Spring 2013

Statistical Properties of the Mc-Dagum and Related Distributions

Sasith Rajasooriya

STATISTICAL PROPERTIES OF THE MC-DAGUM AND RELATED DISTRIBUTIONS

by

SASITH RAJASOORIYA

(Under the Direction of Broderick O. Oluyede)

ABSTRACT

In this thesis, we present a new class of distributions called Mc-Dagum distribution. This class of distributions contains several distributions such as beta-Dagum, beta-Burr III, beta-Fisk and Dagum distributions as special cases. The hazard function, reverse hazard function, moments and mean residual life function are obtained. Inequality measures, entropy and Fisher information are presented. Maximum likelihood estimates of the model parameters are given.

KEY WORDS: Mc-Dagum Distribution, Beta-Dagum Distribution, Inequality Measures, Information Matrix

2009 *Mathematics Subject Classification*: 62E15,60E05

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B.S in Business Administration

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial Fulfillment of the Requirement for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

2013

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Electronic Version Approved:

May 2013

DEDICATION

This thesis is dedicated to my beloved parents and specially my beloved wife "Pubudu" for her exceptional support and encouragement.

ACKNOWLEDGMENTS

My deepest gratitude extends to Dr. Broderick Oluyede for his guidance and encouragement given to me throughout this thesis and also my entire studies at GSU. It is undoubtedly his exceptional support and direction what made this work a success. It was truly a privilege to have him as my advisor. My special thank goes to my committee members, Dr. Charles Champ and Dr. Hani Samawi for their valuable contribution towards the success of this work. It is my pleasure to acknowledge all the members of the faculty and staff in the Department of Mathematical Sciences at Georgia Southern University, specially Dr. Sharon Taylor, Department Chair, Martha Abell, Dean and former Department Chair and Dr. Yan Wu, former Graduate Program Director, for all their utmost commitments towards the success of the students and the department.

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CHAPTER 1

INTRODUCTION

1.1 Review for the Dagum and Mc-Donald Generalized Distributions

1.1.1 Dagum Distribution

Dagum distribution was proposed by Camilo Dagum in 1970's (Dagum 1977-1980). His proposals enable the development of statistical distributions used to fit empirical income and wealth data, that could accommodate both heavy tails in empirical income and wealth distributions, and also permit interior mode. Dagum distribution has both Type-I and Type-II specification, where Type-I is the three parameter specifications and Type-II deal with four parameter specification.

Dagum in 1977 motivated his model from empirical observation that the income elasticity $\eta(F, x)$ of the cumulative distribution function (cdf) F of income into a decreasing and bounded function of F.

The cdf and pdf of Dagum (Type-I) distribution are given by

$$G(x; \lambda, \delta, \beta) = \left(1 + \lambda x^{-\delta}\right)^{-\beta}, \tag{1.1}$$

and

$$g(x; \lambda, \delta, \beta) = \beta\lambda\delta x^{-\delta-1}\left(1 + \lambda x^{-\delta}\right)^{-\beta-1}, \quad \text{for } \lambda,\ \delta,\ \beta > 0, \tag{1.2}$$

respectively, where λ is a scale parameter, and δ and β are shape parameters.

Dagum (1980) refers to his model as the generalized logistic-Burr distribution. Actually when $\beta = 1$, Dagum distribution was also referred to as the log-logistic distribution. Also, generalized (log-) logistic distributions arise naturally in Burr's

(1942) system of distributions. The most popular Burr distributions are Burr-XLL distribution, often called Burr distribution with cdf,

$$F\left(x;\delta,\beta\right)=1-\left(1+x^{-\delta}\right)^{-\beta},\quad \text{for } x>0,\ \delta,\ \beta>0, \tag{1.3}$$

and more importantly the Burr-III distribution with cdf

$$F\left(x;\delta,\beta\right)=\left(1+x^{-\delta}\right)^{-\beta},\quad \text{for } x>0 \text{ and } \delta,\ \beta>0. \tag{1.4}$$

Thus, these distributions are more popular in economics, after the introduction of an additional parameter (λ as we can see above in the Dagum cdf and pdf). It is clear that the Dagum distribution is a Burr III distribution with an additional scale parameter (λ).

The k^{th} raw or noncentral moments of Dagum distribution are given by

$$\begin{aligned} E\left(X^{k}\right) &= \int_{0}^{\infty} x^{k}\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}dx \\ &= \beta\lambda^{\frac{k}{\delta}}B\left(\beta+\frac{k}{\delta},1-\frac{k}{\delta}\right), \end{aligned} \tag{1.5}$$

for $\delta>k,\lambda,\delta,\beta>0$, where B(.,.) is the beta function, (by setting $t=(1+\lambda x^{-\delta})^{-1}$).

The mean, mode and variance of the Dagum distribution are given by

$$\mu_{X}=\frac{\frac{1}{\lambda}\,\Gamma(\beta+\frac{1}{\delta})\,\Gamma\left(1-\frac{1}{\delta}\right)}{\Gamma\left(\beta\right)}, \tag{1.6}$$

$$Mode=\frac{1}{\lambda}\left(\frac{\delta\beta-1}{\delta+1}\right)^{\frac{1}{\delta}}, \tag{1.7}$$

and

$$\sigma_{X}^{2}=\frac{\frac{2}{\lambda^{\delta}}}{\Gamma^{2}\left(\beta\right)}\left[\Gamma\left(\beta\right)\Gamma\left(\beta+\frac{2}{\delta}\right)\Gamma\left(1-\frac{2}{\delta}\right)-\Gamma^{2}\left(\beta+\frac{1}{\delta}\right)\Gamma^{2}\left(1-\frac{1}{\delta}\right)\right], \tag{1.8}$$

respectively. The q^{th} percentile of the Dagum distribution is

$$x(q)=\lambda^{\frac{1}{\delta}}\left(q^{\frac{-1}{\beta}}-1\right)^{\frac{-1}{\delta}}. \tag{1.9}$$

1.1.2 Mc-Donald Generalized Distribution

Consider an arbitrary parent cdf $G(x)$. The probability density function (pdf) $f(x)$ of the new class of distributions called the Mc-Donald generalized distribution is given by

$$f(x;a,b,c) = \frac{cg(x)}{B(a,b)} G^{ac-1}(x)\left(1-G^{c}(x)\right)^{b-1}, \quad \text{for } a>0,\ b>0, \text{ and } c>0. \quad (1.10)$$

See Corderio et al.(2012) for additional details.

Note that $g(x)$ is the pdf of parent distribution , $g(x) = dG(x)/dx$, and a,b and c are additional shape parameters. Introduction of this additional shape parameters is specially to introduce skewness. Also, this allows us to vary tail weight. It is important to note that for c=1 we obtain a sub-model of this generalization which is a beta-generalization and for a=1, we have the Kumaraswamy (Kw),[Kumaraswamy (1980)] generalized distributions. For random variable X with density function given above in (1.10), we write $X\sim$Mc-G(a,b,c).

The cdf for this generalization is given by,

$$F(x;a,b,c) = I_{G(x)^c}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)^c} \omega^{a-1}(1-\omega)^{b-1} d\omega, \quad (1.11)$$

where $I_{G^c(x)}(a,b) = B(a,b)^{-1} \int_0^{G(x)^c} \omega^{a-1}(1-\omega)^{b-1} d\omega$ denotes incomplete beta function ratio (Gradshteyn and Ryzhik, 2000). The same equation can be expressed as follows:

$$F(x;a,b,c) = \frac{G(x)^{ac}}{aB(a,b)} \left[{}_2F_1\left(a, 1-b; a+1; G(x)^c\right)\right], \quad (1.12)$$

where

$${}_2F_1(a,b;c;x) = B(b,c-b)^{-1} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt, \quad (1.13)$$

is the well known hypergeometric function (Gradshteyn and Ryzhik, 2000), and

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{1.14}$$

One important benefit of this class is its ability to fit skewed data that cannot properly be fitted by many other existing distributions. Mc-G family of densities allows for higher levels of flexibility of its tails and has a lot of applications in various fields including economics, finance, reliability and medicine.

1.1.3 Hazard and Reverse Hazard Functions

In this section, some basic utility notions are presented. Suppose the distribution of a continuous random variable X has the parameter set $\theta^* = \{\theta_1, \theta_2, \cdots, \theta_n\}$. Let the probability density function (pdf) of X be given by $f(x;\theta^*)$. The cumulative distribution function of X, is defined to be

$$F(x;\theta^*) = \int_{-\infty}^{x} f(t;\theta^*)\, dt. \tag{1.15}$$

The hazard function of X can be interpreted as the instantaneous failure rate or the conditional probability density of failure at time x, given that the unit has survived until time x. The hazard function $h(x;\theta^*)$ is defined to be

$$h(x;\theta^*) = \lim_{\Delta x \to 0} \frac{P\left(x \leq X \leq x + \Delta x\right)}{\Delta x[1 - F(x;\theta^*)]} = \frac{-\bar{F}'(x;\theta^*)}{\bar{F}(x;\theta^*)} = \frac{f(x;\theta^*)}{1 - F(x;\theta^*)}, \tag{1.16}$$

where $\bar{F}(x;\theta^*)$ is the survival or reliability function.

