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CUSUM Generalized Variance Charts

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CUSUM GENERALIZED VARIANCE CHARTS

by

YUXIANG LI

(Under the Direction of Charles W. Champ)

ABSTRACT

The commonly recommended charts for monitoring the mean vector are affected by a shift in the covariance matrix. As in the univariate case, a chart for monitoring for a change in the covariance matrix should be examined first before examining the chart used to monitor for a change in the mean vector. A variety of charts used to monitor for a shift in the process covariance matrix have been introduced into the literature. A group of these charts are based on the sample generalized variance $|\mathbf{S}|$, where $\mathbf{S}$ is the sample covariance matrix. We examine the multivariate Shewhart and cumulative sum (CUSUM) charts based on function of the generalized variance $|\mathbf{S}|$ and the $\ln(|\mathbf{S}|)$.. The performance of these chart is based on an analysis of the chart's run length distribution. We give closed form expressions for the distribution of these statistics. Properties of the run length distribution are given as solutions to various integral equations. A method for obtaining approximate solutions to these integral equaltions is discussed.

Key Words: average run length, integral equations, Meijer G function, moments, run length distribution

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DEDICATION

I dedicate this this to my parents. It is their unconditional love that motivates me to obtain my goals. I also dedicate this thesis to my wife, Ting Peng, who has provided me with a strong love shield that always surrounds me which never lets any sadness to enter my life.

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CHAPTER 1

INTRODUCTION

1.1 Quality Control Charts

The quality control chart introduced by Walter A. Shewhart (see Shewhart (1931)) in the early 1920's. He described two kinds of variability in a quality measurement X found in a process which he labelled as "natural" and "assignable." Natural causes of variability are inherent to the process whereas assignable ones when removed improve the quality of the process. A process that is operating in which the only natural causes of variability are present is said to be in a "in-control" state. When an assignable cause(s) is present, then the process is in a "out-of-control" state. Duncan (1986) discussed the three uses of the control chart with respect to natural and assignable causes of variability. He summarized that a control chart can be used by the practitioner as an aid in (1) bringing a process into a state of statistical in control, (2) defining what is meant by the process being in a state of statistical in control, and (3) monitoring for a change in the process. That is, the control chart is an aid to the practitioner in discovering assignable causes of variability, as an aid in defining what is meant by a process being in an in-control state, and detecting when an assignable cause of variability has changed the process. Cases (1) and (2) are part of the first phase of controlling a process. Charts used in this phase are referred to as Phase I or retrospective control charts. In the monitoring phase, the charts are called Phase II or prospective control charts.

When there is a $p \times 1$ vector $\mathbf{X}$ of quality measurements, generally, the main interest of a practitioner is to control the $p \times 1$ mean vector μ of the distribution of $\mathbf{X}$. The most popular charts used for this purpose both in Phase I and Phase II are not only affected by a change in the process mean vector μ but also by a change in the process covariance matrix $\mathbf{\Sigma}$. For example, Champ, Jones-Farmer, and Rigdon

(2005) show that the Hotelling's T^2 chart can be affected by changes in the covariance matrix. It is then typically recommended that a chart for controlling for a change in $\mathbf{\Sigma}$ be use. This chart should be examined first. If there is no evidence in a change in the covariance matrix $\mathbf{\Sigma}$, then one can assume that any change indicated by the chart for controlling the mean vector μ is due to a change in μ.

1.2 Model and Sampling Method

The model that we will use in this study is (1) the $p \times 1$ quality vector $\mathbf{X}$ has a multivariate normal distribution with a $p \times 1$ mean vector μ and a $p \times p$ positive definite covariance matrix $\mathbf{\Sigma}$. When the process is in a state of statistical in-control, then $\mu = \mu_0$ and $\mathbf{\Sigma} = \mathbf{\Sigma}_0$, where μ_0 and $\mathbf{\Sigma}_0$ are fixed and typically unknown. In Phase I, the researcher will have available the quality measurements on m samples of each having n items produced. We represent the measurements on these items produced by the process by $\mathbf{X}_{i,1}, \ldots, \mathbf{X}_{i,n}$ for $i = 1, \ldots, m$. These measurements are assumed to be m independent random samples with $\mathbf{X}_{i,j} \sim N_p(\mu_0, \mathbf{\Sigma}_0)$ for $j = 1, \ldots, n$ and $i = 1, \ldots, m$. We will refer to this model for the data as the "independent normal model."

We define estimators $\widehat{\mu}_0$ and $\widehat{\mathbf{\Sigma}}_0$ for μ_0 and $\mathbf{\Sigma}_0$ by

$$\widehat{\mu}_0 = \overline{\overline{\mathbf{X}}}_0 = \frac{1}{mn} \sum\nolimits_{i=1}^{m} \sum\nolimits_{j=1}^{n} \mathbf{X}_{i,j} \text{ and } \widehat{\mathbf{\Sigma}}_0 = \overline{\mathbf{S}}_0 = \frac{1}{m} \sum\nolimits_{i=1}^{m} \mathbf{S}_i,$$

where

$$\mathbf{S}_i = \frac{1}{n-1} \sum\nolimits_{j=1}^{n} \left(\mathbf{X}_{i,j} - \overline{\mathbf{X}}_i\right) \left(\mathbf{X}_{i,j} - \overline{\mathbf{X}}_i\right)^{\mathbf{T}}.$$

It is shown in Anderson (2003) under our independent normal model that

$$\widehat{\mu}_0 \sim N_p\left(\mu_0, \frac{1}{mn}\mathbf{\Sigma}_0\right) \text{ and } \widehat{\mathbf{\Sigma}}_0 \sim Wishart\left(m(n-1), \frac{1}{m(n-1)}\mathbf{\Sigma}_0\right);$$

and $\widehat{\mu}_0$ and $\widehat{\mathbf{\Sigma}}_0$ are stochastically independent.

In Phase II, the researcher will periodically take the $p \times 1$ quality vector $\mathbf{X}$ on n items from the output of the process. We assume that these sets of measurements, $\mathbf{X}_{t,1}, \ldots, \mathbf{X}_{t,n}$, are independent random samples with common $N_p(\mu, \mathbf{\Sigma})$ distribution, for $t = 1, 2, 3, \ldots$. Further, we assume the measurements to be taken in Phases I and II are independent. The estimators we will use in this phase for μ and $\mathbf{\Sigma}$ are

$$\widehat{\mu} = \overline{\mathbf{X}}_t = \frac{1}{n}\sum\nolimits_{j=1}^{n} \mathbf{X}_{t,j} \text{ and } \widehat{\mathbf{\Sigma}} = \mathbf{S}_t = \frac{1}{n-1}\sum\nolimits_{j=1}^{n} \left(\mathbf{X}_{t,j} - \overline{\mathbf{X}}_t\right)\left(\mathbf{X}_{t,j} - \overline{\mathbf{X}}_t\right)^{\mathbf{T}}.$$

In Anderson (2003), it is shown that

$$\widehat{\mu} \sim N_p\left(\mu, \frac{1}{n}\mathbf{\Sigma}\right) \text{ and } \widehat{\mathbf{\Sigma}} \sim Wishart\left(n-1, \frac{1}{n}\mathbf{\Sigma}\right);$$

and $\widehat{\mu}$ and $\widehat{\mathbf{\Sigma}}$ are stochastically independent. It is not difficult to see that $\widehat{\mu}_0$, $\widehat{\mathbf{\Sigma}}_0$, $\widehat{\mu}$, and $\widehat{\mathbf{\Sigma}}$ are stochastically independent.

1.3 Types of Shifts in the Process Generalized Variance

The process generalized variance is the determinant $|\mathbf{\Sigma}|$ of the covariance matrix $\mathbf{\Sigma}$. A change in $\mathbf{\Sigma}$ may result in a change in the process parameter $|\mathbf{\Sigma}|$. It is our interest to control for a change in this process parameter $|\mathbf{\Sigma}|$ from its in-control value $|\mathbf{\Sigma}_0|$. Under the assumption the covariance matrix $\mathbf{\Sigma}$ ($\mathbf{\Sigma}_0$) is positive definite, then the associated correlation matrix $\mathbf{\Psi}$ ($\mathbf{\Psi}_0$) is also positive definite. Note that the covariance matrix can be expressed as

$$\mathbf{\Sigma} = \mathbf{\Gamma\Psi\Gamma^T} \ (\mathbf{\Sigma}_0 = \mathbf{\Gamma_0\Psi_0\Gamma_0^T}),$$

where $\mathbf{\Gamma} = Diagonal\,(\sigma_1, \ldots, \sigma_p)$ ($\mathbf{\Gamma}_0 = Diagonal\,(\sigma_{1,0}, \ldots, \sigma_{p,0})$). One can show that the eigenvalues of a positive definite matrix are all positive real numbers. It then follows that $\mathbf{\Psi}$ ($\mathbf{\Psi}_0$) can be expressed as

$$\mathbf{\Psi} = \mathbf{VCV^T} \ (\mathbf{\Psi}_0 = \mathbf{V}_0\mathbf{C}_0\mathbf{V}_0^{\mathbf{T}}),$$

where $\mathbf{C} = \mathit{Diagonal}\,(\xi_1, \ldots, \xi_p)$ ($\mathbf{C}_0 = \mathit{Diagonal}\,(\xi_{1,0}, \ldots, \xi_{p,0})$) is the diagonal matrix of eigenvalues with of $\mathbf{\Psi}$ ($\mathbf{\Psi}_0$) with the associated normalized eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ ($\mathbf{v}_{1,0}, \ldots, \mathbf{v}_{p,0}$) the columns of the matrix $\mathbf{V}$ ($\mathbf{V}_0$). Define the matrix

$$\mathbf{P} = \mathbf{\Gamma} \mathbf{V} \mathbf{C}^{1/2} \; (\mathbf{P}_0 = \mathbf{\Gamma}_0 \mathbf{V}_0 \mathbf{C}_0^{1/2}).$$

Further define

$$\mathbf{\Lambda} = \mathbf{P}_0^{-1} \mathbf{P}.$$

The square λ^2 of the determinant of $\mathbf{\Lambda}$ is

$$\lambda^2 = |\mathbf{\Lambda}|^2 = \left|\mathbf{\Lambda}\mathbf{\Lambda}^{\mathbf{T}}\right| = \left|\left(\mathbf{P}_0^{-1}\mathbf{P}\right)\left(\mathbf{P}_0^{-1}\mathbf{P}\right)^{\mathbf{T}}\right| = \left|\left(\mathbf{P}_0\mathbf{P}_0^{\mathbf{T}}\right)^{-1}\right| \left|\mathbf{P}\mathbf{P}^{\mathbf{T}}\right|$$
$$= \left|\mathbf{\Sigma}_0^{-1}\mathbf{\Sigma}\right| = \frac{|\mathbf{\Sigma}|}{|\mathbf{\Sigma}_0|} = \frac{\left|\mathbf{\Gamma}\mathbf{\Psi}\mathbf{\Gamma}^{\mathbf{T}}\right|}{\left|\mathbf{\Gamma_0}\mathbf{\Psi_0}\mathbf{\Gamma}_0^{\mathbf{T}}\right|}.$$

Suppose that $\mathbf{\Psi} = \mathbf{\Psi}_0$. It would follow that

$$\lambda^2 = \frac{|\mathbf{\Gamma}|^2}{|\mathbf{\Gamma}_0|^2} = \frac{\prod_{i=1}^p \sigma_i^2}{\prod_{i=1}^p \sigma_{i,0}^2} = \prod\nolimits_{i=1}^p \frac{\sigma_i^2}{\sigma_{i,0}^2} = \prod\nolimits_{i=1}^p \lambda_i^2,$$

where $\lambda_i^2 = \sigma_i^2/\sigma_{i,0}^2$ for $i = 1, \ldots, p$. We see that $\lambda^2 = 1$ if the process is in-control and $\lambda^2 \neq 1$ if the process is in an out-of-control state with respect to the process generalized variance. Suppose only one of the variances has shifted, say, σ_1^2 has shifted from its in-control value of $\sigma_{1,0}^2$. It would follow that

$$\lambda^2 = |\mathbf{\Lambda}| = \frac{\sigma_1^2}{\sigma_{1,0}^2} = \lambda_1^2.$$

We can also see that

$$\lambda^2 = \frac{\left|\mathbf{\Gamma}\mathbf{V}\mathbf{C}\mathbf{V}^{\mathbf{T}}\mathbf{\Gamma}\right|}{\left|\mathbf{\Gamma_0}\mathbf{V_0}\mathbf{C_0}\mathbf{V_0^T}\mathbf{\Gamma}_0\right|} = \frac{|\mathbf{\Gamma}\mathbf{C}\mathbf{\Gamma}|}{|\mathbf{\Gamma_0}\mathbf{C_0}\mathbf{\Gamma}_0|} = \prod\nolimits_{i=1}^p \frac{\sigma_i^2 \xi_i}{\sigma_{i,0}^2 \xi_{i,0}}.$$

If a shift has occurred but not with the variances, then their has been a change in the correlation structure of the process covariance. It would follow that

$$\lambda^2 = |\mathbf{\Lambda}| = \prod\nolimits_{i=1}^p \frac{\xi_i}{\xi_{i,0}} = \prod\nolimits_{i=1}^p \zeta_i,$$

where $\zeta_i = \xi_i / \xi_{i,0}$ for $i = 1, \ldots, p$.

We note that the type of shift Healy (1987) assumed is the special case of the one presented here in which $\mathbf{\Sigma} = c\mathbf{\Sigma}_0$. That is, the standard deviation of each component of the vector of quality measurements shifts the same proportion of their in-control values. For this type of shift in the covariance matrix,

$$\lambda^2 = \left|\mathbf{\Sigma}_0^{-1}\mathbf{\Sigma}\right| = \left|\mathbf{\Sigma}_0^{-1}\left(c\mathbf{\Sigma}_0\right)\right| = c^p.$$

1.4 Thesis Prospectus

In this thesis, we will discuss analytical methods for analyzing the performance of the Shewhart and CUSUM $|\mathbf{S}|$ and $\ln\left(|\mathbf{S}|\right)$ charts. First we examine closed form expressions presented in the literature for the probability and cumulative density functions describing the distribution of the sample generalized variance $|\mathbf{S}|$ and $\ln\left(|\mathbf{S}|\right)$ under the independent multivariate normal model. We provide a new closed form expression for the probability density function of $|\mathbf{S}|$. It is pointed out that each of these methods suffers from the “curse of dimensionality.” We discuss the use of the methods found in the literature for dealing with the dimension problem for large values of p to evaluating the probability density function describing the distributions of $|\mathbf{S}|$ and $\ln\left(|\mathbf{S}|\right)$. An outline is given for using these results in the analytical methods for analyzing the run length performance of the Shewhart and CUSUM $|\mathbf{S}|$ and $\ln\left(|\mathbf{S}|\right)$ charts. Some analytical results are given in the bivariate case. Examples are given to illustrate the procedure. Some recommendations for further research are given.

CHAPTER 2

DISTRIBUTIONS OF FUNCTIONS OF |S|

2.1 Introduction

A measure of the variability in the distribution of a $p \times 1$ multivariate measurement $\mathbf{X}$ is the population generalized variance $|\mathbf{\Sigma}|$, where $\mathbf{\Sigma}$ is the $p \times p$ matrix of variances and covariances of the components of $\mathbf{X}$. It can be shown that the sample covariance $\mathbf{S}$ based on a random sample (independence data model) $\mathbf{X}_1, \ldots, \mathbf{X}_n$ from the distribution of $\mathbf{X}$ is an unbiased estimator of $\mathbf{\Sigma}$. An estimator of $|\mathbf{\Sigma}|$ is the sample generalized variance $|\mathbf{S}|$. In general, the sample generalized variance is a biased estimator of the population generalized variance. Under the multivariate normal model for the distribution of $\mathbf{X}$, it is shown in Anderson (2003) that for $n > p$

$$|\mathbf{S}| \sim \frac{|\mathbf{\Sigma}|}{(n-1)^p} \prod_{i=1}^{p} \chi^2_{(n-1)-(i-1)},$$

where $\chi^2_{n-1}, \ldots, \chi^2_{n-p}$ are independent chi square random variables with respective degrees of freedom $n-1, \ldots, n-p$. Thus under the independent multivariate normal model

$$\frac{|\mathbf{S}|}{\prod_{i=1}^{p} \left(\frac{n-i}{n-1}\right)}$$

is an unbiased estimator of $|\mathbf{\Sigma}|$.

Anderson (2003) states "If $p = 1$, $|\mathbf{S}|$ has the distribution of $|\mathbf{\Sigma}| \cdot \chi^2_{n-1} / (N-1)$. If $p = 2$, $|\mathbf{S}|$ has the distribution of $|\mathbf{\Sigma}| \, \chi^2_{N-1} \cdot \chi^2_{N-2} / (N-1)^2$. It follows from Problem 7.15 or 7.37 that when $p = 2$, $|\mathbf{S}|$ has the distribution of $|\mathbf{\Sigma}| \left(\chi^2_{2N-4}\right)^2 / (2N-2)^2$." Here his capital N is our lower case n which is the sample size. It follow that when $p = 1$ $|\mathbf{\Sigma}| = \sigma^2$ (the process variance) and $|\mathbf{S}| = S^2$ (the sample variance). We see that

$$\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right| \sim \begin{cases} \chi^2_{n-1}, & \text{if } p = 1; \\ \left(\chi^2_{2n-4}\right)^2 / 4, & \text{if } p = 2. \end{cases}$$

It also follows that

$$\ln\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right) \sim \begin{cases} \ln\left(\chi^2_{n-1}\right), & \text{if } p=1; \\ 2\left(\ln\left(\chi^2_{2n-4}\right)^2 - \ln(2)\right), & \text{if } p=2. \end{cases}$$

In this article, we are interested in the distribution of the sample generalized variance $|\mathbf{S}|$ and $\ln\left(|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}|\right)$ under the independent multivariate normal model.

2.2 Some Distributional Results

A random variable X is said to have Chi Square distribution with $\nu > 0$ degrees of freedom if the probability density function (pdf) $f_{\chi^2_\nu}(x)$ describing the distribution of X has the form

$$f_{\chi^2_\nu}(x) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}} x^{\nu/2-1} e^{-x/2} I_{(0,\infty)}(x),$$

where $I_{(0,\infty)}(x)$ is an indicator function having the value 1 if $x \in (0,\infty)$ and 0 otherwise. A graph of $f_{\chi^2_5}(x)$ is given in the following figure.

