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Rademacher-Type Series

V. Kiria-Kaiserberg

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RADEMACHER-TYPE SERIES

by

V. KIRIA-KAISERBERG

(Under the Direction of Andrew Sills)

ABSTRACT

In this thesis we use the classical circle method to find the coefficients $\tilde{p}(n)$ of $\sum_{n=0}^{\infty} \tilde{p}(n)q^n = \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^{2m})^2}$. We introduce a Mathematica package which automates steps in the the circle method and computes the Rademacher-type series expressions for the Fourier coefficients of other Ramanujan-type series.

INDEX WORDS: Partitions, q-series, Circle method.

RADEMACHER-TYPE SERIES

by

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RADEMACHER-TYPE SERIES

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CHAPTER 1
PARTITIONS AND GENERATING FUNCTIONS

1.1 Partitions

A *partition* of a positive integer n is a representation of n as a sum of positive integers where order of summands (parts) does not matter. For example, all the partitions of 4 are

4

3+1

2+2

2+1+1

1+1+1+1,

so, we have five partitions of 4. The eleven partitions of 6 are

6

5+1

4+2

4+1+1

3+3

3+2+1

3+1+1+1

2+2+2

2+2+1+1

2+1+1+1+1

1+1+1+1+1+1.

Let $p(n)$ represent the number of partitions of n . Some values of $p(n)$ are given below:

$$p(0) = 1$$

$$p(1) = 1$$

$$p(2) = 2$$

$$p(3) = 3$$

$$p(4) = 5$$

$$p(5) = 7$$

$$p(6) = 11$$

$$p(7) = 15$$

$$p(8) = 22$$

$$p(9) = 30$$

$$p(10) = 42$$

$$p(100) = 190, 569, 192$$

$$p(200) = 3, 972, 999, 029, 388$$

As could be seen from the above list, $p(n)$ grows rapidly as n increases. However, the pattern of growth is not obvious and a question was raised whether there is a closed formula for the partition function $p(n)$. G. H. Hardy and S. Ramanujan [9] in 1918 obtained the following asymptotic expression for the partition function

$$p(n) \sim \frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}}, \quad \text{as } n \rightarrow \infty, \quad (1.1)$$

where $a(n) \sim b(n)$ as $n \rightarrow \infty$ means

$$\lim_{n \rightarrow \infty} \frac{a(n)}{b(n)} = 1.$$

Later, in 1937, H. Rademacher [10, 11] was able to express $p(n)$ as a convergent series:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k} \sqrt{\frac{2}{3}} \left(n - \frac{1}{24}\right)\right)}{\sqrt{n - \frac{1}{24}}}\right), \quad (1.2)$$

where

$$A_k(n) = \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} e^{\pi i(s(h,k) - 2nh/k)},$$

is a Kloosterman sum, and

$$s(h, k) = \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right)$$

is a Dedekind sum.

In 2011, Ken Ono and Jan Bruinier [4] announced a new formula that expresses $p(n)$ as a *finite* sum.

1.2 Generating functions

In the theory of integer partitions an important role is played by *generating functions*. The *generating function* of a sequence b_n is a formal power series in one unknown, whose coefficients are the b_n , i.e.,

$$b_0 + b_1x + b_2x^2 + b_3x^3 + \dots = \sum_{n=0}^{\infty} b_n x^n \quad (1.3)$$

is a generating function for the sequence $\{b_n\}_{n=0}^{\infty}$. L. Euler proved [6] that the generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}. \quad (1.4)$$

From elementary calculus we know that

$$\sum_{n=0}^{\infty} (x^k)^n = \frac{1}{1-x^k}, \quad |x| < 1, k \in \mathbb{N}. \quad (1.5)$$

Expanding right hand side of (1.4) we have that

$$\begin{aligned}
\prod_{k=1}^{\infty} \frac{1}{1-x^k} &= \left(\frac{1}{1-x}\right) \left(\frac{1}{1-x^2}\right) \left(\frac{1}{1-x^3}\right) \left(\frac{1}{1-x^4}\right) \cdots \\
&= \left(\sum_{n=0}^{\infty} x^n\right) \left(\sum_{n=0}^{\infty} x^{2n}\right) \left(\sum_{n=0}^{\infty} x^{3n}\right) \left(\sum_{n=0}^{\infty} x^{4n}\right) \cdots \\
&= (1+x+x^2+x^3+x^4+x^5+x^6+\cdots) \\
&\quad \times (1+x^2+x^4+x^6+x^8+x^{10}+x^{12}+\cdots) \\
&\quad \times (1+x^3+x^6+x^9+x^{12}+x^{15}+x^{18}+\cdots) \\
&\quad \times (1+x^4+x^8+x^{12}+x^{16}+x^{20}+x^{24}+\cdots) \cdots \\
&= (1+x+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+x^{1+1+1+1+1}+\cdots) \\
&\quad \times (1+x^2+x^{2+2}+x^{2+2+2}+x^{2+2+2+2}+x^{2+2+2+2+2}+\cdots) \\
&\quad \times (1+x^3+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+x^{3+3+3+3+3}+\cdots) \\
&\quad \times (1+x^4+x^{4+4}+x^{4+4+4}+x^{4+4+4+4}+x^{4+4+4+4+4}+\cdots) \cdots \\
&= 1+x+(x^{1+1}+x^2)+(x^{1+1+1}+x^{1+2}+x^3) \\
&\quad + (x^{1+1+1+1}+x^{1+1+2}+x^{2+2}+x^{1+3}+x^4)+\cdots \\
&= 1+x+2x^2+3x^3+5x^4+7x^5+11x^6+\cdots \\
&= \sum_{n=0}^{\infty} p(n)x^n. \tag{1.6}
\end{aligned}$$

This is exactly the generating function for $p(n)$ because coefficient of x^n is the number of partions of n . As could be seen from this example, generating functions by themselves tell us nothing about how to find the coefficient of x^n in the series expansion. We need some additional tools to extract coefficients for each n . The Circle Method, developed by G. Hardy and S. Ramanujan and improved by H. Rademacher, will allow us to do precisely this. In chapter 2 we use this method to find the formula for the coefficient of x^n for one of the Ramanujan-type identities.

CHAPTER 2

Formula for $\tilde{p}(n)$

Let

$$f(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1 - q^m} \quad (2.1)$$

be Euler's generating function for $p(n)$. H. Rademacher used the classical circle method to find the coefficients of q^n . There are many other infinite products to which this method could be applied. We introduce one of these infinite products here and derive the formula for the coefficients of q^n . Define

$$G(q) := \prod_{m=1}^{\infty} \frac{(1 - q^m)}{(1 - q^{2m})^2} = \frac{(f(q^2))^2}{f(q)}. \quad (2.2)$$

Let $\tilde{p}(j)$ denote the coefficient of q^j in the expansion of $G(q)$, i.e.

$$G(q) = \sum_{j=0}^{\infty} \tilde{p}(j)q^j. \quad (2.3)$$

Our goal is to find a closed expression for $\tilde{p}(j)$. We are about to use the classical circle method to achieve this goal. It is a long, tedious calculation, many steps of which have been implemented in a Mathematica package written by the author and his advisor. See Appendix A for the Mathematica code for this package. At key points of this chapter, the Mathematica package is demonstrated with input and output enclosed in a box.

We begin with some preliminaries.

Farey fractions. The sequence \mathcal{F}_N of *proper Farey fractions of order N* is the set of all h/k with $(h, k) = 1$ and $0 \leq h/k < 1, k \leq N$ arranged in increasing order, where (h, k) denotes the $\text{gcd}(h, k)$. Thus, e.g., $\mathcal{F}_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$.

For a given N , let h_p, h_s, k_p and k_s be such that $\frac{h_p}{k_p}$ is the immediate predecessor of

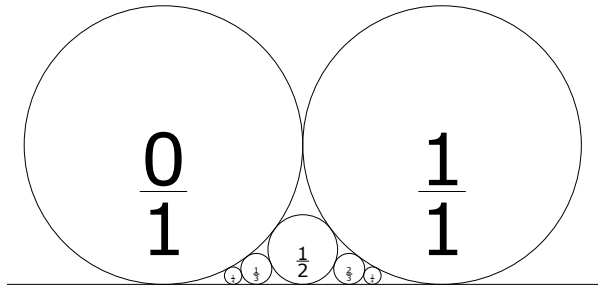


Figure 2.1: Ford Circles of order 4.

$\frac{h}{k}$ and $\frac{h_s}{k_s}$ is the immediate successor of $\frac{h}{k}$ in \mathcal{F}_N and $\frac{h}{k} = \frac{h_p + h_s}{k_p + k_s}$.

Ford circles and Rademacher path. Let h and k be as above. The *Ford circle* $C(h, k)$ [7] is the circle in \mathbb{C} of radius $\frac{1}{2k^2}$ centered at the point

$$\frac{h}{k} + \frac{1}{2k^2}i.$$

The *upper arc* $\gamma(h, k)$ of the Ford circle $C(h, k)$ is those points of $C(h, k)$ from the initial point

$$I_{h,k} := \frac{h}{k} - \frac{k_p}{k(k^2 + k_p^2)} + \frac{1}{k^2 + k_p^2}i \quad (2.4)$$

to the terminal point

$$T_{h,k} := \frac{h}{k} + \frac{k_s}{k(k^2 + k_s^2)} + \frac{1}{k^2 + k_s^2}i \quad (2.5)$$

traversed in the negative direction. We note that

$$I_{0,1} = T_{N-1,N}.$$

Every Ford circle is in the upper half plane. For $\frac{h_1}{k_1}, \frac{h_2}{k_2} \in \mathcal{F}_N$, $C(h_1, k_1)$ and $C(h_2, k_2)$ are either tangent or do not intersect [7].

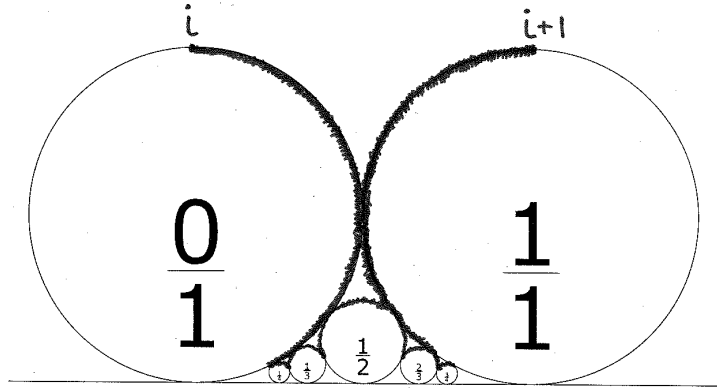


Figure 2.2: Rademacher path of order 4.

The *Rademacher path* $P(N)$ of order N is the path in the upper half of the τ -plane from i to $i + 1$ consisting of

$$\bigcup_{\frac{h}{k} \in \mathcal{F}_N} \gamma(h, k) \quad (2.6)$$

traversed left to right in negative direction.

We note that the left half of the Ford circle $C(0, 1)$ and the corresponding upper arc $\gamma(0, 1)$ are shifted to the right by 1 unit. This is justified since the function that is to be integrated over Rademacher path is periodic.