Reverse Hazard function can be interpreted as an approximate probability of a failure in $[x, x + dx]$, given that the failure had occurred in $[0, x]$. The reverse hazard function $\tau(x;\theta^*)$ is defined to be

$$\tau(x;\theta^*) = \frac{f(x;\theta^*)}{F(x;\theta^*)}. \tag{1.17}$$

Some useful functions that are employed in subsequent sections are given below. The gamma function is given by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{1.18}$$

The digamma function is defined by

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.19}$$

where

$$\Gamma'(x) = \int_0^\infty t^{x-1} (\log t) e^{-t}\, dt$$

is the first derivative of the gamma function. The second derivative of the gamma function is

$$\Gamma''(x) = \int_0^\infty t^{x-1} (\log t)^2 e^{-t}\, dt.$$

The lower incomplete and upper incomplete gamma functions are

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t}\, dt \quad \text{and} \quad \Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t}\, dt \tag{1.20}$$

respectively.

The hazard function (hf) and reverse hazard functions (rhf) of the Mc-G distribution are given by

$$h_F(x) = \frac{cg(x)\, G^{ac-1}(x)\, \{1 - G^c(x)\}^{b-1}}{B(a,b)\left\{1 - I_{G(x)^c}(a,b)\right\}}, \tag{1.21}$$

and

$$\tau_F(x) = \frac{cg(x)\, G^{ac-1}(x)\, \{1 - G^c(x)\}^{b-1}}{B(a,b)\, I_{G^c(x)}(a,b)}, \tag{1.22}$$

respectively.

1.1.4 Outline of Results

The outline of this thesis as follows: In chapter 2, the Mc-Dagum distribution and related family of distributions are introduced. The expansion for the density, hazard and reverse hazard functions, and other properties are presented. Chapter 3 presents the moments, and inequality measures. Chapter 4 contains entropy measures of the Mc-Dagum distribution. Chapter 5 contains inference for the model parameters as well applications of the results presented in earlier chapters.

CHAPTER 2

INTRODUCING MC-DAGUM DISTRIBUTION

In this chapter, a new class of distribution, called Mc-Dagum distribution is introduced. Considering the properties and some useful features of both Dagum and Mc-Donald distributions, a broad range of generalization is possible by combining these distributions. The new class of distributions possess capabilities widely applicable in several areas as we will show in the next few chapters.

2.1 Mc-Dagum Distribution

In chapter 1, the cdf and pdf of Dagum distribution were given as

$$G\left(x;\lambda,\delta,\beta\right)=\left(1+\lambda x^{-\delta}\right)^{-\beta}, \tag{2.1}$$

and

$$g\left(x;\lambda,\delta,\beta\right)=\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1},\lambda,\delta,\beta>0, \tag{2.2}$$

respectively. The pdf for Mc-Donald distribution is given by

$$f(x;a,b,c)=\tfrac{c}{B(a,b)}g(x)G^{ac-1}(x)\left(1-G^{c}(x)\right)^{b-1},a>0,b>0,c>0, \tag{2.3}$$

and the cdf is

$$\begin{aligned} F(x) &= I_{G(x)^{c}}(a,b) \\ &= \tfrac{1}{B(a,b)}\int_{0}^{G(x)^{c}}\omega^{a-1}(1-\omega)^{b-1}d\omega. \end{aligned} \tag{2.4}$$

Now, combining the densities given in equations (2.2) and (2.3), we obtain the

pdf of the Mc-Dagum distribution as follows:

$$\begin{aligned} f(x;\lambda,\delta,\beta,a,b,c) &= \frac{c}{B(a,b)}\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left[\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{ac-1} \\ &\quad \left[1-\left(1+\lambda x^{-\delta}\right)^{-c\beta}\right]^{b-1} \qquad (2.5) \\ &= \frac{c\beta\lambda\delta x^{-\delta-1}}{B(a,b)}\left(1+\lambda x^{-\delta}\right)^{-\beta ac-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-c\beta}\right]^{b-1}, \end{aligned}$$

for $a, b, c, \lambda, \beta, \delta > 0$.

The cdf of this new distribution is given by

$$\begin{aligned} F(x) &= I_{G(x)^c}(a,b) \\ &= \tfrac{1}{B(a,b)}\int_0^{G(x)^c}\omega^{a-1}(1-\omega)^{b-1}d\omega \\ &= \tfrac{1}{B(a,b)}\int_0^{\left(1+\lambda x^{-\delta}\right)^{-\beta c}}\omega^{a-1}(1-\omega)^{b-1}d\omega \\ &= I_{\left(1+\lambda x^{-\delta}\right)^{-\beta c}}(a,b), \end{aligned} \qquad (2.6)$$

where

$$I_y(a,b) = \tfrac{1}{B(a,b)}\int_0^y \omega^{a-1}(1-\omega)^{b-1}d\omega \qquad (2.7)$$

is the incomplete beta function. The cdf can also be written as follows:

$$F(x) = \frac{\left(1+\lambda x^{-\delta}\right)^{-\beta ac}}{aB(a,b)}\left[{}_2F_1\left(a,1-b;a+1;(1+\lambda x^{-\delta})^{-\beta c}\right)\right], \qquad (2.8)$$

where

$${}_2F_1(a,b;c;x) = \tfrac{1}{B(b,c-b)}\int_0^1 \tfrac{y^{b-1}(1-y)^{c-b-1}}{(1-y^z)^a}dy, \qquad (2.9)$$

is the well-known hypergeometric function, (Gradshteyn and Ryzhik,(2000)).

2.2 Hazard and Reverse Hazard Functions

The failure rate function or hazard function and reverse hazard function are given by

$$\begin{aligned} h_F\left(x;a,b,c,\lambda,\beta,\delta\right) &= \frac{cg(x)G^{ac-1}(x)[1-G^c(x)]^{b-1}}{B(a,b)\left[1-I_{G^c(x)}(a,b)\right]} \\ &= \frac{c\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta ac-1}\left[1-(1+\lambda x^{-\delta})^{-c\beta}\right]^{b-1}}{B(a,b)\left[1-I_{\left[(1+\lambda x^{-\delta})^{-\beta c}\right]}(a,b)\right]}, \end{aligned} \tag{2.10}$$

and

$$\tau_F\left(x;a,b,c,\lambda,\beta,\delta\right) = \frac{c\beta\lambda x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta ac-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-c\beta}\right]^{b-1}}{B(a,b)I_{\left(1+\lambda x^{-\delta}\right)^{-\beta c}}(a,b)} \tag{2.11}$$

for $a>0, b>0, c>0, \lambda>0, \beta>0, \delta>0$, respectively.

2.3 Expansion of Distribution

In this section, we present a series expansion of the Mc-Dagum cdf and pdf. Consider the Mc-Dagum cdf given by

$$\begin{aligned} F(x;\lambda,\beta,\delta,a,b,c) &= I_{G(x)^c}(a,b) \\ &= \tfrac{1}{B(a,b)}\int_0^{G(x)^c}\omega^{a-1}(1-\omega)^{b-1}d\omega \\ &= \tfrac{1}{B(a,b)}\int_0^{\left(1+\lambda x^{-\delta}\right)^{-\beta c}}\omega^{a-1}(1-\omega)^{b-1}d\omega. \end{aligned} \tag{2.12}$$

Note that for $|\omega|<1$,

$$(1-\omega)^{b-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(b)}{\Gamma(b-j)\,j!}\omega^j.$$

Therefore, the cdf can be expanded to obtain:

$$\begin{aligned} F(x;\lambda,\beta,\delta,a,b,c) &= \tfrac{1}{B(a,b)} \int_0^{\left(1+\lambda x^{-\delta}\right)^{-\beta c}} \omega^{a-1} \textstyle\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} d\omega \\ &= \textstyle\sum_{j=0}^{\infty} \frac{(-1)^j}{B(a,b)} \frac{\Gamma(b)}{\Gamma(b-j)j!} \int_0^{G(x;\lambda,\beta,\delta)^c} \omega^{a+j-1} d\omega \\ &= \textstyle\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{B(a,b)\Gamma(b-j)j!} \left[\frac{\omega^{a+j-1+1}}{a+j-1+1}\right]_0^{G(x;\lambda,\beta,\delta)^c} \\ &= \textstyle\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{B(a,b)\Gamma(b-j)j!} \frac{[G(x;\lambda,\beta,\delta)]^{c(a+j)}}{(a+j)} \\ &= \textstyle\sum_{j=0}^{\infty} p_j G\left(x;\lambda,\beta c(a+j),\delta\right), \end{aligned} \tag{2.13}$$

for $b > 0$, real non-integer, where $p_j = \frac{(-1)^j \Gamma(a+b)}{j!\Gamma(a)\Gamma(b-j)(a+j)}$.

