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One-Dimensional Fractal Wave Equations

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ONE-DIMENSIONAL FRACTAL WAVE EQUATIONS

by

F. CHAN

(Under the Direction of Dr. Sze-Man Ngai)

ABSTRACT

We study one-dimensional wave equations defined by a class of fractal Laplacians. These Laplacians are defined by fractal measures generated by iterated function systems with overlaps, such as the well-know infinite Bernoulli convolution associated with golden ratio and the 3-fold convolution of the Cantor measure. The iterated function systems defining these measures do not satisfy the open set condition or the post-critically finite condition, and therefore the existing theory, introduced by Kigami and developed by many other mathematicians, cannot be applied. First, by using a weak formulation of the problem, we prove the existence, uniqueness and regularity of weak solutions of these wave equations. Second, we study numerical computations of the solutions. By using the second-order self-similar identities introduced by Strichartz et al., we discretize the equation and use the finite element method and central difference method to obtain numerical solutions. Last, we also prove that the numerical solutions converge to the weak solution, and obtain estimates for the convergence of this approximation scheme.

INDEX WORDS: Fractal, wave equation, iterated function system, second-order self-similar identities, weak solution, finite element method

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by

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0.1 Notation

Let us introduce some notation and symbols used in this thesis.

Let Δ_μ be Laplacian with respect to μ.

Let $C_c^\infty(a,b)$ be the set of infinitely differentiable functions with compact supports $[a,b]$. Let X be a Banach space and $\|\cdot\|_X$ denote the corresponding norm.

Let $L^p([0,T];X)$ be the space of all measurable functions $u:[0,T]\to X$.

Let Xbe a Banach space. The space $C([0,T];X)$ comprises all *continuous* functions $u:[0,T]\to X$. The space $C^1([0,T];X)$ comprises all C^1 functions $u:[0,T]\to X$. The space $C^2([0,T];X)$ comprises all C^2 functions $u:[0,T]\to X$.

Let $\mathrm{Dom}\,(\mathcal{E}) := H_0^1(a,b)$ be the Sobolev space of all functions $f:[a,b]\to\mathbb{R}$ in $L^2([a,b],dx)$ with $f(a)=f(b)=0$.

Let $(\mathrm{Dom}\,(\mathcal{E}))'$ denote the dual space of $\mathrm{Dom}\,(\mathcal{E})$. Then each $u\in(\mathrm{Dom}\,(\mathcal{E}))'$ defines a continuous linear functional $< u,v >:= (u,v)_\mu$.

$\mathcal{E}(u,v)=\int_a^b \nabla u\nabla v\,dx$ with domain $\mathrm{Dom}\,(\mathcal{E})$.

$(u,v)_\mu := \int_a^b uv\,d\mu$ denotes the inner product in $L_\mu^2[a,b]$ and let $\|u\|_\mu$ denote the corresponding norm.

Let $u\in X$, where X is $\mathrm{Dom}\,(\mathcal{E})$, $L^2([a,b],dx)$ or $L_\mu^2[a,b]$. Then ∇u is the distributional derivative with respect to x.

Let $u:[0,T]\to X$. Then $\dot{u}$ denotes the strong derivative. See Appendix G.

Let $\alpha:[0,T]\to\mathbb{R}$. Then α' denotes the strong derivative. See Appendix G.

Let $u : [0,T] \to X$, where X be $\mathrm{Dom}\,(\mathcal{E})$ or $L^2_\mu[a,b]$. Then $\frac{\partial u}{\partial t}$ denotes the partial derivative. See Appendix G.

Let $u : [0,T] \to \mathrm{Dom}\,(\mathcal{E})$ or Let $u : [0,T] \to L^2_\mu[a,b]$. Then u_t and $(u_m)_t$ denote the weak derivative.

CHAPTER 1
INTRODUCTION

In the real world, many geometric objects are best modeled by fractals, such as trees, coastlines, mountains, and clouds. In 1975, Mandelbrot[19] coined the term of fractals and argued the theory of fractals.

Analysis on fractals [13] actors of Laplacian on the Sierpinski gasket, which is first introduced by J. Kigami in [12], and eigenvalues and eigenfunctions of Laplacian on post critically finite self-similar sets. In this work, Kigami also explain how to construct Dirichlet forms, harmonic functions, Green's functions and Laplacians on the post critically finite self-similar sets.

Dalrymple et al. [6] studied analogues of some classical differential equations, such as heat and wave equations on the Sierpinski gasket.

In this thesis, we study one-dimensional wave equations defined by a class of fractal Laplacians. These Laplacians are defined by fractal measures generated by iterated function systems with overlaps. We proved the existence and uniqueness of the weak solution of the hyperbolic initial/boundary value problem of the non-homogeneous wave equation. (see chapter 2.) We also solve the homogeneous wave equation numerically. Finally, we prove that the numerical solutions converge to the weak solution.

1.1 Preliminaries

In this section, we will introduce known results related to our projects.

1.1.1 Fractal measures

Let D be a non-empty compact subset of $\mathbb{R}^d$. A function $S : D \to D$ is called *contraction* on D if there is a number c with $0 < c < 1$ such that $|S(x) - S(y)| \leq c|x - y|$ for all $x, y \in D$. An *iterated function system(IFS)* is a finite collection of contractive functions $\{S_i\}_{i=0}^m$. In this thesis, we are mainly interested in IFS of contractive similitudes on $\mathbb{R}^d$. These functions are of the form

$$S_i(x) = \rho_i R_i x + b_i, \quad i = 0, 1, ..., m, \tag{1.1.1}$$

where $0 < \rho_i < 1, R_i$ is an orthogonal transformation and $b_i \in \mathbb{R}^d$ (see [8]).

To each set of probability weights $\{p_i\}_{i=0}^m$, where $p_i \geq 0$ and $\sum_{i=0}^m p_i = 1$, there exists a unique probability measure, called a *self-similar measure*, satisfying the identity

$$\mu = \sum_{i=0}^{m} p_i \mu \circ S_i^{-1} \tag{1.1.2}$$

(see [10]).

We say that $\{S_i\}_{i=0}^m$ satisfies the *open set condition (OSC)* if the exist a nonempty bounded open set U such that $\cup_{i=0}^m S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all $i \neq j$ (see [8]). Let$\{S_i\}_{i=0}^m$ and μ be of the form (1.1.1) and (1.1.2), respectively. Also, we assume that $\text{supp}(\mu) = [a, b]$. Define

$$T_j(x) = \rho^{n_j} x + d_j, \quad j = 1, 2, \ldots, N, \tag{1.1.3}$$

where $n_j \in \mathbb{N}$ and $d_j \in \mathbb{R}^d$. μ is said to satisfy a family of *second-order self-similar identities* (or simply *second-order identities*) with respect to $\{T_j\}_{j=1}^N$ (see [17]) if

(i) $\text{supp}(\mu) \subseteq \bigcup_{j=1}^N T_j(\text{supp}(\mu))$, and

(ii) for each Borel set $A \subseteq \text{supp}(\mu)$ and $0 \le i, j \le N$, $\mu(T_i \circ T_j A)$ can be expressed as a linear combination of $\{\mu(T_k A) : k = 1, \ldots, L\}$ as

$$\mu(T_i \circ T_j A) = \sum_{k=0}^{m} c_k \mu(T_k A), \tag{1.1.4}$$

where $c_k = c_k(i, j)$ are independent of A.

For our purposes, $\{T_j\}_{i=0}^{N}$ has to satisfy the OSC.

1.1.2 Operator Δ_μ

Let μ be a continuous positive finite Borel measure on $\mathbb{R}$ with $\text{supp}(\mu) \subseteq [a, b]$. It is well known (see e.g., [3, 9]) that μ defines a Dirichlet Laplacian Δ_μ on $L^2_\mu[a, b]$, described as follows. Let $H^1(a, b)$ be the Sobolev space of all functions in $L^2([a, b], dx)$ whose *distributional derivatives* (see Appendix G) belong to $L^2[a, b]$, with the inner product

$$(u, v)_{H^1(a,b)} := \int_a^b uv\, dx + \int_a^b \nabla u \nabla v\, dx.$$

Let $H^1_0(a, b)$ be the completion of $C^\infty_c(a, b)$ in $H^1(a, b)$. $H^1_0(a, b)$ is a dense subspace of $L^2_\mu[a, b]$. Define a quadratic form on $L^2_\mu[a, b]$,

$$\mathcal{E}(u, v) = \int_a^b \nabla u \nabla v\, dx, \tag{1.1.5}$$

with domain equal to $H^1_0(a, b)$ in the Dirichlet case. Since the embeddings $H^1_0(a, b) \hookrightarrow L^2_\mu[a, b]$ and $H^1(a, b) \hookrightarrow L^2_\mu[a, b]$ are compact, $\mathcal{E}$ is closed. Thus there exists a non-negative self-adjoint operator T on $L^2_\mu[a, b]$ such that $\text{Dom}\,(\mathcal{E}) = \text{Dom}\,(T^{1/2})$ and

$$E(u, v) = (T^{1/2}u, T^{1/2}v)_\mu \qquad \text{for all } u, v \in \text{Dom}\,(\mathcal{E}),$$

where

$$(u, v)_\mu := \int uv \, d\mu$$

denotes the inner product in $L^2([a, b], d\mu)$. Let $\| \cdot \|_\mu$ denote the corresponding norm.

We define $\Delta_\mu := -T$ and call it the *Dirichlet Laplacian with respect to* μ if $\mathrm{Dom}\,(\mathcal{E}) = H_0^1(a, b)$.

Let $u \in \mathrm{Dom}\,(\mathcal{E})$ and $f \in L^2_\mu[a, b]$. It is known that $u \in \mathrm{Dom}\,(\Delta_\mu)$ and $\Delta_\mu u = f$ if and only if $\Delta u = f d\mu$ in the sense of distribution, i.e.,

$$\int_a^b \nabla u \nabla v \, dx = \int_a^b (-\Delta u) v \, dx = \int_a^b (-fu) v \, d\mu = \int_a^b (-\Delta_\mu u) v \, d\mu$$

for all $v \in C_c^\infty(a, b)$.

It is known (see, e.g., [3, 9]) that there exists an orthonormal basis of eigenfunctions of Δ_μ and the eigenvalues $\{\lambda_n\}$ are discrete and satisfy $0 \le \lambda_1 < \lambda_2 < \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$.

The eigenvalues λ and eigenfunctions u are defined by the following equation:

$$\int \nabla u(x) \, \nabla v(x) \, dx = \lambda \int u(x) \, v(x) \, d\mu(x), \tag{1.1.6}$$

where the equation holds for all $v \in C_0^\infty(a, b)$.

The operators Δ_μ and their generalizations have been studied in connection with spectral functions of the string and diffusion processes . More recently, they have been studied in connection with fractal measures (see [3, 9, 20]).

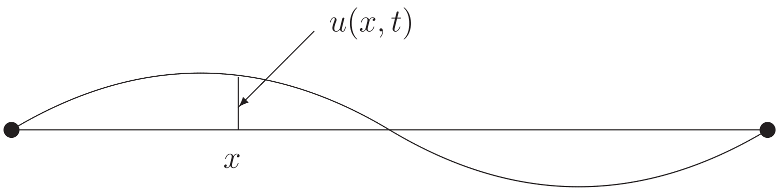
1.2 Main problems in our project

We first give a heuristic derivation of the general wave equation of a vibrating string with non-homogeneous mass density.(see e.g. [25]) In later chapters, we will study the properties of one-dimensional fractal wave equations, such as weak solutions and their regularity. Moreover, we will use numerical methods to find approximations to the weak solution of this type of equations.

We use the following notation: Let $u(x,t)$ be vertical displacement from equilibrium position at position x and time t.

Let $T(x,t)$ be magnitude of tension.

Let μ be measure representing mass density (non-constant).



Applying Newton's second law of motion, we get

$$\frac{T(x,t)}{\sqrt{1+u_x^2}}\Big|_{x=x_1} - \frac{T(x,t)}{\sqrt{1+u_x^2}}\Big|_{x=x_0} = 0, \tag{1.2.7}$$

and

$$\frac{T(x,t)u_x}{\sqrt{1+u_x^2}}\Big|_{x=x_1} - \frac{T(x,t)u_x}{\sqrt{1+u_x^2}}\Big|_{x=x_0} = \int_{x_0}^{x_1} u_{tt}\, d\mu. \tag{1.2.8}$$

Assume that the vertical displacement is small, i.e. $|u_x| << 1$. Then $\sqrt{1+u_x^2} = 1 + O(u_x^2)$. Therefore, (1.2.7) implies $T(x_1,t) \approx T(x_0,t)$, i.e., $T \approx$ constant, and (1.2.8) implies

$$T \cdot u_x\Big|_{x=x_1} - T \cdot u_x\Big|_{x=x_0} \quad = \quad \int_{x_0}^{x_1} u_{tt}\, d\mu.$$

$$T \int_{x_0}^{x_1} u_{xx}\, dx = \int_{x_0}^{x_1} u_{tt}\, d\mu.$$

Therefore,

$$Tu_{xx} = u_{tt}\, d\mu.$$

Normalize so that T=1. Suppose we let $\Delta_\mu u = f \quad \Leftrightarrow \quad \Delta u = f d\mu$

Then,

$$\Delta_\mu u \quad = \quad u_{tt}\,.$$

If μ is Lebesgue measure, this reduces to the standard wave equation $\Delta u = u_{tt}$.

1.3 Main result

The main purpose of this thesis is to study the one-dimensional wave equations defined by a class of fractal Laplacians.

In Chapter 2, we proved that the existent and uniqueness of solution of these kind equations. The most important result of Chapter 2 is as follows:

Theorem 1.3.1. *There exists a constant $C > 0$, depending only on U and T, such* *that for all* $m \in \mathbb{N}$,

$$\begin{aligned}&\max_{0\le t\le T}\Big(\|u_m(x,t)\|_{\mathrm{Dom}\,(\mathcal{E})}+\|(u_m(x,t))_t\|_\mu\Big)+\|(u_m(x,t))_{tt}\|_{L^2([0,T],(\mathrm{Dom}\,(\mathcal{E}))')}\\ \le& C(\|f\|_{L^2([0,T],L^2_\mu[a,b])}+\|g\|^2_{\mathrm{Dom}\,(\mathcal{E})}+\|h\|^2_\mu).\end{aligned} \tag{1.3.9}$$

This helps us prove the existent and uniqueness of solution. In Chapter 3, we used the finite element method to approximate our weak solution. There is a main result in chapter 3:

If $u^m(x,t)$ be defined by

$$u^m(x,t) := \sum_{j=0}^{N^m}\beta_j(t)\phi_j(x), \quad \text{where } \beta_j(t) := \beta_{m,j}(t) \text{ and } \phi_j(x) := \phi_{m,j}(x) \tag{1.3.10}$$

where $\phi_0, \phi_1, \ldots, \phi_{N^m}$ are the standard piecewise linear finite element basis functions defined as:

$$\phi_k(x) = \begin{cases} \frac{x-x_{k-1}}{x_k-x_{k-1}} & \text{if} \quad x_{k-1}\le x\le x_k, \quad k=1,\ldots,N^m \\ \frac{x-x_{k+1}}{x_k-x_{k+1}} & \text{if} \quad x_k\le x\le x_{k+1}, \quad k=0,\ldots,N^m-1 \\ 0 & \text{otherwise.}\end{cases} \tag{1.3.11}$$

and satisfy:

$$-\int_a^b \nabla u^m(x,t)\nabla\phi_i(x)\,dx = \int_a^b {u^m}_t t(x,t)\phi_i(x)\,d\mu, \text{ for } i=0,\cdots,N^m. \tag{1.3.12}$$

Moreover,

$$\begin{cases} \mathbf{M}\dfrac{d^2\mathbf{w}}{dt^2} & = -\mathbf{K}\mathbf{w}, \quad t>0 \\ \mathbf{w}(0) & = \mathbf{w}_0, \quad \mathbf{w}'(0)=\mathbf{w}'_0. \end{cases} \tag{1.3.13}$$

Theorem 1.3.2. *Let* μ *be defined by (1.1.2) on* $\mathbb{R}$ *with* $\operatorname{supp}[\mu] = [a,b]$*and satisfies a family of second-order self-similar identities. Assume* $\mathbf{M}$ *is invertible. Then,* *(1.3.12) could be discretized into a systme of second order ordinary differential equations (1.3.13). Thus, it could be solved numerically by the finite element method.*

In Chapter 6, we proved the convergence of the approximation solution. The main result is as follows:

Theorem 1.3.3. *Let v be a absolute continuous function on $[a,b]$. Let $\pi_m = \{x_i\}_{i=0}^{N_m}$ be any partition of $[a,b]$. Then, $\left|v_(x) - \mathcal{P}_m v(x)\right| \le C \left\|\pi_m\right\|^{1/2}$ for all $x \in [a,b]$.*

This helps us to show: u^m defined in(1.3.10), converges in $L^2_\mu[a,b]$ to u.

CHAPTER 2

EXISTENCE AND UNIQUENESS OF WEAK SOLUTION

In this section, we modify the proof in [7], and replace the standard Laplacian Δ by the μ-Laplacian Δ_μ. Then, Our goal is to prove the existence and uniqueness of a weak solution of the following non-homogeneous hyperbolic initial/boundary value problem (IVBP):

$$\begin{cases} u_{tt} - \Delta_\mu u = f & \text{on } U_T = [a,b] \times [0,T], \\ u = 0 & \text{on } \{a,b\} \times [0,T], \\ u = g, \ u_t = h & \text{on } U \times \{t = 0\}. \end{cases} \tag{2.0.1}$$

Let $\mathcal{E}(u,v)$ be the quadratic form defined in (1.1.5), with domain $H_0^1(a,b)$. First of all, we will discuss the solution of abstract homogeneous wave equations.

Theorem 2.0.4. *(Shinbrot [22]) Let H be a complex Hilbert space. Define $u : \mathbb{R} \to H$, and let A be a self-adjoint operator on H satisfying*

$$(Au(t), u(t)) \geq 0 \text{ for each } t \text{ such that } u(t) \in \operatorname{Dom}(A). \tag{2.0.2}$$

Let $g \in \operatorname{Dom}(A), h \in \operatorname{Dom}(A^{\frac{1}{2}})$. Then the initial value problem

$$\ddot{u}(t) + Au(t) = 0, \tag{2.0.3}$$

$$u(0) = g, \quad \dot{u}(0) = h, \tag{2.0.4}$$

has a unique solution, given by $u(t) = \int_0^\infty \cos(t\sqrt{\lambda}) dE_\lambda g + \int_0^\infty \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} dE_\lambda h$, where $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is the spectral family associated with A .

Proof. See [22]. For completeness we also include some details in Appendix G. $\square$

Proposition 2.0.5. *If* $g \in \mathrm{Dom}\,(-\Delta_\mu)$ *and* $h \in \mathrm{Dom}\,((-\Delta_\mu)^{\frac{1}{2}})$ *then (2.0.1) has a unique solution, given by* $u(t) = \int_0^\infty \cos(t\sqrt{\lambda})dE_\lambda g + \int_0^\infty \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}dE_\lambda h$, *where* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *is the spectral family associated with* $-\Delta_\mu$ *.*

Proof. Since g and h are supported on $[a, b]$ and vanish at the end points a and b. The proposition follows from theorem 2.0.4. □

Remark: In general, the classical wave equation may not have a solution unless f has the proper Darboux structure. (See [4]).

Definition 2.0.1. *(i) A function* $s : [0, T] \to X$ *is called* simple *if it has the form*

$$s(t) = \sum_{m=1}^{N} \chi_{E_m}(t)u_m \text{ for } t \in [0, T], \tag{2.0.5}$$

where each E_m *is a Lebesgue measurable subset of* $[0, T]$ *and* $u_m \in X (m = 1, \ldots, N)$.

(ii) A function $u : [0, T] \to X$ *is* strongly measurable *if there exist simple function* $s_N : [0, T] \to X$ *such that*

$$s_N(t) \to u(t) \text{ for Labesgue a.e. } (0 \le t \le T).$$

(iii) A function $u : [0, T] \to X$ *is* weakly measurable *if for each* $u^* \in X^*$, *the mapping* $t \mapsto \langle u^*, u(t)\rangle$ *is Lebesgue measurable.*

Definition 2.0.2. *We say* $u : [0, T] \to X$ *is* almost separably valued *if there exists a subset* $E \subset [0, T]$, *with* $\mathcal{L}(E) = 0$, *such that the set* $\{u(t)|t \in [0, T]\backslash E\}$ *is separable.*

Theorem 2.0.6. *(Pettis [21]). The mapping* $u : [0, T] \to X$ *is strongly measurable if and only if* u *is weakly measurable and is almost separably valued.*

Definition 2.0.3. *Let X be a separable Banach space with norm $\| \ \|_X$. Then define $L^p([0,T];X)$ to be the space of all* measurable *functions $u : [0,T] \to X$ satisfying*

(i) $\|u\|_{L^p([0,T];X)} := (\int_0^T \|u(\cdot,t)\|^p \, dt)^{\frac{1}{p}} < \infty$ *for* $1 \le p < \infty$, *and*

(ii) $\|u\|_{L^\infty([0,T];X)} := \operatorname{ess\,sup}_{0 \le t \le T} \|u(t)\|_X < \infty$.

Remark: For $1 \le p \le \infty$, $L^p([0,T];X)$ are Banach spaces. (See Appendix B.)

Definition 2.0.4. $g, h \in \mathrm{Dom}\,(\mathcal{E})$, $f \in L^2([0,T];\mathrm{Dom}\,(\mathcal{E}))$, *A function* $u \in L^2([0,T];\mathrm{Dom}\,(\mathcal{E}))$, *with* $u_t \in L^2([0,T];L^2_\mu[a,b])$ *and* $u_{tt} \in L^2([0,T];(\mathrm{Dom}\,(\mathcal{E}))')$ *is a* weak solution *of IBVP (2.0.1) if the following conditions are satisfied:*

(i) $\langle u_{tt}, v\rangle + \mathcal{E}(u,v) = (f,v)_\mu$ *for each* $v \in \mathrm{Dom}\,(\mathcal{E})$, $f \in L^2_\mu[a,b]$ *and Lebesgue a.e.* $t \in [0,T]$;

(ii) $u(x,0) = g(x)$ *and* $u_t(x,0) = h(x)$ *for all* $x \in [a,b]$.

Let $\{w_k\}_{k=1}^\infty \subset C^1[a,b]$ be an orthonormal basis of $L^2_\mu[a,b]$ consisting of the eigenfunctions of $-\Delta_\mu$ with eigenvalues $\{\lambda_k\}_{k=1}^\infty$. Then

$$\int_a^b \nabla w_k \nabla v \, dx = \lambda_k \int_a^b w_k \, v \, d\mu, \quad \forall v \in \mathrm{Dom}\,(\mathcal{E}).$$

The existence of such an orthonormal basis of eigenfunctions of Δ_μ can be found in [3].

Fix a positive integer m, and define

$$u_m(x,t) := \sum_{k=1}^m \alpha_{m,k}(t) w_k(x), \tag{2.0.6}$$

where we will show that the coefficients $\{\alpha_{m,k}(t)\}_{k=1}^{m}$ can be chosen to belong to $C^2(0,T)$ and satisfy

$$\alpha_{m,k}(0) = (g, w_k)_\mu, \qquad k = 1, \dots, m \tag{2.0.7}$$

$$\alpha'_{m,k}(0) = (h, w_k)_\mu, \qquad k = 1, \dots, m \tag{2.0.8}$$

and

$$\Big((u_m)_{tt}, w_k\Big)_\mu + \mathcal{E}(u_m, w_k) = (f, w_k)_\mu, \qquad 0 \le t \le T, \quad k = 1, \dots, m. \tag{2.0.9}$$

Theorem 2.0.7. *For each $m \in \mathbb{N}$, there exists a unique function u_m of the form (2.0.6) satisfying (2.0.7)-(2.0.9).*

Proof. Let $u_m(x,t)$ be defined in (2.0.6). Then, we have

$$\Big((u_m)_{tt}, w_k\Big)_\mu = \alpha''_{m,k}(t). \tag{2.0.10}$$

Moreover, $\mathcal{E}(u_m, w_k) = \sum_{j=1}^{m} \mathcal{E}(w_j, w_k)\alpha_{m,j}(t)$, $j,k = 1, \dots, m$, and $f_k := (f, w_k)_\mu$. Consequently, (2.0.9) is discretized into the linear system of ODEs:

$$\alpha''_{m,k}(t) - \sum_{j=1}^{m} \mathcal{E}(w_j, w_k)\alpha_{m,j}(t) = f_k, \qquad 0 \le t \le T,\ k = 1, \dots, m,$$

or

$$\alpha''_{m,k}(t) - \lambda_k \alpha_{m,k}(t) = f_k, \qquad 0 \le t \le T,\ k = 1, \dots, m, \tag{2.0.11}$$

with the initial conditions (2.0.7) and (2.0.8). There exists a unique C^2 vector-valued function $\alpha''_m(t) = (\alpha''_{m,1}(t), \dots, \alpha''_{m,m}(t))$ satisfying (2.0.7), (2.0.8) and (2.0.11) for $0 \le t \le T$, namely

$$\alpha_{m,k}(t) = (g, w_k)\cos(\sqrt{\lambda_k} t) + P_{m,k}(t),$$

where $P_{m,k}(t)$ is the particular solution that depends on $f(t,x)$. □

We now need to take the limit as $m \to \infty$. To this end we need the following estimates for u_m, together with its partial derivatives with respect to time, that are uniform in m.