Convergence and Cauchy Residue Theorem.

Considering q as a complex variable in

$$\prod_{m=1}^{\infty} \frac{(1 - q^m)}{(1 - q^{2m})^2} = \prod_{m=1}^{\infty} \frac{1}{(1 + q^m)(1 - q^{2m})} = \prod_{m=1}^{\infty} \frac{1}{(1 - (-q^m))(1 - q^{2m})} \quad (2.7)$$

we see from the right hand side that infinite product and thus also infinite series are convergent for $|q| < 1$ since

$$\sum_{n=0}^{\infty} (q^k)^n = \frac{1}{(1 - q^k)} \quad (2.8)$$

is a geometric series which converges for $|q| < 1$ for any fixed $k \geq 1$.

Next, we note that from

$$G(q) = \sum_{j=0}^{\infty} \tilde{p}(j)q^j, \quad (2.9)$$

we get that

$$\frac{G(q)}{q^{n+1}} = \sum_{j=0}^{\infty} \frac{\tilde{p}(j)q^j}{q^{n+1}} \quad \text{if } 0 < |q| < 1. \quad (2.10)$$

The series on the right of (2.10) is a Laurent series of $\frac{G(q)}{q^{n+1}}$. It has a pole of order $n + 1$ at $q = 0$ with residue $\tilde{p}(n)$. Applying Cauchy's Residue Theorem we get that

$$\tilde{p}(n) = \frac{1}{2\pi i} \int_C \frac{G(q)}{q^{n+1}} dq = \frac{1}{2\pi i} \int_C \frac{(f(q^2))^2}{f(q)q^{n+1}} dq, \quad (2.11)$$

where C is any positively oriented simple closed countour lying inside the unit circle. The change of the variable $q = e^{2\pi i\tau}$ maps the unit disk $|q| < 1$ into an infinite vertical strip of width 1 in the τ -plane. To see this we note that from $q = e^{2\pi i\tau}$ we get $\log q = 2\pi i\tau$, so $\tau = \frac{\log q}{2\pi i}$. Choosing the branch cut to be $[0,1]$, we get

$$\tau = \frac{\log |q|}{2\pi i} + \frac{\text{Arg}(q)}{2\pi}. \quad (2.12)$$

As q traverses a circle centered at $q = 0$ of radius $e^{-2\pi}$ in the positive direction, the point τ varies from i to $i + 1$ along a horizontal segment as could be easily deduced from (2.12).

Replacing the segment by the Rademacher path composed of upper arcs of the Ford circles formed by the Farey series \mathcal{F}_N , (2.11) becomes

$$\tilde{p}(n) = \frac{1}{2\pi i} \int_i^{i+1} \frac{(f(e^{4\pi i\tau}))^2 2\pi i e^{2\pi i\tau}}{f(e^{2\pi i\tau}) e^{2\pi i\tau(n+1)}} d\tau, \quad (2.13)$$

which simplifies to

$$\tilde{p}(n) = \int_i^{i+1} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n} d\tau, \quad (2.14)$$

$$= \int_{P(N)} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n} d\tau, \quad (2.15)$$

The above can be written as

$$\int_{P(N)} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n} d\tau = \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \int_{\gamma(h,k)} \frac{(f(e^{4\pi i\tau}))^2}{f(e^{2\pi i\tau})} e^{-2\pi i\tau n} d\tau \quad (2.16)$$

where $\gamma(h, k)$ is the upper arc of the Ford circle $C(h, k)$. Consider another change of variable

$$\tau = \frac{h}{k} + \frac{iz}{k}, \quad (2.17)$$

so that

$$z = -ik \left(\tau - \frac{h}{k} \right) \quad (2.18)$$

$$dz = -ik d\tau. \quad (2.19)$$

Under this transformation the Ford circle $C(h, k)$ in the τ -plane with center at $\frac{h}{k} + i\frac{1}{2k^2}$ and radius $\frac{1}{2k^2}$ is mapped to a negatively oriented circle C_k in the z -plane with center at $\frac{1}{2k}$ and radius $\frac{1}{2k}$. This follows from the fact that any point on the Ford circle $C(h, k)$ is given by

$$\tau = \left(\frac{h}{k} + i\frac{1}{2k^2} \right) + \frac{1}{2k^2} e^{i\theta}, \quad 0 \leq \theta < 2\pi. \quad (2.20)$$

Substitution of (2.20) into (2.18) gives

$$z = \frac{1}{2k} + \frac{1}{2k} (-ie^{i\theta}) \quad (2.21)$$

which is a circle centered at $\frac{1}{2k}$ with radius $\frac{1}{2k}$. Now we make change of variable in (2.16). This gives

$$\tilde{p}(n) = i \sum_{k=1}^N k^{-1} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \frac{(f(e^{4\pi i h/k - 4\pi i z/k}))^2}{f(e^{2\pi i h/k - 2\pi z/k})} e^{2\pi n z/k} dz \quad (2.22)$$

where the initial point

$$s_{h,k} = \frac{k}{k^2 + k_p^2} + \frac{k_p}{k^2 + k_p^2} i \quad (2.23)$$

and the terminal point

$$t_{h,k} = \frac{k}{k^2 + k_s^2} - \frac{k_s}{k^2 + k_s^2} i \quad (2.24)$$

both obtained from (2.4), (2.5) and (2.18) through change of variable.

Next, we note that

$$f(q) = f(e^{2\pi i \tau}) = \frac{e^{\pi i \tau/12}}{\eta(\tau)}, \quad (2.25)$$

where $\eta(\tau)$ is the Dedekind eta function. Rewriting modular functional equation [3, p. 96] for $\eta(\tau)$ in terms of $f(q) = f(e^{2\pi i \tau}) = f(e^{2\pi i h/k - 2\pi z/k})$ we get

$$f(e^{2\pi i h/k - 2\pi z/k}) = \omega(h, k) \exp\left(\frac{\pi(z^{-1} - z)}{12k}\right) \sqrt{z} f\left(\exp\left(2\pi i \frac{iz^{-1} + H}{k}\right)\right) \quad (2.26)$$

where

$$\omega(h, k) = \exp\left(\pi i \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right)\right), \quad (2.27)$$

and $hH \equiv -1 \pmod{k}$, $(h, k) = 1$.

To evaluate (2.22) we would like to express

$$G(q) = G(e^{2\pi i \tau}) = G(e^{2\pi i h/k - 2\pi z/k}) = \frac{(f(e^{4\pi i h/k - 4\pi i z/k}))^2}{f(e^{2\pi i h/k - 2\pi z/k})} \quad (2.28)$$

in the same way we did for $f(q)$ above. Two cases have to be considered: $(k, 2) = 1$ and $(k, 2) = 2$. When $(k, 2) = 1$ we will replace h by $2h$ and z by $2z$, and when $(k, 2) = 2$, k will be replaced by $k/2$ in order to obtain $f(q^2)$ from $f(q)$. Hence, we have

$$G(e^{2\pi ih/k - 2\pi z/k}) = \begin{cases} \frac{\omega^2(2h, k) e^{\frac{\pi((2z)^{-1} - 2z)}{6k}} 2z f^2(e^{2\pi i(i(2z)^{-1} + H_2)/k})}{\omega(h, k) e^{\frac{\pi(z^{-1} - z)}{12k}} \sqrt{z} f(e^{2\pi i(iz^{-1} + H_2)/k})}, & \text{if } (k, 2) = 1, \\ \frac{\omega^2(h, k/2) e^{\frac{\pi(z^{-1} - z)}{3k}} z f^2(e^{4\pi i(iz^{-1} + H_1)/k})}{\omega(h, k) e^{\frac{\pi(z^{-1} - z)}{12k}} \sqrt{z} f(e^{2\pi i(iz^{-1} + H_1)/k})}, & \text{if } (k, 2) = 2, \end{cases} \quad (2.29)$$

which simplifies to

$$G(e^{2\pi ih/k - 2\pi z/k}) = \begin{cases} 2 \frac{\omega^2(2h, k)}{\omega(h, k)} e^{-\frac{\pi z}{4k}} \sqrt{z} \frac{f^2(e^{2\pi i(i(2z)^{-1} + H_2)/k})}{f(e^{2\pi i(iz^{-1} + H_2)/k})}, & \text{if } (k, 2) = 1, \\ \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{\frac{\pi(z^{-1} - z)}{4k}} \sqrt{z} \frac{f^2(e^{4\pi i(iz^{-1} + H_1)/k})}{f(e^{2\pi i(iz^{-1} + H_1)/k})}, & \text{if } (k, 2) = 2, \end{cases} \quad (2.30)$$

or, equivalently

$$G(e^{2\pi ih/k - 2\pi z/k}) = \begin{cases} 2 \frac{\omega^2(2h, k)}{\omega(h, k)} e^{-\frac{\pi z}{4k}} \sqrt{z} \frac{f^2(e^{2\pi i(i(2z)^{-1} + H_2)/k})}{f(e^{2\pi i(iz^{-1} + H_2)/k})}, & \text{if } (k, 2) = 1, \\ \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{\frac{\pi(z^{-1} - z)}{4k}} \sqrt{z} G(e^{2\pi i(iz^{-1} + H_1)/k}), & \text{if } (k, 2) = 2, \end{cases} \quad (2.31)$$

where $hH_j \equiv -1 \pmod{k}$ and $j|H_j$ for $j = 1, 2$.

The implementation of the above in Mathematica is:

```

In[1]:= ModularRHS[h, k, z, ω, f, qprime, {{1,-1},{2,2}}]

Out[1]=  $\left( \begin{array}{l} 1, \left\{ \frac{\omega(2h,k)^2}{\omega(h,k)}, 2\sqrt{z}, -\frac{\pi z}{4k}, \frac{f(\sqrt{qprime})^2}{f(qprime)} \right\} \\ 2, \left\{ \frac{\omega(h, \frac{k}{2})^2}{\omega(h,k)}, \sqrt{z}, \frac{\pi}{4kz} - \frac{\pi z}{4k}, \frac{f(qprime^2)^2}{f(qprime)} \right\} \end{array} \right)$ 

```

We return to evaluation of (2.22). To proceed we note that

$$G(e^{2\pi i(iz^{-1}+H_1)/k}) = 1 + \{G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1\}. \quad (2.32)$$

