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ON DIAMOND-ALPHA DYNAMIC EQUATIONS AND INEQUALITIES

by

NURIYE ATASEVER

(Under the Direction of Billûr Kaymakçalan)

ABSTRACT

In view of the recently developed theory of calculus for dynamic equations on time scales (which unifies discrete and continuous systems), in this project we give some of the basics of the extension of the theory to the combined delta (forward) and nabla (backward) derivatives. In this set up the newly developed theory of diamond-alpha derivatives are analyzed through some equation and inequality properties. In particular Opial type Diamond-alpha dynamic Inequalities are discussed in this context and recently developed results and their improved versions are given in this work.

Key Words: Discrete, Continuous, Time Scales, Dynamic Equations, Delta Derivatives, Nabla Derivatives, Diamond-alpha Derivatives, Opial's inequality.

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iii

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Difference equations appear as natural descriptions of observed evolution phenomena, because most measurements of time evolving variables are discrete and as such, these equations are in their own right important mathematical models. More importantly, since nonlinear problems cannot be solved, for such equations discretization methods which make use of difference equations are employed. Several results in difference equations have been obtained as more or less natural discrete analogs of corresponding results of differential equations. Furthermore, the application of the theory of difference equations to various fields such as numerical analysis, control theory, finite mathematics, computer science, probability theory, queuing problems, statistical problems, stochastic time series, etc. is rapidly increasing. Therefore, difference equations are not merely the discrete analogs of differential equations, in fact they have led the way for the development of the latter. In [1] several examples from the diverse fields have been illustrated which are sufficient to convey the importance of the serious qualitative as well as quantitative study of difference equations.

In recent years discrete time dynamical systems have experienced rapidly growing popularity thereby paving its way into an independent discipline [1, 51]. They are no longer only auxiliary systems for continuous time dynamical systems through discretization but rather they have developed their own independent right to exist. Despite this tendency of independence; however, there is a well-known striking similarity or even duality between the two concepts and therefore the discrete systems are commonly treated "along the lines" of continuous time theory. Nevertheless, the two kinds of systems have a number of significant differences mainly due to the topological fact that in one case the time scale and the corresponding trajectories are

connected while in the other case they are not. All the investigations on the two time scales show that much of the analysis is analogous but, at the same time, usually additional assumptions are needed in the discrete case in order to overcome the topological deficiency of lacking connectedness.

In view of many well-known analogies in the concepts of difference calculus with the difference operator

$$(\Delta_h f)(t) = \frac{f(t+h) - f(t)}{h}$$

on one hand and differential calculus with the differential operator

$$(\frac{df}{dt})(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

on the other hand, it is an obvious desire to have a better way of dealing with problems treated in a parallel manner in both continuous and discrete time scales by establishing a method for theory which allows us to systematically handle both time scales simultaneously. In the context of such a theory it should also be possible to get some insight and better understanding of the sometimes subtle differences between the two types of systems. To create the desired theory, first of all we require to set up a certain structure $\mathbb{T}$ which is to play the role of the time scale generalizing $\mathbb{R}$ and $\mathbb{Z}$. Furthermore, an operation on the space of functions from $\mathbb{T}$ to the state space has to be defined generalizing the differential and difference operations.

Perhaps from a modeling point of view, it is more realistic to model a phenomenon by a dynamic system which incorporates both continuous and discrete times, namely, time as an arbitrary closed set of reals. It is therefore natural to ask whether it is possible to provide a framework which allows us to handle both dynamic systems simultaneously so that we can get some insight and a better understanding of the subtle differences of these two systems. The answer is affirmative and the recently

developed theory of "dynamic systems on time scales" "or dynamic systems on measure chains", (by a measure chain we mean the union of disjoint closed intervals of $\mathbb{R}$) offers the desired unified approach.

In [13, 14, 15, 34, 35, 36], Aulbach and Hilger have initiated the development of this theory with the aim of treating dynamic problems from a qualitative point of view. A calculus on measure chains meeting the requirements described above is developed and in this general framework some basic results for linear dynamic systems are given.

Later in [39, 41, 43, 48, 49] Kaymakçalan and others have extended this theory to a unified analysis of nonlinear systems from the point of view of investigating qualitative and quantitative behaviors of such systems. Most of these results are contained in the monograph [42], which is the first book containing extensive coverage of up-to-date results in the area of "Time-Scales" or "Measure Chains". Recently, many new results in the area have been obtained, and the two monographs [19, 20] by Bohner and Peterson, give very thorough insight into some of the more contemporary developments in the area.