Theorem 2.0.8. *There exists a constant $C > 0$, depending only on U and T, such that for all $m \in \mathbb{N}$,*

$$\max_{0\le t\le T}\Big(\|u_m(x,t)\|_{\mathrm{Dom}\,(\mathcal{E})} + \|(u_m(x,t))_t\|_\mu\Big) + \|(u_m(x,t))_{tt}\|_{L^2([0,T],(\mathrm{Dom}\,(\mathcal{E}))')}$$
$$\le C(\|f\|_{L^2([0,T],L^2_\mu[a,b])} + \|g\|^2_{\mathrm{Dom}\,(\mathcal{E})} + \|h\|^2_\mu). \tag{2.0.12}$$

Proof. Multiplying equality (2.0.9) by $\alpha'_{m,k}(t)$, we have

$$\Big((u_m)_{tt}, \alpha'_{m,k}(t)w_k\Big)_\mu + \mathcal{E}\Big(u_m, \alpha'_{m,k}(t)w_k\Big) = \Big(f, \alpha'_{m,k}(t)w_k\Big)_\mu, \qquad k = 1,\dots,m.$$

Summing the k equations up, we have

$$\Big((u_m)_{tt}, \sum_{k=1}^m \alpha'_{m,k}(t)w_k\Big)_\mu + \mathcal{E}\Big(u_m, \sum_{k=1}^m \alpha'_{m,k}(t)w_k\Big) = \Big(f, \sum_{k=1}^m \alpha'_{m,k}(t)w_k\Big)_\mu.$$

That is,

$$\Big((u_m)_{tt}, (u_m)_t\Big)_\mu + \mathcal{E}\Big(u_m, (u_m)_t\Big) = \Big(f, (u_m)_t\Big)_\mu. \tag{2.0.13}$$

Since

$$\frac{d}{dt}(\frac{1}{2}\|(u_m)_t\|^2_\mu) = ((u_m)_{tt}, (u_m)_t)_\mu$$

and

$$\frac{d}{dt}(\frac{1}{2}\mathcal{E}(u_m, u_m)) = \mathcal{E}(u_m, (u_m)_t),$$

from (2.0.13) we have

$$\frac{d}{dt}\Big(\frac{1}{2}\|(u_m)_t\|^2_\mu + \frac{1}{2}\mathcal{E}(u_m,u_m)\Big) = \Big(f, (u_m)_t\Big)_\mu \le \|(u_m)_t\|_\mu \|f\|_\mu \le \frac{1}{2}\|(u_m)_t\|^2_\mu + \frac{1}{2}\|f\|^2_\mu.$$

Consuquently,

$$\frac{d}{dt}\Big(\|(u_m)_t\|_\mu^2 + \mathcal{E}(u_m, u_m)\Big) \le \|(u_m)_t\|_\mu^2 + \mathcal{E}(u_m, u_m) + \|f\|_\mu^2 .$$

Since $\|(u_m)_t\|_\mu^2$ and $\mathcal{E}(u_m, u_m(x,t))$ are absolutely continuous functions (see Appdeix C1), Gronwall's inequality, yields the estimate:

$$\begin{aligned} &\|(u_m(x,t))_t\|_\mu^2 + \mathcal{E}(u_m(x,t), u_m(x,t)) \\ &\qquad \le e^t\Big(\|(u_m(x,0))_t\|_\mu^2 + \mathcal{E}(u_m(x,0), u_m(x,0)) + \int_0^t \|f(s)\|_\mu^2 \, ds\Big). \end{aligned} \tag{2.0.14}$$

Moreover, for the first term on right hand side of (2.0.14), we have

$$\begin{aligned} \|(u_m(x,0))_t\|_\mu^2 &= \int_a^b \Big(\sum_{k=1}^m \alpha'_{m,k}(0) w_k(x)\Big)^2 d\mu = \sum_{k=1}^m \int_a^b (\alpha'_{m,k}(0))^2 (w_k(x))^2 \, d\mu \\ &= \sum_{k=1}^m (\alpha'_{m,k}(0))^2. \end{aligned}$$

By expressing $h(x) = \sum_{k=1}^\infty \alpha'_{m,k}(0) w_k(x)$, we have for all $m \in \mathbb{N}$,

$$\begin{aligned} \|h\|_\mu^2 &= \lim_{n\to\infty} \Big(\sum_{k=1}^n \alpha'_{n,k}(0) w_k(x), \sum_{k=1}^n \alpha'_{n,k}(0) w_k(x)\Big) = \lim_{n\to\infty} \sum_{k=1}^n (\alpha'_{n,k}(0))^2 \\ &\ge \|(u_m(x,0))_t\|_\mu^2 . \end{aligned} \tag{2.0.15}$$

Also, for the second term on right hand side of (2.0.14), we have

$$\begin{aligned} \mathcal{E}(u_m(x,0), u_m(x,0)) =& \mathcal{E}\Big(\sum_{k=1}^m \alpha_{m,k}(0) w_k, \sum_{k=1}^m \alpha_{m,k}(0) w_k\Big) \\ =& \sum_{k=1}^m \mathcal{E}(\alpha_{m,k}(0) w_k, \alpha_{m,k}(0) w_k) = \sum_{k=1}^m (\alpha_{m,k}(0))^2 \lambda_k. \end{aligned} \tag{2.0.16}$$

By expressing $g(x) = \sum_{k=1}^\infty \alpha_{m,k}(0) w_k(x)$, we obtain

$$\begin{aligned} \mathcal{E}(g,g) &= \lim_{n\to\infty} \mathcal{E}\Big(\sum_{k=1}^n \alpha_{n,k}(0) w_k, \sum_{k=1}^n \alpha_{n,k}(0) w_k\Big) = \lim_{n\to\infty} \sum_{k=1}^n (\alpha_{n,k}(0))^2 \mathcal{E}(w_k, w_k) \\ &= \lim_{n\to\infty} \sum_{k=1}^n (\alpha_{n,k}(0))^2 \lambda_k. \end{aligned} \tag{2.0.17}$$

Combining (2.0.16) and (2.0.17), we have, for all $m \in \mathbb{N}$,

$$\mathcal{E}(u_m(x,0), u_m(x,0)) \leq \mathcal{E}(g,g). \tag{2.0.18}$$

Now, combining (2.0.14), (2.0.15) and (2.0.18), we get, for all $m \in \mathbb{N}$,

$$\|(u_m(x,t))_t\|_\mu^2 + \mathcal{E}\big(u_m(x,t), u_m(x,t)\big) \leq e^t \Big(\|h\|_\mu^2 + \mathcal{E}(g,g) + \|f(t)\|^2_{L^2(0,T,L^2_\mu[a,b])} \Big).$$

Since $0 \leq t \leq T$ was arbitrary, we see from this estimate that

$$\max_{0 \leq t \leq T} \Big(\|(u_m(x,t))_t\|_\mu^2 + \mathcal{E}(u_m(x,t), u_m(x,t)) \Big) \leq e^t (\|h\|_\mu^2 + \mathcal{E}(g,g) + \|f(t)\|^2_{L^2(0,T,L^2_\mu[a,b])}). \tag{2.0.19}$$

Next, we fix any $v \in \operatorname{Dom}(\mathcal{E})$, with $\|v\|_{\operatorname{Dom}(\mathcal{E})} \leq 1$, and write $v = v_1 + v_2$, where $v_1 \in$ span $\{w_k\}_{k=1}^m$ and $(v_2, w_k)_\mu = 0$ $(k = 1, 2, \ldots, m)$. Then, $\|v_1\|_{\operatorname{Dom}(\mathcal{E})} \leq 1$, (2.0.6) and (2.0.9) imply

$$\langle (u_m)_{tt}, v\rangle := \Big((u_m)_{tt}, v\Big)_\mu = \Big((u_m)_{tt}, v_1\Big)_\mu = (f, v_1)_\mu - \mathcal{E}(u_m, v_1).$$

So, $|\langle (u_m)_{tt}, v\rangle| \leq |(f, v_1)_\mu - \mathcal{E}(u_m, v_1)|$, which implies that

$$\|\langle (u_m)_{tt}, v\rangle\|_{\operatorname{Dom}(\mathcal{E}))'} \leq C\,\|f\|_\mu + \sqrt{\mathcal{E}(u_m, u_m)}.$$

Thus, we have

$$\|(u_m)_{tt}\|^2_{(\operatorname{Dom}(\mathcal{E}))'} \leq C(\|f\|_\mu^2 + \mathcal{E}(u_m, u_m)).$$

From (2.0.19), this implies

$$\int_0^T \|(u_m)_{tt}\|^2_{(\operatorname{Dom}(\mathcal{E}))'}\, dt \leq C(\|f\|_{L^2([0,T],L^2_\mu[a,b])} + \|h\|_\mu^2 + \mathcal{E}(g,g)). \tag{2.0.20}$$

Remark □

Definition 2.0.5. *Let X be a Banach space.*

(i) The space $C([0,T];X)$ *comprises all* continuous *functions* $u : [0,T] \to X$ *with*

$$\|u\|_{C([0,T];X)} := \max_{0 \le t \le T} \|u\|_X < \infty,$$

(ii) The space $C^1([0,T];X)$ *comprises all* C^1 *functions* $u : [0,T] \to X$ *with*

$$\|\dot{u}\|_{C([0,T];X)} := \max_{0 \le t \le T} \|\dot{u}\|_X < \infty,$$

(iii) The space $C^2([0,T];X)$ *comprises all* C^2 *functions* $u : [0,T] \to X$ *with*

$$\|\ddot{u}\|_{C([0,T];X)} := \max_{0 \le t \le T} \|\ddot{u}\|_X < \infty.$$

Definition 2.0.6. *Let* $u \in L^1([0,T];X)$. *We say* $v \in L^1([0,T];X)$ *is the* weak derivative *of* u, *written*

$$u_t = v,$$

if

$$\int_0^T \phi_t(t)u(t)dt = -\int_0^T \phi(t)u_t(t)dt.$$

for all scalar test function $\phi(t) \in C_c^\infty(0,T)$.

Definition 2.0.7. *Let* X *be a Banach space. We say a sequence* $\{u_m\}_{m=1}^\infty \subset X$ converges weakly to $u \in X$, *written*

$$u_m \rightharpoonup u,$$

if

$$\langle u^*, u_m \rangle \to \langle u^*, u \rangle$$

for each bounded linear functional $u^* \in X^*$.

Theorem 2.0.9. *Assume* $g \in \mathrm{Dom}\,(\mathcal{E}), h \in L^2_\mu[a,b]$ *and* $f \in L^2([0,T];L^2_\mu[a,b])$. *Then the IBVP (2.0.1) has a weak solution.*

Proof. From the energy estimate (2.0.12), we know that $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^2([0,T];\mathrm{Dom}\,(\mathcal{E}))$, $\{(u_m)_t\}_{m=1}^{\infty}$ is bounded in $L^2([0,T];L^2_\mu[a,b])$, and $\{(u_m)_{tt}\}_{m=1}^{\infty}$ is bounded in $L^2(([0,T];\mathrm{Dom}\,(\mathcal{E}))')$.

So, by Banach Alaoglu theorem, there exists a subsequence $\{u_{m_l}(x,t)\}_{l=1}^{\infty}$ and $u \in L^2([0,T];\mathrm{Dom}\,(\mathcal{E}))$, with $u_t \in L^2([0,T];L^2_\mu[a,b])$, and $u_{tt} \in L^2([0,T];(\mathrm{Dom}\,(\mathcal{E}))')$ such that

$$\begin{cases} u_{m_l}(x,t) & \rightharpoonup \quad u(x,t) \quad \text{in } L^2([0,T];\mathrm{Dom}\,(\mathcal{E})), \\ \\ (u_{m_l}(x,t))_t & \rightharpoonup \quad u_{tt}(x,t) \quad \text{in } L^2([0,T];L^2_\mu[a,b]), \\ \\ (u_{m_l}(x,t))_{tt} & \rightharpoonup \quad u_{tt}(x,t) \quad \text{in } L^2([0,T];(\mathrm{Dom}\,(\mathcal{E}))'). \end{cases} \tag{2.0.21}$$

Remark: Originally, $\frac{\partial u_{m_l}(x,t)}{\partial t}$ weakly converge to some γ in $L^2([0,T];L^2_\mu[a,b])$ and $\frac{\partial^2 u_{m_l}(x,t)}{\partial t^2}$ weakly converge to some σ in $L^2([0,T];(\mathrm{Dom}\,(\mathcal{E}))')$. It can be proved that $\frac{\partial u(x,t)}{\partial t} = \gamma$ and $\frac{\partial^2 u(x,t)}{\partial t^2} = \sigma$. (See Appendix E).

Now, we fix an integer N and choose a function $v \in C^1([0,T];\mathrm{Dom}\,(\mathcal{E}))$ of the form

$$v(x,t) = \sum_{k=1}^{N} \alpha_k(t) w_k(x), \text{ where } \{\alpha_k\}_{k=1}^{N} \subset C^2[0,T]. \tag{2.0.22}$$

We select $m \geq N$, multiply (2.0.9) by $\alpha(t)$, sum $k = 1, \ldots, N$, and then integrate with respect to t to get

$$\int_0^T \Big(\langle (u_m(x,t))_{tt}, v(x,t) \rangle + \mathcal{E}(u_m(x,t), v(x,t)) \Big)\, dt = \int_0^T (f(x,t), v(x,t))_\mu \, dt. \tag{2.0.23}$$

Setting $m = m_l$, letting l tend to ∞, and using (2.0.21), we have

$$\int_0^T \Big(\langle (u_{tt}(x,t)), v(x,t) \rangle + \mathcal{E}(u(x,t), v(x,t)) \Big)\, dt = \int_0^T (f(x,t), v(x,t))_\mu \, dt. \tag{2.0.24}$$

Since $\{w_k\}_{k=1}^{\infty}$ is a basis of $\mathrm{Dom}\,(\mathcal{E})$, the set of functions of the form (2.0.22) is dense in $L^2([0,T];\mathrm{Dom}\,(\mathcal{E}))$, and thus (2.0.24) holds for all $v \in L^2([0,T];\mathrm{Dom}\,(\mathcal{E}))$. Then, (2.0.24) implies

$$\langle u_{tt}(x,t), v(x,t)\rangle + \mathcal{E}(u(x,t), v(x,t)) = (f(x,t), v(x,t))_\mu$$

for all $v \in \mathrm{Dom}\,(\mathcal{E})$ and a.e. $t \in [0,T]$.

Next we will verify

$$u(x,0) = g(x) \qquad \text{and} \qquad u_t(x,0) = h(x). \tag{2.0.25}$$

For this, in (2.0.24), we choose any function $v \in C^2([0,T];\mathrm{Dom}\,(\mathcal{E}))$, with $v(T) = v_t(T) = 0$. Integrating by parts twice with respect to t for the first term of (2.0.24), we get

$$\begin{aligned}&\int_0^T \Big(\langle u(x,t), v_{tt}(x,t)\rangle + \mathcal{E}(u(x,t), v(x,t))\Big)\, dt \\ = &\int_0^T (f(x,t), v(x,t))_\mu\, dt - (u(x,0), v_t(x,0)) + (u_t(x,0), v(x,0)).\end{aligned} \tag{2.0.26}$$

Similar to (2.0.23),

$$\begin{aligned}&\int_0^T (\langle u_m(x,t), v_t(x,t)\rangle + \mathcal{E}(u_m(x,t), v(x,t)))\, dt \\ = &\int_0^T (f(x,t), v(x,t))_\mu\, dt - (u_m(x,0), v_t(x,0)) + (u_{m,t}(x,0), v(x,0)).\end{aligned}$$

Setting $m = m_l$ and combining (2.0.7), (2.0.8) and (2.0.21), we have

$$\begin{aligned}&\int_0^T \Big(\langle u(x,t), \frac{\partial^2 v(x,t)}{\partial t^2}\rangle + \mathcal{E}(u(x,t), v(x,t))\Big)\, dt \\ = &\int_0^T (f, v(x,t))_\mu\, dt - (g(x), v_t(x,0)) + (h(x), v(x,0)).\end{aligned} \tag{2.0.27}$$

Comparing (2.0.26) and (2.0.27), and noting that $v(x,0)$ and $v_t(x,0)$ were arbitrary, we conclude that (2.0.25) holds. Therefore, $u(x,t)$ is a weak solution of (2.0.1). $\square$

Theorem 2.0.10. *Assume the same hypotheses of Theorem 2.0.9. Then the weak solution of the* IBVP *(2.0.1) is unique.*

Proof. To show this, it suffices to show that the only weak solution of (2.0.1) with $g(x) = h(x) = f(x,t) = 0$ is $u \equiv 0$ in $L^2([0,T]; \mathrm{Dom}\,(\mathcal{E}))$. To show this, fix $0 \le r \le T$ and set

$$v(x,t) := \begin{cases} \int_t^s u(x,\tau)d\tau & \text{if} \quad 0 \le t \le s, \\ 0 & \text{if} \quad s \le t \le T. \end{cases}$$

Then $v(x,t) \in \mathrm{Dom}\,(\mathcal{E})$ for each $t \in [0,T]$ and so

$$\int_0^s \Big(\langle u_{tt}(x,t), v(x,t) \rangle + \mathcal{E}(u(x,t), v(x,t)) \Big)\, dt = 0. \tag{2.0.28}$$

We have $u_t(x,0) = 0$ and by the definition of $v(x,t)$, we have $v(x,s) = 0$. Integrating by parts in the first term of (2.0.28), we obtain:

$$-\int_0^s \Big(\langle u_{tt}(x,t), v_t(x,t) \rangle + \mathcal{E}(u(x,t), v(x,t)) \Big)\, dt = 0.$$

Moreover, we have $v_t(x,t) = -u(x,t)$ for $0 \le t \le s$. Hence

$$\begin{aligned} & \int_0^s \Big(\langle u_t(x,t), u(x,t) \rangle - \mathcal{E}(v_t(x,t), v(x,t)) \Big)\, dt = 0 \\ \Rightarrow \quad & \int_0^s \frac{\partial}{\partial t} \Big(\frac{1}{2} \|u(x,t)\|_\mu^2 - \frac{1}{2} \mathcal{E}(v(x,t), v(x,t))) \Big)\, dt = 0 \\ \Rightarrow \quad & \Big[\frac{1}{2} \|u(x,t)\|_\mu^2 - \frac{1}{2} \mathcal{E}(v(x,t), v(x,t)) \Big]_0^s = 0 \quad \text{for a.e. } t \in [0,T]. \end{aligned}$$

$$\|u(x,s)\|_\mu^2 - \mathcal{E}(v(x,s), v(x,s))) = \|u(x,0)\|_\mu^2 - \mathcal{E}(v(x,0), v(x,0)).$$

Since $u(x,0) = v(x,s) = 0$, $\|u(x,s)\|_\mu^2 + \mathcal{E}(v(x,0), v(x,0)) = 0$. This implies

$$u(x,s) = 0 \quad \text{for a.e. } x \in [a,b] \text{ and for all } s \in [0,T]$$

and $u(x,t)$ is continuous in $\mathrm{Dom}\,(\mathcal{E})$. Thus $u(x,t) \equiv 0$. □

CHAPTER 3

THE FINITE ELEMENT METHOD

In this section, we use the finite element method to solve the homogeneous IBVP (2.0.1).

Multiplying the first equation in (2.0.1), where $f(x,t) = 0$, by $v \in \mathrm{Dom}\,(\mathcal{E})$, integrating both sides, and then using integration by parts, we obtain

$$-\int_a^b \nabla u(x,t)\,\nabla v(x,t)\,dx = \int_a^b u_{tt}(x,t)\,v(x,t)\,d\mu. \tag{3.0.1}$$

Next, we apply the finite element method to approximate $u(x,t)$ by

$$u^m(x,t) = \sum_{j=0}^{N^m} \beta_j(t)\phi_j(x), \qquad \text{where } \beta_j(t) := \beta_{m,j}(t) \text{ and } \phi_j(x) := \phi_{m,j}(x) \tag{3.0.2}$$

and $\phi_0, \phi_1, \ldots, \phi_{N^m}$ are the standard piecewise linear finite element basis functions defined as:

$$\phi_0(x) = \begin{cases} \frac{x-x_1}{x_0-x_1} & \text{if} \quad x_0 \le x \le x_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_k(x) = \begin{cases} \frac{x-x_{k-1}}{x_k-x_{k-1}} & \text{if} \quad x_{k-1} \le x \le x_k, \qquad 1 \le k \le N^m - 1 \\ \frac{x-x_{k+1}}{x_k-x_{k+1}} & \text{if} \quad x_k \le x \le x_{k+1} \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_{N^m}(x) = \begin{cases} \frac{x-x_{N^m-1}}{x_{N^m}-x_{N^m-1}} & \text{if} \quad x_{N^m-1} \le x \le x_{N^m} \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently,

$$\phi_k(x) = \begin{cases} \frac{x-x_{k-1}}{x_k-x_{k-1}} & \text{if} \quad x_{k-1} \le x \le x_k, \quad k = 1, \ldots, N^m \\ \frac{x-x_{k+1}}{x_k-x_{k+1}} & \text{if} \quad x_k \le x \le x_{k+1}, \quad k = 0, \ldots, N^m - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.0.3}$$

We require $u^m(x,t)$ to satisfy:

$$-\int_a^b \nabla u^m(x,t)\nabla\phi_i(x)\,dx = \int_a^b u^m_{tt}(x,t)\phi_i(x)\,d\mu, \text{ for } i = 0,\cdots,N^m. \qquad (3.0.4)$$

Substituting (3.0.2) into (3.0.4) and using $u(a,t) = u(b,t) = 0$, we get

$$\sum_{j=1}^{N^m-1} \beta_j(t)\left\{-\int_a^b \nabla\phi_j(x)\,\nabla\phi_i(x)\,dx\right\} = \sum_{j=1}^{N^m-1} \beta_j''(t)\left\{\int_a^b \phi_j(x)\phi_i(x)\,d\mu\right\}. \qquad (3.0.5)$$

We define the matrices **M** and **K** (the *mass* and *stiffness matrices*, respectively) by

$$\mathbf{M}_{ij} = \int_a^b \phi_j(x)\phi_i(x)\,d\mu, \qquad \mathbf{K}_{ij} = -\int_a^b \frac{d\phi_j}{dx}(x)\frac{d\phi_i}{dx}(x)\,dx, \qquad (3.0.6)$$

and the vector-valued function $\mathbf{w}(t)$ by

$$\mathbf{w}(t) = \begin{bmatrix} \beta_1(t) \\ \vdots \\ \beta_{N^m-1}(t) \end{bmatrix}.$$

Then (3.0.5) can be put in a matrix form as

$$\mathbf{M}\mathbf{w}'' + \mathbf{K}\mathbf{w} = 0. \qquad (3.0.7)$$

Equivalently,

$$\mathbf{M}\mathbf{w}'' = -\mathbf{K}\mathbf{w}. \qquad (3.0.8)$$

We have a system of second-order linear ODEs (3.0.8) with constant coefficients. We need two initial conditions, and they are obtained from the initial conditions in (2.0.1). We have

$$u(x,0) = g(x), \qquad 0 < x < 1,$$

and $g(x)$ can be approximated by its linear interpolant:

$$\tilde{g}(x) = \sum_{i=1}^{N^m-1} g(x_i)\phi_i(x).$$

Therefore, we set

$$\mathbf{w}_i(0) = g(x_i).$$

Similarly, we set

$$\mathbf{w}_i'(0) = h(x_i).$$

These lead to the initial conditions

$$\mathbf{w}(0) = \mathbf{w}_0 = \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_{N^m-1}) \end{bmatrix}, \qquad \mathbf{w}'(0) = \mathbf{w}_0' = \begin{bmatrix} h(x_1) \\ \vdots \\ h(x_{N^m-1}) \end{bmatrix}$$

Therefore, we get the IVP

$$\begin{cases} \mathbf{M}\dfrac{d^2\mathbf{w}}{dt^2} &= -\mathbf{K}\mathbf{w}, \quad t > 0 \\ \\ \mathbf{w}(0) &= \mathbf{w}_0, \quad \mathbf{w}'(0) = \mathbf{w}_0'. \end{cases} \tag{3.0.9}$$

Theorem 3.0.11. *Define* $\mathbf{M}$ *by (3.0.6). If for* $i = 1, \ldots, N^m - 1, \mathbf{M}_{i,i} - \mathbf{M}_{i,i-1} - \mathbf{M}_{i,i+1} \geq 0$*, then* $\mathbf{M}$ *is strictly diagonally dominant matrices. Thus, (3.0.9) has a unique solution* $\mathbf{w}(t)$.

In the case of the infinite Bernoulli convolution associated with the golden ratio, and the 3-fold convolutions of the Cantor measure, we prove $\mathbf{M}$ is strictly diagonally dominant. (See [27].) The proof is shown in Appendix A.3.

Remarks: From this theorem, we know that for $j = 1, \cdots, N^m - 1, \beta_j(t)$ are C^2 functions on $(0, T)$.

Remarks: will move to below of the following paragraph.

In the rest of this thesis we will consider self-similar measures μ defined by IFSs of the form $\{S_i\}_{i=1}^N$ as given in (1.1.1) and (1.1.2). We assume that μ satisfies a family of second-order self-similar identities with respect to $\{T_j\}_{j=1}^N$.

Since $T_J[a,b]$ can represent each m-level interval $T_J[a,b]$, where $J=(j_1,\ldots,j_m)$ and $j_k \in \{1,\ldots,N\}$, can be written as $[x_{i-1},x_i]$, where the index k can be obtained directly from J as follows (see [5]):

$$i = i(J) := (j_1-1)N^{m-1} + (j_2-1)N^{m-2} + \cdots + (j_m-1)N^0 + 1. \; .$$

For example, if $J=(1,\ldots,1)$, then $i(J)=1$, and if $J=(N,\ldots,N)$, then $i(J)=N^m$. It follows that

$$T_{J_i} := T_J[a,b] = [x_{i-1},x_i], \quad \text{or} \quad T_{J_i}(x) := T_J(x) = (x_i - x_{i-1})\frac{x-a}{b-a} + x_{i-1}. \quad (3.0.10)$$

We define $c_i^j := c_J^j$ for $j=1,\ldots,N$ and $i=1,\ldots,N^m$.

By assumption, we can evaluate the measure of each 1-level interval, *i.e.*, $\mu(T_j[a,b]) = \int d\mu\circ T_j$, and the integrals of $\int x\,d\mu\circ T_j$ and $\int x^2\,d\mu\circ T_j$, $j=1,\ldots,N$.

Theorem 3.0.12. *Let μ be defined by (1.1.2) on $\mathbb{R}$ with $\operatorname{supp}[\mu]=[a,b]$ and assume that μ satisfies a family of second-order self-similar identities. Assume the mass matrix $\mathbf{M}$ is invertible. Then (3.0.4) can be discretized into a system of second order ordinary differential equations0 (3.0.9). Thus, it can be solved numerically by the finite element method.*

We remark that

$$\mathbf{M}_{ii} = \int_a^b \phi_i(x)\phi_i(x)\,d\mu,$$

while,

$$\mathbf{M}_{ii} = \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{x_i - x_{i-1}} \frac{x - x_{i-1}}{x_i - x_{i-1}} \, d\mu + \int_{x_i}^{x_{i+1}} \frac{x - x_{i+1}}{x_i - x_{i+1}} \frac{x - x_{i+1}}{x_i - x_{i+1}} \, d\mu$$
$$= \int_a^b (\frac{x-a}{b-a})^2 \, d\mu(T_{J_i}(x)) + \int_a^b (-\frac{x-a}{b-a} + 1)^2 \, d\mu(T_{J_{i+1}}(x)).$$

For change of variable, formula, see Appendeix $A.3$.

Method 1. We let $\mathbf{w}_n := \mathbf{w}(t_n), \quad n \geq -1$ and use the *central difference method (CDM)* to solve the IVP (3.0.9).