Rewriting (2.22) in terms of (2.31) and (2.32) we obtain

$$\begin{aligned}
\tilde{p}(n) &= i \sum_{\substack{k=1 \\ (k,2)=1}}^N k^{-1} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \frac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})} e^{\frac{\pi z}{k}(2n-\frac{1}{4})} dz \\
&+ i \sum_{\substack{k=1 \\ (k,2)=2}}^N k^{-1} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \{1 + (G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1)\} \\
&\times e^{(\frac{\pi z}{k}(2n-\frac{1}{4}) + \frac{\pi}{4zk})} dz \\
&= 2i \sum_{\substack{k=1 \\ (k,2)=1}}^N k^{-1} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \frac{f^2(e^{2\pi i(i(2z)^{-1}+H_2)/k})}{f(e^{2\pi i(iz^{-1}+H_2)/k})} e^{\frac{\pi z}{k}(2n-\frac{1}{4})} dz \\
&+ i \sum_{\substack{k=1 \\ (k,2)=2}}^N k^{-1} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h,k)} e^{-2\pi i n h/k} (J_1(h,k) + J_2(h,k)), \quad (2.33)
\end{aligned}$$

where

$$J_1(h,k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} e^{(\frac{\pi z}{k}(2n-\frac{1}{4}) + \frac{\pi}{4zk})} dz \quad (2.34)$$

and

$$J_2(h, k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \{G(e^{2\pi i(iz^{-1}+H_1)/k}) - 1\} e^{(\frac{\pi z}{k}(2n-\frac{1}{4})+\frac{\pi}{4zk})} dz. \quad (2.35)$$

We will estimate the first term in (2.33) and will show that it is small for large N . To do this we change variable again by letting $\xi = zk$. Then the first term in (2.33) becomes

$$2i \sum_{k=1}^N k^{-5/2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(2h, k)}{\omega(h, k)} e^{-2\pi i h/k} \int_{s_{h,k}^*}^{t_{h,k}^*} \sqrt{\xi} \frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})} e^{\frac{\pi \xi}{k^2}(2n-\frac{1}{4})} d\xi \quad (2.36)$$

where

$$s_{h,k}^* = \frac{k^2}{k^2 + k_p^2} + \frac{kk_p}{k^2 + k_p^2} i \quad (2.37)$$

and

$$t_{h,k}^* = \frac{k^2}{k^2 + k_s^2} - \frac{kk_s}{k^2 + k_s^2} i \quad (2.38)$$

are initial and terminal points obtained from (2.23) and (2.24) respectively. Under this change of variable circle C_k in z -plane with center at $\frac{1}{2k}$ and radius $\frac{1}{2k}$ is mapped to a circle C_k^* in ξ -plane centered at $\frac{1}{2}$ with radius $\frac{1}{2}$. Note also that the mapping $w = \frac{1}{\xi}$ maps the circle C_k^* and its interior onto a half-plane $\Re(w) \geq 1$ (where $\Re(w)$ denotes the real part of complex variable w and $\Im(w)$ is the imaginary part). For, from elementary complex analysis we have that $\Re(w) = \frac{x}{x^2 + y^2}$ and $\Im(w) = \frac{-y}{x^2 + y^2}$, where $x + iy = \xi$. It is readily seen that the segment $0 < x \leq 1$ in the ξ -plane is mapped to an infinite strip $[1, \infty)$ in the w -plane. So, it follows that inside and on the circle C_k^* we have that $0 < \Re(\xi) \leq 1$ and $\Re\left(\frac{1}{\xi}\right) \geq 1$. We now show that $\Re\left(\frac{1}{\xi}\right) = 1$

on the circle C_k^* . To see this note that in the polar form $\xi = \frac{1}{2} + \frac{1}{2}e^{i\theta}$ on C_k^* , $0 \leq \theta \leq 2\pi$. From this we get that

$$\begin{aligned}
\frac{1}{\xi} &= \frac{2}{1 + e^{i\theta}} = \frac{2}{(1 + \cos \theta) + i \sin \theta} \\
&= \frac{2[(1 + \cos \theta) - i \sin \theta]}{(1 + \cos \theta)^2 + \sin^2 \theta} \\
&= \frac{2(1 + \cos \theta)}{2 + 2 \cos \theta} - i \frac{2 \sin \theta}{2 + 2 \cos \theta} \\
&= 1 - i \frac{\sin \theta}{1 + \cos \theta}.
\end{aligned} \tag{2.39}$$

so, $\Re\left(\frac{1}{\xi}\right) = 1$.

Furthermore, we may move path of integration from the arc joining $s_{h,k}^*$ and $t_{h,k}^*$ to a segment connecting these two points on the circle C_k^* . By [3, p. 104, Theorem 5.9] the length of the path of integration is bounded by $2\sqrt{2}k/N$, and on the segment connecting $s_{h,k}^*$ and $t_{h,k}^*$, $|\xi| < \sqrt{2}k/N$.

Next, let us define $\tilde{p}^*(m)$ by

$$\sum_{m=0}^{\infty} \tilde{p}^*(m)q^m = \frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1} + H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1} + H_2)/k})}. \tag{2.40}$$

which is a part of the integrand in (2.36). Then, estimating the integrand in (2.36)

we get

$$\begin{aligned}
& \left| \sqrt{\xi} e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})} \right| \times \left| -1 + \frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})} \right| \\
&= |\xi|^{1/2} \left| e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})} \right| \times \left| \sum_{m=1}^{\infty} \tilde{p}^*(m) \exp\left(\frac{2\pi im(i(\frac{\xi}{k})^{-1}+H_2)}{k}\right) \right| \\
&= |\xi|^{1/2} \left| e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})} \right| \times \left| \sum_{m=1}^{\infty} \tilde{p}^*(m) \exp\left(-\frac{2\pi m}{\xi}\right) \exp\left(\frac{2\pi imH_2}{k}\right) \right| \\
&\leq |\xi|^{1/2} e^{\frac{\pi}{k^2}(2n-\frac{1}{4})\Re(\xi)} \left| \sum_{m=1}^{\infty} \tilde{p}^*(m) \exp\left(-\frac{2\pi m}{\xi}\right) \exp\left(\frac{2\pi imH_2}{k}\right) \right| \\
&\leq |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^*(m)| \exp\left(-2\pi m \Re\left(\frac{1}{\xi}\right)\right) \\
&\leq |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^*(m)| e^{-2\pi m} \\
&= |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^*(m)| y^m, \quad (\text{where } y = e^{-2\pi}) \\
&= c|\xi|^{1/2}, \tag{2.41}
\end{aligned}$$

where

$$c = e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^*(m)| y^m. \tag{2.42}$$

Note that c does not depend on ξ or N . It depends on n , but n remains fixed in the above analysis. So,

$$\begin{aligned}
\left| \int_{s^*(h,k)}^{t^*(h,k)} \sqrt{\xi} \frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1}+H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1}+H_2)/k})} e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})} d\xi \right| &\leq c|\xi|^{1/2} \leq c \left(\frac{\sqrt{2}k}{N}\right)^{1/2} \frac{2\sqrt{2}N}{N} \\
&< \alpha k^{3/2} N^{-3/2} \tag{2.43}
\end{aligned}$$

for some constant α and we have that

$$\begin{aligned}
& \left| 2i \sum_{k=1}^N k^{-5/2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(2h, k)}{\omega(h, k)} e^{-2\pi i n h/k} \int_{s^*(h,k)}^{t^*(h,k)} \sqrt{\xi} \frac{f^2(e^{2\pi i(i(\frac{2\xi}{k})^{-1} + H_2)/k})}{f(e^{2\pi i(i(\frac{\xi}{k})^{-1} + H_2)/k})} e^{\frac{\pi \xi}{k^2}(2n - \frac{1}{4})} d\xi \right| \\
& \leq 2 \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \alpha k^{-1} N^{-3/2} \\
& \leq 2\alpha N^{-3/2} \sum_{k=1}^N 1 = 2\alpha N^{-1/2}. \tag{2.44}
\end{aligned}$$

This completes the estimation of the first term in (2.33). We proceed to the second term. First, we will show that

$$J_2(h, k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} \{G(e^{2\pi i(iz^{-1} + H_1)/k}) - 1\} e^{(\frac{\pi z}{k}(2n - \frac{1}{4}) + \frac{\pi}{4zk})} dz$$

is small for large N . Making change of variable $\xi = zk$ as before, we get that

$$J_2(h, k) = k^{-3/2} \int_{s_{h,k}^*}^{t_{h,k}^*} \sqrt{\xi} \{G(e^{2\pi i(i(\xi/k)^{-1} + H_1)/k}) - 1\} e^{(\frac{\pi \xi}{k^2}(2n - \frac{1}{4}) + \frac{\pi}{4\xi})} d\xi$$

where $s_{h,k}^*$ and $t_{h,k}^*$ are as in (2.37) and (2.38) respectively. As before, we define $\tilde{p}^{**}(m)$ by

$$\sum_{m=0}^{\infty} \tilde{p}^{**}(m) q^m = G(e^{2\pi i(i(\xi/k)^{-1} + H_1)/k}) - 1. \tag{2.45}$$

Then, estimating the integrand, we see that

$$\begin{aligned}
& \left| \sqrt{\xi} e^{\left(\frac{\pi\xi}{k^2}(2n-\frac{1}{4})+\frac{\pi}{4\xi}\right)} \right| \times \left| G(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k}) - 1 \right| \\
&= |\xi|^{1/2} \left| e^{\frac{\pi\xi}{k^2}(2n-\frac{1}{4})} e^{\frac{\pi}{4\xi}} \right| \times \left| \sum_{m=0}^{\infty} \tilde{p}^{**}(m) \exp\left(\frac{2\pi im\left(\frac{ik}{\xi} + H_1\right)}{k}\right) - 1 \right| \\
&\leq |\xi|^{1/2} e^{\frac{\pi}{k^2}(2n-\frac{1}{4})\Re(\xi)} e^{\frac{\pi}{4}\Re\left(\frac{1}{\xi}\right)} \left| \sum_{m=1}^{\infty} \tilde{p}^{**}(m) \exp\left(\frac{-2\pi m}{\xi}\right) \exp\left(\frac{2\pi imH_1}{k}\right) \right| \\
&\leq |\xi|^{1/2} e^{2\pi n} e^{\frac{\pi}{4}\Re\left(\frac{1}{\xi}\right)} \sum_{m=1}^{\infty} |\tilde{p}^{**}(m)| \exp\left(-2\pi m\Re\left(\frac{1}{\xi}\right)\right) \\
&= |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(m)| \exp\left(\left(-2\pi m + \frac{\pi}{4}\right)\Re\left(\frac{1}{\xi}\right)\right) \\
&\leq |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(m)| \exp\left(\left(-2\pi m + \frac{\pi}{4}\right)\right) \\
&= |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(m)| e^{-\frac{\pi}{4}(8m-1)} \\
&\leq |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(8m-1)| e^{-\frac{\pi}{4}(8m-1)} \\
&= |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(8m-1)| x^{8m-1}, \quad (\text{where } x = e^{-\frac{\pi}{4}}) \\
&= b|\xi|^{1/2}, \tag{2.46}
\end{aligned}$$

where

$$b = e^{2\pi n} \sum_{m=1}^{\infty} |\tilde{p}^{**}(8m-1)| x^{8m-1}.$$

Note that b does not depend on ξ or N . It depends on n , but n is fixed. It follows, therefore, that

$$|J_2(h, k)| \leq b \left(\frac{\sqrt{2}k}{N}\right)^{1/2} \frac{2\sqrt{2}N}{N} < \beta k^{3/2} N^{-3/2} \tag{2.47}$$

for some constant β . Then we have that

$$\left| i \sum_{k=1}^N k^{-5/2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} J_2(h, k) \right| < \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \beta k^{-1} N^{-3/2} \\ \leq \beta N^{-3/2} \sum_{k=1}^N 1 = \beta N^{-1/2}. \quad (2.48)$$

