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THEORETICAL PROPERTIES AND ESTIMATION IN WEIGHTED WEIBULL AND RELATED DISTRIBUTIONS

by

RYAN ROMAN

(Under the Direction of Broderick O. Oluyede)

ABSTRACT

The Weibull distribution is a well known and common distribution. In this thesis, theoretical properties of weighted Weibull distributions are presented. Properties of the non-weighted Weibull distribution are also reiterated for comparison. The probability density functions, cumulative distribution functions, survival functions, hazard functions and reverse hazard funtions are given for each distribution. In addition, Glaser's Lemma is applied to determine the behavior of the hazard functions. The standardized moments, differential entropy, Fisher information and results based on the likelihood function are given for each distribution as well. These results are also shown for the Rayleigh distribution, a special case of the Weibull distribution, and its weighted versions.

Key Words: Weighted distribution; Weibull distribution; Weighted Weibull distribution

2009 *Mathematics Subject Classification*: 62N05, 62B10, 62N02

THEORETICAL PROPERTIES AND ESTIMATION IN WEIGHTED WEIBULL AND RELATED DISTRIBUTIONS

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RYAN ROMAN

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THEORETICAL PROPERTIES AND ESTIMATION IN WEIGHTED WEIBULL AND RELATED DISTRIBUTIONS

by

RYAN ROMAN

Major Professor: Broderick O. Oluyede

Committee: Charles Champ
Hani Samawi

Electronic Version Approved:

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DEDICATION

This thesis is dedicated to my parents for their unconditional love and support. I could not be blessed with a better mother and father.

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I wish to thank Dr. Oluyede for his support as an advisor. I also wish to thank Dr. Samawi and Dr. Champ for their valuable suggestions and everyone in the math departments at GSU and AASU for caring about their students.

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CHAPTER 1

INTRODUCTION

The usefulness and applications of parametric distributions including Weibull, Raleigh in various areas including reliability, renewal theory, and branching processes can be seen in recent papers by several authors including Oluyede (2006) and in references therein. The Weibull and inverse Weibull distributions are very useful models that can be used to describe the degradation phenomena of mechanical components such as pistons, crank shaft of diesel engines. These models also provide a reasonably good fit to data on times to breakdown of an insulating fluid, subject to constant tension, Nelson (1982).

Applications of weighted distribution to biased samples in various areas including medicine, ecology, reliability, and branching processes can be seen in Patil and Rao (1978), Gupta and Kirmani (1990), Gupta and Keating (1985), Oluyede (1999) and in references therein. In a weighted distribution problem, a realization x of X enters into the investigators record with probability proportional to a weight function $W(t)$. The recorded x is not an observation of X, but rather an observation on a weighted random variable X_W. Thus, the focus of this thesis is weighted Weibull (Rayleigh) distributions with weight functions $w(t) = t^c$ (which can account for certain biasedness') and $w(x) = \frac{1}{\lambda_X(x)}$, where $\lambda_X(x)$ is the hazard function of X (this weight function yields the renewal distribution).

An introduction to the distributions mentioned above is provided in chapter 2. The probability density functions, cumulative distribution functions, survival functions, hazard functions and reverse hazard functions are given. In addition, Glaser's Lemma is introduced and applied to determine the behavior of the hazard functions for the weighted Weibull (Rayleigh) and renewal Weibull (Rayleigh) distributions.

The moments are given in chapter 3. Using these results the mean, variance, standard deviation and coefficients of variation, skewness and kurtosis are computed. The latter three are graphed for comparison.

Differential entropy and Fisher information are presented in chapter 4. Rigorous calculations are provided to justify each result. Conclusions are made about the behavior of each result with respect to the parameters.

The results on likelihood functions are given in chapter 5. The maximum likelihood estimates are calculated for each distribution. Likelihood ratio tests for comparing parent distributions and their weighted counterparts are presented along with some test statistics.

CHAPTER 2

DISTRIBUTIONS AND PROPERTIES

2.1 Basic Notions

Suppose the distribution of a continuous random variable X has parameter set $\theta^* = \{\theta_1, \theta_2, ..., \theta_n\}$. Let the probability density function (pdf) and cumulative density function (cdf) of X be given by $f(x;\theta^*) = f(x)$ and $F(x;\theta^*) = F(x)$, respectively. The survival function of X is given by

$$S_X(x) = P(X > x) = 1 - P(X \leq x) = 1 - F(x). \tag{2.1}$$

Note that $S_X(x)$ is monotonically decreasing and, if X is non-negative,

$$\int_0^{\infty} S_X(x)dx = E_X(X). \tag{2.2}$$

The hazard function and the reverse hazard function of X are given by

$$\lambda_X(x) = \frac{f(x)}{S_X(x)} \tag{2.3}$$

and

$$\tau_X(x) = \frac{f(x)}{F(x)}, \tag{2.4}$$

respectively.

If $\lambda_X(x)$ is monotonically increasing (decreasing), then the distribution of X has a(n) increasing (decreasing) failure rate, denoted IFR (DFR). If $\lambda_X(x)$ is constant, then the distribution of X has a constant failure rate (CFR). Furthermore, if $\lambda_X(x)$ is constant and X is non-negative, then $E_X(X) = \frac{1}{\lambda_X(x)}$. This can be seen by multiplying both sides of equation (2.3) by $S_X(x)$ and integrating both sides over the support of X.

If the distribution of random variable Y is the weighted distribution of (non-negative, continuous random variable) X, with weight function $w(t) > 0$, then the pdf of Y is given by

$$g(y;\theta^*|w(t)) = g(y) = \frac{w(y)f(y;\theta^*)}{E_X[w(X)]}, 0 < E_X[w(X)] < \infty. \tag{2.5}$$

The weight function $w(t) = t^c$ can account for certain biases in the underlying distribution (for example, $w(t) = t$ is used if the underlying distribution is length biased). A special case of a weighted distribution is the renewal distribution. The renewal distribution occurs when $w(t) = \frac{1}{\lambda_X(t)}$. If Z has the renewal distribution of X, then the pdf of Z is given by

$$f_R\left(z;\theta^*|w(t) = \frac{1}{\lambda_X(t)}\right) = f_R(z) = \frac{S_X(z)}{E_X(X)}. \tag{2.6}$$

2.1.1 Useful Functions

In this section some useful functions are presented. The gamma function is given by

$$\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}dt. \tag{2.7}$$

Two important properties of the Gamma function are

$$\Gamma(k+1) = k\Gamma(k), \tag{2.8}$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{2.9}$$

Denote $\Gamma\left(\frac{j}{\beta}+1\right)$, where β is the scale parameter of the Weibull distribution ($\beta = 2$ for the Rayleigh distribution), by Γ_j. The lower and upper incomplete gamma functions are given by

$$\gamma(k,x) = \int_0^x t^{k-1}e^{-t}dt \tag{2.10}$$

and

$$\Gamma(k,x) = \int_x^\infty t^{k-1}e^{-t}dt, \tag{2.11}$$

respectively. Note that $\Gamma(k) = \gamma(k,x) + \Gamma(k,x)$.

The error function is given by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^t dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{n!(2n+1)}. \tag{2.12}$$

A special relation of the lower incomplete gamma function and the error function is

$$\gamma\left(\frac{1}{2}, x\right) = \sqrt{\pi} erf(x). \tag{2.13}$$

Although some functions presented later (for example, the mgf of the Rayleigh distribution) can be written in terms of the error function, this thesis will not include the error function outside of this section.

2.2 Glaser's lemma

Lemma 2.2.1. *Let $f(x)$ be a twice differentiable probability density function of a continuous random variable X. Define $\eta(x) = \frac{-f'(x)}{f(x)}$, where $f'(x)$ is the first derivative of $f(x)$ with respect to x. Furthermore, suppose the first derivative of $\eta(x)$ exist.*

1. If $\eta'(x) < 0$, for all $x > 0$, then the hazard function is monotonically decreasing (DFR).

2. If $\eta'(x) > 0$, for all $x > 0$, then the hazard function is monotonically increasing (IFR).

3. If there exist x_0 such that $\eta'(x) > 0$, for all $0 < x < x_0$;, $\eta'(x_0) = 0$ and $\eta'(x) < 0$ for all $x > x_0$. In addition, $\lim_{x\to 0} f(x) = 0$, then the hazard function is upside down bathtub shape (UBT).

4. If there exist x_0 such that $\eta'(x) < 0$, for all $0 < x < x_0$;, $\eta'(x_0) = 0$ and $\eta'(x) > 0$ for all $x > x_0$. In addition, $\lim_{x\to 0} f(x) = \infty$, then the hazard function is bathtub shape (BT).

5. If $\eta'(x) = 0$, for all $x > 0$, then the hazard function is constant(CFR).

Note that

$$\eta_F(x) = \frac{-f'(x)}{f(x)} = -\frac{d}{dx}\ln[f(x)]. \tag{2.14}$$

It follows that for a weighted distribution with pdf $g(y)$,

$$\begin{aligned}\eta_G(y) &= -\frac{d}{dy}\ln[g(y)] \\ &= -\frac{d}{dy}\ln[w(y)] - \frac{d}{dy}\ln[f(y)] + \frac{d}{dy}\ln(E_X[w(X)]) \\ &= \eta_W(y) + \eta_F(y) \qquad (2.15)\end{aligned}$$

and

$$\eta'_G(y) = \eta'_W(y) + \eta'_F(y). \tag{2.16}$$

For a renewal distribution with pdf $f_R(z)$

$$\eta_{F_R}(z) = \frac{-\frac{S'_X(z)}{E_X(X)}}{\frac{S_X(z)}{E_X(X)}} = \frac{f(z)}{S_X(z)} = \lambda_X(z). \tag{2.17}$$

2.3 Weibull Distribution

Let X have a Weibull distribution with shape parameter θ and scale parameter β (denoted by $X \sim Weibull(\theta, \beta)$). The probability density function (pdf) and cumulative density function (cdf) are given by

$$f(x) = \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-(\frac{x}{\theta})^\beta}, \quad x \geq 0,\ \theta > 0,\ \beta > 0, \tag{2.18}$$

and

$$F(x) = \int_0^x \frac{\beta}{\theta^\beta} t^{\beta-1} e^{-(\frac{x}{\theta})^\beta} dt = 1 - e^{-(\frac{x}{\theta})^\beta}, \quad x \geq 0,\ \alpha > 0,\ \beta > 0, \tag{2.19}$$

respectively. It follows that the corresponding survival function, hazard and reverse hazard functions are given by

$$S_X(x) = 1 - F(x) = e^{-(\frac{x}{\theta})^\beta}, \tag{2.20}$$

$$\lambda_X(x) = \frac{f(x)}{S_X} = \frac{\beta}{\theta^\beta} x^{\beta-1}, \tag{2.21}$$

and

$$\tau_X(x) = \frac{f(x)}{F(x)} = \frac{\beta x^{\beta-1}}{\theta^\beta (e^{(\frac{x}{\theta})^\beta} - 1)}, \tag{2.22}$$

respectively.

By observing the fact that

$$\lambda'_X(x) = \frac{\beta(\beta-1)}{\theta^\beta} x^{\beta-2}, \tag{2.23}$$

it is easy to see the following:

(1)The Weibull distribution has a DFR if $\beta < 1$.

(2)The Weibull distribution has a CFR if $\beta = 1$.

(3)The Weibull distribution has an IFR if $\beta > 1$.

If Glaser's Lemma is used to determine the behavior of $\lambda_X(x)$ then

$$\begin{aligned} f'(x) &= \frac{\beta(\beta-2)}{\theta^\beta} x^{\beta-1} e^{-(\frac{x}{\theta})^\beta} - \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-(\frac{x}{\theta})^\beta} \frac{\beta}{\theta^\beta} x^{\beta-1} \\ &= f(x) \left(\frac{\beta-1}{x} - \frac{\beta}{\theta^\beta} x^{\beta-1} \right), \end{aligned} \tag{2.24}$$

therefore,

$$\eta_F(x) = \frac{-f'(x)}{f(x)} = \frac{\beta}{\theta^\beta} x^{\beta-1} - \frac{\beta-1}{x}, \tag{2.25}$$

and

$$\eta_F'(x) = \frac{\beta(\beta-1)}{\theta^\beta}x^{\beta-2} + \frac{\beta-1}{x^2}. \tag{2.26}$$

Equation (2.26) yields the same conclusions reached before. Although Glaser's Lemma was unnecessary in this particular case, there are many cases in which the behavior of the hazard function is not easily seen.

