{X} = \mathrm{G}(x_0)$ *is given at* $x = g(x_0)$ *by either one of the formulas*

$$\sigma(\delta x, \delta' x) = \left\langle x_0, \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g\right\rangle \tag{2.81}$$

$$= \left\langle\, x\,, \delta g \delta'[g^{-1}] - \delta' g \delta[g^{-1}]\right\rangle. \tag{2.82}$$

(*It is understood that* (2.81) *and* (2.82) *only depend on* $g, \delta g, \delta' g$ *via* $x, \delta x, \delta' x$.)

Proof. Formula (2.81) is immediate from (2.80) and (2.55). For (2.82) we note that

$$\begin{aligned}
\sigma(\delta x, \delta' x) &= \langle g(x_0), [\delta' g.g^{-1}, \delta g.g^{-1}]\rangle && \text{by (2.79)}\\
&= \langle g(x_0), \delta' g.g^{-1}\delta g.g^{-1} - \delta g.g^{-1}\delta' g.g^{-1}\rangle\\
&= \langle x, \delta g\delta'[g^{-1}] - \delta' g\delta[g^{-1}]\rangle && \text{by (2.13).}
\end{aligned}$$

□

Example. Compute the 2-form on the coadjoint orbits of the rotation group $\mathrm{G} = \mathrm{SO}(3) = \{\mathrm{A} \in \mathfrak{gl}(3) : \overline{\mathrm{A}}\mathrm{A} = \mathbb{1}, \det(\mathrm{A}) = 1\}$.

Answer. The Lie algebra (2.65) of G is

$$\mathfrak{g} = \left\{ j(\boldsymbol{\alpha}) := \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix} : \boldsymbol{\alpha} \in \mathbb{R}^3 \right\}. \tag{2.83}$$

We identify $\mathfrak{g}^*$ with $\mathbb{R}^3$, paired to $\mathfrak{g}$ by

$$\langle \mathbf{L}, j(\boldsymbol{\alpha}) \rangle = \langle \mathbf{L}, \boldsymbol{\alpha} \rangle \tag{2.84}$$

where the second bracket is the ordinary euclidean scalar product of $\mathbb{R}^3$. Under this identification, the relation $\mathrm{A}j(\boldsymbol{\alpha})\mathrm{A}^{-1} = j(\mathrm{A}\boldsymbol{\alpha})$ shows that the coadjoint action (2.75) of G on $\mathfrak{g}^*$ is just the ordinary rotation action of SO(3) on $\mathbb{R}^3$. So each orbit is either the origin $\{0\}$ or a sphere of some radius s,

$$\mathrm{X} = \mathrm{G}(s\boldsymbol{e}_3) = \{\mathbf{L} = s\boldsymbol{u} : \|\boldsymbol{u}\| = 1\}. \tag{2.85}$$

To compute the 2-form on each sphere, formula (2.82) turns out to be the most convenient. Writing $\mathrm{A} = (\boldsymbol{v}\,\boldsymbol{w}\,\boldsymbol{u})$ so that $\boldsymbol{u} = \mathrm{A}\boldsymbol{e}_3$, we obtain

$$\sigma(\delta\mathbf{L}, \delta'\mathbf{L}) = \left\langle s\boldsymbol{u}, \delta\mathrm{A}\overline{\delta'\mathrm{A}} - \delta'\mathrm{A}\overline{\delta\mathrm{A}} \right\rangle \tag{2.86}$$

$$= \left\langle s\boldsymbol{u}, (\delta\boldsymbol{v}\,\delta\boldsymbol{w}\,\delta\boldsymbol{u}) \begin{pmatrix} \overline{\delta'\boldsymbol{v}} \\ \overline{\delta'\boldsymbol{w}} \\ \overline{\delta'\boldsymbol{u}} \end{pmatrix} - (\delta'\boldsymbol{v}\,\delta'\boldsymbol{w}\,\delta'\boldsymbol{u}) \begin{pmatrix} \overline{\delta\boldsymbol{v}} \\ \overline{\delta\boldsymbol{w}} \\ \overline{\delta\boldsymbol{u}} \end{pmatrix} \right\rangle$$

$$= \left\langle s\boldsymbol{u}, \delta\boldsymbol{v}\overline{\delta'\boldsymbol{v}} - \delta'\boldsymbol{v}\overline{\delta\boldsymbol{v}} + \delta\boldsymbol{w}\overline{\delta'\boldsymbol{w}} - \delta'\boldsymbol{w}\overline{\delta\boldsymbol{w}} + \delta\boldsymbol{u}\overline{\delta'\boldsymbol{u}} - \delta'\boldsymbol{u}\overline{\delta\boldsymbol{u}} \right\rangle$$

$$= \left\langle s\boldsymbol{u}, j(\delta'\boldsymbol{v} \times \delta\boldsymbol{v}) + j(\delta'\boldsymbol{w} \times \delta\boldsymbol{w}) + j(\delta'\boldsymbol{u} \times \delta\boldsymbol{u}) \right\rangle$$

$$= s \left\langle \boldsymbol{u}, \delta'\boldsymbol{u} \times \delta\boldsymbol{u} \right\rangle \tag{2.87}$$

since $\delta'\boldsymbol{v} \times \delta\boldsymbol{v}$ and $\delta'\boldsymbol{w} \times \delta\boldsymbol{w}$ are parallel to $\boldsymbol{v}$ (resp. $\boldsymbol{w}$), hence perpendicular to $\boldsymbol{u}$. So σ is $(-s)$ times the ordinary area 2-form of the unit sphere S^2.

CHAPTER 3
THE CROSS SECTION LEMMA

Chapter 2 has shown us that to find all homogeneous symplectic manifolds of a given group, G, one need look no further than coadjoint orbits of central extensions of G. However, classifying the orbits of any G-action is a challenging problem that we will need tools to address. In this chapter we provide such a tool, and apply it to several examples which will turn up as building blocks in the case, ultimately relevant to us, of the Galilei group.

3.1 Building Cross Sections

Let a group G act on a space W. In practice, the orbit classification problem amounts to exhibiting a subset, $\mathrm{W}^{\mathrm{o}} \subset \mathrm{W}$, that intersects each orbit exactly once. Such a subset is called a (set-theoretic) ***cross section*** for the action. For example, in (2.85) we considered the standard action of the rotation group SO(3) on $\mathrm{W} = \mathbb{R}^3$. Here the orbits are (1) all spheres $\{\mathbf{L} \mid \|\mathbf{L}\| = s\}$ and (2) the origin $\{0\}$. The vertical half-line $\mathrm{W}^{\mathrm{o}} = \{s\boldsymbol{e}_3 \mid s \geqslant 0\}$ meets each exactly once, so it is a cross section.

To sort out more complicated cases, the following lemma will prove useful. It breaks the classification problem down into two simpler ones: first, find a cross section U^{o} for the action of G on a smaller space U (the 'base'); then, for each $u \in \mathrm{U}^{\mathrm{o}}$, find a cross section for the action on a smaller space V (the 'fiber') of a smaller group G_u (the 'stabilizer' of u).

Proposition 3.1 (Cross Section Lemma). *Suppose that the action of a group* G *on a product space* $\mathrm{W} = \mathrm{V} \times \mathrm{U}$ *has the form*

$$g\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} g_u(v) \\ g(u) \end{pmatrix} \tag{3.1}$$

where $g(u)$ *does not depend on* v. *Then the second row defines an action of* G *on* U, *and for each* $u \in \mathrm{U}$ *the first row defines an action on* V *of the stabilizer* $\mathrm{G}_u := \{g \in \mathrm{G} \mid g(u) = u\}$. *Moreover, if*

$$\begin{aligned} \mathrm{U}^{\mathrm{o}} \subset \mathrm{U} \quad & \textit{is a cross section for the action of } \mathrm{G} \textit{ on } \mathrm{U}, \textit{ and} \\ (\forall u \in \mathrm{U}^{\mathrm{o}}) \quad \mathrm{V}^{\mathrm{o}}_u \subset \mathrm{V} \quad & \textit{is a cross section for the action of } \mathrm{G}_u \textit{ on } \mathrm{V}, \end{aligned}$$

then

$$\mathrm{W}^{\mathrm{o}} := \left\{ \begin{pmatrix} v \\ u \end{pmatrix} \,\middle|\, \begin{matrix} v \in \mathrm{V}^{\mathrm{o}}_u \\ u \in \mathrm{U}^{\mathrm{o}} \end{matrix} \right\} \quad \textit{is a cross section for the action of } \mathrm{G} \textit{ on } \mathrm{W}. \tag{3.2}$$

Proof. Since G acts on W we have the relation $(gg')\left(\begin{smallmatrix} v \\ u \end{smallmatrix}\right) = g\left(g'\left(\begin{smallmatrix} v \\ u \end{smallmatrix}\right)\right)$ which expands as

$$\begin{pmatrix} (gg')_u(v) \\ (gg')(u) \end{pmatrix} = \begin{pmatrix} g_{g'(u)}(g'_u(v)) \\ g(g'(u)) \end{pmatrix}. \tag{3.3}$$

Here the second row, $(gg')(u) = g(g'(u))$, shows that we have an action of G on U. Moreover if g and g' belong to the stabilizer G_u, then so does gg' and the first row writes $(gg')_u(v) = g_u(g'_u(v))$, showing that we have indeed an action of G_u on V. Next we show that every G-orbit $\mathrm{X} = \mathrm{G}\left(\begin{smallmatrix} v_0 \\ u_0 \end{smallmatrix}\right)$ intersects W^{o}. Since U^{o} is a cross section, we can find $g \in \mathrm{G}$ such that $g(u_0) \in \mathrm{U}^{\mathrm{o}}$. Write $u = g(u_0)$ and $v_1 = g_{u_0}(v_0)$ so that we have $g\left(\begin{smallmatrix} v_0 \\ u_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} v_1 \\ u \end{smallmatrix}\right)$. Since $\mathrm{V}^{\mathrm{o}}_u$ is a cross section, we can find $h \in \mathrm{G}_u$ such that $h_u(v_1) \in \mathrm{V}^{\mathrm{o}}_u$. Putting $v = h_u(v_1)$, we obtain

$$h\left(g\begin{pmatrix} v_0 \\ u_0 \end{pmatrix}\right) = \begin{pmatrix} h_u(v_1) \\ h(u) \end{pmatrix} = \begin{pmatrix} v \\ u \end{pmatrix} \in \mathrm{W}^{\mathrm{o}}. \tag{3.4}$$

So X intersects W^{o}, as claimed. Finally we show that each orbit intersects W^{o} at most once. In fact, assume that

$$\begin{pmatrix} v \\ u \end{pmatrix} \in \mathrm{W}^{\mathrm{o}} \qquad \text{and} \qquad g\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} g_u(v) \\ g(u) \end{pmatrix} \in \mathrm{W}^{\mathrm{o}}. \tag{3.5}$$

Then both u and $g(u)$ belong to U^{o}. Since this is a cross section, it follows that $g(u) = u$ and hence $g \in \mathrm{G}_u$. Now both v and $g_u(v)$ belong to $\mathrm{V}^{\mathrm{o}}_u$. Since this is a cross section, it follows that $g_u(v) = v$. Hence the two points (3.5) coincide, as claimed. □

We will summarize the setting of this proposition with the diagram

$$\overset{\mathrm{G}}{\curvearrowright}\!\!\!\!\!\!\mathrm{W} \xrightarrow[\mathrm{V}]{} \overset{\mathrm{G}}{\curvearrowright}\!\!\!\!\!\!\mathrm{U}. \tag{3.6}$$

Here the map is the projection $\pi : \left(\begin{smallmatrix} v \\ u \end{smallmatrix}\right) \mapsto u$ of the total space W onto the base U, and the decorations emphasize that π has typical fiber $\simeq$ V and keeps two G-actions in lockstep. (Precisely: $\pi^{-1}(u) = \mathrm{V} \times \{u\}$ and, as (3.1) shows, $\pi(g(w)) = g(\pi(w))$.) In this setting, every G-orbit in the total space "sits above" a G-orbit in the base, and the proposition shows how classifying the former boils down to classifying (1) G-orbits in the base and (2) stabilizer orbits in selected fibers.

3.2 Example 1: The Heisenberg Group

In this section we classify and describe the coadjoint orbits of the ***Heisenberg group***

$$\mathrm{G} = \left\{ g = \begin{pmatrix} 1 & b & f \\ & 1 & c \\ & & 1 \end{pmatrix} \;\middle|\; b, c, f \in \mathbb{R} \right\}. \tag{3.7}$$

Its Lie algebra (2.65) is

$$\mathfrak{g} = \left\{ \mathrm{Z} = \begin{pmatrix} 0 & \beta & \varphi \\ & 0 & \gamma \\ & & 0 \end{pmatrix} \;\middle|\; \beta, \gamma, \varphi \in \mathbb{R} \right\}. \tag{3.8}$$

This has dimension 3, so we may identify the dual space $\mathfrak{g}^*$ with $\mathbb{R}^3$:

$$\mathfrak{g}^* = \left\{ x = \begin{bmatrix} \mathrm{P} \\ \mathrm{Q} \\ \mathrm{R} \end{bmatrix} \;\middle|\; \mathrm{P}, \mathrm{Q}, \mathrm{R} \in \mathbb{R} \right\} \tag{3.9}$$

paired to $\mathfrak{g}$ by

$$\langle x, \mathrm{Z} \rangle := \mathrm{P}\gamma - \mathrm{Q}\beta - \mathrm{R}\varphi. \tag{3.10}$$

Using this notation, we have:

Proposition 3.2. *The coadjoint action of* G *is given by*

$$g \begin{bmatrix} \mathrm{P} \\ \mathrm{Q} \\ \mathrm{R} \end{bmatrix} = \begin{bmatrix} \mathrm{P} + \mathrm{R}b \\ \mathrm{Q} + \mathrm{R}c \\ \mathrm{R} \end{bmatrix}. \tag{3.11}$$

A cross section for this action is the set $\mathrm{S}_1 \cup \mathrm{S}_2$*, where*

$$\mathrm{S}_1 = \left\{ x_0 = \begin{bmatrix} p \\ q \\ 0 \end{bmatrix} \;\middle|\; p, q \in \mathbb{R} \right\}, \qquad \mathrm{S}_2 = \left\{ x_0 = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} \;\middle|\; r \neq 0 \right\}. \tag{3.12}$$

Proof. Let us begin by computing the group action on the dual to the Lie algebra. We have

$$\begin{aligned} \langle g(x), \mathrm{Z} \rangle = \langle x, g^{-1}\mathrm{Z}g \rangle &= \left\langle \begin{bmatrix} \mathrm{P} \\ \mathrm{Q} \\ \mathrm{R} \end{bmatrix}, \begin{pmatrix} 1 & -b & -f + bc \\ & 1 & -c \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \beta & \varphi \\ & 0 & \gamma \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & b & f \\ & 1 & c \\ & & 1 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} \mathrm{P} \\ \mathrm{Q} \\ \mathrm{R} \end{bmatrix}, \begin{pmatrix} 0 & \beta & \varphi + \beta c - b\gamma \\ & 0 & \gamma \\ & & 0 \end{pmatrix} \right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{P}\gamma - \mathrm{Q}\beta - \mathrm{R}(\varphi + \beta c - b\gamma) \\
&= (\mathrm{P} + \mathrm{R}b)\gamma - (\mathrm{Q} + \mathrm{R}c)\beta - \mathrm{R}\varphi \\
&= \left\langle \begin{bmatrix} \mathrm{P} + \mathrm{R}b \\ \mathrm{Q} + \mathrm{R}c \\ \mathrm{R} \end{bmatrix}, \begin{pmatrix} 0 & \beta & \varphi \\ & 0 & \gamma \\ & & 0 \end{pmatrix} \right\rangle .
\end{aligned}
$$

This establishes (3.11). To classify the orbits, we observe that the action (3.11) leaves R invariant. So we have a very simple instance of diagram (3.6):

$$
\begin{array}{ccc}
\overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} \\
\mathfrak{g}^* & \xrightarrow[\mathbb{R}^2]{} & \mathbb{R} \\
\begin{bmatrix} \mathrm{P} \\ \mathrm{Q} \\ \mathrm{R} \end{bmatrix} & \longmapsto & \mathrm{R}
\end{array}
$$

and Proposition 3.1 gives us an algorithm to classify the G-orbits in $\mathfrak{g}^*$. First we find a cross section U^{o} for the action of G on the base variable $u = \mathrm{R}$. From (3.11), this action is trivial: $g(\mathrm{R}) = \mathrm{R}$, so we must take

$$\mathrm{U}^{\mathrm{o}} = \{ r \mid r \in \mathbb{R} \} . \tag{3.13}$$

Next, for each $u \in \mathrm{U}^{\mathrm{o}}$ we find the stabilizer G_u and read off on (3.11) its action on the fiber variable $v = \left[\begin{smallmatrix} \mathrm{P} \\ \mathrm{Q} \end{smallmatrix}\right]$. There are two cases according to the nature of the base orbit, if $\mathrm{G}(u)$ is 0 or not:

$u \in (3.13)$	$\mathrm{G}(u)$	G_u	$g_u(v)$
0	$\{0\}$	G	$\left[\begin{smallmatrix} \mathrm{P} \\ \mathrm{Q} \end{smallmatrix}\right]$
$r \quad (r \neq 0)$	$\{r\}$	G	$\left[\begin{smallmatrix} \mathrm{P}+rb \\ \mathrm{Q}+rc \end{smallmatrix}\right]$

There remains to classify the orbits of the action of $\mathrm{G}_u = \mathrm{G}$ on $v = \left[\begin{smallmatrix} \mathrm{P} \\ \mathrm{Q} \end{smallmatrix}\right]$ given in the rightmost column. The action in the first row is trivial, so its orbits are points and $\mathrm{V}_u^{\mathrm{o}} = \{ \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] \mid p, q \in \mathbb{R} \}$ is a cross section. In the second row we have the action of $\mathbb{R}^2$ on itself by translation which is transitive: so the origin $\mathrm{V}_u^{\mathrm{o}} = \{ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \}$ is a cross section. Putting U^{o} and the $\mathrm{V}_u^{\mathrm{o}}$'s together as prescribed by (3.2), we obtain (3.12). □

We now describe the structure of each orbit as a (symplectic) manifold:

Proposition 3.3. *Splitting cases as in Proposition 3.2:*

(1) *If* $x_0 \in \mathrm{S}_1$, *the coadjoint orbit* $\mathrm{G}(x_0)$ *is the point* $\{x_0\}$ *with the trivial 2-form.*

(2) *If* $x_0 \in \mathrm{S}_2$, *the coadjoint orbit* $\mathrm{G}(x_0)$ *is the plane*

$$\left\{ x = \begin{bmatrix} \mathrm{P} \\ \mathrm{Q} \\ r \end{bmatrix} \;\middle|\; \mathrm{P}, \mathrm{Q} \in \mathbb{R} \right\} \tag{3.14}$$

with 2-form

$$\sigma(\delta x, \delta' x) = \frac{1}{r}(\delta \mathrm{P}\, \delta' \mathrm{Q} - \delta' \mathrm{P}\, \delta \mathrm{Q}). \tag{3.15}$$

Proof. Case (1) is clear. For case (2), we act on the base point x_0 (3.12) by (3.11). We find

$$\mathrm{G}(x_0) = \{g(x_0) | g \in \mathrm{G}\} = \left\{ x = \begin{bmatrix} rb \\ rc \\ r \end{bmatrix} \;\middle|\; b, c \in \mathbb{R} \right\}.$$

Thus we see that $g\,(x_0)$ only depends on g via the variables $(\mathrm{P}, \mathrm{Q}) := (rb, rc)$, giving (3.14). Now we express σ in terms of (P, Q). By (2.81):[1]

$$\begin{aligned}
\sigma(\delta x, \delta' x) &= \left\langle x_0, \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g \right\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}, \delta \begin{pmatrix} 1 & -b & -f+bc \\ & 1 & -c \\ & & 1 \end{pmatrix} \delta' \begin{pmatrix} 1 & b & f \\ & 1 & c \\ & & 1 \end{pmatrix} - \delta' \begin{pmatrix} 1 & -b & -f+bc \\ & 1 & -c \\ & & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & b & f \\ & 1 & c \\ & & 1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}, \begin{pmatrix} 0 & -\delta b & * \\ & 0 & -\delta c \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & \delta' b & \delta' f \\ & 0 & \delta' c \\ & & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\delta' b & * \\ & 0 & -\delta' c \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & \delta b & \delta f \\ & 0 & \delta c \\ & & 0 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}, \begin{pmatrix} 0 & 0 & -\delta b\, \delta' c + \delta' b\, \delta c \\ & 0 & 0 \\ & & 0 \end{pmatrix} \right\rangle \\
&= r(\delta b\, \delta' c - \delta' b\, \delta c) \\
&= \frac{1}{r}\left(\delta \mathrm{P}\, \delta' \mathrm{Q} - \delta' \mathrm{P}\, \delta \mathrm{Q}\right).
\end{aligned}$$

This gives (3.15). □

3.3 Example 2: The Euclidean Group in Three Dimensions

As our next example, we describe the coadjoint orbits of the ***Euclidean group***

$$\mathrm{G} = \left\{ g = \begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} \mathrm{A} \in \mathrm{SO}(3) \\ \boldsymbol{c} \in \mathbb{R}^3 \end{array} \right\}. \tag{3.16}$$

Its Lie algebra (2.65) is, using the notation j from (2.83),

$$\mathfrak{g} = \left\{ \mathrm{Z} = \begin{pmatrix} j(\boldsymbol{\alpha}) & \boldsymbol{\gamma} \\ 0 & 0 \end{pmatrix} \;\middle|\; \boldsymbol{\alpha}, \boldsymbol{\gamma} \in \mathbb{R}^3 \right\}. \tag{3.17}$$

[1] In what follows a star '$*$' shall denote a matrix entry whose value does not matter.

This has dimension 6, so we may identify the dual space $\mathfrak{g}^*$ with $\mathbb{R}^6$:

$$\mathfrak{g}^* = \left\{ x = \begin{bmatrix} \mathbf{L} \\ \mathbf{P} \end{bmatrix} \;\middle|\; \mathbf{L}, \mathbf{P} \in \mathbb{R}^3 \right\}, \tag{3.18}$$

paired to $\mathfrak{g}$ by

$$\langle x, \mathrm{Z} \rangle := \langle \mathbf{L}, \boldsymbol{\alpha} \rangle + \langle \mathbf{P}, \boldsymbol{\gamma} \rangle. \tag{3.19}$$

Using this notation, we have:

Proposition 3.4. *The coadjoint action of* G *is given by*

$$g \begin{bmatrix} \mathbf{L} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathrm{A}\mathbf{L} + \boldsymbol{c} \times \mathrm{A}\mathbf{P} \\ \mathrm{A}\mathbf{P} \end{bmatrix}. \tag{3.20}$$

A cross section for this action is the set $\mathrm{S}_1 \cup \mathrm{S}_2 \cup \mathrm{S}_3$*, where*

$$\mathrm{S}_1 = \{0\}, \qquad \mathrm{S}_2 = \left\{ x_2 = \begin{bmatrix} s\boldsymbol{e}_3 \\ 0 \end{bmatrix} \;\middle|\; s > 0 \right\}, \qquad \mathrm{S}_3 = \left\{ x_3 = \begin{bmatrix} s\boldsymbol{e}_3 \\ k\boldsymbol{e}_3 \end{bmatrix} \;\middle|\; \begin{matrix} s \in \mathbb{R} \\ k > 0 \end{matrix} \right\}. \tag{3.21}$$

Proof. A euclidean transformation (3.16) can always be written as the product of a translation and a rotation:

$$\underbrace{\begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix}}_{g} = \underbrace{\begin{pmatrix} \mathbb{1} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix}}_{g_1} \underbrace{\begin{pmatrix} \mathrm{A} & 0 \\ 0 & 1 \end{pmatrix}}_{g_2}.$$

Let us compute the action of each factor and then compose the results. We have

$$\begin{aligned}
\langle g_1(x), \mathrm{Z} \rangle = \langle x, g_1^{-1} \mathrm{Z} g_1 \rangle &= \left\langle \begin{bmatrix} \mathbf{L} \\ \mathbf{P} \end{bmatrix}, \begin{pmatrix} \mathbb{1} & -\boldsymbol{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j(\boldsymbol{\alpha}) & \boldsymbol{\gamma} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} \mathbf{L} \\ \mathbf{P} \end{bmatrix}, \begin{pmatrix} j(\boldsymbol{\alpha}) & \boldsymbol{\alpha} \times \boldsymbol{c} + \boldsymbol{\gamma} \\ 0 & 0 \end{pmatrix} \right\rangle \\
&= \langle \mathbf{L}, \boldsymbol{\alpha} \rangle + \langle \mathbf{P}, \boldsymbol{\alpha} \times \boldsymbol{c} + \boldsymbol{\gamma} \rangle \\
&= \langle \mathbf{L} + \boldsymbol{c} \times \mathbf{P}, \boldsymbol{\alpha} \rangle + \langle \mathbf{P}, \boldsymbol{\gamma} \rangle \\
&= \left\langle \begin{bmatrix} \mathbf{L} + \boldsymbol{c} \times \mathbf{P} \\ \mathbf{P} \end{bmatrix}, \begin{pmatrix} j(\boldsymbol{\alpha}) & \boldsymbol{\gamma} \\ 0 & 0 \end{pmatrix} \right\rangle
\end{aligned}$$

and

$$\langle g_2(x), \mathrm{Z} \rangle = \langle x, g_2^{-1} \mathrm{Z} g_2 \rangle = \left\langle \begin{bmatrix} \mathbf{L} \\ \mathbf{P} \end{bmatrix}, \begin{pmatrix} \mathrm{A}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j(\boldsymbol{\alpha}) & \boldsymbol{\gamma} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{A} & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

$$
\begin{aligned}
&= \left\langle \begin{bmatrix} \mathbf{L} \\ \mathbf{P} \end{bmatrix}, \begin{pmatrix} j(\mathrm{A}^{-1}\boldsymbol{\alpha}) & \mathrm{A}^{-1}\boldsymbol{\gamma} \\ 0 & 0 \end{pmatrix} \right\rangle \\
&= \langle \mathbf{L}, \mathrm{A}^{-1}\boldsymbol{\alpha}\rangle + \langle \mathbf{P}, \mathrm{A}^{-1}\boldsymbol{\gamma}\rangle \\
&= \langle \mathrm{A}\mathbf{L}, \boldsymbol{\alpha}\rangle + \langle \mathrm{A}\mathbf{P}, \boldsymbol{\gamma}\rangle \\
&= \left\langle \begin{bmatrix} \mathrm{A}\mathbf{L} \\ \mathrm{A}\mathbf{P} \end{bmatrix}, \begin{pmatrix} j(\boldsymbol{\alpha}) & \boldsymbol{\gamma} \\ 0 & 0 \end{pmatrix} \right\rangle.
\end{aligned}
$$

Thus we get $g_1(x) = \left[\begin{smallmatrix} \mathbf{L}+\boldsymbol{c}\times\mathbf{P} \\ \mathbf{P} \end{smallmatrix}\right]$ and $g_2(x) = \left[\begin{smallmatrix} \mathrm{A}\mathbf{L} \\ \mathrm{A}\mathbf{P} \end{smallmatrix}\right]$; composing the two, we obtain (3.20). To classify the orbits, we observe that the action (3.20) transforms the components of $\mathbf{P}$ among themselves. So we have an instance of diagram (3.6):

$$
\begin{array}{ccc}
\overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} \\
\mathfrak{g}^* & \xrightarrow[\mathbb{R}^3]{} & \mathbb{R}^3 \\
\begin{bmatrix} \mathbf{L} \\ \mathbf{P} \end{bmatrix} & \longmapsto & \mathbf{P}
\end{array}
$$

and Proposition 3.1 gives us an algorithm to classify the G-orbits in $\mathfrak{g}^*$. First we find a cross section U^{o} for the action of G on the base variable $u = \mathbf{P}$. From (3.20), this action writes $g(\mathbf{P}) = \mathrm{A}\mathbf{P}$, so we can take

$$
\mathrm{U}^{\mathrm{o}} = \{k\boldsymbol{e}_3 \mid k \geqslant 0\}. \tag{3.22}
$$

Next, for each $u \in \mathrm{U}^{\mathrm{o}}$ we find the stabilizer G_u and read off on (3.20) its action on the fiber variable $v = \mathbf{L}$. There are two cases according as the base orbit $\mathrm{G}(u)$ is $\{0\}$ or a sphere:

$u \in (3.22)$	$\mathrm{G}(u)$	G_u	$g_u(v)$
0	$\{0\}$	G	$\mathrm{A}\mathbf{L}$
$k\boldsymbol{e}_3 \quad (k > 0)$	a sphere	$\{g \in \mathrm{G} \mid \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3\}$	$\mathrm{A}\mathbf{L} + k\boldsymbol{c} \times \boldsymbol{e}_3$

There remains to classify the orbits of the G_u-action on $\mathbf{L} \in \mathbb{R}^3$ given in the rightmost column. In the first row we have the standard action of the full rotation group; its orbits are spheres and $\{0\}$, so the half-line $\mathrm{V}^{\mathrm{o}}_u = \{s\boldsymbol{e}_3 \mid s \geqslant 0\}$ is a cross section. In the second row the action rotates $\mathbf{L}$ about the vertical axis and adds $k\boldsymbol{c} \times \boldsymbol{e}_3$, which can be any horizontal vector; so the orbits are all horizontal planes, and the vertical line $\mathrm{V}^{\mathrm{o}}_u = \{s\boldsymbol{e}_3 \mid s \in \mathbb{R}\}$ is a cross section. Now, putting U^{o} and the $\mathrm{V}^{\mathrm{o}}_u$'s together as prescribed by (3.2), we obtain the cross section (3.21). □

We now describe the structure of each orbit as a (symplectic) manifold:

Proposition 3.5. *Splitting cases as in Proposition 3.4:*

(1) *If* $x_0 \in \mathrm{S}_1$*, the coadjoint orbit* $\mathrm{G}(x_0)$ *is the origin* $\{0\}$ *with the trivial 2-form.*

(2) *If* $x_0 \in \mathrm{S}_2$*, the coadjoint orbit* $\mathrm{G}(x_0)$ *is a copy of the 2-sphere*

$$\left\{ x = \begin{bmatrix} s\boldsymbol{u} \\ 0 \end{bmatrix} \;\middle|\; \|\boldsymbol{u}\| = 1 \right\} \tag{3.23}$$

with 2-form

$$\sigma(\delta x, \delta' x) = s\langle \boldsymbol{u}, \delta'\boldsymbol{u} \times \delta\boldsymbol{u} \rangle. \tag{3.24}$$

(3) *If* $x_0 \in \mathrm{S}_3$*, the coadjoint orbit* $\mathrm{G}(x_0)$ *is a copy of* TS^2

$$\left\{ x = \begin{bmatrix} \boldsymbol{r} \times k\boldsymbol{u} + s\boldsymbol{u} \\ k\boldsymbol{u} \end{bmatrix} \;\middle|\; \begin{matrix} \langle \boldsymbol{r}, \boldsymbol{u} \rangle = 0 \\ \|\boldsymbol{u}\| = 1 \end{matrix} \right\} \tag{3.25}$$

with 2-form

$$\sigma(\delta x, \delta' x) = k\left[\langle \delta\boldsymbol{u}, \delta'\boldsymbol{r} \rangle - \langle \delta'\boldsymbol{u}, \delta\boldsymbol{r} \rangle\right] + s\langle \boldsymbol{u}, \delta'\boldsymbol{u} \times \delta\boldsymbol{u} \rangle. \tag{3.26}$$

Proof. Case (1) in clear. In case (3) we find

$$\mathrm{G}(x_0) = \{g(x_0) \mid g \in \mathrm{G}\} = \left\{ x = \begin{bmatrix} s\mathrm{A}\boldsymbol{e}_3 + \boldsymbol{c} \times k\mathrm{A}\boldsymbol{e}_3 \\ k\mathrm{A}\boldsymbol{e}_3 \end{bmatrix} \;\middle|\; \begin{matrix} \boldsymbol{c} & \in \mathbb{R}^3 \\ \mathrm{A} & \in \mathrm{SO}(3) \end{matrix} \right\}.$$