Using the approximations

$$\frac{d^2\mathbf{w}(t_n)}{dt^2} \approx \frac{\mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1}}{(\Delta t)^2} \qquad \text{and} \qquad \mathbf{w}'(t_n) \approx \frac{\mathbf{w}_{n+1} - \mathbf{w}_{n-1}}{2\Delta t}. \tag{3.0.11}$$

Substituting (3.0.11) into (3.0.8) yields

$$\frac{\mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1}}{(\Delta t)^2} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{w}_n$$
$$\mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1} = -(\Delta t)^2\mathbf{M}^{-1}\mathbf{K}\mathbf{w}_n$$
$$\mathbf{w}_{n+1} = (2\mathbf{I} - (\Delta t)^2\mathbf{M}^{-1}\mathbf{K})\mathbf{w}_n - \mathbf{w}_{n-1}.$$

Moreover, we have

$$\mathbf{w}_1 = (2\mathbf{I} - (\Delta t)^2\mathbf{M}^{-1}\mathbf{K})\mathbf{w}_0 - \mathbf{w}_{-1}, \quad \mathbf{w}'_0 = \frac{\mathbf{w}_1 - \mathbf{w}_{-1}}{2\Delta t}$$

or

$$\mathbf{w}_1 = (2\mathbf{I} - (\Delta t)^2\mathbf{M}^{-1}\mathbf{K})\mathbf{w}_0 - \mathbf{w}_{-1}, \quad \mathbf{w}_{-1} = \mathbf{w}_1 - 2\Delta t\mathbf{w}'_0$$

From the last two equations we get

$$\mathbf{w}_1 = \Big(\mathbf{I} - \frac{(\Delta t)^2}{2}\mathbf{M}^{-1}\mathbf{K}\Big)\mathbf{w}_0 + \Delta t\mathbf{w}'_0.$$

Therefore, the IVP becomes, by the CDM approximation:

$$\begin{cases} \mathbf{w}_{n+1} = (2\mathbf{I} - (\Delta t)^2\mathbf{M}^{-1}\mathbf{K})\mathbf{w}_n - \mathbf{w}_{n-1}, \quad n = 1, 2, ... \\ \mathbf{w}(t_0) = \mathbf{w}_0 \\ \mathbf{w}(t_1) = \mathbf{w}_1 = \Big(\mathbf{I} - \frac{(\Delta t)^2}{2}\mathbf{M}^{-1}\mathbf{K}\Big)\mathbf{w}_0 + \Delta t \mathbf{w}_0' \\ \quad t_n = n\Delta t. \end{cases} \tag{3.0.12}$$

Method 2. We transform the second-order system of ODEs to an equivalent first-order system.

Let $\mathbf{y}(t) = \mathbf{w}'(t)$, and thus $\mathbf{y}'(t) = \mathbf{w}''(t)$, and let

$$\mathbf{Y}(\mathbf{t}) = \begin{bmatrix} \mathbf{y}(\mathbf{t}) \\ \mathbf{y}'(\mathbf{t}) \end{bmatrix}.$$

Then (3.0.9) becomes the following equivalent first-order system:

$$\mathbf{Y}'(\mathbf{t}) = \mathbf{A}\mathbf{Y}(\mathbf{t}), \quad \mathbf{Y}(\mathbf{t_0}) = \begin{bmatrix} \mathbf{w}(\mathbf{t_0}) \\ \mathbf{w}'(\mathbf{t_0}) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix}. \tag{3.0.13}$$

This system can be solved by using standard ODE theory.

CHAPTER 4

THE CENTRAL DIFFERENCE METHOD

In this section, we use the central difference method to solve the wave equation:

$$\begin{cases} u_{xx} &= u_{tt}\, d\mu \\ u(a,t) &= u(b,t) = 0 \\ u(x,0) &= g(x), \quad u_t(x,0) = h(x). \end{cases} \tag{4.0.1}$$

Let $\mathcal{P} = \mathcal{P}(\{x_j\}_{j=1}^{J}, \{t_n\}_{n=0}^{N})$ be a partition of the rectangle $[a,b] \times [0,T]$, i.e.,

$$a = x_0 < x_1 < \cdots < x_J = b \quad \text{and} \quad 0 = t_0 < t_1 < \cdots < t_N = T.$$

Also, let

$$\Delta x_j = x_j - x_{j-1} \quad \text{and} \quad \Delta t_n = t_n - t_{n-1}.$$

Define $\bar{u}_x = \bar{u}_x(\mathcal{P})$ on $\{((x_{j-1}+x_j)/2, t_n) : 1 \le j \le J,\ 0 \le n \le N\}$ by

$$\bar{u}_x\Big(\frac{x_{j-1}+x_j}{2}, t_n\Big) := \frac{u(x_j,t_n) - u(x_{j-1},t_n)}{\Delta x_j}.$$

$\bar{u}_x$ approximates the partial derivative u_x at the points $((x_{j-1}+x_j)/2, t_n)$.

Next, define $\bar{u}_{xx} = \bar{u}_{xx}(\mathcal{P})$ on $\{x_j, t_n) : 1 \le j \le J-1,\ 0 \le n \le N\}$ by

$$\begin{aligned} \bar{u}_{xx}(x_j,t_n) :&= \frac{\bar{u}_x((x_j+x_{j+1})/2, t_n) - \bar{u}_x((x_{j-1}+x_j)/2, t_n)}{(\Delta x_j + \Delta_{j+1})/2} \\ &= \frac{2\Big[\frac{u(x_{j+1},t_n)}{\Delta x_{j+1}} - \Big(\frac{1}{\Delta x_j} + \frac{1}{\Delta x_{j+1}}\Big)u(x_j,t_n) + \frac{1}{\Delta x_j}u(x_{j-1},t_n)\Big]}{\Delta x_j + \Delta x_{j+1}}. \end{aligned} \tag{4.0.2}$$

Thus, $\bar{u}_{xx}(x_j,t_n)$ approximates the second-order partial derivative u_{xx} at (x_j,t_n).

In a similar fashion, we define $\bar{u}_t = \bar{u}_t(\mathcal{P})$ on $\{(x_j, (t_{n-1}+t_n)/2) : 0 \le j \le J,\ 1 \le n \le N\}$ by

$$\bar{u}_t\Big(x_j, \frac{t_{n-1}+t_n}{2}\Big) := \frac{u(x_j,t_n) - u(x_j,t_{n-1})}{\Delta t_n},$$

and define $\bar{u}_{tt} = \bar{u}_{tt}(\mathcal{P})$ on $\{(x_j, t_n) : 0 \le j \le J,\ 1 \le n \le N-1\}$ by

$$\begin{aligned}\bar{u}_{tt} &:= \frac{\bar{u}_t(x_j, (t_n + t_{n+1})/2) - \bar{u}_t(x_j, (t_n + t_{n-1})/2)}{(\Delta t_n + \Delta_{n+1})/2} \\ &= \frac{2\Big[\frac{1}{\Delta t_{n+1}} u(x_j, t_{n+1}) - \Big(\frac{1}{\Delta t_n} + \frac{1}{\Delta t_{n+1}}\Big) u(x_j, t_n) + \frac{1}{\Delta t_n} u(x_j, t_{n-1})\Big]}{\Delta t_n + \Delta t_{n+1}}.\end{aligned} \tag{4.0.3}$$

Obviously, $\bar{u}_t$ and $\bar{u}_{tt}$ approximate the partial derivatives u_t and u_{tt}, respectively.

In all of our computations, we set $\Delta t_n = \Delta t$ for all $n = 1, \dots, N$. In this case, equation (4.0.3) simplifies to

$$\bar{u}_{tt}(x_j, t_n) = \frac{u(x_j, t_{n+1}) - 2u(x_j, t_n) + u(x_j, t_{n-1})}{(\Delta t)^2}. \tag{4.0.4}$$

Substituting (4.0.2) and (4.0.4) into (4.0.1) leads to the following discretized wave equation:

$$\begin{aligned}&\frac{2(\frac{1}{\Delta x_{j+1}} \tilde{u}(x_{j+1}, t_n) - (\frac{1}{\Delta x_j} + \frac{1}{\Delta x_{j+1}}) \tilde{u}(x_j, t_n) + \frac{1}{\Delta x_j} \tilde{u}(x_{j-1}, t_n))}{\Delta x_{j+1} + \Delta x_j} \\ =&\frac{\tilde{u}(x_j, t_{n+1}) - 2\tilde{u}(x_j, t_n) + \tilde{u}(x_j, t_{n-1})}{(\Delta t)^2} \frac{\mu([x_{j-1}, x_{j+1}])}{(\Delta x_j + \Delta x_{j+1})}.\end{aligned} \tag{4.0.5}$$

We consider the two cases with the same or different Δx_j.

Case 1: $\Delta x_j = \Delta x$ for all $1 \le j \le J$. In this case equation (4.0.5) becomes

$$\begin{aligned}&\frac{\tilde{u}(x_{j+1}, t_n) - 2\tilde{u}(x_j, t_n) + \tilde{u}(x_{j-1}, t_n)}{(\Delta x)^2} \\ =&\frac{\tilde{u}(x_j, t_{n+1}) - 2\tilde{u}(x_j, t_n) + \tilde{u}(x_j, t_{n-1}}{(\Delta t)^2} \frac{\mu([x_{j-1}, x_{j+1}])}{2(\Delta x)}.\end{aligned}$$

Let

$$s_j = s_j(\mathcal{P}) := \frac{2(\Delta t)^2}{\Delta x(\mu[x_{j-1}, x_{j+1}])}, \qquad j = 1, \dots, J-1.$$

Then we get

$$s_j\big(\tilde{u}(x_{j+1}, t_n) - 2\tilde{u}(x_j, t_n) + \tilde{u}(x_{j-1}, t_n)\big) = \tilde{u}(x_j, t_{n+1}) - 2\tilde{u}(x_j, t_n) + \tilde{u}(x_j, t_{n-1}),$$

or

$$\tilde{u}(x_j, t_{n+1}) + \tilde{u}(x_j, t_{n-1}) = s_j \tilde{u}(x_{j+1}, t_n) - 2(s_j - 1)\tilde{u}(x_j, t_n) + s_j \tilde{u}(x_{j-1}, t_n). \quad (4.0.6)$$

Moreover, we have

$$\tilde{u}(x_j, t_1) - \tilde{u}(x_j, t_{-1}) = \tilde{u}_t(x_j, t_0) \cdot (2\Delta t) + \mathcal{O}(\Delta t)^2 \approx u_t(x_j, t_0)(2\Delta t) \approx h_j(2\Delta t). \quad (4.0.7)$$

Therefore, adding (4.0.6) (for $n = 0$) and (4.0.7), we get

$$\tilde{u}(x_j, t_{0+1}) \approx \frac{s_j}{2} \cdot \tilde{u}(x_{j+1}, 0) - (s_j - 1) \cdot \tilde{u}(x_j, 0) + \frac{s_j}{2} \tilde{u}(x_{j-1}, 0) + h_j \cdot (\Delta t). \quad (4.0.8)$$

Finally,

$$\tilde{u}(x_j, t_0) = \tilde{u}(x_j, t_0) = g(x_j) = g_j, \quad \tilde{u}(x_0, t) = \tilde{u}(a, t) \quad \text{and} \quad \tilde{u}(x_J, t) = \tilde{u}(b, t) = 0. \quad (4.0.9)$$

Therefore, using the central difference method, we can approximate the solution of the system (4.0.1) by the following formulas:

$$\begin{cases} \tilde{u}(x_j, t_{n+1}) \approx s_j \cdot \tilde{u}(x_{j+1}, t_n) + 2(s_j - 1)\tilde{u}(x_j, t_n) + s_j \tilde{u}(x_{j-1}, t_n) - \tilde{u}(x_j, t_{n-1}) \\ \tilde{u}(x_j, t_1) \approx \frac{s_j}{2} \cdot \tilde{u}(x_{j+1}, t_0) + (s_j - 1) \cdot \tilde{u}(x_j, t_0) + \frac{s_j}{2} \tilde{u}(x_{j-1}, t_0) + h_j \cdot (\Delta t) \\ \tilde{u}(x_j, t_0) = \tilde{u}(x_j, 0) = g(x_j) = g_j \\ \tilde{u}(x_0, t) = \tilde{u}(a, t) = 0, \quad \tilde{u}(x_J, t) = \tilde{u}(b, t) = 0. \end{cases} \quad (4.0.10)$$

Case 2. Δx_j is not constant.

If $x_{j+1} - x_j \neq x_j - x_{j-1}$ and let $\Delta x_j = x_j - x_{j-1}$, where $j = 1, 2, \ldots$. Then from (4.0.5), we have

$$\begin{aligned} & \frac{2\left(\frac{1}{\Delta x_{j+1}} \tilde{u}(x_{j+1}, t_n) - \left(\frac{1}{\Delta x_j} + \frac{1}{\Delta x_{j+1}}\right)\tilde{u}(x_j, t_n) + \frac{1}{\Delta x_j} \tilde{u}(x_{j-1}, t_n)\right)}{\Delta x_{j+1} + \Delta x_j} \\ = & \frac{u_j^{n+1} - 2\tilde{u}(x_j, t_n) + \tilde{u}(x_j, t_{n-1})}{(\Delta t)^2} \frac{\mu([x_{j-1}, x_{j+1}])}{(\Delta x_j + \Delta x_{j+1})}, \end{aligned}$$

which is simplified to

$$s_j\Big(\frac{1}{\Delta x_{j+1}}\tilde{u}(x_{j+1},t_n) - (\frac{1}{\Delta x_j} + \frac{1}{\Delta x_{j+1}})\tilde{u}(x_j,t_n) + \frac{1}{\Delta x_j}\tilde{u}(x_{j-1},t_n)\Big)$$
$$=\tilde{u}(x_j,t_{n+1}) - 2\tilde{u}(x_j,t_n) + \tilde{u}(x_j,t_{n-1}),$$

or

$$\begin{aligned}&\tilde{u}(x_j,t_{n+1}) + \tilde{u}(x_j,t_{n-1})\\ =&\frac{s_j}{\Delta x_{j+1}}\cdot\tilde{u}(x_{j+1},t_n) - (s_j(\frac{1}{\Delta x_j} + \frac{1}{\Delta x_{j+1}}) - 2)\tilde{u}(x_j,t_n) + \frac{s}{\Delta x_j}\tilde{u}(x_{j-1},t_n).\end{aligned} \tag{4.0.11}$$

Moreover, from (2.12), if $n = 0$, we have

$$\begin{aligned}\tilde{u}(x_j,t_1) - \tilde{u}(x_j,t_{-1}) = \tilde{u}_t(x_j,t_0)\cdot(2\Delta t) + \mathcal{O}(\Delta t)^2 \approx \tilde{u}_t(x_j,t_0)(2\Delta t)\\ \approx \tilde{u}(x_j,t_1) - \tilde{u}(x_j,t_{-1}) \approx h_j(2\Delta t).\end{aligned} \tag{4.0.12}$$

By summing (4.0.11) (for $n = 0$) and (4.0.12), we get

$$\begin{aligned}\tilde{u}(x_j,t_1) \approx \frac{s}{2\Delta x_{j+1}}\cdot\tilde{u}(x_{j+1},0) - (s(\frac{1}{\Delta x_j} + \frac{1}{\Delta x_{j+1}})/2 - 1)\tilde{u}(x_j,t_0)\\ +\frac{s_j}{2\Delta x_j}\tilde{u}(x_{j-1},t_0) + h_j\cdot(\Delta t).\end{aligned} \tag{4.0.13}$$

Moreover,

$$\tilde{u}(x_j,t_0) = u(x_j,0) = g(x_j) = g_j, \quad \tilde{u}(x_0,t) = u(a,t) = 0, \quad \tilde{u}(x_J,t) = u(b,t) = 0. \tag{4.0.14}$$

Therefore, we can approximate the solution of the system(4.0.1) by the following scheme:

$$\begin{cases}\tilde{u}(x_j,t_{n+1}) = \frac{s_j}{\Delta x_{j+1}}\cdot\tilde{u}(x_{j+1},t_n) - (s_j(\frac{1}{\Delta x_j} + \frac{1}{\Delta x_{j+1}}) - 2)\tilde{u}(x_j,t_n)\\ \qquad\qquad + \frac{s_j}{\Delta x_j}\tilde{u}(x_{j-1},t_n) - \tilde{u}(x_j,t_{n-1})\\ \tilde{u}(x_j,t_1) = \frac{s_j}{2\Delta x_{j+1}}\cdot\tilde{u}(x_{j+1},0) - (s_j(\frac{1}{\Delta x_j} + \frac{1}{\Delta x_{j+1}})/2 - 1)\tilde{u}(x_j,t_0)\\ \qquad\qquad + \frac{s_j}{2\Delta x_j}\tilde{u}(x_{j-1},t_0) + h_j\cdot(\Delta t)\\ \tilde{u}(x_j,0) = g(x_j) = g_j\\ \tilde{u}(a,t) = 0, \quad \tilde{u}(b,t) = 0.\end{cases} \tag{4.0.15}$$

CHAPTER 5

FRACTAL MEASURES DEFINED BY ITERATED FUNCTION SYSTEMS

In this chapter, we solve the homogeneous IBVP (2.0.1) numerically for three different measures namely, the weighted Lebesgue measure, the infinite Bernoulli convolution associated with the golden ratio, and the 3-fold convolutions of the Cantor measure.

Let $\{S_i\}_{i=1}^N$ be an IFS of contractive similitudes on $\mathbb{R}$ of the form

$$S_i(x) = \rho x + b_i, \quad i = 1, \ldots, N, \tag{5.0.1}$$

and let μ be an associated *self-similar measure* with *supp*$[\mu] = [a, b]$ satisfying the following *self-similar identity*:

$$\mu = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1}, \tag{5.0.2}$$

where $0 < p_i < 1$ and $\sum_{i=1}^N p_i = 1$.

5.0.1 Weighted Lebesgue measure

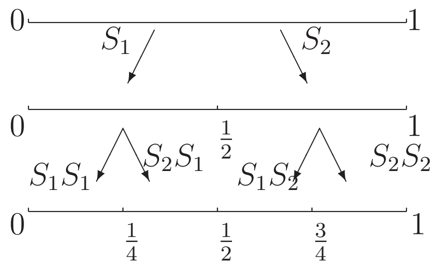
The weighted Lebesgue measure is defined by the system of

$$S_1(x) = \frac{1}{2}x, \qquad S_2(x) = \frac{1}{2}x + \frac{1}{2}$$

and

$$\mu = p\mu \circ S_1^{-1} + (1 - p)\mu \circ S_2^{-1}.$$

In view of [3], we will choose the weight $p = 2 - \sqrt{3}$.



For any Borel subset $A \subseteq [0,1]$, we have:

$$\begin{bmatrix} \mu(S_1T_iA) \\ \mu(S_2T_iA) \end{bmatrix} = M_i \begin{bmatrix} \mu(S_1A) \\ \mu(S_2A) \end{bmatrix}, \quad i = 1, 2,$$

where

$$M_1 = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1-p & 0 \\ 0 & (1-p) \end{bmatrix},$$

Let $J = j_1j_2\ldots j_m$, $j_i = 1, 2$ or 3. Then

$$\mu(S_JA) = c_J \begin{bmatrix} \mu(S_1A) \\ \mu(S_2A) \end{bmatrix},$$

where

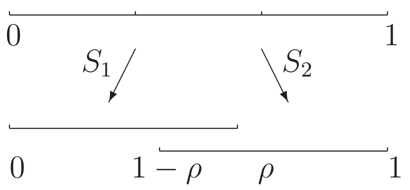
$$c_J = \boldsymbol{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2).$$

5.1 Infinite Bernoulli convolution associated with the golden ratio

The infinite Bernoulli convolution associated with the golden ratio satisfies a family of second-order self-similar identities. This was first pointed out by Strichartz et al. [26]. We can make use of this to calculate the measure of suitable subintervals.

The infinite Bernoulli convolution associated with the golden ratio is defined the by the IFS

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1-\rho), \qquad \rho = \frac{\sqrt{5}-1}{2}.$$



For each $0 < p < 1$, we call the corresponding self-similar measures

$$\mu(A) = p\mu \circ S_1^{-1}(A) + (1-p)\mu \circ S_2^{-1}(A).$$

the *weighted infinite Bernoulli convolution associated with the golden ratio.* If $p = 1/2$, we get the classical one.

Define

$$T_1(x) = \rho^2 x. \quad T_2(x) = \rho^3 x + \rho^2, \quad T_3(x) = \rho^2 x + \rho.$$

Then μ satisfies the following second-order self-similar identities (see [17]): for any Borel subset $A \subseteq [0,1]$, we have:

$$\begin{bmatrix} \mu(T_1T_iA) \\ \mu(T_2T_iA) \\ \mu(T_3T_iA) \end{bmatrix} = M_i \begin{bmatrix} \mu(T_1A) \\ \mu(T_2A) \\ \mu(T_3A) \end{bmatrix}, \quad i = 1,2,3,$$

where

$$M_1 = \begin{bmatrix} p^2 & 0 & 0 \\ (1-p)p^2 & (1-p)p & 0 \\ 0 & 1-p & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & p^2 & 0 \\ 0 & (1-p)p & 0 \\ 0 & (1-p)^2 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & p & 0 \\ 0 & (1-p)p & (1-p)^2 p \\ 0 & 0 & (1-p)^2 \end{bmatrix}.$$

If $p = 1/2$, we obtain

$$M_1 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}, \quad M_2 = \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \frac{1}{8}\begin{bmatrix} 0 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Let $J = j_1 j_2 \dots j_m$, $j_i = 1, 2$ or 3. Then

$$\mu(T_J A) = c_J \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix},$$

where

$$c_J = \boldsymbol{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2, c_J^3).$$

Moreover, we have

$$\int_a^b f(x)\, d\mu = p \int_a^b f(S_1(x))\, d\mu + (1-p) \int_a^b f(S_2(x))\, d\mu. \tag{5.1.3}$$

Then, we use (5.1.3) to evaluate the measure of each interval $\mu(T_j[0,1]) = \int_0^1 d\mu \circ T_j$, $\int_0^1 x\, d\mu \circ T_j$ and $\int_0^1 x^2\, d\mu \circ T_j, j = 1, 2, 3$ for any probability weights p and $1-p$. The following result is the matrix with $p = \frac{1}{2}$:

$$\begin{bmatrix} \int_0^1 d\mu \circ T_1 & \int_0^1 d\mu \circ T_2 & \int_0^1 d\mu \circ T_3 \\ \int_0^1 x\, d\mu \circ T_1 & \int_0^1 x\, d\mu \circ T_2 & \int_0^1 x\, d\mu \circ T_3 \\ \int_0^1 x^2\, d\mu \circ T_1 & \int_0^1 x^2\, d\mu \circ T_2 & \int_0^1 x^2\, d\mu \circ T_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6(3\rho-1)} & \frac{1}{6} & \frac{1}{6(3\rho^2+3)} \\ \frac{5\rho+4}{6(\rho+8)} & \frac{\rho+5}{6(\rho+8)} & \frac{2-\rho}{6(\rho+8)} \end{bmatrix}. \tag{5.1.4}$$

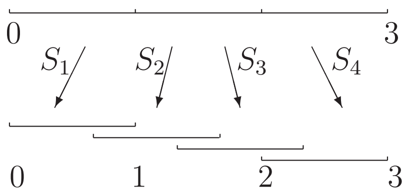
We will apply this matrix to calculate the value of the entries of $\mathbf{M}$ in (3.0.12), which is very important for finding out the measure of each interval on the level m of the finite element method.

5.2 3-fold convolution of the Cantor measure

The 3-fold convolutions of the Cantor measure satisfies a family of second-order identities. It is defined by the IFS

$$S_i(x) = \frac{1}{3}x + \frac{2}{3}(i-1), \quad \text{for} \quad i = 1, 2, 3, 4,$$

does not satisfy the OSC.



Its corresponding self-similar measures

$$\mu = \frac{1}{8}\mu \circ S_1^{-1} + \frac{3}{8}\mu \circ S_2^{-1} + \frac{3}{8}\mu \circ S_3^{-1} + \frac{1}{8}\mu \circ S_4^{-1}$$

Define,

$$T_1(x) = \frac{1}{3}x, \quad T_2(x) = \frac{1}{3}x + 1, \quad T_3(x) = \frac{1}{3}x + 2$$

Then μ satisfies the following second-order self-similar identities (see [17]): for any Borel subset $A \subseteq [0, 3]$,

$$\begin{bmatrix} \mu(T_{1j}A) \\ \mu(T_{2j}A) \\ \mu(T_{3j}A) \end{bmatrix} = M_j \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix}, \quad j = 1, 2, 3,$$

where the coefficient matrices M_j are given by

$$M_1 = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad M_2 = \frac{1}{8}\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \frac{1}{8}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $J = j_1 j_2 \dots j_m$, $j_i = 1, 2$ or 3. Then

$$\mu(T_J A) = c_J \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix},$$

where

$$c_J = \boldsymbol{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2, c_J^3).$$

We can also evaluate the measure of each interval T_J. They are given by

$$\begin{bmatrix} \int_0^3 d\mu \circ T_1 & \int_0^3 d\mu \circ T_2 & \int_0^3 d\mu \circ T_3 \\ \int_0^3 x d\mu \circ T_1 & \int_0^3 x d\mu \circ T_2 & \int_0^3 x d\mu \circ T_3 \\ \int_0^3 x^2 d\mu \circ T_1 & \int_0^3 x^2 d\mu \circ T_2 & \int_0^3 x^2 d\mu \circ T_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{27}{70} & \frac{9}{10} & \frac{3}{14} \\ \frac{5517}{6440} & \frac{11943}{6440} & \frac{63}{184} \end{bmatrix}. \qquad (5.2.5)$$

We will apply this matrix to calculate the value of entries of **M** in (3.0.12), that is very important for finding out the measure of each interval on level m of the finite element method.

CHAPTER 6

CONVERGENCE OF THE APPROXIMATION SCHEME

In this chapter, we proved the convergence for the approximation scheme for the homogeneous IBVP (2.0.1). Some of our results are obtained by modifying similarly ones in [24].

Let V_m be the set of end-points of all the m-level intervals, and rearrange its elements so that $V_m = \{x_k : k = 0, 1, \ldots, N^m\}$ with $x_k < x_{k+1}$ for $k = 0, 1, \ldots, N^m - 1$. Let S^m be the space of continuous piecewise linear functions with nodes V_m. The functions in S^m are bounded; moreover, $\dim S^m = N^m + 1$. Let

$$S_D^m := \{u \in S^m : u(a) = u(b) = 0\}$$

be the subspace of S^m consisting of functions satisfying the Dirichlet boundary conditions. Then

$$\dim S_D^m = \#V_m - 2 = N^m - 1.$$

We will choose the basis of S^m consisting of the tent functions defined in (3.0.3):

Then fix $t \in [0, T]$, any $u^m(x, t) \in S^m$ defined in(3.0.2)

Definition 6.0.1. *The linear map $\mathcal{P}_m : \mathrm{Dom}\,(\mathcal{E}) \to S_D^m$ is defined by*

$$\mathcal{P}_m := \sum_{i=0}^{N^m - 1} u(x_i)\phi_i(x) \text{ for all } u \in \mathrm{Dom}\,(\mathcal{E}).$$

is called the Rayleigh-Ritz projection.