2.3.1 Rayleigh Distribution

If random variable $X \sim Rayleigh(\sigma)$, then $X \sim Weibull(\sqrt{2}\sigma, 2)$. Therefore, the Rayleigh distribution has an IHR and the pdf, cdf, survival function, hazard function and reversed hazard function are given by

$$f(x) = \frac{xe^{-\frac{1}{2}(\frac{x}{\sigma})^2}}{\sigma^2}, \quad x \geq 0,\, \sigma > 0, \tag{2.27}$$

$$F(x) = 1 - e^{-\frac{1}{2}(\frac{x}{\sigma})^2}, \quad x \geq 0,\, \sigma > 0, \tag{2.28}$$

$$S_X(x) = e^{-\frac{1}{2}(\frac{x}{\sigma})^2}, \tag{2.29}$$

$$\lambda_X(x) = \frac{x}{\sigma^2}, \tag{2.30}$$

and

$$\tau_X(x) = \frac{x}{\sigma^2(e^{\frac{1}{2}(\frac{x}{\sigma})^2} - 1)}, \tag{2.31}$$

respectively.

2.4 Weighted Weibull Distribution

Let random variable Y have a weighted Weibull distribution with shape parameter θ, scale parameter β and weight function $w(t)$ (denoted by $Y \sim Weibull_W(\theta, \beta; w(y))$).

If $w(t) = t^c$, then the pdf and cdf are given by

$$g(y) = \frac{\beta}{\theta^{\beta+c}\Gamma_c} y^{c+\beta-1} e^{-(\frac{y}{\theta})^\beta}, \quad y \geq 0,\ \theta > 0,\ \beta > 0, \tag{2.32}$$

and

$$\begin{aligned} G(y) &= \int_0^y g(y)dy \\ &= \frac{1}{\Gamma_c}\int_0^{(\frac{y}{\theta})^\beta} u^{\frac{c}{\beta}} e^{-u} du \\ &= \frac{\gamma(\frac{c}{\beta}+1, (\frac{y}{\theta})^\beta)}{\Gamma_c}, \quad y \geq 0,\ \theta > 0,\ \beta > 0, \end{aligned} \tag{2.33}$$

respectively. The survival function, hazard function and reverse hazard function are given by

$$S_Y(y) = \frac{\Gamma(\frac{c}{\beta}+1, (\frac{y}{\theta})^\beta)}{\Gamma_c}, \tag{2.34}$$

$$\lambda_Y(y) = \frac{\beta}{\theta^{\beta+c}\Gamma(\frac{c}{\beta}+1, (\frac{y}{\theta})^\beta)} y^{c+\beta-1} e^{-(\frac{y}{\theta})^\beta} \tag{2.35}$$

and

$$\tau_Y(y) = \frac{\beta}{\theta^{\beta+c}\gamma(\frac{c}{\beta}+1, (\frac{y}{\theta})^\beta)} y^{c+\beta-1} e^{-(\frac{y}{\theta})^\beta}. \tag{2.36}$$

respectively.

The behavior of $\lambda_Y(y)$ is not as simple to determine as $\lambda_X(x)$. Using the fact that

$$\eta'_W(y) = -\frac{d^2}{dy^2}\ln[w(y)] = \frac{c}{y^2}, \tag{2.37}$$

along with (2.26) and Glaser's Lemma results in

$$\eta'_G(y) = \frac{c}{y^2} + \frac{\beta(\beta-1)}{\theta^\beta} y^{\beta-2} + \frac{\beta-1}{y^2} = \frac{c+\beta-1+\frac{\beta}{\theta^\beta}(\beta-1)y^\beta}{y^2}. \tag{2.38}$$

Setting (2.38) to zero and solving for y yields

$$y_0 = \theta\left(\frac{1-c-\beta}{\beta(\beta-1)}\right)^{\frac{1}{\beta}}. \tag{2.39}$$

Equations (2.38) and (2.39) lead to the following conclusions:

(1) If $c+\beta \le 1$ the distribution has a DFR.

(2) Since $y < y_0$ implies $\eta'_G(y_0) > 0$ and $y > y_0$ implies $\eta'_G(y_0) < 0$, the weighted Weibull has a UBTFR if $c+\beta > 1$ and $\beta < 1$ (see figure 1.5).

(3) If $\beta \ge 1$ the distribution has a IFR.

2.4.1 Weighted Rayleigh distribution

Let random variable $Y \sim Rayleigh_W(\sigma; w(y))$, then $Y \sim Weibull_W(\sqrt{2}\sigma, 2; w(y))$. Therefore, if $w(t) = t^c$, the weighted Rayleigh distribution has an IHR and the pdf, cdf, survival function, hazard function and reverse hazard function are given by

$$g(y) = \frac{1}{2^{\frac{c}{2}}\sigma^{c+2}\Gamma_c} y^{c+1} e^{-\frac{1}{2}(\frac{y}{\sigma})^2}, \quad y \ge 0,\ \sigma > 0, \tag{2.40}$$

$$G(y) = \frac{\gamma(\frac{c}{2}+1, \frac{1}{2}(\frac{y}{\sigma})^2)}{\Gamma_c}, \quad y \ge 0,\ \sigma > 0, \tag{2.41}$$

$$S_Y(y) = \frac{\Gamma(\frac{c}{2}+1, \frac{1}{2}(\frac{y}{\sigma})^2)}{\Gamma_c}, \tag{2.42}$$

$$\lambda_Y(y) = \frac{1}{2^{\frac{c}{2}}\sigma^{c+2}\Gamma(\frac{c}{2}+1, \frac{1}{2}(\frac{y}{\sigma})^2)} y^{c+1} e^{-\frac{1}{2}(\frac{y}{\sigma})^2}, \tag{2.43}$$

and

$$\tau_Y(y) = \frac{1}{2^{\frac{c}{2}}\sigma^{c+2}\gamma(\frac{c}{2}+1, \frac{1}{2}(\frac{y}{\sigma})^2)} y^{c+1} e^{-\frac{1}{2}(\frac{y}{\sigma})^2}, \tag{2.44}$$

respectively.

2.5 Weibull Renewal Distribution

Let random variable Z have a Weibull renewal distribution with shape parameter θ and scale parameter β (denoted by $Z \sim Weibull_R(\theta, \beta)$). The pdf and cdf of the

Weibull renewal distribution are given by

$$f_R(z) = \frac{\beta e^{-(\frac{z}{\theta})^\beta}}{\theta\Gamma(\frac{1}{\beta})}, \quad z \geq 0,\ \theta > 0,\ \beta > 0, \tag{2.45}$$

and

$$\begin{aligned} F_R(z) &= \int_0^z f_R(t)dt \\ &= \frac{1}{\Gamma(\frac{1}{\beta})}\int_0^{(\frac{z}{\theta})^\beta} u^{\frac{1}{\beta}+1}e^{-u}du \\ &= \frac{\gamma(\frac{1}{\beta},(\frac{z}{\theta})^\beta)}{\Gamma(\frac{1}{\beta})}, \quad z \geq 0,\ \theta > 0,\ \beta > 0, \end{aligned} \tag{2.46}$$

respectively. The survival function, hazard function and reverse hazard functions are given by

$$S_Z(z) = \frac{\Gamma(\frac{1}{\beta},(\frac{z}{\theta})^\beta)}{\Gamma(\frac{1}{\beta})}, \tag{2.47}$$

$$\lambda_Z(z) = \frac{\beta e^{-(\frac{z}{\theta})^\beta}}{\theta\Gamma(\frac{1}{\beta},(\frac{z}{\theta})^\beta)}, \tag{2.48}$$

and

$$\tau_Z(z) = \frac{\beta e^{-(\frac{z}{\theta})^\beta}}{\theta\gamma(\frac{1}{\beta},(\frac{z}{\theta})^\beta)}. \tag{2.49}$$

respectively.

Based on the results from (2.17) and section 2.3 the following conclusions are made:

(1)The Weibull renewal distribution has a DFR if $\beta < 1$.

(2)The Weibull renewal distribution has a CFR if $\beta = 1$.

(3)The Weibull renewal distribution has an IFR if $\beta > 1$.

2.5.1 Rayleigh Renewal Distribution

If random variable $Z \sim Rayleigh_R(\sigma)$, then $Z \sim Weibull_R(\sqrt{2}\sigma, 2)$. Therefore, the Rayleigh renewal distribution has an IHR and the pdf, cdf, survival function, hazard function and reverse hazard function are given by

$$f_R(z) = \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}(\frac{z}{\sigma})^2}}{\sigma}, \quad z \geq 0,\ \sigma > 0, \tag{2.50}$$

$$F_R(z) = \frac{\gamma(\frac{1}{2}, \frac{1}{2}(\frac{z}{\sigma})^2)}{\sqrt{\pi}}, \quad z \geq 0,\ \sigma > 0, \tag{2.51}$$

$$S_Z(z) = \frac{\Gamma(\frac{1}{2}, \frac{1}{2}(\frac{z}{\sigma})^2)}{\sqrt{\pi}}, \tag{2.52}$$

$$\lambda_Z(z) = \frac{\sqrt{2}e^{-\frac{1}{2}(\frac{z}{\sigma})^2}}{\sigma\Gamma(\frac{1}{2}, \frac{1}{2}(\frac{z}{\sigma})^2)}, \tag{2.53}$$

and

$$\tau_Z(z) = \frac{\sqrt{2}e^{-\frac{1}{2}(\frac{z}{\sigma})^2}}{\sigma\gamma(\frac{1}{2}, \frac{1}{2}(\frac{z}{\sigma})^2)}, \tag{2.54}$$

respectively.