Thus we see that $g\,(x_0)$ only depends on g via the variables

$$\begin{cases} \boldsymbol{u} := \mathrm{A}\boldsymbol{e}_3 \\ \boldsymbol{r} := [\mathbb{1} - \mathrm{A}\boldsymbol{e}_3\overline{\mathrm{A}\boldsymbol{e}_3}]\boldsymbol{c}. \end{cases}$$

and we obtain (3.25). Now let us express σ in terms of $(\boldsymbol{u}, \boldsymbol{r})$. By (2.81):

$$\begin{aligned}
\sigma(\delta x, \delta' x) &= \left\langle x_0, \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g \right\rangle \\
&= \left\langle \begin{bmatrix} s\boldsymbol{e}_3 \\ k\boldsymbol{e}_3 \end{bmatrix}, \delta \begin{pmatrix} \overline{\mathrm{A}} & -\overline{\mathrm{A}}\boldsymbol{c} \\ 0 & 1 \end{pmatrix} \delta' \begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} - \delta' \begin{pmatrix} \overline{\mathrm{A}} & -\overline{\mathrm{A}}\boldsymbol{c} \\ 0 & 1 \end{pmatrix} \delta \begin{pmatrix} \mathrm{A} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} s\boldsymbol{e}_3 \\ k\boldsymbol{e}_3 \end{bmatrix}, \begin{pmatrix} \overline{\delta\mathrm{A}} & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta'\mathrm{A} & \delta'\boldsymbol{c} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \overline{\delta'\mathrm{A}} & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta\mathrm{A} & \delta\boldsymbol{c} \\ 0 & 0 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} s\boldsymbol{e}_3 \\ k\boldsymbol{e}_3 \end{bmatrix}, \begin{pmatrix} \overline{\delta\mathrm{A}}\delta'\mathrm{A} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{c} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \overline{\delta'\mathrm{A}}\delta\mathrm{A} & \delta'\mathrm{A}\delta\boldsymbol{c} \\ 0 & 0 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} s\boldsymbol{e}_3 \\ k\boldsymbol{e}_3 \end{bmatrix}, \begin{pmatrix} \overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{c} - \overline{\delta'\mathrm{A}}\delta\boldsymbol{c} \\ 0 & 0 \end{pmatrix} \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle s\boldsymbol{e}_3,\ j^{-1}\left(\overline{\delta \mathrm{A}}\delta' \mathrm{A} - \overline{\delta' \mathrm{A}}\delta \mathrm{A}\right)\rangle + k\langle \boldsymbol{e}_3, \overline{\delta \mathrm{A}}\delta' \boldsymbol{c} - \overline{\delta' \mathrm{A}}\delta \boldsymbol{c}\rangle \\
&= \langle s\boldsymbol{e}_3,\ j^{-1}\left(\overline{\delta \mathrm{A}}\delta' \mathrm{A} - \overline{\delta' \mathrm{A}}\delta \mathrm{A}\right)\rangle + k\left[\langle \delta\left[\mathrm{A}\boldsymbol{e}_3\right], \delta' \boldsymbol{c}\rangle - \langle \delta'\left[\mathrm{A}\boldsymbol{e}_3\right], \delta \boldsymbol{c}\rangle\right] \\
&= \ s\left\langle \boldsymbol{u}, \delta' \boldsymbol{u} \times \delta \boldsymbol{u}\right\rangle + k\left[\langle \delta \boldsymbol{u}, \delta' \boldsymbol{r}\rangle - \langle \delta' \boldsymbol{u}, \delta \boldsymbol{r}\rangle\right]. \qquad (3.27)
\end{aligned}
$$

In the first term, we have recognized the 2-form (2.81, 2.82, 2.86, 2.87) on a coadjoint orbit of SO(3), a sphere with radius s. This proves (3.26). Case (2) is similar to case (3), but $k = 0$ and the k term drops out giving (3.23) and (3.24). □

3.4 Example 3: The Bargmann Group in 1+1 Space-time Dimensions

As a final example we describe the coadjoint orbits of the ***Bargmann group*** in 1+1 space-time dimensions,

$$
\mathrm{G} = \left\{ g = \begin{pmatrix} 1 & b & \frac{1}{2}b^2 & f \\ & 1 & b & c \\ & & 1 & e \\ & & & 1 \end{pmatrix} \ \middle|\ b, c, e, f \in \mathbb{R} \right\}. \qquad (3.28)
$$

Its Lie algebra (2.65) is

$$
\mathfrak{g} = \left\{ \mathrm{Z} = \begin{pmatrix} 0 & \beta & 0 & \varphi \\ & 0 & \beta & \gamma \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \ \middle|\ \beta, \gamma, \varepsilon, \varphi \in \mathbb{R} \right\}. \qquad (3.29)
$$

This has dimension 4, so we may identify the dual space $\mathfrak{g}^*$ with $\mathbb{R}^4$:

$$
\mathfrak{g}^* = \left\{ x = \begin{bmatrix} \mathrm{G} \\ \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix} \ \middle|\ \mathrm{G}, \mathrm{E}, \mathrm{P}, \mathrm{M} \in \mathbb{R} \right\},^{2} \qquad (3.30)
$$

paired to $\mathfrak{g}$ by

$$
\langle x, \mathrm{Z}\rangle := \mathrm{P}\gamma - \mathrm{G}\beta - \mathrm{E}\varepsilon - \mathrm{M}\varphi. \qquad (3.31)
$$

Proposition 3.6. *The coadjoint action of* G *is given by*

$$
g \begin{bmatrix} \mathrm{G} \\ \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix} = \begin{bmatrix} (\mathrm{G} - \mathrm{P}e) + \mathrm{M}(c - be) \\ \mathrm{E} + b\mathrm{P} + \frac{1}{2}\mathrm{M}b^2 \\ \mathrm{P} + \mathrm{M}b \\ \mathrm{M} \end{bmatrix}. \qquad (3.32)
$$

[2] This double use of the letter G is traditional and should not cause confusion.

A cross section for this action is the set $\mathrm{S}_1 \cup \mathrm{S}_2 \cup \mathrm{S}_3$*, where*

$$\mathrm{S}_1 = \left\{ x_0 = \begin{bmatrix} n \\ c \\ 0 \\ 0 \end{bmatrix} \;\middle|\; n, c \in \mathbb{R} \right\}, \qquad \mathrm{S}_2 = \left\{ x_0 = \begin{bmatrix} 0 \\ 0 \\ k \\ 0 \end{bmatrix} \;\middle|\; k \neq 0 \right\},$$

$$\mathrm{S}_3 = \left\{ x_0 = \begin{bmatrix} 0 \\ c \\ 0 \\ m \end{bmatrix} \;\middle|\; \begin{matrix} c \in \mathbb{R} \\ m \neq 0 \end{matrix} \right\}. \tag{3.33}$$

Proof. We begin by looking at g as a composition of a "translation" g_1 and a "boost" g_2:

$$\underbrace{\begin{pmatrix} 1 & b & \frac{1}{2}b^2 & f \\ & 1 & b & c \\ & & 1 & e \\ & & & 1 \end{pmatrix}}_{g} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & f \\ & 1 & 0 & c \\ & & 1 & e \\ & & & 1 \end{pmatrix}}_{g_1} \underbrace{\begin{pmatrix} 1 & b & \frac{1}{2}b^2 & 0 \\ & 1 & b & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}}_{g_2}.$$

We have

$$\begin{aligned}
\langle g_1(x), \mathrm{Z} \rangle &= \langle x, g_1^{-1} \mathrm{Z} g_1 \rangle \\
&= \left\langle \begin{bmatrix} \mathrm{G} \\ \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix}, \begin{pmatrix} 1 & 0 & 0 & -f \\ & 1 & 0 & -c \\ & & 1 & -e \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & \beta & 0 & \varphi \\ & 0 & \beta & \gamma \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & f \\ & 1 & 0 & c \\ & & 1 & e \\ & & & 1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} \mathrm{G} \\ \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix}, \begin{pmatrix} 0 & \beta & 0 & \varphi + \beta c \\ & 0 & \beta & \gamma + \beta e \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \right\rangle \\
&= \mathrm{P}(\gamma + \beta e) - \mathrm{G}\beta - \mathrm{E}\varepsilon - \mathrm{M}(\varphi + \beta c) \\
&= \mathrm{P}\gamma - (\mathrm{G} - \mathrm{P}e + \mathrm{M}c)\beta - \mathrm{E}\varepsilon - \mathrm{M}\varphi \\
&= \left\langle \begin{bmatrix} \mathrm{G} - \mathrm{P}e + \mathrm{M}c \\ \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix}, \mathrm{Z} \right\rangle.
\end{aligned}$$

and

$$\langle g_2(x), \mathrm{Z} \rangle = \langle x, g_2^{-1} \mathrm{Z} g_2 \rangle$$

$$
\begin{aligned}
&= \left\langle \begin{bmatrix} \mathrm{G} \\ \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix}, \begin{pmatrix} 1 & -b & \frac{1}{2}b^2 & 0 \\ & 1 & -b & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & \beta & 0 & \varphi \\ & 0 & \beta & \gamma \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \begin{pmatrix} 1 & b & \frac{1}{2}b^2 & 0 \\ & 1 & b & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} \mathrm{G} \\ \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix}, \begin{pmatrix} 0 & \beta & 0 & \varphi - b\gamma + \frac{1}{2}b^2\varepsilon \\ & 0 & \beta & \gamma - b\varepsilon \\ & & 0 & \varepsilon \\ & & & \end{pmatrix} \right\rangle \\
&= \mathrm{P}(\gamma - b\varepsilon) - \mathrm{G}\beta - \mathrm{E}\varepsilon - \mathrm{M}(\varphi - b\gamma + \tfrac{1}{2}b^2\varepsilon) \\
&= (\mathrm{P} + \mathrm{M}b)\gamma - \mathrm{G}\beta - (\mathrm{E} + b\mathrm{P} + \tfrac{1}{2}\mathrm{M}b^2)\varepsilon - \mathrm{M}\varphi \\
&= \left\langle \begin{bmatrix} \mathrm{G} \\ \mathrm{E} + b\mathrm{P} + \frac{1}{2}\mathrm{M}b^2 \\ \mathrm{P} + \mathrm{M}b \\ \mathrm{M} \end{bmatrix}, \mathrm{Z} \right\rangle
\end{aligned}
$$

Whence

$$
\begin{aligned}
g(x) &= g_1(g_2(x)) \\
&= g_1 \begin{bmatrix} \mathrm{G} \\ \mathrm{E} + b\mathrm{P} + \frac{1}{2}\mathrm{M}b^2 \\ \mathrm{P} + \mathrm{M}b \\ \mathrm{M} \end{bmatrix} \\
&= \begin{bmatrix} (\mathrm{G} - \mathrm{P}e) + \mathrm{M}(c - be) \\ \mathrm{E} + \mathrm{P}b + \frac{1}{2}\mathrm{M}b^2 \\ \mathrm{P} + \mathrm{M}b \\ \mathrm{M} \end{bmatrix}.
\end{aligned}
$$

This proves (3.32). Next, we again use Proposition 3.1 to find a cross section of the orbits. From looking at (3.32) we see layers of invariance in the coadjoint action. The simplest is the invariance of M. Next we see that the two variables P, M transform among themselves, and so do the three variables E, P, M. Thus, we have a tower of four actions sitting above each other:

$$
\begin{array}{ccccccc}
\overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} \\
\mathfrak{g}^* & \xrightarrow[\mathbb{R}]{} & \mathbb{R}^3 & \xrightarrow[\mathbb{R}]{} & \mathbb{R}^2 & \xrightarrow[\mathbb{R}]{} & \mathbb{R} \\
\begin{bmatrix} \mathrm{G} \\ \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \begin{bmatrix} \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \begin{bmatrix} \mathrm{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \mathrm{M}.
\end{array}
\tag{3.34}
$$

Step 1: We start at the right of diagram (3.35).

$$\begin{array}{ccc} \overset{G}{\curvearrowright} & & \overset{G}{\curvearrowright} \\ \mathbb{R}^2 & \xrightarrow[\mathbb{R}]{} & \mathbb{R} \\ \begin{bmatrix} \mathrm{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \mathrm{M}. \end{array} \tag{3.35}$$

We first find a cross section for the action of G on the base variable $u = \mathrm{M}$. The action is trivial: $g(\mathrm{M}) = \mathrm{M}$, so we must take

$$\mathrm{U}^{\mathrm{o}} = \{m \mid m \in \mathbb{R}\}. \tag{3.36}$$

Next, for each $u \in \mathrm{U}^{\mathrm{o}}$, we find the stabilizer G_u and compute its action on the fiber variable $v = \mathrm{P}$ as described in (3.32). There are two cases according as the base orbit $\mathrm{G}(u)$ is $\{0\}$ or not:

$u \in (3.36)$	$\mathrm{G}(u)$	G_u	$g_u(v)$
0	$\{0\}$	G	P
$m \quad (m \neq 0)$	$\{m\}$	G	$\mathrm{P} + mb$

There remains to classify the orbits of the action of $\mathrm{G}_u = \mathrm{G}$ on $v = \mathrm{P}$ given in the rightmost column. The action in the first row is trivial, so its orbits are points and $\mathrm{V}_u^{\mathrm{o}} = \{k \mid k \in \mathbb{R}\}$ is a cross section. In the second row we have the action of $\mathbb{R}$ on itself by translation which is transitive: so the origin $\mathrm{V}_u^{\mathrm{o}} = \{0\}$ is a cross section. Putting U^{o} and the $\mathrm{V}_u^{\mathrm{o}}$ together as prescribed by (3.2), we get the following cross section:

$$\left\{ \begin{bmatrix} k \\ 0 \end{bmatrix} \mid k \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 \\ m \end{bmatrix} \mid m \neq 0 \right\}. \tag{3.37}$$

Step 2: We now move to the second arrow from the right in diagram (3.34).

$$\begin{array}{ccc} \overset{G}{\curvearrowright} & & \overset{G}{\curvearrowright} \\ \mathbb{R}^3 & \xrightarrow[\mathbb{R}]{} & \mathbb{R}^2 \\ \begin{bmatrix} \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \begin{bmatrix} \mathrm{P} \\ \mathrm{M} \end{bmatrix}. \end{array} \tag{3.38}$$

We now redefine the base, fiber, and actions for our new setting. We have a cross section for the action of G on the base variable $u = \left[\begin{smallmatrix} \mathrm{P} \\ \mathrm{M} \end{smallmatrix}\right]$ as in (3.37). Next, for each $u \in \mathrm{U}^{\mathrm{o}}$ we find the stabilizer and read off its action on the fiber variable $v = \mathrm{E}$. There are three cases according to the nature of the base orbit $\mathrm{G}(u)$:

$u \in (3.37)$	$\mathrm{G}(u)$	G_u	$g_u(v)$
$\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$	the origin	G	E
$\left[\begin{smallmatrix}k\\0\end{smallmatrix}\right]$ $(k \neq 0)$	a point	G	$\mathrm{E} + kb$
$\left[\begin{smallmatrix}0\\m\end{smallmatrix}\right]$ $(m \neq 0)$	a line	$\{g \in \mathrm{G} \mid b = 0\}$	E

There remains to classify the orbits of the action of G_u on $v = \mathrm{E}$ given in the rightmost column. In the first and third rows we have trivial actions, so $\mathrm{V}_u^o = \{c \mid c \in \mathbb{R}\}$ is a cross section. The action in the second row is transitive, so $\mathrm{V}_u^o = \{0\}$ is a cross section. Putting U^o and the V_u^os together as prescribed by (3.2) we get the following cross section:

$$\left\{ \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \middle| c \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix} \middle| k \neq 0 \right\} \cup \left\{ \begin{bmatrix} c \\ 0 \\ m \end{bmatrix} \middle| \begin{matrix} c \in \mathbb{R} \\ m \neq 0 \end{matrix} \right\}. \tag{3.39}$$

Step 3: We now move to the leftmost arrow in (3.34) :

$$\begin{matrix} \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} \\ \mathfrak{g}^* & \xrightarrow[\mathbb{R}]{} & \mathbb{R}^3 \\ \begin{bmatrix} \mathrm{G} \\ \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \begin{bmatrix} \mathrm{E} \\ \mathrm{P} \\ \mathrm{M} \end{bmatrix}. \end{matrix} \tag{3.40}$$

We now redefine the base, fiber, and actions for our new setting. We have a cross section for the action of G on the base variable $u = \left[\begin{smallmatrix}\mathrm{E}\\\mathrm{P}\\\mathrm{M}\end{smallmatrix}\right]$ as in (3.39). Next, for each $u \in \mathrm{U}^o$ we find the stabilizer G_u and compute its action on the fiber variable $v = \mathrm{G}$ from (3.32). There are three cases according to the nature of the base orbit $\mathrm{G}(u)$:

$u \in (3.39)$	$\mathrm{G}(u)$	G_u	$g_u(v)$
$\left[\begin{smallmatrix}c\\0\\0\end{smallmatrix}\right]$	a point	G	G
$\left[\begin{smallmatrix}0\\k\\0\end{smallmatrix}\right]$ $(k \neq 0)$	a line	$\{g \in \mathrm{G} \mid b = 0\}$	$\mathrm{G} - ke$
$\left[\begin{smallmatrix}c\\0\\m\end{smallmatrix}\right]$ $(m \neq 0)$	a parabola	$\{g \in \mathrm{G} \mid b = 0\}$	$\mathrm{G} + mc$

There remains to classify the orbits of the action of G_u on $v = \mathrm{G}$ given in the rightmost column. In the first row we have the trivial action, so $\mathrm{V}_u^o = \{n \mid n \in \mathbb{R}\}$ is a cross section. The action in the second and third rows is transitive, so $\mathrm{V}_u^o = \{0\}$ is a cross section. Putting U^o and the V_u^o's together as prescribed by (3.2), we obtain (3.33).□

We now describe the structure of each orbit as a (symplectic) manifold:

Proposition 3.7. *Splitting cases as in Proposition 3.6:*

(1) *If $x_0 \in \mathrm{S}_1$, the coadjoint orbit $\mathrm{G}(x_0)$ is the point $\{x_0\}$ with the trivial 2-form.*