Lemma 6.0.1. *Let v be a absolutely continuous function on $[a, b]$. Then $|v(x) - v(y)| \leq |x - y|^{1/2} \|v\|_{H_0^1}$ for all $x, y \in [a, b]$.*

Proof. Since v is a absolutely continuous function on $[a,b]$, and for all $x, y \in [a,b]$,

$$\begin{aligned} |v(x)-v(y)| &= \left|\int_0^x \nabla v(s)\,ds - \int_0^y \nabla v(s)\,ds\right| = \left|\int_y^x \nabla v(s)\,ds\right| \\ &\leq \int_y^x |\nabla v(s)|\ ds \leq |x-y|^{1/2}\,\|v\|_{H_0^1}.(See e.g[7].) \end{aligned} \tag{6.0.1}$$

□

Lemma 6.0.2. *Let* $\mathcal{P}_m$ *be the* Rayleigh-Ritz projection,*and* $u \in \mathrm{Dom}(\mathcal{E})$. *Then,* $\mathcal{P}_m u$ *is its component in the subspace* S_D^m, $u - \mathcal{P}_m u$ *vanishes on the boundary and*

$$\mathcal{E}(u - \mathcal{P}_m u, \bar{u}^{(m)}) = 0 \quad \textit{for all } \bar{u}^{(m)} \in S_D^m.(See[24]).$$

Proof.

$$\mathcal{E}(u - \mathcal{P}_m u, \phi_{m,i}(x)) = \int_a^b \nabla(u - \sum_{i=1}^n u(x_i)\phi_{m,i}(x))\nabla\phi_{m,i}(x)\,dx$$

$$\begin{aligned}
=&\int_{x_{i-1}}^{x_i} \nabla(u - u(x_{i-1})\phi_{m,i-1}(x) - u(x_i)\phi_{m,i}(x))\,dx \\
&- \int_{x_i}^{x_{i+1}} \nabla(u - u(x_i)\phi_{m,i}(x) - u(x_{i+1})\phi_{m,i+1}(x))\,dx \\
=&\Big(u - u(x_{i-1})\phi_{m,i-1}(x) - u(x_i)\phi_{m,i}(x)\Big)\Big|_{x_i}^{x_{i-1}} \\
&- \Big(u - u(x_i)\phi_{m,i}(x) - u(x_{i+1})\phi_{m,i+1}(x)\Big)\Big|_{x_{i+1}}^{x_i} \\
=&\Big(u(x_i) - u(x_{i-1})\phi_{m,i-1}(x_i) - u(x_i)\phi_{m,i}(x_i)\Big) \\
&- \Big(u(x_{i-1}) - u(x_{i-1})\phi_{m,i-1}(x_{i-1}) - u(x_i)\phi_{m,i}(x_{i-1})\Big) \\
&- \Big(u(x_{i+1}) - u(x_i)\phi_{m,i}(x_{i+1}) - u(x_{i+1})\phi_{m,i+1}(x_{i+1})\Big) \\
&+ \Big(u(x_i) - u(x_i)\phi_{m,i}(x_i) - u(x_{i+1})\phi_{m,i+1}(x_i)\Big) \\
=&(u(x_i) - u(x_i)) - (u(x_{i-1}) - u(x_{i-1})) - (u(x_{i+1}) - u(x_{i+1})) + (u(x_i) - u(x_i)) \\
=&0.
\end{aligned}$$

Since $\{\phi_{m,i}\}_{i=1}^{N^m-1}$ is a basis of S_D^m, $\mathcal{E}(u - \mathcal{P}_m u, \bar{u}^{(m)}) = 0$. □

Lemma 6.0.3. *Let v be in $\mathrm{Dom}\,(\mathcal{E})$, and let $\mathcal{P}_m v$ be the Rayleigh-Ritz projection of v to the subspace S_D^m of piecewise linear functions with any partition $\pi_m = \{x_i\}_{i=0}^{N_m}$. Then*

$$v|_{\pi_m} = \mathcal{P}_m v|_{\pi_m}.$$

Proof. Similar to that of [5, Lemma 5.3]. □

Theorem 6.0.4. *Let v be a absolute continuous function on $[a, b]$. Let $\pi_m = \{x_i\}_{i=0}^{N_m}$ be any partition of $[a, b]$. Then, $\left|v_(x) - \mathcal{P}_m v(x)\right| \leq C \left\|\pi_m\right\|^{1/2}$ for all $x \in [a, b]$.*

Proof. Since $v(x)$ be a absolute continuous function on $[a, b]$, and $x \in [a, b]$, there exists $i \in \{1, \ldots, N^m\}$ such that $x \in [x_{i-1}, x_i]$ and lemma 6.0.1 and 6.0.3 . Thus,

$$\begin{aligned}
&|v(x) - \mathcal{P}_m v(x)| \leq |v(x) - v(x_{i-1}) + v(x_{i-1}) - \mathcal{P}_m v(x)| \\
&\leq |v(x) - v(x_{i-1})| + |v(x_{i-1}) - \mathcal{P}_m v(x)| \\
&\leq \Big(\int_{x_{i-1}}^{x} |\nabla v|^2 dx\Big)^{1/2}\Big(x - x_{i-1}\Big)^{1/2} + \Big(\int_{x_{i-1}}^{x_i} |\nabla v|^2 dx\Big)^{1/2}\Big(x_i - x_{i-1}\Big)^{1/2} \\
&\leq 2M\Big(x_i - x_{i-1}\Big)^{1/2}, \text{ where } M = \|v\|_{\mathrm{Dom}\,(\mathcal{E})}.
\end{aligned}$$

Let $h = \max\{x_i - x_{i-1}\}$. Then,

$$|v(x) - \mathcal{P}_m v(x)| < 2Mh^{1/2} \leq 2M \left\|\pi_m\right\|^{1/2}, \text{ for all } x \in [a, b]. \tag{6.0.2}$$

□

Let $g, h \in \mathrm{Dom}\,(\mathcal{E})$, $f \in L^2([0, T]; \mathrm{Dom}\,(\mathcal{E}))$, and u be defined by in the IVBP (2.0.1). Then we have,

$$< u_{tt}, v > +\mathcal{E}(u, v) = (f, v)_\mu \text{ for all } v \in \mathrm{Dom}\,(\mathcal{E}) \tag{6.0.3}$$

Lemma 6.0.5. *Fix m. Let $u \in L^2([0, T], \mathrm{Dom}\,(\mathcal{E}))$ be a weak solution of IBVP (2.0.1). Let u^m be defined in (3.0.2). We can choose it to satisfy:*

(i)

$$(u^m_{tt}, v^m)_\mu + \mathcal{E}(u^m, v^m) = (f, v^m)_\mu \text{ for all } v^m \in S^m_D, \tag{6.0.4}$$

(ii) $u^m(x,0) = \sum_{i=0}^{N^m-1} g_(x_i)\phi_i(x)$, *and* $u^m_t(x,0) = \sum_{i=0}^{N^m-1} h_(x_i)\phi_i(x)$.

Define $e(t) := Pu(t) - u^m(t)$. *Then,* $(e_{tt}, e_t)_\mu + \mathcal{E}(e, e_t) = ([Pu - u]_{tt}, e_t)_\mu$.

Proof. Since $e_t \in S^m_D$, substituting e_t for v in (6.0.3) and e_t for v^m in (6.0.4) respectively. Using the definition of pairing and subtracting these equations, we get

$$(u_{tt} - u^m_{tt}, e_t)_\mu + \mathcal{E}(u - u^m, e_t) = 0.$$

Equivalently,

$$(u_{tt} - (\mathcal{P}_m u)_{tt} + (\mathcal{P}_m u)_{tt} - u^m_{tt}, e_t)_\mu + \mathcal{E}(u - \mathcal{P}_m u + \mathcal{P}_m u - u^m, e_t) = 0,$$

which imply

$((\mathcal{P}_m u)_{tt} - u^m_{tt}, e_t)_\mu + \mathcal{E}(\mathcal{P}_m u - u^m, e_t) = ((\mathcal{P}_m u)_{tt} - u_{tt}, e_t)_\mu$, because $\mathcal{E}(u - \mathcal{P}_m u, e_t) = 0$. By the definition of $e(t)$, this becomes

$$(e_{tt}, e_t)_\mu + \mathcal{E}(e, e_t) = ([\mathcal{P}_m u - u]_{tt}, e_t)_\mu. \tag{6.0.5}$$

□

Proposition 6.0.6. *Fix t. Then* u^m *converges in* $L^2_\mu[a,b]$ *to* u.

Proof. Let $E(t) := \frac{1}{2}(e_t, e_t)_\mu + \frac{1}{2}\mathcal{E}(e, e) = \frac{1}{2}\|e_t(t)\|^2_\mu + \frac{1}{2}\|e(t)\|^2_{\mathrm{Dom}\,\mathcal{E}}$. Then

$$\|e_t\|^2_\mu \le \sqrt{2}\sqrt{E(t)}, \tag{6.0.6}$$

$$\|e\|^2_{\mathrm{Dom}\,\mathcal{E}} \le \sqrt{2}\sqrt{E(t)}, \tag{6.0.7}$$

$$E(t) \leq \frac{1}{2}\Big(\|e_t\|_\mu + \|e\|_{\mathrm{Dom}\,\mathcal{E}} \Big)^2. \tag{6.0.8}$$

Left hand side of (6.0.5) is equal to $(\frac{1}{2}\|e_t\|_\mu^2)_t + (\frac{1}{2}\|e\|^2_{\mathrm{Dom}\,\mathcal{E}})_t = E_t(t)$.

For the right hand side of (6.0.5), it follows Cauchy Swartz inequality and (6.0.6), that's

$$([\mathcal{P}_m u - u]_{tt}, e_t)_\mu \leq \|[\mathcal{P}_m u - u]_{tt}\|_\mu \|e_t\|_\mu \leq \|[\mathcal{P}_m u - u]_{tt}\|_\mu \sqrt{2}\sqrt{E(t)}.$$

So combine (6.0.7) and (6.0.8),

$$\begin{aligned} E_t(t) &\leq \|[\mathcal{P}_m u - u]_{tt}\|_\mu \sqrt{2}\sqrt{E(t)}, \\ \frac{E_t(t)}{\sqrt{E(t)}} &\leq \|[\mathcal{P}_m u - u]_{tt}\|_\mu \sqrt{2}, \\ 2\sqrt{E(s)} - 2\sqrt{E(0)} &\leq \sqrt{2}\int_0^s \|[\mathcal{P}_m u - u]_{tt}\|_\mu \, dt, \\ 2\sqrt{E(s)} &\leq 2 + \sqrt{2}\int_0^s \|[\mathcal{P}_m u - u]_{tt}\|_\mu \, dt, \text{ because } \sqrt{E(0)} = 0. \end{aligned} \tag{6.0.9}$$

From (6.0.7) and (6.0.8), we have

$$\begin{aligned} \frac{2}{\sqrt{2}}\|e(s)\|_{\mathrm{Dom}\,\mathcal{E}} \leq 2\sqrt{E(s)} &\leq \sqrt{2}\int_0^s \|[\mathcal{P}_m u - u]_{tt}\|_\mu \, dt, \\ \|e(s)\|_{\mathrm{Dom}\,\mathcal{E}} &\leq \sqrt{2}\int_0^s \|[\mathcal{P}_m u - u]_{tt}\|_\mu \, dt, \\ \frac{2}{\sqrt{2}}\|e(s)\|_\mu &\leq \sqrt{2}\int_0^s \|[\mathcal{P}_m u - u]_{tt}\|_\mu \, dt. \end{aligned} \tag{6.0.10}$$

$$\begin{aligned} \|e(s)\|_\mu &\leq C_s \int_0^s \|[\mathcal{P}_m u - u]_{tt}\|_\mu \, dt \leq C_s \int_0^T \|[\mathcal{P}_m u - u]_{tt}\|_\mu \, dt \\ &\leq C_s(\int_0^T \|[\mathcal{P}_m u - u]_{tt}\|^2_\mu \, dt)^{\frac{1}{2}}\sqrt{T} \leq C_s(\int_0^T 2\|u_{tt}\|^2_\mu \rho^m dt)^{\frac{1}{2}}\sqrt{T} \\ &\leq C_s\sqrt{2T}(\rho^m)^{\frac{1}{2}}\|u_{tt}\|_{L^2([0,T],\mathrm{Dom}\,(\mathcal{E})}. \end{aligned} \tag{6.0.11}$$

Therefore, fix t,

$$\begin{aligned} \|u^m - u\|_\mu &\leq \|u^m - \mathcal{P}_m u\|_\mu + \|\mathcal{P}_m u - u\|_\mu \\ &\leq C_s\sqrt{2T}(\rho^{m/2})\|u_{tt}\|_{L^2([0,T],\mathrm{Dom}\,(\mathcal{E}))} + 2(\rho^{m/2})\|u\|_{\mathrm{Dom}\,(\mathcal{E})} \to 0 \text{ as } m \leftarrow \infty. \end{aligned} \tag{6.0.12}$$

□

CHAPTER 7

NUMERICAL RESULTS

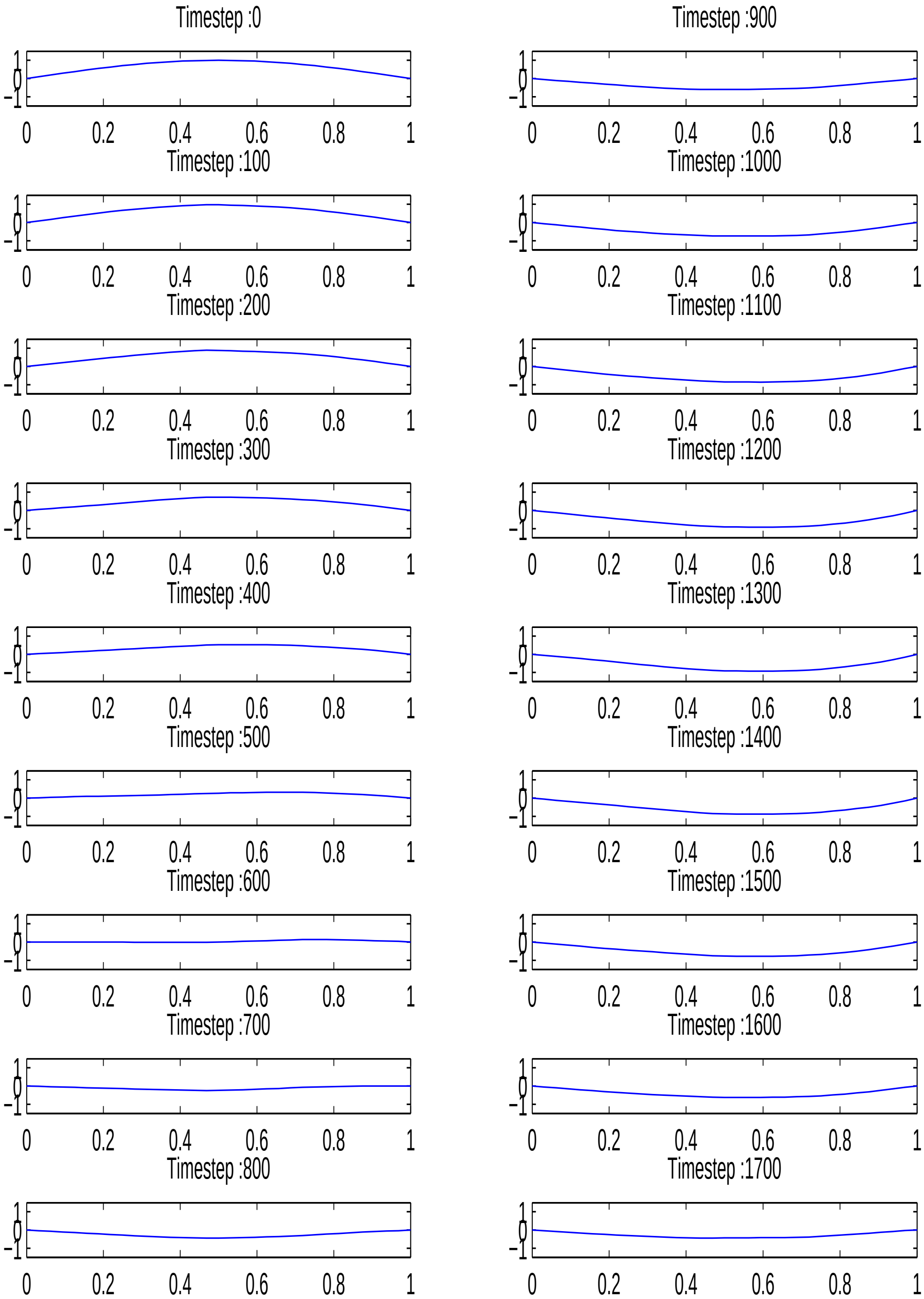
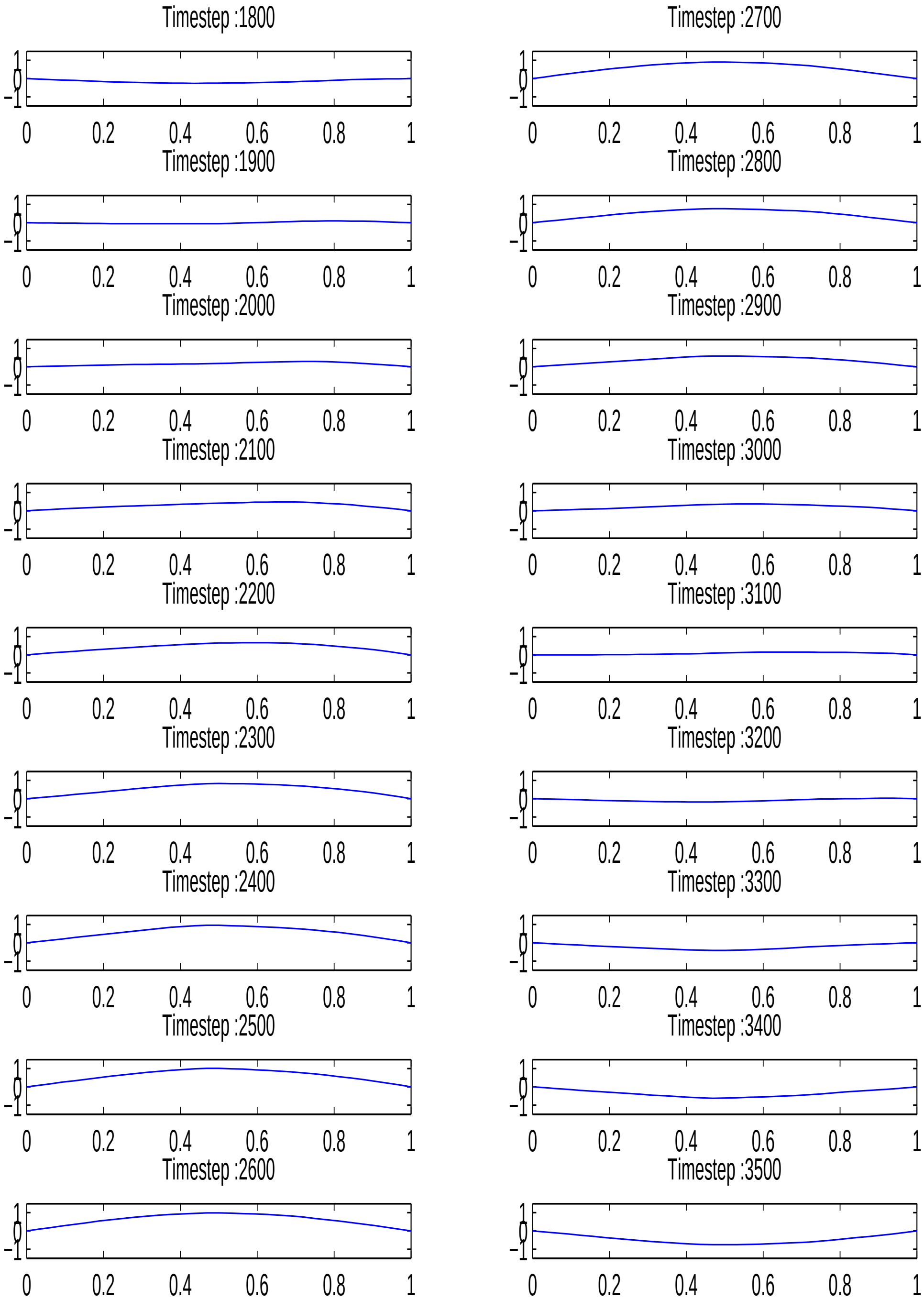


Figure 1: Dirichlet boundary condition for the weighted Lebesque measure associated with the weight $p = 2 - \sqrt{3}$ and $1 - p = \sqrt{3} - 1$.



Continuation of Fig. 1

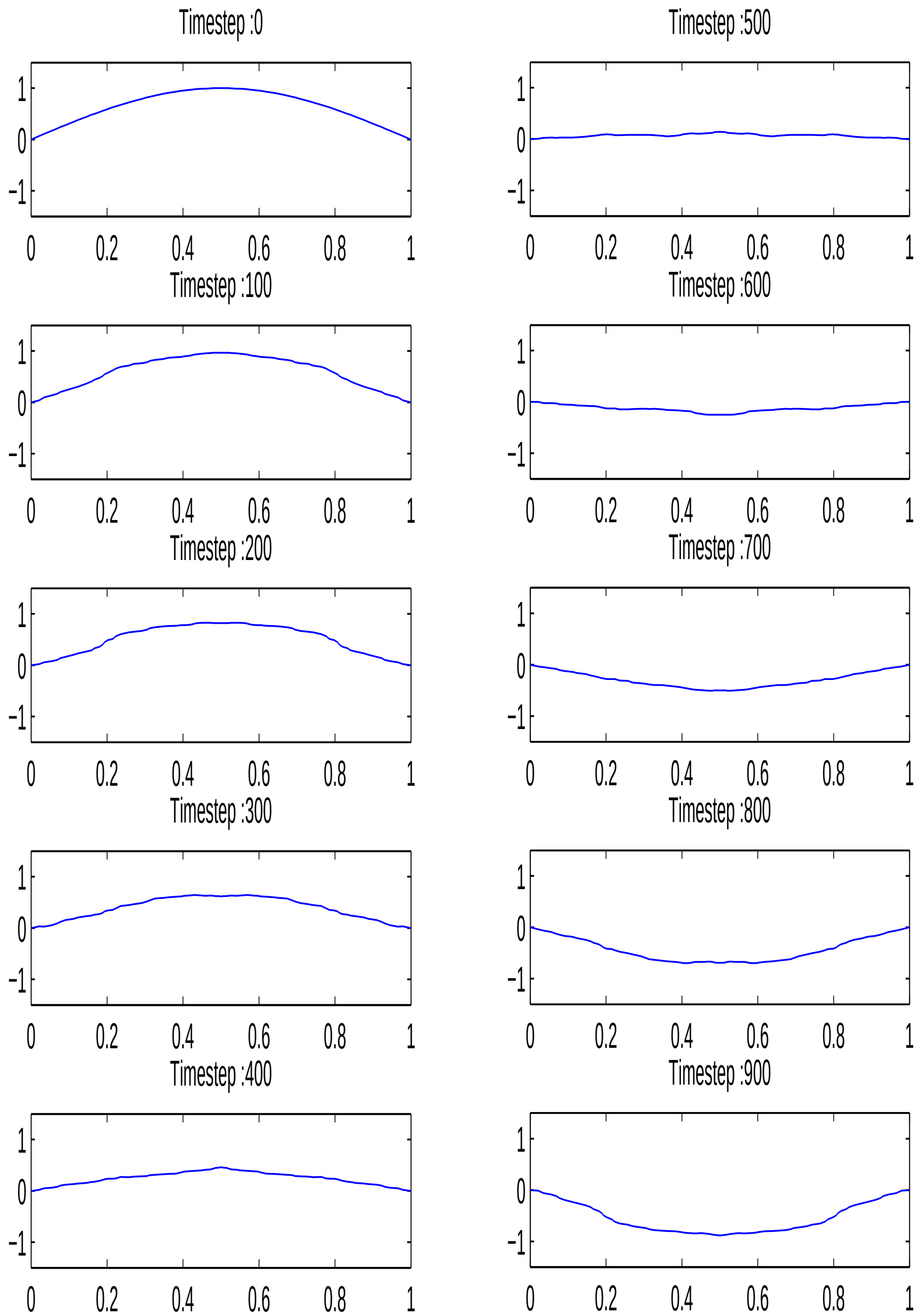
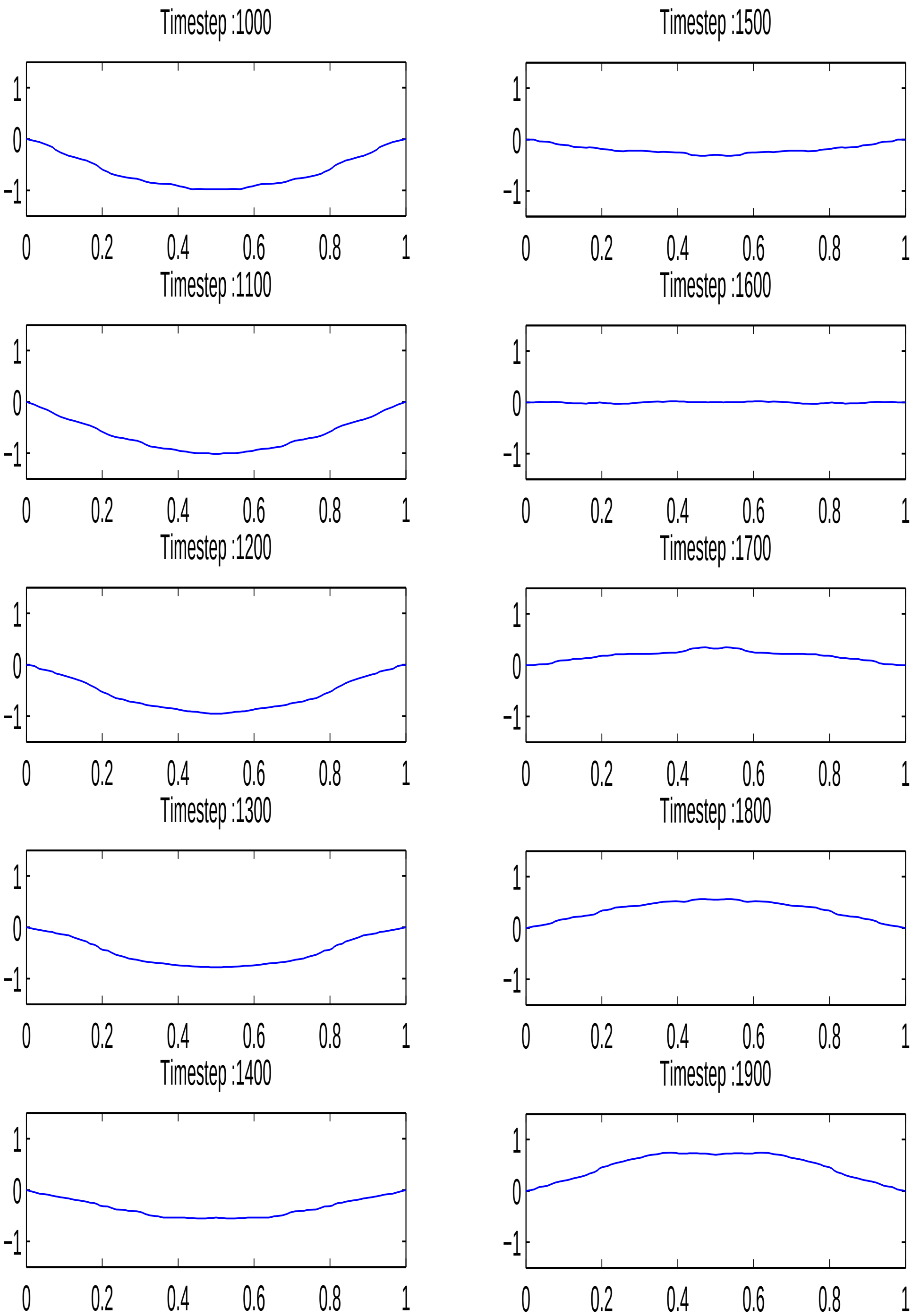


Figure 2: Dirichlet boundary conditions for the infinite Bernoulli convolution associated with the golden ratio.