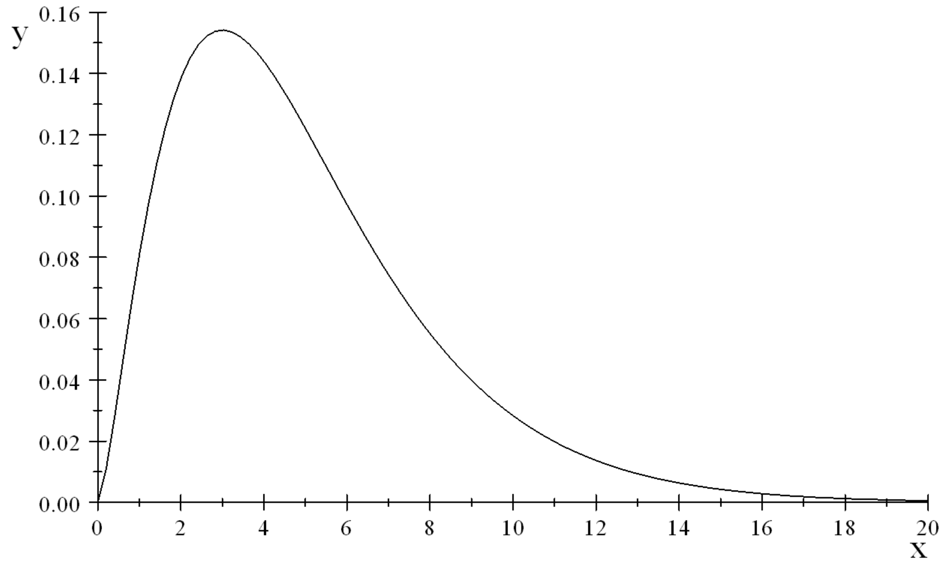


Figure 2.1: Plot of $f_{\chi^2_5}(x)$

The kth moment of a Chi Square distribution is given in the following theorem.

Theorem 2.1: If $X \sim \chi^2_\nu$, then $E\left(X^k\right)$ is

$$E\left(X^k\right) = \frac{\Gamma\left(\frac{\nu}{2}+k\right)2^k}{\Gamma\left(\frac{\nu}{2}\right)},$$

for $k = 1, 2, 3, \ldots$.

Proof of Theorem 2.1: We see that

$$\begin{aligned} E\left(X^k\right) &= \int_0^\infty x^k \frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}} x^{\nu/2-1} e^{-x/2} dx \\ &= \frac{\Gamma\left(\frac{\nu+2k}{2}\right)2^{(\nu+2k)/2}}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}} \int_0^\infty \frac{1}{\Gamma\left(\frac{\nu+2k}{2}\right)2^{(\nu+2k)/2}} x^{(\nu+2k)/2-1} e^{-x/2} dx \\ &= \frac{\Gamma\left(\frac{\nu}{2}+k\right)2^k}{\Gamma\left(\frac{\nu}{2}\right)} \end{aligned}$$

■

An interesting general family of transformations of χ^2_ν random variable is defined by $Y = \log_b\left(\chi_\nu^2\right)$, where $b > 0$ and $b \neq 1$. The cdf describing the distribution of Y is

$$F_{\log_b(\chi_\nu^2)}(y) = P\left(\log_b(X) \leq y\right) = P\left(\log_b(X) \leq y\right) = P\left(b^{\log_b(X)} \leq b^y\right)$$
$$= P\left(X \leq b^y\right) = F_{\chi_\nu^2}\left(b^y\right).$$

It then follows that the pdf of the distribution of X is given by

$$f_{\log_b(\chi_\nu^2)}(y) = b^y f_X\left(b^y;\nu\right) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}}e^{-(b^y-\nu\ln(b)y)/2}.$$

The following figure displays the plot of $f_{\log_e\left(\chi_5^2\right)}(y)$ and $f_{\log_2\left(\chi_5^2\right)}(y)$.

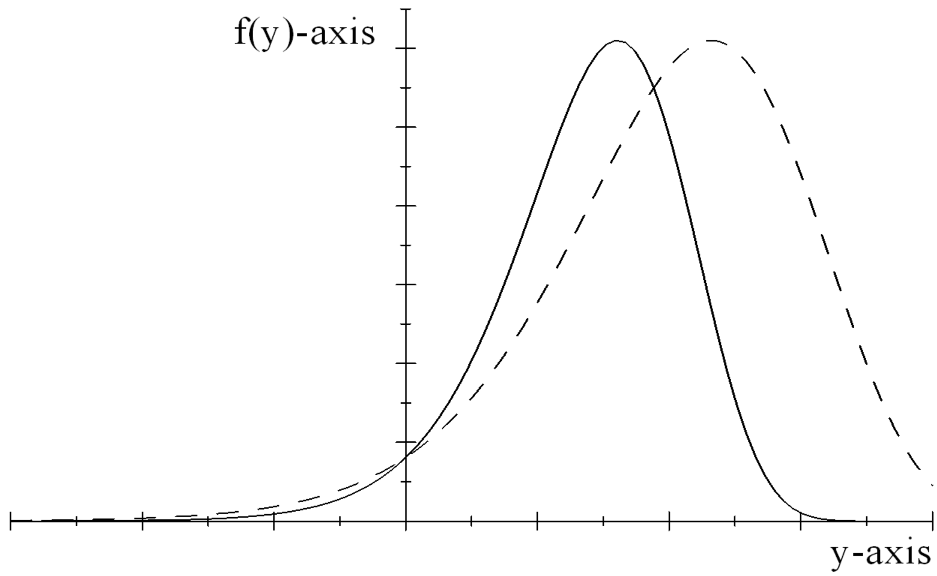


Figure 2.2: Plot of $f_{\log_e\left(\chi_5^2\right)}(y)$ (solid curve), $f_{\log_2\left(\chi_5^2\right)}(y)$ (dashed curve)

Also of interest are transformations of X defined by $Y = -\log_b(\chi_\nu^2)$. The cdf describing the distribution of Y is

$$\begin{aligned} F_{-\log_b(\chi_\nu^2)}(y) &= P(-\log_b(X) \leq y) = P(\log_b(X) \geq -y) = P\left(X \geq b^{-y}\right) \\ &= 1 - P\left(X \leq b^{-y}\right) = 1 - F_{\chi_\nu^2}\left(b^{-y}\right). \end{aligned}$$

It then follows that the pdf of the distribution of X is given by

$$f_{-\log_b(\chi_\nu^2)}(y) = b^{-y} f_X\left(b^{-y}; \nu\right) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right) 2^{\nu/2}} e^{-\left(b^{-y} + \nu \ln(b) y\right)/2}.$$

The following figure is displays the plot of $f_{-\log_e\left(\chi_5^2\right)}(y)$ and $f_{-\log_2\left(\chi_5^2\right)}(y)$.

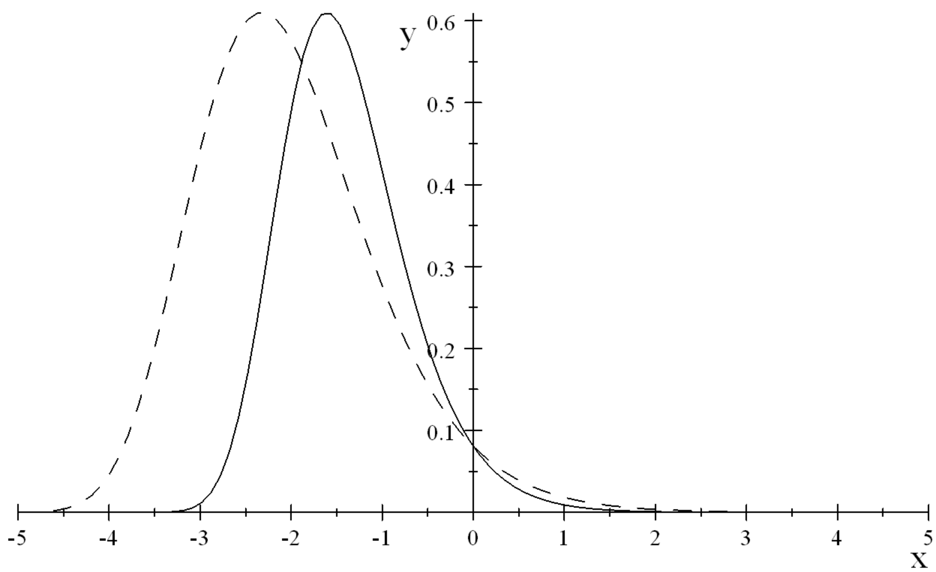


Figure 2.3: Plot of $f_{-\log_e\left(\chi_5^2\right)}(y)$ (solid curve), $f_{-\log_2\left(\chi_5^2\right)}(y)$ (dashed curve)

The following theorem is useful.

Theorem 2.2: $f_{-\log_b(\chi_\nu^2)}(-y) = f_{\log_b(\chi_\nu^2)}(y)$ and $f_{\log_b(\chi_\nu^2)}(-y) = f_{-\log_b(\chi_\nu^2)}(y)$.

Proof of Theorem 2.2: We see that

$$\begin{aligned} f_{-\log_b(\chi_\nu^2)}(-y) &= \frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}} e^{-\left(b^{-(-y)}+\nu\ln(b)(-y)\right)/2} \\ &= \frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}} e^{-(b^{y}-\nu\ln(b)y)/2} \\ &= f_{\log_b(\chi_\nu^2)}(y)\,. \end{aligned}$$

Further, we see that

$$\begin{aligned} f_{\log_b(\chi_\nu^2)}(-y) &= \frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}} e^{-\left(b^{-y}-\nu\ln(b)(-y)\right)/2} \\ &= \frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}} e^{-\left(b^{-y}+\nu\ln(b)y\right)/2} \\ &= f_{-\log_b(\chi_\nu^2)}(y)\,. \end{aligned}$$

■

Another useful theorem is as follows.

Theorem 2.3: $f_{\chi^2_\nu}(x) = x^{-1}\log_b(e)\, f_{\ln(\chi^2_\nu)}(\log_b(x))$.

Proof of Theorem 2.3: We see that

$$F_{\chi^2_\nu}(x) = P\left(\log_b\left(\chi^2_\nu\right) \leq \log_b(x)\right) = F_{\log_b(\chi^2_\nu)}(\log_b(x)).$$

Thus,

$$f_{\chi^2_\nu}(x) = \frac{d}{dx} F_{\log_b(\chi^2_\nu)}(\log_b(x)) = x^{-1}\log_b(e)\, f_{\ln(\chi^2_\nu)}(\log_b(x)).$$

■

2.3 Distribution of $|\mathbf{S}|$

Grigoryan and He (2005) derived a closed form expression for the probability density function $f_{|\mathbf{S}|}(w)$ describing the distribution of $|\mathbf{S}|$ under the independent multivariate normal model using the results in Anderson (2003). First they derived a closed form expresssion the probability density function $f_{|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|}(w)$ describing the distribution of

$$\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right| \sim \prod\nolimits_{i=1}^{p} \chi^2_{n-i},$$

where $\chi^2_{n-1}, \ldots, \chi^2_{n-p}$ are independent chi square random variables with respective degrees of freedom $n-1, \ldots, n-p$. Let $X_i = \chi^2_{n-i}$ and make the transformation

$$W_1 = X_1, W_2 = X_1 X_2, \ldots, W_p = X_1 \cdots X_p.$$

The inverse of this transformation is

$$X_1 = W_1, X_2 = W_2/W_1, \ldots, X_p = W_p/W_{p-1}$$

with Jacobian

$$J = \frac{1}{W_1 W_2 \cdots W_{p-1}}.$$

It follows that the joint probability density function $f_{W_1,W_2,\ldots,W_p}(w_1, w_2, \ldots, w_p)$ describing the joint distribution of $W_1, W_2, \ldots, W_p$ is given by

$$\begin{aligned} f_{W_1,W_2,\ldots,W_p}(w_1, w_2, \ldots, w_p) &= f_{X_1}(w_1)\, f_{X_2}(w_2/w_1) \cdots f_{X_p}(w_p/w_{p-1}) \frac{1}{\prod_{i=1}^{p-1} w_i} \\ &= \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{(n-1)/2}} w_1^{(n-1)/2-1} e^{-w_1/2} \\ &\times \frac{1}{\Gamma\left(\frac{n-2}{2}\right) 2^{(n-2)/2}} (w_2/w_1)^{(n-2)/2-1} e^{-(w_2/w_1)/2} \\ &\cdots \frac{1}{\Gamma\left(\frac{n-p}{2}\right) 2^{(n-p)/2}} (w_p/w_{p-1})^{(n-2)/2-1} e^{-(w_p/w_{p-1})/2} \frac{1}{\prod_{i=1}^{p-1} w_i} \\ &= \frac{w_p^{(n-2)/2-1} \prod_{i=1}^{p-1} w_i^{-1/2}}{\prod_{i=1}^{p} \Gamma\left(\frac{n-i}{2}\right) 2^{p(2n-p-1)/4}} e^{-w_1/2} e^{-\sum_{i=2}^{p}(w_i/w_{i-1})/2} \end{aligned}$$

It now follows that the probability density function $f_{|\mathbf{S}|}(w_p)$ as given in Grigoryan and He (2005) describing the marginal distribution of $W_p = |(n-1)\, \boldsymbol{\Sigma}^{-1}\mathbf{S}|$ is

$$\begin{aligned} f_{|(n-1)\boldsymbol{\Sigma}^{-1}\mathbf{S}|}(w_p) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty & \frac{w_p^{(n-2)/2-1} \prod_{i=1}^{p-1} w_i^{-1/2}}{\prod_{i=1}^{p} \Gamma\left(\frac{n-i}{2}\right) 2^{p(2n-p-1)/4}} \\ & \times e^{-w_1/2} e^{-\sum_{i=2}^{p}(w_i/w_{i-1})/2} dw_1 \ldots dw_{p-1}. \end{aligned}$$

Observing that

$$\begin{aligned} F_{|\mathbf{S}|}(w) = P(|\mathbf{S}| \leq w) &= P\left(\left|(n-1)\, \boldsymbol{\Sigma}^{-1}\mathbf{S}\right| \leq (n-1)^p \left|\boldsymbol{\Sigma}^{-1}\right| w\right) \\ &= F_{|(n-1)\boldsymbol{\Sigma}^{-1}\mathbf{S}|}\left((n-1)^p \left|\boldsymbol{\Sigma}^{-1}\right| w\right). \end{aligned}$$

The probability density function describing the distribution $f_{|\mathbf{S}|}(w)$ of $|\mathbf{S}|$ is

$$\begin{aligned} f_{|\mathbf{S}|}(w) &= (n-1)^p \left|\boldsymbol{\Sigma}^{-1}\right| f_{|(n-1)\boldsymbol{\Sigma}^{-1}\mathbf{S}|}\left((n-1)^p \left|\boldsymbol{\Sigma}^{-1}\right| w\right) \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{(n-1)^p |\boldsymbol{\Sigma}^{-1}| \left((n-1)^p |\boldsymbol{\Sigma}^{-1}| w\right)^{(n-2)/2-1} \prod_{i=1}^{p-1} w_i^{-1/2}}{\prod_{i=1}^{p} \Gamma\left(\frac{n-i}{2}\right) 2^{p(2n-p-1)/4}} \\ &\times e^{-w_1/2} e^{-\left(\sum_{i=2}^{p-1}(w_i/w_{i-1}) + \left((n-1)^p \left|\boldsymbol{\Sigma}^{-1}\right| w/w_{p-1}\right)\right)/2} dw_1 \ldots dw_{p-1}. \end{aligned}$$

Pham-Gia and Turkkan (2010) show that the distribution of $\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|$ can be expressed in terms of the Meijer G function. The Meijer G function is defined by

$$G_{p,q}^{m\ r}\left(x\ \middle|\ \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j - s\right)\prod_{j=1}^{r}\Gamma\left(1-a_j+s\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+s\right)\prod_{j=r+1}^{p}\Gamma\left(a_j - s\right)} x^s ds,$$

where the integral is along the complex contour L of a ratio of products of gamma functions. On page 936, they express the pdf describing the distribution of $\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|$ with some modification is given as follows

$$f_{\left|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}\right|}\left(w\right) = \frac{1}{2^p}\left(\prod_{j=1}^{p}\frac{1}{\Gamma\left(\frac{n-j}{2}\right)}\right) G_{0,p}^{p\ 0}\left(\frac{w}{2^p}\ \middle|\ \begin{matrix} \frac{n-1}{2},\ldots,\frac{n-p}{2} \\ \frac{n-1}{2},\ldots,\frac{n-p}{2} \end{matrix}\right) I_{(0,\infty)}\left(w\right).$$

It would then follow that the pdf describing the distribution of $\ln\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right)$ would have the form

$$\begin{aligned} f_{\ln\left(\left|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right)}\left(w\right) &= e^w f_{\left|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}\right|}\left(e^w\right) \\ &= \frac{e^w}{2^p}\left(\prod_{j=1}^{p}\frac{1}{\Gamma\left(\frac{n-j}{2}\right)}\right) G_{0,p}^{p\ 0}\left(\frac{e^w}{2^p}\ \middle|\ \begin{matrix} \frac{n-1}{2},\ldots,\frac{n-p}{2} \\ \frac{n-1}{2},\ldots,\frac{n-p}{2} \end{matrix}\right). \end{aligned}$$

The Meijer G function has been implimented in both MATLAB and Mathematica. The MATLAB code is

$$\begin{aligned} &G_{p,q}^{m\ r}\left(x\ \middle|\ \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) \\ &= meijerG([[a_1,\ldots,a_r],[a_{r+1},\ldots,a_p]],[[b_1,\ldots,b_m],[b_{m+1},\ldots,b_q]],x). \end{aligned}$$

It follows that

$$G_{0,p}^{p\ 0}\left(x\ \middle|\ \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right)$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{p}\Gamma\left(b_j - s\right)\prod_{j=1}^{0}\Gamma\left(1-a_j+s\right)}{\prod_{j=p+1}^{p}\Gamma\left(1-b_j+s\right)\prod_{j=0+1}^{0}\Gamma\left(a_j - s\right)} x^s ds$$
$$= \frac{1}{2\pi i}\int_L \prod_{j=1}^{p}\Gamma\left(b_j - s\right) x^s ds$$
$$= meijerG([[a_1,\ldots,a_0],[a_{0+1},\ldots,a_0]],[[b_1,\ldots,b_p],[b_{p+1},\ldots,b_p]],x)$$
$$= meijerG([[],[]],[[b_1,\ldots,b_p],[]],x).$$