(2) *If $x_0 \in \mathrm{S}_2$, the coadjoint orbit $\mathrm{G}(x_0)$ is the plane*

$$\left\{ x = \begin{bmatrix} kt \\ \mathrm{E} \\ k \\ 0 \end{bmatrix} \;\middle|\; \mathrm{E}, t \in \mathbb{R} \right\} \tag{3.41}$$

with 2-form

$$\sigma(\delta x, \delta' x) = \delta \mathrm{E}\, \delta' t - \delta' \mathrm{E}\, \delta t. \tag{3.42}$$

(3) *If $x_0 \in \mathrm{S}_3$, the coadjoint orbit $\mathrm{G}(x_0)$ is the paraboloid*

$$\left\{ x = \begin{bmatrix} mr \\ \frac{1}{2}mv^2 + c \\ mv \\ m \end{bmatrix} \;\middle|\; r, v \in \mathbb{R} \right\} \tag{3.43}$$

with 2-form

$$\sigma(\delta x, \delta' x) = m\,(\delta v\, \delta' r - \delta' v\, \delta r)\,. \tag{3.44}$$

Proof. Case (1) is clear. For case (2), we act on the base point x_0 (3.33) by (3.32) and then compute the 2-form using (2.81). We find

$$\mathrm{G}(x_0) = \{g(x_0) | g \in \mathrm{G}\} = \left\{ x = \begin{bmatrix} -ke \\ kb \\ k \\ 0 \end{bmatrix} \mid b, c \in \mathbb{R} \right\}. \tag{3.45}$$

Thus we see that $g\,(x_0)$ only depends on g via the variables $(t, \mathrm{E}) := (-e, kb)$, giving (3.41). Now we express σ in terms of (t, E). By (2.81):

$$\begin{aligned}
\sigma(\delta x, \delta' x) &= \left\langle x_0, \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g \right\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ 0 \\ k \\ 0 \end{bmatrix}, \delta \begin{pmatrix} 1 & -b & \frac{1}{2}b^2 & -f+bc-\frac{1}{2}eb^2 \\ & 1 & -b & -c+be \\ & & 1 & -e \\ & & & 1 \end{pmatrix} \delta' \begin{pmatrix} 1 & b & \frac{1}{2}b^2 & f \\ & 1 & b & c \\ & & 1 & e \\ & & & 1 \end{pmatrix} - \delta' \begin{pmatrix} 1 & -b & \frac{1}{2}b^2 & -f+bc-\frac{1}{2}eb^2 \\ & 1 & -b & -c+be \\ & & 1 & -e \\ & & & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & b & \frac{1}{2}b^2 & f \\ & 1 & b & c \\ & & 1 & e \\ & & & 1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ 0 \\ k \\ 0 \end{bmatrix}, \begin{pmatrix} 0 & -\delta b & b\delta b & * \\ & 0 & -\delta b & * \\ & & 0 & * \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & \delta' b & b\delta' b & \delta' f \\ & 0 & \delta' b & \delta' c \\ & & 0 & \delta' e \\ & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\delta' b & b\delta' b & * \\ & 0 & -\delta' b & * \\ & & 0 & * \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & \delta b & b\delta b & \delta f \\ & 0 & \delta b & \delta c \\ & & 0 & \delta e \\ & & & 0 \end{pmatrix} \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle \begin{bmatrix} 0 \\ 0 \\ k \\ 0 \end{bmatrix}, \begin{pmatrix} 0 & 0 & * & * \\ & 0 & * & -\delta b \delta' e + \delta' b \delta e \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \right\rangle \\
&= k\left(-\delta b \, \delta e + \delta' b \, \delta e\right) \\
&= \delta \mathrm{E} \delta' t - \delta' \mathrm{E} \delta t.
\end{aligned}
$$

This gives (3.42).

For case (3), we act on the base point x_0 (3.33) by (3.32) and then compute the 2-form using (2.81).

$$
\begin{aligned}
\mathrm{G}(x_3) &= \{g(x_0) \mid g \in \mathrm{G}\} && (3.46) \\
&= \left\{ x = \begin{bmatrix} m(c-be) \\ \frac{1}{2}mb^2 + c \\ mb \\ m \end{bmatrix} \mid b, c, e \in \mathbb{R} \right\}. && (3.47)
\end{aligned}
$$

Thus we see that $g\left(x_3\right)$ only depends on g via the variables $(r, v) := (c - be, b)$, giving (3.43). Now we express σ in terms of (r, v). By (2.81):

$$
\begin{aligned}
\sigma(\delta x, \delta' x) &= \left\langle x_0, \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g \right\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ c \\ 0 \\ m \end{bmatrix}, \delta \begin{pmatrix} 1 & -b & \frac{1}{2}b^2 & -f+bc-\frac{1}{2}eb^2 \\ & 1 & -b & -c+be \\ & & 1 & -e \\ & & & 1 \end{pmatrix} \delta' \begin{pmatrix} 1 & b & \frac{1}{2}b^2 & f \\ & 1 & b & c \\ & & 1 & e \\ & & & 1 \end{pmatrix} - \delta' \begin{pmatrix} 1 & -b & \frac{1}{2}b^2 & -f+bc-\frac{1}{2}eb^2 \\ & 1 & -b & -c+be \\ & & 1 & -e \\ & & & 1 \end{pmatrix} \delta \begin{pmatrix} 1 & b & \frac{1}{2}b^2 & f \\ & 1 & b & c \\ & & 1 & e \\ & & & 1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ c \\ 0 \\ m \end{bmatrix}, \begin{pmatrix} 0 & -\delta b & b\delta b & * \\ & 0 & -\delta b & * \\ & & 0 & * \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & \delta' b & b\delta' b & \delta' f \\ & 0 & \delta' b & \delta' c \\ & & 0 & \delta' e \\ & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\delta' b & b\delta' b & * \\ & 0 & -\delta' b & * \\ & & 0 & * \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & \delta b & b\delta b & \delta f \\ & 0 & \delta b & \delta c \\ & & 0 & \delta e \\ & & & 0 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ c \\ 0 \\ m \end{bmatrix}, \begin{pmatrix} 0 & 0 & * & (-\delta b\delta' c + b\delta b\delta' e) - (-\delta' b\delta c + b\delta' b\delta e) \\ & 0 & * & * \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \right\rangle \\
&= -m\left(\left(-\delta b \ \delta' c + b \ \delta b \ \delta' e\right) - \left(-\delta' b \ \delta c + b\delta' b \ \delta e\right)\right) \\
&= m\left(\left(\delta b \ \delta' c - b \ \delta b \ \delta' e\right) - \left(\delta' b \ \delta c - b\delta' b \ \delta e\right)\right) \\
&= m\left(\delta b \ \delta' \left[c - be\right] - \delta' b \ \delta \left[c - be\right]\right) \\
&= m\left(\delta v \ \delta' r - \delta' v \ \delta r\right)
\end{aligned}
$$

This gives (3.44). □

CHAPTER 4
THE CLASSIFICATION

In this chapter we make inductive use of our cross section lemma 3.1 to classify the coadjoint orbits of the Bargmann group in 3+1 space-time dimensions, which V. Bargmann showed is the essentially unique central extension of the Galilei group. By the Kirillov-Kostant-Souriau theorem 2.12, these are, up to possible coverings, the homogeneous symplectic manifolds of the Galilei group. In a second section we sketch the description of their discrete quotients, which are the sought homogeneous spaces.

4.1 Coadjoint Orbits of the Bargmann Group

V. Bargmann [3] has shown that the Galilei group (1.12) has an essentially unique central extension, the ***Bargmann group***

$$\mathrm{G} = \left\{ g = \begin{pmatrix} 1 & \overline{\boldsymbol{b}}\mathrm{A} & \frac{1}{2}\|\boldsymbol{b}\|^2 & f \\ & \mathrm{A} & \boldsymbol{b} & \boldsymbol{c} \\ & & 1 & e \\ & & & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} \mathrm{A} \in \mathrm{SO}(3) \\ \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3 \\ e, f \in \mathbb{R} \end{array} \right\} \tag{4.1}$$

which projects onto the Galilei group by forgetting the first row and column. As a consequence we know from Theorem 2.12 that the homogeneous symplectic manifolds of the Galilei group are, up to possible coverings, exactly the coadjoint orbits of G. We now endeavor to classify these. As the Lie algebra (2.65) of G we find

$$\mathfrak{g} = \left\{ \mathrm{Z} = \begin{pmatrix} 0 & \overline{\boldsymbol{\beta}} & 0 & \varphi \\ & j(\boldsymbol{\alpha}) & \boldsymbol{\beta} & \boldsymbol{\gamma} \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^3 \\ \varepsilon, \varphi \in \mathbb{R} \end{array} \right\} \tag{4.2}$$

where $j(\boldsymbol{\alpha})$ is as in (2.83). This has dimension 11, so we may identify the dual space $\mathfrak{g}^*$ with $\mathbb{R}^{11}$:

$$\mathfrak{g}^* = \left\{ x = \begin{bmatrix} \mathbf{L} \\ \mathbf{G} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix} \;\middle|\; \begin{array}{l} \mathbf{L}, \mathbf{G}, \mathbf{P} \in \mathbb{R}^3 \\ \mathrm{E}, \mathrm{M} \in \mathbb{R} \end{array} \right\}, \tag{4.3}$$

paired to $\mathfrak{g}$ by

$$\langle x, \mathrm{Z} \rangle = \langle \mathbf{L}, \boldsymbol{\alpha} \rangle - \langle \mathbf{G}, \boldsymbol{\beta} \rangle + \langle \mathbf{P}, \boldsymbol{\gamma} \rangle - \mathrm{E}\varepsilon - \mathrm{M}\varphi. \tag{4.4}$$

(The quantities $\mathbf{L}$, $\mathbf{G}$, E, $\mathbf{P}$, M are known respectively as angular momentum, first mass moment, energy, linear momentum, and mass.) Using this notation, we have:

Theorem 4.1. *The coadjoint action of* G *is given by*

$$g\begin{bmatrix}\mathbf{L}\\ \mathbf{G}\\ \mathrm{E}\\ \mathbf{P}\\ \mathrm{M}\end{bmatrix} = \begin{bmatrix}\mathrm{A}\mathbf{L}+\mathrm{A}\mathbf{G}\times\boldsymbol{b}+\boldsymbol{c}\times\mathrm{A}\mathbf{P}+\mathrm{M}\boldsymbol{c}\times\boldsymbol{b}\\ \mathrm{A}\left(\mathbf{G}-\mathbf{P}e\right)+\mathrm{M}(\boldsymbol{c}-\boldsymbol{b}e)\\ \mathrm{E}+\langle\boldsymbol{b},\mathrm{A}\mathbf{P}\rangle+\frac{1}{2}\mathrm{M}\left\|\boldsymbol{b}\right\|^2\\ \mathrm{A}\mathbf{P}+\mathrm{M}\boldsymbol{b}\\ \mathrm{M}\end{bmatrix}. \tag{4.5}$$

A cross section for this action is the set $\mathrm{S}_1\cup\ldots\cup\mathrm{S}_7$, *where*

subset	S_1	S_2	S_3	S_4	S_5	S_6	S_7
typical element x_0	$\begin{bmatrix}0\\0\\c\\0\\0\end{bmatrix}$	$\begin{bmatrix}s\boldsymbol{e}_3\\0\\c\\0\\0\end{bmatrix}$	$\begin{bmatrix}s\boldsymbol{e}_3\\n\boldsymbol{e}_3\\c\\0\\0\end{bmatrix}$	$\begin{bmatrix}s\boldsymbol{e}_3\\0\\0\\k\boldsymbol{e}_3\\0\end{bmatrix}$	$\begin{bmatrix}0\\n\boldsymbol{e}_1\\0\\k\boldsymbol{e}_3\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\c\\0\\m\end{bmatrix}$	$\begin{bmatrix}s\boldsymbol{e}_3\\0\\c\\0\\m\end{bmatrix}$
conditions	$c\in\mathbb{R}$	$s>0$, $c\in\mathbb{R}$	$s\in\mathbb{R}$, $n>0$, $c\in\mathbb{R}$	$s\in\mathbb{R}$, $k>0$	$n>0$, $k>0$	$c\in\mathbb{R}$, $m\neq 0$	$s>0$, $c\in\mathbb{R}$, $m\neq 0$

(4.6)

Proof. We begin by looking at g as a composition of a "translation" g_1, "boost" g_2, and a "rotation" g_3:

$$\underbrace{\begin{pmatrix}1 & \overline{\boldsymbol{b}}\mathrm{A} & \frac{1}{2}\left\|\boldsymbol{b}\right\|^2 & f\\ & \mathrm{A} & \boldsymbol{b} & \boldsymbol{c}\\ & & 1 & e\\ & & & 1\end{pmatrix}}_{g} = \underbrace{\begin{pmatrix}1 & 0 & 0 & f\\ & \mathbb{1} & 0 & \boldsymbol{c}\\ & & 1 & e\\ & & & 1\end{pmatrix}}_{g_1}\underbrace{\begin{pmatrix}1 & \overline{\boldsymbol{b}} & \frac{1}{2}\left\|\boldsymbol{b}\right\|^2 & 0\\ & \mathbb{1} & \boldsymbol{b} & 0\\ & & 1 & 0\\ & & & 1\end{pmatrix}}_{g_2}\underbrace{\begin{pmatrix}1 & 0 & 0 & 0\\ & \mathrm{A} & 0 & 0\\ & & 1 & 0\\ & & & 1\end{pmatrix}}_{g_3}.$$

We have

$$\langle g_3(x),\mathrm{Z}\rangle = \langle x, g_3^{-1}\mathrm{Z}g_3\rangle$$

$$= \left\langle x, \begin{pmatrix} 1 & 0 & 0 & 0 \\ & \overline{\mathrm{A}} & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & \overline{\boldsymbol{\beta}} & 0 & \varphi \\ & j(\boldsymbol{\alpha}) & \boldsymbol{\beta} & \boldsymbol{\gamma} \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ & \mathrm{A} & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} \mathbf{L} \\ \mathbf{G} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix}, \begin{pmatrix} 0 & \overline{\boldsymbol{\beta}}\mathrm{A} & 0 & \varphi \\ & j(\overline{\mathrm{A}}\boldsymbol{\alpha}) & \overline{\mathrm{A}}\boldsymbol{\beta} & \overline{\mathrm{A}}\gamma \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \right\rangle$$

$$= \left\langle \mathbf{L}, \overline{\mathrm{A}}\boldsymbol{\alpha} \right\rangle - \left\langle \mathbf{G}, \overline{\mathrm{A}}\boldsymbol{\beta} \right\rangle + \left\langle \mathbf{P}, \overline{\mathrm{A}}\boldsymbol{\gamma} \right\rangle - \mathrm{E}\varepsilon - \mathrm{M}\varphi$$

$$= \langle \mathrm{A}\mathbf{L}, \boldsymbol{\alpha} \rangle - \langle \mathrm{A}\mathbf{G}, \boldsymbol{\beta} \rangle + \langle \mathrm{A}\mathbf{P}, \boldsymbol{\gamma} \rangle - \mathrm{E}\varepsilon - \mathrm{M}\varphi$$

$$= \left\langle \begin{bmatrix} \mathrm{A}\mathbf{L} \\ \mathrm{A}\mathbf{G} \\ \mathrm{E} \\ \mathrm{A}\mathbf{P} \\ \mathrm{M} \end{bmatrix}, \mathrm{Z} \right\rangle.$$

$$\langle g_2(x), \mathrm{Z} \rangle = \langle x, g_2^{-1} \mathrm{Z} g_2 \rangle$$

$$= \left\langle x, \begin{pmatrix} 1 & -\overline{\boldsymbol{b}} & \frac{1}{2}\|\boldsymbol{b}\|^2 & 0 \\ & 1 & -\boldsymbol{b} & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & \overline{\boldsymbol{\beta}} & 0 & \varphi \\ & j(\boldsymbol{\alpha}) & \boldsymbol{\beta} & \boldsymbol{\gamma} \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \begin{pmatrix} 1 & \overline{\boldsymbol{b}} & \frac{1}{2}\|\boldsymbol{b}\|^2 & 0 \\ & 1 & \boldsymbol{b} & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} \mathbf{L} \\ \mathbf{G} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix}, \begin{pmatrix} 0 & \overline{\boldsymbol{\beta}} - \overline{\boldsymbol{b}} j(\boldsymbol{\alpha}) & 0 & \varphi - \overline{\boldsymbol{b}}\boldsymbol{\gamma} + \frac{1}{2}\|\boldsymbol{b}\|^2 \varepsilon \\ & j(\boldsymbol{\alpha}) & j(\boldsymbol{\alpha})\boldsymbol{b} + \boldsymbol{\beta} & \boldsymbol{\gamma} - \boldsymbol{b}\varepsilon \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \right\rangle$$

$$= \langle \mathbf{L}, \boldsymbol{\alpha} \rangle - \langle \mathbf{G}, \boldsymbol{\alpha} \times \boldsymbol{b} + \boldsymbol{\beta} \rangle + \langle \mathbf{P}, \boldsymbol{\gamma} - \boldsymbol{b}\varepsilon \rangle - \mathrm{E}\varepsilon - \mathrm{M}(\varphi - \overline{\boldsymbol{b}}\boldsymbol{\gamma} + \tfrac{1}{2}\|\boldsymbol{b}\|^2 \varepsilon)$$