Similarly, the pdf is given by

$$f(x) = \sum_{j=0}^{\infty} p_j g(x;\lambda,\beta c(a+j),\delta). \tag{2.14}$$

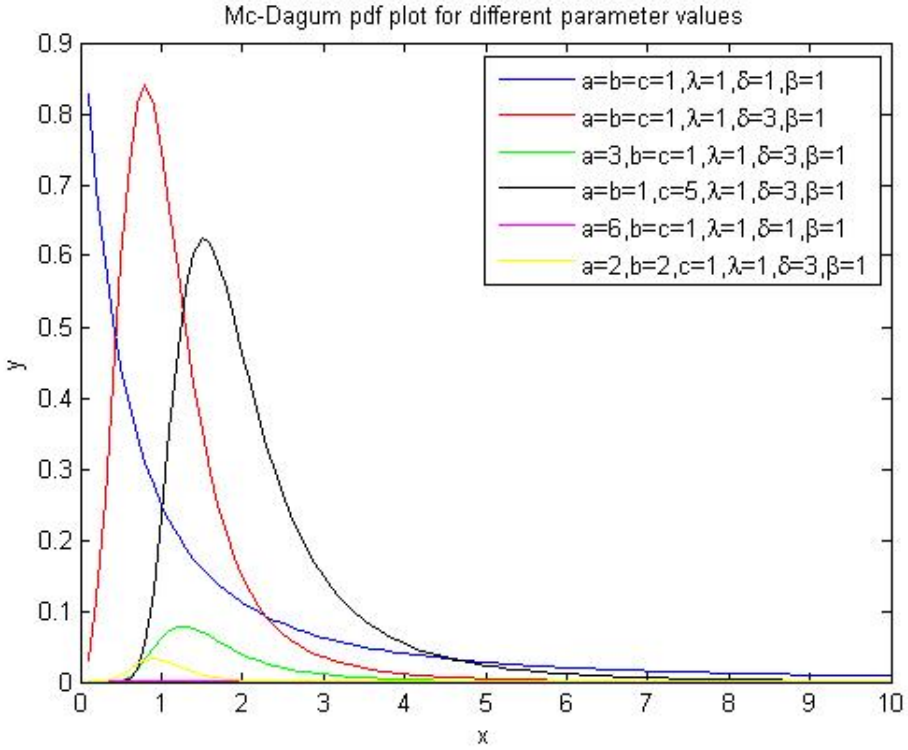
If $b > 0$ is an integer, then

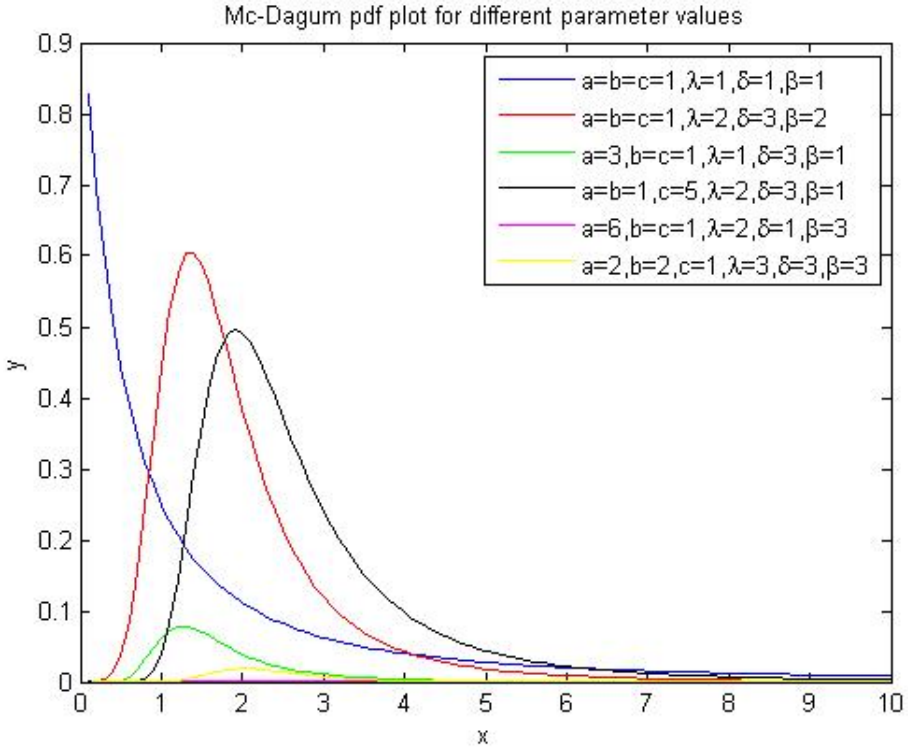
$$F\left(x;\lambda,\beta,\delta,a,b,c\right) = \sum_{j=0}^{b-1} p_j G\left(x;\beta c(a+j),\lambda,\delta\right), \tag{2.15}$$

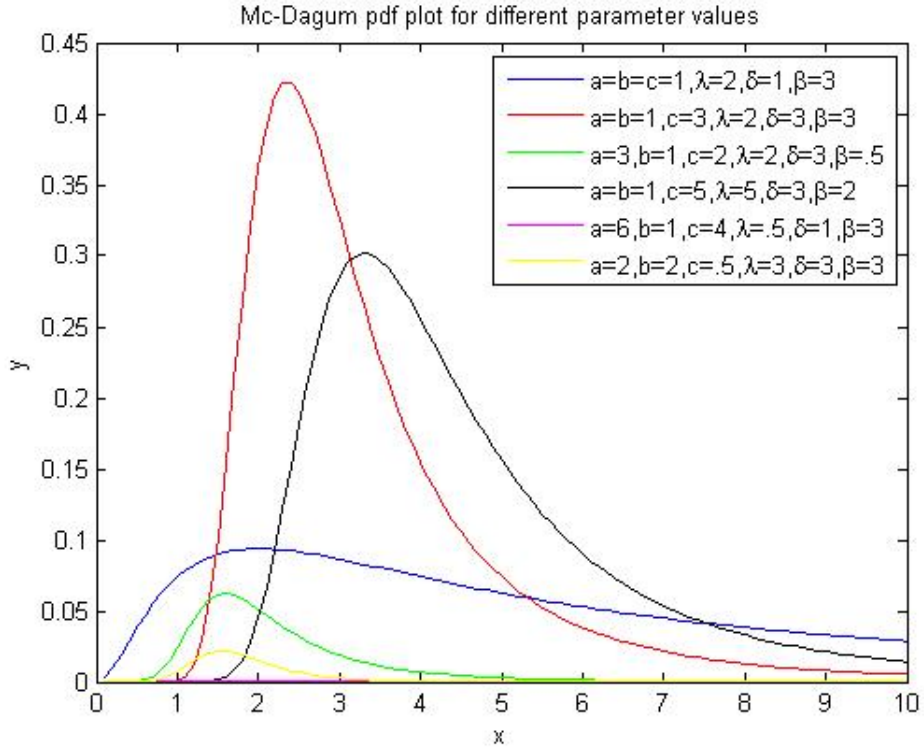
and

$$f(x;\lambda,\beta,\delta,a,b,c) = \sum_{j=0}^{b-1} p_j g(x;\beta c\left(a+j\right),\lambda,\delta). \tag{2.16}$$

This is a finite mixture of Dagum distributions with parameters $\lambda, \beta c(a+j) and \delta$. The graphs below are the pdf of the Mc-Dagum distribution for different values of parameters $\lambda, \delta, \beta, a, b$, and c.


Mc-Dagum pdf plot for different parameter values
a=b=c=1,λ=1,δ=1,β=1
a=b=c=1,λ=1,δ=3,β=1
a=3,b=c=1,λ=1,δ=3,β=1
a=b=1,c=5,λ=1,δ=3,β=1
a=6,b=c=1,λ=1,δ=1,β=1
a=2,b=2,c=1,λ=1,δ=3,β=1
y
x


Mc-Dagum pdf plot for different parameter values
a=b=c=1,λ=1,δ=1,β=1
a=b=c=1,λ=2,δ=3,β=2
a=3,b=c=1,λ=1,δ=3,β=1
a=b=1,c=5,λ=2,δ=3,β=1
a=6,b=c=1,λ=2,δ=1,β=3
a=2,b=2,c=1,λ=3,δ=3,β=3
y
x