It follows that

$$f_{|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|}\left(w\right) = \frac{1}{2^p}\left(\prod_{j=1}^{p}\frac{1}{\Gamma\left(\frac{n-j}{2}\right)}\right) meijerG([[],[]],[[\frac{n-1}{2},\ldots,\frac{n-p}{2}],[]],w/2^p).$$

In Mathematica, the Meijer -function is implemented as

$$G_{p,q}^{m\ r}\left(x\ \middle|\ \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = MeijerG[a1,...,ar,a(r{+}1),...,ap,b1,...,bm,b(m{+}1),...,bq,x].$$

We then have

$$f_{|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|}\left(w\right) = \frac{1}{2^p}\left(\prod_{j=1}^{p}\frac{1}{\Gamma\left(\frac{n-j}{2}\right)}\right) MeijerG([[],[]],[[\frac{n-1}{2},\ldots,\frac{n-p}{2}],[]],w/2^p).$$

2.4 Distribution of $\ln\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right)$

The cumulative distribution function $F_U\left(u\right)$, where $U = \ln\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right)$ is given by

$$F_U\left(u\right) = P\left(\ln\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right) \leq u\right) = P\left(|\mathbf{S}| \leq |\mathbf{\Sigma}|\, e^u/\left(n-1\right)^p\right)$$
$$= F_{|\mathbf{S}|}\left(|\mathbf{\Sigma}|\, e^u/\left(n-1\right)^p\right).$$

Hence, the probability density function describing the distribution of U can be expressed in terms of the probability density function describing the distribution of $|\mathbf{S}|$

as

$$f_U(u) = |\mathbf{\Sigma}|\, e^u / (n-1)^p \, f_{|\mathbf{S}|}\left(|\mathbf{\Sigma}|\, e^u / (n-1)^p\right).$$

Using the expression derived by Grigoryan and He (2005), we can express the probability density function $f_U(u)$ as

$$f_U(u) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\left(|\mathbf{\Sigma}|\, e^u / (n-1)^p\right)^{(n-2)/2-1} \left(\prod_{i=1}^{p-1} w_i^{-1/2}\right)}{\prod_{i=1}^{p} \Gamma\left(\frac{n-i}{2}\right) 2^{p(2n-p-1)/4}}$$
$$\times\, e^{-w_1/2} e^{-\left(\sum_{i=2}^{p-1} (w_i/w_{i-1}) + (|\mathbf{\Sigma}| e^u/(n-1)^p / w_{p-1})\right)/2} dw_1 \dots dw_{p-1}$$

From the expression by Pham-Gia and Turkkan (2010) for the probability density function describing the distribution of $|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}|$, we have

$$f_{\ln(|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|)}(w) = e^w f_{|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|}(e^w)$$
$$= \frac{e^w}{2^p} \left(\prod_{j=1}^{p} \frac{1}{\Gamma\left(\frac{n-j}{2}\right)}\right) G_{0,p}^{p\,0}\left(\frac{e^w}{2^p} \,\middle|\, \begin{matrix} \frac{n-1}{2}, \dots, \frac{n-p}{2} \\ \frac{n-1}{2}, \dots, \frac{n-p}{2} \end{matrix}\right).$$

Here we give another method for deriving the probability density function

$$f_U(u) = f_{\ln(|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|)}(u)$$

describing the distribution of

$$U = \ln\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right) \sim \sum_{i=1}^{p} \ln\left(\chi_{n-i}^2\right),$$

where $n > p$ are positive integers, χ_{n-i}^2 is a random variable with a Chi Square distribution with $n-i$ degrees of freedom, and $\chi_{n-1}^2, \dots, \chi_{n-p}^2$ are independent. The pdf of U is given in the following theorem.

Theorem 2.4: If $\chi_{n-1}^2, \dots, \chi_{n-p}^2$ are independent Chi Square random variables with degrees of freedom $n-1, \dots, n-p$, respectively, with $n > p$, then the probability density function $f_U(u)$ describing the distribution of

$$U = \sum_{i=1}^{p} \ln\left(\chi_{n-i}^2\right)$$

can be expressed as

$$f_U(u) = \int_0^\infty \cdots \int_0^\infty f_{\ln\left(\chi^2_{n-1}\right)}(u - x_2 - \ldots - x_p) f_{\ln\left(\chi^2_{n-2}\right)}(x_2)$$
$$\cdots f_{\ln\left(\chi^2_{n-p}\right)}(x_p)\, dx_2 \cdots dx_p$$
$$= \int_0^\infty \cdots \int_0^\infty e^u f_{\chi^2_{n-1}}\left(e^{u-x_2-\ldots-x_p}\right) f_{\chi^2_{n-2}}\left(e^{x_2}\right)$$
$$\cdots f_{\chi^2_{n-p}}\left(e^{x_p}\right) dx_2 \cdots dx_p.$$

Also, the cumulative distribution function $F_U(u)$ can be expressed as

$$F_U(u) = \int_0^\infty \cdots \int_0^\infty F_{\ln\left(\chi^2_{n-1}\right)}(u - x_2 - \ldots - x_p) f_{\ln\left(\chi^2_{n-2}\right)}(x_2)$$
$$\cdots f_{\ln\left(\chi^2_{n-p}\right)}(x_p)\, dx_2 \cdots dx_p$$
$$= \int_0^\infty \cdots \int_0^\infty e^u F_{\chi^2_{n-1}}\left(e^{u-x_2-\ldots-x_p}\right) f_{\chi^2_{n-2}}\left(e^{x_2}\right)$$
$$\cdots f_{\chi^2_{n-p}}\left(e^{x_p}\right) dx_2 \cdots dx_p.$$

Proof of Theorem 2.4: For convenience, define $Y_i = \ln\left(\chi^2_{n-i}\right)$ for $i = 1, \ldots, p$ for any integer $p < n$. Also, we let

$$c_i = \frac{1}{\Gamma\left(\frac{n-i}{2}\right) 2^{(n-i)/2}}.$$

Consider the one-to-one transformation

$$U_i = \sum\nolimits_{j=1}^{i} Y_j$$

for $i = 1, \ldots, p$. We see that $U = U_p$. The inverse transformation is

$$Y_1 = U_1 \text{ and } Y_i = U_i - U_{i-1}$$

for $i = 2, \ldots, p$ with Jacobian $J = 1$. The joint probability density function of $U_1, \ldots, U_p$ for

$$u_1 < u_2 < \ldots < u_p$$

is given by

$$f_{U_1,\ldots,U_p}(u_1,\ldots,u_p) = f_{Y_1}(u_1)\, f_{Y_2}(u_2-u_1)\cdots f_{Y_p}(u_p-u_{p-1}).$$

For $p=2$ and $n>p$, we have

$$\begin{aligned}
f_{U_1,U_2}(u_1,u_2) &= f_{Y_1}(u_1)\, f_{Y_2}(u_2-u_1) \\
&= c_1 e^{-(e^{u_1}-(n-1)u_1)/2} c_2 e^{-\left(e^{u_2-u_1}-(n-2)(u_2-u_1)\right)/2} \\
&= c_1 c_2 e^{-e^{u_1}/2} e^{-\left(e^{u_2-u_1}-(n-2)(u_2-u_1)-(n-1)u_1\right)/2} \\
&= c_1 c_2 \left(\sum_{i=0}^{\infty} \frac{(-1)^i e^{iu_1}}{2^i i!}\right) e^{-\left(e^{u_2-u_1}-(n-2)(u_2-u_1)-(n-1)u_1\right)/2} \\
&= c_1 c_2 \sum_{i=0}^{\infty} \frac{(-1)^i}{2^i i!} e^{-\left(e^{u_2-u_1}-(n-2)(u_2-u_1)-(n-1)u_1-2iu_1\right)/2} \\
&= c_1 c_2 e^{(n-1)u_2/2} \sum_{i=0}^{\infty} \frac{(-1)^i e^{iu_2}}{2^i i!} e^{-\left(e^{u_2-u_1}+(2i+1)(u_2-u_1)\right)/2}
\end{aligned}$$

It follows that

$$\begin{aligned}
f_{U_2}(u_2) &= \int_{-\infty}^{u_2} f_{Y_1}(u_1)\, f_{Y_2}(u_2-u_1)\, du_1 \\
&= c_1 c_2 e^{(n-1)u_2/2} \sum_{i=0}^{\infty} \frac{(-1)^i e^{iu_2}}{2^i i!} \int_{-\infty}^{u_2} e^{-\left(e^{u_2-u_1}+(2i+1)(u_2-u_1)\right)/2} du_1
\end{aligned}$$

Making the transformation $x_2 = u_1 - u_2$ with $dx_2 = du_1$, we have

$$\begin{aligned}
f_{U_2}(u_2) &= c_1 c_2 e^{(n-1)u_2/2} \sum_{i=0}^{\infty} \frac{(-1)^i e^{iu_2}}{2^i i!} \int_{-\infty}^{0} e^{-\left(e^{-x_2}-(2i+1)x_2\right)/2} dx_2 \\
&= c_1 c_2 e^{(n-1)u_2/2} \int_{-\infty}^{0} \sum_{i=0}^{\infty} \frac{(-1)^i e^{i(u_2+x_2)}}{2^i i!} e^{-\left(e^{-x_2}-x_2\right)/2} dx_2 \\
&= \int_{-\infty}^{0} c_1 e^{-\left(e^{u_2+x_2}-(n-1)(u_2+x_2)\right)/2} c_2 e^{-\left(e^{-x_2}+(n-2)x_2\right)/2} dx_2 \\
&= \int_{-\infty}^{0} f_{\ln\left(\chi^2_{n-1}\right)}(u_2+x_2)\, f_{-\ln\left(\chi^2_{n-2}\right)}(x_2)\, dx_2 \\
&= \int_{0}^{\infty} f_{\ln\left(\chi^2_{n-1}\right)}(u_2-x_2)\, f_{\ln\left(\chi^2_{n-2}\right)}(x_2)\, dx_2
\end{aligned}$$

Hence, the theorem is true for $p=2$ and $n>p$. Now suppose that the theorem is true for $p>2$ and $n>p$. For $n>p+1$ and $u_1<u_2<\ldots<u_{p+1}$, we have

$$f_{U_1,\ldots,U_{p+1}}(u_1,\ldots,u_{p+1}) = f_{Y_1}(u_1)\, f_{Y_2}(u_2-u_1)\cdots f_{Y_p}(u_p-u_{p-1})\, f_{Y_{p+1}}(u_{p+1}-u_p).$$

It follows that

$$
\begin{aligned}
f_{U_{p+1}}(u_{p+1}) &= \int_{-\infty}^{u_{p+1}} \int_{-\infty}^{u_p} \cdots \int_{-\infty}^{u_2} f_{Y_1}(u_1) f_{Y_2}(u_2 - u_1) \cdots \\
&\times f_{Y_p}(u_p - u_{p-1}) f_{Y_{p+1}}(u_{p+1} - u_p)\, du_1 \cdots du_{p-1} du_p \\
&= \int_{-\infty}^{u_{p+1}} \int_{-\infty}^{u_p} \cdots \int_{-\infty}^{u_2} f_{Y_1}(u_1) f_{Y_2}(u_2 - u_1) \cdots f_{Y_p}(u_p - u_{p-1}) \\
&\times du_1 \cdots du_{p-1} f_{Y_{p+1}}(u_{p+1} - u_p)\, du_p \\
&= \int_{-\infty}^{u_{p+1}} f_{U_p}(u_p) f_{Y_{p+1}}(u_{p+1} - u_p)\, du_p.
\end{aligned}
$$

We now have

$$
\begin{aligned}
f_{U_{p+1}}(u_{p+1}) &= \int_{-\infty}^{u_{p+1}} (\int_0^\infty \cdots \int_0^\infty f_{\ln\left(\chi^2_{n-1}\right)}(u_p - x_2 - \ldots - x_p) \\
&\times f_{\ln\left(\chi^2_{n-2}\right)}(x_2) \cdots f_{\ln\left(\chi^2_{n-p}\right)}(x_p)\, dx_2 \cdots dx_p) \\
&\times f_{Y_{p+1}}(u_{p+1} - u_p)\, du_p \\
&= \int_0^\infty \cdots \int_0^\infty (\int_{-\infty}^{u_{p+1}} f_{\ln\left(\chi^2_{n-1}\right)}(u_p - x_2 - \ldots - x_p) \\
&\times f_{Y_{p+1}}(u_{p+1} - u_p)\, du_p) \\
&\times f_{\ln\left(\chi^2_{n-2}\right)}(x_2) \cdots f_{\ln\left(\chi^2_{n-p}\right)}(x_p)\, dx_2 \cdots dx_p.
\end{aligned}
$$

Observe that

$$\int_{-\infty}^{u_{p+1}} f_{\ln\left(\chi_{n-1}^2\right)}\left(u_p - x_2 - \ldots - x_p\right) f_{Y_{p+1}}\left(u_{p+1} - u_p\right) du_p$$
$$= \int_{-\infty}^{u_{p+1}} c_1 e^{-\left(e^{u_p - x_2 - \ldots - x_p} - (n-1)(u_p - x_2 - \ldots - x_p)\right)/2}$$
$$\times c_{p+1} e^{-\left(e^{u_{p+1} - u_p} - (n-p-1)(u_{p+1} - u_p)\right)/2} du_p$$
$$= c_1 c_{p+1}$$
$$\times \int_{-\infty}^{u_{p+1}} e^{-e^{u_p - x_2 - \ldots - x_p}/2} e^{-\left(e^{u_{p+1} - u_p} - (n-p-1)(u_{p+1} - u_p) - (n-1)(u_p - x_2 - \ldots - x_p)\right)/2} du_p$$
$$= c_1 c_{p+1} \int_{-\infty}^{u_{p+1}} \sum_{i=0}^{\infty} \frac{(-1)^i e^{(n-1+2i)(u_{p+1} - x_2 - \ldots - x_p)/2}}{2^i i!}$$
$$\times e^{-\left(e^{u_{p+1} - u_p} + (p-1+2i)(u_{p+1} - u_p)\right)/2} du_p$$
$$= c_1 c_{p+1} e^{(n-1)(u_{p+1} - x_2 - \ldots - x_p)/2}$$
$$\times \sum_{i=0}^{\infty} \frac{(-1)^i e^{i(u_{p+1} - x_2 - \ldots - x_p)}}{2^i i!} \int_{-\infty}^{u_{p+1}} e^{-\left(e^{u_{p+1} - u_p} + (p+2i)(u_{p+1} - u_p)\right)/2} du_p$$

Making the transformation $x_{p+1} = u_{p+1} - u_p$ with $dx_{p+1} = du_p$, we have

$$\int_{-\infty}^{u_{p+1}} f_{\ln\left(\chi_{n-1}^2\right)}\left(u_p - x_2 - \ldots - x_p\right) f_{Y_{p+1}}\left(u_{p+1} - u_p\right) du_p$$
$$= c_1 c_{p+1} e^{(n-1)(u_{p+1} - x_2 - \ldots - x_p)/2} \sum_{i=0}^{\infty} \frac{(-1)^i e^{i(u_{p+1} - x_2 - \ldots - x_p)}}{2^i i!}$$
$$\times \int_{-\infty}^{0} e^{-\left(e^{-x_{p+1}} - (p+2i)x_{p+1}\right)/2} dx_{p+1}$$
$$= c_1 c_{p+1} e^{(n-1)(u_{p+1} - x_2 - \ldots - x_p)/2} \int_{-\infty}^{0} \sum_{i=0}^{\infty} \frac{(-1)^i e^{i(u_{p+1} - x_2 - \ldots - x_p + x_{p+1})}}{2^i i!}$$
$$\times e^{-\left(e^{-x_{p+1}} - (p+2i)x_{p+1}\right)/2} dx_{p+1}$$
$$= \int_{-\infty}^{0} c_1 e^{-\left(e^{u_{p+1} - x_2 - \ldots - x_p - x_{p+1}} - (n-1)(u_{p+1} - x_2 - \ldots - x_p + x_{p+1})\right)/2}$$
$$\times c_{p+1} e^{-\left(e^{-x_{p+1}} + (n-p)x_{p+1}\right)/2} dx_{p+1}$$
$$= \int_{-\infty}^{0} f_{\ln\left(\chi_{n-1}^2\right)}\left(u_{p+1} - x_2 - \ldots - x_p + x_{p+1}\right) f_{-\ln\left(\chi_{n-p-1}^2\right)}\left(x_{p+1}\right) dx_{p+1}$$
$$= \int_{0}^{\infty} f_{\ln\left(\chi_{n-1}^2\right)}\left(u_{p+1} - x_2 - \ldots - x_p - x_{p+1}\right) f_{\ln\left(\chi_{n-p-1}^2\right)}\left(x_{p+1}\right) dx_{p+1}$$

It now follows that

$$f_{U_{p+1}}\left(u_{p+1}\right) = \int_{-\infty}^{u_{p+1}} (\int_0^\infty \cdots \int_0^\infty f_{\ln\left(\chi^2_{n-1}\right)}\left(u_p - x_2 - \ldots - x_p\right) f_{\ln\left(\chi^2_{n-2}\right)}\left(x_2\right) \cdots$$

$$\times f_{\ln\left(\chi^2_{n-p}\right)}\left(x_p\right) dx_2 \cdots dx_p) f_{Y_{p+1}}\left(u_{p+1} - u_p\right) du_p$$

$$= \int_0^\infty \cdots \int_0^\infty f_{\ln\left(\chi^2_{n-1}\right)}\left(u_{p+1} - x_2 - \ldots - x_p - x_{p+1}\right)$$

$$\times f_{\ln\left(\chi^2_{n-2}\right)}\left(x_2\right) \cdots f_{\ln\left(\chi^2_{n-p-1}\right)}\left(x_{p+1}\right) dx_2 \cdots dx_{p+1}.$$

Hence, the theorem is true for $p+1$. By the Axiom of Induction, the results holds for all n and p such that $n > p$.