$$= \langle \mathbf{L} + \mathbf{G} \times \boldsymbol{b}, \boldsymbol{\alpha} \rangle - \langle \mathbf{G}, \boldsymbol{\beta} \rangle + \langle \mathbf{P} + \mathrm{M}\boldsymbol{b}, \boldsymbol{\gamma} \rangle - (\mathrm{E} + \langle \boldsymbol{b}, \mathbf{P} \rangle + \tfrac{1}{2}\mathrm{M}\|\boldsymbol{b}\|^2)\varepsilon - \mathrm{M}\varphi$$

$$= \left\langle \begin{bmatrix} \mathbf{L} + \mathbf{G} \times \boldsymbol{b} \\ \mathbf{G} \\ \mathrm{E} + \langle \boldsymbol{b}, \mathbf{P} \rangle + \frac{1}{2}\mathrm{M}\|\boldsymbol{b}\|^2 \\ \mathbf{P} + \mathrm{M}\boldsymbol{b} \\ \mathrm{M} \end{bmatrix}, \mathrm{Z} \right\rangle$$

and

$$\langle g_1(x), \mathrm{Z} \rangle = \left\langle x, g_1^{-1} \mathrm{Z} g_1 \right\rangle$$

$$
\begin{aligned}
&= \left\langle x, \begin{pmatrix} 1 & 0 & 0 & -f \\ & 1 & 0 & -\boldsymbol{c} \\ & & 1 & -e \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & \overline{\boldsymbol{\beta}} & 0 & \varphi \\ & j(\boldsymbol{\alpha}) & \boldsymbol{\beta} & \boldsymbol{\gamma} \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & f \\ & 1 & 0 & \boldsymbol{c} \\ & & 1 & e \\ & & & 1 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} \mathbf{L} \\ \mathbf{G} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix}, \begin{pmatrix} 0 & \overline{\boldsymbol{\beta}} & 0 & \varphi + \langle \boldsymbol{\beta}, \boldsymbol{c} \rangle \\ & j(\boldsymbol{\alpha}) & \boldsymbol{\beta} & \boldsymbol{\gamma} + \boldsymbol{\beta} e + j(\boldsymbol{\alpha})\boldsymbol{c} \\ & & 0 & \varepsilon \\ & & & 0 \end{pmatrix} \right\rangle \\
&= \langle \mathbf{L}, \boldsymbol{\alpha} \rangle - \langle \mathbf{G}, \boldsymbol{\beta} \rangle + \langle \mathbf{P}, \boldsymbol{\gamma} + \boldsymbol{\alpha} \times \boldsymbol{c} + \boldsymbol{\beta} e \rangle - \mathrm{E}\varepsilon - \mathrm{M}\left(\varphi + \langle \boldsymbol{c}, \boldsymbol{\beta} \rangle\right) \\
&= \langle \mathbf{L} + \boldsymbol{c} \times \mathbf{P}, \boldsymbol{\alpha} \rangle - \langle \mathbf{G} + \mathrm{M}\boldsymbol{c} - \mathbf{P}e, \boldsymbol{\beta} \rangle + \langle \mathbf{P}, \boldsymbol{\gamma} \rangle - \mathrm{E}\varepsilon - \mathrm{M}\varphi \\
&= \left\langle \begin{bmatrix} \mathbf{L} + \boldsymbol{c} \times \mathbf{P} \\ \mathbf{G} - \mathbf{P}e + \mathrm{M}\boldsymbol{c} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix}, \mathrm{Z} \right\rangle .
\end{aligned}
$$

Whence

$$
\begin{aligned}
g(x) &= g_1(g_2(g_3(x))) \\
&= g_1\left(g_2\left(\begin{bmatrix} \mathrm{A}\mathbf{L} \\ \mathrm{A}\mathbf{G} \\ \mathrm{E} \\ \mathrm{A}\mathbf{P} \\ \mathrm{M} \end{bmatrix} \right) \right) \\
&= g_1\left(\begin{bmatrix} \mathrm{A}\mathbf{L} + \mathrm{A}\mathbf{G} \times \boldsymbol{b} \\ \mathrm{A}\mathbf{G} \\ \mathrm{E} + \langle \boldsymbol{b}, \mathrm{A}\mathbf{P} \rangle + \frac{1}{2}\mathrm{M}\left\|\boldsymbol{b}\right\|^2 \\ \mathrm{A}\mathbf{P} + \mathrm{M}\boldsymbol{b} \\ \mathrm{M} \end{bmatrix} \right) \\
&= \begin{bmatrix} \mathrm{A}\mathbf{L} + \mathrm{A}\mathbf{G} \times \boldsymbol{b} + \boldsymbol{c} \times (\mathrm{A}\mathbf{P} + \mathrm{M}\boldsymbol{b}) \\ \mathrm{A}\mathbf{G} + \mathrm{M}\boldsymbol{c} - (\mathrm{A}\mathbf{P} + \mathrm{M}\boldsymbol{b})e \\ \mathrm{E} + \langle \boldsymbol{b}, \mathrm{A}\mathbf{P} \rangle + \frac{1}{2}\mathrm{M}\left\|\boldsymbol{b}\right\|^2 \\ \mathrm{A}\mathbf{P} + \mathrm{M}\boldsymbol{b} \\ \mathrm{M} \end{bmatrix}
\end{aligned}
$$

which proves (4.5).

Next, we again use Proposition 3.1 to find a cross section of the orbits. From looking at (4.5) we see layers of invariance in the coadjoint action. The simplest is the invariance of M. Next we see that the two variables $\mathbf{P}, \mathrm{M}$ transform among themselves, then $\mathrm{E}, \mathbf{P}, \mathrm{M}$, and $\mathbf{G}, \mathrm{E}, \mathbf{P}, \mathrm{M}$. Thus, we have a tower of 5 actions sitting above each other:

$$\begin{array}{ccccccccc} \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} \\ \mathfrak{g}^* & \xrightarrow[\mathbb{R}^3]{} & \mathbb{R}^8 & \xrightarrow[\mathbb{R}^3]{} & \mathbb{R}^5 & \xrightarrow[\mathbb{R}^1]{} & \mathbb{R}^4 & \xrightarrow[\mathbb{R}^3]{} & \mathbb{R} \\ \begin{bmatrix} \mathbf{L} \\ \mathbf{G} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \begin{bmatrix} \mathbf{G} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \begin{bmatrix} \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \begin{bmatrix} \mathbf{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \mathrm{M}. \end{array} \tag{4.7}$$

Step 1: We start at the right of the diagram (4.7).

$$\begin{array}{ccc} \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} \\ \mathbb{R}^4 & \xrightarrow[\mathbb{R}^3]{} & \mathbb{R} \\ \begin{bmatrix} \mathbf{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \mathrm{M}. \end{array} \tag{4.8}$$

We first find a cross section for the action of G on the base variable $u = \mathrm{M}$. The action is trivial: $g(\mathrm{M}) = \mathrm{M}$, so we can take

$$\mathrm{U}^{\mathrm{o}} = \{m \mid m \in \mathbb{R}\}. \tag{4.9}$$

Next, for each $u \in \mathrm{U}^{\mathrm{o}}$, we find the stabilizer G_u and read from (4.5) its action on the fiber variable $v = \mathbf{P}$. There are two cases according to the nature of the base orbit $\mathrm{G}(u) = 0$ or $\mathrm{G}(u) \neq 0$.

$u \in (4.9)$	$\mathrm{G}(u)$	G_u	$g_u(v)$
0	$\{0\}$	G	$\mathrm{A}\mathbf{P}$
$m \quad (m \neq 0)$	$\{m\}$	G	$\mathrm{A}\mathbf{P} + m\boldsymbol{b}$

There remains to classify the orbits of the action of $\mathrm{G}_u = \mathrm{G}$ on $v = \mathbf{P}$ given in the rightmost column. In the first row, we again have the standard action of the full standard rotation group and its orbits are the origin and spheres and $\mathrm{V}_u^{\mathrm{o}} = \{k\boldsymbol{e}_3 \mid k \geqslant 0\}$ is a cross section. In the second row we have a transitive action of a copy of Euclid's group (3.16) on $\mathbb{R}^3$, so the origin is a cross section, $\mathrm{V}_u^{\mathrm{o}} = \{0\}$. Putting U^{o} and the $\mathrm{V}_u^{\mathrm{o}}$ together as prescribed by (3.2), we get the following cross section:

$$\left\{ \begin{bmatrix} k\boldsymbol{e}_3 \\ 0 \end{bmatrix} \middle| k \geqslant 0 \right\} \cup \left\{ \begin{bmatrix} 0 \\ m \end{bmatrix} \mid m \neq 0 \right\}. \tag{4.10}$$

Step 2: We now move to the second arrow from the right in diagram (4.7).

$$\begin{array}{ccc} \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} \\ \mathbb{R}^5 & \xrightarrow[\mathbb{R}]{} & \mathbb{R}^4 \\ \begin{bmatrix} \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \begin{bmatrix} \mathbf{P} \\ \mathrm{M} \end{bmatrix} \end{array} \tag{4.11}$$

We now redefine the base, fiber, and actions for our new setting. We have a cross section U^{o} for the action of G on the base variable $u = \left[\begin{smallmatrix} \mathbf{P} \\ \mathrm{M} \end{smallmatrix}\right]$ as in (4.10). Next, for each $u \in \mathrm{U}^{\mathrm{o}}$ we find the stabilizer G_u and read from (4.5) its action on the fiber variable $v = \mathrm{E}$. There are three cases according to the nature of the base point u:

$u \in (4.10)$	G_u	$g_u(v)$
$\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$	G	E
$\left[\begin{smallmatrix} k\boldsymbol{e}_3 \\ 0 \end{smallmatrix}\right]$ $(k > 0)$	$\{g \in \mathrm{G} \mid \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3\}$	$\mathrm{E} + \langle k\boldsymbol{e}_3, \boldsymbol{b}\rangle$
$\left[\begin{smallmatrix} 0 \\ m \end{smallmatrix}\right]$ $(m \neq 0)$	$\{g \in \mathrm{G} \mid \boldsymbol{b} = 0\}$	E

There remains to classify the orbits of the action of G_u on $v = \mathrm{E}$ given in the rightmost column. The action in the first and third rows is trivial on E, so the orbits are points and $\mathrm{V}_u^{\mathrm{o}} = \{c \mid c \in \mathbb{R}\}$ is a cross section. In the second row, the action is transitive on V, so $\mathrm{V}_u^{\mathrm{o}} = \{0\}$ is a cross section. Putting U^{o} and the $\mathrm{V}_u^{\mathrm{o}}$s together as prescribed by (3.2) we get the following cross section:

$$\left\{ \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \middle| c \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 \\ k\boldsymbol{e}_3 \\ 0 \end{bmatrix} \middle| k > 0 \right\} \cup \left\{ \begin{bmatrix} c \\ 0 \\ m \end{bmatrix} \middle| \begin{matrix} c \in \mathbb{R} \\ m \neq 0 \end{matrix} \right\}. \tag{4.12}$$

Step 3: We now move to the third arrow from the right in (4.7) :

$$\begin{array}{ccc} \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} \\ \mathbb{R}^8 & \xrightarrow[\mathbb{R}^3]{} & \mathbb{R}^5 \\ \begin{bmatrix} \mathbf{G} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \begin{bmatrix} \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix}. \end{array} \tag{4.13}$$

We now redefine the base, fiber, and actions for our new setting. We have a cross section U^{o} for the action of G on the base variable $u = \left[\begin{smallmatrix} \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{smallmatrix}\right]$ as in (4.12). Next, for each $u \in \mathrm{U}^{\mathrm{o}}$ we find the stabilizer G_u and read from (4.5) its action on the fiber variable $v = \mathbf{G}$. There are three cases according to the nature of the base orbit $\mathrm{G}(u)$.

$u \in (4.12)$	$\mathrm{G}(u)$	G_u	$g_u(v)$
$\begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$	a point	G	$\mathrm{A}\mathbf{G}$
$\begin{bmatrix} 0 \\ k\boldsymbol{e}_3 \\ 0 \end{bmatrix}$ $(k > 0)$	a line	$\left\{ g \in \mathrm{G} \,\middle\vert\, \begin{array}{r} \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3 \\ \boldsymbol{b} \perp \boldsymbol{e}_3 \end{array} \right\}$	$\mathrm{A}\mathbf{G} - ek\boldsymbol{e}_3$
$\begin{bmatrix} c \\ 0 \\ m \end{bmatrix}$ $(m \neq 0)$	a paraboloid	$\{g \in \mathrm{G} \mid \boldsymbol{b} = 0\}$	$\mathrm{A}\mathbf{G} + m\boldsymbol{c}$

There remains to classify the orbits of the action of G_u on $v = \mathbf{G}$ given in the rightmost column. In the first row we have the action of the full rotation group on $\mathbb{R}^3$ and the orbits are spheres and the origin, so $\mathrm{V}_u^{\mathrm{o}} = \{n\boldsymbol{e}_3 \mid n \geqslant 0\}$ is a cross section. In the second row, we have rotations about the vertical axis and translations along it; so the orbits are cylinders and a cross section is the half line $\mathrm{V}_u^{\mathrm{o}} = \{n\boldsymbol{e}_1 \mid n \geqslant 0\}$. In the third row we have a transitive action of Euclid's group (3.16), so the origin $\mathrm{V}_u^{\mathrm{o}} = \{0\}$ is a cross section. Putting U^{o} and the $\mathrm{V}_u^{\mathrm{o}}$'s together as prescribed by (3.2), we obtain the following cross section:

$$\left\{ \begin{bmatrix} n\boldsymbol{e}_3 \\ c \\ 0 \\ 0 \end{bmatrix} \,\middle\vert\, \begin{array}{c} n \geqslant 0 \\ c \in \mathbb{R} \end{array} \right\} \cup \left\{ \begin{bmatrix} n\boldsymbol{e}_1 \\ 0 \\ k\boldsymbol{e}_3 \\ 0 \end{bmatrix} \,\middle\vert\, \begin{array}{c} n \geqslant 0 \\ k > 0 \end{array} \right\} \cup \left\{ \begin{bmatrix} 0 \\ c \\ 0 \\ m \end{bmatrix} \,\middle\vert\, \begin{array}{c} c \in \mathbb{R} \\ m \neq 0 \end{array} \right\}. \tag{4.14}$$

Step 4: We now move to the fourth arrow from the right in (4.7).

$$\begin{array}{ccc} \overset{\mathrm{G}}{\curvearrowright} & & \overset{\mathrm{G}}{\curvearrowright} \\ \mathbb{R}^{11} & \xrightarrow[\mathbb{R}^3]{} & \mathbb{R}^8 \\ \begin{bmatrix} \mathbf{L} \\ \mathbf{G} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix} & \longmapsto & \begin{bmatrix} \mathbf{G} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix}. \end{array} \tag{4.15}$$

We now redefine the base, fiber, and actions for our new setting. We have a cross section U^{o} for the action of G on the base variable $u = \begin{bmatrix} \mathbf{G} \\ \mathrm{E} \\ \mathbf{P} \\ \mathrm{M} \end{bmatrix}$ as in (4.14). Next, for each $u \in \mathrm{U}^{\mathrm{o}}$ we find the stabilizer G_u and read from (4.5) its action on the fiber variable $v = \mathbf{L}$. There are five cases according to the nature of the base point u.