Combining the results from (2.44) and (2.48) we have that

$$\begin{aligned} \tilde{p}(n) &= i \sum_{k=1}^N k^{-5/2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} J_1(h, k) + O(\beta N^{-1/2} + 2\alpha N^{-1/2}) \\ &= i \sum_{k=1}^N k^{-5/2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} J_1(h, k) + O(N^{-1/2}). \end{aligned} \quad (2.49)$$

Finally, we turn our attention to

$$J_1(h, k) = k^{-3/2} \int_{s_{h,k}^*}^{t_{h,k}^*} \sqrt{\xi} e^{\left(\frac{\pi \xi}{k^2} \left(2n - \frac{1}{4}\right) + \frac{\pi}{4\xi}\right)} d\xi. \quad (2.50)$$

We note that

$$J_1(h, k) = \int_{C_k^*} - \int_{s_{h,k}^*}^0 - \int_0^{t_{h,k}^*} = \int_{C_k^*} - S_1 - S_2, \quad (2.51)$$

where C_k^* is a circle in the ξ -plane centered at $\frac{1}{2}$ with radius $\frac{1}{2}$, as before. It is easily seen that the length of the arc connecting 0 and $s_{h,k}^*$ is less than

$$2\pi \frac{|s_{h,k}^*|}{2} \leq \pi |s_{h,k}^*| \leq \pi \sqrt{2} \frac{k}{N} \quad (2.52)$$

From the discussion above we know that $\Re\left(\frac{1}{\xi}\right) = 1$ and $0 < \Re(\xi) \leq 1$ on C_k^* . So, the integrand in S_1 could be estimated as

$$\begin{aligned}
\left| \sqrt{\xi} e^{\left(\frac{\pi\xi}{k^2}\left(2n-\frac{1}{4}\right)+\frac{\pi}{4\xi}\right)} \right| &= |\xi|^{1/2} \left| e^{\frac{\pi\xi}{k^2}\left(2n-\frac{1}{4}\right)} \right| \left| e^{\frac{\pi}{4\xi}} \right| \\
&= |\xi|^{1/2} e^{\frac{\pi}{k^2}\left(2n-\frac{1}{4}\right)\Re(\xi)} e^{\frac{\pi}{4}\Re\left(\frac{1}{\xi}\right)} \\
&\leq 2^{1/4} \frac{k^{1/2}}{N^{1/2}} e^{2\pi n} e^{\frac{\pi}{4}}.
\end{aligned} \tag{2.53}$$

Combining the results in (2.52) and (2.53) we get

$$|S_1| < \gamma k^{3/2} N^{-3/2}, \tag{2.54}$$

where γ is a constant. We can obtain similar estimate for S_2 and, as before, we get an error term $O(N^{-1/2})$ in the formula for $\tilde{p}(n)$. Therefore, we can write

$$\tilde{p}(n) = i \sum_{k=1}^N k^{-5/2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \int_{C_k^*} \sqrt{\xi} e^{\left(\frac{\pi\xi}{k^2}\left(2n-\frac{1}{4}\right)+\frac{\pi}{4\xi}\right)} d\xi + O(N^{-1/2}). \tag{2.55}$$

Letting $N \rightarrow \infty$ we have that

$$\tilde{p}(n) = i \sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \int_{C_k^*} \sqrt{\xi} e^{\left(\frac{\pi\xi}{k^2}\left(2n-\frac{1}{4}\right)+\frac{\pi}{4\xi}\right)} d\xi. \tag{2.56}$$

The above could be implemented in Mathematica as:

```
In[2]:=ZToZetaIntegrandComponents[n, h, k, z, ξ, ω, {{1,-1},{2,2}}]
Out[2]= 
$$\left( \begin{array}{l} 1, \quad \left\{ \frac{\omega(2h,k)^2}{\omega(h,k)}, 2\sqrt{\frac{\zeta}{k}}, \frac{2n\pi\zeta}{k^2} - \frac{\pi\zeta}{4k^2} \right\} \\ 2, \quad \left\{ \frac{\omega(h,\frac{k}{2})^2}{\omega(h,k)}, \sqrt{\frac{\zeta}{k}}, \frac{2n\pi\zeta}{k^2} - \frac{\pi\zeta}{4k^2} + \frac{\pi}{4\zeta} \right\} \end{array} \right)$$

```

We introduce another change of variable

$$\xi = \frac{1}{w}, \quad d\xi = -\frac{1}{w^2}.$$

Then (2.56) becomes

$$\tilde{p}(n) = \frac{1}{i} \sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \int_{1-\infty i}^{1+\infty i} w^{-5/2} e^{(\frac{\pi}{k^2}(2n-\frac{1}{4})\frac{1}{w} + \frac{\pi w}{4})} dw. \quad (2.57)$$

Let $t = \frac{\pi w}{4}$ in (2.57), then the above becomes

$$\tilde{p}(n) = 2\pi \left(\frac{\pi^{3/2}}{8} \right) \sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \frac{1}{2\pi i} \int_{\pi/4-\infty i}^{\pi/4+\infty i} t^{-5/2} e^{(t + \frac{\pi^2}{4k^2}(2n-\frac{1}{4})\frac{1}{t})} dt. \quad (2.58)$$

The implementation of the above in Mathematica is:

```
In[3]:=IntegrandInT[n, h, k, ξ, dξ, ω, t, dt, {{1,-1},{2,2}}]
Out[3]= 
$$\left( 2, \left\{ -\frac{1}{8}i e^{-\frac{2ihn\pi}{k}} \left(\frac{1}{k}\right)^{5/2} \pi^{3/2}, \frac{8n\pi^2 - \pi^2}{16k^2 t} + t, -\frac{5}{2}, \frac{\omega(h,\frac{k}{2})^2}{\omega(h,k)}, dt \right\} \right)$$

```

In Watson's Treatise on Bessel functions [13, p. 181] we find a formula equivalent to the following:

$$I_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{2\pi i} \int_{c-\infty i}^{c+\infty i} t^{-\nu-1} e^{t+(z^2/4t)} dt \quad , (\text{if } c > 0, \quad \Re(\nu) > 0). \quad (2.59)$$

Let

$$\frac{z}{2} = \left\{ \frac{\pi^2}{4k^2} \left(2n - \frac{1}{4} \right) \right\}^{1/2} \quad (2.60)$$

and $\nu = 3/2$. Then we have

$$\begin{aligned} \tilde{p}(n) &= 2\pi \left(\frac{\pi^{3/2}}{8} \right) \sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \frac{\pi^{-3/2} \left(2n - \frac{1}{4} \right)^{-3/4}}{4^{-3/4} k^{-3/2}} I_{\frac{3}{2}} \left(\frac{\pi}{k} \sqrt{2n - \frac{1}{4}} \right) \\ &= \frac{2\pi \left(2n - \frac{1}{4} \right)^{-\frac{3}{4}}}{\sqrt{8}} \sum_{k=1}^{\infty} k^{-1} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} I_{\frac{3}{2}} \left(\frac{\pi}{k} \sqrt{2n - \frac{1}{4}} \right). \end{aligned} \quad (2.61)$$

The implementation of the above in Mathematica is:

```
In[4]:= RawFormula[n, h, k, ω, {{1,-1},{2,2}}]
Out[4]=  $\left( 2, \frac{2e^{-\frac{2ihn\pi}{k}} \pi \text{BESSELI}\left(\frac{3}{2}, \frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{2}\right)^2}{k(8n-1)^{3/4} \omega(h,k)} \right)$ 
```

Note that Bessel functions of this order can be expressed as

$$I_{\frac{3}{2}}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left(\frac{\sinh z}{z} \right). \quad (2.62)$$

Expanding (2.62) we have that

$$I_{\frac{3}{2}}(z) = \sqrt{\frac{2z}{\pi}} \left(\frac{\cosh z}{z} - \frac{\sinh z}{z^2} \right) \quad (2.63)$$

Substituting (2.60) into (2.63), we get

$$\begin{aligned} I_{\frac{3}{2}}(z) &= I_{\frac{3}{2}} \left(\frac{\pi}{k} \left(2n - \frac{1}{4} \right)^{\frac{1}{2}} \right) = I_{\frac{3}{2}} \left(\frac{\pi \sqrt{8n-1}}{2k} \right) \\ &= \sqrt{\frac{2 \left(\frac{\pi \sqrt{8n-1}}{2k} \right)}{\pi}} \left(\frac{\cosh \left(\frac{\pi \sqrt{8n-1}}{2k} \right)}{\left(\frac{\pi \sqrt{8n-1}}{2k} \right)} - \frac{\sinh \left(\frac{\pi \sqrt{8n-1}}{2k} \right)}{\left(\frac{\pi \sqrt{8n-1}}{2k} \right)^2} \right) \\ &= \frac{(8n-1)^{1/4}}{\sqrt{k}} \left(\frac{2 \cosh \left(\frac{\pi \sqrt{8n-1}}{2k} \right)}{\frac{\pi}{k} \sqrt{8n-1}} - \frac{\frac{4k}{\pi} \sinh \left(\frac{\pi \sqrt{8n-1}}{2k} \right)}{\frac{\pi}{k} (8n-1)} \right) \\ &= \frac{1}{\pi \sqrt{\frac{\sqrt{8n-1}}{k}}} \left(2 \cosh \left(\frac{\pi \sqrt{8n-1}}{2k} \right) - \frac{4k \sinh \left(\frac{\pi \sqrt{8n-1}}{2k} \right)}{\pi \sqrt{8n-1}} \right) \end{aligned} \quad (2.64)$$

Multiplying (2.64) by

$$\frac{2\pi \left(2n - \frac{1}{4} \right)^{-\frac{3}{4}}}{k\sqrt{8}} = \frac{2\pi}{k(8n-1)^{\frac{3}{4}}} \quad (2.65)$$

we get

$$\frac{2 \left(2 \cosh \left(\frac{\pi \sqrt{8n-1}}{2k} \right) - \frac{4k \sinh \left(\frac{\pi \sqrt{8n-1}}{2k} \right)}{\pi \sqrt{8n-1}} \right)}{k(8n-1)^{3/4} \sqrt{\frac{\sqrt{8n-1}}{k}}}. \quad (2.66)$$

Finally, we rewrite (2.61) in terms of (2.66) to get

$$\tilde{p}(n) = \sum_{k=1}^{\infty} k^{-1} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\omega^2(h, k/2)}{\omega(h, k)} e^{-2\pi i n h/k} \frac{2 \left(2 \cosh \left(\frac{\pi \sqrt{8n-1}}{2k} \right) - \frac{4k \sinh \left(\frac{\pi \sqrt{8n-1}}{2k} \right)}{\pi \sqrt{8n-1}} \right)}{k(8n-1)^{3/4} \sqrt{\frac{\sqrt{8n-1}}{k}}} \quad (2.67)$$

which is in agreement with the formula obtained using our Mathematica package.