The cumulative distribution function $F_U\left(u\right)$ describing the distribution of U is has the form

$$F_U\left(u\right) = \int_{-\infty}^{u} f_U\left(t\right) dt$$

$$= \int_{-\infty}^{u} \int_0^\infty \cdots \int_0^\infty f_{\ln\left(\chi^2_{n-1}\right)}\left(t - x_2 - \ldots - x_p\right) f_{\ln\left(\chi^2_{n-2}\right)}\left(x_2\right)$$

$$\cdots f_{\ln\left(\chi^2_{n-p}\right)}\left(x_p\right) dx_2 \cdots dx_p dt$$

$$= \int_0^\infty \cdots \int_0^\infty \left(\int_{-\infty}^{u} f_{\ln\left(\chi^2_{n-1}\right)}\left(t - x_2 - \ldots - x_p\right) dt\right) f_{\ln\left(\chi^2_{n-2}\right)}\left(x_2\right)$$

$$\cdots f_{\ln\left(\chi^2_{n-p}\right)}\left(x_p\right) dx_2 \cdots dx_p$$

$$= \int_0^\infty \cdots \int_0^\infty \left(\int_{-\infty}^{u - x_2 - \ldots - x_p} f_{\ln\left(\chi^2_{n-1}\right)}\left(t\right) dt\right) f_{\ln\left(\chi^2_{n-2}\right)}\left(x_2\right)$$

$$\cdots f_{\ln\left(\chi^2_{n-p}\right)}\left(x_p\right) dx_2 \cdots dx_p$$

$$= \int_0^\infty \cdots \int_0^\infty F_{\ln\left(\chi^2_{n-1}\right)}\left(u - x_2 - \ldots - x_p\right) f_{\ln\left(\chi^2_{n-2}\right)}\left(x_2\right)$$

$$\cdots f_{\ln\left(\chi^2_{n-p}\right)}\left(x_p\right) dx_2 \cdots dx_p$$

$$= \int_0^\infty \cdots \int_0^\infty e^u F_{\chi^2_{n-1}}\left(e^{u - x_2 - \ldots - x_p}\right) f_{\chi^2_{n-2}}\left(e^{x_2}\right)$$

$$\cdots f_{\chi^2_{n-p}}\left(e^{x_p}\right) dx_2 \cdots dx_p.$$

■

The distribution of $|\mathbf{S}|$ expressed in terms of the distribution of $\ln\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right)$ under the independent multivariate normal model is given in the following theorem.

Theorem 2.5: If $\chi^2_{n-1},\ldots,\chi^2_{n-p}$ are independent Chi Square random variables with degrees of freedom $n-1,\ldots,n-p$, respectively, with $n>p$, then the probability density function $f_{|\mathbf{S}|}(u)$ and cumulative distribution function $F_{|\mathbf{S}|}(u)$ describing the distribution of $|\mathbf{S}|$ are

$$f_{|\mathbf{S}|}(w) = w^{-1} f_{\ln(|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|)}\left(\ln\left(\frac{(n-1)^p\,w}{|\mathbf{\Sigma}|}\right)\right) \text{ and}$$
$$F_{|\mathbf{S}|}(w) = F_{\ln(|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|)}\left(\ln\left(\frac{(n-1)^p\,w}{|\mathbf{\Sigma}|}\right)\right).$$

Proof of Theorem 2.5: Observe that

$$\begin{aligned}F_{|\mathbf{S}|}(w) &= P\left(|\mathbf{S}|\le w\right) = P\left(\ln\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right)\le \ln\left(\frac{(n-1)^p\,w}{|\mathbf{\Sigma}|}\right)\right)\\ &= F_{\ln(|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|)}\left(\ln\left(\frac{(n-1)^p\,w}{|\mathbf{\Sigma}|}\right)\right).\end{aligned}$$

Hence, their probability density function $f_{|\mathbf{S}|}(w)$ is

$$f_{|\mathbf{S}|}(w) = \frac{d}{dw}F_{|\mathbf{S}|}(w) = w^{-1} f_{\ln(|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|)}\left(\ln\left(\frac{(n-1)^p\,w}{|\mathbf{\Sigma}|}\right)\right).$$

■

Theorem 2.6: The kth moment of the distribution of $|\mathbf{S}|$ for $n>p$ under the independent normal model is

$$E\left(|\mathbf{S}|^k\right) = \frac{2^{pk}\,|\mathbf{\Sigma}|^k}{(n-1)^{pk}}\prod_{i=1}^{p}\frac{\Gamma\left(\frac{n-i}{2}+k\right)}{\Gamma\left(\frac{n-i}{2}\right)}.$$

Proof: Recall that

$$\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right| \sim \prod_{i=1}^{p}\chi^2_{n-i},$$

$\chi^2_{n-1},\ldots,\chi^2_{n-p}$ are independent Chi Square random variables with degrees of freedom $n-1,\ldots,n-p$, respectively, with $n>p$. Thus,

$$E\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|^k\right) = E\left[\left(\prod_{i=1}^{p}\chi^2_{n-i}\right)^k\right] = \prod_{i=1}^{p}E\left[\left(\chi^2_{n-i}\right)^k\right],$$

since the random variables $\chi^2_{n-1}, \ldots, \chi^2_{n-p}$ are independent. From Theorem 2.1, we have

$$E\left[\left(\chi^2_{n-i}\right)^k\right] = \frac{\Gamma\left(\frac{n-i}{2}+k\right)2^k}{\Gamma\left(\frac{n-i}{2}\right)}.$$

Hence, the kth moment of W is expressed as

$$E\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|^k\right) = 2^{pk}\prod_{i=1}^{p}\frac{\Gamma\left(\frac{n-i}{2}+k\right)}{\Gamma\left(\frac{n-i}{2}\right)}.$$

It follows that the kth moment of $|\mathbf{S}|$ is

$$E\left(|\mathbf{S}|^k\right) = \frac{2^{pk}\,|\mathbf{\Sigma}|^k}{(n-1)^{pk}}\prod_{i=1}^{p}\frac{\Gamma\left(\frac{n-i}{2}+k\right)}{\Gamma\left(\frac{n-i}{2}\right)}.$$

■

Note that the distribution of $|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}|$ can be expressed in terms of the distribuiton of $\ln\left(|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}|\right)$ by

$$f_{|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|}(w) = w^{-1}f_{\ln(|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|)}\left(\ln(w)\right)I_{(0,\infty)}(w).$$

The kth moment of $|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}|$ can then be determined by

$$\begin{aligned}
E\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|^k\right) &= \int_0^\infty w^k f_{|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|}(w)\,dw \\
&= \int_0^\infty w^k w^{-1} f_{\ln(|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|)}\left(\ln(w)\right)dw \\
&= \int_0^\infty w^{k-1}\int_0^\infty \cdots \int_0^\infty e^{\ln(w)} f_{\chi^2_{n-1}}\left(e^{\ln(w)-x_2-\ldots-x_p}\right) f_{\chi^2_{n-2}}\left(e^{x_2}\right) \\
&\quad \cdots f_{\chi^2_{n-p}}\left(e^{x_p}\right)dx_2\cdots dx_p dw \\
&= \int_0^\infty \cdots \int_0^\infty \left(\int_0^\infty w^k f_{\chi^2_{n-1}}\left(e^{\ln(w)-x_2-\ldots-x_p}\right)dw\right) f_{\chi^2_{n-2}}\left(e^{x_2}\right) \\
&\quad \cdots f_{\chi^2_{n-p}}\left(e^{x_p}\right)dx_2\cdots dx_p.
\end{aligned}$$

Making the change of variable

$$x_1 = \ln(w) - x_2 - \ldots - x_p \text{ or } w = e^{x_2+\ldots+x_p}e^{x_1} \text{ with } dw = e^{x_2+\ldots+x_p}e^{x_1}dx_1,$$

we have

$$E\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|^{k}\right)=\int_{0}^{\infty}\cdots\int_{0}^{\infty}\left(\int_{0}^{\infty}\left(e^{x_2+\ldots+x_p}e^{x_1}\right)^{k}f_{\chi^2_{n-1}}\left(e^{x_1}\right)e^{x_2+\ldots+x_p}e^{x_1}dx_1\right)$$

$$\times f_{\chi^2_{n-2}}\left(e^{x_2}\right)\cdots f_{\chi^2_{n-p}}\left(e^{x_p}\right)dx_2\cdots dx_p$$

$$=\int_{0}^{\infty}\cdots\int_{0}^{\infty}\left(\int_{0}^{\infty}\left(e^{x_1}\right)^{k}e^{x_1}f_{\chi^2_{n-1}}\left(e^{x_1}\right)dx_1\right)$$

$$\times\left(e^{x_2}\right)^{k}e^{x_2}f_{\chi^2_{n-2}}\left(e^{x_2}\right)\cdots\left(e^{x_p}\right)^{k}e^{x_p}f_{\chi^2_{n-p}}\left(e^{x_p}\right)dx_2\cdots dx_p$$

$$=\prod_{i=1}^{p}\left(\int_{0}^{\infty}\left(e^{x_i}\right)^{k}e^{x_i}f_{\chi^2_{n-i}}\left(e^{x_i}\right)dx_i\right).$$

Making the change of variables

$$t_i=e^{x_i}\text{ with }dt=e^{x_i}dx_i,$$

we have

$$E\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|^{k}\right)=\prod_{i=1}^{p}\left(\int_{0}^{\infty}t^{k}f_{\chi^2_{n-i}}(t)\,dt\right).$$

Using the results from Theorem 2.1, we have

$$E\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|^{k}\right)=\prod_{i=1}^{p}\left(\frac{\Gamma\left(\frac{n-i}{2}+k\right)2^{k}}{\Gamma\left(\frac{n-i}{2}\right)}\right)$$

$$=2^{pk}\prod_{i=1}^{p}\left(\frac{\Gamma\left(\frac{n-i}{2}+k\right)}{\Gamma\left(\frac{n-i}{2}\right)}\right),\text{ and}$$

$$E\left(|\mathbf{S}|^{k}\right)=\frac{2^{pk}\left|\mathbf{\Sigma}\right|^{k}}{(n-1)^{pk}}\prod_{i=1}^{p}\frac{\Gamma\left(\frac{n-i}{2}+k\right)}{\Gamma\left(\frac{n-i}{2}\right)}.$$

When the moment generating function of the distribution of a random variable exists, then one can show that it uniquely determine the distribution. Bain and Engelhardt (1992) state and prove that the moment generating function can be determined from its moments about zero. By comparing moments, we have verified that Theorem 2.4 holds.

2.5 Evaluating $f_{|\mathbf{S}|}(w)$ and $f_{\ln(|(n-1)\mathbf{\Sigma}^{-1}\mathbf{S}|)}(u)$ Numerically

Define

$$h_u(\mathbf{x}) = (2\pi)^{(p-1)/2} e^{\mathbf{x}^{\mathbf{T}}\mathbf{x}/2} f_{\ln\left(\chi^2_{n-1}\right)}(u - x_2 - \ldots - x_p) f_{\ln\left(\chi^2_{n-2}\right)}(x_2) \cdots f_{\ln\left(\chi^2_{n-p}\right)}(x_p),$$

where $n > p$ with

$$\mathbf{x} = [x_2, \ldots, x_p]^{\mathbf{T}}.$$

We can then write the probability density function describing the distribution of

$$\begin{aligned} f_U(u) &= (2\pi)^{-(p-1)/2} \int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-\mathbf{x}^{\mathbf{T}}\mathbf{x}/2} h_u(\mathbf{x})\, dx_2 \cdots dx_p \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{1}{(2\pi)^{(p-1)/2} |\mathbf{I}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{0})^{\mathbf{T}}\mathbf{I}^{-1}(\mathbf{x}-\mathbf{0})} h_u(\mathbf{x})\, dx_2 \cdots dx_p \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \phi(\mathbf{x})\, h_u(\mathbf{x})\, dx_2 \cdots dx_p, \end{aligned}$$

where $\mathbf{0}^{(p-1)\times 1}$ vector of zeros and $\mathbf{I}^{(p-1)\times(p-1)}$ identity matrix.

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{(p-1)/2} |\mathbf{I}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{0})^{\mathbf{T}}\mathbf{I}^{-1}(\mathbf{x}-\mathbf{0})}$$

with Genz and Monahan (1999) give a numerical method for evaluating $I(h_u)$, where

$$I(h_u) = (2\pi)^{-(p-1)/2} \int_{-\infty}^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty e^{-\mathbf{x}^{\mathbf{T}}\mathbf{x}/2} h_u(\mathbf{x})\, dx_2 \cdots dx_p.$$

FORTRAN code to evaluate $I(f)$ is available from Dr. Alan Genz, Department of Mathematics, Washington State University, Pullman, WA.

According to Dr. Genz, his method can be adapted to evaluate $f_U(u)$. First we need to substitute $\mathbf{x}$ with the vector

$$\mathbf{y} = [y_2, \ldots, y_p]^{\mathbf{T}},$$

where $y_i = |x_i|$ for $i = 2, \ldots, p$ and divide by 2^{p-1}. The method used by Genz and Monahan (1999) to evaluate

$$f_U(u) = \frac{1}{2^{p-1}} \int_{-\infty}^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \phi(\mathbf{y})\, h_u(\mathbf{y})\, dy_2 \cdots dy_p$$

make a change of variables to a spherical-radial coordinate system to describe the stochastic spherical-radial rules. Defining $\mathbf{y} = r\mathbf{z}$, with $\mathbf{z}^{\mathbf{T}}\mathbf{z} = 1$, such that $\mathbf{y}^{\mathbf{T}}\mathbf{y} = r^2$, $r \geq 0$ results in

$$f_U\left(u\right) = \pi^{-(p-1)/2}2^{-(p-1)}\int_{\mathbf{z}^{\mathbf{T}}\mathbf{z}=1}\int_0^{\infty} e^{-r^2/2}r^{p-1}h_u\left(r\mathbf{z}\right)drd\mathbf{z}.$$

2.6 Conclusion

In this chapter, we presented and derived some useful distributional results for describing the distribution of the generalized variance $|\mathbf{S}|$ and the distribution of $\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}\right|\right)$.

CHAPTER 3

GENERALIZED VARIANCE SHEWHART CHART

3.1 Introduction

The Phase I Shewhart quality control chart is a plot of a statistic

$$\widehat{\theta}_i = \widehat{\theta}_i \left(\mathbf{X}_{i,1}, \ldots, \mathbf{X}_{i,n} \right)$$

versus the sample number i for each of the m preliminary samples. The plot usually includes a center line (CL) along with a lower (LCL) and an upper (UCL) control limits. The chart is said to signal a potential out-of-control process if either $\widehat{\theta}_i \leq LCL$ or $\widehat{\theta}_i \geq UCL$. These charts are used in a Phase I study of the process in an attempt to remove what Shewhart (1931) referred to as "assignable" causes of variability that when detected and removed from the process results in a better quality process. The data collected in this phase in which there is evidence it is from an in-control process is used to estimate the in-control process parameters that are used to design a Phase II chart.

Once the process is believed to be in a state of statistical in-control, it is desirable to monitor for a change in the process. The monitoring phase is often referred to as Phase II. In this phase, the practitioner plots a statistic

$$\widehat{\theta}_t = \widehat{\theta}_t \left(\mathbf{X}_{t,1}, \ldots, \mathbf{X}_{t,n} \right)$$

versus the sample number t for $t = 1, 2, 3, \ldots$. This plot typically includes a CL (center line), a LCL (lower control limit), and an UCL (upper control limit). The charts signals a potential out-of-control process at time (sampling stage) t if either $\widehat{\theta}_t \leq LCL$ or $\widehat{\theta}_t \geq UCL$. It is our interest in this chapter to examine Phase II Shewhart chart based on the statistic

$$\widehat{\theta}_t = \left| \mathbf{S}_t \right|.$$

The Phase II generalized variance Shewhart chart is a plot of the points $(t, |\mathbf{S}_t|)$ for $t = 1, 2, 3, \ldots$. The chart signals a potential change in the process from a in-control state to an out-of-control state at sampling stage t if either $|\mathbf{S}_t| \leq LCL$ or $|\mathbf{S}_t| \geq UCL$.

3.2 Literature Review

Montgomery and Wadsworth (1972) developed a Shewhart $|\mathbf{S}|$ chart using an asymtotic normal approximation to the distribution of the generalized variance. They assumed that the $p \times 1$ vector $\mathbf{X}$ of quality measurements has a $N_p(\mu, \mathbf{\Sigma})$ distribution. Further, they assumed that periodically the practitioner would have available independent random samples $\mathbf{X}_{t,1}, \ldots, \mathbf{X}_{t,n}$ in Phase II from the distribution of $\mathbf{X}$. Recall that the two previous assumptions are referred to as the independent multivariate normal model. Under this model, Anderson (2003) shows that

$$\sqrt{n-1}\left(|\mathbf{S}| / |\mathbf{\Sigma}| - 1\right)$$

is asymptotically normally distributed with mean 0 and variance $2p$. It follows that

$$\frac{\sqrt{n-1}\left(|\mathbf{S}| / |\mathbf{\Sigma}| - 1\right)}{\sqrt{2p}}$$

has an asymptotically standard normal distribution.

To determine the asymtotic control limits, we observed that asymtotically

$$\alpha = 1 - P\left(-z_{\alpha/2} < \frac{\sqrt{n-1}\left(|\mathbf{S}| / |\mathbf{\Sigma}| - 1\right)}{\sqrt{2p}} < z_{\alpha/2}\right).$$

Next observe that we can write

$$\alpha = 1 - P\left[-\left(\frac{z_{\alpha/2}\sqrt{2p}}{\sqrt{n-1}} - 1\right)|\mathbf{\Sigma}| < |\mathbf{S}| < \left(\frac{z_{\alpha/2}\sqrt{2p}}{\sqrt{n-1}} + 1\right)|\mathbf{\Sigma}|\right].$$

Hence, the asymptotic center line (CL), lower (LCL) and upper (UCL) probability control limits when $\mathbf{\Sigma}_0$ is known are given by

$$LCL = -\left(z_{\alpha/2}\frac{\sqrt{2p}}{\sqrt{n-1}} - 1\right)|\mathbf{\Sigma}_0|\,;\; CL = \left(\prod_{i=1}^{p}\frac{n-i}{n-1}\right)|\mathbf{\Sigma}_0|\,;\text{ and}$$
$$UCL = \left(z_{\alpha/2}\frac{\sqrt{2p}}{\sqrt{n-1}} + 1\right)|\mathbf{\Sigma}_0|\,,$$

where $0 < \alpha < 0.5$. Replacing $z_{\alpha/2}$ with 3 gives the asymptotic 3-sigma limits suggested by Shewhart (1931). If $\mathbf{\Sigma}_0$ is unknown, then it is recommended that $|\mathbf{\Sigma}_0|$ be replaced by $\left|\overline{\mathbf{S}}_0\right|$.