Having the basics of the Theory of Time-Scales at our disposal, in this thesis we want to focus our attention to obtaining unified results in the context of recently developed theory of combined derivatives, namely diamond-alpha derivatives, diamond-alpha equations and inequalities. In particular we will focus on inequalities which involve integrals of functions and derivatives, namely Opial Type inequalities. It has been shown that these types of inequalities are of great importance in many areas of mathematics with applications in the theory of differential equations, approximations and probability. For detailed investigations of Opial's type inequalities

their several generalizations, extensions and discretizations along with applications such as their role in establishing the existence and uniqueness of solutions to initial and boundary value problems for ordinary and partial differential equations as well as difference equations, the recently published monograph [7], which is the first book dedicated to the theory of Opial type inequalities, serves as an excellent reference. In addition to the above mentioned applications, many qualitative behaviours such as oscillation, non-oscillation, boundedness, have also been discussed [7] in the light of Opial's inequality .

We begin this work, with giving the basics of Time-Scales in Chapter 2. Throughout $\mathbb{T}$ denotes a time-scale (any closed subset of $\mathbb{R}$ with order and topological structure defined in a canonical way) with $t_0 \geq 0$ as a minimal element. In Section 2.1, some of the essential features of the order and topological structure of time-scales are introduced, with the concepts of rd-continuity and ld-continuity being given in Section 2.2. The concepts and results given in sections 2.3, 2.4, 2.5, 2.6 and 2.7 form the basics of the calculus developed and include those features necessary for our Opial inequality unification purpose.

In Chapter 3 basics of the recently developed theory of dynamic equations using diamond-alpha derivatives, which are convex combinations of the delta and nabla derivatives are given. Along with the corresponding antiderivative and integral notions, the special features of the diamond-alpha exponential functions are studied. In particular the main difference compared to the delta or nabla exponential functions being not able to represent a diamond-alpha exponential function as the unique solution of a corresponding IVP is emphasized in detail. In order to discuss about such an unique solution corresponding to a diamond-alpha exponential function, the importance of either two initial conditions or two boundary conditions being provided

is also stressed.

In Chapter 4, a survey of some recently obtained Dynamic Inequalities in view of the newly developed theory of Diamond-alpha derivatives and integrals are given.

In Chapter 5, along with the basic Opial inequality given by Zdzidlaw Opial in 1960, we state some basic analogous discrete and continuous inequalities and results pertaining to them. Then in Section 5.1, the introduced inequalities are given in the time scale set-up along with generalizations. In these unifications the calculus on time-scales plays a vital role, the theorems are modified in a manner to reflect the differences of the discrete and continuous results at the same time. In Section 5.6 a sequence of Opial-type inequalities for first-order Diamond-alpha derivatives on time scales are given and in Section 5.8 the inequalities given in Section 5.6 are improved by removing some of the restrictions.

CHAPTER 2

BASIC CONCEPTS OF TIME SCALES

2.1 Order and Topological Structure

In this chapter we give the main definitions and characteristics of the calculus on time scales initiated by Aulbach and Hilger [13, 14, 15, 34, 35, 36] which comprise those features of the differential and difference calculus as they are relevant for the development of a qualitative theory of dynamical systems. We note that the contents of such a development of some higher ranging calculus is quite extensive. So we suffice only with giving the essentials necessary for the further aims of this work and refer to [13, 14, 19, 42] for more details.

Throughout this chapter we remark the many similarities and differences in considering the Time Scale as in the $\mathbb{R}$ or $\mathbb{Z}$ set-up.

We begin this chapter with highlighting the basics of the order and topological structure of Time Scale. In this context the special features of openness brings additional considerations into account. Next, we continue by giving the special type of Induction Principle that is used as a main tool in the arguments. We proceed by paying special attention to concepts such as Continuity, Rd-Continuity, Differentiability which possess important roles in the analysis of discrete and continuous scales in a unified manner. As a result of these concepts, we end this chapter by considering Integrals and Exponential Functions for Time Scales, and give some so-called Useful Time Scales Formulas that will be employed throughout the thesis.

As subsets of $\mathbb{R}$, time scales carry an order structure in a canonical way. A time scale $\mathbb{T}$ may be bounded above or below. As a consequence of the embedding of $\mathbb{T}$ in $\mathbb{R}$, all order theoretical notions such as bounds, least upper bounds, greatest lower

bounds and intervals are available in $\mathbb{T}$ as they are in $\mathbb{R}$.

We note that the order structure of $\mathbb{R}$ induces an order structure on each time-scale $\mathbb{T}$. On time scales, there exist primarily two order structures that should be distinguished from each other. But this distinction is easily seen to be only figurative; since it can be shown that due to the closedness assumption of time-scales, the $\mathbb{R}$ or $\mathbb{T}$ suffixes need not be mentioned. Hence, when order theoretical concepts are concerned the $\mathbb{R}$ or $\mathbb{T}$ specifications can be dropped and for instance; bound, boundedness or supremum concepts can be used, instead of $\mathbb{R}$-boundedness, $\mathbb{T}$-boundedness, $\mathbb{R}$-supremum, $\mathbb{T}$-supremum, etc.