In [5] :=RTF[$\frac{\text{QPochhammer}[q, q]}{\text{QPochhammer}[q^2, q^2]^2}$, p, 3.14]

$$\text{If } \prod_{n=1}^{\infty} \frac{1 - q^n}{(1 - q^{2n})^2} = \sum_{n=0}^{\infty} p_{3.14}(n) q^n$$

, then . . .

Out [5] = $p_{3.14}(n)$

$$= \sum_{k=1}^{\infty} \sum_{h=0}^{k-1} \frac{2e^{-\frac{2ihn\pi}{k}} \left\lfloor \frac{1}{\gcd(h,k)} \right\rfloor \left(2 \cosh \left(\frac{\sqrt{8n-1}\pi}{2k} \right) - \frac{4k \sinh \left(\frac{\sqrt{8n-1}\pi}{2k} \right)}{\sqrt{8n-1}\pi} \right)}{\sqrt{k}(8n-1)\omega(h, k)} \\ \times \chi(\gcd(2, k) = 2) \omega \left(h, \frac{k}{2} \right)^2$$

CHAPTER 3

RADEMACHER-TYPE FORMULAE FOR FOURIER COEFFICIENTS OF RAMANUJAN-TYPE SERIES

3.1 Ramanujan-type identities

There are numerous Ramanujan-type infinite products. Each product could be represented as an infinite series in variable q . We use our Mathematica package to extract the coefficients of q^n for the given infinite product.

In his lost notebook [1, 2], Ramanujan recorded these identities:

$$\sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q; q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1 - q^{6n})^2}{(1 - q^n)(1 - q^{12n})} \quad [1, \text{Entry 11.3.1, p. 254}] \quad (3.1)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2; q^2)_n}{(q; q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1 - q^{3n})(1 - q^{12n})}{(1 - q^n)(1 - q^{6n})} \quad [1, \text{Entry 11.3.2, p. 254}] \quad (3.2)$$

$$\sum_{n=0}^{\infty} \frac{q^n(-q^2; q^2)_n}{(q; q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2(1 - q^{12n})^2}{(1 - q^n)^2(1 - q^{6n})(1 - q^{4n})} \quad [1, \text{Entry 11.3.3, p. 254}] \quad (3.3)$$

$$\sum_{n=0}^{\infty} \frac{q^n(-q; q)_{2n}}{(q; q)_n(-q; q)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^{6n})^2}{(1 - q^n)(1 - q^{3n})} \quad [1, \text{Entry 11.3.4, p. 255}] \quad (3.4)$$

$$\sum_{n=0}^{\infty} \frac{q^n(-q; q^2)_n}{(q; q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1 - q^{4n})(1 - q^{6n})^5}{(1 - q^n)(1 - q^{2n})(1 - q^{3n})^2(1 - q^{12n})^2} \quad [1, \text{Entry 11.3.5, p. 255}] \quad (3.5)$$

$$\sum_{n=0}^{\infty} \frac{(-q)^n(q; q^2)_n}{(-q; -q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^2(1 - q^n)(1 - q^{4n})^2}{(1 - q^{2n})^4(1 - q^{6n})} \quad [1, \text{Entry 11.3.5, p. 255}] \quad (3.6)$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q; q^2)_{2n}^2}{(q^2; q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^2}{(1 - q^{2n})(1 - q^{6n})} \quad [2, \text{Entry 5.3.3, p. 102}] \quad (3.7)$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2} (q^3; q^6)_n}{(q; q^2)_n (q^2; q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1 - q^{9n})(1 - q^{6n})^2}{(1 - q^{2n})(1 - q^{3n})(1 - q^{18n})} \quad [2, \text{Entry 5.3.4, p. 103}] \quad (3.8)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(q^4; q^4)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})(1 - q^{3n})^2}{(1 - q^n)(1 - q^{4n})(1 - q^{6n})} \quad [2, \text{Entry 5.3.6, p. 104}] \quad (3.9)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (-q; q^2)_n}{(q^4; q^4)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - q^{4n})} \quad [2, \text{Entry 5.3.6, p. 104}] \quad (3.10)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n} (-q; q^2)_n}{(q^4; q^4)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^{6n})^2}{(1 - q^{3n})(1 - q^{4n})} \quad [2, \text{Entry 5.3.7, p. 105}] \quad (3.11)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2} (-q^3; q^6)_n}{(q^2; q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1 - q^{8n})(1 - q^{2n})^2(1 - q^{12n})^2}{(1 - q^{24n})(1 - q^{4n})^2(1 - q^n)(1 - q^{6n})} \quad [2, \text{Entry 5.3.8, p. 105}] \quad (3.12)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2} (q^3; q^6)_n}{(q^4; q^4)_n (q; q^2)_n^2} = \prod_{n=1}^{\infty} \frac{(1 - q^{4n})(1 - q^{6n})}{(1 - q^n)(1 - q^{12n})} \quad [2, \text{Entry 5.3.9, p. 106}] \quad (3.13)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(q^2; q^2)_n^2} = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - q^{2n})^2} \quad [2, \text{Entry 4.2.6, p. 84}] \quad (3.14)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-q; q^2)_n q^{n^2}}{(q^4; q^4)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - q^{4n})} \quad [2, \text{Entry 5.3.6, p. 104}] \quad (3.15)$$

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n^2 q^{2n^2}}{(q^2; q^2)_{2n}} = \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^2}{(1 - q^{2n})(1 - q^{6n})} \quad [2, \text{Entry 5.3.3, p. 102}] \quad (3.16)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(q^4; q^4)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^n)(1 - q^{6n})^5}{(1 - q^{3n})^2(1 - q^{2n})^2(1 - q^{12n})^2} \quad [2, \text{Entry 4.2.7, p. 85}] \quad (3.17)$$

The following identities are due to Slater:

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2; q^2)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - q^{2n})} \quad [8, \text{p. 152, Eq. (3, 23)}] \quad (3.18)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q; q)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^n)} \quad [8, \text{p. 152, Eq. (2)}] \quad (3.19)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n+1)}}{(q^2; q^2)_n (-q; q^2)_{n+1}} = \prod_{n=1}^{\infty} \frac{(1 - q^n)(1 - q^{4n})}{(1 - q^{2n})^2} \quad [8, \text{p. 152, Eq. (5)}] \quad (3.20)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2; q^2)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^{4n})}{(1 - q^{2n})} \quad [8, \text{p. 152, Eq. (7)}] \quad (3.21)$$

$$\sum_{n=0}^{\infty} \frac{(-1; q^2)_n q^{n(n+1)}}{(q; q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1 - q^{4n})(1 - q^{6n})^5}{(1 - q^{2n})^2 (1 - q^{3n})^2 (1 - q^{12n})^2} \quad [8, \text{p. 152, Eq. (48)}] \quad (3.22)$$

The following identities are due to Bowman, McLaughlin and Sills:

$$\sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{2n^2}}{(q; q^2)_n (q^4; q^4)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^{6n})^5}{(1 - q^{2n})(1 - q^{3n})^2 (1 - q^{12n})^2} \quad [5, \text{p. 314, Eq. (2.19)}] \quad (3.23)$$

$$\sum_{n=0}^{\infty} \frac{(q; q^2)_n q^{2n^2}}{(-q; q)_{2n} (q^2; q^2)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^2}{(1 - q^{6n})(1 - q^{2n})} \quad [5, \text{p. 314, Eq. (2.19)}] \quad (3.24)$$

The following identity is due to Starcher

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} (-1; q)_n}{(q; q)_n^2} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^n)^2} \quad [12, \text{p. 805, Eq. (3.6)}] \quad (3.25)$$

3.2 Formulae for the Fourier Coefficients of the series presented in 3.2

Let $\Pi_j(q)$ represent the right hand side of Eq.(j), where j runs from (3.1) to (3.25) and let $\tilde{p}_j(n)$ be defined by

$$\sum_{n=0}^{\infty} \tilde{p}_j(n) q^n = \Pi_j(q).$$

Our Mathematica package conjectures the following Rademacher-type formulae. These formulae appear to be new:

$$\begin{aligned}
\tilde{p}_{3.1}(n) = & \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{4\sqrt{3}k}\right) \omega(12h, k) \omega(h, k)}{k\sqrt{24n-1} \omega(6h, k)^2} \\
& + \sum_{\substack{k=1 \\ (12,k)=12}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{6k}\right) \omega\left(h, \frac{k}{12}\right) \omega(h, k)}{k\sqrt{24n-1} \omega\left(h, \frac{k}{6}\right)^2} \\
& + \sum_{\substack{k=1 \\ (12,k)=4}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{6k}\right) \omega\left(3h, \frac{k}{4}\right) \omega(h, k)}{\sqrt{3}k\sqrt{24n-1} \omega\left(3h, \frac{k}{2}\right)^2}. \tag{3.26}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_{3.2}(n) = & \sum_{\substack{k=1 \\ (12,k)=6}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right) \omega\left(h, \frac{k}{6}\right) \omega(h, k)}{2k\sqrt{3n+1} \omega\left(h, \frac{k}{3}\right) \omega\left(2h, \frac{k}{6}\right)} \\
& + \sum_{\substack{k=1 \\ (12,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right) \omega\left(3h, \frac{k}{2}\right) \omega(h, k)}{2\sqrt{3}k\sqrt{3n+1} \omega(3h, k) \omega\left(6h, \frac{k}{2}\right)} \\
& + \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{3n+1}\pi}{\sqrt{6}k}\right) \omega(6h, k) \omega(h, k)}{4k\sqrt{3n+1} \omega(3h, k) \omega(12h, k)}. \tag{3.27}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_{3.3}(n) = & \sum_{\substack{k=1 \\ (12,k)=3}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\sqrt{5}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{\frac{5}{2}}\sqrt{3n+2}\pi}{3k}\right) \omega\left(2h, \frac{k}{3}\right) \omega(4h, k) \omega(h, k)^2}{8k\sqrt{3n+2} \omega(2h, k)^2 \omega\left(4h, \frac{k}{3}\right)^2} \\
& + \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\sqrt{\frac{5}{3}}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{\frac{5}{2}}\sqrt{3n+2}\pi}{3k}\right) \omega(4h, k) \omega(6h, k) \omega(h, k)^2}{8k\sqrt{3n+2} \omega(2h, k)^2 \omega(12h, k)^2}. \tag{3.28}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_{3.4}(n) &= \sum_{\substack{k=1 \\ (6,k)=3}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right) \omega\left(h, \frac{k}{3}\right) \omega(h, k)}{2\sqrt{2}k\sqrt{3n+1} \omega\left(2h, \frac{k}{3}\right)^2} \\
&+ \sum_{\substack{k=1 \\ (6,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{3n+1}\pi}{3k}\right) \omega(3h, k) \omega(h, k)}{2\sqrt{6}k\sqrt{3n+1} \omega(6h, k)^2}. \tag{3.29}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_{3.5}(n) &= \sum_{\substack{k=1 \\ (12,k)=3}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{12k}\right) \omega\left(h, \frac{k}{3}\right)^2 \omega(h, k) \omega(2h, k) \omega\left(4h, \frac{k}{3}\right)^2}{2k\sqrt{120n+5} \omega\left(2h, \frac{k}{3}\right)^5 \omega(4h, k)} \\
&+ \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{12k}\right) \omega(h, k) \omega(2h, k) \omega(3h, k)^2 \omega(12h, k)^2}{2\sqrt{3}k\sqrt{120n+5} \omega(4h, k) \omega(6h, k)^5}. \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_{3.6}(n) &= \sum_{\substack{k=1 \\ (12,k)=6}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{6k}\right) \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{2}\right)^4}{k\sqrt{120n+5} \omega\left(h, \frac{k}{3}\right)^2 \omega(h, k) \omega\left(2h, \frac{k}{2}\right)^2} \\
&+ \sum_{\substack{k=1 \\ (12,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{5\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{120n+5}\pi}{6k}\right) \omega\left(3h, \frac{k}{2}\right) \omega\left(h, \frac{k}{2}\right)^4}{\sqrt{3}k\sqrt{120n+5} \omega(h, k) \omega\left(2h, \frac{k}{2}\right)^2 \omega(3h, k)^2}. \tag{3.31}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_{3.7}(n) &= \sum_{\substack{k=1 \\ (6,k)=6}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{2}\right)}{k\sqrt{12n-1} \omega\left(h, \frac{k}{3}\right)^2} \\
&+ \sum_{\substack{k=1 \\ (6,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(3h, \frac{k}{2}\right) \omega\left(h, \frac{k}{2}\right)}{\sqrt{3}k\sqrt{12n-1} \omega(3h, k)^2}. \tag{3.32}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_{3.8}(n) &= \sum_{\substack{k=1 \\ (18,k)=18}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h, \frac{k}{18}\right) \omega\left(h, \frac{k}{3}\right) \omega\left(h, \frac{k}{2}\right)}{k\sqrt{12n-1} \omega\left(h, \frac{k}{9}\right) \omega\left(h, \frac{k}{6}\right)^2} \\
&+ \sum_{\substack{k=1 \\ (18,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{10\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{60n-5}\pi}{9k}\right) \omega(3h, k) \omega\left(9h, \frac{k}{2}\right) \omega\left(h, \frac{k}{2}\right)}{3\sqrt{3}k\sqrt{60n-5} \omega\left(3h, \frac{k}{2}\right)^2 \omega(9h, k)} \\
&+ \sum_{\substack{k=1 \\ (18,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\sqrt{\frac{2}{3}}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{2}\sqrt{12n-1}\pi}{9k}\right) \omega(2h, k) \omega(3h, k) \omega(18h, k)}{3k\sqrt{12n-1} \omega(6h, k)^2 \omega(9h, k)}.
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
\tilde{p}_{3.9}(n) &= \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{4\sqrt{3}k}\right) \omega(4h, k) \omega(6h, k) \omega(h, k)}{3k\sqrt{8n-1} \omega(2h, k) \omega(3h, k)^2} \\
&+ \sum_{\substack{k=1 \\ (12,k)=12}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{4}\right) \omega(h, k)}{k\sqrt{8n-1} \omega\left(h, \frac{k}{3}\right)^2 \omega\left(h, \frac{k}{2}\right)} \\
&+ \sum_{\substack{k=1 \\ (12,k)=4}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{4}\right) \omega\left(3h, \frac{k}{2}\right) \omega(h, k)}{\sqrt{3}k\sqrt{8n-1} \omega\left(h, \frac{k}{2}\right) \omega(3h, k)^2}.
\end{aligned} \tag{3.34}$$

$$\tilde{p}_{3.10}(n) = \sum_{\substack{k=1 \\ (4,k)=4}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2e^{-\frac{2i\pi hn}{k}} \pi I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{4}\right)}{k\sqrt{8n-1} \omega(h, k)}. \tag{3.35}$$

$$\begin{aligned}
\tilde{p}_{3.11}(n) &= \sum_{\substack{k=1 \\ (12,k)=4}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+5\pi}}{2\sqrt{3k}}\right) \omega\left(h, \frac{k}{4}\right) \omega(3h, k)}{k\sqrt{24n+5} \omega\left(3h, \frac{k}{2}\right)^2} \\
&+ \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+5\pi}}{12k}\right) \omega(4h, k) \omega(3h, k)}{\sqrt{3k}\sqrt{24n+5} \omega(6h, k)^2} \\
&+ \sum_{\substack{k=1 \\ (12,k)=3}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+5\pi}}{12k}\right) \omega\left(h, \frac{k}{3}\right) \omega(4h, k)}{k\sqrt{24n+5} \omega\left(2h, \frac{k}{3}\right)^2}. \tag{3.36}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_{3.12}(n) &= \sum_{\substack{k=1 \\ (24,k)=24}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1\pi}}{2k}\right) \omega\left(h, \frac{k}{24}\right) \omega\left(h, \frac{k}{6}\right) \omega(h, k) \omega\left(h, \frac{k}{4}\right)^2}{k\sqrt{8n-1} \omega\left(h, \frac{k}{12}\right)^2 \omega\left(h, \frac{k}{8}\right) \omega\left(h, \frac{k}{2}\right)^2} \\
&+ \sum_{\substack{k=1 \\ (24,k)=4}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\sqrt{5}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{5\sqrt{8n-1\pi}}}{6k}\right) \omega(h, k) \omega\left(3h, \frac{k}{2}\right) \omega\left(6h, \frac{k}{4}\right) \omega\left(h, \frac{k}{4}\right)^2}{3k\sqrt{8n-1} \omega\left(h, \frac{k}{2}\right)^2 \omega\left(2h, \frac{k}{4}\right) \omega\left(3h, \frac{k}{4}\right)^2} \\
&+ \sum_{\substack{k=1 \\ (24,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\sqrt{\frac{5}{2}}\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{5\sqrt{8n-1\pi}}}{12k}\right) \omega(h, k) \omega(4h, k)^2 \omega(6h, k) \omega(24h, k)}{3k\sqrt{8n-1} \omega(2h, k)^2 \omega(8h, k) \omega(12h, k)^2} \\
&+ \sum_{\substack{k=1 \\ (24,k)=3}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1\pi}}{4k}\right) \omega(h, k) \omega\left(2h, \frac{k}{3}\right) \omega(4h, k)^2 \omega\left(8h, \frac{k}{3}\right)}{\sqrt{2k}\sqrt{8n-1} \omega(2h, k)^2 \omega\left(4h, \frac{k}{3}\right)^2 \omega(8h, k)}. \tag{3.37}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_{3.13}(n) &= \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1\pi}}{3\sqrt{2k}}\right) \omega(12h, k) \omega(h, k)}{3k\sqrt{8n-1} \omega(4h, k) \omega(6h, k)} \\
&+ \sum_{\substack{k=1 \\ (12,k)=12}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1\pi}}{2k}\right) \omega\left(h, \frac{k}{12}\right) \omega(h, k)}{k\sqrt{8n-1} \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{4}\right)}. \tag{3.38}
\end{aligned}$$

$$\tilde{p}_{3.14}(n) = \sum_{\substack{k=1 \\ (2,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2e^{-\frac{2i\pi hn}{k}} \omega\left(h, \frac{k}{2}\right)^2 \left(2 \cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right) - \frac{4k \sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\pi\sqrt{8n-1}}\right)}{k(8n-1)^{3/4} \sqrt{\frac{\sqrt{8n-1}}{k}} \omega(h, k)}. \quad (3.39)$$

$$\tilde{p}_{3.15}(n) = \sum_{\substack{k=1 \\ (4,k)=4}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{4}\right)}{k\sqrt{8n-1} \omega(h, k)}. \quad (3.40)$$

$$\begin{aligned} \tilde{p}_{3.16}(n) &= \sum_{\substack{k=1 \\ (6,k)=6}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{2}\right)}{k\sqrt{12n-1} \omega\left(h, \frac{k}{3}\right)^2} \\ &+ \sum_{\substack{k=1 \\ (6,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(3h, \frac{k}{2}\right) \omega\left(h, \frac{k}{2}\right)}{\sqrt{3k}\sqrt{12n-1} \omega(3h, k)^2}. \end{aligned} \quad (3.41)$$

$$\begin{aligned} \tilde{p}_{3.17}(n) &= \sum_{\substack{k=1 \\ (12,k)=4}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2e^{-\frac{2ihn\pi}{k}} \pi I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(3h, \frac{k}{4}\right)^2 \omega(3h, k)^2 \omega\left(h, \frac{k}{2}\right)^2}{\sqrt{3k}\sqrt{8n-1} \omega(h, k) \omega\left(3h, \frac{k}{2}\right)^5} \\ &+ \sum_{\substack{k=1 \\ (12,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{4e^{-\frac{2ihn\pi}{k}} \pi I_1\left(\frac{\sqrt{8n-1}\pi}{2\sqrt{3}k}\right) \omega(3h, k)^2 \omega\left(6h, \frac{k}{2}\right)^2 \omega\left(h, \frac{k}{2}\right)^2}{3k\sqrt{8n-1} \omega(h, k) \omega\left(3h, \frac{k}{2}\right)^5} \\ &+ \sum_{\substack{k=1 \\ (12,k)=12}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2e^{-\frac{2ihn\pi}{k}} \pi I_1\left(\frac{\sqrt{8n-1}\pi}{2k}\right) \omega\left(h, \frac{k}{12}\right)^2 \omega\left(h, \frac{k}{3}\right)^2 \omega\left(h, \frac{k}{2}\right)^2}{k\sqrt{8n-1} \omega\left(h, \frac{k}{6}\right)^5 \omega(h, k)} \end{aligned} \quad (3.42)$$

$$\tilde{p}_{3.18}(n) = \sum_{\substack{k=1 \\ (2,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n-1}\pi}{6k}\right) \omega\left(h, \frac{k}{2}\right)}{k\sqrt{24n-1} \omega(h, k)}. \quad (3.43)$$

$$\tilde{p}_{3.19}(n) = \sum_{\substack{k=1 \\ (2,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+1}\pi}{6\sqrt{2k}}\right) \omega(h, k)}{k\sqrt{24n+1} \omega(2h, k)}. \quad (3.44)$$

$$\tilde{p}_{3.20}(n) = \sum_{\substack{k=1 \\ (4,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{24n+1}\pi}{3\sqrt{2k}}\right) \omega\left(h, \frac{k}{2}\right)^2}{k\sqrt{24n+1} \omega(h, k) \omega\left(2h, \frac{k}{2}\right)}. \quad (3.45)$$

$$\begin{aligned} \tilde{p}_{3.21}(n) &= \sum_{\substack{k=1 \\ (4,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n+1}\pi}{3\sqrt{2k}}\right) \omega\left(h, \frac{k}{2}\right)}{k\sqrt{12n+1} \omega\left(2h, \frac{k}{2}\right)} \\ &+ \sum_{\substack{k=1 \\ (4,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n+1}\pi}{6\sqrt{2k}}\right) \omega(2h, k)}{2k\sqrt{12n+1} \omega(4h, k)}. \end{aligned} \quad (3.46)$$