Mc-Dagum pdf plot for different parameter values
a=b=c=1,λ=2,δ=1,β=3
a=b=1,c=3,λ=2,δ=3,β=3
a=3,b=1,c=2,λ=2,δ=3,β=.5
a=b=1,c=5,λ=5,δ=3,β=2
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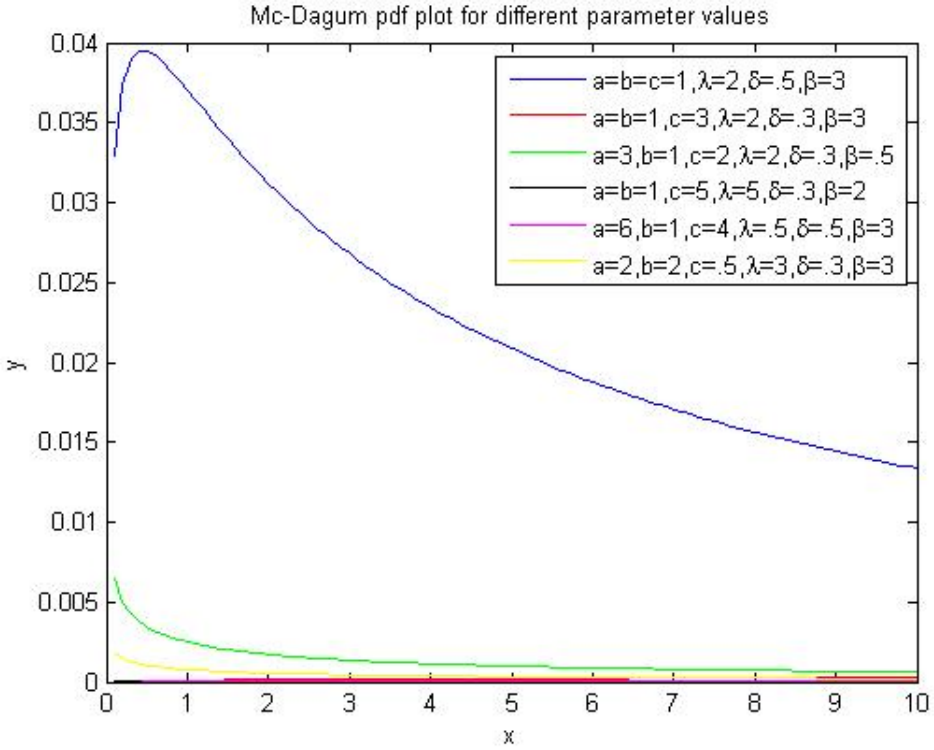
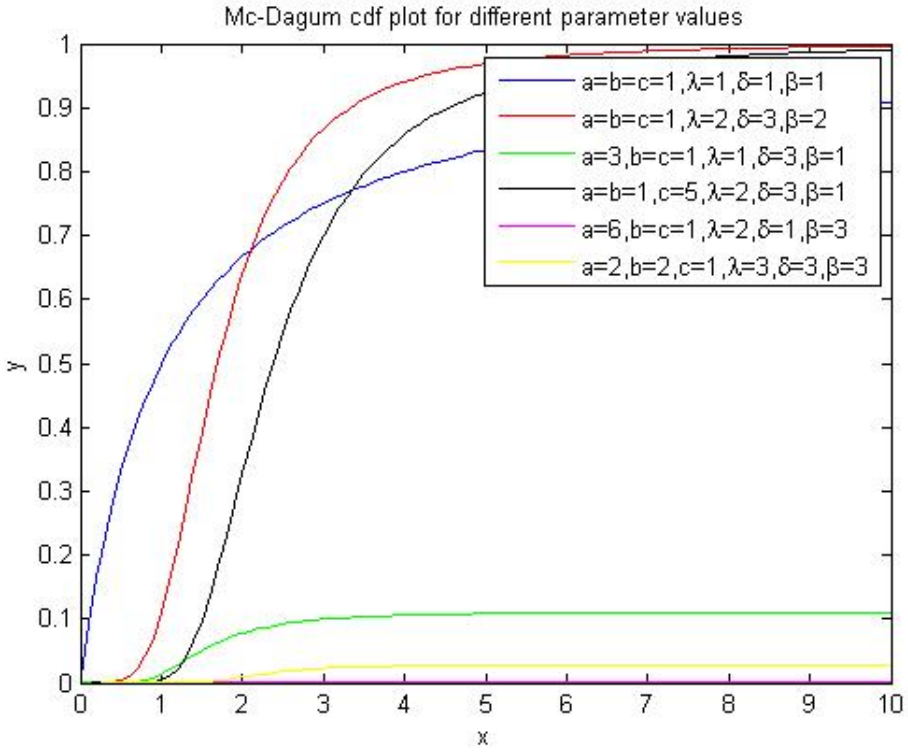
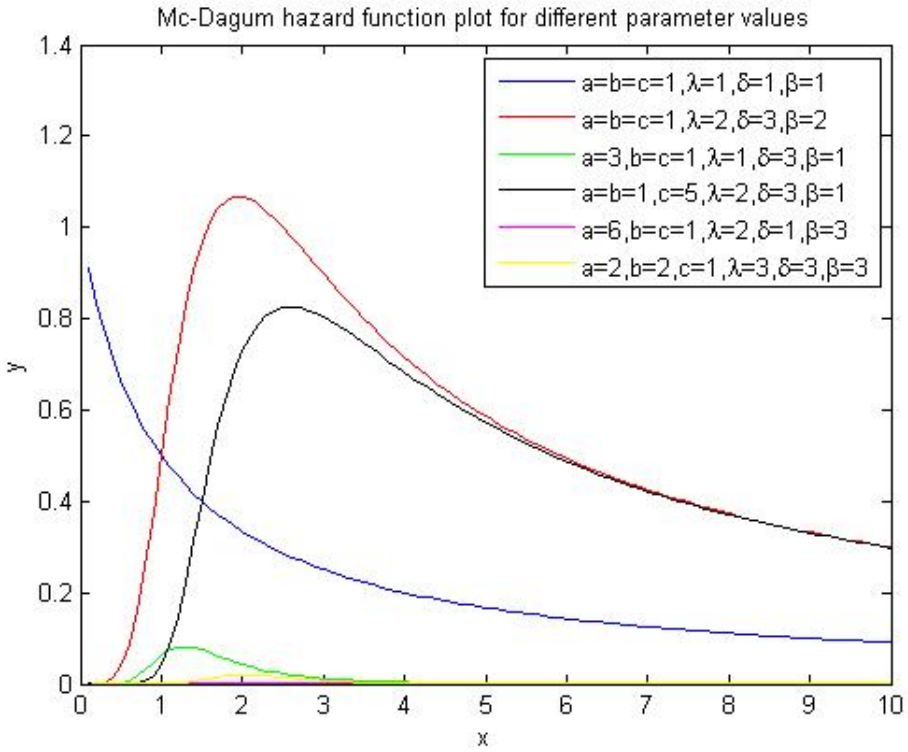
y
x


Mc-Dagum pdf plot for different parameter values
a=b=c=1,λ=2,δ=.5,β=3
a=b=1,c=3,λ=2,δ=.3,β=3
a=3,b=1,c=2,λ=2,δ=.3,β=.5
a=b=1,c=5,λ=5,δ=.3,β=2
a=6,b=1,c=4,λ=.5,δ=.5,β=3
a=2,b=2,c=.5,λ=3,δ=.3,β=3
y
x

The graphs below are the cdf and hazard functions of the Mc-Dagum distribution for different values of parameters $a, b, c, \lambda, \delta, \beta$.





2.3.1 Submodels

With this generalization, we have several submodels that can be obtained with specific values of the parameters λ, β, a, b and c.

1. When $c = 1$, the Mc-Dagum distribution is the beta-Dagum distribution, with the density given by:

$$f(x; \lambda, \beta, \delta, a, b) = \frac{\beta\lambda\delta x^{-\delta-1}}{B(a,b)} \left(1 + \lambda x^{-\delta}\right)^{-\beta a-1} \left[1 - (1 + \lambda x^{-\delta})^{-\beta}\right]^{b-1}, \quad (2.17)$$

for $x > 0$, $\lambda > 0$, $\beta > 0$, $\delta > 0$, $a > 0$, and $b > 0$.

2. If $a = b = c = 1$, we have the Dagum distribution with the pdf,

$$f_D(x; \lambda, \delta, \beta) = \beta\lambda\delta x^{-\delta-1} \left(1 + \lambda x^{-\delta}\right)^{-\beta-1}, \quad (2.18)$$

for $\lambda, \delta, \beta > 0$.

3. If $b = c = 1$ and $a > 0$, then we have the Dagum distribution with parameters $\beta a, \lambda$ and δ. The pdf is

$$f\left(x; \beta a, \lambda, \delta, \right) = \beta a \lambda \delta x^{-\delta-1} \left(1 + \lambda x^{-\delta}\right)^{-\beta a - 1}, \tag{2.19}$$

for $\lambda, \delta, \beta > 0$.

4. If $a = c = 1$ and $b > 0$, we have another Beta-Dagum distribution with parameters $b, \beta, \lambda, \delta$ and the pdf is given by

$$f_{BD}\left(x; \lambda, \delta, \beta, b\right) = b\beta\lambda\delta x^{-\delta-1} \left(1 + \lambda x^{-\delta}\right)^{-\beta-1} \left[1 - \left(1 + \lambda x^{-\delta}\right)^{-\beta}\right]^{b-1}, \tag{2.20}$$

for λ, δ and $\beta > 0$.

5. If $a = c = \lambda = 1$, then we have the beta-Burr III distribution with parameters b, β, δ and the pdf is given by

$$f_{BB}\left(x; \delta, \beta b, \right) = b\beta\delta x^{-\delta-1} \left(1 + x^{-\delta}\right)^{-\beta-1} \left[1 - \left(1 + x^{-\delta}\right)^{-\beta}\right]^{b-1}, \tag{2.21}$$

for $b, \delta, \beta > 0$.

6. If $c = \beta = 1$, then we have the beta-Fisk distribution with parameters a, b, λ, δ and the pdf is given by

$$f_{BF}\left(x; \lambda, \delta, a, b\right) = \frac{\lambda\delta x^{-\delta-1}}{B\left(a, b\right)} \left(1 + \lambda x^{-\delta}\right)^{-a-1} \left[1 - \left(1 + \lambda x^{-\delta}\right)^{-1}\right]^{b-1}, \tag{2.22}$$

for $a, b, \lambda, \delta > 0$.

2.3.2 Kum-Dagum Distribution

Kumaraswamy in his paper (1980) proposed a two-parameter distribution (Kumaraswamy distribution) defined in (0, 1). Here we will refer to it as Kum distribution. Its cdf is

given by:

$$F(x;a;b) = 1-(1-x^a)^b\,, \quad x \in (0,1),\, a>0,\, b>0. \tag{2.23}$$

The parameters a and b are the shape parameters. The Kum distribution has the probability density function (pdf) given by:

$$f(x;a,b) = abx^{a-1}(1-x^a)^{b-1}, \quad x \in (0,1),\, a>0,\, b>0. \tag{2.24}$$

Note that the Kumaraswamy distribution can be derived from the beta distribution. The beta distribution has the pdf:

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad \text{where } x \in (0\ 1), \alpha>0, \beta>0. \tag{2.25}$$

Combining cdf of Kum distribution with the Dagum distribution discussed in chapter 1, we obtain Kum-Dagum distribution with the cdf and pdf for this distribution given by

$$F_{Kum}(x) = 1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta a}\right]^b, \tag{2.26}$$

and

$$\begin{aligned} f_{kum}(x) &= ab\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-1}\left[1+\lambda x^{-\delta}\right]^{-\beta(a-1)}\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta a}\right]^b \\ &= ab\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta-\beta a+\beta-1}\left(1-\left[1+\lambda x^{-\delta}\right]^{-\beta a}\right)^{-\beta-1} \\ &= ab\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta a-1}\left(1-\left[1+\lambda x^{-\delta}\right]^{-\beta a}\right)^{-\beta-1}, \end{aligned} \tag{2.27}$$

for $a, b, \beta, \lambda, \delta > 0$, respectively. We do not study the properties of the Kum-Dagum distribution in this thesis.