2.6 Useful Graphs

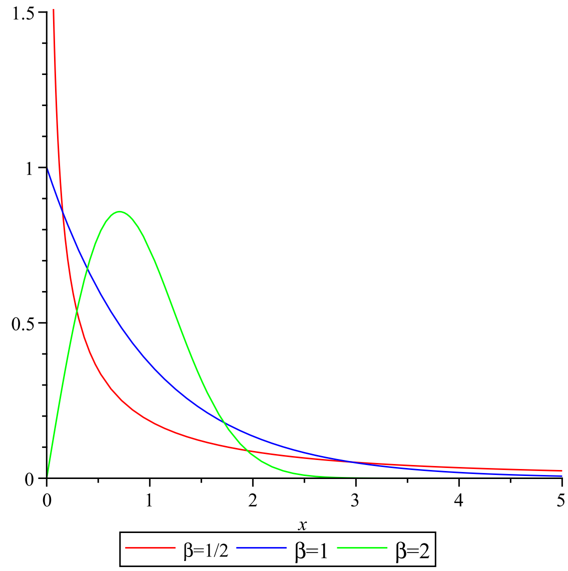

Figure 2.1: Weibull pdf with $\theta = 1$

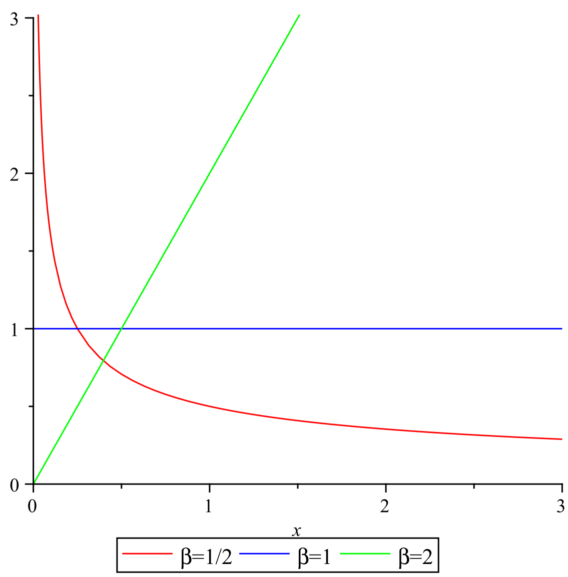

Figure 2.2: $\lambda_X(x)$ with $\theta = 1$

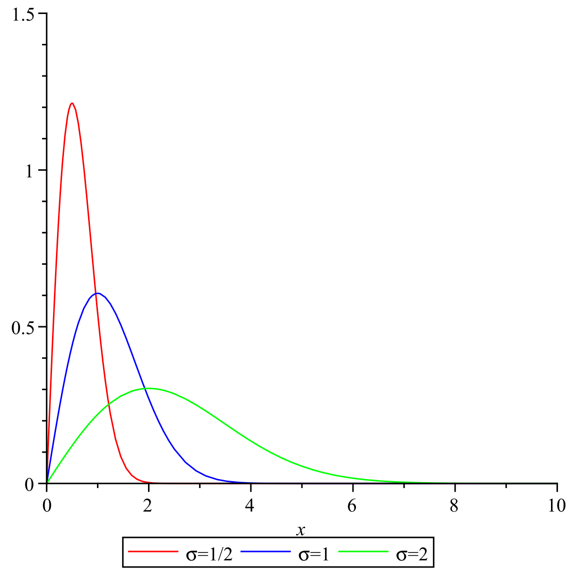

Figure 2.3: Rayleigh pdf

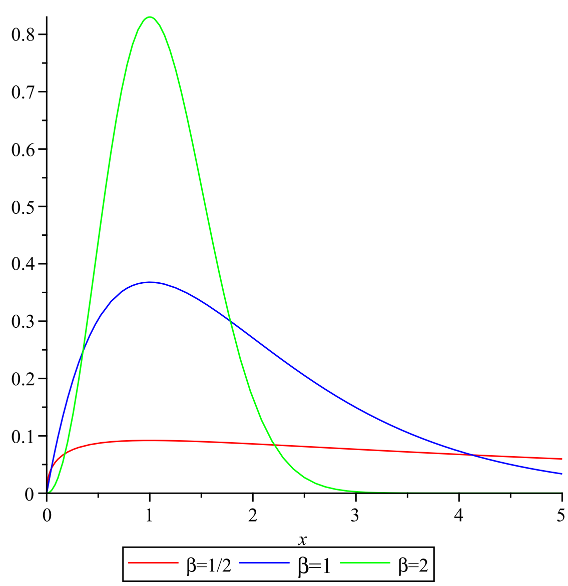

Figure 2.4: Length biased Weibull pdf with $\theta = 1$

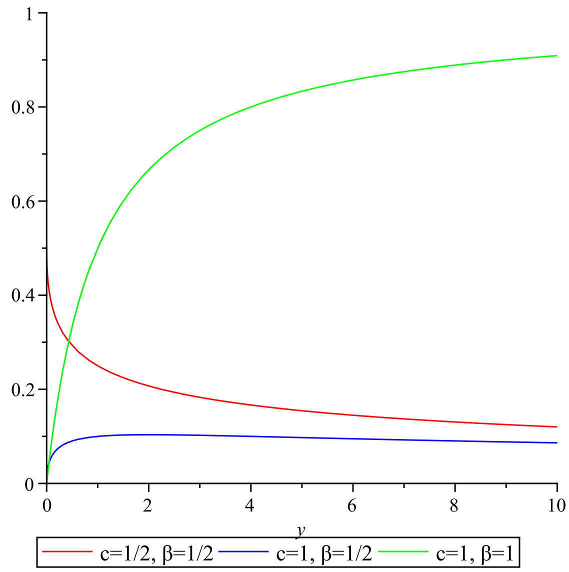

Figure 2.5: $\lambda_Y(y)$ with $\theta = 1$

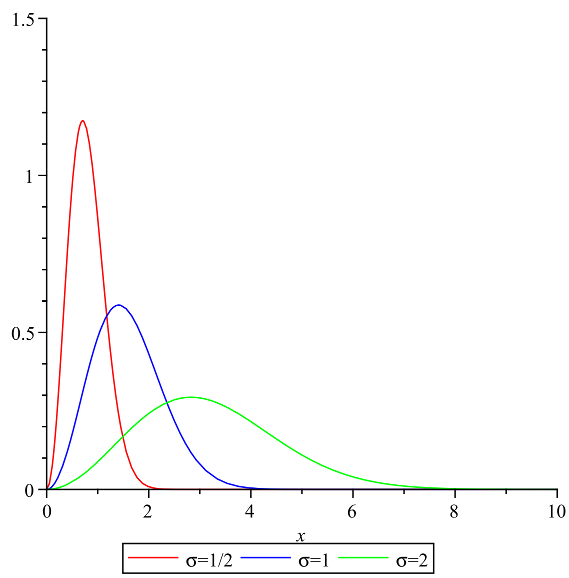

Figure 2.6: Length biased Rayleigh pdf

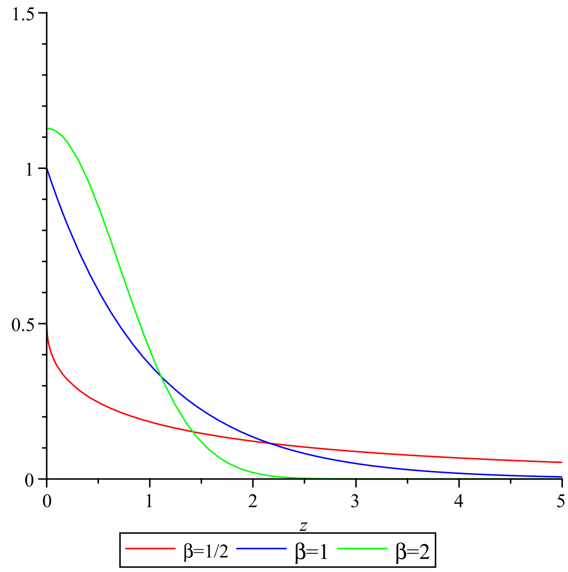

Figure 2.7: Weibull renewal pdf with $\theta = 1$

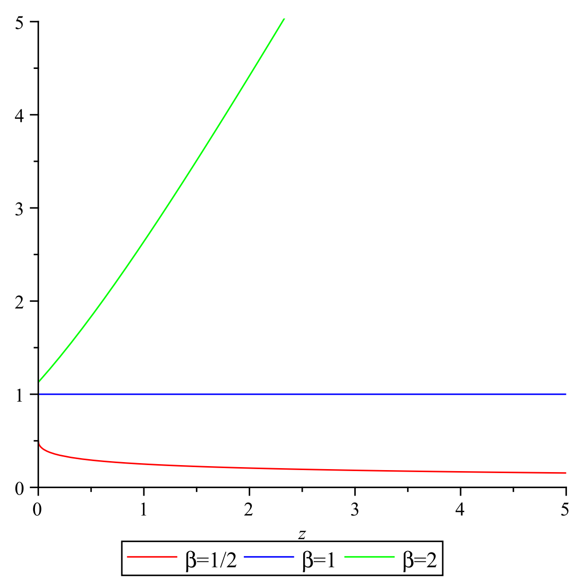

Figure 2.8: $\lambda_Z(y)$ with $\theta = 1$

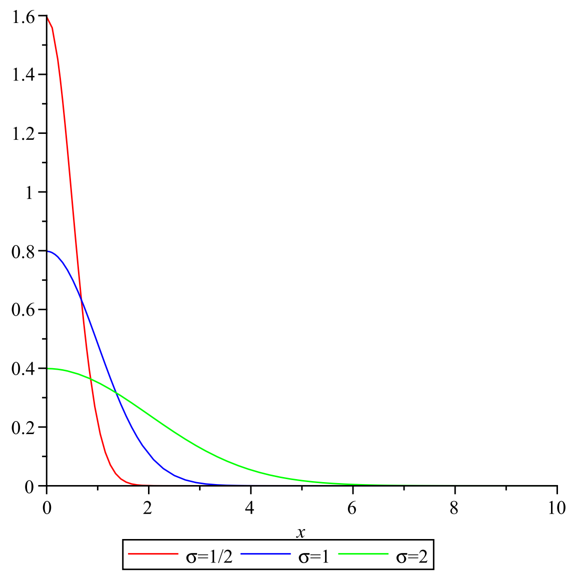

Figure 2.9: Rayleigh renewal pdf

CHAPTER 3

MOMENTS, MOMENT GENERATING FUNCTION IN WEIGHTED WEIBULL DISTRIBUTION AND ITS VARIANTS

3.1 Basic Notions

Recall that the n^{th} central moment is of random variable X is given by

$$\mu_{X,n} = E_X[(X - \mu_X)^n]. \tag{3.1}$$

The coefficients of variation (CV), skewness (CS) and kurtosis (CK) are given by

$$CV = \frac{\sigma_X}{\mu_X}, \quad \mu_X \neq 0, \tag{3.2}$$

$$CS = \frac{\mu_{X,3}}{\sigma_X^3}, \tag{3.3}$$

and

$$CK = \frac{\mu_{X,4}}{\sigma_X^4}, \tag{3.4}$$

respectively. Kurtosis and excess kurtosis are easily mistaken for one another. In this paper, only kurtosis is computed. However, excess kurtosis (EK) is simply given by $EK = CK - 3$.

3.2 Weibull Distribution

Let $X \sim Weibull(\theta, \beta)$. The moments of X are given by

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k f(x) dx \\ &= \theta^k \int_0^\infty u^{\frac{k}{\beta}+1} e^{-u} du \\ &= \theta^k \Gamma_k. \end{aligned} \tag{3.5}$$

The moment generating function (MGF) of X is given by

$$\begin{aligned} M_X(t) = E(e^{tx}) &= \int_0^\infty e^{tx} f(x)dx \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^\infty x^j f(x)dx \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!} \theta^j \Gamma_j. \end{aligned} \tag{3.6}$$

The mean, variance and standard deviation are given by

$$\mu_X = \theta\Gamma_1, \tag{3.7}$$

$$\sigma_X^2 = \theta^2(\Gamma_2 - \Gamma_1^2), \tag{3.8}$$

and

$$\sigma_X = \theta\sqrt{\Gamma_2 - \Gamma_1^2}, \tag{3.9}$$

respectively.

The coefficients of variation, skewness and kurtosis are given by

$$CV = \frac{\sqrt{\Gamma_2 - \Gamma_1^2}}{\Gamma_1}, \tag{3.10}$$

$$CS = \frac{\Gamma_3 - 3\Gamma_2\Gamma_1 + 2\Gamma_1^3}{(\Gamma_2 - \Gamma_1^2)^{\frac{3}{2}}}, \tag{3.11}$$

and

$$CK = \frac{\Gamma_4 - 4\Gamma_3\Gamma_1 + 6\Gamma_2\Gamma_1^2 - 3\Gamma_1^4}{(\Gamma_2 - \Gamma_1^2)^2}, \tag{3.12}$$

respectively.

3.2.1 Rayleigh Distribution

Let $X \sim Rayleigh(\sigma)$. The moments of X are given by

$$E(X^k) = 2^{\frac{k}{2}} \sigma^k \Gamma_k. \tag{3.13}$$

The MGF of X is given by

$$M_X(t) = \sum_{j=0}^{\infty} 2^{\frac{j}{2}} \sigma^j \Gamma_j. \tag{3.14}$$

The mean, variance and standard deviation are given by

$$\mu_X = \sigma \frac{\sqrt{2\pi}}{2}, \tag{3.15}$$

$$\sigma_X^2 = \sigma^2 \left(2 - \frac{\pi}{2}\right), \tag{3.16}$$

and

$$\sigma_X = \sigma \sqrt{\left(2 - \frac{\pi}{2}\right)}, \tag{3.17}$$

respectively.

The coefficients of variation, skewness and kurtosis are given by

$$CV = 2\sqrt{\frac{1}{\pi} - \frac{1}{4}} \approx 0.5227, \tag{3.18}$$

$$CS = \frac{2\sqrt{\pi}(\pi - 3)}{(4 - \pi)^{\frac{3}{2}}} \approx 0.6311, \tag{3.19}$$

and

$$CK = \frac{32 - 3\pi^2}{(4 - \pi)^2} \approx 3.2451, \tag{3.20}$$

respectively.

3.3 Weighted Weibull Distribution

Let random variable Y have the weighted distribution of random variable X. If X and Y have pdfs $f(x)$ and $g(y|w(y) = y^c)$, respectively, then

$$\begin{aligned} E_Y(Y^k) &= \int_0^{\infty} y^k g(y) dy \\ &= \frac{1}{E_X(X^c)} \int_0^{\infty} y^{k+c} f(y) dy \\ &= \frac{E_X(X^{c+k})}{E_X(X^c)}. \end{aligned} \tag{3.21}$$

Therefore, if $Y \sim Weibull_W(\theta, \beta | w(t) = t^c)$, then the moments of Y are given by

$$E_Y(Y^k) = \theta^k \frac{\Gamma_{c+k}}{\Gamma_c}. \tag{3.22}$$

The MGF of Y is given by

$$M_Y(t) = \frac{1}{\Gamma_c} \sum_{j=0}^{\infty} \frac{t^j}{j!} \theta^k \Gamma_{c+j}. \tag{3.23}$$

The mean, variance and standard deviation are given by

$$\mu_Y = \theta \frac{\Gamma_{c+1}}{\Gamma_c}, \tag{3.24}$$

$$\sigma_Y^2 = \left(\frac{\theta}{\Gamma_c}\right)^2 (\Gamma_{c+2}\Gamma_c - \Gamma_{c+1}^2), \tag{3.25}$$

and

$$\sigma_Y = \frac{\theta}{\Gamma_c} \sqrt{(\Gamma_{c+2}\Gamma_c - \Gamma_{c+1}^2)}, \tag{3.26}$$

respectively.