$$\begin{aligned} \tilde{p}_{3.22}(n) &= \sum_{\substack{k=1 \\ (12,k)=3}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{n}\pi}{\sqrt{2k}}\right) \omega\left(h, \frac{k}{3}\right)^2 \omega(2h, k)^2 \omega\left(4h, \frac{k}{3}\right)^2}{4k\sqrt{n} \omega\left(2h, \frac{k}{3}\right)^5 \omega(4h, k)} \\ &+ \sum_{\substack{k=1 \\ (12,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{\frac{2}{3}}\sqrt{n}\pi}{k}\right) \omega\left(h, \frac{k}{2}\right)^2 \omega(3h, k)^2 \omega\left(6h, \frac{k}{2}\right)^2}{3k\sqrt{n} \omega\left(2h, \frac{k}{2}\right) \omega\left(3h, \frac{k}{2}\right)^5} \\ &+ \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{n}\pi}{\sqrt{2k}}\right) \omega(2h, k)^2 \omega(3h, k)^2 \omega(12h, k)^2}{4\sqrt{3}k\sqrt{n} \omega(4h, k) \omega(6h, k)^5}. \end{aligned} \quad (3.47)$$

$$\begin{aligned}
\tilde{p}_{3.23}(n) &= \sum_{\substack{k=1 \\ (12,k)=3}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{6k}\right) \omega(2h, k) \omega\left(4h, \frac{k}{3}\right)^2 \omega\left(h, \frac{k}{3}\right)^2}{k\sqrt{12n-1} \omega\left(2h, \frac{k}{3}\right)^5} \\
&+ \sum_{\substack{k=1 \\ (12,k)=12}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h, \frac{k}{12}\right)^2 \omega\left(h, \frac{k}{2}\right) \omega\left(h, \frac{k}{3}\right)^2}{k\sqrt{12n-1} \omega\left(h, \frac{k}{6}\right)^5} \\
&+ \sum_{\substack{k=1 \\ (12,k)=4}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h, \frac{k}{2}\right) \omega\left(3h, \frac{k}{4}\right)^2 \omega(3h, k)^2}{\sqrt{3k}\sqrt{12n-1} \omega\left(3h, \frac{k}{2}\right)^5} \\
&+ \sum_{\substack{k=1 \\ (12,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{6k}\right) \omega(2h, k) \omega(3h, k)^2 \omega(12h, k)^2}{\sqrt{3k}\sqrt{12n-1} \omega(6h, k)^5}. \tag{3.48}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_{3.24}(n) &= \sum_{\substack{k=1 \\ (6,k)=6}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(h, \frac{k}{6}\right) \omega\left(h, \frac{k}{2}\right)}{k\sqrt{12n-1} \omega\left(h, \frac{k}{3}\right)^2} \\
&+ \sum_{\substack{k=1 \\ (6,k)=2}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{2\pi e^{-\frac{2i\pi hn}{k}} I_1\left(\frac{\sqrt{12n-1}\pi}{3k}\right) \omega\left(3h, \frac{k}{2}\right) \omega\left(h, \frac{k}{2}\right)}{\sqrt{3k}\sqrt{12n-1} \omega(3h, k)^2}. \tag{3.49}
\end{aligned}$$

$$\tilde{p}_{3.25}(n) = \sum_{\substack{k=1 \\ (2,k)=1}}^{\infty} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{e^{-\frac{2ihn\pi}{k}} \left(2 \cosh\left(\frac{\sqrt{n}\pi}{k}\right) - \frac{2k \sinh\left(\frac{\sqrt{n}\pi}{k}\right)}{\sqrt{n}\pi}\right) \omega(h, k)^2}{8\sqrt{k}n\omega(2h, k)}. \tag{3.50}$$

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Appendix A

RADEMACHER.M MATHEMATICA PACKAGE

```
Print["The Rademacher Package"]
```

```
Print["Version of January 19, 2011"]
```

```
Print["V. Kiria and A. Sills"]
```

```
FindL[ simplifiedinput_List] :=
```

```
Apply[LCM,PowersOfQ[simplifiedinput]]
```

```
PowersOfQ[simplifiedinput_List] :=
```

```
Table[ simplifiedinput[[i,1]] ,{i,1,Length[simplifiedinput]}]
```

```
PowersOfP[simplifiedinput_List] :=
```

```
Table[ simplifiedinput[[i,2]] ,{i,1,Length[simplifiedinput]}]
```

```
BigOmega[d_Integer,h_Symbol,k_Symbol,
```

```
omega_Symbol,simplifiedinput_List] :=
```

```
Module[{numfactors,ppowers,qpowers},
```

```
numfactors=Length[simplifiedinput];
```

```
ppowers=PowersOfP[simplifiedinput];
```

```
qpowers=PowersOfQ[simplifiedinput];
```

```
Product[ (omega[h,k ]/.{
```

```
k-> k/GCD[d,qpowers[[i]]],
```

```
h-> qpowers[[i]]*h/GCD[d,qpowers[[i]]]
```

```
}]
```

```
)^ppowers[[i]]      ,{i,1,numfactors}]
]
```

```
BigZ[ d_Integer, z_Symbol, simplifiedinput_List]:=
Module[{numfactors,ppowers,qpowers},
numfactors=Length[simplifiedinput];
ppowers=PowersOfP[simplifiedinput ];
qpowers=PowersOfQ[simplifiedinput];
Product[ (Sqrt[z]/.{
z->  qpowers[[i]]*z/GCD[d,qpowers[[i]]]
}
)^ppowers[[i]]      ,{i,1,numfactors}]
]
```

```
BigExp[d_Integer,z_Symbol,k_Symbol,simplifiedinput_List]:=
Module[{numfactors,ppowers,qpowers},
numfactors=Length[simplifiedinput];
ppowers=PowersOfP[simplifiedinput ];
qpowers=PowersOfQ[simplifiedinput];
Collect[Simplify[
Sum[
ppowers[[i]]*(Pi (1/z - z)/12/k)/.
{k-> k/GCD[d,qpowers[[i]]},
z->  qpowers[[i]]*z/GCD[d,qpowers[[i]]}]
,{i,1,numfactors}
```

```

]
]
,{z,z^(-1)}
]
]

```

```

BigP[d_Integer,P_Symbol,qprime_Symbol,simplifiedinput_List]:=
Module[{numfactors,ppowers,qpowers},
numfactors=Length[simplifiedinput];
ppowers=PowersOfP[simplifiedinput];
qpowers=PowersOfQ[simplifiedinput];
Product[ P[qprime^(GCD[d,qpowers[[i]]]^2/qpowers[[i]])]^ppowers[[i]]
,{i,1,numfactors}]
]

```

```

OnePieceModularRHS[d_Integer,h_Symbol,k_Symbol,z_Symbol,omega_Symbol,
P_Symbol,qprime_Symbol,simplifiedinput_List ]:=
{BigOmega[d,h,k,omega,simplifiedinput],BigZ[ d, z, simplifiedinput],
BigExp[d,z,k,simplifiedinput],BigP[d,P,qprime,simplifiedinput]}

```

```

ModularRHS[h_Symbol,k_Symbol,z_Symbol,omega_Symbol,
P_Symbol,qprime_Symbol,simplifiedinput_List ]:=
Module[{d,L},
L=FindL[ simplifiedinput];
d=Divisors[L];

```

```
Table[{d[[i]],OnePieceModularRHS[ d[[i]], h,k,z,omega,
P,qprime,simplifiedinput]},{i,1, Length[d]}]
]
```

```
ZToZetaIntegrandComponents[n_,h_,k_,z_,
zeta_,omega_,simplifiedinput_]:=
Module[{extractchange,transstuff,P,Q},
transstuff=ModularRHS[h,k,z,omega,P,Q,simplifiedinput];
extractchange =
Table[
{transstuff[[i,1]],
{transstuff[[i,2,1]],
transstuff[[i,2,2]],2 Pi n z/k + transstuff[[i,2,3]]
}/.{z-> zeta/k}
} ,{i,Length[transstuff]}]
]
```

```
ExtractNonTrivialCases[n_,h_,k_,z_,zeta_,
omega_,simplifiedinput_]:=Module[{tmp},
tmp=ZToZetaIntegrandComponents[n,h,k,z,
zeta,omega,simplifiedinput];
Complement[
Table[
If[ Coefficient[tmp[[i,2,3]], zeta,-1]>0,
tmp[[i]]
```

```

]
, {i,1,Length[tmp]}],
{Null}
]
]

ZetaToTTransformation[ expo_, zeta_, t_]:=
zeta-> Coefficient[ expo, zeta,-1]/t

DZetaToDt[expo_,zeta_,dzeta_,t_,dt_]:=
dzeta-> D[Extract[ZetaToTTransformation[
expo, zeta, t],2],t]dt

IntegrandInT[n_,h_,k_,zeta_,dzeta_,omega_,
t_,dt_, simplifiedinput_]:= Module[{z, zetacases,tmp},
zetacases=ExtractNonTrivialCases[
n,h,k,z,zeta,omega,simplifiedinput];
(*Print[zetacases];*)
tmp=
Table[
{zetacases[[i,1]],
{I/k^2 *Exp[-2 Pi I n h/k ],
zetacases[[i,2,3]]/.ZetaToTTransformation[
zetacases[[i,2,3]], zeta, t] ,
zetacases[[i,2,1]],

```



```

zetacases[[i,2,2]]/.ZetaToTTransformation[
  zetacases[[i,2,3]], zeta, t],
dzeta/.DZetaToDt[zetacases[[i,2,3]],zeta,dzeta,t,dt]
}
},{i,1,Length[zetacases]}
];
(*Print[tmp];*)
Table[
{tmp[[i,1]] ,
{tmp[[i,2,1]]*(tmp[[i,2,5]]*tmp[[i,2,4]])/.{t->1,dt-> 1},
Collect[Together[tmp[[i,2,2]]],{t,t^(-1)}],
Exponent[ tmp[[i,2,5]] ,t]+
Exponent[PowerExpand[tmp[[i,2,4]] ],t],
tmp[[i,2,3]],
dt
}
},
,{i,1,Length[tmp]}]
]

```

```

RawFormula[n_,h_,k_,omega_,simplifiedinput_]:=
Module[{i,temp,zeta,dzeta,t,dt},temp=IntegrandInT[
n,h,k,zeta,dzeta,omega,t,dt, simplifiedinput];
(*Print[temp];*)
Table[{temp[[i,1]],

```

```

PowerExpand[
temp[[i,2,1]]*2*Pi*I/
PowerExpand[Simplify[(Sqrt[Coefficient[temp[[i,2,2]],
t^(-1)]))^(-(1+temp[[i,2,3]])]
*temp[[i,2,4]]*
BESSELI[-(1+temp[[i,2,3])],(2PowerExpand[Simplify[
Sqrt[Coefficient[temp[[i,2,2]],t^(-1)]])]]
]
}
,{i,1,Length[temp]}]
]

FormattedSummand[ n_,h_,k_,omega_,simplifiedinput_]:=
Module[{i,tmp,L},
L=FindL[simplifiedinput];
tmp=RawFormula[n,h,k,omega,simplifiedinput];
Sum[
tmp[[i,2]]*Floor[1/GCD[h,k]]*\[Chi][GCD[k,L]==tmp[[i,1]] ]
,{i,1,Length[tmp]}]//PowerExpand
]

DedekindSum[h_,k_]:=
Sum[(\[Mu]/k-Floor[\[Mu]/k]-1/2)*(h*\[Mu]/k-Floor[
h*\[Mu]/k]-1/2),{\[Mu],1,k-1}]