2.4 Concluding Remarks

In this chapter, we introduced a new class of distributions called the Mc-Dagum distribution. We obtained the pdf, cdf, hazard function, reverse hazard function

for this class of distributions. We obtained the series expansion of the distribution and presented plots of pdf, cdf and hazards function for different parameter values. Through these graphs we see that the distribution possesses the ability to fit for a large range of data sets. We noted that there are several submodels for selected values of the Mc-Dagum model parameters. Additionally we introduced another new distribution called "Kum-Dagum distribution" but we do not discuss its properties in this thesis.

CHAPTER 3

MOMENTS AND INEQUALITY MEASURES

In this chapter, we present moments and inequality measures for the Mc-Dagum distribution. Income distribution and its variation is an important concern for economists. We use the results presented in chapter 2 which we obtained by expanding the pdf.

3.1 Moments

We can derive the k^{th} moment of a Mc-Dagum distribution using properties of the mixture distribution. The k^{th} raw or non-central moments are given by,

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k \tfrac{c\beta\lambda x^{-\delta-1}}{B(a,b)} \left(1+\lambda x^{-\delta}\right)^{-\beta ac-1} \left(1-\left(1+\lambda x^{-\delta}\right)^{\beta c}\right)^{b-1} dx \\ &= \tfrac{c\beta\lambda}{B(a,b)} \int_0^\infty x^{k-\delta-1} \left(1+\lambda x^{\delta}\right)^{-\beta ac-1} \left(1-\left(1+\lambda x^{-\delta}\right)^{-\beta c}\right)^{b-1} dx. \end{aligned} \tag{3.1}$$

Now let, $y^{-1} = \left(1+\lambda x^{-\delta}\right)$, then $x = (1-y)^{\frac{-1}{\delta}}(\lambda y)^{\frac{1}{\delta}}$, and we have

$$E(X^k) = \tfrac{c\beta}{\delta B(a,b)} \int_0^1 (1-y)^{\frac{-k}{\delta}} (\lambda y)^{\frac{k}{\delta}} y^{\beta ac-1}(1-y^{\beta c})^{b-1} dy. \tag{3.2}$$

Using the fact that $(1-y^{\beta c})^{b-1} = \sum_{j=1}^\infty \frac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!}(y^{\beta c})^j$, and for $p_j = \frac{(-1)^j\Gamma(a+b)}{j!\Gamma(a)\Gamma(b-j)(a+j)}$,

$$\begin{aligned} E(X^k) &= \tfrac{\lambda^{\frac{k}{\delta}} c\beta}{\delta B(a,b)} \sum_0^\infty \tfrac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!} \int_0^1 y^{\frac{k}{\delta}+\beta ac+\beta cj-1}(1-y)^{1-\frac{k}{\delta}-1} dy \\ &= \tfrac{\lambda^{\frac{k}{\delta}} c\beta}{\delta B(a,b)} \sum_0^\infty \tfrac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!} B(\beta c(a+j)+\tfrac{k}{\delta}, 1-\tfrac{k}{\delta}) \\ &= \tfrac{\sum_{j=0}^\infty p_j \beta c(a+j)\lambda^{\frac{k}{\delta}}}{\delta} B(\beta c(a+j)+\tfrac{k}{\delta}, 1-\tfrac{k}{\delta}), \delta > k. \end{aligned} \tag{3.3}$$

and $|\mathrm{y}^{\beta c} weobtain$ We can obtain the k^{th} incomplete moment for a Mc-Dagum distribution as follows:

$$\begin{aligned} E\left[X^k | X \le x\right] &= E_{X\le x}\left[X^k; \lambda, \beta, \delta, a, b, c\right] \\ &= \int_0^x u^k f(u)\, du \\ &= \int_0^x u^k \sum_{j=0}^\infty p_j f(u; \beta c(a+j), \lambda, \delta) du \\ &= \sum_{j=0}^\infty p_j \int_0^x u^k f(u; \beta c(a+j), \lambda, \delta) du \\ &= \sum_0^\infty p_j \tfrac{\beta c(a+j)\lambda^{\frac{k}{\delta}}}{\delta} B((1+\lambda x^{-\delta})^{-1}; \beta c(a+j)+\tfrac{k}{\delta}, 1-\tfrac{k}{\delta}), \end{aligned} \tag{3.4}$$

for $\delta > k$, where $B(t; c_1, c_2) = \int_0^t y^{c_1-1}(1-y)^{c_2-1}dy$.

The mean residual life (MRF) function denoted by $\mu(x; \lambda, \beta, \delta, a, b, c) = \mu(x)$ is given by

$$\begin{aligned}\mu(x) &= E[X - x|X \geq x] \\ &= \frac{E(X)-E(X|X\leq x)}{1-F(x)} - x \\ &= \frac{\frac{\sum_{j=0}^{\infty} p_j \beta c(a+j)\lambda^{\frac{k}{\delta}}}{\delta} B(\beta c(a+j)+\frac{k}{\delta}, 1-\frac{k}{\delta}) - \sum_0^{\infty} p_j \frac{\beta c(a+j)\lambda^{\frac{k}{\delta}}}{\delta} B((1+\lambda x^{-\delta})^{-1}; \beta c(a+j)+\frac{k}{\delta}, 1-\frac{k}{\delta})}{1-\sum_{j=0}^{\infty} p_j G(x;\lambda,\beta c(a+j),\delta)} - x.\end{aligned} \tag{3.5}$$

3.2 Inequality Measures

Lorenz and Bonferroni curves are the most widely used inequality measures in income and wealth distribution (Kleiber, 2004). Zenga curve was presented by Zenga in 2007. In this section, we will derive Lorenz, Bonferroni and Zenga curves for the Mc-Dagum distribution.

The Lorenz, Bonferroni and Zenga curves are defined by

$$\begin{aligned} L_F(x) &= \frac{\int_0^x t f(t)dt}{E(X)} \\ &= \frac{E_{X\leq x}(X)}{E(X)}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} B(F(x)) &= \frac{\int_0^x t f(t)dt}{F(x)E(X)} \\ &= \frac{E_{X\leq x}(X)}{F(x)E(X)} \\ &= \frac{L_F(x)}{F(x)}, \end{aligned} \tag{3.7}$$

and

$$A(x) = 1 - \frac{\mu^-(x)}{\mu^+(x)}, \tag{3.8}$$

respectively, where $\mu^{-}(x) = \frac{\int_0^x tf(t)dt}{F(x)} = \frac{E_X(x)}{F(x)}$ and $\mu^{+}(x) = \frac{\int_x^\infty tf(t)dt}{1-F(x)} = \frac{E(X)-E_{X>x}(x)}{1-F(x)}$ are the lower and upper means. For Mc-Dagum distribution, using these results, we obtain the curves. Lorenz curve for Mc-Dagum distribution is given by

$$L_{F_G}(x;\lambda,\beta,\delta,a,b,c) = \frac{\sum_{j=0}^{\infty} p_j \beta c(a+j)\lambda^{\frac{1}{\delta}} B((1+\lambda x^{-\delta})^{-1};\beta c(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})}{\sum_{j=0}^{\infty} p_j \beta c(a+j)\lambda^{\frac{1}{\delta}} B(\beta c(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})}. \tag{3.9}$$

Bonferroni curve for Mc-Dagum distribution is given by

$$B(F_G(x;\lambda,\beta,\delta,a,b,c)) = \frac{\sum_{j=0}^{\infty} p_j \beta c(a+j)\lambda^{\frac{1}{\delta}} B((1+\lambda x^{-\delta})^{-1};\beta c(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})}{\sum_{j=0}^{\infty} p_j G(x;\lambda,\beta c(a+j),\delta) \sum_{j=0}^{\infty} p_j \beta c(a+j)\lambda^{\frac{1}{\delta}} B(\beta c(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})}. \tag{3.10}$$

Zenga curve for the Mc-Dagum distribution is given by

$$\begin{aligned} A(x;\lambda,\beta,\delta,a,b,c) &= 1 - \left[\frac{\frac{E(X|X\leq x)}{F(x)}}{\frac{E(X)-E(X\leq x)}{1-F(x)}}\right] \\ &= 1 - \frac{(1-F(x))E[X|X\leq x]}{F(x)[E(X)-E(X|X\leq x)]}, \end{aligned} \tag{3.11}$$

where $E[X|X \leq x] = \sum_0^x p_j \frac{\beta c(a+j)\lambda^{\frac{1}{\delta}}}{\delta} B((1+\lambda x^{-\delta})^{-1};\beta c(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})$,
$E(X) = \frac{\sum_{j=0}^{\infty} p_j \beta c(a+j)\lambda^{\frac{1}{\delta}}}{\delta} B(\beta c(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})$, and
$F(x) = \sum_{j=0}^{\infty} p_j G(x;\lambda,\beta c(a+j),\delta)$.