The coefficients of variation, skewness and kurtosis are given by

$$CV = \frac{\sqrt{\Gamma_{c+2}\Gamma_c - \Gamma_{c+1}^2}}{\Gamma_{c+1}}, \tag{3.27}$$

$$CS = \frac{\Gamma_{c+3}\Gamma_c^2 - 3\Gamma_{c+2}\Gamma_{c+1}\Gamma_c + 2\Gamma_{c+1}^3}{(\Gamma_{c+2}\Gamma_c - \Gamma_{c+1}^2)^{\frac{3}{2}}}, \tag{3.28}$$

and

$$CK = \frac{\Gamma_{c+4}\Gamma_c^3 - 4\Gamma_{c+3}\Gamma_{c+1}\Gamma_c^2 + 6\Gamma_{c+2}\Gamma_{c+1}^2\Gamma_c - 3\Gamma_{c+1}^4}{(\Gamma_{c+2}\Gamma_c - \Gamma_{c+1}^2)^2}, \tag{3.29}$$

respectively.

3.3.1 Weighted Rayleigh Distribution

If Y has a weighted Rayleigh distribution, then the equations for the moments, MGF, mean, variance, standard deviation, CV, CS and CK are the same as equations (3.21-29), respectively, setting $\theta = \sqrt{2}\sigma$ and $\beta = 2$. If Y has a length bias Rayleigh distribution ($Y \sim Rayleigh_W(\sigma|w(t) = t)$), then the moments of Y are given by

$$E_Y(Y^k) = 2\sigma^k \sqrt{\frac{2^k}{\pi}} \Gamma_{k+1}. \tag{3.30}$$

The MGF of Y is given by

$$M_Y(t) = \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} 2^{\frac{j}{2}} \sigma^j \Gamma_{j+1}. \tag{3.31}$$

The mean, variance and standard deviation are given by

$$\mu_Y = 2\sigma \sqrt{\frac{2}{\pi}}, \tag{3.32}$$

$$\sigma_Y^2 = \sigma^2 \left(3 - \frac{8}{\pi} \right), \tag{3.33}$$

and

$$\sigma_Y = \sigma \sqrt{3 - \frac{8}{\pi}}, \tag{3.34}$$

respectively.

The coefficients of variation, skewness and kurtosis are given by

$$CV = \sqrt{\frac{3}{8}\pi - 1} \approx 0.4220, \tag{3.35}$$

$$CS = \sqrt{2} \left(\frac{32 - 10\pi}{(3\pi - 8)^{\frac{3}{2}}} \right) \approx 0.4857, \tag{3.36}$$

and

$$CK = \frac{15\pi^2 + 16\pi - 192}{(3\pi - 8)^2} \approx 3.1081, \tag{3.37}$$

respectively.

3.4 Weibull Renewal Distribution

Let random variable Z have the renewal distribution of random variable X. If X and Z have pdfs $f(x)$ and $f_R(z)$, respectively, then

$$\begin{aligned} E_Z(Z^k) &= \int_0^\infty z^k r(z)dz \\ &= \frac{1}{E_X(X)}\int_0^\infty z^k S_X(z)dz \\ &= \frac{1}{E_X(X)}\left(\frac{z^{k+1}}{k+1}S_X(z)|_{z=0}^\infty + \int_0^\infty \frac{z^{k+1}}{k+1}f_X(z)dz\right) \\ &= \frac{E_X(X^{k+1})}{(k+1)E_X(X)}. \end{aligned} \tag{3.38}$$

Therefore, if $Z \sim Weibull_R(\theta, \beta)$, then the moments of Z are given by

$$E_Z(Z^k) = \theta^k \frac{\Gamma(\frac{k+1}{\beta})}{\Gamma(\frac{1}{\beta})}. \tag{3.39}$$

The MGF of Z is given by

$$M_Z(t) = \frac{1}{\beta}\sum_{j=0}^\infty \theta^j \Gamma\left(\frac{j+1}{\beta}\right). \tag{3.40}$$

The mean, variance and standard deviation are given by

$$\mu_Z = \theta\frac{\Gamma(\frac{2}{\beta})}{\Gamma(\frac{1}{\beta})}, \tag{3.41}$$

$$\sigma_Z^2 = \left(\frac{\theta}{\Gamma(\frac{1}{\beta})}\right)^2\left[\Gamma\left(\frac{3}{\beta}\right)\Gamma\left(\frac{1}{\beta}\right) - \Gamma^2\left(\frac{2}{\beta}\right)\right], \tag{3.42}$$

and

$$\sigma_Z = \left(\frac{\theta}{\Gamma(\frac{1}{\beta})}\right)\sqrt{\Gamma\left(\frac{3}{\beta}\right)\Gamma\left(\frac{1}{\beta}\right) - \Gamma^2\left(\frac{2}{\beta}\right)}, \tag{3.43}$$

respectively.

The coefficients of variation, skewness and kurtosis are given by

$$CV = \frac{\sqrt{\Gamma(\frac{3}{\beta})\Gamma(\frac{1}{\beta}) - \Gamma^2(\frac{2}{\beta})}}{\Gamma(\frac{2}{\beta})}, \tag{3.44}$$

$$CS = \frac{\Gamma(\frac{4}{\beta})\Gamma^2(\frac{1}{\beta}) - 3\Gamma(\frac{3}{\beta})\Gamma(\frac{2}{\beta})\Gamma(\frac{1}{\beta}) + 2\Gamma^3(\frac{2}{\beta})}{(\Gamma(\frac{3}{\beta})\Gamma(\frac{1}{\beta}) - \Gamma^2(\frac{2}{\beta}))^{\frac{3}{2}}}, \tag{3.45}$$

and

$$CK = \frac{\Gamma(\frac{5}{\beta})\Gamma^3(\frac{1}{\beta}) - 4\Gamma(\frac{4}{\beta})\Gamma(\frac{2}{\beta})\Gamma^2(\frac{1}{\beta}) + 6\Gamma(\frac{3}{\beta})\Gamma^2(\frac{2}{\beta})\Gamma(\frac{1}{\beta}) - 3\Gamma^4(\frac{2}{\beta})}{(\Gamma(\frac{3}{\beta})\Gamma(\frac{1}{\beta}) - \Gamma^2(\frac{2}{\beta}))^2}, \tag{3.46}$$

respectively.

3.4.1 Rayleigh Renewal Distribution

Let $Z \sim Rayleigh_R(\sigma)$. The moments of Z are given by

$$E_Z(Z^k) = \sqrt{\frac{2^k}{\pi}}\sigma^k\Gamma\left(\frac{k+1}{2}\right). \tag{3.47}$$

The MGF of Z is

$$M_Z(t) = \frac{1}{\sqrt{\pi}}\sum_{i=0}^{\infty} 2^{\frac{j}{2}}\theta^j\Gamma\left(\frac{j+1}{2}\right). \tag{3.48}$$

The mean, variance and standard deviation are given by

$$\mu_Z = \sqrt{\frac{2}{\pi}}\sigma, \tag{3.49}$$

$$\sigma_Z^2 = \sigma^2\left(1 - \frac{2}{\pi}\right), \tag{3.50}$$

and

$$\sigma_Z = \sigma\sqrt{1 - \frac{2}{\pi}}, \tag{3.51}$$

respectively.

The coefficients of variation, skewness and kurtosis are given by

$$CV = \sqrt{\frac{\pi}{2} - 1} \approx 0.7055, \tag{3.52}$$

$$CS = \sqrt{2}\left(\frac{4-\pi}{(\pi-2)^{\frac{3}{2}}}\right) \approx 0.9953, \tag{3.53}$$

and

$$CK = \frac{3\pi^2 - 4\pi - 12}{(\pi - 2)^2} \approx 3.8692, \tag{3.54}$$

respectively.

3.5 Useful Graphs

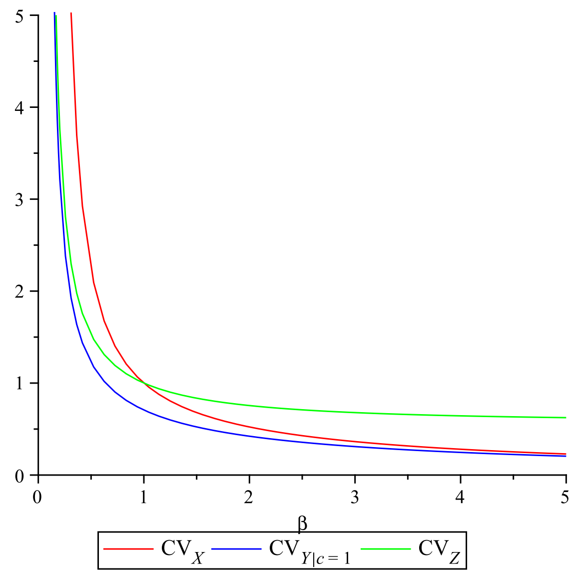

Figure 3.1: CV

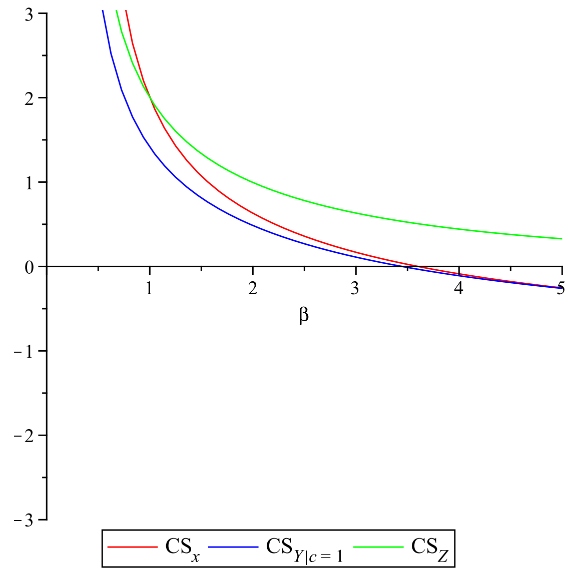

Figure 3.2: CS

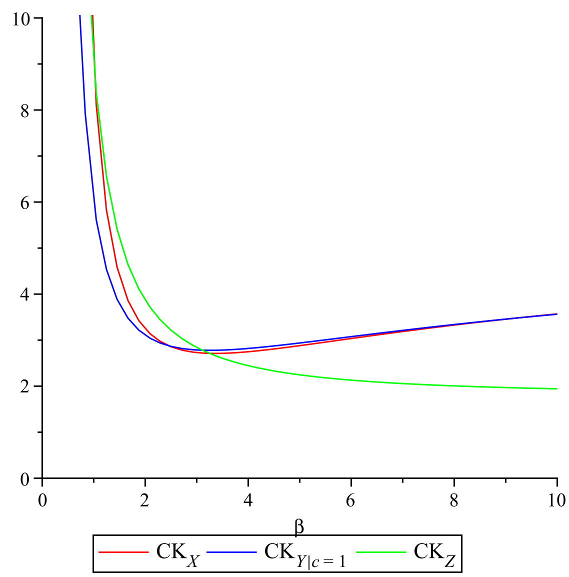

Figure 3.3: CK

CHAPTER 4

ENTROPY AND INFORMATION

4.1 Basic notions

In this chapter (differential) entropy and Fisher Information (FI) are computed for the Weibull (Rayleigh) distribution and its weighted and renewal versions. The entropy of a random variable measures its uncertainty. The FI measures the amount of information that random variable carries about the distribution's unknown parameter(s). The formal definitions for entropy and FI are given below:

Definition 4.1.1. *Let the distribution of random variable X have pdf $f(x)$. The entropy of X is given by*

$$h(X) = -E_X(\ln[f(X)]) = -\int_{-\infty}^{\infty} \ln[f(x)]f(x)dx \tag{4.1}$$

Definition 4.1.2. *The Fisher Information of a continuous distribution (satisfying standard regularity conditions) with parameter θ and pdf $f(x)$ is given by*

$$I_X(\theta) = -E_X\left(\frac{d^2}{d\theta^2}\ln[f(X)]\right). \tag{4.2}$$

If the distribution has parameter set θ^*, the Fisher information is a matrix (FIM) with entries

$$[I_X(\theta_i, \theta_j)]_{(i,j)} = -E_X\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln[f(X)]\right]. \tag{4.3}$$