$u \in (4.14)$	G_u	$g_u(v)$
$\begin{bmatrix} 0 \\ c \\ 0 \\ 0 \end{bmatrix}$	G	$\mathrm{A}\mathbf{L}$
$\begin{bmatrix} n\boldsymbol{e}_3 \\ c \\ 0 \\ 0 \end{bmatrix}$ $(n > 0)$	$\{g \in \mathrm{G} \mid \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3\}$	$\mathrm{A}\mathbf{L} + n\boldsymbol{e}_3 \times \boldsymbol{b}$
$\begin{bmatrix} 0 \\ 0 \\ k\boldsymbol{e}_3 \\ 0 \end{bmatrix}$ $(k > 0)$	$\left\{g \in \mathrm{G} \mid \begin{smallmatrix} \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3 \\ \boldsymbol{b} \perp \boldsymbol{e}_3 \\ e = 0 \end{smallmatrix}\right\}$	$\mathrm{A}\mathbf{L} + \boldsymbol{c} \times k\boldsymbol{e}_3$
$\begin{bmatrix} n\boldsymbol{e}_1 \\ 0 \\ k\boldsymbol{e}_3 \\ 0 \end{bmatrix}$ $(n, k > 0)$	$\left\{g \in \mathrm{G} \mid \begin{smallmatrix} \mathrm{A} = \mathbb{1} \\ \boldsymbol{b} \perp \boldsymbol{e}_3 \\ e = 0 \end{smallmatrix}\right\}$	$\mathbf{L} + n\boldsymbol{e}_1 \times \boldsymbol{b} + \boldsymbol{c} \times k\boldsymbol{e}_3$
$\begin{bmatrix} 0 \\ c \\ 0 \\ m \end{bmatrix}$ $(m \neq 0)$	$\{g \in \mathrm{G} \mid \boldsymbol{b} = \boldsymbol{c} = 0\}$	$\mathrm{A}\mathbf{L}$

There remains to classify the orbits of the action of G_u on $v = \mathbf{L}$ given in the rightmost column. In the first and fifth rows we have the action of the full rotation group on $\mathbf{L}$ and the orbits are spheres and the origin, so $\mathrm{V}_u^{\mathrm{o}} = \{s\boldsymbol{e}_3 \mid s \geqslant 0\}$ is a cross section. In the second and third rows, we have rotations about the vertical axis and the addition of any vector perpendicular to it; so the orbits are planes of constant L_3 and $\mathrm{V} = \{s\boldsymbol{e}_3 \mid s \in \mathbb{R}\}$ is a cross section. In the fourth row we add an arbitrary vector instead, so the action is transitive and $\mathrm{V}_u^{\mathrm{o}} = \{0\}$ is a cross section. Putting U^{o} and the $\mathrm{V}_u^{\mathrm{o}}$'s together as prescribed by (3.2), we obtain (4.6). (We have split rows one and five into two for reasons that will become clear below.) □

We now describe the structure of each orbit as a (symplectic) manifold:

Theorem 4.2. *Splitting cases as in Theorem 4.1:*

(1) *If* $x_0 \in \mathrm{S}_1$*, the coadjoint orbit* $\mathrm{G}(x_0)$ *is the point* $\{x_0\}$ *with the trivial 2-form.*

(2) *If* $x_0 \in \mathrm{S}_2$*, the coadjoint orbit* $\mathrm{G}(x_0)$ *is a copy of the 2-sphere* S^2

$$\left\{ x = \begin{bmatrix} s\boldsymbol{u} \\ 0 \\ c \\ 0 \\ 0 \end{bmatrix} \;\middle|\; \|\boldsymbol{u}\| = 1 \right\} \tag{4.16}$$

with 2-form

$$\sigma\,(\delta x, \delta' x) = s\langle \boldsymbol{u}, \delta'\boldsymbol{u} \times \delta\boldsymbol{u}\rangle. \tag{4.17}$$

(3) *If* $x_0 \in \mathrm{S}_3$, *the coadjoint orbit* $\mathrm{G}(x_0)$ *is a copy of* TS^2

$$\left\{ x = \begin{bmatrix} n\boldsymbol{u} \times \boldsymbol{v} + s\boldsymbol{u} \\ n\boldsymbol{u} \\ c \\ 0 \\ 0 \end{bmatrix} \;\middle|\; \begin{array}{r} \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0 \\ \|\boldsymbol{u}\| = 1 \end{array} \right\} \tag{4.18}$$

with 2-form

$$\sigma(\delta x, \delta' x) = -n\,[\langle \delta\boldsymbol{u}, \delta'\boldsymbol{v}\rangle - \langle \delta'\boldsymbol{u}, \delta\boldsymbol{v}\rangle] + s\langle \boldsymbol{u}, \delta'\boldsymbol{u} \times \delta\boldsymbol{u}\rangle. \tag{4.19}$$

(4) *If* $x_0 \in \mathrm{S}_4$, *the coadjoint orbit* $\mathrm{G}(x_0)$ *is a copy of* $\mathrm{TS}^2 \times \mathbb{R}^2$

$$\left\{ x = \begin{bmatrix} \boldsymbol{r} \times k\boldsymbol{u} + s\boldsymbol{u} \\ -kt\boldsymbol{u} \\ \mathrm{E} \\ k\boldsymbol{u} \\ 0 \end{bmatrix} \;\middle|\; \begin{array}{r} \langle \boldsymbol{r}, \boldsymbol{u} \rangle = 0 \\ \|\boldsymbol{u}\| = 1 \\ \mathrm{E}, t \in \mathbb{R} \end{array} \right\} \tag{4.20}$$

with 2-form

$$\begin{aligned} \sigma(\delta x, \delta' x) =& k\,[\langle \delta\boldsymbol{u}, \delta'\boldsymbol{r}\rangle - \langle \delta'\boldsymbol{u}, \delta\boldsymbol{r}\rangle] - (\delta\mathrm{E}\,\delta' t - \delta'\mathrm{E}\,\delta t) \\ &+ s\,\langle \boldsymbol{u}, \delta'\boldsymbol{u} \times \delta\boldsymbol{u}\rangle\,. \end{aligned} \tag{4.21}$$

(5) *If* $x_0 \in \mathrm{S}_5$, *the coadjoint orbit* $\mathrm{G}(x_0)$ *is a copy of* $\mathrm{TSO}(3) \times \mathbb{R}^2$

$$\left\{ x = \begin{bmatrix} \boldsymbol{\ell} \\ n\boldsymbol{v} - kt\boldsymbol{u} \\ \mathrm{E} \\ k\boldsymbol{u} \\ 0 \end{bmatrix} \;\middle|\; \begin{array}{l} \mathrm{U} = (\boldsymbol{v}\,\boldsymbol{w}\,\boldsymbol{u}) \in \mathrm{SO}(3) \\ \mathrm{L} = j(\boldsymbol{\ell})\mathrm{U} \in \mathrm{T_U SO}(3) \\ \mathrm{E}, t \in \mathbb{R} \end{array} \right\} \tag{4.22}$$

with 2-form

$$\sigma(\delta x, \delta' x) = \tfrac{1}{2}\mathrm{Tr}(\overline{\delta\mathrm{L}}\delta'\mathrm{U} - \overline{\delta'\mathrm{L}}\delta\mathrm{U}) - (\delta\mathrm{E}\,\delta' t - \delta'\mathrm{E}\,\delta t). \tag{4.23}$$

(6) *If* $x_0 \in \mathrm{S}_6$, *the coadjoint orbit* $\mathrm{G}(x_0)$ *is a copy of* $\mathbb{R}^6$

$$\left\{ x = \begin{bmatrix} m\boldsymbol{r} \times \boldsymbol{v} \\ m\boldsymbol{r} \\ \frac{1}{2}m\|\boldsymbol{v}\|^2 + c \\ m\boldsymbol{v} \\ m \end{bmatrix} \;\middle|\; \boldsymbol{r}, \boldsymbol{v} \in \mathbb{R}^3 \right\} \tag{4.24}$$

with 2-form

$$\sigma(\delta x, \delta' x) = m\,[\langle \delta\boldsymbol{v}, \delta'\boldsymbol{r}\rangle - \langle \delta'\boldsymbol{v}, \delta\boldsymbol{r}\rangle]\,. \tag{4.25}$$

(7) *If* $x_0 \in \mathrm{S}_7$*, the coadjoint orbit* $\mathrm{G}(x_0)$ *is a copy of* $\mathbb{R}^6 \times \mathrm{S}^2$

$$\left\{ x = \begin{bmatrix} m\boldsymbol{r} \times \boldsymbol{v} + s\boldsymbol{u} \\ m\boldsymbol{r} \\ \frac{1}{2}m\|\boldsymbol{v}\|^2 + c \\ m\boldsymbol{v} \\ m \end{bmatrix} \;\middle|\; \begin{matrix} \|\boldsymbol{u}\| = 1 \\ \boldsymbol{r}, \boldsymbol{v} \in \mathbb{R}^3 \end{matrix} \right\} \tag{4.26}$$

with 2-form

$$\sigma(\delta x, \delta' x) = m\left[\langle \delta\boldsymbol{v}, \delta'\boldsymbol{r}\rangle - \langle \delta'\boldsymbol{v}, \delta\boldsymbol{r}\rangle\right] + s\,\langle \boldsymbol{u}, \delta'\boldsymbol{u} \times \delta\boldsymbol{u}\rangle\,. \tag{4.27}$$

Proof. As a preliminary, we will compute the term $\delta\left[g^{-1}\right]\delta' g - \delta'\left[g^{-1}\right]\delta g$ of (2.81) once and for all, to be later plugged into the 2-form. First we compute the inverse. We will use the method of composition used to compute the coadjoint action. Then we finish the computation with substitution.

$$\begin{aligned}
g &= g_1\, g_2\, g_3 \\
g^{-1} &= (g_1\, g_2\, g_3)^{-1} \\
&= (g_3)^{-1}\; (g_2)^{-1}\; (g_1)^{-1} \\
&= \begin{pmatrix} 1 & & & \\ & \overline{\mathrm{A}} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -\overline{\boldsymbol{b}} & \frac{1}{2}\|\boldsymbol{b}\|^2 & \\ & \mathbb{1} & -\boldsymbol{b} & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & -f \\ & \mathbb{1} & & -\boldsymbol{c} \\ & & 1 & -e \\ & & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -\overline{\boldsymbol{b}} & \frac{1}{2}\|\boldsymbol{b}\|^2 & -f + \overline{\boldsymbol{b}}\boldsymbol{c} - \frac{1}{2}\|\boldsymbol{b}\|^2 e \\ & \overline{\mathrm{A}} & -\overline{\mathrm{A}}\boldsymbol{b} & -\overline{\mathrm{A}}\,(\boldsymbol{c} - \boldsymbol{b}e) \\ & & 1 & -e \\ & & & 1 \end{pmatrix}
\end{aligned} \tag{4.28}$$

Therefore,

$$\begin{aligned}
\delta\left[g^{-1}\right]\delta' g &= \delta \begin{pmatrix} 1 & -\overline{\boldsymbol{b}} & \frac{1}{2}\|\boldsymbol{b}\|^2 & -f + \overline{\boldsymbol{b}}\boldsymbol{c} - \frac{1}{2}\|\boldsymbol{b}\|^2 e \\ & \overline{\mathrm{A}} & -\overline{\mathrm{A}}\boldsymbol{b} & -\overline{\mathrm{A}}\,(\boldsymbol{c} - \boldsymbol{b}e) \\ & & 1 & -e \\ & & & 1 \end{pmatrix} \delta' \begin{pmatrix} 1 & \overline{\boldsymbol{b}}\mathrm{A} & \frac{1}{2}\|\boldsymbol{b}\|^2 & f \\ & \mathrm{A} & \boldsymbol{b} & \boldsymbol{c} \\ & & 1 & e \\ & & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \delta\left(-\overline{\boldsymbol{b}}\right) & \overline{\boldsymbol{b}}\delta\boldsymbol{b} & * \\ & \delta\left(\overline{\mathrm{A}}\right) & \delta\left(-\overline{\mathrm{A}}\boldsymbol{b}\right) & * \\ & & 0 & * \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & * & * & * \\ & \delta'\mathrm{A} & \delta'\boldsymbol{b} & \delta'\boldsymbol{c} \\ & & 0 & \delta' e \\ & & & 0 \end{pmatrix}
\end{aligned}$$

$$
= \begin{pmatrix} 0 & -\overline{\delta\boldsymbol{b}}\delta'\mathrm{A} & -\overline{\delta\boldsymbol{b}}\delta'\boldsymbol{b} & -\overline{\delta\boldsymbol{b}}\delta'\boldsymbol{c} + \overline{\boldsymbol{b}}\delta\boldsymbol{b}\delta' e \\ & \overline{\delta\mathrm{A}}\delta'\mathrm{A} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{b} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{c} - \delta\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta' e \\ & & 0 & \\ & & & 0 \end{pmatrix}
$$

and

$$
\delta\left[g^{-1}\right]\delta' g - \delta'\left[g^{-1}\right]\delta g = \begin{pmatrix} 0 & -\overline{\delta\boldsymbol{b}}\delta'\mathrm{A} + \overline{\delta'\boldsymbol{b}}\delta\mathrm{A} & 0 & -\overline{\delta\boldsymbol{b}}\delta'\boldsymbol{c} + \overline{\boldsymbol{b}}\delta\boldsymbol{b}\delta' e + \overline{\delta'\boldsymbol{b}}\delta\boldsymbol{c} - \overline{\boldsymbol{b}}\delta'\boldsymbol{b}\delta e \\ & \overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{b} - \overline{\delta'\mathrm{A}}\delta\boldsymbol{b} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{c} - \delta\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta' e - \overline{\delta'\mathrm{A}}\delta\boldsymbol{c} + \delta'\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta e \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \tag{4.29}
$$

Case (1) is clear. In case (2) we find

$$
\begin{aligned}
\mathrm{G}(x_0) &= \{g(x_0) \mid g \in \mathrm{G}\} \\
&= \left\{ x = \begin{bmatrix} s\mathrm{A}\boldsymbol{e}_3 \\ 0 \\ c \\ 0 \\ 0 \end{bmatrix} \;\middle|\; \mathrm{A} \in \mathrm{SO}(3) \right\}
\end{aligned}
$$

Thus we see that $g\left(x_0\right)$ only depends on g via the variable $\boldsymbol{u} := \mathrm{A}\boldsymbol{e}_3$, giving (4.16). Now we express σ in terms of $\boldsymbol{u}$. By (2.81, 4.29):

$$
\begin{aligned}
\sigma(\delta x, \delta' x) &= \left\langle x_0, \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g \right\rangle \\
&= \left\langle \begin{bmatrix} s\boldsymbol{e}_3 \\ 0 \\ c \\ 0 \\ 0 \end{bmatrix}, \begin{pmatrix} 0 & -\overline{\delta\boldsymbol{b}}\delta'\mathrm{A} + \overline{\delta'\boldsymbol{b}}\delta\mathrm{A} & 0 & -\overline{\delta\boldsymbol{b}}\delta'\boldsymbol{c} + \overline{\boldsymbol{b}}\delta\boldsymbol{b}\delta' e + \overline{\delta'\boldsymbol{b}}\delta\boldsymbol{c} - \overline{\boldsymbol{b}}\delta'\boldsymbol{b}\delta e \\ & \overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{b} - \overline{\delta'\mathrm{A}}\delta\boldsymbol{b} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{c} - \delta\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta' e - \overline{\delta'\mathrm{A}}\delta\boldsymbol{c} + \delta'\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta e \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \right\rangle \\
&= \left\langle s\boldsymbol{e}_3, j^{-1}\left(\overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A}\right) \right\rangle
\end{aligned}
$$

From this we recognize the 2-form (2.81, 2.82, 2.86, 2.87) for S^2 from SO(3), giving (4.17). In case (3), we find

$$
\begin{aligned}
\mathrm{G}(x_0) &= \{g(x_0) \mid g \in \mathrm{G}\} \\
&= \left\{ x = \begin{bmatrix} n\mathrm{A}\boldsymbol{e}_3 \times \boldsymbol{b} + s\mathrm{A}\boldsymbol{e}_3 \\ n\mathrm{A}\boldsymbol{e}_3 \\ c \\ 0 \\ 0 \end{bmatrix} \;\middle|\; \begin{matrix} \mathrm{A} \in \mathrm{SO}(3) \\ \boldsymbol{b} \in \mathbb{R}^3 \end{matrix} \right\}.
\end{aligned}
$$