Alt and Smith (1988) introduced a generalized variance $|\mathbf{S}|$ Shewhart chart. They assume the independent multivariate normal model. At each time t, the point $(t, |\mathbf{S}_t|)$ is plotted for $t = 1, 2, 3, \ldots$. On this plot a center line (CL), lower control (LCL) and upper control (UCL) limits are drawn with

$$LCL = \max\left\{0, E\left(|\mathbf{S}|\,|\mathbf{\Sigma} = \mathbf{\Sigma}_0\right) - 3\sqrt{V\left(|\mathbf{S}|\,|\mathbf{\Sigma} = \mathbf{\Sigma}_0\right)}\right\};$$
$$CL = E\left(|\mathbf{S}|\,|\mathbf{\Sigma} = \mathbf{\Sigma}_0\right);\text{ and}$$
$$UCL = E\left(|\mathbf{S}|\,|\mathbf{\Sigma} = \mathbf{\Sigma}_0\right) + 3\sqrt{V\left(|\mathbf{S}|\,|\mathbf{\Sigma} = \mathbf{\Sigma}_0\right)}.$$

It is shown in Anderson (2003) that under the given model

$$E\left(|\mathbf{S}|\,|\mathbf{\Sigma} = \mathbf{\Sigma}_0\right) = b_1|\mathbf{\Sigma}_0| \text{ and } V\left(|\mathbf{S}|\,|\mathbf{\Sigma} = \mathbf{\Sigma}_0\right) = b_2|\mathbf{\Sigma}_0|^2,$$

where

$$b_1 = \frac{\prod_{i=1}^{p}(n-i)}{(n-1)^p} \text{ and } b_2 = \frac{\prod_{i=1}^{p}(n-i)}{(n-1)^p}\left[\prod_{i=1}^{p}\frac{2+n-i}{n-1} - \prod_{i=1}^{p}\frac{n-i}{n-1}\right].$$

It follows that the center line and control limits can be expressed as

$$LCL = \max\left\{0, |\mathbf{\Sigma}_0|\left(b_1 - 3\sqrt{b_2}\right)\right\};\; CL = b_1|\mathbf{\Sigma}_0|\,;\text{ and}$$
$$UCL = |\mathbf{\Sigma}_0|\left(b_1 + 3\sqrt{b_2}\right).$$

Note that under the independent multivariate normal model and assuming $\mathbf{\Sigma}$ is positive definite, Dykstra (1970) proved that the sample covariance matrix $\mathbf{S}$ is positive definite with probability one. It then follows that the sample generalized variance $|\mathbf{S}|$ is positive with probability one. Thus the LCL can be set to zero if the "three sigma" lower control limit is negative. The chart is said to "signal" at sampling stage t when either $|\mathbf{S}_t| \leq LCL$ or $|\mathbf{S}_t| \geq UCL$.

Aparisi et al. (1999) studied the properties of the run length distributon of the $|\mathbf{S}|$ chart assuming and the in-control covariance matrix $\mathbf{\Sigma}_0$ is known. They considered more general control limits of the form

$$LCL = \max\left\{0, |\mathbf{\Sigma}_0| \left(b_1 - k_L\sqrt{b_2}\right)\right\}; \; CL = b_1 \left|\mathbf{\Sigma}_0\right|; \text{ and}$$
$$UCL = |\mathbf{\Sigma}_0| \left(b_1 + k_U\sqrt{b_2}\right),$$

where $k_L, k_U > 0$ and k_L is chosen so that the LCL is nonnegative.

The control limits of the Shewhart $|\mathbf{S}|$ chart can be expressed in terms percentage points of the distribution of the generalized variance $|\mathbf{S}|$. Note that the distribution of

$$W = \left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right| \sim \prod\nolimits_{i=1}^{p} \chi^2_{(n-1)-(i-1)}$$

only depends on n and p. Letting $w_{n,p,1-\gamma}$ be the $100\,(1-\gamma)$th percentile of the distribution of W, then the $100\,(1-\gamma)$th percentile of the distribution of $|\mathbf{S}|$ can be expressed as $|\mathbf{\Sigma}|\, w_{n,p,1-\gamma}/\,(n-1)^p$. Hence, the lower and upper control limits for the Shewhart $|\mathbf{S}|$ chart is given by

$$LCL = |\mathbf{\Sigma}_0|\, w_{n,p,1-\tau}/\,(n-1)^p \text{ and } UCL = |\mathbf{\Sigma}_0|\, w_{n,p,\alpha-\tau}/\,(n-1)^p\,,$$

where $0 < \tau < \alpha < 0.5$. Typically, the value of τ is selected to be $\alpha/2$.

3.3 The Run Length Distribution

The run length T of the chart is the number of the sample in Phase II in which the chart first signals. For the chart proposed by Montgomery and Wadsworth (1972), the distribution of the run length based on the asymptotic control limit is

$$LCL = -\left(z_{\alpha/2}\frac{\sqrt{2p}}{\sqrt{n-1}} - 1\right)|\mathbf{\Sigma}_0|\,;$$
$$CL = \left(\prod\nolimits_{i=1}^{p}\frac{n-i}{n-1}\right)|\mathbf{\Sigma}_0|\,;\text{ and}$$
$$UCL = \left(z_{\alpha/2}\frac{\sqrt{2p}}{\sqrt{n-1}} + 1\right)|\mathbf{\Sigma}_0|\,,$$

$$\begin{aligned}\alpha &= \alpha\left(\lambda^2, n, p\right) = P\left(T=1\right)\\ &= 1 - P\left[LCL < |\mathbf{S}_1| < UCL\right]\\ &= 1 - P\left[B < \frac{\sqrt{n-1}\left(|\mathbf{S}_1|\,/\,|\mathbf{\Sigma}| - 1\right)}{\sqrt{2p}} < A\right]\\ &= 1 - \Phi\left(A\right) + \Phi\left(B\right),\end{aligned}$$

where

$$B = \frac{\left(-z_{\alpha/2}\sqrt{2p} + \sqrt{n-1}\right)\lambda^{-2} - \sqrt{n-1}}{\sqrt{2p}}\text{ and}$$
$$A = \frac{\left(z_{\alpha/2}\sqrt{2p} + \sqrt{n-1}\right)\lambda^{-2} - \sqrt{n-1}}{\sqrt{2p}}.$$

For $t > 1$, the event $\{T = t\}$ can be expressed as

$$\{T=t\} = \left(\bigcap\nolimits_{i=1}^{t-1}\left\{B < \frac{\sqrt{n-1}\left(|\mathbf{S}_i|\,/\,|\mathbf{\Sigma}| - 1\right)}{\sqrt{2p}} < A\right\}\right) \cap \left\{B < \frac{\sqrt{n-1}\left(|\mathbf{S}_t|\,/\,|\mathbf{\Sigma}| - 1\right)}{\sqrt{2p}} < A\right\}^{\mathbf{c}}.$$

We see that $\{T = t\}$ has been expressed as the intersection of t independent events. Hence,

$$P\left(T=t\right) = \prod\nolimits_{i=1}^{t-1} P\left(B < \frac{\sqrt{n-1}\left(|\mathbf{S}_i|\,/\,|\mathbf{\Sigma}| - 1\right)}{\sqrt{2p}} < A\right) \times P\left(\left\{B < \frac{\sqrt{n-1}\left(|\mathbf{S}_t|\,/\,|\mathbf{\Sigma}| - 1\right)}{\sqrt{2p}} < A\right\}^{\mathbf{c}}\right).$$

It then follows that

$$P\left(T=t\right) = \prod\nolimits_{i=1}^{t-1}\left(1-\alpha\right)\times\alpha = \alpha\left(1-\alpha\right)^{t-1}.$$

Thus, the asymtotic run length distribution is a geometric distribution with parameter α. The expection $E(T)$ is commonly referred to as the average run length (ARL). It follows for this chart that

$$ARL = ARL\left(\lambda^2, n, p\right) = \frac{1}{\alpha\left(\lambda^2, n, p\right)}.$$

The process is in-control when $\lambda^2 = 1$. The asymtotic in-control average run length is

$$ARL = ARL\left(1, n, p\right) = \frac{1}{\alpha\left(1, n, p\right)}.$$

In the more general setting when the control limits for the Shewhart $|\mathbf{S}|$ are expressed in as percentage points of the distribution of $|\mathbf{S}|$ when $\mathbf{\Sigma}_0$, we see that

$$\begin{aligned}\alpha &= 1 - P\left(\frac{w_{n,p,1-\tau}}{(n-1)^p}\left|\mathbf{\Sigma}_0\right| < |\mathbf{S}| < \frac{w_{n,p,\alpha-\tau}}{(n-1)^p}\left|\mathbf{\Sigma}_0\right|\right) \\ &= 1 - P\left(\frac{w_{n,p,1-\tau}}{\lambda^2} < \left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right| < \frac{w_{n,p,\alpha-\tau}}{\lambda^2}\right) \\ &= 1 - F_W\left(\frac{w_{n,p,\alpha-\tau}}{\lambda^2}\right) + F_W\left(\frac{w_{n,p,1-\tau}}{\lambda^2}\right).\end{aligned}$$

The ARL of the chart can then be expressed as

$$ARL = \frac{1}{1 - F_W\left(\frac{w_{n,p,\alpha-\tau}}{\lambda^2}\right) + F_W\left(\frac{w_{n,p,1-\tau}}{\lambda^2}\right)}.$$

As we have seen, the control limits of the Shewhart $|\mathbf{S}|$ are or can be expressed as a function of $|\mathbf{\Sigma}_0|$. If $|\mathbf{\Sigma}_0|$ is replaced with $\left|\overline{\mathbf{S}}_0\right|$, then the distribution of the run length is no longer a geometric distribution. However, the distribution of T conditioned on $\left|\overline{\mathbf{S}}_0\right|$ has a geometric distribution. For the Shewhart $|\mathbf{S}|$ with estimated control limits

$$\begin{aligned}LCL &= \left|\overline{\mathbf{S}}_0\right| w_{n,p,1-\tau}/\left(n-1\right)^p \text{ and} \\ UCL &= \left|\overline{\mathbf{S}}_0\right| w_{n,p,\alpha-\tau}/\left(n-1\right)^p\end{aligned}$$

the parameter for this geometric distribution is

$$\begin{aligned}
&\alpha\left(\lambda^2, m, n, p\,|W_0 = w_0\right) \\
&= 1 - P\left(\frac{\left|\overline{\mathbf{S}}_0\right| w_{n,p,1-\tau}}{(n-1)^p} < |\mathbf{S}| < \frac{\left|\overline{\mathbf{S}}_0\right| w_{n,p,\alpha-\tau}}{(n-1)^p}\,|W_0 = w_0\right) \\
&= 1 - P\left(\frac{w_0 w_{n,p,1-\tau}}{m^p (n-1)^p \lambda^2} < W < \frac{w_0 w_{n,p,\alpha-\tau}}{m^p (n-1)^p \lambda^2}\right) \\
&= 1 - F_W\left(\frac{w_0 w_{n,p,\alpha-\tau}}{m^p (n-1)^p \lambda^2}\right) + F_W\left(\frac{w_0 w_{n,p,1-\tau}}{m^p (n-1)^p \lambda^2}\right),
\end{aligned}$$

where

$$W = \left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right| \text{ and } W_0 = \left|m\,(n-1)\,\mathbf{\Sigma}_0^{-1}\overline{\mathbf{S}}_0\right|.$$

The ARL of the chart given $W_0 = w_0$ is

$$\begin{aligned}
ARL\left(\lambda^2, m, n, p\,|W_0 = w_0\right) &= \frac{1}{1 - F_W\left(\frac{w_0 w_{n,p,\alpha-\tau}}{m^p (n-1)^p \lambda^2}\right) + F_W\left(\frac{w_0 w_{n,p,1-\tau}}{m^p (n-1)^p \lambda^2}\right)} \\
&= \frac{1}{1 - F_W\left(\frac{w_0}{m^p (n-1)^p}\frac{w_{n,p,\alpha-\tau}}{\lambda^2}\right) + F_W\left(\frac{w_0}{m^p (n-1)^p}\frac{w_{n,p,1-\tau}}{\lambda^2}\right)}.
\end{aligned}$$

The unconditional ARL of the chart is then given by

$$ARL\left(\lambda^2, m, n, p\right) = \int_0^\infty \frac{1}{1 - F_W\left(\frac{w_0 w_{n,p,\alpha-\tau}}{m^p (n-1)^p \lambda^2}\right) + F_W\left(\frac{w_0 w_{n,p,1-\tau}}{m^p (n-1)^p \lambda^2}\right)} f_{W_0}\left(w_0\right) dw_0.$$

We have that the mean of W_0 is

$$E\left(W_0\right) = \prod_{i=1}^{p} \left(m\,(n-1) - (i-1)\right).$$

which coverges in probability to $m^p\,(n-1)^p$ as m goes to infinity. Thus, as m increases, the unconditional ARL approaches the ARL when $\mathbf{\Sigma}_0$ is known.

A Shewhart chart based on the $\ln\left(|\mathbf{S}|\right)$ chart is equivalent to a chart based on $|\mathbf{S}|$. Observe that at sampling stage t, the event defined by the inequality

$$\frac{\left|\overline{\mathbf{S}}_0\right| w_{n,p,1-\tau}}{(n-1)^p} < |\mathbf{S}| < \frac{\left|\overline{\mathbf{S}}_0\right| w_{n,p,\alpha-\tau}}{(n-1)^p}.$$

This inequality is equivalent to the inequality

$$\ln\left(\frac{\left|\overline{\mathbf{S}}_0\right| w_{n,p,1-\tau}}{(n-1)^p}\right) < \ln\left(|\mathbf{S}|\right) < \ln\left(\frac{\left|\overline{\mathbf{S}}_0\right| w_{n,p,\alpha-\tau}}{(n-1)^p}\right).$$

Hence, a Shewhart $\ln\left(|\mathbf{S}|\right)$ chart with control limits

$$LCL = \ln\left(\frac{\left|\overline{\mathbf{S}}_0\right| w_{n,p,1-\tau}}{(n-1)^p}\right) \text{ and } UCL = \ln\left(\frac{\left|\overline{\mathbf{S}}_0\right| w_{n,p,\alpha-\tau}}{(n-1)^p}\right)$$

is equivalent to the Shewhart $|\mathbf{S}|$ chart with control limits

$$LCL = \frac{\left|\overline{\mathbf{S}}_0\right| w_{n,p,1-\tau}}{(n-1)^p} \text{ and } UCL = \frac{\left|\overline{\mathbf{S}}_0\right| w_{n,p,\alpha-\tau}}{(n-1)^p}.$$

Other measures of the performance of the chart are the standard deviation ($SDRL$) of the run length distribution and percentiles of the run length distribution. The $SDRL$ for the chart with known parameters is

$$SDRL\left(\lambda^2, n, p\right) = \frac{\sqrt{1-\alpha\left(\lambda^2, n, p\right)}}{\alpha\left(\lambda^2, n, p\right)}.$$

where

$$\alpha\left(\lambda^2, n, p\right) = 1 - F_W\left(\frac{w_{n,p,\alpha-\tau}}{\lambda^2}\right) + F_W\left(\frac{w_{n,p,1-\tau}}{\lambda^2}\right).$$

In the estimated parameters case, the conditional $SDRL$ given $W_0 = w_0$ is

$$SDRL\left(\lambda^2, m, n, p \,|W_0 = w_0\right) = \frac{\sqrt{1-\alpha\left(\lambda^2, m, n, p\,|W_0 = w_0\right)}}{\alpha\left(\lambda^2, m, n, p\,|W_0 = w_0\right)},$$

where

$$\begin{aligned}\alpha\left(\lambda^2, m, n, p\,|W_0 = w_0\right) = 1 &- F_W\left(\frac{W_0 w_{n,p,\alpha-\tau}}{m^p\left(n-1\right)^p\lambda^2}\,|W_0 = w_0\right)\\ &+ F_W\left(\frac{W_0 w_{n,p,1-\tau}}{m^p\left(n-1\right)^p\lambda^2}\,|W_0 = w_0\right).\end{aligned}$$

The uncondition $SDRL\left(\lambda^2, m, n, p\right)$ is

$$SDRL\left(\lambda^2, m, n, p\right) = \int_0^\infty \frac{\sqrt{1-\alpha\left(\lambda^2, m, n, p\,|W_0 = w_0\right)}}{\alpha\left(\lambda^2, m, n, p\,|W_0 = w_0\right)} f_{W_0}\left(w_0\right) dw_0,$$

In general, the 100γth is the value $T_{1-\gamma}$ of T that satisfies the following inequalities

$$P\left(T \leq T_{1-\gamma}\right) \geq \gamma \text{ and } P\left(T < T_{1-\gamma}\right) < \gamma.$$

For a random variable T having geometric distribution with parameter α, it can be shown that

$$P\left(T > t\right) = \left(1-\alpha\right)^{t}.$$

Hence,

$$P\left(T \leq t\right) = 1-\left(1-\alpha\right)^{t} \text{ and } P\left(T < t\right) = 1-\left(1-\alpha\right)^{t-1} \text{ (for } t > 1).$$

The 100γth percentile $T_{1-\gamma}$ of a geometric distribution is the solution to the set of inequalities

$$1-\left(1-\alpha\right)^{T_{1-\gamma}} \geq \gamma \text{ and } 1-\left(1-\alpha\right)^{T_{1-\gamma}-1} < \gamma.$$

Observe that

$$\begin{aligned}
&1-\gamma \geq \left(1-\alpha\right)^{T_{1-\gamma}} \text{ and } 1-\gamma < \left(1-\alpha\right)^{T_{1-\gamma}-1} \text{ implies}\\
&\ln\left(1-\gamma\right) \geq T_{1-\gamma}\ln\left(1-\alpha\right) \text{ and } \ln\left(1-\gamma\right) < \left(T_{1-\gamma}-1\right)\ln\left(1-\alpha\right) \text{ implies}\\
&\frac{\ln\left(1-\gamma\right)}{\ln\left(1-\alpha\right)} \leq T_{1-\gamma} \text{ and } 1+\frac{\ln\left(1-\gamma\right)}{\ln\left(1-\alpha\right)} > T_{1-\gamma} \text{ implies}\\
&\frac{\ln\left(1-\gamma\right)}{\ln\left(1-\alpha\right)} \leq T_{1-\gamma} < 1+\frac{\ln\left(1-\gamma\right)}{\ln\left(1-\alpha\right)}.
\end{aligned}$$