3.2.1 Inequality Measures for Some Sub models

For various submodels that we introduced in chapter 2, we can generate Lorenz, Bonferroni and Zenga curves. Let ξ_1=$(\lambda,\beta,\delta,a,b)$, ξ_2=(λ,β,δ,b), ξ_3=(λ,δ,a,b) and E=$\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{\frac{1}{\delta}} B(\beta(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})$

1. If $c = 1$, we obtain the Lorenz and Bonferroni curves for the beta-Dagum distribution:

$$L_{F_G}(x;\xi_1) = \frac{\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{\frac{1}{\delta}} B((1+\lambda x^{-\delta})^{-1};\beta(a+j)+\frac{1}{\delta},1-\frac{1}{\delta})}{E},$$

and

$$B(F_G(x;\xi_1)) = \frac{\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{\frac{1}{\delta}} B((1+\lambda x^{-\delta})^{-1}; \beta(a+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})}{\sum_{j=0}^{\infty} p_j G\left(x;\lambda,\beta(a+j),\delta\right) E},$$

respectively.

2. If $a = c = 1$ and $b > 0$, then Lorenz and Bonferroni curves for another Beta-Dagum distribution with parameters $b, \beta, \lambda, \delta$ given by

$$L_{F_G}(x;\xi_2) = \frac{\sum_{j=0}^{\infty} p_j \beta(1+j)\lambda^{\frac{1}{\delta}} B((1+\lambda x^{-\delta})^{-1}; \beta(1+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})}{\sum_{j=0}^{\infty} p_j \beta(1+j)\lambda^{\frac{1}{\delta}} B(\beta(1+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})},$$

and

$$B(F_G(x;\xi_2)) = \frac{\sum_{j=0}^{\infty} p_j \beta(a+j)\lambda^{\frac{1}{\delta}} B((1+\lambda x^{-\delta})^{-1}; \beta(1+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})}{\sum_{j=0}^{\infty} p_j G\left(x;\lambda,\beta(1+j),\delta\right) E},$$

respectively.

3. If $a = c = \lambda = 1$, then we obtain the Lorenz and Bonferroni curves for the beta-Burr III distribution with parameters b, β, δ, that is

$$L_{F_G}(x;\beta,\delta,b) = \frac{\sum_{j=0}^{\infty} p_j \beta(1+j) B((1+x^{-\delta})^{-1}; \beta(1+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})}{\sum_{j=0}^{\infty} p_j \beta(1+j) B(\beta(1+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})},$$

and

$$B(F_G(x;\beta,\delta,b)) = \frac{\sum_{j=0}^{\infty} p_j \beta(a+j) B((1+x^{-\delta})^{-1}; \beta(1+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})}{\sum_{j=0}^{\infty} p_j G\left(x;\beta(1+j),\delta\right) E}.$$

4. If $c = \beta = 1$, then we obtain the Lorenz and Bonferroni curves for the beta-Fisk distribution with parameters a, b, λ, δ.

$$L_{F_G}(x;\xi_3) = \frac{\sum_{j=0}^{\infty} p_j (a+j)\lambda^{\frac{1}{\delta}} B((1+\lambda x^{-\delta})^{-1}; (a+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})}{E},$$

and

$$B(F_G(x;\xi_3)) = \frac{\sum_{j=0}^{\infty} p_j (a+j) B((1+\lambda x^{-\delta})^{-1}; (a+j)+\frac{1}{\delta}, 1-\frac{1}{\delta})}{\sum_{j=0}^{\infty} p_j G\left(x;\lambda,(a+j),\delta\right) E}$$

for $a, b, \lambda, \delta > 0$.

3.3 Concluding Remarks

In this chapter, we presented the raw moments and the k^{th} incomplete moments for the Mc-Dagum distribution. Inequality measures for the distribution are derived using well known Lorenz and Bonferroni curves. Additionally, Zenga curve was also obtained. Lorenz curve and Bonferroni curves for some submodels of this class of distributions are also obtained.

CHAPTER 4

ENTROPY

In this chapter, we discuss the Renyi entropy, Shannon entropy and $\widetilde{\beta}$-entropy for the Mc-Dagum distribution. The entropy of a random variable X is a measure of variation of the uncertainty.

4.1 Renyi and Shanon Entropy

For a pdf $f(x)$, Renyi entropy (Renyi, 1961) is given by

$$H_R(f) = \frac{log}{1-s}\left(\int_0^\infty f^s(x)dx\right), s > 0, s \neq 1. \tag{4.1}$$

As $s \to 1$, we obtain the Shanon entropy. Note that,

$$\begin{aligned}
\mathrm{f}^s(x) &= \tfrac{(c\beta\lambda\delta)^s x^{-s\delta-s}}{B^s(a,b)}\left(1+\lambda x^{-\delta}\right)^{-\beta acs-s}\left[1-\left(1+\lambda x^{-\delta}\right)^{-c\beta}\right]^{bs-s} \\
\text{and } \int_0^\infty f^s(x)dx &= \tfrac{(c\beta\lambda\delta)^s}{B^s(a,b)}\int_0^\infty x^{-s\delta-s}\left(1+\lambda x^{-\delta}\right)^{-\beta acs-s}\left[1-\left(1+\lambda x^{-\delta}\right)^{-c\beta}\right]^{bs-s}dx \\
&= \tfrac{(c\beta\lambda\delta)^s}{B^s(a,b)}\int_0^1 \left[(1-y)^{\frac{-1}{\delta}}(\lambda y)^{\frac{1}{\delta}}\right]^{-s\delta-s}\frac{y^{\beta acs+s}\left(1-y^{\beta c}\right)^{bs-s}}{y^2\lambda\delta\left[(1-y)^{\frac{-1}{\delta}}(\lambda y)^1\delta\right]^{-\delta-1}}dy \\
&= \tfrac{(c\beta\lambda\delta)^s}{B^s(a,b)}\int_0^1 \lambda^{-s\delta-s}y^{\frac{-s\delta-s}{\delta}+\beta acs+s-2+\frac{1}{\delta}+1}(1-y^{\beta c})^{sb-s}(1-y)^{s-1+\frac{s-1}{\delta}}dy.
\end{aligned} \tag{4.2}$$

Using the fact that, $(1-\omega)^{b-1} = \sum_{j=0}^\infty \frac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!}\omega^j$, and setting
$y = (1+\lambda x^{-\delta})^{-1}$, so that $x^{-\delta} = \frac{y^{-1}-1}{\lambda} = \frac{1-y}{\lambda y}$, and $\lambda\delta x^{-\delta-1}dx = y^{-2}dy$, we obtain

$$\begin{aligned}
\int_0^\infty f^s(x)dx &= \tfrac{(c\beta\lambda\delta)^s}{B^s(a,b)}\int_0^1 \lambda^{-s\delta-s}y^{\frac{-s\delta-s}{\delta}+\beta acs+s-2+\frac{1}{\delta}+1}(1-y^{\beta c})^{sb-s}(1-y)^{s+\frac{s}{\delta}-\frac{1}{\delta}-1}dy \\
&= \tfrac{(c\beta\lambda\delta)^s\lambda^{-s\delta-s}\lambda^{1+\frac{1}{\delta}}}{B^s(a,b)}\int_0^1 y^{\beta acs+s-s-\frac{s}{\delta}+\frac{1}{\delta}+1-2}(1-y)^{s+\frac{s}{\delta}-\frac{1}{\delta}-1}(1-y^{\beta c})^{sb-s}dy \\
&= \tfrac{(c\beta\lambda\delta)^s\lambda^{1+\frac{1}{\delta}-s\delta-s}}{B^s(a,b)}\int_0^1 y^{\beta acs-\frac{s}{\delta}+\frac{1}{\delta}-1}(1-y)^{s+\frac{s}{\delta}-\frac{1}{\delta}-1}\sum_{j=0}^\infty \tfrac{(-1)^j\Gamma(sb-s+1)y^{\beta cj}}{\Gamma(sb-s+1-j)j!} \\
&= \tfrac{(c\beta\lambda\delta)^s\lambda^{1+\frac{1}{\delta}-s\delta-s}}{B^s(a,b)}\sum_{j=0}^\infty \tfrac{(-1)^j\Gamma(sb-s+1)}{\Gamma(sb-s+1-j)j!}\int_0^1 y^{\beta cj+\beta acs-\frac{s}{\delta}+1\delta-1}(1-y)^{s+\frac{s}{\delta}-\frac{1}{\delta}-1}dy \\
&= \tfrac{(c\beta\lambda\delta)^s\lambda^{1+\frac{1}{\delta}-s\delta-s}}{B^s(a,b)}\sum_{j=0}^\infty \tfrac{(-1)^j\Gamma(sb-s+1)}{\Gamma(sb-s+1-j)j!}B(\beta cj+\beta acs-\tfrac{s}{\delta}+\tfrac{1}{\delta}, s+\tfrac{s}{\delta}-\tfrac{1}{\delta}).
\end{aligned} \tag{4.3}$$

Therefore, Renyi entropy for the Mc-Dagum distribution is

$$H_R(f) = \frac{log}{1-s}\left[\frac{(c\beta\lambda\delta)^s\lambda^{1+\frac{1}{\delta}-s\delta-s}}{B^s(a,b)}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(sb-s+1)}{\Gamma(sb-s+1-j)j!}\right. \\ \left. B\left(\beta cj+\beta acs-\frac{s}{\delta}+\frac{1}{\delta},s+\frac{s}{\delta}-\frac{1}{\delta}\right)\right] \tag{4.4}$$

for $s>0$ and $s\neq 1$. If $bs-s$ is a positive integer, then the sum in the Renyi entropy stops at $bs-s$.