Continuation of Fig. 2

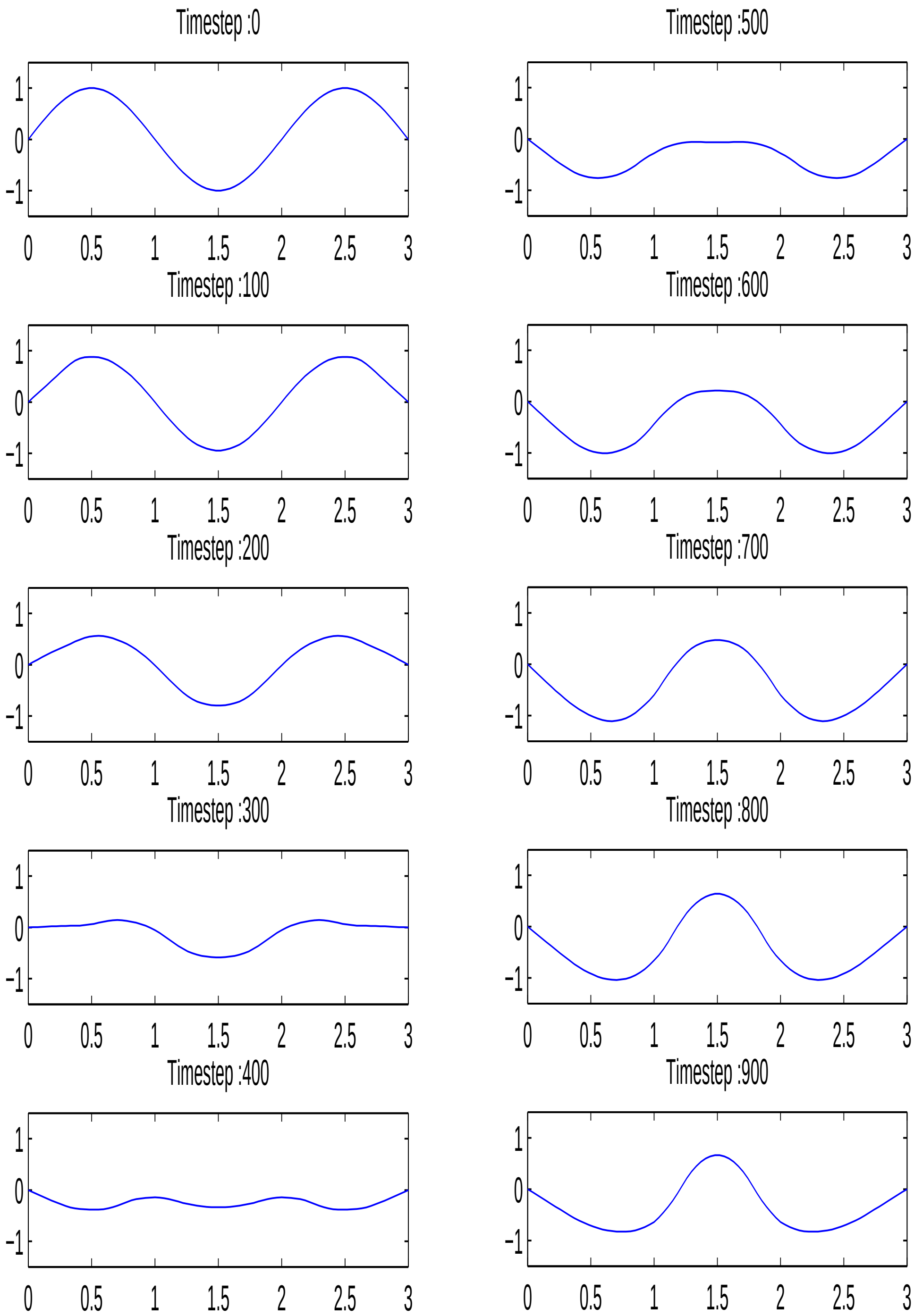


Figure 3: Dirichlet boundary c ondition for the 3-fold convolution of the Cantor measure.

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APPENDIX A

FRACTAL MEASURES

A.1 BERNOULLI CONVOLUTION ASSOCIATED WITH THE GOLDEN RATIO

$S_1(x) = \rho x \Longrightarrow S_1^{-1}(x) = \rho^{-1}x$

$S_2(x) = \rho x + (1-\rho) = \rho x + \rho^2 \Longrightarrow S_2^{-1}(x) = \rho^{-1}x - \rho$

$T_1(x) = \rho^2 x \Longrightarrow T_1^{-1}(x) = \rho^{-2}x$

$T_2(x) = \rho^3 x + \rho^2 \Longrightarrow T_2^{-1}(x) = \rho^{-3}x - (1+\rho)$

$T_3(x) = \rho^2 x + \rho \Longrightarrow T_3^{-1}(x) = \rho^{-2}x - (1+\rho)$

$\mathrm{Supp}(\mu) = [0,1] \& \mu = p\mu \circ S_1^{-1} + (1-p)\mu \circ S_2^{-1}$

$$\int_A f(x)\,d\mu = p\int_{S_1^{-1}[A]} f(S_1 x)\,d\mu + (1-p)\int_{S_2^{-1}[A]} f(S_2 x)\,d\mu$$

and

$$\int_A f(x)\,d\mu \circ T_i = \int_{T_i[A]} f(T_i^{-1}x)\,d\mu$$

$\rho^2 = 1-\rho, \quad \rho^3 = 2\rho - 1, \quad \rho^4 = 2 - 3\rho, \quad \rho^5 = 5\rho - 3, \quad \rho^6 = 5 - 8\rho.$

Let $p = 1/2$. Then, our goal is to claim that (5.1.4) is true.

Proof. Consider the first row:

$$\begin{aligned}\int_0^{\rho^2} d\mu &= 1/2 \int_{S_1^{-1}[0,\rho^2]} d\mu + 1/2 \int_{S_2^{-1}[0,\rho^2]} d\mu \\ &= 1/2 \int_0^{\rho} d\mu + 1/2 \int_{-\rho}^{0} d\mu \\ &= 1/2 \int_0^{\rho} d\mu\end{aligned}$$

Therefore,

$$\int_0^{\rho^2} d\mu = 1/2 \int_0^{\rho^2} d\mu + 1/2 \int_{\rho^2}^{\rho} d\mu \Rightarrow \int_0^{\rho^2} d\mu = \int_0^{\rho} d\mu. \tag{A.1.1}$$

Moreover,

$$\begin{aligned}\int_{\rho^2}^{\rho} d\mu &= \int_{S_1^{-1}[\rho^2,\rho]} d\mu + 1/2 \int_{S_2^{-1}[\rho^2,\rho]} d\mu \\ &=1/2 \int_{\rho}^{1} d\mu + 1/2 \int_0^{\rho^2} d\mu \\ &=1/2 \int_{\rho}^{1} d\mu + 1/2 \int_{\rho^2}^{\rho} d\mu \; by (A.1.1)\end{aligned}$$

Then, by $(A.1.1)$

$$\int_{\rho^2}^{\rho} d\mu = \int_{\rho}^{1} d\mu = \int_0^{\rho^2} d\mu \text{ and } \int_0^1 d\mu = 1$$

$$\int_0^{\rho^2} d\mu = \int_{\rho^2}^{\rho} d\mu = \int_{\rho}^{1} d\mu = 1/3 \int_0^1 d\mu = 1/3$$

Therefore,

$$\int_0^{\rho^2} d\mu \circ T_1 = \int_{T_1[0,1]} d\mu = \int_0^{\rho^2} d\mu = 1/3$$

$$\int_0^{\rho^2} d\mu \circ T_2 = \int_{T_2[0,1]} d\mu = \int_{\rho^2}^{\rho} d\mu = 1/3$$
$$\int_0^{\rho^2} d\mu \circ T_3 = \int_{T_3[0,1]} d\mu = \int_{\rho}^{1} d\mu = 1/3$$

Now consider the 2nd row

$$\begin{aligned}\int_0^1 x d\mu =& \int_{S_1^{-1}[0,1]} S_1(x) d\mu + 1/2 \int_{S_2^{-1}[0,1]} S_2(x) d\mu \\ =& 1/2 \int_0^{1+\rho} \rho x d\mu + 1/2 \int_{-\rho}^{1} (\rho x + \rho^2) d\mu \\ =& 1/2 \int_0^1 \rho x d\mu + 1/2 \int_0^1 \rho x d\mu + 1/2 \int_0^1 \rho^2 d\mu.\end{aligned}$$

Therefore,

$$(1-\rho)\int_0^1 x d\mu = 1/2\rho^2 \int_0^1 d\mu \quad \Rightarrow \quad \int_0^1 x d\mu = 1/2 \int_0^1 d\mu = 1/2.$$

$$\begin{aligned}\int_0^{\rho^2} x d\mu =& 1/2 \int_{S_1^{-1}[0,\rho^2]} S_1(x) d\mu + 1/2 \int_{S_2^{-1}[0,\rho^2]} S_2(x) d\mu \\ =& 1/2 \int_0^{\rho} \rho x d\mu + 1/2 \int_{-\rho}^{0} (\rho x + \rho^2) d\mu \\ =& 1/2\rho \int_0^{\rho^2} x d\mu + 1/2\rho \int_{\rho^2}^{\rho} x d\mu.\end{aligned}$$

$$\int_0^{\rho^2} x d\mu = \frac{\frac{1}{2}\rho}{1-\frac{1}{2}\rho} \int_{\rho^2}^{\rho} x d\mu = \frac{\rho}{2-\rho} \int_{\rho^2}^{\rho} x d\mu. \tag{A.1.2}$$

$$\begin{aligned}\int_{\rho^2}^{\rho} x d\mu =& 1/2 \int_{S_1^{-1}[\rho^2,\rho]} S_1(x) d\mu + 1/2 \int_{S_2^{-1}[\rho^2,\rho]} S_2(x) d\mu \\ =& 1/2 \int_{\rho}^{1} \rho x d\mu + 1/2 \int_0^{\rho^2} (\rho x + \rho^2) d\mu \\ =& \frac{1}{2}\rho \int_{\rho}^{1} x d\mu + \frac{1}{2}\rho \int_0^{\rho^2} x d\mu + \frac{1}{2}\rho^2 \int_0^{\rho^2} d\mu.\end{aligned}$$

Thus,

$$\frac{2+\rho}{2}\int_{\rho^2}^{\rho} x d\mu = \frac{1}{2}\rho\int_0^1 x d\mu + \frac{1}{2}\rho^2\int_0^{\rho^2} x d\mu = \frac{1}{2}\rho\frac{1}{2} + \frac{1}{2}\rho^2\frac{1}{3} = \frac{1}{4}\rho + \frac{1}{6}\rho^2.$$

$$\begin{aligned}\int_{\rho^2}^{\rho} x d\mu =& \frac{2}{2+\rho}(\frac{1}{4}\rho + \frac{1}{6}\rho^2) = \frac{1}{2+\rho}(\frac{1}{2}\rho + \frac{1}{3}\rho^2) \\ =& \frac{1}{2+\rho}(\frac{3\rho+2\rho^2}{6}) = \frac{1}{2+\rho}\frac{2+\rho^2}{6} = \frac{1}{6}.\end{aligned} \tag{A.1.3}$$

Therefore, $(A.1.2)$ becomes

$$\int_0^{\rho^2} x d\mu = \frac{\rho}{6(2-\rho)}. \tag{A.1.4}$$

$$\begin{aligned}\int_{\rho}^{1} x d\mu =& 1/2\int_{S_1^{-1}[\rho,1]} S_1(x) d\mu + 1/2\int_{S_2^{-1}[\rho,1]} S_2(x) d\mu \\ =& 1/2\int_1^{1+\rho} \rho x d\mu + 1/2\int_{\rho^2}^{1} (\rho x + \rho^2) d\mu \\ =& \frac{1}{2}\rho\int_{\rho^2}^{1} x d\mu + \frac{1}{2}\rho^2\int_{\rho^2}^{1} x d\mu. \\ \Rightarrow & (1-\frac{1}{2}\rho)\int_{\rho}^{1} x d\mu = \frac{1}{2}\int_{\rho^2}^{\rho} \rho x d\mu + \frac{\rho^2}{3}.\end{aligned}$$

$$\Rightarrow \int_{\rho}^{1} x d\mu = \frac{2}{2-\rho}(\frac{\rho}{12} + \frac{\rho^2}{3}) = \frac{2}{2-\rho}(\frac{\rho+4\rho^2}{12}) = \frac{1+3\rho^2}{6(2-\rho)} = \frac{4-3\rho}{6(2-\rho)}$$

Therefore,

$$\begin{aligned}\int_0^1 x d\mu \circ T_1 =& \int_{T_1[0,1]} T_1^{-1}(x) d\mu = \int_0^{\rho^2} \rho^{-2} d\mu = \rho^{-2}\frac{\rho}{6(2-\rho)} \\ =& \frac{1}{6\rho(2-\rho)} = \frac{1}{6(3\rho-1)}\end{aligned}$$

$$\int_0^1 x d\mu \circ T_2 = \int_{T_2[0,1]} T_2^{-1}(x) d\mu = \int_{\rho^2}^{\rho} [\rho^{-3}x - (1+\rho)] d\mu = \int_{\rho^2}^{\rho} \rho^{-3} x d\mu - \int_{\rho^2}^{\rho} (1+\rho) d\mu$$

$$\Rightarrow \int_0^1 x d\mu \circ T_2 = \rho^{-3}\frac{1}{6} - (1+\rho)\frac{1}{3} = \frac{2+\rho}{6\rho} - \frac{1+\rho}{3} = \frac{1}{6}$$

$$\begin{aligned}\int_0^1 x d\mu \circ T_3 &= \int_{T_3[0\ 1]} T_3^{-1}(x) d\mu = \int_\rho^1 [\rho^{-2}x - (1+\rho)] d\mu \\ &= \int_\rho^1 \rho^{-2} x d\mu - \int_\rho^1 (1+\rho) d\mu\end{aligned}$$

$$\begin{aligned}\Rightarrow \int_0^1 x d\mu \circ T_3 =& \rho^{-2}\frac{4-3\rho}{6(2-\rho)} - (1+\rho)\frac{1}{3} = \frac{5+\rho}{6(2-\rho)} - \frac{1+\rho}{3} \\ =& \frac{3\rho^2}{6(2-\rho)} = \frac{\rho}{2(1+2\rho)} = \frac{1}{2(3+\rho)}\end{aligned}$$

Finally, we consider the third row

$$\begin{aligned}\int_0^1 x^2 d\mu =& \frac{1}{2}\int_{S_1^{-1}[0,1]} (S_1(x))^2 d\mu + \frac{1}{2}\int_{S_2^{-1}[0,1]} (S_2(x))^2 d\mu \\ =& \frac{1}{2}\int_0^1 \rho^2 x^2 d\mu + \frac{1}{2}\int_0^1 (\rho x + \rho^2)^2 d\mu \\ =& \frac{1}{2}\int_0^1 \rho^2 x^2 d\mu + \frac{1}{2}\int_0^1 \rho^2 x^2 d\mu + \rho^3 \int_0^1 x d\mu + \frac{1}{2}\rho^4 \int_0^1 d\mu\end{aligned}$$

$$\Rightarrow \quad (1-\rho^2)\int_0^1 x^2 d\mu = \frac{1}{2}(2\rho - 1) + \frac{1}{2}(2-3\rho)1 = \frac{1-\rho}{2}$$

$$\Rightarrow \quad \int_0^1 x^2 d\mu = \frac{1-\rho}{2(1-\rho^2)} = \frac{\rho}{2}$$

Moreover,

$$\begin{aligned}\int_0^{\rho^2} x^2 d\mu =& \frac{1}{2}\int_{S_1^{-1}[0,\rho^2]} (S_1(x))^2 d\mu + \frac{1}{2}\int_{S_2^{-1}[0,\rho^2]} (S_2(x))^2 d\mu \\ =& \frac{1}{2}\int_0^\rho \rho^2 x^2 d\mu + \frac{1}{2}\int_{-\rho}^0 (\rho x + \rho^2)^2 d\mu = \frac{1}{2}\int_0^{\rho^2} \rho^2 x^2 d\mu + \frac{1}{2}\int_{\rho^2}^\rho \rho^2 x^2 d\mu\end{aligned}$$

$$\Rightarrow \quad (2-\rho^2)\int_0^{\rho^2} x^2 d\mu = \rho^2 \int_{\rho^2}^\rho x^2 d\mu$$

$$\Rightarrow \quad \int_0^{\rho^2} x^2 d\mu = \frac{\rho^2}{2-\rho^2}\int_{\rho^2}^{\rho} x^2 d\mu = (2\rho-1)\int_{\rho^2}^{\rho} x^2 d\mu$$

Moreover,

$$\begin{aligned}
\int_{\rho^2}^{\rho} x^2 d\mu =& \frac{1}{2}\int_{S_1^{-1}[\rho^2,\rho]} (S_1(x))^2 d\mu + \frac{1}{2}\int_{S_2^{-1}[\rho^2,\rho]} (S_2(x))^2 d\mu \\
=& \frac{1}{2}\int_{\rho}^{1} \rho^2 x^2 d\mu + \frac{1}{2}\int_0^{\rho^2} (\rho x+\rho^2)^2 d\mu \\
=& \frac{1}{2}\int_{\rho}^{1} \rho^2 x^2 d\mu + \frac{1}{2}\int_0^{\rho^2} \rho x^2 d\mu + \rho^3\int_0^{\rho^2} x d\mu + \frac{1}{2}\rho^4\int_0^{\rho^2} d\mu
\end{aligned}$$

$$\Rightarrow (2+\rho^2)\int_{\rho^2}^{\rho} x^2 d\mu = \rho^2\int_0^1 x^2 d\mu + 2\rho^3\int_0^1 x d\mu + \rho^4\int_0^1 d\mu$$

$$\Rightarrow (2+\rho^2)\int_{\rho^2}^{\rho} x^2 d\mu = \rho^2\frac{\rho}{2} + 2(2\rho-1)\frac{\rho}{6(2-\rho)} + \frac{2-3\rho}{3}$$

$$\Rightarrow (2+\rho^2)\int_{\rho^2}^{\rho} x^2 d\mu = \frac{2\rho-1}{2} + \frac{4\rho^2-2\rho}{6(2-\rho)} + \frac{2-3\rho}{3}$$

$$\begin{aligned}
\Rightarrow (2+\rho^2)\int_{\rho^2}^{\rho} x^2 d\mu =& \frac{3(2\rho-1)(2-\rho)+4\rho^2-2\rho+2(2-3\rho)(2-\rho)}{6(2-\rho)} \\
=& \frac{2-3\rho+4\rho^2}{6(2-\rho)}
\end{aligned}$$

$$\Rightarrow \int_{\rho^2}^{\rho} x^2 = \frac{6-7\rho}{6(2+\rho^2)(2-\rho)} = \frac{6-7\rho}{6(3-\rho)(2-\rho)} = \frac{6-7\rho}{6(7-6\rho)}$$

Therefore,

$$\int_0^{\rho^2} x^2 = \frac{1-\rho}{1+\rho}\int_{\rho^2}^{\rho} x^2 = \frac{1-\rho}{1+\rho}\,\frac{6-7\rho}{6(3-\rho)(2-\rho)} = \frac{6-7\rho-6\rho+7rho^2}{6(7-6\rho+7\rho-6\rho^2)} = \frac{13-20\rho}{6(1+7\rho)}$$

And,

$$\begin{aligned}
\int_{\rho}^{1} x^2 d\mu =& \frac{1}{2}\int_{S_1^{-1}[\rho,1]} (S_1(x))^2 d\mu + \frac{1}{2}\int_{S_2^{-1}[\rho,1]} (S_2(x))^2 d\mu \\
=& \frac{1}{2}\int_{1}^{1+\rho} \rho^2 x^2 d\mu + \frac{1}{2}\int_{\rho^2}^{1} (\rho x+\rho^2)^2 d\mu
\end{aligned}$$

$$\Rightarrow \int_{\rho}^{1} x^2 d\mu = \frac{1}{2}\int_{\rho}^{1} \rho^2 x^2 d\mu + \frac{1}{2}\int_{\rho^2}^{\rho} \rho^2 x^2 d\mu + \rho^3\int_{\rho^2}^{1} x d\mu + \frac{1}{2}\rho^4\int_{\rho^2}^{1} d\mu$$

$$\Rightarrow \int_\rho^1 x^2 d\mu = \frac{1}{2}\int_\rho^1 \rho^2 x^2 d\mu + \frac{1}{2}\int_{\rho^2}^\rho \rho^2 x^2 d\mu + \rho^3(\int_0^1 x d\mu - \int_0^{\rho^2} x d\mu) + \frac{1}{3}\rho^4$$

$$\Rightarrow (1-\frac{1}{2}\rho^2)\int_\rho^1 x^2 d\mu = \frac{1}{2}\rho^2 \ \frac{6-7\rho}{6(7-6\rho)} + \rho^3(\frac{1}{2} - \frac{\rho}{6(2-\rho)}) + \frac{1}{3}\rho^4$$

$$\Rightarrow (1-\frac{1}{2}\rho^2)\int_\rho^1 x^2 d\mu = \frac{1}{2}\rho^2 \ \frac{6-7\rho}{6(7-6\rho)} + \frac{1}{2}(2\rho-1) - \frac{2-3\rho}{6(2-\rho)}) + \frac{1}{3}(2-3\rho)$$

$$\Rightarrow (1-\frac{1}{2}\rho^2)\int_\rho^1 x^2 d\mu = \frac{1}{2}\rho^2 \ \frac{6-7\rho}{6(7-6\rho)} + \frac{1}{6} - \frac{2-3\rho}{6(2-\rho)})$$

$$\Rightarrow (1-\frac{1}{2}\rho^2)\int_\rho^1 x^2 d\mu = \frac{\rho^2(6-7\rho)(2-\rho)+2(7-6\rho)(2-\rho)-2(2-3\rho)(7-6\rho)}{12(7-6\rho)(2-\rho)}$$

$$\Rightarrow (1-\frac{1}{2}\rho^2)\int_\rho^1 x^2 d\mu = \frac{(19-46\rho+27\rho^2)+(-14\rho+12\rho^2+42\rho-36\rho^2)}{12(14-19\rho+6\rho^2)}$$

$$\Rightarrow (1-\frac{1}{2}\rho^2)\int_\rho^1 x^2 d\mu = \frac{19-18\rho-3\rho^2}{12(20-25\rho)} = \frac{22-21\rho}{12(20-25\rho)}$$

$$\Rightarrow \int_\rho^1 x^2 d\mu = \frac{22-21\rho}{6(20-25\rho)(2-\rho^2)} = \frac{22-21\rho}{6(20-25\rho)(1+\rho)}$$

$$\Rightarrow \int_\rho^1 x^2 d\mu = \frac{22-21\rho}{6(20-5\rho-25\rho^2)} = \frac{22-21\rho}{6(20\rho-5)}$$

Therefore,

$$\int_0^1 x^2 d\mu \circ T_1 = \int_{T_1[0,1]} (T_1^{-1}(x))^2 d\mu = \int_0^{\rho^2} \rho^{-4} x^2 d\mu = \frac{1}{2-3\rho}\frac{13-20\rho}{6(1+7\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_1 = \frac{13-20\rho}{6(2+11\rho-21\rho^2)} = \frac{13-20\rho}{6(32\rho-19)} = \frac{6-7\rho}{6(13\rho-6)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_1 = \frac{-1+6\rho}{6(7-6\rho)} = \frac{5\rho+4}{6(8+\rho)}$$

$$\int_0^1 x^2 d\mu \circ T_2 = \int_{T_2[0,1]} (T_2^{-1}(x))^2 d\mu = \int_{\rho^2}^\rho (\rho^{-3}x - (1+\rho)^2 d\mu$$

$$= \int_{\rho^2}^\rho (\rho^{-6}x^2 - 2\rho^3(1+\rho)x + (1+\rho)^2 d\mu$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_2 = \frac{11}{5-8\rho}\frac{6-7\rho}{6(7-6\rho)} - \frac{2(1+\rho)}{2\rho-1}\frac{1}{6} + (1+\rho)^2\frac{1}{3}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_2 = \frac{6-7\rho}{6(35-86\rho+48\rho^2)} - \frac{(1+\rho)}{3(2\rho-1)} + \frac{2+\rho}{3}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_2 = \frac{6-7\rho}{6(83-134\rho)} - \frac{(1+\rho)-(2-\rho)(2\rho-1)}{3(2\rho-1)}$$

$$= \frac{-1+6\rho}{6(-51+83\rho)} - \frac{1}{3(2\rho-1)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_2 = \frac{5-\rho}{6(32-51\rho)} - \frac{1}{3(2\rho-1)} = \frac{(5-\rho)(2\rho-1)-2(32-51\rho)}{6(32-51\rho)(2\rho-1)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_2 = \frac{-69+113\rho-2\rho^2}{6(-32+115\rho-102\rho^2)} = \frac{-71+115\rho}{6(-134+217\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_2 = \frac{44-71\rho}{6(83-134\rho)} = \frac{-27+44\rho}{6(-51+83\rho)} = \frac{17-27\rho}{6(32-51\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_2 = \frac{-10+17\rho}{6(-19+32\rho)} = \frac{7-10\rho}{6(13-19\rho)} = \frac{-3+7\rho}{6(-6+13\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_2 = \frac{4-3\rho}{6(7-6\rho)} = \frac{1+4\rho}{6(1+7\rho)} = \frac{5+\rho}{6(8+\rho)}$$

$$\int_0^1 x^2 d\mu \circ T_3 = \int_{T_3[0,1]} (T_3^{-1}(x))^2 d\mu = \int_\rho^1 (\rho^{-2}x - (1+\rho)^2 d\mu$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \int_\rho^1 (\rho^{-4}x^2 - 2\rho^{-2}(1+\rho)x + (1+\rho)^2 d\mu$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{1}{2-3\rho}\frac{22-21\rho}{6(20\rho-5)} - \frac{2(1+\rho)}{1-\rho}\frac{4-3\rho}{6(2-\rho)} + (1+\rho)^2\frac{1}{3}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{22-21\rho}{6(-10+55\rho-60\rho^2)} - \frac{(1+\rho)(4-3\rho)}{3(1-\rho)(2-\rho)} + \frac{2+\rho}{3}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{22-21\rho}{6(-70+115\rho)} - \frac{(1+4\rho)}{3((2-3\rho+\rho^2)} + \frac{2+\rho}{3}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{22-21\rho}{6(-70+115\rho)} - \frac{(1+4\rho)+(2+\rho)(3-4\rho)}{3(3-4\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{1+22\rho}{6(45-70\rho)} - \frac{-5+9\rho+4\rho^2}{3(3-4\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{1+22\rho}{6(45-70\rho)} - \frac{-1+5\rho}{3(3-4\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{(1+22\rho)(3-4\rho) - 2(-1+5\rho)(45-70\rho)}{6(45-70\rho)(3-4\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{(3-62\rho-88\rho^2) - 2(-45+295\rho-350\rho^2)}{6(135-390\rho+280\rho^2)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{5-440\rho-700\rho^2}{6(415-670\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{705-1140\rho}{6(415-670\rho)} = \frac{-435+705\rho}{6(-255+415\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{270-435\rho}{6(160-255\rho)} = \frac{54-87\rho}{6(32-51\rho)} = \frac{-33+54\rho}{6(-19+32\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{21-33\rho}{6(13-19\rho)} = \frac{-12+21\rho}{6(-6+13\rho)} = \frac{9-12\rho}{6(7-6\rho)}$$

$$\Rightarrow \int_0^1 x^2 d\mu \circ T_3 = \frac{-3+9\rho}{6(1+7\rho)} = \frac{6-3\rho}{6(8+\rho)} = \frac{2-\rho}{2(8+\rho)}$$

This is to complete of proof of the claim of (5.1.4). □

A.2 THREE-FOLD CONVOLUTION OF THE CANTOR MEASURE

$$S_1(x) = \frac{1}{3}x, \quad S_2(x) = \frac{1}{3}x + \frac{2}{3}, \quad S_3(x) = \frac{1}{3}x + \frac{4}{3}, \quad S_4(x) = \frac{1}{3}x + 2$$

$$S_1^{-1}(x) = 3x, \quad S_2^{-1}(x) = 3x - 2, \quad S_3^{-1}(x) = 3x - 4, \quad S_4^{-1}(x) = 3x - 6$$

$$T_1(x) = \frac{1}{3}x, \quad T_2(x) = \frac{1}{3}x + 1, \quad T_3(x) = \frac{1}{3}x + 2$$

$$T_1^{-1}(x) = 3x, \quad T_2^{-1}(x) = 3x - 3, \quad T_3^{-1}(x) = 3x - 6$$

$$\int_A f(x) d\mu = \frac{1}{8}\int_{S_1^{-1}[A]} f(S_1 x) d\mu + \frac{3}{8}\int_{S_2^{-1}[A]} f(S_2 x) d\mu$$

$$+ \frac{3}{8}\int_{S_3^{-1}[A]} f(S_3 x) d\mu + \frac{1}{8}\int_{S_4^{-1}[A]} f(S_4 x) d\mu$$

$$\int_A f(x)d\mu \circ T_i = \int_{T_i[A]} f(T_i^{-1}x)d\mu$$

But, we have

$$\int_\Omega f(Tx)d\mu = \int_{\Omega'} f(y)d\mu \circ T^{-1}(y)$$

Our goal is to claim that (5.2.5) is true.