Thus, it follows that

$$T_{1-\gamma} = \left\lceil \frac{\ln\left(1-\gamma\right)}{\ln\left(1-\alpha\right)} \right\rceil,$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x. In the known parameters case, we have

$$T_{1-\gamma} = T_{n,p,\lambda^2,\alpha,\tau,1-\gamma} = \left\lceil \frac{\ln\left(1-\gamma\right)}{\ln\left(F_W\left(\frac{w_{n,p,\alpha-\tau}}{\lambda^2}\right) - F_W\left(\frac{w_{n,p,1-\tau}}{\lambda^2}\right)\right)} \right\rceil.$$

In the estimated parameters case, we see that the 100γth percentile can be expressed as

$$\begin{aligned} T_{1-\gamma} &= T_{m,n,p,\lambda^2,\alpha,\tau,1-\gamma} \\ &= \left[\int_0^\infty \frac{\ln(1-\gamma) f_{W_0}(w_0)\, dw_0}{\ln\left(F_W\left(\frac{W_0 w_{n,p,\alpha-\tau}}{m^p(n-1)^p\lambda^2} \middle| W_0 = w_0 \right) - F_W\left(\frac{W_0 w_{n,p,1-\tau}}{m^p(n-1)^p\lambda^2} \middle| W_0 = w_0 \right)\right)} \right]. \end{aligned}$$

3.4 The Bivariate Case

From the results given by Anderson (2003) for $p = 2$ under the independent multivariate normal model, we have

$$\begin{aligned} W &= \left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right| \sim \left(\chi^2_{2n-4}\right)^2/4 \text{ and} \\ W_0 &= \left|m\,(n-1)\,\mathbf{\Sigma}_0^{-1}\overline{\mathbf{S}}_0\right| \sim \left(\chi^2_{2m(n-1)-2}\right)^2/4 \end{aligned}$$

When $\mathbf{\Sigma}_0$ is known, then the ARL is determined by

$$ARL = \frac{1}{1 - F_{\left(\chi^2_{2n-4}\right)^2/4}\left(\frac{\left(\chi^2_{2n-4,\alpha-\tau}\right)^2/4}{\lambda^2} \right) + F_{\left(\chi^2_{2n-4}\right)^2/4}\left(\frac{\left(\chi^2_{2n-4,1-\tau}\right)^2/4}{\lambda^2} \right)},$$

where

$$w_{n,2,1-\gamma} = \left(\chi^2_{2n-4,1-\gamma}\right)^2/4.$$

Note that

$$\begin{aligned} F_{\left(\chi^2_{2n-4}\right)^2/4}(y) &= P\left(\left(\chi^2_{2n-4}\right)^2/4 \le y \right) = P\left(\chi^2_{2n-4} \le \sqrt{4y} \right) \\ &= F_{\chi^2_{2n-4}}\left(2\sqrt{y}\right). \end{aligned}$$

Thus,

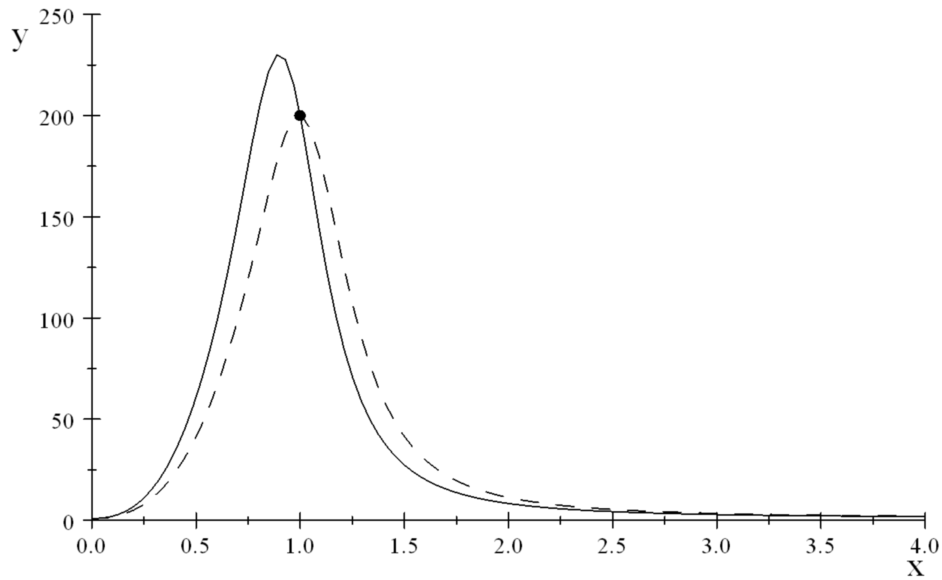
$$ARL = \frac{1}{1 - F_{\chi^2_{2n-4}}\left(2\sqrt{\frac{\left(\chi^2_{2n-4,\alpha-\tau}\right)^2/4}{\lambda^2}} \right) + F_{\chi^2_{2n-4}}\left(2\sqrt{\frac{\left(\chi^2_{2n-4,1-\tau}\right)^2/4}{\lambda^2}} \right)}.$$

The ARL can be numerically evaluated using the Scientific WorkPlace functions ChiSquareInv and ChiSquareDist with

$$\chi^2_{\nu,1-\gamma} = \text{ChiSquareInv}\,(\gamma;\nu) \text{ and } F_{\chi^2_\nu}\,(y) = \text{ChiSquareDist}\,(y;\nu)\,.$$

A graph of the ARL versus λ is given in the following figure for $n = 5$, $\alpha = 0.005$, and $\tau = 0.0025$ (solid line) along with the case in which $\tau = 0.0038$.

Figure 3.1: Plot of ARL versus λ



$\alpha = 0.005$, $\tau = 0.0025$ (solid line) and $\tau = 0.0038$ (dashed line)

We can see for the case in which τ is chosen to be $\alpha/2$ there are values for λ in which the out-of-control ARL is greater than the in-control ARL. A chart that has this property is referred to an ARL biased chart. ARL biased charts were first studied by Krumbholz (1992). An ARL unbiased chart (in-control ARL greater than any out-of-control ARL) can be designed by selecting τ such that $ARL\,(1) > ARL\,(\lambda)$ for $\lambda \neq 1$ $(\lambda > 0)$. As one can see from the Figure 3.1, selecting $\tau = 0.0038$ results in a chart that is “close to” being an unbiased chart. The method presented in Champ (2001) could be used to determine the exact value τ that would result in an unbised Shewhart $|\mathbf{S}|$ chart.

The following table gives the average run length, standard deviations of the run length, and various percentiles for a Shewhart $|\mathbf{S}|$ chart with a known value for $\mathbf{\Sigma}_0$.

Table 3.1: ARL, $SDRL$, $T_{5,2,\lambda,0.005,0.0038,1-\gamma}$

λ	ARL	SDRL	0.01	0.05	0.25	0.50	0.75	0.95	0.99
0.50	41.15	40.65	1	3	12	29	57	122	188
0.60	66.01	65.51	1	4	19	46	91	197	302
0.70	99.20	98.70	1	6	29	69	137	296	455
0.80	140.24	139.74	2	8	41	97	194	419	644
0.90	181.43	180.93	2	10	53	126	251	543	834
1.00	200.00	199.50	3	11	58	139	277	598	919
1.10	176.63	176.13	2	10	51	123	245	528	812
1.20	129.95	129.45	2	7	38	90	180	388	597
1.30	88.26	87.76	1	5	26	61	122	263	405
1.40	59.67	59.17	1	4	18	42	83	178	273
1.50	41.49	40.99	1	3	12	29	57	123	189

When $\mathbf{\Sigma}_0$ is unknown and is to be estimated with $\overline{\mathbf{S}}_0$, then the run length performance of the Shewhart $|\mathbf{S}|$ chart is measured by the unconditional ARL which is given by

$$ARL = \int_0^\infty \frac{f_{\left(\chi^2_{2m(n-1)-2}\right)^2/4}(w_0)\,dw_0}{1 - F_{\left(\chi^2_{2n-4}\right)^2/4}\left(\frac{w_0 w_{n,2,\alpha-\tau}}{m^2(n-1)^2\lambda^2}\right) + F_{\left(\chi^2_{2n-4}\right)^2/4}\left(\frac{w_0 w_{n,2,1-\tau}}{m^2(n-1)^2\lambda^2}\right)}.$$

Note that

$$F_{\left(\chi^2_{2m(n-1)-2}\right)^2/4}(y) = F_{\chi^2_{2m(n-1)-2}}(2\sqrt{y})\,;\text{ and}$$

$$f_{\left(\chi^2_{2m(n-1)-2}\right)^2/4}(y) = 2y^{-1/2} f_{\chi^2_{2m(n-1)-2}}(2\sqrt{y})\,.$$

Thus, we can write

$$ARL\left(\lambda^2, m, n\right) = \int_0^\infty \frac{w_0^{-1/2} f_{\chi^2_{2m(n-1)-2}}\left(2\sqrt{w_0}\right) dw_0}{1 - F_{\chi^2_{2n-4}}\left(2\sqrt{\frac{w_0\left(\chi^2_{2n-4,\alpha-\tau}\right)^2/4}{m^2(n-1)^2\lambda^2}}\right) + F_{\chi^2_{2n-4}}\left(2\sqrt{\frac{w_0\left(\chi^2_{2n-4,1-\tau}\right)^2/4}{m^2(n-1)^2\lambda^2}}\right)}.$$

The results in Table 3.2 illustrate that as m increases the unconditional ARL of the Shewhart $|\mathbf{S}|$ chart with estimated parameters increases to the ARL of the chart with parameters known.

Table 3.2: ARL,$n = 5, \alpha = 0.005, \tau = \alpha/2, \lambda = 1$

m	ARL	m	ARL	m	ARL
5	137.08	25	179.27	45	187.34
10	159.11	30	182.14	50	188.45
15	169.28	35	184.29	100	193.84
20	175.28	40	185.98	400	198.37

Obtaining a parameters estimated chart having the same in-control ARL as the parameters known chart for a given value of m can be done by decreasing the value α. For example, for $m = 10$ with $n = 5$, if one choose $\alpha = 0.00395$ with $\tau = \alpha/2$, then the in-control value of the ARL of the chart is 200.01. But as we can see from Table 3.3 the chart will not on average detect changes in $|\mathbf{\Sigma}|$.

Table 3.3: ARLs,$m = 10, n = 5, \alpha = 0.00395, \tau = \alpha/2$

λ	ARL(Know Parameters)	ARL(Estimated Parameters)
0.7	147.73	200.72
0.8	202.02	234.57
0.9	230. 48	232.56
1.0	200.00	200.01
1.1	138.88	154.48
1.2	88.89	111.30
1.3	57.45	77.49

The following table gives the average run lengths, standard deviations of the run length, and various percentiles for a Shewhart $|\mathbf{S}|$ chart when $\mathbf{\Sigma}_0$ is estimated with $\overline{\mathbf{S}}_0$.

Table 3.4: *ARL*, *SDRL*, for $m = 10$, $n = 5$, $\alpha = 0.005$, and $\tau = 0.0038$

λ	ARL	SDRL
0.50	49.79	49.28
0.60	79.39	78.89
0.70	114.29	113.78
0.80	145.81	145.31
0.90	162.85	162.35
1.00	159.92	159.41
1.10	140.62	140.12
1.20	113.60	113.10
1.30	86.65	86.15
1.40	64.05	63.54
1.50	46.87	46.37

Both in-control and out-of-control *ARL*s are less than their counterparts when $\mathbf{\Sigma}_0$ is assumed known.

3.5 AN EXAMPLE

Montgomery (2001) gives an example in which tensile (X_1) and diameter (X_2) of a textile fiber are the quality measurements of interest. At each sampling stage, random samples of size $n = 10$ are taken from the output of the process. The practitioner has available $m = 20$ samples each of size $n = 10$ for the process when the process was believed to be in-control. The estimates for the in-control mean vector and covariance matrix using these data are

$$\bar{\bar{\mathbf{x}}} = \begin{bmatrix} \bar{\bar{x}}_1 \\ \bar{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 115.59 \\ 1.06 \end{bmatrix} \text{ and } \overline{\mathbf{S}}_0 = \begin{bmatrix} 1.23 & 0.79 \\ 0.79 & 0.83 \end{bmatrix}.$$

It follows that

$$\left|\overline{\mathbf{S}}_0\right| = \left|\begin{bmatrix} 1.23 & 0.79 \\ 0.79 & 0.83 \end{bmatrix}\right| = 0.3968.$$

For $\alpha = 0.004305$ with $\tau = \alpha/2$, the unconditional average run length of the chart is 199.9$\underline{9}$. The control limits for the Phase II Shewhart $|\mathbf{S}|$ chart are

$$\begin{aligned} LCL &= \left|\overline{\mathbf{S}}_0\right| w_{n,p,1-\tau}/\left(n-1\right)^p = \frac{(0.3968)\left(\chi^2_{2(10)-4,1-0.004305/2}\right)^2/4}{(10-1)^2} \\ &= \frac{(0.3968)\left(\text{ChiSquareInv}\left(0.004305/2; 2\left(10\right)-4\right)\right)^2}{4\left(10-1\right)^2} \\ &= 0.02\underline{4} \text{ and} \\ UCL &= \left|\overline{\mathbf{S}}_0\right| w_{n,p,\alpha-\tau}/\left(n-1\right)^p = \frac{(0.3968)\left(\chi^2_{2(10)-4,1-0.004305/2}\right)^2/4}{(10-1)^2} \\ &= 1.66\underline{9}. \end{aligned}$$

The center line (CL) for this chart is

$$CL = \frac{\left|\overline{\mathbf{S}}_0\right|}{\left(n-1\right)^2}\left(n-1\right)\left(n-2\right) = \frac{(0.3968)}{(10-1)^2}\left(10-1\right)\left(10-2\right) = 0.35\underline{3}.$$

The following table gives summary information from the process in Phase II.

Table 3.5: Phase II Data Summary

t	$\overline{x}_1$	$\overline{x}_2$	s_1^2	s_2^2	$s_{1,2}$	$\|\mathbf{S}\|$
1	115.25	1.04	1.25	0.87	0.80	0.4475
2	115.05	1.09	1.30	0.90	0.82	0.4976
3	115.90	1.07	1.16	0.73	0.80	0.2068
4	114.98	1.05	1.25	0.78	0.75	0.4125
5	116.15	1.09	1.19	0.87	0.83	0.3464
6	115.75	0.99	1.45	0.79	0.78	1.208475
7	116.01	1.05	1.26	0.55	0.72	0.39285
8	115.29	1.11	1.23	2.0025	1.23	0.950175

Unknown to the practitioner, the variance for measurement X_2 made a substained shift at time $t = 6$ by a value of 1.5^2. The following plot of the sample generalized variance versus the sample number indicates there may have been a change in the process at time $t = 8$, but the chart did not signal.

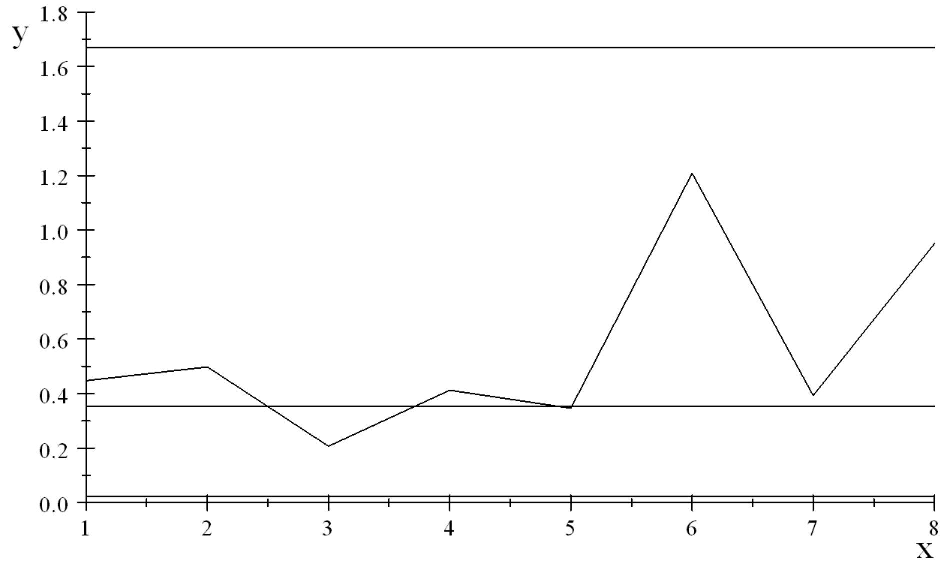


Figure 3.2: Phase II Plot of $|\mathbf{S}|$ versus t

It would be interesting to see how a CUSUM $\ln\left(\left|(n-1)\,\overline{\mathbf{S}}_0^{-1}\mathbf{S}\right|\right)$ chart would compare to Shewhart $\ln\left(\left|(n-1)\,\overline{\mathbf{S}}_0^{-1}\mathbf{S}\right|\right)$ chart which is equivalent to Shewhart $|\mathbf{S}|$ with estimated parmaeters.

3.6 Conclusion

Control charting procedures with a center line and control limits were introduced by Walter A. Shewhart (see Shewhart (1931) and have been referred to as Shewhart charts. Over time there have bee several Shewhart charts introduced into the literature by a variety of authors. The Shewhart generalized variance chart is one of those. We have show that the Shewhart $|\mathbf{S}|$ and $\ln(|\mathbf{S}|)$ charts are equivalent in the sense that when one of the charts signals the other also signals a potential out-of-control process. Also, we have show how to analyze each of these charts when parameters are estimated.

CHAPTER 4

GENERALIZED VARIANCE CUSUM CHART

4.1 Introduction

Page (1956) in his seminal work proposed the cumulative sum (CUSUM) quality control chart for monitoring for a change in the process mean. Ewan and Kemp (1960) recommended a tabular form of the CUSUM consisting of a lower and upper sided CUSUM sum procedures. In the design of a CUSUM chart in which the statistic whose values are to be "acumulated" is always positive, one must adjust their method.