We see that $g\left(x_0\right)$ only depends on g via the variables

$$
\begin{cases} \boldsymbol{u} := \mathrm{A}\boldsymbol{e}_3 \\ \boldsymbol{v} := [\mathbb{1} - \mathrm{A}\boldsymbol{e}_3\overline{\mathrm{A}\boldsymbol{e}_3}]\boldsymbol{b} \end{cases}
$$

giving (4.18). Now we express σ in terms of $(\boldsymbol{u}, \boldsymbol{v})$. By (2.81, 4.29):

$$\begin{aligned}
\sigma(\delta x, \delta' x) &= \left\langle x_0, \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g \right\rangle \\
&= \left\langle \begin{bmatrix} s\boldsymbol{e}_3 \\ n\boldsymbol{e}_3 \\ \mathrm{E} \\ 0 \\ 0 \end{bmatrix}, \begin{pmatrix} 0 & -\overline{\delta \boldsymbol{b}}\delta'\mathrm{A} + \overline{\delta' \boldsymbol{b}}\delta\mathrm{A} & 0 & -\overline{\delta \boldsymbol{b}}\delta'\boldsymbol{c} + \overline{\boldsymbol{b}}\delta\boldsymbol{b}\delta' e + \overline{\delta'\boldsymbol{b}}\delta\boldsymbol{c} - \overline{\boldsymbol{b}}\delta'\boldsymbol{b}\delta e \\ & \overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{b} - \overline{\delta'\mathrm{A}}\delta\boldsymbol{b} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{c} - \delta\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta' e - \overline{\delta'\mathrm{A}}\delta\boldsymbol{c} + \delta'\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta e \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \right\rangle \\
&= \left\langle s\boldsymbol{e}_3, j^{-1}\left(\overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A}\right)\right\rangle - \left\langle n\boldsymbol{e}_3, \overline{\delta\mathrm{A}}\delta'\boldsymbol{b} - \overline{\delta'\mathrm{A}}\delta\boldsymbol{b}\right\rangle \\
&= \left\langle s\boldsymbol{e}_3, j^{-1}\left(\overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A}\right)\right\rangle + n\left[\left\langle \boldsymbol{e}_3, \overline{\delta'\mathrm{A}}\delta\boldsymbol{b}\right\rangle - \left\langle \boldsymbol{e}_3, \overline{\delta\mathrm{A}}\delta'\boldsymbol{b}\right\rangle\right] \\
&= \left\langle s\boldsymbol{e}_3, j^{-1}\left(\overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A}\right)\right\rangle + n\left[\langle \delta'\mathrm{A}\boldsymbol{e}_3, \delta\boldsymbol{b}\rangle - \langle \delta\mathrm{A}\boldsymbol{e}_3, \delta'\boldsymbol{b}\rangle\right] \\
&= s\langle \boldsymbol{u}, \delta'\boldsymbol{u} \times \delta\boldsymbol{u}\rangle + n\left[\langle \delta'\boldsymbol{u}, \delta\boldsymbol{v}\rangle - \langle \delta\boldsymbol{u}, \delta'\boldsymbol{v}\rangle\right]
\end{aligned}$$

which we recognize as the 2-form (3.27) for the coadjoint orbit TS^2 of Euclid's group, giving (4.19). In case (4), we find

$$\begin{aligned}
\mathrm{G}(x_0) &= \{g(x_0) \mid g \in \mathrm{G}\} \\
&= \left\{ x = \begin{bmatrix} \boldsymbol{c} \times k\mathrm{A}\boldsymbol{e}_3 + s\mathrm{A}\boldsymbol{e}_3 \\ -k\mathrm{A}\boldsymbol{e}_3 e \\ \langle k\mathrm{A}\boldsymbol{e}_3, \boldsymbol{b}\rangle \\ k\mathrm{A}\boldsymbol{e}_3 \\ 0 \end{bmatrix} \;\middle|\; \begin{array}{l} \mathrm{A} \in \mathrm{SO}(3) \\ \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3 \\ e \in \mathbb{R} \end{array} \right\}.
\end{aligned}$$

We see that $g\,(x_0)$ only depends on g via the variables

$$\begin{cases} \boldsymbol{u} := \mathrm{A}\boldsymbol{e}_3 \\ \boldsymbol{r} := [\mathbb{1} - \mathrm{A}\boldsymbol{e}_3\overline{\mathrm{A}\boldsymbol{e}_3}]\boldsymbol{c} \\ \mathrm{E} := \langle \mathrm{A}\boldsymbol{e}_3, k\boldsymbol{b}\rangle \\ t := e \end{cases}$$

giving (4.20). Now we express σ in terms of $(\boldsymbol{u}, \boldsymbol{r}, \mathrm{E}, t)$. By (2.81, 4.29):

$$\begin{aligned}
\sigma(\delta x, \delta' x) &= \left\langle x_0, \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g \right\rangle \\
&= \left\langle \begin{bmatrix} s\boldsymbol{e}_3 \\ 0 \\ 0 \\ k\boldsymbol{e}_3 \\ 0 \end{bmatrix}, \begin{pmatrix} 0 & -\overline{\delta \boldsymbol{b}}\delta'\mathrm{A} + \overline{\delta' \boldsymbol{b}}\delta\mathrm{A} & 0 & -\overline{\delta \boldsymbol{b}}\delta'\boldsymbol{c} + \overline{\boldsymbol{b}}\delta\boldsymbol{b}\delta' e + \overline{\delta'\boldsymbol{b}}\delta\boldsymbol{c} - \overline{\boldsymbol{b}}\delta'\boldsymbol{b}\delta e \\ & \overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{b} - \overline{\delta'\mathrm{A}}\delta\boldsymbol{b} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{c} - \delta\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta' e - \overline{\delta'\mathrm{A}}\delta\boldsymbol{c} + \delta'\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta e \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \right\rangle \\
&= \left\langle s\boldsymbol{e}_3, j^{-1}\left(\overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A}\right)\right\rangle + \left\langle k\boldsymbol{e}_3, \overline{\delta\mathrm{A}}\delta'\boldsymbol{c} - \delta\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta' e - \overline{\delta'\mathrm{A}}\delta\boldsymbol{c} + \delta'\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta e\right\rangle \\
&= \left\langle s\boldsymbol{e}_3, j^{-1}\left(\overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A}\right)\right\rangle + \left\langle k\boldsymbol{e}_3, \overline{\delta\mathrm{A}}\delta'\boldsymbol{c} - \overline{\delta'\mathrm{A}}\delta\boldsymbol{c}\right\rangle \\
&\quad - \left\langle k\boldsymbol{e}_3, \delta\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta' e - \delta'\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta e\right\rangle \\
&= \left\langle s\boldsymbol{e}_3, j^{-1}\left(\overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A}\right)\right\rangle + k\left[\langle \delta\mathrm{A}\boldsymbol{e}_3, \delta'\boldsymbol{c}\rangle - \langle \delta'\mathrm{A}\boldsymbol{e}_3, \delta\boldsymbol{c}\rangle\right] \\
&\quad - \left\langle k\boldsymbol{e}_3, \overline{\delta\mathrm{A}}\boldsymbol{b} + \overline{\mathrm{A}}\delta\boldsymbol{b}\right\rangle \delta' e + \left\langle k\boldsymbol{e}_3, \overline{\delta'\mathrm{A}}\boldsymbol{b} + \overline{\mathrm{A}}\delta'\boldsymbol{b}\right\rangle \delta e
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle s\boldsymbol{e}_3, j^{-1}\left(\overline{\delta \mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A}\right)\right\rangle + k\left[\langle \delta\mathrm{A}\boldsymbol{e}_3, \delta'\boldsymbol{c}\rangle - \langle \delta'\mathrm{A}\boldsymbol{e}_3, \delta\boldsymbol{c}\rangle\right] \\
&\quad - \left(\langle k\delta\mathrm{A}\boldsymbol{e}_3, \boldsymbol{b}\rangle + \langle k\mathrm{A}\boldsymbol{e}_3, \delta\boldsymbol{b}\rangle\right)\delta' e + \left(\langle k\delta'\mathrm{A}\boldsymbol{e}_3, \boldsymbol{b}\rangle + \langle k\mathrm{A}\boldsymbol{e}_3, \delta'\boldsymbol{b}\rangle\right)\delta e \\
&= \left\langle s\boldsymbol{e}_3, j^{-1}\left(\overline{\delta \mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A}\right)\right\rangle + k\left[\langle \delta\mathrm{A}\boldsymbol{e}_3, \delta'\boldsymbol{c}\rangle - \langle \delta'\mathrm{A}\boldsymbol{e}_3, \delta\boldsymbol{c}\rangle\right] \\
&\quad - \delta\,\langle k\mathrm{A}\boldsymbol{e}_3, \boldsymbol{b}\rangle\,\delta' e + \delta'\,\langle k\mathrm{A}\boldsymbol{e}_3, \boldsymbol{b}\rangle\,\delta e \\
&= s\,\langle \boldsymbol{u}, \delta'\boldsymbol{u} \times \delta\boldsymbol{u}\rangle + k\left[\langle \delta\boldsymbol{u}, \delta'\boldsymbol{r}\rangle - \langle \delta'\boldsymbol{u}, \delta\boldsymbol{r}\rangle\right] - \delta\mathrm{E}\delta' t + \delta'\mathrm{E}\delta t
\end{aligned}
$$

This proves (4.21). In case (5), we find

$$
\begin{aligned}
\mathrm{G}(x_0) &= \{g(x_0) \mid g \in \mathrm{G}\} \\
&= \left\{ x = \begin{bmatrix} \mathrm{A}\boldsymbol{e}_1 \times n\boldsymbol{b} + k\boldsymbol{c} \times \mathrm{A}\boldsymbol{e}_3 \\ n\mathrm{A}\boldsymbol{e}_1 - ke\mathrm{A}\boldsymbol{e}_3 \\ \langle \mathrm{A}\boldsymbol{e}_3, k\boldsymbol{b}\rangle \\ k\mathrm{A}\boldsymbol{e}_3 \\ 0 \end{bmatrix} \;\middle|\; \begin{array}{l} \mathrm{A} \in \mathrm{SO}(3) \\ \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3 \\ e \in \mathbb{R} \end{array} \right\}.
\end{aligned}
$$

We see that $g\,(x_0)$ only depends on g via the variables

$$
\begin{cases}
(\boldsymbol{v}\,\boldsymbol{w}\,\boldsymbol{u}) := (\mathrm{A}\boldsymbol{e}_1\,\mathrm{A}\boldsymbol{e}_2\,\mathrm{A}\boldsymbol{e}_3) \\
\boldsymbol{\ell} := \mathrm{A}\boldsymbol{e}_1 \times n\boldsymbol{b} + k\boldsymbol{c} \times \mathrm{A}\boldsymbol{e}_3 \\
\mathrm{E} := \langle \mathrm{A}\boldsymbol{e}_3, k\boldsymbol{b}\rangle \\
t := e.
\end{cases}
\tag{4.30}
$$

This proves (4.22). To express σ in terms of the variables (4.30), we recall from (2.80) that the pull-back of σ to G by the map $\pi : g \mapsto g(x_0)$ is the exterior derivative of the 1-form (where Θ is the Maurer-Cartan 1-form on G)

$$
\begin{aligned}
\theta(\delta g) &:= \langle x_0, \Theta(\delta g)\rangle \\
&= \langle x_0, g^{-1}\delta g\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ n\boldsymbol{e}_1 \\ 0 \\ k\boldsymbol{e}_3 \\ 0 \end{bmatrix}, \begin{pmatrix} 1 & -\overline{\boldsymbol{b}} & \frac{1}{2}\|\boldsymbol{b}\|^2 & -f+\overline{\boldsymbol{b}}\boldsymbol{c}-\frac{1}{2}\|\boldsymbol{b}\|^2 e \\ & \overline{\mathrm{A}} & -\overline{\mathrm{A}}\boldsymbol{b} & -\overline{\mathrm{A}}(\boldsymbol{c}-\boldsymbol{b}e) \\ & & 1 & -e \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & \delta[\overline{\boldsymbol{b}}\mathrm{A}] & \delta[\frac{1}{2}\|\boldsymbol{b}\|^2] & \delta f \\ & \delta\mathrm{A} & \delta\boldsymbol{b} & \delta\boldsymbol{c} \\ & & 0 & \delta e \\ & & & 0 \end{pmatrix} \right\rangle && \text{by (4.28)} \\
&= \left\langle \begin{bmatrix} 0 \\ n\boldsymbol{e}_1 \\ 0 \\ k\boldsymbol{e}_3 \\ 0 \end{bmatrix}, \begin{pmatrix} 0 & * & 0 & * \\ & * & \overline{\mathrm{A}}\delta\boldsymbol{b} & \overline{\mathrm{A}}(\delta\boldsymbol{c}-\boldsymbol{b}\delta e) \\ & & 0 & * \\ & & & 0 \end{pmatrix} \right\rangle \\
&= \langle k\boldsymbol{e}_3, \overline{\mathrm{A}}(\delta\boldsymbol{c} - \boldsymbol{b}\delta e)\rangle - \langle n\boldsymbol{e}_1, \overline{\mathrm{A}}\delta\boldsymbol{b}\rangle \\
&= \langle k\mathrm{A}\boldsymbol{e}_3, \delta\boldsymbol{c}\rangle - \langle k\mathrm{A}\boldsymbol{e}_3, \boldsymbol{b}\rangle\delta e - \langle n\mathrm{A}\boldsymbol{e}_1, \delta\boldsymbol{b}\rangle \\
&= \langle k\boldsymbol{u}, \delta\boldsymbol{c}\rangle - \langle n\boldsymbol{v}, \delta\boldsymbol{b}\rangle - \mathrm{E}\delta t && \text{by (4.30)} \\
&= \langle n\boldsymbol{b}, \delta\boldsymbol{v}\rangle - \langle k\boldsymbol{c}, \delta\boldsymbol{u}\rangle - \mathrm{E}\delta t + \delta[\langle k\boldsymbol{u}, \boldsymbol{c}\rangle - \langle n\boldsymbol{v}, \boldsymbol{b}\rangle];
\end{aligned}
$$

now, writing $\psi = \langle k\boldsymbol{u}, \boldsymbol{c}\rangle - \langle n\boldsymbol{v}, \boldsymbol{b}\rangle$ and $\mathrm{U} = (\boldsymbol{v}\,\boldsymbol{w}\,\boldsymbol{u})$, and introducing the vector $\boldsymbol{\omega} = j^{-1}(\delta\mathrm{U}\overline{\mathrm{U}})$ which is such that $(\delta\boldsymbol{v}\,\delta\boldsymbol{w}\,\delta\boldsymbol{u}) = (\boldsymbol{\omega}\times\boldsymbol{v}\ \ \boldsymbol{\omega}\times\boldsymbol{w}\ \ \boldsymbol{\omega}\times\boldsymbol{u})$, this is

$$
\begin{aligned}
&= \langle n\boldsymbol{b}, \boldsymbol{\omega}\times\boldsymbol{v}\rangle - \langle k\boldsymbol{c}, \boldsymbol{\omega}\times\boldsymbol{u}\rangle - \mathrm{E}\delta t + \delta\psi \\
&= \langle \boldsymbol{v}\times n\boldsymbol{b} + k\boldsymbol{c}\times\boldsymbol{u}, \boldsymbol{\omega}\rangle - \mathrm{E}\delta t + \delta\psi \\
&= \langle \boldsymbol{\ell}, \boldsymbol{\omega}\rangle - \mathrm{E}\delta t + \delta\psi \\
&= -\tfrac{1}{2}\mathrm{Tr}(j(\boldsymbol{\ell})j(\boldsymbol{\omega})) - \mathrm{E}\delta t + \delta\psi \\
&= -\tfrac{1}{2}\mathrm{Tr}(j(\boldsymbol{\ell})\delta\mathrm{U}\overline{\mathrm{U}}) - \mathrm{E}\delta t + \delta\psi \\
&= \tfrac{1}{2}\mathrm{Tr}(\overline{j(\boldsymbol{\ell})\mathrm{U}}\delta\mathrm{U}) - \mathrm{E}\delta t + \delta\psi.
\end{aligned}
$$

This means that $\theta = \pi^*\theta_0 + d\psi$ where $\theta_0 = \frac{1}{2}\mathrm{Tr}(\overline{j(\boldsymbol{\ell})\mathrm{U}}d\mathrm{U}) - \mathrm{E}dt$, being expressed in terms of the variables (4.30), is a 1-form on the orbit $\mathrm{G}(x_0)$. By (2.47, 2.51) it follows that $\pi^*\sigma = d\theta = \pi^*d\theta_0$, whence $\sigma = d\theta_0$ since π is a submersion. Computing $d\theta_0$ by (2.44), we obtain (4.23). In case (6), we find

$$
\begin{aligned}
\mathrm{G}(x_0) &= \{g(x_0) \mid g\in\mathrm{G}\} \\
&= \left\{ x = \begin{bmatrix} m\boldsymbol{c}\times\boldsymbol{b} \\ m\,(\boldsymbol{c}-\boldsymbol{b}e) \\ \frac{1}{2}m\|\boldsymbol{b}\|^2 + c \\ m\boldsymbol{b} \\ m \end{bmatrix} \;\middle|\; \begin{array}{l} \boldsymbol{b},\boldsymbol{c}\in\mathbb{R}^3 \\ e\in\mathbb{R} \end{array} \right\}.
\end{aligned}
$$

We see that $g\,(x_0)$ only depends on g via $(\boldsymbol{v},\boldsymbol{r}) := (\boldsymbol{b}, \boldsymbol{c}-\boldsymbol{b}e)$, giving (4.24). Now we express σ in terms of $(\boldsymbol{v},\boldsymbol{r})$. By (2.81, 4.29):

$$
\begin{aligned}
\sigma(\delta x, \delta' x) &= \left\langle x_0, \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g\right\rangle \\
&= \left\langle \begin{bmatrix} 0 \\ 0 \\ c \\ 0 \\ m \end{bmatrix}, \begin{pmatrix} 0 & -\overline{\delta\boldsymbol{b}}\delta'\mathrm{A} + \overline{\delta'\boldsymbol{b}}\delta\mathrm{A} & 0 & -\overline{\delta\boldsymbol{b}}\delta'\boldsymbol{c} + \overline{\boldsymbol{b}}\delta\boldsymbol{b}\delta' e + \overline{\delta'\boldsymbol{b}}\delta\boldsymbol{c} - \overline{\boldsymbol{b}}\delta'\boldsymbol{b}\delta e \\ & \overline{\delta\mathrm{A}}\delta'\mathrm{A} - \overline{\delta'\mathrm{A}}\delta\mathrm{A} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{b} - \overline{\delta'\mathrm{A}}\delta\boldsymbol{b} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{c} - \delta\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta' e - \overline{\delta'\mathrm{A}}\delta\boldsymbol{c} + \delta'\left(\overline{\mathrm{A}}\boldsymbol{b}\right)\delta e \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \right\rangle \\
&= -m\left(-\overline{\delta\boldsymbol{b}}\delta'\boldsymbol{c} + \overline{\boldsymbol{b}}\delta\boldsymbol{b}\delta' e + \overline{\delta'\boldsymbol{b}}\delta\boldsymbol{c} - \overline{\boldsymbol{b}}\delta'\boldsymbol{b}\delta e\right) \\
&= m\left[\langle\delta\boldsymbol{b}, \delta'\boldsymbol{c} - \boldsymbol{b}\delta' e\rangle - \langle\delta'\boldsymbol{b}, \delta\boldsymbol{c} - \boldsymbol{b}\delta e\rangle\right] \\
&= m\left[\langle\delta\boldsymbol{b}, \delta'(\boldsymbol{c} - \boldsymbol{b}e)\rangle - \langle\delta'\boldsymbol{b}, \delta(\boldsymbol{c} - \boldsymbol{b}e)\rangle\right] \\
&= m\left[\langle\delta\boldsymbol{v}, \delta'\boldsymbol{r}\rangle - \langle\delta'\boldsymbol{v}, \delta\boldsymbol{r}\rangle\right].
\end{aligned}
$$