4.1.1 Useful functions

Two special functions are common throughout the computations in this chapter. Recall that

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt.$$

The following result is well known

$$\Gamma^{(n)}(x) = \frac{d^n}{dx^n}\Gamma(x) = \frac{d^n}{dx^n}\int_0^\infty t^{x-1}e^{-t}dt$$
$$= \int_0^\infty \frac{d^n}{dx^n} t^{x-1}e^{-t}dt$$
$$= \int_0^\infty t^{x-1}\ln^n(t)e^{-t}dt. \quad (4.4)$$

Another common function is the digamma function given by

$$\Psi(x) = \frac{d}{dx}\ln[\Gamma(x)] = \frac{\Gamma'(x)}{\Gamma(x)}. \quad (4.5)$$

Dentote $\Gamma^{(n)}(\frac{c}{\beta}+1)$ and $\Psi^{(n)}(\frac{c}{\beta}+1)$ by Γ_c^n and Ψ_c^n, respectively ($\beta = 2$ in the Rayliegh sections. The Euler-Mascheroni constant, γ, is also part of several results for this chapter. It is given by

$$\gamma = -\Gamma'(1) = -\Psi(1) = -\int_0^\infty \ln(x)e^{-x}dx \approx 0.5772. \quad (4.6)$$

4.2 Weibull Distribution

Let $X \sim Weibull(\theta, \beta)$ with pdf $f(x)$. It follows that

$$\ln[f(x)] = \ln\left(\frac{\beta}{\theta^\beta}\right) + (\beta - 1)\ln(x) - \left(\frac{x}{\theta}\right)^\beta. \quad (4.7)$$

The entropy of the Weibull distribution is given by

$$\begin{aligned}
h(X) = -E_X(\ln[f(X)]) &= -\int_0^\infty \ln[f(x)]f(x)dx \\
&= \int_0^\infty \left(\frac{x}{\theta}\right)^\beta f(x)dx + (1-\beta)\int_0^\infty \ln(x)f(x)dx - \ln\left(\frac{\beta}{\theta^\beta}\right) \\
&= \int_0^\infty ue^{-u}du + (1-\beta)\int_0^\infty \ln(\theta u^{\frac{1}{\beta}})e^{-u}du - \ln\left(\frac{\beta}{\theta^\beta}\right) \\
&= 1 + (1-\beta)\ln(\theta)\int_0^\infty e^{-u}du + \left(\frac{1}{\beta}-1\right)\int_0^\infty \ln(u)e^{-u}du \\
&- \ln\left(\frac{\beta}{\theta^\beta}\right) \\
&= 1 + \ln(\theta) - \beta\ln(\theta) + \left(\frac{1}{\beta}-1\right)\Gamma'(1) - \ln(\beta) + \beta\ln(\theta) \\
&= 1 + \ln\left(\frac{\theta}{\beta}\right) + \left(1-\frac{1}{\beta}\right)\gamma. \qquad (4.8)
\end{aligned}$$

It is easy to see that $h(X)$ is a monotonically increasing function of θ when β is known or fixed. Since

$$\frac{\partial}{\partial\beta}h(X) = \frac{\gamma}{\beta^2} - \frac{1}{\beta}, \qquad (4.9)$$

then $\frac{\partial}{\partial\beta}h(X) = 0$ if and only if $\beta = \gamma$. It follows that $h(X)$ is a(n) increasing(decreasing) function of β when $\beta \leq \gamma(\beta \geq \gamma)$ and θ is known or fixed.

The following results will be used to help compute the FIM:

$$\frac{\partial}{\partial\theta}\ln[f(x)] = -\frac{\beta}{\theta} + \frac{\beta}{\theta^{\beta+1}}x^\beta = \frac{\beta}{\theta}\left[\left(\frac{x}{\theta}\right)^\beta - 1\right], \qquad (4.10)$$

$$\frac{\partial^2}{\partial\theta^2}\ln[f(x)] = \frac{\beta}{\theta^2} - \frac{(\beta+1)\beta}{\theta^{\beta+2}}x^\beta = \frac{\beta}{\theta^2}\left[1-(\beta+1)\left(\frac{x}{\theta}\right)^\beta\right], \qquad (4.11)$$

$$\frac{\partial}{\partial\beta}\ln[f(x)] = \frac{1}{\beta} - \ln(\theta) + \ln(x) - \left(\frac{x}{\theta}\right)^\beta \ln\left(\frac{x}{\theta}\right), \qquad (4.12)$$

$$\frac{\partial^2}{\partial\beta^2}\ln[f(x)] = -\frac{1}{\beta^2} - \left(\frac{x}{\theta}\right)^\beta \ln^2\left(\frac{x}{\theta}\right) = -\frac{1}{\beta^2}\left(1 + \left(\frac{x}{\theta}\right)^\beta \ln^2\left[\left(\frac{x}{\theta}\right)^\beta\right]\right), \qquad (4.13)$$

$$\frac{\partial^2}{\partial\beta\partial\theta}\ln[f(x)] = \frac{1}{\theta}\left[\left(\frac{x}{\theta}\right)^\beta \ln\left[\left(\frac{x}{\theta}\right)^\beta\right] + \left(\frac{x}{\theta}\right)^\beta - 1\right]. \qquad (4.14)$$

The entries of the FIM are given by

$$
\begin{aligned}
[I_X(\theta,\beta)]_{1,1} &= -E_X\left(\frac{\partial^2}{\partial\theta^2}\ln[f(X)]\right) \\
&= -\int_0^\infty \left(\frac{\beta}{\theta^2}\left[1-(\beta+1)\left(\frac{x}{\theta}\right)^\beta\right]\right) f(x)dx \\
&= \frac{\beta}{\theta^2}\left[(\beta+1)\int_0^\infty \left(\frac{x}{\theta}\right)^\beta f(x)dx - \int_0^\infty f(x)dx\right] \\
&= \frac{\beta}{\theta^2}[(\beta+1)\int_0^\infty ue^{-u}du - 1] \\
&= \frac{\beta}{\theta^2}[(\beta+1)-1] = \left(\frac{\beta}{\theta}\right)^2,
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
[I_X(\theta,\beta)]_{2,2} &= -E_X\left(\frac{\partial^2}{\partial\beta^2}\ln[f(X)]\right) \\
&= \int_0^\infty \frac{1}{\beta^2}\left(1+\left(\frac{x}{\theta}\right)^\beta \ln^2\left[\left(\frac{x}{\theta}\right)^\beta\right]\right) f(x)dx \\
&= \frac{1}{\beta^2}\left(\int_0^\infty f(x)dx + \int_0^\infty \left(\frac{x}{\theta}\right)^\beta \ln^2\left[\left(\frac{x}{\theta}\right)^\beta\right] f(x)dx\right) \\
&= \frac{1}{\beta^2}\left(1+\int_0^\infty u\ln^2(u)e^{-u}du\right) \\
&= \frac{1}{\beta^2}[1+\Gamma''(2)] \\
&= \frac{1}{\beta^2}\left(1+\frac{\pi^2}{6}-2\gamma+\gamma^2\right),
\end{aligned} \tag{4.16}
$$

and

$$
\begin{aligned}
[I_X(\theta,\beta)]_{1,2} = [I_X(\theta,\beta)]_{2,1} &= -E_X\left(\frac{\partial^2}{\partial\theta\partial\beta}\ln[f(X)]\right) \\
&= -\int_0^\infty \frac{1}{\theta}\left[\left(\frac{x}{\theta}\right)^\beta \ln\left[\left(\frac{x}{\theta}\right)^\beta\right] + \left(\frac{x}{\theta}\right)^\beta - 1\right] f(x)dx \\
&= -\frac{1}{\theta}\left(\int_0^\infty \left(\frac{x}{\theta}\right)^\beta \ln\left[\left(\frac{x}{\theta}\right)^\beta\right] f(x)dx + \int_0^\infty \left(\frac{x}{\theta}\right)^\beta f(x)dx - 1\right) \\
&= -\frac{1}{\theta}\left(\int_0^\infty u\ln(u)e^{-u}du + \int_0^\infty ue^{-u}du - 1\right) \\
&= -\frac{1}{\theta}(1+\Gamma'(2)-1) \\
&= -\frac{1}{\theta}\Gamma'(2) = \frac{1}{\theta}(\gamma-1).
\end{aligned} \tag{4.17}
$$

4.2.1 Rayleigh Distribution

Let $X \sim Rayleigh(\sigma)$ with pdf $f(x)$. To find the entropy of X, simply set $\theta = \sqrt{2}\sigma$ and $\beta = 2$ which yields

$$h(X) = -E_X(\ln[f(X)]) = 1 + \ln\left(\frac{\sqrt{2}}{2}\sigma\right) + \frac{\gamma}{2}. \tag{4.18}$$

Note that $h(X)$ is a monotonically increasing function of σ.

Every computation thus far for the Raleigh distribution and its weighted and renewal versions have been simple substitutions. If the same substitution is applied to obtain the FI then

$$I_X(\sigma) = \left(\frac{2}{\sqrt{2}\sigma}\right)^2 = \frac{2}{\sigma^2}.$$

However, this is not the FI for the Rayleigh distribution. Note that

$$\frac{d^2}{d\sigma^2}\ln[f(x)] = -\frac{3x^2}{\sigma^4} + \frac{2}{\sigma^2} = -\frac{2}{\sigma^2}\left(\frac{3x^2}{2\sigma^2} - 1\right). \tag{4.19}$$

Therefore, the FI is given by

$$\begin{aligned} I_X(\sigma) &= -E_X\left(\frac{d^2}{d\sigma^2}\ln[f(x)]\right) \\ &= \frac{2}{\sigma^2}\left(\int_0^\infty \frac{3x^2}{2\sigma^2} f(x)dx - \int_0^\infty f(x)dx\right) \\ &= \frac{2}{\sigma^2}\left(3\int_0^\infty ue^{-u}du - 1\right) \\ &= \frac{2}{\sigma^2}(3-1) = \frac{4}{\sigma^2}. \end{aligned} \tag{4.20}$$

This is because the information the random variable carries about its parameter(s) is not affected by the constant multiple.

4.3 Weighted Weibull Distribution

Let $Y \sim Weibull_W(\theta, \beta)$ with pdf $g(y|w(t) = t^c)$. It follows that

$$\ln[g(y)] = \ln\left(\frac{\beta}{\theta^{\beta+c}\Gamma_c}\right) + (c+\beta-1)\ln(y) - \left(\frac{y}{\theta}\right)^{\beta}. \tag{4.21}$$

The entropy for the weighted Weibull is

$$\begin{aligned}
-E_Y(\ln[g(Y)]) &= -\int_0^\infty \ln[g(y)]g(y)dy \\
&= \int_0^\infty \left(\frac{y}{\theta}\right)^{\beta} g(y)dy + (1-c-\beta)\int_0^\infty \ln(y)g(y)dy - ln\left(\frac{\beta}{\theta^{\beta+c}\Gamma_c}\right) \\
&= \frac{1}{\Gamma_c}\int_0^\infty u^{\frac{c}{\beta}+1}e^{-u}du + \frac{(1-c-\beta)}{\Gamma_c}\int_0^\infty \ln(\theta u^{\frac{1}{\beta}})u^{\frac{c}{\beta}}e^{-u}du + \ln\left(\frac{\theta^{\beta+c}\Gamma_c}{\beta}\right) \\
&= \frac{1}{\Gamma_c}\int_0^\infty u^{\frac{c}{\beta}+1}e^{-u}du + \frac{(1-c-\beta)}{\Gamma_c}\Bigg(\ln(\theta)\int_0^\infty u^{\frac{c}{\beta}}e^{-u}du \\
&+ \frac{1}{\beta}\int_0^\infty \ln(u)u^{\frac{c}{\beta}}e^{-u}du\Bigg) + \ln\left(\frac{\theta^{\beta+c}\Gamma_c}{\beta}\right) \\
&= \frac{\Gamma(\frac{c}{\beta}+2)}{\Gamma_c} + \frac{(1-c-\beta)}{\Gamma_c}\left(\ln(\theta)\Gamma_c + \frac{\Gamma'_c}{\beta}\right) + \ln\left(\frac{\theta^{\beta+c}\Gamma_c}{\beta}\right) \\
&= \frac{c}{\beta} + 1 + (1-c-\beta)\left(\ln(\theta) + \frac{\Psi_c}{\beta}\right) + \ln\left(\frac{\theta^{\beta+c}\Gamma_c}{\beta}\right) \\
&= \frac{c}{\beta} + 1 + (1-c-\beta)\frac{\Psi_c}{\beta} + \ln\left(\frac{\theta\Gamma_c}{\beta}\right).
\end{aligned} \tag{4.22}$$

Clearly, $h(Y)$ is a increasing function of θ when β is known or fixed. Note that

$$\frac{\partial}{\partial\beta}h(Y) = \frac{(c^2-1)\Psi'_c}{\beta^3} + \frac{c\Psi'_c - \Psi_c - c}{\beta^2} - \frac{1}{\beta}. \tag{4.23}$$

Figure 4.2 and (4.23) suggest there exists $c^* \leq 1$ such that for all $c < c^*$, $\frac{\partial}{\partial\beta}h(Y) = 0$, $h(Y)$ has solution β_c. Therefore, if $c < c^*$, $h(Y)$ is an increasing(decreasing) function of β when $\beta < \beta_c(\beta \geq \beta_c)$ and θ is known or fixed. If $c \geq 1$, then $h(Y)$ is a decreasing function of β when θ is known or fixed.