For the lower-sided CUSUM chart base on the generalized variance $|\mathbf{S}|$ with head start value c^- as proposed by Lucas (1985), the CUSUM statistic has the form

$$C_0^- = c^- \text{ and } C_t^- = \min\left\{0, C_{t-1}^- + |\mathbf{S}_t| - k^-\right\},$$

for $t = 1, 2, 3, \ldots$, where $k^- > 0$. The chart is a plot of the points $\left(t, C_t^-\right)$ for $t = 1, 2, 3, \ldots$. The chart signals at the first sampling stage t in which $C_t^- \leq h^- \leq 0$. The head start value is selected to be in the interval $(h^-, 0]$. The values c^-, k^-, and h^- are referred to as chart parameters. The upper-sided CUSUM chart based on the generalized variance $|\mathbf{S}|$ with head start value c^+ is a plot of the points $\left(t, C_t^+\right)$, where

$$C_0^+ = c^+ \text{ and } C_t^+ = \max\left\{0, C_{t-1}^+ + |\mathbf{S}_t| - k^+\right\},$$

for $t = 1, 2, 3, \ldots$, where $k^+ > 0$. The chart is a plot of the points $\left(t, C_t^+\right)$ for $t = 1, 2, 3, \ldots$. The chart signals at the first sampling stage t in which $C_t^+ \geq h^+ \geq 0$. The head start value c^+ is selected to be in the interval $[0, h^+)$. The two-sided CUSUM chart based on the generalized variance $|\mathbf{S}|$ is the chart in which the points $\left(t, C_t^-\right)$ and $\left(t, C_t^+\right)$ are plotted on the same graph. Note that setting h^- and h^+ both to zero in the two-sided CUSUM $|\mathbf{S}|$ results in a Shewhart $|\mathbf{S}|$ chart with $LCL = k^-$ and $UCL = k^+$. That is the Shewhart chart is a special case of the CUSUM chart.

Healy (1987) used what he called "rescaling" to obtain an equivalent form of the chart in the sense that the charts either both signal or not at each time t. One rescaling of the lower and upper CUSUM $|\mathbf{S}|$ charts is accomplished by defining

$$C_t^{*-} = \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\right| C_t^{-} \text{ and } C_t^{*+} = \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\right| C_t^{+}.$$

The CUSUM statistics and chart parameters for this "rescaled" chart become

$$\begin{aligned}
&C_0^{*-} = c^{*-} \text{ and } C_t^{*-} = \min\left\{0, C_{t-1}^{*-} + \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}_t\right| - k^{*-}\right\} \text{ and} \\
&C_0^{*+} = c^{*+} \text{ and } C_t^{*+} = \max\left\{0, C_{t-1}^{*+} + \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}_t\right| - k^{*+}\right\}
\end{aligned}$$

with

$$\begin{aligned}
&c^{*-} = \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\right| c^{-},\ k^{*-} = \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\right| k^{-}, \\
&h^{*-} = \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\right| h^{-},\ c^{*+} = \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\right| c^{+}, \\
&k^{*+} = \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\right| k^{+}, \text{ and } h^{*+} = \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\right| h^{+}.
\end{aligned}$$

If the in-control covariance matrix $\mathbf{\Sigma}_0$ is unknown, then the rescaling is done by replacing $\mathbf{\Sigma}_0$ with $\overline{\mathbf{S}}_0$. Note that we can express

$$\begin{aligned}
\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}_t\right| &= \left|\mathbf{\Sigma}_0^{-1}\mathbf{\Sigma}\right| \left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}_t\right| = \lambda^2 \left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}_t\right| = \\
&\lambda^2 W_t,
\end{aligned}$$

when parameters are known, where $W_t = |(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}_t|$. In the parameters estimated case, we see that

$$\begin{aligned}
\left|(n-1)\,\overline{\mathbf{S}}_0^{-1}\mathbf{S}\right| &= m^p\,(n-1)^p \left|\mathbf{\Sigma}_0^{-1}\mathbf{\Sigma}\right| \left(\left|m\,(n-1)\,\mathbf{\Sigma}_0^{-1}\overline{\mathbf{S}}_0\right|\right)^{-1} \left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right| \\
&= m^p\,(n-1)^p \left(\left|m\,(n-1)\,\mathbf{\Sigma}_0^{-1}\overline{\mathbf{S}}_0\right|\right)^{-1} \left(\lambda^2 \left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right) \\
&= m^p\,(n-1)^p\,W_0^{-1}\left(\lambda^2 W_t\right) \\
&= \left(\frac{W_0}{m^p\,(n-1)^p}\right)^{-1} \left(\lambda^2 W_t\right),
\end{aligned}$$

where $W_0 = \left| m (n-1) \boldsymbol{\Sigma}_0^{-1} \overline{\mathbf{S}}_0 \right|$. We see that replacing $m^p (n-1)^p W_0^{-1}$ with 1 in the parameters estimated case gives the parameters known case.

Another CUSUM chart that could be used to monitor for a change in the process generalized variance is a CUSUM chart based on the $\ln(|\mathbf{S}|)$ or equivalently the $\ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|\right)$ if $\boldsymbol{\Sigma}_0$ is known or $\ln\left(\left|(n-1)\overline{\mathbf{S}}_0^{-1}\mathbf{S}\right|\right)$ when $\boldsymbol{\Sigma}_0$ is unknown. The lower and upper CUSUM $\ln(|\mathbf{S}|)$ charts are defined by

$$C_0^- = c^- \text{ and } C_t^- = \min\left\{0, C_{t-1}^- + \ln(|\mathbf{S}_t|) - k^-\right\} \text{ and}$$
$$C_0^+ = c^+ \text{ and } C_t^+ = \max\left\{0, C_{t-1}^+ + \ln(|\mathbf{S}_t|) - k^+\right\}.$$

Each of these charts are equivalent to a CUSUM $\ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}_t\right|\right)$ chart when $\boldsymbol{\Sigma}_0$ is known and a CUSUM $\ln\left(\left|(n-1)\overline{\mathbf{S}}_0^{-1}\mathbf{S}_t\right|\right)$ when $\boldsymbol{\Sigma}_0$ is estimated by $\overline{\mathbf{S}}_0$. This can be seen to hold, for example, for the upper one-sided CUSUM $\ln(|\mathbf{S}_t|)$ chart by observing that the statistic C_t^+ can be expressed as

$$\begin{aligned} C_t^+ &= \max\left\{0, C_{t-1}^+ + \ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}_t\right|\right) - \left(k^+ + \ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\right|\right)\right)\right\}. \\ &= \max\left\{0, C_{t-1}^+ + \ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}_t\right|\right) - k^{*+}\right\} \end{aligned}$$

Defining

$$k^{*+} = \left(k^+ + \ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\right|\right)\right),$$

the upper one-sided CUSUM $\ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}_t\right|\right)$ chart is defined by the sequence

$$C_0^+ = c^+ \text{ and } C_t^+ = \max\left\{0, C_{t-1}^+ + \ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}_t\right|\right) - k^{*+}\right\}$$

for $t = 1, 2, 3, \ldots$. This would hold for the lower one-sided chart and the charts with estimated parameters. Note that

$$\ln\left(\left|(n-1)\boldsymbol{\Sigma}_0^{-1}\mathbf{S}_t\right|\right) = \ln\left(\lambda^2 \left|(n-1)\boldsymbol{\Sigma}^{-1}\mathbf{S}_t\right|\right)$$

Other CUSUM charts for monitoring for a change in the process covariance matrix have been proposed. A chart derived by Healy (1987) after rescaling is based on the CUSUM statistics

$$C_i = \max\left\{0, C_{i-1} + \sum\nolimits_{j=1}^{n} \left(\mathbf{X}_{i,j} - \mu_0\right)^{\mathbf{T}} \boldsymbol{\Sigma}_0^{-1} \left(\mathbf{X}_{i,j,} - \mu_0\right) - k\right\},$$

where $k = pnc \ln(c) / (c-1)$ and $0 \leq C_0 < h$. This chart was designed specifically to detect a change in $\boldsymbol{\Sigma}$ from $\boldsymbol{\Sigma}_0$ to $c\boldsymbol{\Sigma}_0$ for $c > 0$ given. Pignatiello, Runger, and Korpela (1986) proposed using the CUSUM chart to monitor for a change in the mean vector to monitor for a change in the covariance matrix. Crosier (1986) proposed using the CUSUM chart based on the CUSUM statistics

$$C_i = \max\left\{0, C_{i-1} + \sqrt{\sum\nolimits_{j=1}^{n} \left(\mathbf{X}_{i,j} - \mu_0\right)^{\mathbf{T}} \boldsymbol{\Sigma}_0^{-1} \left(\mathbf{X}_{i,j,} - \mu_0\right)} - k\right\},$$

when $n = 1$ without the restriction Healy (1987) placed on the chart parameter k. It is not difficult to show that all the CUSUM charts discussed in this section have as a special case a Shewhart chart.

4.2 Run Length Distribution

The run length T of a CUSUM chart is the first sampling stage t in which the chart first signals a potential out-of-control process. The three most commonly used methods for evaluating the run length distribution are (1) simulation, (2) approximating the chart as a discrete state Markov chain, and (3) expressing a parameter of the run length distribution, such as the ARL, as the solution to an integral equation. Champ and Rigdon (1991) demonstrated that the Markov chain and the integral equation approach for the CUSUM $\overline{X}$ chart are equivalent.

Champ, Rigdon, and Scharnagl (2001) derive various integral equations whose exact solutions are parameters of the run length distribution. Recall that the lower and upper one-sided CUSUM $|\mathbf{S}|$ chart when $\mathbf{\Sigma}_0$ is known are defined by the sequences of statistics

$$C_0^- = c^- \text{ and } C_t^- = \min\left\{0, C_{t-1}^- + |\mathbf{S}_t| - k^-\right\} \text{ and}$$
$$C_0^+ = c^+ \text{ and } C_t^+ = \max\left\{0, C_{t-1}^+ + |\mathbf{S}_t| - k^+\right\}$$

with $k^-, k^+ > 0$ and control limits $h^- \leq 0$ and $h^+ \geq 0$ for $t = 1, 2, 3, \ldots$. It is convenient to use the following notation.

$$pr^-\left(t\left|c^-\right.\right) = P\left(T^- = t\left|C_0^- = c^-\right.\right) \text{ and}$$
$$pr^+\left(t\left|c^+\right.\right) = P\left(T^+ = t\left|C_0^+ = c^+\right.\right),$$

where T^- and T^+ are the run lengths of the lower and upper one-sided CUSUM charts, respectively. The distribution of the run length of both the lower and upper one-sided CUSUM $|\mathbf{S}|$ charts can be determined iteratively using integral equations determined from the results in Champ, Rigdon, and Scharnagl (2001). We have

$$pr^-\left(1\left|c^-\right.\right) = 1 - F_{|\mathbf{S}|}\left(h^- - c^- + k^-\right) \text{ and.}$$

$$pr^-\left(t\left|c^-\right.\right) = \left\{\begin{array}{ll} pr^-\left(t-1\left|0\right.\right) F_{|\mathbf{S}|}\left(h^- - c^- + k^-\right) & \\ \quad + \int_{h^-}^{0} pr^-\left(t-1\left|y\right.\right) f_{|\mathbf{S}|}\left(y - c^- + k^-\right) dy, & \text{if } -k^- \leq c^- \leq 0; \\ 1 + \int_{h^-}^{k^-+c^-} pr^-\left(t-1\left|y\right.\right) f_{|\mathbf{S}|}\left(y - c^- + k^-\right) dy, & \text{if } h^- < c^- < -k^-, \end{array}\right\}$$

and

$$pr^+\left(1\left|c^+\right.\right) = 1 - F_{|\mathbf{S}|}\left(h^+ - c^+ + k^+\right) \text{ and.}$$

$$pr^+\left(t\left|c^+\right.\right) = \left\{\begin{array}{ll} pr^+\left(t-1\left|0\right.\right) F_{|\mathbf{S}|}\left(h^+ - c^+ + k^+\right) & \\ \quad + \int_{0}^{h^+} pr^+\left(t-1\left|y\right.\right) f_{|\mathbf{S}|}\left(y - c^+ + k^+\right) dy, & \text{if } 0 \leq c^+ \leq k^+; \\ 1 + \int_{k^+-c^+}^{h^+} pr^+\left(t-1\left|y\right.\right) f_{|\mathbf{S}|}\left(y - c^+ + k^+\right) dy, & \text{if } k^+ < c^+ < h^+, \end{array}\right\}$$

for $t > 1$. The integral equations whose exact solution is the average run length

ARL^- of the lower one-sided chart is

$$ARL^-\left(c^-\right) = \left\{ \begin{array}{ll} 1 + ARL^-\left(0\right) F_{|\mathbf{S}|}\left(k^- - c^-\right) & \text{if } k^- \leq c^- \leq 0; \\ + \int_{h^-}^{0} ARL^-\left(y\right) f_{|\mathbf{S}|}\left(y - c^- + k^-\right) dy, & \\ 1 + \int_{h^-}^{k^- + c^-} ARL^-\left(y\right) f_{|\mathbf{S}|}\left(y - c^- + k^-\right) dy, & \text{if } h^- < c^- < k^-. \end{array} \right\}.$$

For the upper one-sided chart, the average run length ARL^+ is the exact solution to the integral equation

$$ARL^+\left(c^+\right) = \left\{ \begin{array}{ll} 1 + ARL^+\left(0\right) F_{|\mathbf{S}|}\left(k^+ - c^+\right) & \text{if } 0 \leq c^+ \leq k^+; \\ + \int_{0}^{h^+} ARL^+\left(y\right) f_{|\mathbf{S}|}\left(y - c^+ + k^+\right) dy, & \\ 1 + \int_{k^+ - c^+}^{h^+} ARL^+\left(y\right) f_{|\mathbf{S}|}\left(y - c^+ + k^+\right) dy, & \text{if } k^+ < c^+ < h^+. \end{array} \right\}.$$

Knoth (1998) gives a method for solving integral equation of this form.

Woodall (1983) proved that for CUSUM charts the tail probabilities could be approximated by a geometric distribution. The tail probabilities $pr^-\left(t^{*-} + t \left| c^-\right.\right)$ and $pr^+\left(t^{*+} + t \left| c^+\right.\right)$ for "large" value of t^{*-} and t^{*+} are approximated by

$$\begin{aligned} pr^-\left(t^{*-} + t \left| c^-\right.\right) &\approx \left(\widehat{\theta}^-\right)^t pr^-\left(t^{*-} \left| c^-\right.\right) \text{ and} \\ pr^+\left(t^{*+} + t \left| c^+\right.\right) &\approx \left(\widehat{\theta}^+\right)^t pr^+\left(t^{*+} \left| c^+\right.\right). \end{aligned}$$

It follows that the average run lengths can be approximated by

$$ARL^-\left(c^-\right) \approx \sum_{t=1}^{t^{*-}} t \times pr^-\left(t \left| c^-\right.\right) + \widehat{\theta}^- pr\left(t^{*-} \left| c^-\right.\right) \left(\frac{t^{*-}}{1 - \widehat{\theta}^-} + \frac{1}{\left(1 - \widehat{\theta}^-\right)^2} \right) \text{ and}$$

$$ARL^+\left(c^+\right) \approx \sum_{t=1}^{t^{*+}} t \times pr^+\left(t \left| c^+\right.\right) + \widehat{\theta}^+ pr\left(t^{*+} \left| c^+\right.\right) \left(\frac{t^{*+}}{1 - \widehat{\theta}^+} + \frac{1}{\left(1 - \widehat{\theta}^+\right)^2} \right).$$

Woodall (1983) shows that

$$E\left(T^{-2}\left|c^{-}\right.\right) \approx \sum\nolimits_{t=1}^{t^{*-}} t^{2} \times pr^{-}\left(t\left|c^{-}\right.\right) + \widehat{\theta}^{-} pr\left(t^{*-}\left|c^{-}\right.\right)\left(\frac{\left(t^{*-}\right)^{2}}{1-\widehat{\theta}^{-}} + \frac{2t^{*-}-1}{\left(1-\widehat{\theta}^{-}\right)^{2}} + \frac{2}{\left(1-\widehat{\theta}^{-}\right)^{3}}\right) \text{ and}$$

$$E\left(T^{+2}\left|c^{+}\right.\right) \approx \sum\nolimits_{t=1}^{t^{*+}} t^{2} \times pr^{+}\left(t\left|c^{+}\right.\right) + \widehat{\theta}^{+} pr\left(t^{*+}\left|c^{+}\right.\right)\left(\frac{\left(t^{*+}\right)^{2}}{1-\widehat{\theta}^{+}} + \frac{2t^{*+}-1}{\left(1-\widehat{\theta}^{+}\right)^{2}} + \frac{2}{\left(1-\widehat{\theta}^{+}\right)^{3}}\right).$$

The standard deviation of the run length distributions can then be approximated by

$$SDRL^{-}\left(c^{-}\right) \approx \sqrt{E\left(T^{-2}\left|c^{-}\right.\right) - \left[E\left(T^{-}\left|c^{-}\right.\right)\right]^{2}} \text{ and}$$

$$SDRL^{+}\left(c^{+}\right) \approx \sqrt{E\left(T^{+2}\left|c^{+}\right.\right) - \left[E\left(T^{+}\left|c^{+}\right.\right)\right]^{2}}.$$

In the case in which $\gamma \leq \sum_{t=1}^{t^{*-}} pr^{-}\left(t\left|c^{-}\right.\right)$ ($\gamma \leq \sum_{t=1}^{t^{*+}} pr^{+}\left(t\left|c^{+}\right.\right)$), then the 100γth percentage point of the distribution of T^{-} (T^{+}) can be determined exactly. For the case in which $\gamma > \sum_{t=1}^{t^{*-}} pr^{-}\left(t\left|c^{-}\right.\right)$ ($\gamma > \sum_{t=1}^{t^{*+}} pr^{+}\left(t\left|c^{+}\right.\right)$), then the percentage point $T^{-}_{n,p,\lambda^{2},\alpha,\tau,1-\gamma}$ ($T^{+}_{n,p,\lambda^{2},\alpha,\tau,1-\gamma}$) can be approximated by

$$T^{-}_{c^{-},1-\gamma} \approx t^{*-} - 1 + \frac{\ln\left(\left(\frac{\left(1-\widehat{\theta}^{-}\right)\left(\sum_{t=1}^{t^{*-}} pr^{-}\left(t\left|c^{-}\right.\right)-\gamma\right)}{pr\left(t^{*-}\left|c^{-}\right.\right)}\right) + \widehat{\theta}^{-}\right)}{\ln\left(\widehat{\theta}^{-}\right)} \text{ and}$$

$$T^{+}_{c^{+},1-\gamma} \approx t^{*+} - 1 + \frac{\ln\left(\left(\frac{\left(1-\widehat{\theta}^{+}\right)\left(\sum_{t=1}^{t^{*+}} pr^{+}\left(t\left|c^{+}\right.\right)-\gamma\right)}{pr\left(t^{*+}\left|c^{+}\right.\right)}\right) + \widehat{\theta}^{+}\right)}{\ln\left(\widehat{\theta}^{+}\right)}$$

Woodall (1983) gives a method for checking how accurate the approximations of $\widehat{\theta}^{-}$ and $\widehat{\theta}^{+}$ are for the respective values of t^{*-} and t^{*+}. For each choice of t^{*-} and (t^{*+}) approximate $ARL^{-}\left(c^{-}\right)$ ($ARL^{+}\left(c^{+}\right)$) using each of the following values for $\widehat{\theta}^{-}$

$(\widehat{\theta}^{+})$:

$$\frac{pr^{-}\left(t^{*-}\left|c^{-}\right.\right)}{pr^{-}\left(t^{*-}-1\left|c^{-}\right.\right)} \text{ and } \frac{1-\sum_{t=1}^{t^{*-}} pr^{-}\left(t\left|c^{-}\right.\right)}{1-\sum_{t=1}^{t^{*-}-1} pr^{-}\left(t\left|c^{-}\right.\right)}$$

$$\left(\frac{pr^{+}\left(t^{*+}\left|c^{+}\right.\right)}{pr^{+}\left(t^{*+}-1\left|c^{+}\right.\right)} \text{ and } \frac{1-\sum_{t=1}^{t^{*+}} pr^{+}\left(t\left|c^{+}\right.\right)}{1-\sum_{t=1}^{t^{*+}-1} pr^{+}\left(t\left|c^{+}\right.\right)}\right).$$

If these approximations are not "close" choose a larger value for t^{*-} and (t^{*+}).