4.2 $\widetilde{\beta}$-entropy

We also obtain $\widetilde{\beta}$-entropy for the Mc-Dagum density as follows.

$$\mathrm{H}_{\widetilde{\beta}}(f) = \begin{cases} \frac{1}{\widetilde{\beta}-1}[1-\int_0^{\infty} f^{\widetilde{\beta}}(x)dx]\ , & \text{if } \widetilde{\beta}\neq 1, \widetilde{\beta}>0, \\ E[-log(f(X))], & \text{if } \widetilde{\beta}=1. \end{cases} \tag{4.5}$$

Therefore, if $\widetilde{\beta}\neq 1$, $\widetilde{\beta}>0$,

$$H_{\widetilde{\beta}}(f) = \frac{1}{\widetilde{\beta}-1}\left[1-\frac{(c\beta\lambda\delta)^{\widetilde{\beta}}\lambda^{1+\frac{1}{\delta}-\widetilde{\beta}\delta-\widetilde{\beta}}}{B^{\widetilde{\beta}}(a,b)}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\widetilde{\beta}b-\widetilde{\beta}+1)}{\Gamma(\widetilde{\beta}b-\widetilde{\beta}+1-j)j!}\right. \\ \times \left. B\left(\beta cj+\beta ac\widetilde{\beta}-\frac{\widetilde{\beta}}{\delta}+\frac{1}{\delta},\widetilde{\beta}+\frac{\widetilde{\beta}}{\delta}-\frac{1}{\delta}\right)\right].$$

4.3 Concluding Remarks

In chapter 4, measures of uncertainty, including Renyi, Shanon and $\widetilde{\beta}$-entropy for the Mc-Dagum distribution were presented.

CHAPTER 5

INFERENCE

5.1 Maximum Likelihood Estimates (MLE)

Let $\Theta = (\lambda, \beta, \delta, a, b, c)^T$. In order to estimate the parameters $\lambda, \beta, \delta, a, b$ and c of the Mc-Dagum distribution, we use the method of maximum likelihood estimation. Let $X_1, X_2, ..., X_n$ be a random sample from $f(x; \lambda, \beta, \delta, a, b, c)$. The log-likelihood function $L(\lambda, \beta, \delta, a, b, c)$ is:

$$\begin{aligned} L(\lambda, \beta, \delta, a, b, c) &= nlog\left(\frac{c\beta\lambda\delta}{B(a,b)}\right) + log\left(\prod_{i=1}^{n} x_i^{-\delta-1}\right) + log\left[\prod_{i=1}^{n}\left(1+\lambda x_i^{-\delta}\right)^{-\beta ac-1}\right] \\ &\quad + log\prod_{i=1}^{n}\left[1-\left(1+\lambda x_i^{-\delta}\right)^{-c\beta}\right]^{b-1} \qquad (5.1) \\ &= nlog(c) + nlog(\beta) + nlog(\lambda) + nlog(\delta) - nlogB(a,b) \\ &\quad - (\delta+1)\sum_{i=1}^{n} logx_i - (\beta ac+1)\sum_{1}^{n} log[1+\lambda x_i^{-\delta}] \\ &\quad + (b-1)\sum_{i=1}^{n} log[1-(1+\lambda x_i^{-\delta})^{-c\beta}]. \end{aligned} \qquad (5.2)$$

Differentiating $L(\lambda, \beta, \delta, a, b, c)$ with respect to each parameter $\lambda, \beta, \delta, a, b$ and c and setting the result equals to zero, we obtain maximum likelihood estimates. The partial derivatives of L with respect to each parameter or the score function is given by:

$$U_n(\Theta) = \left(\frac{\partial L}{\partial\lambda}, \frac{\partial L}{\partial\beta}, \frac{\partial L}{\partial\delta}, \frac{\partial L}{\partial a}, \frac{\partial L}{\partial b}, \frac{\partial L}{\partial c}\right), \qquad (5.3)$$

where

$$\frac{\partial L}{\partial\lambda} = \frac{n}{\lambda} - \beta ac\sum_{i=1}^{n}\left(\frac{x_i^{-\delta}}{1+\lambda x_i^{-\delta}}\right) - \sum_{i=1}^{n}\frac{x_i^{-\delta}}{(1+\lambda x_i^{-\delta})} + (b-1)\sum_{i=1}^{n}\frac{c\beta(1+\lambda x_i^{-\delta})^{-c\beta-1}x_i^{-\delta}}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \qquad (5.4)$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} - ac\sum_{i=1}^{n} log(1+\lambda x_i^{-\delta}) + c(b-1)\sum_{i=1}^{n} \frac{(1+\lambda x_i^{-\delta})^{-c\beta} log(1+\lambda x_i^{-\delta})}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \quad (5.5)$$

$$\begin{aligned} \frac{\partial L}{\partial \delta} &= \frac{n}{\delta} - \sum_{i=1}^{n} logx_i + \lambda(\beta ac+1)\sum_{i=1}^{n} \frac{x_i^{-\delta} log(x_i)}{(1+\lambda x_i^{-\delta})} \\ &\quad - \lambda c\beta(b-1)\sum_{i=1}^{n} \frac{x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-c\beta-1} logx_i}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \end{aligned} \quad (5.6)$$

$$\frac{\partial L}{\partial a} = -n(\psi(a) - \psi(a+b)) - \beta c\sum_{i=1}^{n} log(1+\lambda x_i^{-\delta}), \quad (5.7)$$

$$\frac{\partial L}{\partial b} = -n[\psi(b) - \psi(a+b)] + \sum_{i=1}^{n} log[1-(1+\lambda x_i^{-\delta})^{-c\beta}], \quad (5.8)$$

where $\psi(.)$ is digamma function defined by $\psi(x) = \frac{d}{dx} log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, and

$$\frac{\partial L}{\partial c} = \frac{n}{c} - \beta a\sum_{i=1}^{n} log(1+\lambda x_i^{-\delta}) + (b-1)\beta\sum_{i=1}^{n} \frac{(1+\lambda x_i^{-\delta})^{-c\beta} log(1+\lambda x_i^{-\delta})}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}. \quad (5.9)$$

The MLE of the parameters λ,β,δ,a,b and c, say $\hat{\lambda}$,$\hat{\beta}$,$\hat{\delta}$,$\hat{a}$,$\hat{b}$ and $\hat{c}$ are obtained by solving the following equations, $\frac{\partial L}{\partial \lambda} = \frac{\partial L}{\partial \beta} = \frac{\partial L}{\partial \delta} = \frac{\partial L}{\partial a} = \frac{\partial L}{\partial b} = \frac{\partial L}{\partial c} = 0$. There is no closed form solution to these equations, so numerical technique such as Newton-Rapson method must be applied.

5.2 Fishers Information Matrix

To obtain the Fishers information matrix (FIM), we derive the second partial derivatives and cross partial derivatives with respect to each parameter λ,β,δ,a,b and c as follows:

From equation (5.4) we obtain

$$\frac{\partial^2 L}{\partial \lambda^2} = \frac{-n}{\lambda^2} + (\beta ac+1)\sum_{i=1}^{n} \frac{x_i^{-2\delta}}{A_i^2} + (b-1)\sum_{i=1}^{n} \frac{c\beta x_i^{-2\delta} A_i^{-c\beta-2}[A_i^{-c\beta} - c\beta - 1]}{[1-A_i^{-c\beta}]^2}, \quad (5.10)$$

where, A_i=$(1+\lambda x_i^{-\delta})$,

$$\frac{\partial^2 L}{\partial\lambda\partial\beta} = -ac\sum_{i=1}^{n}\frac{x_i^{-\delta}}{A_i} + (b-1)\sum_{i=1}^{n} x_i^{-\delta} c A_i^{-c\beta-1}\frac{[c\beta log(A_i) - 1 + A_i^{-c\beta}]}{[1-A_i^{-c\beta}]^2}, \tag{5.11}$$

$$\frac{\partial^2 L}{\partial\lambda\partial\delta} = (-\beta ac - 1)\sum_{i=1}^{n}\frac{x_i^{-\delta} log(x_i)}{A_i^2} + (b-1)\sum_{i=1}^{n} c\beta x_i^{-\delta} A_i^{-c\beta-1} log(x_i) B_i, \tag{5.12}$$

where, $B_i = \frac{1-\lambda x_i^{-\delta} c\beta A_i^{-1} - \lambda x_i^{-\delta} A_i^{-1} + \lambda x_i^{-\delta} c\beta A_i^{-c\beta-1} - (1+\lambda x_i^{-\delta})^{-c\beta}}{[1-A_i^{-c\beta}]^2}$,

$$\frac{\partial^2 L}{\partial\lambda\partial a} = -\beta c\sum_{i=1}^{n}\frac{x_i^{-\delta}}{(1+\lambda x_i^{-\delta})}, \tag{5.13}$$

$$\frac{\partial^2 L}{\partial\lambda\partial b} = \sum_{i=1}^{n}\frac{c\beta(1+\lambda x_i^{-\delta})^{-c\beta-1} x_i^{-\delta}}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \tag{5.14}$$

and

$$\frac{\partial^2 L}{\partial\lambda\partial c} = -\beta a\sum_{i=1}^{n}\frac{x_i^{-\delta}}{A_i} + (b-1)\sum_{i=1}^{n}\frac{\beta x_i^{-\delta} A_i^{-c\beta-1}[c\beta log A_i - 1 + A_i^{-c\beta}]}{[1-A_i^{-c\beta}]^2}. \tag{5.15}$$