Note that

$$\ln[g(y)] = \ln[w(y)] + \ln[f(y)] - \ln(E_X[w(X)]).$$

It follows that

$$\begin{aligned}
[I_Y(\theta^*)]_{i,j} &= -E_Y\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln[g(Y)]\right] \\
&= -E_Y\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln[w(Y)] + \frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln[f(Y)] - \frac{\partial^2}{\partial\theta_i\partial\theta_j}(E_X[w(X)])\right) \\
&= -E_Y\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln[w(Y)]\right) - E_Y\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln[f(Y)]\right) \\
&+ \frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln(E_X[w(X)]). \qquad (4.24)
\end{aligned}$$

Given $w(t) = t^c$, (4.24) simplifies to

$$[I_Y(\theta^*)]_{i,j} = -E_Y\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln[f(Y)]\right) + \frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln[E_X(X^c)]. \qquad (4.25)$$

The following results will be used to help compute the FIM:

$$\frac{\partial}{\partial\theta}\ln[E_X(X^c)] = \frac{\partial}{\partial\theta}[c\ln(\theta) + \ln(\Gamma_c)] = \frac{c}{\theta}, \qquad (4.26)$$

$$\frac{\partial^2}{\partial\theta^2}\ln[E_X(X^c)] = -\frac{c}{\theta^2}, \qquad (4.27)$$

$$\frac{\partial}{\partial\beta}\ln[E_X(X^c)] = \frac{\partial}{\partial\beta}[c\ln(\theta) + \ln(\Gamma_c)] = -\frac{c}{\beta^2}\Psi_c, \qquad (4.28)$$

$$\frac{\partial^2}{\partial\beta^2}\ln[E_X(X^c)] = \frac{2c}{\beta^3}\Psi_c + \frac{c^2}{\beta^4}\Psi'_c, \qquad (4.29)$$

$$\frac{\partial^2}{\partial\theta\partial\beta}E_X(X^c) = \frac{\partial^2}{\partial\theta\partial\beta}[c\ln(\theta) + \ln(\Gamma_c)] = 0, \qquad (4.30)$$

$$\begin{aligned}
-E_Y\left(\frac{\partial^2}{\partial\theta^2}\ln[f(Y)]\right) &= -\int_0^\infty\left(\frac{\beta}{\theta^2}\left[1-(\beta+1)\left(\frac{y}{\theta}\right)^\beta\right]\right)g(y)dy \\
&= \frac{\beta}{\theta^2}\left[(\beta+1)\int_0^\infty\left(\frac{y}{\theta}\right)^\beta g(y)dy-\int_0^\infty g(y)dy\right] \\
&= \frac{\beta}{\theta^2}\left[\frac{\beta+1}{\Gamma_c}\int_0^\infty u^{\frac{c}{\beta}+1}e^{-u}dy-1\right] \\
&= \frac{\beta}{\theta^2}\left[\frac{\beta+1}{\Gamma_c}\Gamma\left(\frac{c}{\beta}+2\right)-1\right] \\
&= \frac{\beta}{\theta^2}\left[(\beta+1)\left(\frac{c}{\beta}+1\right)-1\right] \\
&= \frac{1}{\theta^2}[(\beta+1)(\beta+c)-\beta] \\
&= \frac{\beta^2+c\beta+c}{\theta^2}, \qquad (4.31)
\end{aligned}$$

$$\begin{aligned}
-E_Y\left(\frac{\partial^2}{\partial\beta^2}\ln[f(Y)]\right) &= \int_0^\infty\frac{1}{\beta^2}\left(1+\left(\frac{y}{\theta}\right)^\beta\ln^2\left[\left(\frac{y}{\theta}\right)^\beta\right]\right)g(y)dy \\
&= \frac{1}{\beta^2}\left(\int_0^\infty g(y)dy+\int_0^\infty\left(\frac{y}{\theta}\right)^\beta\ln^2\left[\left(\frac{y}{\theta}\right)^\beta\right]g(y)dy\right) \\
&= \frac{1}{\beta^2}\left(1+\frac{1}{\Gamma_c}\int_0^\infty u^{\frac{c}{\beta}+1}\ln^2(u)e^{-u}du\right) \\
&= \frac{1}{\beta^2}\left(1+\frac{\Gamma''(\frac{c}{\beta}+2)}{\Gamma_c}\right), \qquad (4.32)
\end{aligned}$$

$$\begin{aligned}
-E_Y\left(\frac{\partial^2}{\partial\theta\partial\beta}\ln[f(Y)]\right) &= -\frac{1}{\theta}\int_0^\infty\left[\left(\frac{y}{\theta}\right)^\beta\ln\left[\left(\frac{y}{\theta}\right)^\beta\right]+\left(\frac{y}{\theta}\right)^\beta-1\right]g(y)dy \\
&= -\frac{1}{\theta}\left(\int_0^\infty\left(\frac{y}{\theta}\right)^\beta\ln\left[\left(\frac{y}{\theta}\right)^\beta\right]g(y)dy+\int_0^\infty\left(\frac{y}{\theta}\right)^\beta g(y)dy-1\right) \\
&= -\frac{1}{\theta}\left(\frac{1}{\Gamma_c}\int_0^\infty u^{\frac{c}{\beta}+1}\ln(u)e^{-u}du+\frac{1}{\Gamma_c}\int_0^\infty u^{\frac{c}{\beta}+1}e^{-u}du-1\right) \\
&= -\frac{1}{\theta}\left(\frac{\Gamma'(\frac{c}{\beta}+2)}{\Gamma_c}+\frac{\Gamma(\frac{c}{\beta}+2)}{\Gamma_c}-1\right) \\
&= -\frac{1}{\theta}\left[\left(\frac{c}{\beta}+1\right)\Psi\left(\frac{c}{\beta}+2\right)+\frac{c}{\beta}\right]. \qquad (4.33)
\end{aligned}$$

Using (4.11), (4.13) and (4.25-33) the FIM entries are given by

$$[I_Y(\theta,\beta)]_{1,1}=\frac{\beta(\beta+c)}{\theta^2}, \qquad (4.34)$$

$$[I_Y(\theta,\beta)]_{2,2} = \frac{1}{\beta^2}\left(1 + \frac{\Gamma''(\frac{c}{\beta}+2)}{\Gamma_c} + \frac{2c}{\beta}\Psi_c + \frac{c^2}{\beta^2}\Psi'_c\right), \tag{4.35}$$

and

$$[I_Y(\theta,\beta)]_{1,2} = [I_Y(\theta,\beta)]_{2,1} = -\frac{1}{\theta}\left[\left(\frac{c}{\beta}+1\right)\Psi\left(\frac{c}{\beta}+2\right) + \frac{c}{\beta}\right]. \tag{4.36}$$

4.3.1 Weighted Rayleigh Distribution

Let $Y \sim Rayleigh_W(\sigma)$ with pdf $g(y)$. The entropy of Y, is a monotonically increasing function of σ given by

$$h(Y) = -E_Y(\ln[g(Y)]) = \frac{c+2-(c+1)\Psi_c}{2} + \ln\left(\frac{\sqrt{2}}{2}\sigma\Gamma_c\right). \tag{4.37}$$

If Y has a length biased distribution, then

$$-E_Y(\ln[g(Y|w(t)=t)]) = -\frac{1}{2} + \gamma + \ln(\frac{\sqrt{2\pi}}{4}\sigma). \tag{4.38}$$

Note that

$$\frac{d^2}{d\sigma^2}\ln[E_X(X^c)] = -\frac{c}{\sigma^2} \tag{4.39}$$

and

$$\begin{aligned} -E_Y\left(\frac{d^2}{d\sigma^2}\ln[f(Y)]\right) &= \frac{2}{\sigma^2}\left(\int_0^\infty \frac{3x^2}{2\sigma^2}g(y)dy - \int_0^\infty g(y)dy\right) \\ &= \frac{2}{\sigma^2}\left(\frac{3}{\Gamma_c}\int_0^\infty u^{\frac{c}{2}+1}e^{-u}du - 1\right) \\ &= \frac{2}{\sigma^2}\left(3\frac{\Gamma(\frac{c}{2}+2)}{\Gamma_c} - 1\right) \\ &= \frac{2}{\sigma^2}\left(3\left(\frac{c}{2}+1\right) - 1\right) = \frac{3c+4}{\sigma^2}. \end{aligned} \tag{4.40}$$

Using (4.24) and (4.38-39) we have

$$I_Y(\sigma) = \frac{2(c+2)}{\sigma^2}. \tag{4.41}$$

Clearly, $I_Y(\sigma)$ is a monotonically decreasing function of σ.

4.4 Weibull Renewal Distribution

Let $X \sim Weibull_R(\theta, \beta)$ with pdf $f_R(z)$. It follows that

$$\ln[f_R(z)] = -\left(\frac{z}{\theta}\right)^{\beta} - \ln(\theta\Gamma_1). \tag{4.42}$$

The entropy of the Weibull renewal distribution is

$$\begin{aligned} -E_Z(\ln[f_R(Z)]) &= -\int_0^{\infty} \ln[f_R(z)] f_R(z) dz \\ &= \int_0^{\infty} \left(\frac{z}{\theta}\right)^{\beta} f_R(z) dx + \ln(\theta\Gamma_1) \int_0^{\infty} f_R(z) dz \\ &= \frac{1}{\Gamma(\frac{1}{\beta})} \int_0^{\infty} u^{\frac{1}{\beta}} e^{-u} du + \ln(\theta\Gamma_1) \\ &= \frac{\Gamma_1}{\Gamma(\frac{1}{\beta})} + \ln(\theta\Gamma_1) \\ &= \frac{1}{\beta} + \ln(\theta\Gamma_1). \end{aligned} \tag{4.43}$$

Clearly, $h(Z)$ is a monotonically increasing function of θ when β is known. Note that

$$\frac{\partial}{\partial\beta} h(Z) = -\frac{1+\Psi_1}{\beta^2}. \tag{4.44}$$

It follows that $h(Z)$ is a decreasing function β when θ is known or fixed (see figure 4.3).