Proof.

$$\begin{bmatrix} \int_0^3 d\mu \circ T_1 & \int_0^3 d\mu \circ T_2 & \int_0^3 d\mu \circ T_3 \\ \\ \int_0^3 xd\mu \circ T_1 & \int_0^3 xd\mu \circ T_2 & \int_0^3 xd\mu \circ T_3 \\ \\ \int_0^3 x^2d\mu \circ T_1 & \int_0^3 x^2d\mu \circ T_2 & \int_0^3 x^2d\mu \circ T_3 \\ \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \\ \frac{27}{70} & \frac{9}{10} & \frac{3}{14} \\ \\ \frac{5517}{6440} & \frac{11943}{6440} & \frac{63}{184} \\ \end{bmatrix}$$

Proof : Consider the first row:

$$\int_0^1 d\mu = \frac{1}{8}\int_{S_1^{-1}[0\ 1]} d\mu + \frac{3}{8}\int_{S_2^{-1}[0\ 1]} d\mu + \frac{3}{8}\int_{S_3^{-1}[0\ 1]} d\mu + \frac{1}{8}\int_{S_4^{-1}[0\ 1]} d\mu$$

$$\Rightarrow \int_0^1 d\mu = \frac{1}{8}\int_0^3 d\mu + \frac{3}{8}\int_{-2}^1 d\mu + \frac{3}{8}\int_{-4}^{-1} d\mu + \frac{1}{8}\int_{-6}^{-3} d\mu$$

$$\Rightarrow \frac{5}{8}\int_0^1 d\mu = \frac{1}{8}\int_0^3 d\mu$$

$$\Rightarrow \int_0^1 d\mu = \frac{1}{5}\int_0^3 d\mu = \frac{1}{5}$$

$$\int_1^2 d\mu = \frac{1}{8}\int_{S_1^{-1}[1\ 2]} d\mu + \frac{3}{8}\int_{S_2^{-1}[1\ 2]} d\mu + \frac{3}{8}\int_{S_3^{-1}[1\ 2]} d\mu + \frac{1}{8}\int_{S_4^{-1}[1\ 2]} d\mu$$

$$\Rightarrow \int_1^2 d\mu = \frac{1}{8}\int_3^6 d\mu + \frac{3}{8}\int_1^4 d\mu + \frac{3}{8}\int_{-1}^2 d\mu + \frac{1}{8}\int_{-3}^0 d\mu$$
$$\Rightarrow \int_1^2 d\mu = \frac{3}{8}(1-\frac{1}{5}) + \frac{3}{8}(\frac{1}{5}+\int_1^2 d\mu)$$
$$\Rightarrow (1-\frac{3}{8})\int_1^2 d\mu = \frac{3}{10} + \frac{3}{40} = \frac{15}{40}$$
$$\Rightarrow \int_1^2 d\mu = (\frac{15}{40})(\frac{8}{5}) = \frac{3}{5}$$

$$\int_2^3 d\mu = \frac{1}{8}\int_{S_1^{-1}[2,3]} d\mu + \frac{3}{8}\int_{S_2^{-1}[2,3]} d\mu + \frac{3}{8}\int_{S_3^{-1}[2,3]} d\mu + \frac{1}{8}\int_{S_4^{-1}[2,3]} d\mu$$
$$\Rightarrow \int_2^3 d\mu = \frac{1}{8}\int_6^9 d\mu + \frac{3}{8}\int_4^7 d\mu + \frac{3}{8}\int_2^5 d\mu + \frac{1}{8}\int_0^3 d\mu$$
$$\Rightarrow \int_2^3 d\mu = \frac{3}{8}\int_2^3 d\mu + \frac{1}{8}$$
$$\Rightarrow \int_2^3 d\mu = \frac{1}{5}$$

Therefore,

$$\int_0^3 d\mu \circ T_1 = \int_{T_1[0,3]} d\mu = \int_0^1 d\mu = \frac{1}{5}$$

$$\int_0^3 d\mu \circ T_2 = \int_{T_2[0,3]} d\mu = \int_1^2 d\mu = \frac{3}{5}$$

$$\int_0^3 d\mu \circ T_3 = \int_{T_3[0,3]} d\mu = \int_2^3 d\mu = \frac{1}{5}$$

Now consider the 2nd row:

$$\begin{aligned}\int_0^3 x d\mu =& \frac{1}{8}\int_{S_1^{-1}[0,3]} S_1(x) d\mu + \frac{3}{8}\int_{S_2^{-1}[0,3]} S_2(x) d\mu \\ &+ \frac{3}{8}\int_{S_3^{-1}[0,3]} S_3(x) d\mu + \frac{1}{8}\int_{S_4^{-1}[0,3]} S_4(x) d\mu\end{aligned}$$

$$\Rightarrow \int_0^3 xd\mu = \frac{1}{8}\int_0^9 \frac{1}{3}xd\mu + \frac{3}{8}\int_{-2}^7 (\frac{1}{3}x + \frac{2}{3})d\mu + \frac{3}{8}\int_{-4}^5 (\frac{1}{3}x + \frac{4}{3})d\mu + \frac{1}{8}\int_{-6}^3 (\frac{1}{3}x + 2)d\mu$$

$$\Rightarrow \int_0^3 xd\mu = \frac{1}{24}\int_0^3 xd\mu + \frac{1}{8}\int_0^3 xd\mu \frac{1}{4} + \frac{1}{8}\int_0^3 xd\mu + \frac{1}{2} + \frac{1}{24}\int_0^3 xd\mu + \frac{1}{4}$$

$$\Rightarrow \int_0^3 xd\mu = \frac{3}{2}$$

$$\int_0^1 xd\mu = \frac{1}{8}\int_0^3 \frac{1}{3}xd\mu + \frac{3}{8}\int_{-2}^1 (\frac{1}{3}x + \frac{2}{3})d\mu + 0$$

$$\Rightarrow \frac{7}{8}\int_0^1 xd\mu = \frac{1}{24}\int_0^3 xd\mu + \frac{1}{4}\int_0^1 d\mu$$

$$\Rightarrow \int_0^1 xd\mu = \frac{1}{21}\int_0^3 xd\mu + (\frac{4}{14})(\frac{1}{5}) = (\frac{1}{21})(\frac{3}{2}) + (\frac{4}{14})(\frac{1}{5}) = \frac{9}{70}$$

Moreover,

$$\int_1^2 xd\mu = 0 + \frac{3}{8}\int_1^4 (\frac{1}{3}x + \frac{1}{3})d\mu + \frac{3}{8}\int_{-1}^2 (\frac{1}{3}x + \frac{4}{3})d\mu + 0$$

$$\int_1^2 xd\mu = \frac{1}{8}\int_1^3 xd\mu + \frac{1}{4}\int_1^3 d\mu + \frac{1}{8}\int_0^2 xd\mu + \frac{1}{2}\int_0^2 d\mu$$

$$\Rightarrow \frac{7}{8}\int_1^2 xd\mu = \frac{1}{8}\int_0^3 xd\mu + \frac{1}{4} + \frac{1}{4}\int_0^1 d\mu + \frac{1}{2}\int_1^2 d\mu$$

$$\Rightarrow \frac{7}{8}\int_1^2 xd\mu = (\frac{1}{8})(\frac{3}{2}) + (\frac{1}{4}) + (\frac{1}{4})(\frac{1}{5}) + (\frac{1}{2})(\frac{3}{5}) = \frac{63}{80}$$

$$\Rightarrow \int_1^2 xd\mu = \frac{9}{10}$$

Moreover,

$$\int_2^3 xd\mu = 0 + \frac{3}{8}\int_2^5 (\frac{1}{3}x + \frac{4}{3})d\mu + \frac{1}{8}\int_0^3 (\frac{1}{3}x + 2)d\mu + 0$$

$$\Rightarrow \int_2^3 xd\mu = \frac{1}{8}\int_2^3 xd\mu + \frac{1}{2}\int_2^3 d\mu + \frac{1}{24}\int_0^3 xd\mu + \frac{1}{4}\int_0^3 d\mu$$

$$\Rightarrow \frac{7}{8}\int_2^3 xd\mu = \frac{1}{2}\ \frac{1}{5} + \frac{1}{24}\ \frac{3}{2} + \frac{1}{4}\ 1 = \frac{33}{80}$$

$$\Rightarrow \int_2^3 xd\mu = \frac{33}{70}$$

Therefore,

$$\int_0^3 xd\mu \circ T_1 = \int_{T_1[0,3]} T_1^{-1}(x)d\mu = \int_0^1 3xd\mu == \frac{27}{70}$$

$$\int_0^3 xd\mu \circ T_2 = \int_{T_2[0,3]} T_2^{-1}(x)d\mu = \int_1^2 (3x-3)d\mu = 3\ (\frac{9}{10} - \frac{3}{5}) = \frac{9}{10}$$

$$\int_0^3 xd\mu \circ T_3 = \int_{T_3[0,3]} T_3^{-1}(x)d\mu = \int_2^3 (3x-6)d\mu = 3\ \frac{33}{70} - 6\ \frac{1}{5} = \frac{3}{14}$$

Consider the third row :

$$\begin{aligned}\int_0^3 x^2 d\mu =& \frac{1}{8}\int_{S_1^{-1}[0,3]} (S_1(x))^2 d\mu + \frac{3}{8}\int_{S_2^{-1}[0,3]} (S_2(x))^2 d\mu \\ &+ \frac{3}{8}\int_{S_3^{-1}[0,3]} (S_3(x))^2 d\mu + \frac{1}{8}\int_{S_4^{-1}[0,3]} (S_4(x))^2 d\mu\end{aligned}$$

$$\begin{aligned}\Rightarrow \int_0^3 x^2 d\mu =& \frac{1}{8}\int_0^9 \left(\frac{1}{3}x\right)^2 d\mu + \frac{3}{8}\int_{-2}^7 (\frac{1}{3}x + \frac{2}{3})^2 d\mu \\ &+ \frac{3}{8}\int_{-4}^5 (\frac{1}{3}x + \frac{4}{3})^2 d\mu + \frac{1}{8}\int_{-6}^3 (\frac{1}{3}x + 2)^2 d\mu\end{aligned}$$

$$\Rightarrow \begin{aligned}\int_0^3 x^2 d\mu =& \frac{1}{72}\int_0^3 x^2 d\mu + \frac{3}{72}\int_0^3 x^2 d\mu + \frac{12}{72}\int_0^3 xd\mu + \frac{12}{72}\int_0^3 d\mu + \frac{3}{72}\int_0^3 x^2 d\mu \\ &+ \frac{24}{72}\int_0^3 xd\mu + \frac{48}{72}\int_0^3 d\mu + \frac{1}{72}\int_0^3 x^2 d\mu + \frac{12}{72}\int_0^3 xd\mu + \frac{36}{72}\int_0^3 d\mu\end{aligned}$$

$$\Rightarrow \int_0^3 x^2 d\mu = \frac{1}{9}\int_0^3 x^2 d\mu + \frac{48}{72}\int_0^3 xd\mu + \frac{12}{72} + \frac{48}{72} + \frac{36}{72}$$

$$\Rightarrow \frac{8}{9}\int_0^3 x^2 d\mu = (\frac{48}{72})(\frac{3}{2}) + \frac{4}{3} = \frac{7}{3}$$

$$\Rightarrow \int_0^3 x^2 d\mu = \frac{21}{8}$$

$$\int_0^1 x^2 d\mu = \frac{1}{8}\int_0^3 (\frac{1}{3}x)^2 d\mu + \frac{3}{8}\int_{-2}^1 (\frac{1}{3}x + \frac{2}{3})^2 d\mu$$

$$\Rightarrow \int_0^1 x^2 d\mu = \frac{1}{72}\int_0^3 x^2 d\mu + \frac{3}{72}\int_0^1 x^2 d\mu + \frac{12}{72}\int_0^1 x d\mu + \frac{12}{72}\int_0^1 d\mu$$

$$\Rightarrow \frac{69}{72}\int_0^1 x^2 d\mu = (\frac{1}{72})(\frac{21}{8}) + (\frac{12}{72})(\frac{9}{10}) + (\frac{12}{72})(\frac{1}{5}) = \frac{613}{6720}$$

$$\Rightarrow \int_0^1 x^2 d\mu = \frac{613}{6440}$$

$$\int_1^2 x^2 d\mu = 0 + \frac{3}{8}\int_1^4 (\frac{1}{3}x + \frac{2}{3})^2 d\mu + \frac{3}{8}\int_{-2}^1 (\frac{1}{3}x + \frac{4}{3})^2 d\mu + 0$$

$$\Rightarrow \int_1^2 x^2 d\mu = \frac{3}{72}\int_1^3 x^2 d\mu + \frac{12}{72}\int_1^3 x d\mu + \frac{12}{72}\int_1^3 d\mu + \frac{3}{72}\int_0^2 x^2 + \frac{24}{72}\int_0^2 x d\mu + \frac{48}{72}\int_0^2 d\mu$$

$$\Rightarrow \int_1^2 x^2 d\mu = \frac{3}{72}\int_1^3 x^2 d\mu + \frac{12}{72}\int_1^3 x d\mu + \frac{12}{72}\ \frac{4}{5} + \frac{3}{72}\int_0^2 x^2 + \frac{24}{72}\int_0^2 x d\mu + \frac{48}{72}\ \frac{4}{5}$$

$$\Rightarrow \frac{69}{72}\int_1^2 x^2 d\mu = (\frac{3}{72})(\frac{21}{8}) + \frac{12}{72}(\frac{3}{2} - \frac{9}{70}) + \frac{24}{72}(\frac{9}{70} + \frac{9}{10}) + (\frac{12}{72})(\frac{4}{5}) + (\frac{48}{72})(\frac{4}{5}) = \frac{1811}{1344}$$

$$\Rightarrow \int_1^2 x^2 d\mu = \frac{1811}{1288}$$

$$\int_2^3 x^2 d\mu = 0 + \frac{3}{8}\int_2^5 (\frac{1}{3}x + \frac{4}{3})^2 d\mu + \frac{1}{8}\int_0^3 (\frac{1}{3}x + 2)^2 d\mu$$

Thus, we have

$$\int_2^3 x^2 d\mu = \frac{3}{72}\int_2^3 x^2 d\mu + \frac{24}{72}\int_2^3 x d\mu + \frac{48}{72}\int_2^3 d\mu + \frac{1}{72}\int_0^3 x^2 d\mu + \frac{12}{72}\int_0^3 x d\mu + \frac{36}{72}\int_0^3 d\mu$$

$$\Rightarrow \frac{69}{72}\int_2^3 x^2 d\mu = (\frac{24}{72})(\frac{33}{70}) + (\frac{48}{72})(\frac{1}{5}) + (\frac{1}{72})(\frac{21}{8}) + (\frac{12}{72})(\frac{3}{2}) + (\frac{36}{72})1 = \frac{7237}{6720}$$

$$\Rightarrow \int_2^3 x^2 d\mu = \frac{7237}{6440}$$

Therefore,

$$\int_0^3 x^2 d\mu \circ T_1 = \int_{T_1[0,3]} (T_1^{-1}(x))^2 d\mu = \int_0^1 (3x)^2 d\mu = 9 \int_0^1 x^2 d\mu = 9(\frac{613}{6440}) = \frac{5517}{6440}$$

$$\int_0^3 x^2 d\mu \circ T_2 = \int_{T_2[0,3]} (T_2^{-1}(x))^2 d\mu = \int_1^2 (3x-3)^2 d\mu$$

$$\Rightarrow \int_0^3 x^2 d\mu \circ T_2 = 9 \int_1^2 (x^2 - 2x + 1) d\mu = 9 \int_1^2 x^2 d\mu - 18 \int_1^2 x d\mu + 9 \int_2^2 d\mu$$

$$\Rightarrow \int_0^3 x^2 d\mu \circ T_2 = 9(\frac{1811}{1288}) - 18(\frac{9}{10}) + 9(\frac{3}{5}) = \frac{11943}{6440}$$

$$\int_0^3 x^2 d\mu \circ T_3 = \int_{T_3[0,3]} (T_3^{-1}(x))^2 d\mu = \int_2^3 (3x-6)^2 d\mu$$

$$\Rightarrow \int_0^3 x^2 d\mu \circ T_3 = 9 \int_2^3 (x^2 - 4x + 4) d\mu = 9 \int_2^3 x^2 d\mu - 36 \int_2^3 x d\mu + 36 \int_2^3 d\mu$$

$$\Rightarrow \int_0^3 x^2 d\mu \circ T_3 = 9(\frac{7237}{6440}) - 36(\frac{33}{70}) + 36(\frac{1}{5}) = \frac{63}{184}$$

This is to complete of proof of the claim of (5.2.5).

□

A.3 INVERTIBILITY OF M

Theorem A.3.1. *(see [2]) If f is nonnegative, then*

$$\int_\Omega f(T\omega)\mu(d\omega) = \int_{\Omega'} f(\omega')\mu T^{-1}(d\omega'). \tag{A.3.5}$$

A function f (not necessarily nonnegative) is integrable with respect to μT^{-1} if and only if fT is integrable with respect to μ, in which case (A.3.5) and

$$\int_{T^{-1}A'} f(T\omega)\mu(d\omega) = \int_{A'} f(\omega')\mu T^{-1}(d\omega') \tag{A.3.6}$$

holds. For nonnegative f, (A.3.6) always holds.

Lemma A.3.2. *From the weighted Lebesgue measure, the infinite Bernoulli convolution associated with the golden ratio, we have show that (5.1.4) is true. For $i = 1,2,3$, $\int_0^1 (2x^2 - x)\, d\mu \circ T_i > 0$ and $\int_0^1 (2x^2 - 3x + 1)\, d\mu \circ T_i > 0$. Thus, $\int_0^1 (2x^2 - x)\, d\mu \circ T_J > 0$ and $\int_0^1 (2x^2 - 3x + 1)\, d\mu \circ T_J > 0$. Then, $\mathbf{M}$ is invertible in this case.*

Proof.

$$\int_0^1 (2x^2 - x)\, d\mu \circ T_1 = \frac{2(5\rho + 4)}{6(\rho + 8)} - \frac{1}{6(3\rho - 1)} = \frac{1}{6}\frac{13\rho - 16 + 30\rho^2}{(3\rho - 1)(\rho + 8)} > 0$$

and

$$\int_0^1 (2x^2 - 3x + 1)\, d\mu \circ T_1 = \frac{2(5\rho + 4)}{6(\rho + 8)} - \frac{3}{6(3\rho - 1)} + \frac{1}{3} = \frac{1}{2}\frac{12\rho^2 + 19\rho - 16}{(3\rho - 1)(\rho + 8)} > 0.$$

$$\int_0^1 (2x^2 - x)\, d\mu \circ T_2 = \frac{2(\rho + 5)}{6(\rho + 8)} - \frac{1}{6} = \frac{1}{6}\frac{\rho + 2}{\rho + 8} > 0$$

and

$$\int_0^1 (2x^2 - 3x + 1)\, d\mu \circ T_2 = \frac{2(\rho + 5)}{6(\rho + 8)} - 3(\frac{1}{6}) + \frac{1}{3} = \frac{1}{6}\frac{\rho + 2}{\rho + 8} > 0.$$

$$\int_0^1 (2x^2 - x)\, d\mu \circ T_3 = \frac{2(2 - \rho)}{6(\rho + 8)} - \frac{1}{6(3\rho^2 + 3)} = -\frac{1}{18}\frac{7\rho - 4 - 12\rho^2 + 6\rho^3}{(\rho^2 + 1)(\rho + 8)} > 0$$

and

$$\int_0^1 (2x^2 - 3x + 1)\, d\mu \circ T_3 = \frac{2(2 - \rho)}{6(\rho + 8)} - \frac{3}{6(3\rho^2 + 3)} + \frac{1}{3} = \frac{1}{6}\frac{20\rho^2 - \rho + 12}{(\rho^2 + 1)(\rho + 8)} > 0.$$

□

Lemma A.3.3. *From the 3-fold convolutions of the Cantor measure, we have show that (5.2.5) is true. For* $i = 1, 2, 3,$ $\int_0^3 (\frac{2}{9}x^2 - \frac{1}{3}x)\, d\mu \circ T_i > 0$ *and* $\int_0^3 (\frac{2}{9}x^2 - x + 1)\, d\mu \circ T_i > 0$*. Thus,* $\int_0^3 (\frac{2}{9}x^2 - \frac{1}{3}x)\, d\mu \circ T_J > 0$ *and* $\int_0^3 (\frac{2}{9}x^2 - x + 1)\, d\mu \circ T_J > 0$*. Then,* $\mathbf{M}$ *is invertible in this case.*

Proof.

$$\int_0^3 (\frac{2}{9}x^2 - \frac{1}{3}x)\, d\mu \circ T_1 = \frac{2}{9}\frac{5517}{6440} - \frac{1}{3}\frac{27}{70} = \frac{199}{3220} > 0,$$

and

$$\int_0^3 (\frac{2}{9}x^2 - x + 1)\, d\mu \circ T_1 = \frac{2}{9}\frac{5517}{6440} - \frac{27}{70} + \frac{1}{5} = \frac{3}{644} > 0.$$

$$\int_0^3 (\frac{2}{9}x^2 - \frac{1}{3}x)\, d\mu \circ T_2 = \frac{2}{9}\frac{11943}{6440} - \frac{1}{3}\frac{9}{10} = \frac{361}{3220} > 0,$$

and

$$\int_0^3 (\frac{2}{9}x^2 - x + 1)\, d\mu \circ T_2 = \frac{2}{9}\frac{11943}{6440} - \frac{9}{10} + \frac{3}{5} = \frac{361}{3220} > 0.$$

$$\int_0^3 (\frac{2}{9}x^2 - \frac{1}{3}x)\, d\mu \circ T_3 = \frac{2}{9}\frac{63}{184} - \frac{1}{3}\frac{3}{14} = \frac{3}{644} > 0,$$

and

$$\int_0^3 (\frac{2}{9}x^2 - x + 1)\, d\mu \circ T_3 = \frac{2}{9}\frac{63}{184} - \frac{3}{14} + \frac{1}{5} = \frac{199}{3220} > 0.$$

□

APPENDIX B

NORMED SPACES INVOLVING TIME

B.1 COMPLETENESS OF $L^2([0,T];X)$

Theorem B.1.1. *Let X be a Banach space and for $1 \leq p \leq \infty$. Then $L^p([0,T];X)$ is a Banach space.*

Proof. For $1 \leq p < \infty$,

(1) For all $f \in L^p([0,T];X)$, we have $\mathcal{I}_f := \int_0^T \|f\|_X^p \, dt \geq 0$. $\mathcal{I}_f = 0$, if and only if $f = 0$ for Lebesgue a.e. $t \in [0,T]$.

(2) For all $f \in L^p([0,T];X)$ and $\alpha \in \mathbb{R}$, we have $\int_0^T \|\alpha f\|_X^p \, dt = \int_0^T |\alpha|^p \|f\|_X^p \, dt = |\alpha|^p \mathcal{I}_f$.