In case (7), we find

$$
\begin{aligned}
\mathrm{G}(x_0) &= \{g(x_0) \mid g\in\mathrm{G}\} \\
&= \left\{ x = \begin{bmatrix} m\boldsymbol{c}\times\boldsymbol{b} + s\mathrm{A}\boldsymbol{e}_3 \\ m\,(\boldsymbol{c}-\boldsymbol{b}e) \\ \frac{1}{2}m\|\boldsymbol{b}\|^2 + c \\ m\boldsymbol{b} \\ m \end{bmatrix} \;\middle|\; \begin{array}{l} \mathrm{A}\in\mathrm{SO}\,(3) \\ \boldsymbol{b},\boldsymbol{c}\in\mathbb{R}^3 \\ e\in\mathbb{R} \end{array} \right\}
\end{aligned}
$$

We see that $g\,(x_0)$ only depends on g via $(\boldsymbol{v},\boldsymbol{r},\boldsymbol{u}) := (\boldsymbol{b},\boldsymbol{c}-\boldsymbol{b}e,\mathrm{A}\boldsymbol{e}_3)$, giving (4.26). Now we express σ in terms of $(\boldsymbol{v},\boldsymbol{r},\boldsymbol{u})$. By (2.81, 4.29):

$$
\begin{aligned}
\sigma(\delta x,\delta' x) &= \left\langle x_0, \delta[g^{-1}]\delta' g - \delta'[g^{-1}]\delta g\right\rangle \\
&= \left\langle \begin{bmatrix} s\boldsymbol{e}_3 \\ 0 \\ c \\ 0 \\ m \end{bmatrix}, \begin{pmatrix} 0 & -\overline{\delta\boldsymbol{b}}\delta'\mathrm{A}+\overline{\delta'\boldsymbol{b}}\delta\mathrm{A} & 0 & -\overline{\delta\boldsymbol{b}}\delta'\boldsymbol{c}+\overline{\boldsymbol{b}}\delta\boldsymbol{b}\delta' e+\overline{\delta'\boldsymbol{b}}\delta\boldsymbol{c}-\overline{\boldsymbol{b}}\delta'\boldsymbol{b}\delta e \\ & \overline{\delta\mathrm{A}}\delta'\mathrm{A}-\overline{\delta'\mathrm{A}}\delta\mathrm{A} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{b}-\overline{\delta'\mathrm{A}}\delta\boldsymbol{b} & \overline{\delta\mathrm{A}}\delta'\boldsymbol{c}-\delta\big(\overline{\mathrm{A}}\boldsymbol{b}\big)\delta' e-\overline{\delta'\mathrm{A}}\delta\boldsymbol{c}+\delta'\big(\overline{\mathrm{A}}\boldsymbol{b}\big)\delta e \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \right\rangle \\
&= \left\langle s\boldsymbol{e}_3, j^{-1}\left(\overline{\delta\mathrm{A}}\delta'\mathrm{A}-\overline{\delta'\mathrm{A}}\delta\mathrm{A}\right)\right\rangle - m\left(-\overline{\delta\boldsymbol{b}}\delta'\boldsymbol{c}+\overline{\boldsymbol{b}}\delta\boldsymbol{b}\delta' e+\overline{\delta'\boldsymbol{b}}\delta\boldsymbol{c}-\overline{\boldsymbol{b}}\delta'\boldsymbol{b}\delta e\right) \\
&= s\,\langle \boldsymbol{u},\delta'\boldsymbol{u}\times\delta\boldsymbol{u}\rangle + m\left[\langle\delta\boldsymbol{v},\delta'\boldsymbol{r}\rangle - \langle\delta'\boldsymbol{v},\delta\boldsymbol{r}\rangle\right].
\end{aligned}
$$

Here we have recognized once more, in the first term, the 2-form (2.87) of the 2-sphere; and the calculation of the second term is just as in case (6). This proves (4.27). □

4.2 Discrete Quotients

The Kirillov-Kostant-Souriau Theorem only provides the homogeneous symplectic manifolds *up to covering.* Thus, if $\mathrm{G}(x_0) = \mathrm{G}/\mathrm{H}$ is one of our spaces in Theorem 4.2 (where H is the stabilizer of x_0), there remains for us to figure out if σ passes to a quotient G/K where K has the same identity component as H. Since H is connected in all cases, as one readily checks, this amounts to asking for which $\mathrm{K} \supset \mathrm{H}$ the 2-form σ *descends* to the deeper quotient G/K, i.e. there is a (necessarily G-invariant) 2-form τ on G/K such that $\sigma = \pi^*\tau$ where π is the projection $\mathrm{G}/\mathrm{H} \to \mathrm{G}/\mathrm{K}$. For this we have the following criterion:

Proposition 4.3. *Let* $\mathrm{G}(x_0) = \mathrm{G}/\mathrm{H}$ *be as above, and let* $\mathrm{K} \subset \mathrm{G}$ *have identity component* H. *Then* K *is contained in the normalizer* $\mathrm{N} = \{n \in \mathrm{G} : n\mathrm{H}n^{-1} = \mathrm{H}\}$. *Moreover, if* σ *descends to* G/K *then* K *is contained in the subgroup*

$$\mathrm{N}_\sigma = \big\{n \in \mathrm{N} : n(x_0) - x_0 \text{ annihilates } [\mathfrak{g},\mathfrak{g}]\big\}. \tag{4.31}$$

Proof. The first assertion is clear since any group K normalizes its identity component. To prove the second, assume that $\sigma = \pi^*\tau$ so that we have

$$\sigma(\mathrm{Z}(g\mathrm{H}),\mathrm{Z}'(g\mathrm{H})) = \tau(\mathrm{Z}(g\mathrm{K}),\mathrm{Z}'(g\mathrm{K})), \tag{4.32}$$

and consider the right action of K on G/H, $k_\mathrm{R}(g\mathrm{H}) := g\mathrm{H}k^{-1} = gk^{-1}\mathrm{H}$, by which G/K is the quotient of G/H. We claim that this action preserves σ. Indeed we have

$$(k_\mathrm{R}^*\sigma)(\mathrm{Z}(g\mathrm{H}),\mathrm{Z}'(g\mathrm{H})) = \sigma(k_{\mathrm{R}*}(\mathrm{Z}(g\mathrm{H})),k_{\mathrm{R}*}(\mathrm{Z}'(g\mathrm{H})))$$

$$
\begin{aligned}
&= \sigma\big(\tfrac{d}{dt}k_{\mathrm{R}}(e^{t\mathrm{Z}}g\mathrm{H})\big|_{t=0}, \tfrac{d}{dt}k_{\mathrm{R}}(e^{t\mathrm{Z}'}g\mathrm{H})\big|_{t=0}\big) \\
&= \sigma\big(\tfrac{d}{dt}e^{t\mathrm{Z}}gk^{-1}\mathrm{H}\big|_{t=0}, \tfrac{d}{dt}e^{t\mathrm{Z}'}gk^{-1}\mathrm{H}\big|_{t=0}\big) \\
&= \sigma(\mathrm{Z}(gk^{-1}\mathrm{H}), \mathrm{Z}'(gk^{-1}\mathrm{H})) \\
&= \tau(\mathrm{Z}(g\mathrm{K}), \mathrm{Z}'(g\mathrm{K})) && \text{by (4.32)} \\
&= \sigma(\mathrm{Z}(g\mathrm{H}), \mathrm{Z}'(g\mathrm{H})) && \text{by (4.32)}
\end{aligned}
$$

as claimed. On the other hand we know that σ is also invariant under the commuting left action $k(g\mathrm{H}) = kg\mathrm{H}$. Therefore it is still invariant under the composed action $\underline{k}(g\mathrm{H}) = kgk^{-1}\mathrm{H}$ which fixes the base point $x_0 = e\mathrm{H}$. Hence we have

$$
\begin{aligned}
\langle x_0, [\mathrm{Z}', \mathrm{Z}]\rangle &= \sigma(\mathrm{Z}(e\mathrm{H}), \mathrm{Z}'(e\mathrm{H})) && \text{by (2.77)} \\
&= (\underline{k}^*\sigma)(\mathrm{Z}(e\mathrm{H}), \mathrm{Z}'(e\mathrm{H})) && \text{since } \underline{k}^*\sigma = \sigma \\
&= \sigma\big(\tfrac{d}{dt}\,\underline{k}(e^{t\mathrm{Z}}\mathrm{H})\big|_{t=0}, \tfrac{d}{dt}\,\underline{k}(e^{t\mathrm{Z}'}\mathrm{H})\big|_{t=0}\big) \\
&= \sigma\big(\tfrac{d}{dt}\,ke^{t\mathrm{Z}}k^{-1}\mathrm{H})\big|_{t=0}, \tfrac{d}{dt}\,ke^{t\mathrm{Z}'}k^{-1}\mathrm{H})\big|_{t=0}\big) \\
&= \sigma((\mathrm{Ad}_k\mathrm{Z})(e\mathrm{H}), (\mathrm{Ad}_k\mathrm{Z}')(e\mathrm{H})) \\
&= \langle x_0, [\mathrm{Ad}_k\mathrm{Z}', \mathrm{Ad}_k\mathrm{Z}]\rangle && \text{by (2.77)} \\
&= \langle x_0, \mathrm{Ad}_k[\mathrm{Z}', \mathrm{Z}]\rangle \\
&= \langle k^{-1}(x_0), [\mathrm{Z}', \mathrm{Z}]\rangle && \text{by (2.75)}
\end{aligned}
$$

for all $k \in \mathrm{K}$ and $[\mathrm{Z}, \mathrm{Z}'] \in [\mathfrak{g}, \mathfrak{g}]$. This completes the proof. □

With this criterion in hand, we can prove:

Theorem 4.4. *The homogeneous symplectic manifolds of the Galilei group comprise the seven classes* (1) *through* (7) *of Theorem 4.2, plus the classes*

(4_a) *For each* $k > 0$, $s \in \mathbb{R}$ *and* $a > 0$, *a copy of* $\mathrm{TS}^2 \times \mathrm{TS}^1$

$$
\left\{ x = \begin{pmatrix} \boldsymbol{r} \times k\boldsymbol{u} + s\boldsymbol{u} \\ -kt\boldsymbol{u} \\ \mathrm{E} \\ k\boldsymbol{u} \end{pmatrix} \;\middle|\; \begin{array}{r} \langle \boldsymbol{r}, \boldsymbol{u}\rangle = 0 \\ \|\boldsymbol{u}\| = 1 \\ \mathrm{E} \in \mathbb{R}/a\mathbb{Z} \\ t \in \mathbb{R} \end{array} \right\} \tag{4.33}
$$

with 2-form

$$
\begin{aligned}
\sigma(\delta x, \delta' x) =& k\,[\langle \delta\boldsymbol{u}, \delta'\boldsymbol{r}\rangle - \langle \delta'\boldsymbol{u}, \delta\boldsymbol{r}\rangle] - (\delta\mathrm{E}\,\delta' t - \delta'\mathrm{E}\,\delta t) \\
&+ s\,\langle \boldsymbol{u}, \delta'\boldsymbol{u} \times \delta\boldsymbol{u}\rangle\,.
\end{aligned} \tag{4.34}
$$

(5_a) *For each* $k, n, a > 0$, *a copy of* $\mathrm{TSO}(3) \times \mathrm{TS}^1$

$$\left\{ x = \begin{pmatrix} \boldsymbol{\ell} \\ n\boldsymbol{v} - kt\boldsymbol{u} \\ \mathrm{E} \\ k\boldsymbol{u} \end{pmatrix} \;\middle|\; \begin{array}{l} \mathrm{U} = (\boldsymbol{v}\,\boldsymbol{w}\,\boldsymbol{u}) \in \mathrm{SO}(3) \\ \mathrm{L} = j(\boldsymbol{\ell})\mathrm{U} \in \mathrm{T_U SO}(3) \\ \mathrm{E} \in \mathbb{R}/a\mathbb{Z} \\ t \in \mathbb{R} \end{array} \right\} \tag{4.35}$$

with 2-form

$$\sigma(\delta x, \delta' x) = \tfrac{1}{2}\mathrm{Tr}(\overline{\delta \mathrm{L}}\delta'\mathrm{U} - \overline{\delta' \mathrm{L}}\delta\mathrm{U}) - (\delta\mathrm{E}\,\delta' t - \delta'\mathrm{E}\,\delta t). \tag{4.36}$$

The action of the Galilei group on (4.33) *and* (4.35) *is the same as the coadjoint action* (4.5) *on* (4.20) *and* (4.22), *except that* E *is taken modulo* a.

Proof. We proceed according to the cases of Theorem 4.2. In case (1), the groups H, N, and N_σ (4.31) all coincide with G. So necessarily K = H, and no new homogeneous symplectic manifolds are obtained. In case (2) we find

$$\begin{aligned} \mathrm{H} &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3\} \\ \mathrm{N} &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \pm\boldsymbol{e}_3\} \\ \mathrm{N}_\sigma &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3\}. \end{aligned}$$

So necessarily K = H, and no new homogeneous symplectic manifolds are obtained. In case (3) we find

$$\begin{aligned} \mathrm{H} &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3, b_1 = b_2 = 0\} \\ \mathrm{N} &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \pm\boldsymbol{e}_3, b_1 = b_2 = 0\} \\ \mathrm{N}_\sigma &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3, b_1 = b_2 = 0\}. \end{aligned}$$

So necessarily K = H, and no new homogeneous symplectic manifolds are obtained. In case (4) we find

$$\begin{aligned} \mathrm{H} &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3, b_3 = c_1 = c_2 = e = 0\} \\ \mathrm{N} &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \pm\boldsymbol{e}_3, c_1 = c_2 = e = 0\} \\ \mathrm{N}_\sigma &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3, c_1 = c_2 = e = 0\}. \end{aligned}$$

So the possible K are the groups

$$\mathrm{K}_a = \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3, kb_3 \in a\mathbb{Z}, c_1 = c_2 = e = 0\} \qquad (a \in \mathbb{R}),$$

giving (4.33) to which σ clearly descends. In case (5) we find

$$\begin{aligned} \mathrm{H} &= \{g \in \mathrm{G} : \mathrm{A} = \mathbb{1}, b_2 = b_3 = c_1 = c_2 = e = 0\} \\ \mathrm{N} &= \{g \in \mathrm{G} : \mathrm{A} = \mathbb{1}, e = 0\} \\ \mathrm{N}_\sigma &= \{g \in \mathrm{G} : \mathrm{A} = \mathbb{1}, b_2 = nb_3 + kc_1 = c_2 = e = 0\}. \end{aligned}$$

So the possible K are the groups

$$\mathrm{K}_a = \{g \in \mathrm{G} : \mathrm{A} = \mathbb{1}, kb_3 \in a\mathbb{Z}, b_2 = nb_3 + kc_1 = c_2 = e = 0\} \qquad (a \in \mathbb{R}),$$

giving (4.35) to which σ clearly descends. In case (6) we find

$$\mathrm{H} = \mathrm{N} = \mathrm{N}_\sigma = \{g \in \mathrm{G} : \boldsymbol{b} = \boldsymbol{c} = 0\}.$$

So necessarily K = H, and no new homogeneous symplectic manifolds are obtained. In case (7) we find

$$\begin{aligned} \mathrm{H} &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3, \boldsymbol{b} = \boldsymbol{c} = 0\} \\ \mathrm{N} &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \pm\boldsymbol{e}_3, \boldsymbol{b} = 0, c_1 = c_2 = 0\} \\ \mathrm{N}_\sigma &= \{g \in \mathrm{G} : \mathrm{A}\boldsymbol{e}_3 = \boldsymbol{e}_3, \boldsymbol{b} = \boldsymbol{c} = 0\}. \end{aligned}$$

So necessarily K = H, and no new homogeneous symplectic manifolds are obtained. This completes the proof. □

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