Using similar methods as (4.23), the entries for the FIM for a renewal distribution can be given by

$$[I(\theta^*)]_{(i,j)} = -E_Z\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j} \ln[f_Z(Z)]\right) = -E_Z\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j} \ln[S_X(Z)]\right) + \frac{\partial^2}{\partial\theta_i\partial\theta_j} \ln[E_X(X)]. \tag{4.45}$$

The following results will be used to help compute the FIM:

$$\frac{\partial}{\partial\theta} \ln[S_X(z)] = \frac{\beta}{\theta}\left(\frac{z}{\theta}\right)^{\beta}, \tag{4.46}$$

$$\frac{\partial^2}{\partial\theta^2}\ln[S_X(z)] = -\frac{(\beta+1)\beta}{\theta^2}\left(\frac{z}{\theta}\right)^{\beta}, \tag{4.47}$$

$$\begin{aligned} -E_Z\left(\frac{\partial^2}{\partial\theta^2}\ln[S_X(Z)]\right) &= \frac{(\beta+1)\beta}{\theta^2}\int_0^\infty \left(\frac{z}{\theta}\right)^{\beta} f_R(z)dz, \\ &= \frac{(\beta+1)\beta}{\theta^2\Gamma(\frac{1}{\beta})}\int_0^\infty u^{\frac{1}{\beta}}e^{-u}du \\ &= \frac{(\beta+1)\beta}{\theta^2\Gamma(\frac{1}{\beta})}\Gamma_1 = \frac{\beta+1}{\theta^2}, \end{aligned} \tag{4.48}$$

$$\frac{\partial}{\partial\beta}\ln[S_X(z)] = -\left(\frac{z}{\theta}\right)^{\beta}\ln\left(\frac{z}{\theta}\right), \tag{4.49}$$

$$\frac{\partial^2}{\partial\beta^2}\ln[S_X(z)] = -\left(\frac{z}{\theta}\right)^{\beta}\ln^2\left(\frac{z}{\theta}\right) = -\frac{1}{\beta^2}\left(\frac{z}{\theta}\right)^{\beta}\ln^2\left[\left(\frac{z}{\theta}\right)^{\beta}\right], \tag{4.50}$$

$$\begin{aligned} -E_Z\left(\frac{\partial^2}{\partial\beta^2}\ln[S_X(Z)]\right) &= \frac{1}{\beta^2}\int_0^\infty \left(\frac{z}{\theta}\right)^{\beta}\ln^2\left[\left(\frac{z}{\theta}\right)^{\beta}\right] f_R(z)dz \\ &= \frac{1}{\beta^2\Gamma(\frac{1}{\beta})}\int_0^\infty u^{\frac{1}{\beta}}\ln^2(u)e^{-u}du \\ &= \frac{\Gamma_1''}{\beta^2\Gamma(\frac{1}{\beta})}, \end{aligned} \tag{4.51}$$

$$\frac{\partial^2}{\partial\theta\partial\beta}\ln[S_X(z)] = \frac{1}{\theta}\left[\beta\left(\frac{z}{\theta}\right)^{\beta}\ln\left(\frac{z}{\theta}\right) + \left(\frac{z}{\theta}\right)^{\beta}\right] = \frac{1}{\theta}\left[\left(\frac{z}{\theta}\right)^{\beta}\ln\left[\left(\frac{z}{\theta}\right)^{\beta}\right] + \left(\frac{z}{\theta}\right)^{\beta}\right], \tag{4.52}$$

$$\begin{aligned} -E_Z\left(\frac{\partial^2}{\partial\theta\partial\beta}\ln[S_X(Z)]\right) &= -\frac{1}{\theta}\int_0^\infty \left[\left(\frac{z}{\theta}\right)^{\beta}\ln\left[\left(\frac{z}{\theta}\right)^{\beta}\right] + \left(\frac{z}{\theta}\right)^{\beta}\right] f_R(z)dz \\ &= -\frac{1}{\theta}\left(\int_0^\infty \left(\frac{z}{\theta}\right)^{\beta} ln\left[\left(\frac{z}{\theta}\right)^{\beta}\right] f_R(z)dz + \int_0^\infty \left(\frac{z}{\theta}\right)^{\beta} f_R(z)dz\right) \\ &= -\frac{1}{\theta}\left(\frac{1}{\Gamma(\frac{1}{\beta})}\int_0^\infty u^{\frac{1}{\beta}} ln(u)e^{-u}du + \frac{1}{\Gamma(\frac{1}{\beta})}\int_0^\infty u^{\frac{1}{\beta}}e^{-u}du\right) \\ &= -\frac{1}{\theta}\left(\frac{\Gamma_1'}{\Gamma(\frac{1}{\beta})} + \frac{\Gamma_1}{\Gamma(\frac{1}{\beta})}\right) = -\frac{\Psi_1+1}{\theta\beta}. \end{aligned} \tag{4.53}$$

Using (4.27), (4.29) and (4.45-53) the entries of the FIM are given by

$$[I_Z(\theta,\beta)]_{1,1} = \frac{\beta}{\theta^2}, \tag{4.54}$$

$$[I_Z(\theta,\beta)]_{2,2} = \frac{1}{\beta^2}\left(\frac{\Gamma_1''}{\Gamma(\frac{1}{\beta})} + \frac{2}{\beta}\Psi_1 + \frac{1}{\beta^2}\Psi_1'\right), \tag{4.55}$$

and

$$[I_Z(\theta,\beta)]_{1,2} = [I_Z(\theta,\beta)]_{1,2} = -\frac{\Psi_1 + 1}{\theta\beta}. \tag{4.56}$$

4.4.1 Rayleigh Renewal Distribution

Let $Z \sim Rayleigh_R(\sigma)$ with pdf $f_R(z)$. The entropy of Z is a monotonically increasing function of σ given by

$$h(Z) = -E_Y(\ln[f_R(Z)]) = \frac{1}{2} + \ln\left(\frac{\sqrt{2\pi}}{2}\sigma\right). \tag{4.57}$$

Note that

$$\frac{d^2}{\partial\sigma^2}\ln[S_X(z)] = -\frac{3x^2}{\sigma^4} = -\frac{2}{\sigma^2}\left(\frac{3x^2}{\sigma^2}\right) \tag{4.58}$$

and

$$\begin{aligned} -E_Z\left(\frac{d^2}{d\sigma^2}\ln[S_X(Z)]\right) &= \frac{2}{\sigma^2}\int_0^\infty \frac{3x^2}{2\sigma^2} f_R(z)dz \\ &= \frac{6}{\sigma^2\sqrt{\pi}}\int_0^\infty u^{\frac{1}{2}}e^{-u}du \\ &= \frac{6}{\sigma^2\sqrt{\pi}}\Gamma_1 = \frac{6}{\sigma^2\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = \frac{3}{\sigma^2}. \end{aligned} \tag{4.59}$$

Using (4.39), (4.45) and (4.58-59) the FI is given by

$$I_Z(\sigma) = \frac{2}{\sigma^2}. \tag{4.60}$$

4.5 Useful Graphs

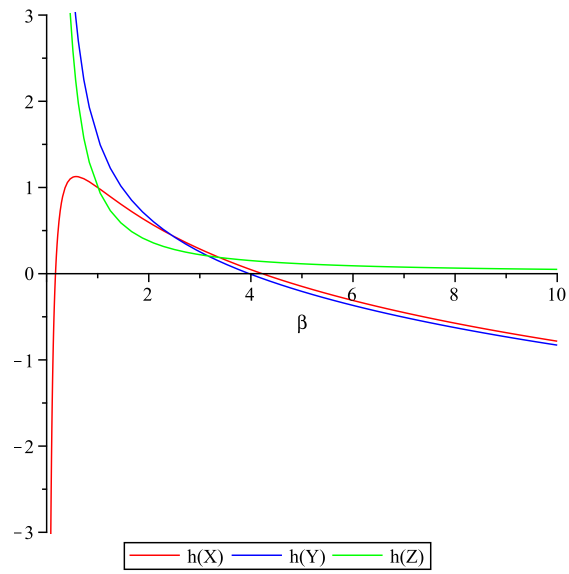

Figure 4.1: Entropy with $\theta = 1$ and $c = 1$

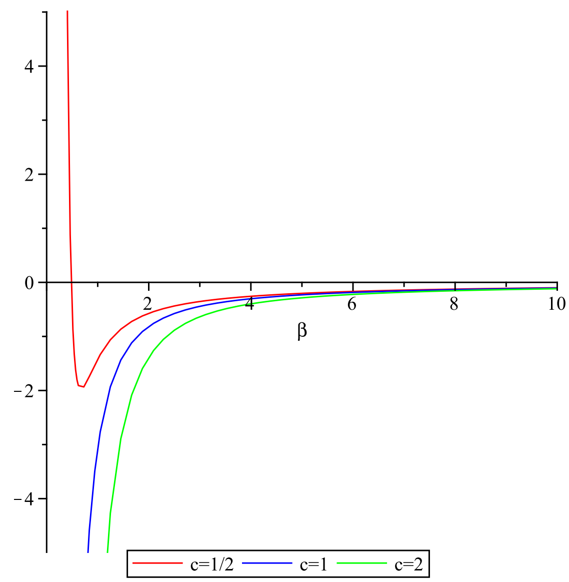

Figure 4.2: $\frac{\partial}{\partial \beta} h(Y)$

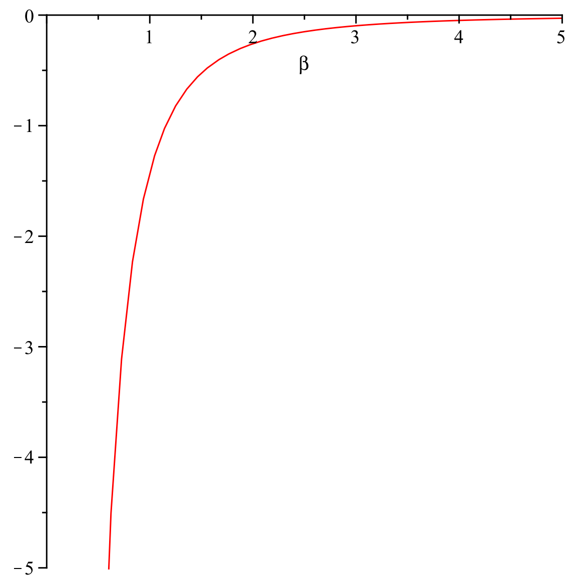

Figure 4.3: $\frac{\partial}{\partial \beta} h(Z)$

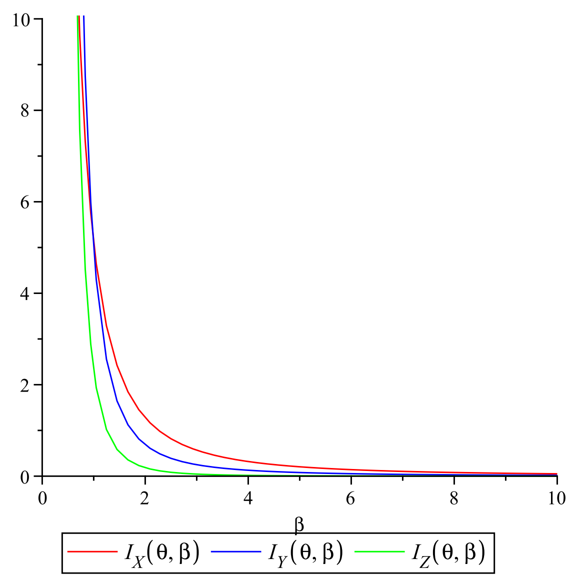

Figure 4.4: (2,2) entries of the FIM with $\theta = 1$ and $c = 1$

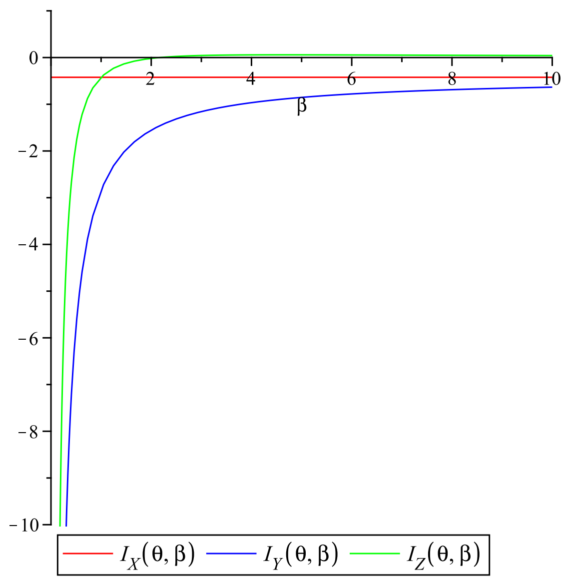

Figure 4.5: (1,2) and (2,1) entries of the FIM with $\theta = 1$ and $c = 1$

CHAPTER 5

TESTS CONCERNING WEIGHTED AND PARENT DISTRIBUTIONS

5.1 Basic Notions

In this chapter, the maximum likelihood estimates of the Weibull (Rayleigh) and its weighted and renewal versions are presented. In section 5.5 the likelihood ratio test is defined and results are presented on tests for the parent distribution versus the weighted distribution.

Definition 5.1.1. *Let random variables $X_1, X_2, ..., X_n$ be iid with pdf $f(x_k; \theta^*)$. The likelihood function of θ^* is given by*

$$L_X(\theta^*|x_1, x_2, ..., x_n) = \Pi_{i=1}^{n} f(x_i|\theta^*) \qquad (5.1)$$

Denote $L_X(\theta^*|x_1, x_2, ..., x_n)$ by $L_X(\theta^*)$. In general, it is much more convenient to work with the log-likelihood function, $\ln[L_X(\theta^*)]$. Maximizing the log-likelihood with respect to the parameters gives us the maximum likelihood estimates.