(3) For all $f, g \in L^p([0,T];X)$, we have $\int_0^T \|f+g\|_X^p \, dt \leq \int_0^T (\|f\|_X + \|g\|_X)^p \, dt \leq \mathcal{I}_f + \mathcal{I}_g$.

These three conditions hold for $p = \infty$ with its norm. Thus, for $1 \leq p \leq \infty$, $L^p([0,T];X)$ are normed spaces.

We modify the proof in [18]. Let $\{u_n\} \subset L^p([0,T];X)$ be a Cauchy sequence. It suffices to show that $\{u_n\}$ has a convergent subsequence. Let $n_1 \in \mathbb{N}$ such that

$$\|u_n - u_{n_1}\|_{L^p([0,T];X)} \leq 1/2 \text{ for all } n \geq n_1.$$

Let $n_2 \in \mathbb{N}$ such that $n_2 \geq n_1$ and

$$\|u_n - u_{n_2}\|_{L^p([0,T];X)} \leq 1/2^2 \text{ for all } n \geq n_2.$$

In general, for each $k \in N$, let $n_k \in N$ such that $n_k \geq n_{k-1}$ and

$$\|u_n - u_{n_k}\|_{L^p([0,T];X)} \leq 1/2^2 \text{ for all } n \geq n_k.$$

Now for each $m \in \mathbb{N}$, define $S_m : [0,T] \to [0,\infty]$ by

$$S_m(t) := \|u_{n_1}(t)\|_X + \sum_{k=1}^{m} \left\|u_{n_{k+1}}(t) - u_{n_k}(t)\right\|_X, t \in [0,T]. \tag{B.1.1}$$

Then $\{S_m\}$ is a monotone increasing sequence of real-valued functions on $[0,T]$. Applying the triangle inequality in the space $L^p([0,T],dt)$ to (B.1.1), yields

$$\begin{aligned}\|S_m\|_{L^p([0,T];X)} &= \|u_{n_1}\|_{L^p([0,T];X)} + \sum_{k=1}^{m} \left\|u_{n_{k+1}}(t) - u_{n_k}(t)\right\|_{L^p([0,T];X)} \\ &\leq \|u_{n_1}\|_{L^p([0,T];X)} + \sum_{k=1}^{m} 1/2^k = \|u_{n_1}\|_{L^p([0,T];X)} + 1 < \infty.\end{aligned} \tag{B.1.2}$$

Define $S : [0,T] \to [0,\infty]$ by

$$S(t) := \lim_{m\to\infty} S_m : [0,T]. \tag{B.1.3}$$

Then for $1 \leq p < \infty$, by the monotone convergence theorem and (B.1.2),

$$\int_0^T |S|^p \, dt = \lim_{m\to\infty} \int_0^T |S_m|^p \, dt < \infty.$$

Hence $S \in L^p([0,T],dt)$, and thus $S(t) < \infty$ for Lebesgue a.e. $t \in [0,T]$. The same holds by $(B.1.2)$ if $p = \infty$. Next, we note that for each $t \in [0,T]$,

$$\begin{aligned}u_{n_{m+1}}(t) =& u_{n_1}(t) + (u_{n_2}(t) - u_{n_1}(t)) + \cdots + ((u_{n_{m+1}}(t) - u_{n_m}(t)) \\ =& u_{n_1}(t) + \sum_{k=1}^{m} ((u_{n_{k+1}}(t) - u_{n_k}(t)).\end{aligned} \tag{B.1.4}$$

Moreover, by (B.1.3), the series $u_{n_1}(t) + \sum_{k=1}^{\infty}((u_{n_{k+1}}(t) - u_{n_k}(t))$ is absolutely summable in X, provided $S(t) < \infty$. Since X is a Banach space, the series is summable in X for such t. Hence for all t such that $S(t) < \infty$, we could define

$$u(t) := u_{n_1}(t) + \sum_{k=1}^{\infty}(u_{n_{k+1}}(t) - u_{n_k}(t)). \tag{B.1.5}$$

Equations (B.1.4) and (B.1.5) imply that

$$u(t) = \lim_{m\to\infty} u_{n_{m+1}}(t) \text{ in X, provided } S(t) < infty. \tag{B.1.6}$$

Moreover, for all such t,

$$\|u(t)\|_X = \lim_{m\to\infty} \left\|u_{n_{m+1}}(t)\right\|_X \leq S(t) < infty.$$

Hence, $\|u(t)\|_X \in L^p([0,T], dt)$. This means $u \in L^p([0,T], X)$. Lastly, we notice that for all t such that $S(t) < infty$,

$$\begin{aligned} \left\|u_{n_{m+1}}(t) - u(t)\right\|_X \leq& \left\|u_{n_{m+1}}(t)\right\|_X + \|u(t)\|_X \\ \leq& S(t) + \|u(t)\|_X \in L^p([0,T], dt). \end{aligned} \tag{B.1.7}$$

Combining (B.1.6), (B.1.6) and using the dominated convergence theorem, we get

$$\lim_{m\to\infty} \int_0^T \left\|u_{n_{m+1}}(t) - u(t)\right\|_X \, dt \quad = \quad 0.$$

That is , $\{u_{n_{m+1}}\}$ converges to u in $L^p([0,T], X)$. Thus, $L^p([0,t]; X)$ is a Banach space. □

Lemma B.1.2. *If X is a Hilbert space, and for all $f, g, h \in L^2([0,T]; X)$ and $< f, g >:= \int_0^T (f(t), g(t))_X \, dt$. Then,*

(1) $< f, f > \geq 0, < f, f >= 0$ if and only if $f = 0$ for a.e. $t \in [0,T]$;

(2) $< f, g >=< g, f >$;

(3) $< \alpha f, g >= \alpha < f, g >$ for all $\alpha \in \mathbb{R}$;

(4) $< f + h, g >=< f, g > + < h, g >$.

Thus, $L^2([0,T];X)$ *is a Hilbert space.*

Proof.

(1) $< f, f >= \int_0^T (f(t), f(t))_X \, dt = \int_0^T \|f(t)\|_X^2 \, dt \geq 0;$

$< f, f >= 0 \Leftrightarrow \int_0^T \|f(t)\|_X^2 \, dt = 0 \Leftrightarrow \|f(t)\|_X = 0$ for a.e. $\in [0,T] \Leftrightarrow f(t) = 0$ for a.e. $t \in [0,T]$;

(2) $< f, g >= \int_0^T (f(t), g(t))_X \, dt = \int_0^T (g(t), f(t))_X \, dt =< g, f >;$

(3) $< \alpha f, g >= \int_0^T (\alpha f(t), g(t))_X \, dt = \int_0^T \alpha (f(t), g(t))_X \, dt = \alpha < f, g >;$

(4) $< f + h, g >= \int_0^T (f(t) + h(t), g(t))_X \, dt = \int_0^T (f(t), g(t))_X + (h(t), g(t))_X \, dt =< f, g > + < h, g >$.

Thus, $L^2([0,T];X)$ is a Hilbert space. □

APPENDIX C

ABSOLUTE CONTINUITY

C.1 ABSOLUTE CONTINUITY OF $\|(u_m(x,t))_t\|_\mu^2$ AND $\mathcal{E}(u_m(x,t), u_m(x,t))$.

The following lemma is used in the proof of the theorem 2.0.8 in Chapter 2.

Lemma C.1.1. *$\|(u_m(x,t))_t\|_\mu^2$ and $\mathcal{E}(u_m(x,t), u_m(x,t))$ are absolute continuous functions on $t \in [0,T]$ for any $x \in \mathbb{R}$.*

Proof. Since

$$\frac{d}{dt}\left(\frac{1}{2}\|(u_m(x,t))_t\|_\mu^2\right) = ((u_m(x,t))_{tt}, (u_m(x,t))_t)_\mu$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\mathcal{E}(u_m(x,t), u_m(x,t))\right) = \mathcal{E}(u_m(x,t), (u_m(x,t))_t),$$

it suffices to show that

$$((u_m(x,t))_{tt}, (u_m(x,t))_t)_\mu$$

and

$$\mathcal{E}(u_m(x,t), (u_m(x,t))_t)$$

are continuous.

By using (2.0.6), (2.0.10) and (2), we have

$$\begin{aligned}\int_a^b (u_m(x,t))_{tt}(u_m(x,t))_t \, d\mu &= \int_a^b \sum_{k=1}^m \alpha''_{m,k}(t) w_k(x) \sum_{j=1}^m \alpha'_{m,j}(t) w_j(x) \, d\mu \\ &= \sum_{k=1}^m \alpha''_{m,k}(t)\alpha'_{m,k}(t).\end{aligned}$$

$$\int_a^b \nabla u_m(x,t)\nabla(u_m(x,t))_t\,dx = \sum_{k=1}^m \alpha_{m,k}(t)\lambda_k \int_a^b w_k(x)\nabla(u_m(x,t))_t\,d\mu$$
$$= \sum_{k=1}^m \lambda_k \alpha_{m,k}(t)\alpha'_{m,k}(t).$$

Thus,

$$\|(u_m(x,t))_t\|_\mu^2 = \|(u_m(x,0))_t\|_\mu^2 + 2((u_m(x,t))_{tt},(u_m(x,t))_t)_\mu$$

and

$$\mathcal{E}(u_m(x,t),u_m(x,t)) = \mathcal{E}(u_m(x,0),u_m(x,0)) + 2\mathcal{E}\Big(u_m(x,t),(u_m(x,t))_t\Big).$$

□

Lemma C.1.2 (Gronwall's inequality)**.** *Let $\eta(\cdot)$ be a non-negative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are non-negative, summable functions on $[0,T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}(\eta(0) + \int_0^t \psi(s)ds)$$

for all $t \in [0,T]$.

APPENDIX D

SEPARABILITY

The following lemma is used in the proof of the theorem 2.0.6 in Chapter 2.

Lemma D.0.3. *Any subset of separable Banach Space X, with norm $\| \ \|$, is separable.*

Proof. Let $Y = \{y_k\}_{k=1}^{\infty}$ be a countable dense subset of X. Let $n \in \mathbb{N}$. Then for each $a \in A$, let $\mathcal{B}(a, \frac{1}{n})$ be the open ball in $(X, \|\cdot\|_X$, with center a and radius $\frac{1}{n}$. For each such open ball, there exists $y \in Y \cap \mathcal{B}(a, \frac{1}{n})$. Let $\{y_n^{(m)}\}_{m=1}^{\infty} = Y \cap (\cup_{a\in A}\mathcal{B}(a, \frac{1}{n}))$. For each m, let $x_n^{(m)} \in A$ such that

$$\left\|x_n^{(m)} - y_n^{(m)}\right\|_X \leq \frac{1}{n}.$$

Let $\epsilon > 0$, be arbitrary and for all $x \in A$ be arbitrary. Let $N_\epsilon := \left\lfloor \frac{2}{\epsilon} \right\rfloor + 1$. Then there exists $y_{N_\epsilon}^{(m_0)} \in \mathcal{B}(x, \frac{1}{N_\epsilon}) \leq B(x, \epsilon)$. Then, there exists $x_{N_\epsilon}^{(m_0)} \in A$ such that

$$\left\|x_{N_\epsilon}^{(m_0)} - y_{N_\epsilon}^{(m_0)}\right\|_X \leq \frac{1}{N_\epsilon}.$$

Therefore,

$$\left\|x_{N_\epsilon}^{(m_0)} - x\right\|_X \leq \left\|x_{N_\epsilon}^{(m_0)} - y_{N_\epsilon}^{(m_0)}\right\|_X \leq \frac{1}{N_\epsilon} + \left\|y_{N_\epsilon}^{(m_0)} - x\right\|_X \leq \frac{1}{N_\epsilon} + \frac{1}{N_\epsilon} = \frac{2}{N_\epsilon} \leq \epsilon.$$

□

APPENDIX E

WEAK DERIVATIVE

E.1 BANACH SPACE VALUED FUNCTIONS

Let X be a seperable Banach space, with norm $\| \ \|$.

Definition E.1.1. *(i) If* $s(t) = \sum_{i=1}^{m} \chi_{E_i}(t) u_i$ *is simple, we define*

$$\int_0^T s(t)\, dt := \sum_{i=1}^{m} |E_i| u_i. \tag{E.1.1}$$

(ii) We say $f : [0, T] \to X$ *is* summable *if there exists a sequence* $\{s_k\}_{k=1}^{\infty}$ *of simple functions such that*

$$\int_0^T \|s_k(t) - f(t)\|\, dt \to 0 \quad as \quad k \to \infty. \tag{E.1.2}$$

(iii) If f is summable, we define

$$\int_0^T f(t)\, dt = \lim_{k\to\infty} \int_0^T s_k(t)\, dt. \tag{E.1.3}$$

Theorem E.1.1 (Bochner)**.** *A strongly measurable function* $u : [0, T] \to X$ *is summable if and only if* $t \mapsto \|u(t)\|_X$ *is summable. In this case*

$$\left\| \int_0^T u(t)\, dt \right\| \leq \int_0^T \|u(t)\|\, dt,$$

and

$$\left\langle u^*, \int_0^T u(t)\, dt \right\rangle \leq \int_0^T \langle u^*, f(t) \rangle\, dt,$$

for each $u^* \in X^*$.

Lemma E.1.2. *If* $\alpha(t) := \sum_{i=1}^{m} \chi_{E_i}(t)\, c_i$ *is simple on* $[0, T]$ *and* $u \in X$*, then* $\int_0^T \alpha(t) u\, dt = u \int_0^T \alpha(t)\, dt$.

Proof.

$$\int_0^T \alpha(t)u\,dt = \int_0^T \Big(\sum_{i=1}^m \chi_{E_i}(t)\,c_i\Big)\,u\,dt = \sum_{i=1}^m |E_i|\,c_i\,u$$
$$= u\sum_{i=1}^m |E_i|\,c_i = u\int_0^T \alpha(t)\,dt.$$

□

Lemma E.1.3. *If $f : [0,T] \to \mathbb{R}$ is a Lebesgue integrable function on $[0,T]$ and $u \in X$, then $\int_0^T f(t)u\,dt = u\int_0^T f(t)\,dt$.*

Proof. Since $f(t)$ is a Lebesgue integrable function on $[0,T]$, there exists a sequence of simple functions $\{\alpha_n\}_{n=1}^{\infty}$ on $[0,T]$ such that $f(t) = \lim_{n\to\infty}\alpha_n(t)$. Hence,

$$\int_0^T \|\alpha_n(t)\,u - f(t)\,u\|\;dt = \int_0^T |\alpha_n(t) - f(t)|\;\|u\|\;dt$$
$$= \|u\|\int_0^T |\alpha_n(t) - f(t)|\;dt$$
$$\to \quad 0 \text{ as } n \to \infty.$$

Therefore, $f(t)u$ is summable. Thus, we have

$$\int_0^T f(t)u\,dt = \lim_{n\to\infty}\int_0^T \alpha_n(t)\,u\,dt = u\lim_{n\to\infty}\int_0^T \alpha_n(t)\,dt = u\int_0^T f(t)\,dt.$$

□

Lemma E.1.4. *If $u_m \rightharpoonup u$ in $L^2([0,T];\mathrm{Dom}\,(\mathcal{E}))$, then $u_m \rightharpoonup u$ in $L^2([0,T];L^2_\mu[a,b])$.*

Proof. Suppose for v^* in the dual space of $L^2([0,T];L^2_\mu[a,b])$ and v is the corresponding Reisz's representation of v^*. Then,

$$
\begin{aligned}
&\left|\langle v^*, u_m\rangle_{L^2([0,T];L^2_\mu[a,b])} - \langle v^*, u\rangle_{L^2([0,T];L^2_\mu[a,b])}\right| \\
=&\left|\int_0^T (v, u_m)_\mu\, dt - \int_0^T (v, u)_\mu\, dt\right| = \left|\int_0^T (v, u_m - u)_\mu\, dt\right| \\
\le& \int_0^T |(v, u_m - u)_\mu|\; dt \le \int_0^T \|v\|_\mu\; \|u_m - u\|_\mu\; dt \\
\le& \Big(\int_0^T \|v\|_\mu^2\; dt\Big)^{\frac{1}{2}} \Big(\int_0^T \|u_m - u\|_\mu^2\; dt\Big)^{\frac{1}{2}} \\
\to&\quad 0 \text{ as } m \to \infty.
\end{aligned}
$$

□

Lemma E.1.5. *For all $u \in L^2([0,T]; \mathrm{Dom}\,(\mathcal{E}))$, we have*

$$\nabla \int_0^T u(x,t)\, dt \;=\; \int_0^T \nabla u(x,t)\, dt, \;\; \textit{where } \nabla = \frac{d}{dx}.$$

Proof. We know that $v(x) \in L^2[a,b]$. In fact, for all $v(x) \in C_c^\infty(a,b)$, we have

$$
\begin{aligned}
\int_a^b v(x)\nabla\Big(\int_0^T u(x,t)\, dt\Big)dx \;&=\; -\int_a^b v'(x)\Big(\int_0^T u(x,t)\, dt\Big)dx \\
&= -\int_0^T\int_a^b v'(x)\, u(x,t)\, dx\, dt \\
&= \int_0^T\int_a^b v(x)\, \nabla u(x,t)\, dx\, dt \\
&= \int_a^b v(x)\Big(\int_0^T \nabla u(x,t)\, dt\Big)dx.
\end{aligned}
$$

Then, we have $\nabla \int_0^T u(x,t)\, dt \;=\; \int_0^T \nabla u(x,t)\, dt$. □

Lemma E.1.6. *Let $u \in L^2([0,T]; X)$, where X is $\mathrm{Dom}\,(\mathcal{E})$ or $L^2_\mu[a,b]$. Define $v(\cdot,t) := \int_0^t u(\cdot,\tau)\, d\tau$. Then $v(x,t) \in L^2([0,T]; \mathrm{Dom}\,(\mathcal{E}))$, and $v_t(\cdot,t) = u(\cdot,t)$.*

Proof. Fix $t \in [0,T]$.Then,

$$\begin{aligned}\int_a^b |\nabla v(x,t)|^2\,dx &\le \int_a^b \Big(\int_0^t |\nabla u(x,\tau)|\;d\tau\Big)^2 dx\\ &\le \int_a^b \Big((\int_0^t |\nabla u(x,\tau)|^2\;d\tau)^{\frac{1}{2}}\,T^{\frac{1}{2}}\Big)^2 dx\\ &\le \int_a^b T\Big(\int_0^t |\nabla u(x,\tau)|^2\;d\tau\Big)dx\\ &\le T\int_0^t\int_a^b |\nabla u(x,\tau)|^2\;dx\,d\tau\\ &\le T\int_0^t \|u(\cdot,\tau)\|^2_{\mathrm{Dom}\,(\mathcal{E})}\;d\tau \le TM.\end{aligned}$$

Thus, for each $t \in [0,T]$, $v(x,t) \in \mathrm{Dom}\,(\mathcal{E})$.

Moreover, let $\varphi(t) \in C_C^\infty(0,T)$. Then,

$$\begin{aligned}&\int_0^T \varphi_t(t)v(x,t)\,dt = \int_0^T \varphi_t(t)\int_0^t u(\cdot,\tau)\,d\tau\,dt\\ &= \int_0^T\int_{t=\tau}^T \varphi_t(t)\,u(\cdot,\tau)\,dt\,d\tau = \int_0^T(\int_{t=\tau}^T \varphi_t(t)\,dt)u(\cdot,\tau)\,d\tau\\ &= \int_0^T -\varphi(t)u(x,t)\,dt.\end{aligned}$$

So, $v_t(\cdot,t) = u(\cdot,t)$. ◻

Lemma E.1.7. *The set* $\{\varphi\psi : \varphi \in C_c^\infty(0,T), \psi(x) \in C_c^\infty(a,b)\}$ *is dense in* $L^2([0,T];L^2_\mu[a,b])$.

Proof. Suppose $u \in L^2([0,T];\mathrm{Dom}\,(\mathcal{E}))$. For each fix t, $u(\cdot,t) \in \mathrm{Dom}\,(\mathcal{E})$. Thus, there exists $\psi_n(x) \in C_c^\infty[a,b]$ such that $\|\psi_n(x) - u\|_{\mathrm{Dom}\,(\mathcal{E})} < \epsilon/2^n$. Now, $\varphi_n(t) := 1$ for $t \in$

$E_n \subset [0,T], \varphi_n(t) :< 1$ for $t \in \bar{E}_n$, where $\bar{E}_n = [0,T]\backslash E_n$, and $\varphi_n(t) \in C_c^\infty(0,T)$.

$$\begin{aligned}
\|\varphi_n(t)\psi_n(x) - u\|^2_{L^2([0,T];\mathrm{Dom}\,(\mathcal{E}))} &= \int_0^T \|\varphi_n(t)\psi_n(x) - u\|^2_{\mathrm{Dom}\,(\mathcal{E})}\,dt \\
=& \int_{E_n} \|\varphi_n(t)\psi_n(x) - u\|^2_{\mathrm{Dom}\,(\mathcal{E})}\,dt + \int_{\bar{E}_n} \|\varphi_n(t)\psi_n(x) - u\|^2_{\mathrm{Dom}\,(\mathcal{E})}\,dt \\
\leq& \frac{\epsilon}{2^n}\mathcal{L}(E_n) + 2\int_{\bar{E}_n} \Big(|\varphi(t)_n|^2\,\|\psi_n(x)\|^2_{\mathrm{Dom}\,(\mathcal{E})} + \|u\|^2_{\mathrm{Dom}\,(\mathcal{E})}\Big)dt \\
\leq& \frac{\epsilon}{2^n}\mathcal{L}(E_n) + 2\int_{\bar{E}_n} \Big(\|\psi_n(x)\|^2_{\mathrm{Dom}\,(\mathcal{E})} + \|u\|^2_{\mathrm{Dom}\,(\mathcal{E})}\Big)dt \\
\leq& \frac{\epsilon}{2^n}\mathcal{L}(E_n) + 2\int_{\bar{E}_n} \Big((\frac{\epsilon}{2^n} + \|u\|)^2_{\mathrm{Dom}\,(\mathcal{E})} + \|u\|^2_{\mathrm{Dom}\,(\mathcal{E})}\Big)dt \\
\leq& \frac{\epsilon}{2^n}\mathcal{L}(E_n) + 2\int_{\bar{E}_n} \Big((2(\frac{\epsilon}{2^n})^2 + 3\,\|u\|^2_{\mathrm{Dom}\,(\mathcal{E})}\Big)dt \\
\leq& 5(\frac{\epsilon}{2^n})^2\mathcal{L}(E_n) + 6\int_{\bar{E}_n} \|u\|^2_{\mathrm{Dom}\,(\mathcal{E})}\,dt \to 0.
\end{aligned}$$

Remark: $\mathcal{L}(E_n) \to 0$ and any f$\in L^1(X), X \subset \mathbb{R}^n$. Then, $\lim_{n\to\infty} \int_{E_n} f dx = 0$. □

Proposition E.1.8. *Suppose* $u_m \rightharpoonup u$ *in* $L^2([0,T];\mathrm{Dom}\,(\mathcal{E}))$, *and* $(u_m)_t \rightharpoonup \gamma$ *in* $L^2([0,T];L^2_\mu[a,b])$. *Then* $(u_m)_t \rightharpoonup u_t$ *in* $L^2([0,T];L^2_\mu[a,b])$.

Proof. Suppose $\beta := \int_0^T \gamma\,dt$, $v^* \in L^2([0,T];L^2_\mu[a,b])$. Then $\langle v^*,(u_m)_t\rangle \to \langle v^*,\gamma\rangle$

$$\Leftrightarrow \int_0^T \Big(\int_a^b v^*\,(u_m)_t\,d\mu\Big)dt \to \int_0^T \Big(\int_a^b v^*\,\gamma\,d\mu\Big)dt.$$

Choosing $\varphi(t) \in C_C^\infty(0,T)$ and $\psi(x) \in C_C^\infty(a,b)$ and replacing $v^*$by $\varphi(t)\psi(x)$ in the above equation. We get

$$\begin{aligned}
L.H.S &= \lim_{m\to\infty} \int_0^T \Big(\int_a^b (u_m)_t\,\varphi(t)\psi(x)\,d\mu\Big)dt \\
&= \lim_{m\to\infty} \int_a^b \Big(\int_0^T \varphi(t)\psi(x)\,(u_m)_t\,dt\Big)d\mu = \lim_{m\to\infty} \int_a^b \psi(x)\Big(\int_0^T \varphi(t)\,(u_m)_t\,dt\Big)d\mu \\
&= \lim_{m\to\infty} \int_a^b \psi(x)\Big(\int_0^T -\varphi_t(t)\,u_m\,dt\Big)d\mu \\
&= \lim_{m\to\infty} \int_0^T \Big(\int_a^b -\varphi_t(t)\psi(x)\,u_m\,d\mu\Big)dt = \int_0^T \Big(\int_a^b -\varphi_t(t)\psi(x)\,u\,d\mu\Big)dt.
\end{aligned}$$

$$R.H.S = \int_0^T \Big(\int_a^b \gamma\,\varphi(t)\psi(x)\,d\mu\Big)dt = \int_a^b \Big(\int_0^T \varphi(t)\psi(x)\,\gamma\,dt\Big)d\mu$$
$$= \int_a^b \psi(x)\Big(\int_0^T \varphi(t)\,\gamma\,dt\Big)d\mu = \int_a^b \psi(x)\Big(\int_0^T -\varphi_t(t)\,\beta\,dt\Big)d\mu$$
$$= \int_0^T \Big(\int_a^b -\varphi_t(t)\psi(x)\,\beta\,d\mu\Big)dt.$$

So, we have

$$\int_0^T \Big(\int_a^b -\varphi_t(t)\psi(x)\,u\,d\mu\Big)dt = \int_0^T \Big(\int_a^b -\varphi_t(t)\psi(x)\,\beta\,d\mu\Big)dt.$$

Since $\{\varphi_t(t)\psi(x)\}$ is dense in $L^2([0,T];L^2_\mu[a,b])$, we get $u=\beta$. This implies $u_t = \gamma$. □

Lemma E.1.9. *Fix $v \in L^2([0,T],\mathrm{Dom}\,(\mathcal{E}))$, and for all $w \in L^2([0,T],\mathrm{Dom}\,(\mathcal{E})')$, define $l(w) := \int_0^T \langle w,v\rangle\;dt$. Then, $l \in (L^2([0,T],\mathrm{Dom}\,(\mathcal{E})'))'$.*

Proof.

$$|l(w)| = \left|\int_0^T \langle w,v\rangle\;dt\right| \le \int_0^T |\langle w,v\rangle|\;dt \le \int_0^T \|w\|_{\mathrm{Dom}\,(\mathcal{E})'}\,\|v\|_{\mathrm{Dom}\,(\mathcal{E})}\;dt$$
$$\le \Big(\int_0^T \|w\|^2_{\mathrm{Dom}\,(\mathcal{E})'}\;dt\Big)^{1/2}\Big(\int_0^T \|v\|^2_{\mathrm{Dom}\,(\mathcal{E})}\;dt\Big)^{1/2}$$
$$\le \|w\|^2_{L^2([0,T],\mathrm{Dom}\,(\mathcal{E})')}\,\|v\|^2_{L^2([0,T],\mathrm{Dom}\,(\mathcal{E}))}.$$

Thus, $l(w)$ is bounded.