```

```

Omega[h_,k_] :=
Exp[Pi*I*DedekindSum[h,k]]

NearlyHeadlessNick[simplifiedinput_,q_,MaxN_] :=
Series[Product[(1/QPochhammer[q^(simplifiedinput[
[i,1]])])^simplifiedinput[[i,2]],
{i,1,Length[simplifiedinput]}],{q,0,MaxN}]

TestFormula[ n_,h_, k_,omega_,
simplifiedinput_, MaxK_,MaxN_] :=
Module[{calcvals,i,tmp,series,
summand,q,seriesvals},
summand=FormattedSummand[n,h,k,omega,
simplifiedinput];
tmp=Sum[ summand ,{k,1,MaxK},{h,0,k-1}];
tmp=tmp/.{BESSELI->
BesselI,omega->Omega, \[Chi][True]-> 1,
\[Chi][False]-> 0 };
(*Print[tmp];*)
calcvals=Table[ N[tmp/.n-> i],{i,1,MaxN}];
series= NearlyHeadlessNick[simplifiedinput,q,MaxN];
seriesvals=Table[ Coefficient[series,q,i] ,{i,1,MaxN}];
{Re[calcvals]-seriesvals, Im[calcvals]}
]

```

```

PrettyFormula[n_,h_,k_,omega_,simplifiedinput_]:=
Module[{fs,rf,L},
L=FindL[simplifiedinput ];
rf=RawFormula[n,h,k,omega,simplifiedinput]/.BESSELI-> BesselI;
(*Sum[
Sum[ HoldForm [TraditionalForm[Evaluate[rf[[i,2]] ] *
\[Chi][ GCD[h,k]==1 && GCD[k,L]==rf[[i,1]] ] ]]
,{k,1,Infinity},{h,0,k-1}]
,{i,Length[rf]} ]*)
(*fs=ToString[FormattedSummand[
n,h,k,omega,simplifiedinput]/.BESSELI->BesselI,
TraditionalForm]*)

fs = PowerExpand[ FormattedSummand[n,h,k,omega,
simplifiedinput]/.BESSELI->BesselI]//TraditionalForm;
(*SUM[ fs, {k,1,Infinity},{h,0,k-1}]*)
fs
]

ProductToList[ expr_, q_:q,f_:f,
psi_:\[Psi], phi_:\[Phi] ]:=
Module[{tmp,tmplist,a,b,c,p,r,numer,
denom,ex,subs2,subs3,subs4,subs6},
tmp=expr;
tmp=tmp/. f[a_,b_]-> QPochhammer[-a, a*b]

```

```

QPochhammer[-b, a*b] QPochhammer[a*b];
tmp= tmp/. psi[-q^a_]->
QPochhammer[q^(2a)]/QPochhammer[-q^a, q^(2a)];
tmp=tmp/. psi[q^a_]->
QPochhammer[q^(2a)]/QPochhammer[q^a, q^(2a)];
tmp=tmp/. phi[-q^a_]->
QPochhammer[q^a]/QPochhammer[-q^a];
tmp=tmp/. phi[q^a_]->
QPochhammer[-q, q^(2a)]^2 QPochhammer[q^(2a)];
tmp=tmp/. QPochhammer[ a_, -q^p_]->
QPochhammer[ a, q^(2p)]*QPochhammer[-a*q^p, q^(2p)];
tmp = tmp/. QPochhammer[-q^r_., q^p_]->
QPochhammer[q^(2r), q^(2p)]/QPochhammer[q^r, q^p];
numer=Numerator[tmp]/. QPochhammer[a_, b_]^c_->
QPochhammer[a, b];
If[numer[[0]]=== Times,
numer=numer/.Times-> List,
numer={numer}];
If[verbose, Print["numer=", numer]];
denom=Denominator[tmp]/. QPochhammer[a_, b_]^c_->
QPochhammer[a, b];
If[denom[[0]]===Times,
denom=denom/.Times-> List,
denom={denom}];
If[verbose, Print["denom=", denom]];

```

```

ex=Union[Table[Exponent[numer[[i,2]],q],{i,1,Length[numer]}],
Table[Exponent[denom[[i,2]],q],{i,1,Length[denom]}
];
If[verbose,Print[ex]];
ex=Union[Flatten[Table[Divisors[ex[[i]]],{i,1,Length[ex]}]];
If[verbose,Print[ex]];
subs2= Table[
QPochhammer[q^(ex[[i]]/2),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/2)]/QPochhammer[q^ex[[i]]]
,{i,1,Length[ex]}];
If[verbose,Print["subs2 = ",subs2]];
tmp=tmp/.subs2;
If[verbose,Print["after subs 2, tmp = ",tmp]];
subs3= Table[
QPochhammer[q^(ex[[i]]/3),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/3)]/QPochhammer[q^ex[[i]]]/
QPochhammer[q^(2ex[[i]]/3),q^ex[[i]]]
,{i,1,Length[ex]}];
If[verbose,Print["subs3 = ",subs3]];
tmp=tmp/.subs3;
If[verbose,Print["after subs3, tmp = ",tmp]];
subs4= Table[
QPochhammer[q^(ex[[i]]/4),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/4),q^(ex[[i]]/2)]
QPochhammer[q^(3ex[[i]]/4),q^ex[[i]]]

```

```

,{i,1,Length[ex]}}];
If[verbose,Print["subs4 = ",subs4]];
tmp=(tmp/.subs4)/.subs2;
If[verbose,Print["after subs4 & subs2, tmp = ",tmp ] ];
subs6= Table[
QPochhammer[q^(ex[[i]]/6),q^ex[[i]]]->
QPochhammer[q^(ex[[i]]/6) ,q^(ex[[i]]/3)]/
QPochhammer[q^(5ex[[i]]/6),q^ex[[i]]]/
QPochhammer[ q^(ex[[i]]/2) ,q^(ex[[i]])]
,{i,1,Length[ex]}}];
If[verbose,Print["subs6 = ",subs6]];
tmp=(tmp/.subs6)/.subs2;
If[verbose,Print["after subs6 & subs2, tmp = ",tmp ] ];
(*tmp=(((tmp/.subs2)/.subs3)/.subs4)/.subs6;
If[verbose,Print["after repeated subs, tmp = ",tmp ] ];*)
numer=Numerator[tmp];
denom=Denominator[tmp];

If[numer[[0]]===Times,numer= numer/.Times->List,numer={numer}];
If[numer==={1},numer={}];
If[verbose,Print["numer = ",numer ] ];
If[denom[[0]]===Times,denom=denom/.Times->List,denom={denom}];
If[denom==={1},denom={}];
If[verbose,Print["denom = ",denom ] ];
Join[(numer/. QPochhammer[a_]^b_ ->

```

```
{Log[a]/Log[q],-b})//PowerExpand,
(denom/. QPochhammer[a_]^b_. ->
{Log[a]/Log[q],b})//PowerExpand
]
]
```

```
RademacherTypeFormula[
prodexpr_,a_:a, eqcode_:{}, n_:n,h_:h,k_:k,omega_:\[Omega]]:=
Module[{pf},
pf=PrettyFormula[
n,h,k,omega,ProductToList[ prodexpr, q,f, \[Psi], \[Phi] ]];
Print["If ", TraditionalForm[prodexpr==
HoldForm[Sum[Subscript[a,eqcode][n] q^n ,{n,0,Infinity}]]]]
Print[" then . . ." ];
TraditionalForm[Subscript[a,eqcode][n]== pf]
]
```

```
LaTeXSummand[prodexpr_,a_:a, eqcode_:{}, n_:n,h_:h,k_:k,
omega_:\[Omega]]:=Module[{pf,simplifiedinput},
simplifiedinput=ProductToList[prodexpr];
TeXForm[Subscript[a,eqcode][n]==
FormattedSummand[ n,h,k,omega,
simplifiedinput]/.{BESSELI->BesselI,Floor[GCD[h, k]^(-1)]->1}
]
]
```



```
(*
RTF[prodepr_,a_:a, eqcode_:{}, n_:n,h_:h,k_:k,omega_:\[Omega]] :=
Module[{pf},
pf=
  Style[\[CapitalSigma]\[CapitalSigma],Large]
  PrettyFormula[n,h,k,omega,ProductToList[
    prodepr, q,f, \[Psi], \[Phi] ]];
Print["If ", TraditionalForm[prodepr==
HoldForm[Sum[Subscript[a,eqcode][n] q^n ,{n,0,Infinity}]]]]
Print[" , then . . ." ];
TraditionalForm[Subscript[a,eqcode][n]== pf]
]
*)
```

```
ListToProduct[simplifiedinput_List,q_:q,n_:n] :=
Product[(1-q^(
simplifiedinput[[i,1]]n))^(-simplifiedinput[[i,2]]),
{i,1,Length[simplifiedinput]} ]
```

```
MakeBoxes[MySum[s_,{k_, k0_, k1_},
{n_, n0_, n1_}],TraditionalForm] :=
RowBox[{UnderoverscriptBox["\[Sum]",RowBox[
{MakeBoxes[k,TraditionalForm], "=",
MakeBoxes[k0,TraditionalForm]}],
MakeBoxes[k1,TraditionalForm]]],
```

```

RowBox[{UnderoverscriptBox["\[Sum]", RowBox[
{MakeBoxes[n, TraditionalForm], "="},
MakeBoxes[n0, TraditionalForm]}],
MakeBoxes[n1, TraditionalForm]],
MakeBoxes[s, TraditionalForm]}]}]

```

```

MakeBoxes[MyProduct[s_, {k_, k0_, k1_}], TraditionalForm] :=
RowBox[{UnderoverscriptBox["\[
Product]", RowBox[{MakeBoxes[k, TraditionalForm], "=",
MakeBoxes[k0, TraditionalForm]}],
MakeBoxes[k1, TraditionalForm]],
MakeBoxes[s, TraditionalForm]}]}]

```

The Rademacher Package

Version of January 19, 2011

V. Kiria and A. Sills

```

RTF[prodexpr_, a_:a, eqcode_: {},
n_:n, h_:h, k_:k, omega_:\[Omega]] :=
Module[{pf},
pf=
TraditionalForm[
MySum[

```

```

    PrettyFormula[n,h,k,omega,ProductToList[
    prodexpr, q,f, \[Psi], \[Phi] ]],
    {k,1,Infinity},{h,0,k-1}
  ]
]
;
Print["If ", TraditionalForm[
  MyProduct[ ListToProduct[ProductToList[prodexpr] ],
  {n,1,Infinity}] ==
HoldForm[Sum[Subscript[a,eqcode][n] q^n ,{n,0,Infinity}]]]
Print[" , then . . ." ];
TraditionalForm[Subscript[a,eqcode][n]== pf]
]

verbose=False;

```