Definition 5.1.2. *Given likelihood function $L_X(\theta^*)$, the solution for θ_k in the system of equations*

$$\frac{\partial}{\partial \theta_i} \ln[L_X(\theta^*)] = 0, \quad 1 \le i \le n, \qquad (5.2)$$

yields the maximum likelihood estimate (MLE) of θ_k, denoted by $\hat{\theta}_k$.

In many cases, the MLE does not have a closed-form solution and (5.2) must be solved using numerical methods.

5.2 Weibull (Rayleigh) Distribution

Let $X \sim Weibull(\theta, \beta)$ with pdf $f(x)$. The likelihood function for X given n independent observations is given by

$$L_X(\theta, \beta) = \frac{\beta^n}{\theta^{n\beta}} e^{-\sum_{i=1}^{n}(\frac{x_i}{\theta})^\beta} \Pi_{i=1}^{n} x_i^{\beta-1}. \tag{5.3}$$

The log-likelihood function is given by

$$\ln[L_X(\theta, \beta)] = n\ln(\beta) + (\beta - 1)\sum_{i=1}^{n}\ln(x_i) - n\beta\ln(\theta) - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^\beta. \tag{5.4}$$

Maximizing (5.4) with respect to β and θ yields

$$\frac{\partial}{\partial\beta}\ln[L_X(\theta, \beta)] = \frac{n}{\beta} + \sum_{i=1}^{n}\ln(x_i) - n\ln(\theta) - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^\beta \ln\left(\frac{x_i}{\theta}\right) = 0 \tag{5.5}$$

and

$$\frac{\partial}{\partial\theta}\ln[L_X(\theta, \beta)] = \frac{\beta}{\theta^{\beta+1}}\sum_{i=1}^{n} x_i^\beta - \frac{n\beta}{\theta} = 0. \tag{5.6}$$

Solving for θ in (5.6) given $\beta = \hat{\beta}$ yields the MLE for θ given by

$$\hat{\theta} = \left(\frac{\sum_{i=1}^{n} x_i^{\hat{\beta}}}{n}\right)^{\frac{1}{\hat{\beta}}}. \tag{5.7}$$

Substituting $\hat{\theta}$ for θ in (5.5) and solving (numerically) yields $\hat{\beta}$. For the Rayleigh distribution, using usual substitution, the MLE for σ is

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}}. \tag{5.8}$$

5.3 Weighted Weibull (Rayleigh) Distribution

Let $Y \sim Weibull_W(\theta, \beta; w(t) = t^c)$ with pdf $g(y)$. The likelihood function for Y given n independent observations is given by

$$L_X(\theta, \beta) = \frac{\beta^n}{\theta^{n(\beta+c)}\Gamma_c^n} e^{-\sum_{i=1}^{n}(\frac{y_i}{\theta})^\beta} \Pi_{i=1}^{n} y_i^{c+\beta-1}. \tag{5.9}$$

The log-likelihood function is given by

$$\ln[L_Y(\theta,\beta)] = n\ln(\beta) + (c+\beta-1)\sum_{i=1}^{n}\ln(y_i) - n(\beta+c)\ln(\theta) - \sum_{i=1}^{n}\left(\frac{y_i}{\theta}\right)^{\beta} - n\ln(\Gamma_c). \tag{5.10}$$

Maximizing (5.10) with respect to β and θ yields

$$\frac{\partial}{\partial\beta}\ln[L_Y(\theta,\beta)] = \frac{n}{\beta} + \sum_{i=1}^{n}\ln(y_i) - n\ln(\theta) - \sum_{i=1}^{n}\left(\frac{y_i}{\theta}\right)^{\beta}\ln\left(\frac{y_i}{\theta}\right) + \frac{nc}{\beta^2}\Psi_c = 0 \tag{5.11}$$

and

$$\frac{\partial}{\partial\theta}\ln[L_Y(\theta,\beta)] = \frac{\beta}{\theta^{\beta+1}}\sum_{i=1}^{n}y_i^{\beta} - \frac{n(\beta+c)}{\theta} = 0. \tag{5.12}$$

Solving for θ in (5.12) given $\beta = \hat{\beta}$ yields the MLE for θ given by

$$\hat{\theta} = \left(\frac{\hat{\beta}\sum_{i=1}^{n}y_i^{\hat{\beta}}}{n(\hat{\beta}+c)}\right)^{\frac{1}{\hat{\beta}}}. \tag{5.13}$$

Substituting $\hat{\theta}$ for θ in (5.11) and solving (numerically) yields $\hat{\beta}$. For the Rayleigh distribution, using usual substitution, the MLE for σ is

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}y_i^2}{n(c+2)}}. \tag{5.14}$$

5.4 Weibull(Rayleigh) Renewal Distribution

Let $Z \sim Weibull_R(\theta,\beta)$ with pdf $f_R(x)$. The likelihood function for Z given n independent observations is given by

$$L_Z(\theta,\beta) = \frac{e^{-\sum_{i=1}^{n}(\frac{z_i}{\theta})^{\beta}}}{\theta^n\Gamma_1^n}. \tag{5.15}$$

The log-likelihood function is given by

$$\ln[L_Z(\theta,\beta)] = -\sum_{i=1}^{n}\left(\frac{z_i}{\theta}\right)^{\beta} - n\ln(\theta) - n\ln(\Gamma_1). \tag{5.16}$$

Maximizing (5.16) with respect to β and θ yields

$$\frac{\partial}{\partial\beta}\ln[L_Z(\theta,\beta)] = \frac{n}{\beta^2}\Psi_1 - \sum_{i=1}^{n}\left(\frac{z_i}{\theta}\right)^{\beta}\ln\left(\frac{z_i}{\theta}\right) = 0 \tag{5.17}$$

and

$$\frac{\partial}{\partial\theta}\ln[L_Z(\theta,\beta)] = \frac{\beta}{\theta^{\beta+1}}\sum_{i=1}^{n} z_i^{\beta} - \frac{n}{\theta} = 0. \tag{5.18}$$

Solving for θ in (5.18) given $\beta = \hat{\beta}$ yields the MLE for θ given by

$$\hat{\theta} = \left(\frac{\hat{\beta}\sum_{i=1}^{n} z_i^{\hat{\beta}}}{n}\right)^{\frac{1}{\hat{\beta}}}. \tag{5.19}$$

Substituting $\hat{\theta}$ for θ in (5.17) and solving (numerically) yields $\hat{\beta}$. For the Rayleigh renewal distribution, using usual substitution, the MLE for σ is given by

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} z_i^2}{n}}. \tag{5.20}$$

5.5 Likelihood Ratio Test

The likelihood-ratio (LHR) test compares the ratios of two likelihood functions that have a different parameter. This test can also be used to see if a sample came from a particular distribution or its weighted distribution. Let θ^* be in parameter space Ω and let Ω_0 be a subset of Ω. The compliment of Ω_0 is Ω_0^c.

Definition 5.5.1. *Let the distribution of random variable X have parameter set θ^*. Given the hypotheses*

$H_0 : \theta^* \in \Omega_0$

$H_A : \theta^* \in \Omega_0^c$

and test statistic

$$\Lambda(x) = \frac{sup\{L_X(\theta^*)|\theta^* \in \Omega_0\}}{sup\{L_X(\theta^*)|\theta^* \in \Omega\}}, \tag{5.21}$$

the likelihood-ratio test is any test with rejection region $\{x|0 \leq \Lambda \leq C \leq 1\}$*, where* C *is constant.*

Note that the test statistic in (5.21) may be written as

$$\Lambda(x) = \frac{sup\{L_X(\theta^*)|\theta^* \in \Omega_0\}}{sup\{L_X(\theta^*)|\theta^* \in \Omega_0^c\}} \tag{5.22}$$

without changing the rejection region.

5.5.1 Testing for Weightedness

To LHR test can be used to see if a sample came from a particular distribution or one of it's weighted versions by observing the LHR using the likelihood function of each. Although this does not seem fit the definition of the LHR test it is the case. The proceeding argument explains why this is true.

Let the distributions of random variables X and Y have pdfs $f(x;\theta^*)$ and $g(y;\theta^*)$, respectively, where Y has the weighted distribution of X. Let distribution of random variable V have pdf

$$g^*(v;\theta^*;k) = \frac{[w(v)]^k f(v)}{(E_X[w(X)])^k}, \quad k \in \{0,1\}. \tag{5.23}$$

Since $g^*(v;k|k=0) = f(v)$ and $g^*(v;k|k=1) = g(y)$, $g^*(v;k)$ is indeed a pdf and k is a parameter of $g^*(v;k)$. A LHR test for k can be done with hypotheses

$H_0 : k = 0$

$H_A : k = 1$

and test statistic (given independent observed values $v_1, v_2, ..., v_n$ of V)

$$\begin{aligned}\Lambda(v) = \frac{L_V(\theta^*;0)}{L_V(\theta^*;1)} &= \frac{\Pi_{i=1}^n g^*(v_i;k|k=0)}{\Pi_{i=1}^n g^*(v_i;k|k=1)} \\ &= \frac{\Pi_{i=1}^n f(v_i)}{\Pi_{i=1}^n g_(v_i)} \\ &= \frac{(E_X[w(X)])^n}{\Pi_{i=1}^n w(v_i)}. \qquad (5.24)\end{aligned}$$

Note that the hypothesis mentioned above is equivalent to

$H_0 : V$ has the same distribution as X

$H_A : V$ has the same distribution as Y.

Given $w(t) = t^c$, the test statistic is

$$\Lambda_W(v) = \frac{(E_X[w(X)])^n}{\Pi_{i=1}^n w(v_i)} = \frac{[E_X(X^c)]^n}{\Pi_{i=1}^n v_i^c}. \qquad (5.25)$$

Recall that a renewal distribution is a weighted distribution with $w(t) = \frac{1}{\lambda_X(t)}$. Therefore, the test statistic is

$$\Lambda_R(v) = \frac{(E_X[w(X)])^n}{\Pi_{i=1}^n w(v_i)} = [E_X(X)]^n \Pi_{i=1}^n \lambda_X(v_i). \qquad (5.26)$$

5.5.2 Tests concerning Weibull (Rayleigh) Distributions

Let $x_1, x_2, ..., x_n$ be n independent observed value of random variable X which has either a Weibull (Rayleigh) distribution or weighted Weibull (Rayleigh) distribution. The test for weight ($w(t) = t^c$) has hypotheses

$H_0 : X \sim Weibull(\theta, \beta)$

$H_A : X \sim Weibull_W(\theta, \beta; X^c)$

and test statistic

$$\Lambda(x) = \frac{(\theta^c \Gamma_c)^n}{\Pi_{i=1}^n x_i^c}. \qquad (5.27)$$

The test statistic for testing for length-biasedness and size-biasedness in a Rayleigh distribution are given by

$$\Lambda_L(x) = \frac{(\sqrt{2\pi}\sigma)^n}{2^n \Pi_{i=1}^n x_i} \tag{5.28}$$

and

$$\Lambda_S(x) = \frac{(2\sigma^2)^n}{\Pi_{i=1}^n x_i}, \tag{5.29}$$

respectively.

Let $z_1, z_2, ..., z_n$ be n independent observed value of random variable Z which has either a Weibull (Rayleigh) distribution or Weibull(Rayleigh) renewal distribution. The test has hypotheses

$H_0 : Z \sim Weibull(\theta, \beta)$

$H_A : Z \sim Weibull_R(\theta, \beta)$

and test statistic

$$\Lambda(z) = (\theta\Gamma_1)^n \Pi_{i=1}^n \frac{\beta}{\theta^\beta} z_i^{\beta-1} = \left(\frac{\beta\Gamma_1}{\theta^{\beta-1}}\right)^n \Pi_{i=1}^n z_i^{\beta-1}. \tag{5.30}$$

The test statistic for testing for the Rayleigh renewal distribution is given by

$$\Lambda_R(z) = \left(\frac{1}{\sigma^2}\sqrt{\frac{\pi}{2}}\right)^n \Pi_{i=1}^n z_i. \tag{5.31}$$

5.6 Conclusion

In this thesis several theoretical properties of two cases of the weighted Weibull distribution were presented. Some areas for further research include:

1. Theoretical properties of combined distributions in the form:

$$k(x) = pf_X(x) + (1-p)g_Y(x),$$

where f_X is the parent distribution, g_Y is the weighted distribution and $0 \leq p \leq 1$.

2. Estimation of parameters based on combined samples. Tests of hypothesis concerning the models parameters based on combined samples.

3. Extensions to multivariate weighted distributions.

4. Stochastic inequalities and dependence results.

Results in these areas could be very useful in real life applications.

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