The lower and upper one-sided CUSUM $\ln\left(\left|(n-1)\,\overline{\mathbf{S}}_0^{-1}\mathbf{S}\right|\right)$ charts are defined by the sequences

$$C_0^{-} = c^{-} \text{ and } C_t^{-} = \min\left\{0, C_{t-1}^{-} + \ln\left(\left|(n-1)\,\overline{\mathbf{S}}_0^{-1}\mathbf{S}\right|\right) + k^{-}\right\} \text{ and}$$
$$C_0^{+} = c^{+} \text{ and } C_t^{+} = \max\left\{0, C_{t-1}^{+} + \ln\left(\left|(n-1)\,\overline{\mathbf{S}}_0^{-1}\mathbf{S}\right|\right) - k^{+}\right\}$$

with $k^{-}, k^{+} > 0$ and control limits $h^{-} < 0$ and $h^{+} > 0$. Evaluating the run length properties of this chart requires a conditional approach by first conditioning on the random variable

$$U_0 = \ln\left(\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\overline{\mathbf{S}}_0\right|\right).$$

Recall that the distribution of U_0 was studied in Chapter 2. It is convenient to represent the probability mass function of the run length distribution of lower and upper one-sided CUSUM $\ln\left(\left|(n-1)\,\overline{\mathbf{S}}_0^{-1}\mathbf{S}\right|\right)$ charts using the notation

$$pr^{-}\left(t\left|c^{-},u\right.\right) = P\left(T=t\left|C_0^{-}=c^{-},U_0=u\right.\right) \text{ and}$$
$$pr^{+}\left(t\left|c^{+},u\right.\right) = P\left(T=t\left|C_0^{+}=c^{+},U_0=u\right.\right).$$

Since the support of the statistic $\ln\left(\left|(n-1)\,\overline{\mathbf{S}}_0^{-1}\mathbf{S}\right|\right)$ is the reals, then the probability mass functions of the charts can be determined iterately by

$$pr^{-}\left(1\left|c^{-},u_0\right.\right) = F_U\left(h^{-}-c^{-}-k^{-}+u_0-\theta\right) \text{ and}$$
$$pr^{-}\left(t\left|c^{-},u_0\right.\right) = pr^{-}\left(t-1\left|0,u_0\right.\right)\left[1-F_U\left(-c^{-}-k^{-}+u_0-\theta\right)\right]$$
$$+\int_{h^{-}}^{0} pr^{-}\left(t-1\left|c_1^{-},u_0\right.\right) f_U\left(c_1^{-}-c^{-}-k^{-}+u_0-\theta\right) dc_1^{-}.$$

for the lower one-sided chart and by

$$pr^{+}\left(1\left|c^{+},u_{0}\right.\right)=1-F_{U}\left(h^{+}-c^{+}+k^{+}+u_{0}-\theta\right)\text{ and}$$
$$pr^{+}\left(t\left|c^{+},u_{0}\right.\right)=pr^{+}\left(t-1\left|0,u_{0}\right.\right)F_{U}\left(-c^{+}+k^{+}+u_{0}-\theta\right)$$
$$+\int_{0}^{h^{+}}pr^{+}\left(t-1\left|c_{1}^{+},u_{0}\right.\right)f_{U}\left(c_{1}^{+}-c^{+}+k^{+}+u_{0}-\theta\right)dc_{1}^{+}.$$

for the upper one-sided chart (see Champ, Rigdon, and Scharnagl (2001)), where

$$U=\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}\right|\right)\text{ and }\theta=\ln\left(m^{p}\,(n-1)^{p}\,\lambda^{2}\right).$$

Recall that the distribution of U was studied in Chapter 2. As it turns out, the probability mass function describing the distribution of run lengths T^{-} and T^{+}, respectively, of the lower and upper one-sided CUSUM charts in the known parameters case are described by removing the variable u_0 from the previous sequences of integral equation and replacing θ with $\ln(\lambda^2)$.

Approximate solutions to the probability mass functions for the lower and upper one-sided CUSUM charts make use of Gaussian quadrature and the method of Woodall (1983) for approximating the tail probabilities. To illustrate, consider the upper one-sided CUSUM chart. Making the transformation

$$c_{1}^{+}=\frac{h^{+}}{2}(x+1)\text{ with }dc_{1}^{+}=\frac{h^{+}}{2}dx.$$

We can then express the probability mass function describing the distribution of T^{+} by

$$pr^{+}\left(1\left|c^{+},u_{0}\right.\right)=1-F_{U}\left(h^{+}-c^{+}+k^{+}+u_{0}-\theta\right)\text{ and}$$
$$pr^{+}\left(t\left|c^{+},u_{0}\right.\right)=pr^{+}\left(t-1\left|0,u_{0}\right.\right)F_{U}\left(-c^{+}+k^{+}+u_{0}-\theta\right)$$
$$+\int_{-1}^{1}pr^{+}\left(t-1\left|\frac{h^{+}}{2}(x+1),u_{0}\right.\right)f_{U}\left(\frac{h^{+}}{2}(x+1)-c^{+}+k^{+}+u_{0}-\theta\right)\frac{h^{+}}{2}dx.$$

Using the abscissas (nodes) and weight factors for η-point Gaussian integration using Legendre polynomials, we obtain an iterative system of matrix equations that can be used to approximate the probability mass functions at the nodes

$$c_i^+ = \frac{h^+}{2}(x_i + 1)$$

for $i = 1, \ldots, \eta$. Letting $c_0^+ = 0$, this sequence of systems of equations have the form

$$\mathbf{p}_1 = \begin{bmatrix} pr^+\left(1 \left| c_0^+, u_0\right.\right) \\ pr^+\left(1 \left| c_1^+, u_0\right.\right) \\ \vdots \\ pr^+\left(1 \left| c_\eta^+, u_0\right.\right) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} F_U\left(h^+ - c_0^+ + k^+ + u_0 - \theta\right) \\ F_U\left(h^+ - c_1^+ + k^+ + u_0 - \theta\right) \\ \vdots \\ F_U\left(h^+ - c_\eta^+ + k^+ + u_0 - \theta\right) \end{bmatrix} \text{ and}$$

$$\mathbf{p}_t = \mathbf{B}\mathbf{p}_{t-1},$$

where

$$b_{i,j} = \begin{cases} F_U\left(-c_i^+ + k^+ + u_0 - \theta\right), & \text{if } j = 0; \\ f_U\left(c_j^+ - c_i^+ + k^+ + u_0 - \theta\right)\frac{h^+}{2}w_j, & \text{if } j = 1, \ldots, \eta, \end{cases}$$

for $i = 0, 1, \ldots, \eta$. Similarly, we can obtain a sequence of system of equations that can be used to approximate the probability mass function of the lower one-sided CUSUM chart.

Defining the $(\eta + 1) \times 1$ vector $\mathbf{M}^+$ by

$$\mathbf{M}^+ = \begin{bmatrix} E\left(T^+ \left| c_0^+, u_0\right.\right) \\ E\left(T^+ \left| c_1^+, u_0\right.\right) \\ \vdots \\ E\left(T^+ \left| c_\eta^+, u_0\right.\right) \end{bmatrix} = \begin{bmatrix} ARL^+\left(c_0^+, u_0\right) \\ ARL^+\left(c_1^+, u_0\right) \\ \vdots \\ ARL^+\left(c_\eta^+, u_0\right) \end{bmatrix},$$

where $E\left(T^+ \left| c_i^+, u_0\right.\right) = ARL^+\left(c_i^+, u_0\right)$ is the average run length of the chart given $C_0^+ = c_i^+$ and $U_0 = u_0$. One can show that

$$\mathbf{M}^+ = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{1}.$$

Further, define the $(\eta+1)\times 1$ vector $\mathbf{M}_2^+$ by

$$\mathbf{M}_2^+ = \begin{bmatrix} E\left((T^+)^2 \left| c_0^+, u_0\right.\right) \\ E\left((T^+)^2 \left| c_1^+, u_0\right.\right) \\ \vdots \\ E\left((T^+)^2 \left| c_\eta^+, u_0\right.\right) \end{bmatrix}.$$

It can be shown that

$$\mathbf{M}_2^+ = (\mathbf{I}-\mathbf{Q})^{-1}(\mathbf{I}+2\mathbf{Q})\,\mathbf{1}.$$

The components of $\mathbf{M}^+$ and $\mathbf{M}_2^+$ can be used to obtain vector of run length variances

$$\begin{bmatrix} V\left(T^+ \left| c_0^+, u_0\right.\right) \\ V\left(T^+ \left| c_1^+, u_0\right.\right) \\ \vdots \\ V\left(T^+ \left| c_\eta^+, u_0\right.\right) \end{bmatrix} = \begin{bmatrix} \left(\mathbf{M}_2^+\right)_0 - \left[(\mathbf{M}^+)_0\right]^2 \\ \left(\mathbf{M}_2^+\right)_1 - \left[(\mathbf{M}^+)_1\right]^2 \\ \vdots \\ \left(\mathbf{M}_2^+\right)_\eta - \left[(\mathbf{M}^+)_\eta\right]^2 \end{bmatrix},$$

where $(\mathbf{M}^+)_i$ and $\left(\mathbf{M}_2^+\right)_i$ are the respective ith components of the vectors $\mathbf{M}^+$ and $\mathbf{M}_2^+$. Similar results hold for the lower one-sided CUSUM chart.

The run length T of the two-sided CUSUM $\ln\left(\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right)$ chart is defined by

$$T = \min\left\{T^-, T^+\right\}.$$

It is shown in Kemp (1961) that

$$\frac{1}{E\left(T\left|0,u_0\right.\right)} \approx \frac{1}{E\left(T^-\left|0,u_0\right.\right)} + \frac{1}{E\left(T^+\left|0,u_0\right.\right)}$$

provides a good approximation the average run length of the two-sided CUSUM chart.

The average run lengths can be approximated by

$$\begin{aligned} ARL^{-}\left(c^{-}\right) &\approx \sum\nolimits_{t=1}^{t^{*-}} t \times pr^{-}\left(t\left|c^{-}, u_0\right.\right) \\ &\quad + \widehat{\theta}^{-} pr\left(t^{*-}\left|c^{-}, u_0\right.\right)\left(\frac{t^{*-}}{1-\widehat{\theta}^{-}} + \frac{1}{\left(1-\widehat{\theta}^{-}\right)^2}\right) \text{ and} \\ ARL^{+}\left(c^{+}\right) &\approx \sum\nolimits_{t=1}^{t^{*+}} t \times pr^{+}\left(t\left|c^{+}, u_0\right.\right) \\ &\quad + \widehat{\theta}^{+} pr\left(t^{*+}\left|c^{+}, u_0\right.\right)\left(\frac{t^{*+}}{1-\widehat{\theta}^{+}} + \frac{1}{\left(1-\widehat{\theta}^{+}\right)^2}\right). \end{aligned}$$

Note that both t^{*-} and t^{*+} depend on the value of u_0. Again from the results in Woodall (1983), we have

$$\begin{aligned} E\left(T^{-2}\left|c^{-}, u_0\right.\right) &\approx \sum\nolimits_{t=1}^{t^{*-}} t^2 \times pr^{-}\left(t\left|c^{-}, u_0\right.\right) \\ &\quad + \widehat{\theta}^{-} pr\left(t^{*-}\left|c^{-}, u_0\right.\right)\left(\frac{\left(t^{*-}\right)^2}{1-\widehat{\theta}^{-}} + \frac{2t^{*-}-1}{\left(1-\widehat{\theta}^{-}\right)^2} + \frac{2}{\left(1-\widehat{\theta}^{-}\right)^3}\right) \text{ and} \\ E\left(T^{+2}\left|c^{+}, u_0\right.\right) &\approx \sum\nolimits_{t=1}^{t^{*+}} t^2 \times pr^{+}\left(t\left|c^{+}, u_0\right.\right) \\ &\quad + \widehat{\theta}^{+} pr\left(t^{*+}\left|c^{+}, u_0\right.\right)\left(\frac{\left(t^{*+}\right)^2}{1-\widehat{\theta}^{+}} + \frac{2t^{*+}-1}{\left(1-\widehat{\theta}^{+}\right)^2} + \frac{2}{\left(1-\widehat{\theta}^{+}\right)^3}\right). \end{aligned}$$

Hence, the standard deviation of the run length distributions can then be approximated by

$$\begin{aligned} SDRL^{-}\left(c^{-}, u_0\right) &\approx \sqrt{E\left(T^{-2}\left|c^{-}, u_0\right.\right) - \left[E\left(T^{-}\left|c^{-}, u_0\right.\right)\right]^2} \text{ and} \\ SDRL^{+}\left(c^{+}, u_0\right) &\approx \sqrt{E\left(T^{+2}\left|c^{+}, u_0\right.\right) - \left[E\left(T^{+}\left|c^{+}, u_0\right.\right)\right]^2}. \end{aligned}$$

The percentage points of the run length distribution can be approximated by

$$T^{-}_{c^{-},1-\gamma,u_0} \approx t^{*-} - 1 + \frac{\ln\left(\left(\frac{\left(1-\widehat{\theta}^{-}\right)\left(\sum_{t=1}^{t^{*-}} pr^{-}\left(t\left|c^{-},u_0\right.\right)-\gamma\right)}{pr\left(t^{*-}\left|c^{-},u_0\right.\right)}\right)+\widehat{\theta}^{-}\right)}{\ln\left(\widehat{\theta}^{-}\right)} \text{ and}$$

$$T^{+}_{c^{+},1-\gamma,u_0} \approx t^{*+} - 1 + \frac{\ln\left(\left(\frac{\left(1-\widehat{\theta}^{+}\right)\left(\sum_{t=1}^{t^{*+}} pr^{+}\left(t\left|c^{+},u_0\right.\right)-\gamma\right)}{pr\left(t^{*+}\left|c^{+},u_0\right.\right)}\right)+\widehat{\theta}^{+}\right)}{\ln\left(\widehat{\theta}^{+}\right)}$$

Note that both $\widehat{\theta}^{-}$ and $\widehat{\theta}^{+}$ are also functions of u_0.

The unconditional probability mass functions for the run length distributions in the parameters estimated case are given by

$$pr^{-}\left(t\left|c^{-}\right.\right) = \int_{-\infty}^{\infty} pr^{-}\left(t\left|c^{-},u_0\right.\right) f_{U_0}\left(u_0\right) du_0 \text{ and}$$

$$pr^{+}\left(t\left|c^{+}\right.\right) = \int_{-\infty}^{\infty} pr^{+}\left(t\left|c^{+},u_0\right.\right) f_{U_0}\left(u_0\right) du_0.$$

In general, if $\xi^{-}\left(c^{-}\left|u_0\right.\right)$ and $\xi^{+}\left(c^{+}\left|u_0\right.\right)$ are parameters of the distributions of T^{-} and T^{+}, then there unconditional values are

$$\xi^{-}\left(c^{-}\right) = \int_{-\infty}^{\infty} \xi^{-}\left(c^{-}\left|u_0\right.\right) f_{U_0}\left(u_0\right) du_0 \text{ and}$$

$$\xi^{+}\left(c^{+}\right) = \int_{-\infty}^{\infty} \xi^{+}\left(c^{+}\left|u_0\right.\right) f_{U_0}\left(u_0\right) du_0.$$

4.3 Conclusion

We have discussed the use of integral equations in analyzing the run length distribution of both the CUSUM $|\mathbf{S}|$ and $\ln(|\mathbf{S}|)$ both with $\mathbf{\Sigma}_0$ known and estimated. The analytical method for evaluating the integral equations for the CUSUM $|\mathbf{S}|$ requires using the method developed by Knoth (1998). We do not provide this analysis in this thesis. The method for analyzing the CUSUM $\ln(|\mathbf{S}|)$ envolve Fredholm equations that can be well approximated by Gaussian quadrature. The run length properties of this chart were obtained with $\mathbf{\Sigma}_0$ known and estimated.

CHAPTER 5

CONCLUSION

5.1 General Conclusions

It has been demonstrated in the literature that control charting procedures designed to monitor for a change in the process mean vector of a multivariate quality measurement are affected by changes in both the process mean vector and covariance matrix. In the univariate case, it has been recommended that a control chart for monitoring the process variance be used and examined if there is a signal on the chart from monitoring the mean to see if the variance may have changed. If not and the process is out-of-control, then it is most likely due to a change in the process mean. A similar strategy should be used when there are several quality measurements on an item. While simulation can be used to study the run length properties of a chart under a given model, a more accurate study can be done using analytical methods. We have outlined a method using integral equations to study the performance of the CUSUM $|\mathbf{S}|$ and $\ln(|\mathbf{S}|)$ charts each of which the Shewhart chart as a special case.

5.2 Areas for Further Research

We are interested in continuing our study the integral equation method for analyzing the run length distribution of the the CUSUM $|\mathbf{S}|$ and $\ln(|\mathbf{S}|)$ charts under the independent multivariate normal model. There are several methods that have been proposed in the literature for monitoring for a change in the process covariance matrix under the independent normal model. We plan to provide a comparison of these methods both when the process in-control covariance matrix $\boldsymbol{\Sigma}_0$ is known and when it is estimated from a Phase I study. Two other areas also interest us: how well the charts perform under multivariate non-normal model and when the multivarite quality measurements on items are autocorrelated.

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