From equation (5.5), we obtain

$$\frac{\partial^2 L}{\partial\beta^2} = \frac{-n}{\beta^2} + c^2(b-1)\sum_{i=1}^{n}\frac{(1+\lambda x_i^{-\delta})^{-c\beta}[log(1+\lambda x_i^{-\delta})]^2}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]^2}, \tag{5.16}$$

$$\frac{\partial^2 L}{\partial\beta\partial\delta} = ac\sum_{i=1}^{n} C_i + (b-1)\sum_{i=1}^{n}\frac{\lambda c A_i^{-c\beta-1} x_i^{-\delta} log x_i[1 - c\beta log A_i - A_i^{-c\beta}]}{[1-A_i^{-c\beta}]^2}, \tag{5.17}$$

where, $C_i = \frac{\lambda x_i^{-\delta} log(x_i)}{A_i}$,

$$\frac{\partial^2 L}{\partial\beta\partial a} = -c\sum_{i=1}^{n} log(1+\lambda x_i^{-\delta}), \tag{5.18}$$

$$\frac{\partial^2 L}{\partial\beta\partial b} = \sum_{i=1}^{n}\frac{c(1+\lambda x_i^{-\delta})^{-c\beta} log(1+\lambda x_i^{-\delta})}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \tag{5.19}$$

and

$$\frac{\partial^2 L}{\partial\beta\partial c} = -a\sum_{i=1}^{n} log A_i + (b-1)\sum_{i=1}^{n}\frac{A_i^{-c\beta} log A_i[c\beta log A_i + A_i^{-c\beta} - 1]}{[1-A_i^{-c\beta}]^2}. \tag{5.20}$$

From equation (5.6), we obtain

$$\frac{\partial^2 L}{\partial \delta^2} = \frac{-n}{\delta^2} + \lambda(\beta ac + 1)\sum_{i=1}^{n} x_i^{-\delta}(logx_i)^2 A_i^{-c\beta-1} D_i \tag{5.21}$$

where $D_i = \frac{[1-\lambda c\beta x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-1} - \lambda x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-1} - (1+\lambda x_i^{-\delta})^{-c\beta} + \lambda x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-c\beta-1}]}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]^2}$.

Also,

$$\frac{\partial^2 L}{\partial \delta \partial a} = \lambda\beta c\sum_{i=1}^{n} \frac{x_i^{-\delta} logx_i}{(1+\lambda x_i^{-\delta})}, \tag{5.22}$$

$$\frac{\partial^2 L}{\partial \delta \partial b} = -\lambda c\beta\sum_{i=1}^{n} \frac{x_i^{-\delta}(1+\lambda x_i^{-\delta})^{-c\beta-1} logx_i}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}, \tag{5.23}$$

and

$$\frac{\partial^2 L}{\partial \delta \partial c} = \lambda\beta c\sum_{i=1}^{n} F_i - \lambda\beta(b-1)\sum_{i=1}^{n} \frac{A_i^{-c\beta-1} x_i^{-\delta} logx_i[c\beta logA_i + A_i^{-c\beta} - 1]}{[1-A_i^{-c\beta}]^2}, \tag{5.24}$$

where $F_i = \frac{x_i^{-\delta} logx_i}{(1+\lambda x_i^{-\delta})}$.

From equation (5.7), we obtain

$$\frac{\partial^2 L}{\partial a^2} = n\left[(\psi(a+b))^2 - \frac{\Gamma''(a+b)}{\Gamma(a+b)} - (\psi(a))^2 + \frac{\Gamma''(a)}{\Gamma(a)}\right], \tag{5.25}$$

$$\frac{\partial^2 L}{\partial a \partial b} = n\left[(\psi(a+b))^2 - \frac{\Gamma''(a+b)}{\Gamma(a+b)}\right], \tag{5.26}$$

and

$$\frac{\partial^2 L}{\partial a \partial c} = -\beta\sum_{i=1}^{n} log(1+\lambda x_i^{-\delta}). \tag{5.27}$$

From equation (5.8), we obtain

$$\frac{\partial^2 L}{\partial a^2} = n\left[(\psi(a+b))^2 - \frac{\Gamma''(a+b)}{\Gamma(a+b)} - (\psi(b))^2 + \frac{\Gamma''(b)}{\Gamma(b)}\right], \tag{5.28}$$

and

$$\frac{\partial^2 L}{\partial b \partial c} = \sum_{i=1}^{n} \frac{\beta(1+\lambda x_i^{-\delta})^{-c\beta} log(1+\lambda x_i^{-\delta})}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]}. \tag{5.29}$$

From equation (5.9), we obtain

$$\frac{\partial^2 L}{\partial c^2} = \frac{-n}{c^2} + \beta(b-1)\sum_{i=1}^{n} \frac{\beta(1+\lambda x_i^{-\delta})^{-c\beta}[log(1+\lambda x_i^{-\delta})]^2}{[1-(1+\lambda x_i^{-\delta})^{-c\beta}]^2}. \tag{5.30}$$

Fisher information matrix for the Mc-Dagum distribution is:

$$I(\theta) = I(\lambda,\beta,\delta,a,b,c) = \begin{bmatrix} I_{\lambda\lambda} & I_{\lambda\beta} & I_{\lambda\delta} & I_{\lambda a} & I_{\lambda b} & I_{\lambda c} \\ I_{\beta\lambda} & I_{\beta\beta} & I_{\beta\delta} & I_{\beta a} & I_{\beta b} & I_{\beta c} \\ I_{\delta\lambda} & I_{\delta\beta} & I_{\delta\delta} & I_{\delta a} & I_{\delta b} & I_{\delta c} \\ I_{a\lambda} & I_{a\beta} & I_{a\delta} & I_{aa} & I_{ab} & I_{ac} \\ I_{b\lambda} & I_{b\beta} & I_{b\delta} & I_{ba} & I_{bb} & I_{bc} \\ I_{c\lambda} & I_{c\beta} & I_{c\delta} & I_{ca} & I_{cb} & I_{cc} \end{bmatrix}. \tag{5.31}$$

where, $I_{\lambda\lambda}$=$-E\left[\frac{\partial^2 L}{\partial \lambda^2}\right]$,.....,$I_{cc}$=$-E\left[\frac{\partial^2 L}{\partial c^2}\right]$.

The elements of the 6 X 6 matrix $I(\lambda,\beta,\delta,a,b,c)$ can be approximated by the elements of the information matrix, where

$$I_{ij}(\theta) = -E\left[\frac{\partial^2 L}{\partial\theta_i \partial\theta_j}\right] \approx \frac{-\partial^2 L}{\partial\theta_i \partial\theta_j}. \tag{5.32}$$

Applying the usual large sample approximation, MLE of Θ, that is $\hat{\Theta}$ is approximately $N_6(\Theta, I_n^{-1}(\Theta))$, where $I_n(\Theta)$ is the 6X6 observed information matrix. Under the regularity conditions and parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}((\hat{\Theta})-\Theta)$ is $N_6(\Theta, I^{-1}(\Theta))$, where $I(\Theta) = \lim_{n\to\infty} n^{-1} I_n(\Theta)$.

Therefore, the approximate 100(1-α)% two-sided confidence intervals for $\lambda, \beta, \delta, a, b$ and c are given by:

$\hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{I^{-1}_{\lambda\lambda}(\hat{\theta})}$, $\hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{I^{-1}_{\beta\beta}(\hat{\theta})}$, $\hat{\delta} \pm Z_{\frac{\alpha}{2}} \sqrt{I^{-1}_{\delta\delta}(\hat{\theta})}$,
$\hat{a} \pm Z_{\frac{\alpha}{2}} \sqrt{I^{-1}_{aa}(\hat{\theta})}$, $\hat{b} \pm Z_{\frac{\alpha}{2}} \sqrt{I^{-1}_{bb}(\hat{\theta})}$ and $\hat{c} \pm Z_{\frac{\alpha}{2}} \sqrt{I^{-1}_{cc}(\hat{\theta})}$,
where, $Z_{\frac{\alpha}{2}}$ is the upper $(\frac{\alpha}{2})^{th}$ percentile of a standard normal distribution.

5.3 Concluding Remarks

In this chapter, we presented log-likelihood function for the Mc-Dagum distribution and obtained partial derivatives with respect to each parameter to estimate the model parameters. We noticed that there are no closed form estimates of the parameters, so numerical methods must be applied. We also obtained Fisher Information matrix;

$$I_{ij}(\theta) = -E\left[\frac{\partial^2 L}{\partial\theta_i \partial\theta_j}\right] \approx \frac{-\partial^2 L}{\partial\theta_i \partial\theta_j}. \tag{5.33}$$

Finally the approximate confidence intervals for each parameter was given.

5.4 Future Works

In the future, we will investigate and obtain results on the Kumaraswamy-Dagum (*Kum-Dagum*) distribution that was mentioned in the Chapter one. We will also work on obtaining estimates of model parameters from the Bayesian viewpoint for both Mc-Dagum and Kum-Dagum distributions and conduct goodness-of-fit tests for these models.

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