Moreover, if $\alpha_1,\alpha_2 \in \mathbb{R}$ and $w_1,w_2 \in L^2([0,T],\mathrm{Dom}\,(\mathcal{E})')$

$$l(\alpha_1 w_1 + \alpha_2 w_2) = \int_0^T \langle \alpha_1 w_1 + \alpha_2 w_2, v\rangle\;dt = \int_0^T \alpha_1\langle w_1,v\rangle + \alpha_2\langle w2,v\rangle\;dt$$
$$= \int_0^T \alpha_1(\tilde{w}_1,v)_\mu\,dt + \int_0^T \alpha_2(\tilde{w}_2,v)_\mu\,dt$$
$$= \alpha_1\int_0^T \langle w_1,v\rangle\;dt + \alpha_2\int_0^T \langle w_2,v\rangle\;dt$$
$$= \alpha_1 l(w_1) + \alpha_2 l(w_2).$$

□

Proposition E.1.10. *Suppose* $(u_m)_t \rightharpoonup u_t$ *in* $L^2([0,T];L^2_\mu[a,b])$ *, and* $l_{(u_m)_{tt}} \rightharpoonup l_\sigma$ *in* $L^2([0,T];\mathrm{Dom}\,(\mathcal{E})')$*. Then* $l_{(u_m)_{tt}} \rightharpoonup l_{(u)_{tt}}$ *in* $L^2([0,T];\mathrm{Dom}\,(\mathcal{E})')$ *.*

Proof. Suppose $\sigma \in L^2([0,T];L^2_\mu[a,b])$ is the representative of l_σ, and define $\gamma := \int_0^T \sigma\,dt$, for any $v \in L^2([0,T];\mathrm{Dom}\,\mathcal{E})$. Then by $l_{(u_m)_{tt}} \rightharpoonup l_\sigma$ in $L^2([0,T];\mathrm{Dom}\,(\mathcal{E})')$ and Lemma E.1.9, we have

$$\int_0^T \left\langle l_{(u_m)_{tt}}, v\right\rangle\,dt \to \int_0^T \left\langle l_\sigma, v\right\rangle\,dt.$$

i.e.

$$\int_0^T \Big(\int_a^b (u_m)_{tt}\,v\,d\mu\Big)dt \to \int_0^T \Big(\int_a^b l_\sigma\,v\,d\mu\Big)dt.$$

Choosing $\varphi(t) \in C_C^\infty(0,T)$ and $\psi(x) \in C_C^\infty(a,b)$ and replacing v by $\varphi(t)\psi(x)$ in the above equation.

$$\begin{aligned}
Lefthandside &= \lim_{m\to\infty}\int_0^T \Big(\int_a^b (u_m)_{tt}\,\varphi(t)\psi(x)\,d\mu\Big)dt\\
&= \lim_{m\to\infty}\int_a^b \Big(\int_0^T \varphi(t)\psi(x)\,(u_m)_{tt}\,dt\Big)d\mu = \lim_{m\to\infty}\int_a^b \psi(x)\Big(\int_0^T \varphi(t)\,(u_m)_{tt}\,dt\Big)d\mu\\
&= \lim_{m\to\infty}\int_a^b \psi(x)\Big(\int_0^T -\varphi(t)'\,(u_m)_t\,dt\Big)d\mu\\
&= \lim_{m\to\infty}\int_0^T \Big(\int_a^b -\varphi_t(t)\psi(x)\,(u_m)_t\,d\mu\Big)dt = \int_0^T \Big(\int_a^b -\varphi_t(t)\psi(x)\,u_t\,d\mu\Big)dt.
\end{aligned}$$

$$\begin{aligned}
Righthandside &= \int_0^T \Big(\int_a^b \sigma\,\varphi(t)\psi(x)\,d\mu\Big)dt = \int_a^b \Big(\int_0^T \varphi(t)\psi(x)\,\sigma\,dt\Big)d\mu\\
&= \int_a^b \psi(x)\Big(\int_0^T \varphi(t)\,\sigma\,dt\Big)d\mu = \int_a^b \psi(x)\Big(\int_0^T -\varphi_t(t)\,\gamma\,dt\Big)d\mu\\
&= \int_0^T \Big(\int_a^b -\varphi_t(t)\psi(x)\,\gamma\,d\mu\Big)dt.
\end{aligned}$$

So, we have

$$\int_0^T \Big(\int_a^b -\varphi_t(t)\psi(x)\,u_t\,d\mu\Big)dt = \int_0^T \Big(\int_a^b -\varphi_t(t)\psi(x)\,\gamma\,d\mu\Big)dt.$$

Since $\{\varphi_t(t)\psi(x)\}$ is dense in $L^2([0,T]; L^2_\mu[a,b])$, we get $u_t = \gamma$. This implies $u_{tt} = \sigma$. ☐

Lemma E.1.11. *Let u, $v \in L^2([0,T]; \mathrm{Dom}\,(\mathcal{E}))$. Then, all Lebesgue a.e. $x \in [a,b]$,*

$$(u(x,t)v(x,t))_t = u(x,t)_t v(x,t) + u(x,t)v_t(x,t).$$

Proof. Let $\varphi(t) \in C_c^\infty$. Then, $\int_0^T \Big(u(x,t)v(t)\Big)_t \varphi(t)\,dt = -\int_0^T \Big(u(x,t)v(t)\Big)\varphi_t(t)dt$.

$$\begin{aligned}
&\int_0^T \Big(u_t(x,t)v(t) + u(x,t)v_t(t)\Big)\varphi(t)dt \\
&= -\int_0^T u(x,t)\Big(v(t)\varphi(t)\Big)_t dt + \int_0^T u(x,t)v_t\varphi(t)dt \\
&= -\int_0^T u(x,t)v(t)\varphi_t(t)\,dt - \int_0^T u(x,t)v_t(t)\varphi(t)\,dt + \int_0^T u(x,t)v_t(t)\varphi(t)dt \\
&= -\int_0^T \Big(u(x,t)v(t)\Big)\varphi_t(t)dt.
\end{aligned}$$

Now, let $v_n(x,\cdot) \in C_c^\infty(0,T)$ such that $v_n(x,\cdot) \to v(x,\cdot)$ in $H_0^1(0,T)$. Then,

$$\begin{aligned}
&\int_0^T \Big(u(x,t)v(x,t)\Big)_t \varphi(t)\,dt = -\int_0^T \Big(u(x,t)v(x,t)\Big)\varphi_t(t)dt \\
&= -\lim_{n\to\infty}\int_0^T u(x,t)v_n(x,t)\varphi_t(t)dt \\
&= \lim_{n\to\infty}\int_0^T \Big(u_t(x,t)v_n(x,t) + u(x,t)(v_n(x,t))_t\Big)\varphi(t)dt \\
&= \int_0^T \Big(u_t(x,t)v(x,t) + u(x,t)v_t(x,t)\Big)\varphi(t)dt.
\end{aligned}$$

For the last equality, we use the fact $v_n(x,\cdot) \to v(x,\cdot)$ in $H_0^1(0,T)$. ☐

APPENDIX F

EMBEDDING OF H_0^1 IN L_μ^2

Proposition F.0.12. *Let $u \in H_0^1(a,b)$ and let $\{\phi_n\} \subset C_c^\infty(a,b)$ such that $\phi_n \to u$ in $H_0^1(a,b)$. Then, there exists a subsequence $\{\phi_{n_k}\}$ such that $\phi_{n_k} \to u_c$ everywhere in $[a,b]$, where u_c is the continuous representative of the equivalence class of u in $H_0^1(a,b)$.*

Proof. Since $\phi_n \to u$ in $H_0^1(a,b)$, there exists a subsequence $\{\phi_{n_k}\}$ converging pointwise Lebesgue a.e. to u_c on (a,b). Now let $x \in (a,b)$, and let $\epsilon > 0$ be arbitrary. We first notice that since ϕ_n is convergent, there exists $C > 0$ such that

$$\|\phi_n\|_{\mathrm{Dom}\,\mathcal{E}} \leq C \text{ for all } n \in \mathbb{N}. \tag{F.0.1}$$

Next, by the continuity of u_c, there exists $0 < \delta_\epsilon < (\frac{\epsilon}{3+C})^2$ such that for all $y \in [a,b]$, with $|y-x| < \delta_\epsilon$, we have

$$|u_c(x) - u_c(y)| < \epsilon/3. \tag{F.0.2}$$

Hence,

$$|\phi_{n_k}(x) - u_c(x)| \leq |\phi_{n_k}(x) - \phi_{n_k}(y)| + |\phi_{n_k}(y) - u_c(y)| + |u_c(y) - u_c(x)|\,. \tag{F.0.3}$$

The first term can be estimated as follows,

$$\begin{aligned} |\phi_{n_k}(x) - \phi_{n_k}(y)| &= \left|\int_y^x \nabla\phi_{n_k}(s)\,ds\right| \leq \Big(\int_y^x |\nabla\phi_{n_k}(s)|^2\,ds\Big)^{1/2} |x-y|^{1/2} \\ &\leq \|\phi_{n_k}\|_{\mathrm{Dom}\,\mathcal{E}}\,|x-y|^{1/2} \leq C(\frac{\epsilon}{3+C})^2 \leq \epsilon/3. \end{aligned} \tag{F.0.4}$$

Substituting (F.0.2) and (F.0.4) into (F.0.3), we get

$$|\phi_{n_k}(x) - u_c(x)| \leq \epsilon/3 + |\phi_{n_k}(y) - u_c(y)| + \epsilon/3,$$

for all $y \in (x - \delta_\epsilon, x + \delta_\epsilon)$.

Let $y \in (a,b)$satisfy $\lim_{k\to\infty} \phi_{n_k}(y) = u_c(y)$. Then, for all k sufficient large, $|\phi_{n_k}(y) - u_c(y)| < \epsilon/3$, and hence $|\phi_{n_k}(x) - u_c(x)| < \epsilon$. Thus, $\lim_{k\to\infty} \phi_{n_k}(x) = u_c(x)$ for al $x \in [a,b]$.

□

Corollary F.0.13. *Let $u \in H_0^1(a,b)$ and let $\bar{u}$ be its unique $L^2_\mu[a,b]$ representative. Then we can take $\bar{u}$ to be u_c.*

Corollary F.0.14. *If* $\mathrm{supp}(u) = [a,b], a < b$, *then* $I : H_0^1(a,b) \to L^2_\mu[a,b]$ *is injective. Consequently,* $\mathrm{Dom}\,(\mathcal{E}) = H_0^1(a,b)$.

Proof. Let $u \in H_0^1(a,b)$ such that $I(u) = 0$. Then we have $\bar{u} = u_c = 0$ in $L^2_\mu[a,b]$. Since $supp(u) = [a,b]$, we have $u_c \equiv 0$ on $[a,b]$. Thus, $u = 0$ Lebesgue a.e on $[a,b]$ □

APPENDIX G

DIFFERENTIABILITY OF DISTRIBUTION AND BANACH SPACE VALUED FUNCTIONS

Let $\mathcal{D}$ denote the collection of all test functions on $[a, b]$.

Definition G.0.2. *(see e.g.[25]) A* distribution *f is a functional : $\mathcal{D} \mapsto \mathbb{R}$ which is* linear *and* continuous *in the following sense. If $\phi \in \mathcal{D}$ is a test function, then we denote the corresponding real number by (f, ϕ).*

By linearity *we mean that*

$$(f, \alpha\phi + \beta\psi) = \alpha(f, \phi) + \beta(f, \psi)$$

for all constants α, β and all test functions ϕ, ψ.

By continuity *we mean following. If $\{\phi_n\}$ is a sequence of test functions that vanish outside a common interval and converge uniformly to a test function ϕ, and if all their derivatives do as well, then*

$$(f, \phi_n) \to (f, \phi) \quad \text{as } n \to \infty.$$

Definition G.0.3. *(see e.g.[25]) For any distribution f, we define its* derivative ∇f *by the formula*

$$(\nabla f, \phi) = (f, \nabla\phi) \quad \text{for all test functions } \phi.$$

Definition G.0.4 (Strong derivative)**.** *(see e.g. [11]) If $f : (a, b) \subset \mathbb{R} \to X$, where X is a Banach space, then it is* differentiable *at $t \in (a, b)$ if $L := \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$ exists in X. The limit L, if it exists, will be denoted by $\dot{f}(t)$.*

Definition G.0.5 (Fréchet derivative)**.** *(see e.g.[15]) Let V and W be Banach spaces, and $U \subset V$ be an open subset of V. A function $f : U \to W$ is called* Fréchet differentiable *at* $t \in U$ *if there exists a bounded linear operator* $A_t : V \to W$ *such that*

$$\lim_{h\to 0} \frac{\|f(t+h) - f(t) - A_t(h)\|_W}{\|h\|_V} = 0$$

Definition G.0.6 (Gâteaux derivative)**.** *(see e.g.[23]) Suppose* X *and* Y *are locally convex topological vector spaces (for example, Banach spaces),* $U \subset X$ *is open, and* $F : X \to Y$*. The* Gâteaux differential $dF(u;\Psi)$ *of* F *at* $u \in U$ *in the direction* $\Psi \to X$ *is defined as*

$$dF(u;\Psi) = \frac{F(u+h\Psi) - F(u)}{h}.$$

Remark: In the case, $f : (a,b) \subset \mathbb{R} \to X$, where X is a Banach space, if f has a strong derivative at $t \in (a,b)$, then f is Fréchet differentiable at t and also Gâteaux derivative at t.

Proposition G.0.15. *If* f *is strongly differentiable at* $t \in (a,b)$*, and* L *be the strong derivative of* f*, then* f *is Fréchet differentiable at* t*.*

Proof.

$$\lim_{h\to 0} \left\| \frac{f(t+h) - f(t)}{h} - L \right\|_X = 0.$$

$$\Leftrightarrow \lim_{h\to 0} \left\| \frac{f(t+h) - f(t) - hL)}{h} \right\|_X = 0.$$

$$\Leftrightarrow \lim_{h\to 0} \frac{\|f(t+h) - f(t) - hL)\|_X}{|h|} = 0.$$

Define $A_t(h) = hL$. Then $A_t(h) : (a,b) \to X$ is a linear operator. Hence f is Fréchet differentiable at t. □

Definition G.0.7 (Partial derivative)**.** $\frac{\partial}{\partial t} u(x,t) = \lim_{h\to 0} \frac{u(x,t+h) - u(x,t)}{h}$ *exists in* $\mathbb{R}$.

Proposition G.0.16. *Suppose for* $t \in (0,T)$. $\dot{u}(t) := \lim_{h\to 0} \frac{u(t+h)-u(t)}{h}$*, the limit exists in* $\mathrm{Dom}\,(\mathcal{E})$*. Then the partial derivative* $\frac{\partial}{\partial t} u(x,t) = \dot{u}(x,t)$ *for Lebesgue a.e.* $x \in [a,b]$*. Consequently, we can regard* $u_t = \dot{u}$ *in* $\mathrm{Dom}\,(\mathcal{E}) = H_0^1(a,b)$.

Proof.

$$\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}-\dot{u}(t)\right\|_{\mathrm{Dom}\,(\mathcal{E})}=0.$$

$$\Leftrightarrow \lim_{h\to 0}\int_a^b \left|\nabla(\frac{u(t+h)-u(t)}{h}-\dot{u}(x,t))\right|^2 dx=0.$$

$$\Leftrightarrow \lim_{h\to 0}\int_a^b \left|\frac{u(t+h)-u(t)}{h}-\dot{u}(x,t)\right|^2 dx=0.$$

$$\Rightarrow \text{for a.e } x\in[a,b], \lim_{h\to 0}\left|\frac{u(t+h)-u(t)}{h}-\dot{u}(x,t)\right|=0.$$

$$\Rightarrow \text{for a.e } x\in[a,b], \frac{\partial}{\partial t}u(x,t)=\dot{u}(x,t).$$

□

Definition G.0.8 (Spectral Family)**.** *(see e.g.[14]) A real spectral family is a one-parameter family* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *of projections* E_λ *defined on a Hilbert space* H *(of any dimension) which depends on a real parameter* λ *and is such that*

(i) $E_\lambda \le E_\eta$*, hence* $E_\lambda E_\eta = E_\eta E_\lambda = E_\lambda$ *if* $\lambda \le \eta$*,*

(ii) $\lim_{\lambda\to-\infty} E_\lambda x = 0, \lim_{\lambda\to+\infty} E_\lambda x = x$ *for any* $x\in H$*,*

(iii) $E_{\lambda+0}x = \lim_{\eta\to\lambda+0} E_\eta x = E_\lambda x.$

Proof of theorem 2.0.4. The uniqueness of (2.0.3) is well known.

Let $v(t) := \int_0^\infty -\sqrt{\lambda}\sin(t\sqrt{\lambda})dE_\lambda g + \int_0^\infty \cos(t\sqrt{\lambda})dE_\lambda h$. Fix $t\in\mathbb{R}$. Then, for any $h\in\mathbb{R}\backslash\{0\}$,

$$\begin{aligned}\frac{u(t+h)-u(t)}{h}-v(t) &= \int_0^\infty \frac{\cos((t+h)\sqrt{\lambda})-\cos(t\sqrt{\lambda})}{h}+\sqrt{\lambda}\sin(t\sqrt{\lambda})dE_\lambda g\\ &+\int_0^\infty \frac{\sin((t+h)\sqrt{\lambda})-\sin(t\sqrt{\lambda})}{h}-\cos(t\sqrt{\lambda})dE_\lambda h.\end{aligned}$$

Hence,

$$
\begin{aligned}
&\left\|\frac{u(t+h)-u(t)}{h}-v(t)\right\| \\
\le&\left\|\int_0^\infty \frac{\cos((t+h)\sqrt{\lambda})-\cos(t\sqrt{\lambda})}{h}+\sqrt{\lambda}\sin(t\sqrt{\lambda})dE_\lambda g\right\| \\
+&\left\|\int_0^\infty \frac{\sin((t+h)\sqrt{\lambda})-\sin(t\sqrt{\lambda})}{h}-\cos(t\sqrt{\lambda})dE_\lambda h\right\| \\
\le&\Big(\int_0^\infty \left|\frac{\cos((t+h)\sqrt{\lambda})-\cos(t\sqrt{\lambda})}{h}+\sqrt{\lambda}\sin(t\sqrt{\lambda})\right|^2 dE_\lambda g\Big)^{1/2} \\
+&\Big(\int_0^\infty \left|\frac{\sin((t+h)\sqrt{\lambda})-\sin(t\sqrt{\lambda})}{h}-\cos(t\sqrt{\lambda})\right|^2 dE_\lambda h\Big)^{1/2}.
\end{aligned}
\tag{G.0.1}
$$

(see, e.g.,Yosida [28], p312, Corollary 2)

Note that for the first term in (G.0.1), we have,

$$
\begin{aligned}
&\left|\frac{\cos((t+h)\sqrt{\lambda})-\cos(t\sqrt{\lambda})}{h}+\sqrt{\lambda}\sin(t\sqrt{\lambda})\right| \\
=&\left|\frac{\cos(t\sqrt{\lambda})\cos(h\sqrt{\lambda})-\sin(t\sqrt{\lambda})\sin(h\sqrt{\lambda})-\cos(t\sqrt{\lambda})}{h}+\sqrt{\lambda}\sin(t\sqrt{\lambda})\right| \\
=&\left|\frac{\cos(t\sqrt{\lambda})(\cos(h\sqrt{\lambda})-1)}{h}-\frac{\sqrt{\lambda}\sin(t\sqrt{\lambda})\sin(h\sqrt{\lambda})}{h\sqrt{\lambda}}+\sqrt{\lambda}\sin(t\sqrt{\lambda})\right| \\
\le&\left|\cos(t\sqrt{\lambda})\right|\left|\sqrt{\lambda}\frac{(\cos(h\sqrt{\lambda})-1)}{h\sqrt{\lambda}}\right|+\left|\sqrt{\lambda}\sin(t\sqrt{\lambda})\right|\left|\frac{\sin(h\sqrt{\lambda})}{h\sqrt{\lambda}}\right|+\left|\sqrt{\lambda}\right| \\
\le&(1)(1)(\sqrt{\lambda})+(\sqrt{\lambda})2+\sqrt{\lambda}=4(\sqrt{\lambda}),
\end{aligned}
\tag{G.0.2}
$$

for all h sufficient small.

Also for the second term in (G.0.1), we have,

$$\begin{aligned}
&\left|\frac{\sin((t+h)\sqrt{\lambda})-\sin(t\sqrt{\lambda})}{h\sqrt{\lambda}}-\cos(t\sqrt{\lambda})\right| \\
=&\left|\frac{\sin(t\sqrt{\lambda})\cos(h\sqrt{\lambda})+\cos(t\sqrt{\lambda})\sin(h\sqrt{\lambda})-\sin(t\sqrt{\lambda})}{h\sqrt{\lambda}}-\cos(t\sqrt{\lambda})\right| \\
=&\left|\frac{\sin(t\sqrt{\lambda})(\cos(h\sqrt{\lambda})-1)}{h\sqrt{\lambda}}-\cos(t\sqrt{\lambda})\frac{\sin(h\sqrt{\lambda})}{h\sqrt{\lambda}}-\cos(t\sqrt{\lambda})\right| \\
\leq&\left|\sin(t\sqrt{\lambda})\right|\left|\frac{\cos(h\sqrt{\lambda}-1)}{h\sqrt{\lambda}}\right|+\left|\cos(t\sqrt{\lambda})\right|\left|\frac{\sin(h\sqrt{\lambda})}{h\sqrt{\lambda}}\right|+\left|\cos(h\sqrt{\lambda})\right| \\
\leq&(1)(1)+(1)(2)+1=4,
\end{aligned} \tag{G.0.3}$$

for all h sufficient small.

Combining (G.0.1),(G.0.2), (G.0.3) and by the dominated convergence theorem, we have $\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}-v(t)\right\|=0$. Hence $\dot{u(t)}=v(t)$. Let

$$w(t):=\int_0^\infty -\lambda\cos(t\sqrt{\lambda})dE_\lambda g-\sqrt{\lambda}\int_0^\infty \sin(t\sqrt{\lambda})dE_\lambda h$$

. Then for any $h\in\mathbb{R}\backslash\{0\}$,

$$\begin{aligned}
&\left\|\frac{\dot{u}(t+h)-\dot{u}(t)}{h}-w(t)\right\| \\
\leq&\left\|\int_0^\infty \frac{-\sqrt{\lambda}\sin((t+h)\sqrt{\lambda})+\sqrt{\lambda}\sin(t\sqrt{\lambda})}{h}+\lambda\cos(t\sqrt{\lambda})dE_\lambda g\right\| \\
+&\left\|\int_0^\infty \frac{\sqrt{\lambda}\cos((t+h)\sqrt{\lambda})-\sqrt{\lambda}\cos(t\sqrt{\lambda})}{h\sqrt{\lambda}}+\sqrt{\lambda}\sin(t\sqrt{\lambda})dE_\lambda h\right\| \\
\leq&\Big(\int_0^\infty \left|\frac{-\sqrt{\lambda}\sin((t+h)\sqrt{\lambda})+\sqrt{\lambda}\sin(t\sqrt{\lambda})}{h}+\lambda\cos(t\sqrt{\lambda})\right|^2 dE_\lambda g\Big)^{1/2} \\
+&\Big(\int_0^\infty \left|\frac{\cos((t+h)\sqrt{\lambda})-\cos(t\sqrt{\lambda})}{h}+\sqrt{\lambda}\sin(t\sqrt{\lambda})\right|^2 dE_\lambda h\Big)^{1/2}.
\end{aligned} \tag{G.0.4}$$

Using (G.0.3), for the first term of (G.0.4), we have

$$\left|\frac{-\sqrt{\lambda}\sin((t+h)\sqrt{\lambda})+\sqrt{\lambda}\sin(t\sqrt{\lambda})}{h}+\lambda\cos(t\sqrt{\lambda})\right|^2\leq(4\lambda)^2 \tag{G.0.5}$$

Similarly, by using (G.0.2), for the second integral of (G.0.4), satisfies

$$\left| \frac{\cos((t+h)\sqrt{\lambda}) - \cos(t\sqrt{\lambda})}{h} + \sqrt{\lambda}\sin(t\sqrt{\lambda}) \right|^2 \leq (4\sqrt{\lambda})^2 = 16\lambda. \tag{G.0.6}$$

Combining (G.0.4),(G.0.5), (G.0.6), by the dominated convergence theorem and letting h tend to 0, we get $\ddot{u}(t) = w(t)$.

Finally,

$$\begin{aligned}
&\langle Au(t), x\rangle = \int_0^\infty \lambda\, d\, \langle E_\lambda u(t), x\rangle = \int_0^\infty \lambda\, d\, \langle u(t), E_\lambda x\rangle \\
&= \int_0^\infty \lambda\, d \left\langle \int_0^\infty \cos(t\sqrt{\eta}) dE_\eta g + \int_0^\infty \frac{\sin(t\sqrt{\eta})}{\sqrt{\eta}} dE_\eta h, E_\lambda x \right\rangle \\
&= \int_0^\infty \lambda\, d\Big(\int_0^\infty \cos(t\sqrt{\eta}) d\, \langle E_\eta g, E_\lambda x\rangle + \int_0^\infty \frac{\sin(t\sqrt{\eta})}{\sqrt{\eta}} d\, \langle E_\eta h, E_\lambda x\rangle \Big) \\
&= \int_0^\infty \lambda\, d\Big(\int_0^\infty \cos(t\sqrt{\eta}) d\, \langle E_\lambda E_\eta g, x\rangle + \int_0^\infty \frac{\sin(t\sqrt{\eta})}{\sqrt{\eta}} d\, \langle E_\lambda E_\eta h, x\rangle \Big) \qquad \text{(G.0.7)} \\
&= \int_0^\infty \lambda\, d\Big(\int_0^\lambda \cos(t\sqrt{\eta}) d\, \langle E_\eta g, x\rangle + \int_0^\lambda \frac{\sin(t\sqrt{\eta})}{\sqrt{\eta}} d\, \langle E_\eta h, x\rangle \Big) \\
&= \int_0^\infty \lambda \cos(t\sqrt{\lambda}) d\, \langle E_\lambda g, x\rangle + \int_0^\lambda \lambda \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} d\, \langle E_\lambda h, x\rangle \Big) \\
&= - \langle \ddot{u}(t), x\rangle .
\end{aligned}$$

Since $\langle E_\lambda h, x\rangle$ is of bounded variation on $[0, \lambda]$, , we get the last equality from the second equality. (See e.g. Apostal [1].)

The initial conditions are obvious. □