As a consequence of the definition of a time-scale $\mathbb{T}$ being a closed subset of $\mathbb{R}$, topological structure of $\mathbb{T}$, especially from the openness point of view has various features. Obviously any subset A of $\mathbb{T}$ which is open in $\mathbb{R}$, is also open in $\mathbb{T}$. The reverse is generally not true, though, as the simple example $\mathbb{T} := \mathbb{Z}$ shows, where any subset in the induced topology is open in $\mathbb{T}$ but not open in $\mathbb{R}$. This is taken care of by distinguishing between $\mathbb{R}$-openness and $\mathbb{T}$-openness. In order to investigate the details of the notion of openness in time-scales, we must define the concept of neighborhood in this set-up. We give two different versions of neighborhood definitions, distinguishing between the concepts of $\mathbb{R}$-neighborhood and $\mathbb{T}$-neighborhood, giving way to the distinction between $\mathbb{R}$-openness and $\mathbb{T}$-openness.

Given a time-scale $\mathbb{T}, t \in \mathbb{T}$ and an $\epsilon > 0$, we denote by

$$\mathbb{R}_\epsilon(t) := \{x \in \mathbb{R} : t - \epsilon < x < t + \epsilon\}$$

$$\mathbb{T}_\epsilon(t) := \{x \in \mathbb{T} : t - \epsilon < x < t + \epsilon\}$$

the ϵ-neighborhoods of t in $\mathbb{R}$ and $\mathbb{T}$, respectively.

An interval, in the time-scale context, is always understood as the intersection

of a real interval with a given time-scale.

The following definition will lead the way to the concept of $\mathbb{T}$-openness.

Definition 2.1.1. Let $\mathbb{T}$ be a time-scale and $t \in \mathbb{T}$. The set $U \subseteq \mathbb{R}$ is called an $\mathbb{R}$-neighborhood of t provided that there is $\epsilon > 0$ with $\mathbb{R}_\epsilon(t) \subseteq U$. The set $V \subseteq \mathbb{T}$ is called a $\mathbb{T}$-neighborhood of t, provided that there is $\epsilon > 0$ with $\mathbb{T}_\epsilon(t) \subseteq V$.

Neighborhood concepts give rise to further topological notions.

Definition 2.1.2. A subset A of a time-scale $\mathbb{T}$ is open in $\mathbb{T}$ if for each $t \in A$ there is an $\epsilon > 0$ such that $\mathbb{T}_\epsilon(t) \subseteq A$.

Remark 2.1.1. For any time-scale $\mathbb{T}$, ϕ and $\mathbb{T}$ are open in $\mathbb{T}$.

The next result enables us to observe a connection between the concepts of $\mathbb{T}$-openness and $\mathbb{R}$-openness.

Theorem 2.1.1. A set $A \subseteq \mathbb{T}$ is open in $\mathbb{T}$ if and only if there is a set $B \subseteq \mathbb{R}$, open in $\mathbb{R}$, with $A = B \cap \mathbb{T}$.

Next, we give the following result which was mentioned earlier and remark once again that the converse is not true.

Theorem 2.1.2. If A is a subset of $\mathbb{T}$, then A is open in $\mathbb{R}$ implies that A is open in $\mathbb{T}$.

Although, due to the closedness of $\mathbb{T}$, the notion of closedness in $\mathbb{T}$ coincides with that of $\mathbb{R}$, also the definition of $\mathbb{T}$-openness gives way in a natural manner to the following concept of closedness.

Definition 2.1.3. A subset A of $\mathbb{T}$ is called closed in $\mathbb{T}$ provided that $\mathbb{T} \setminus A$ is open in $\mathbb{T}$.

Having mentioned the concepts of openness and closedness for time-scales; we can also remark about other topological concepts such as compactness and connectedness and their special features in time-scales.

Definition 2.1.4. A subset A of a time-scale $\mathbb{T}$ is called compact in $\mathbb{T}$ provided that A is bounded and closed in $\mathbb{T}$.

From the connectedness point of view, there can not be one single notion that applies to all time scales, and we conclude that a time-scale $\mathbb{T}$ may or may not be connected. In order to overcome this topological deficiency, the concept of jump operators were introduced [34].

Definition 2.1.5. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, such that

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

are called jump operators. In the case $\mathbb{T}$ is bounded above, we supplement the definition by $\sigma(\max \mathbb{T}) := \max \mathbb{T}$ and accordingly $\rho(\min \mathbb{T}) := \min \mathbb{T}$ if $\mathbb{T}$ is bounded below.

These jump operators enable us to classify the points $\{t\}$ of a time-scale as right-dense, right-scattered, left-dense and left-scattered depending on whether $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively, for any $t \in \mathbb{T}$.

Remark 2.1.2. :
(1)So far, we have not specified any direction in a time-scale and used both (positive

and negative) directions in a symmetric manner. But from now on we have to cease this symmetric consideration, since the time-scale calculus will naturally include the classical difference operation as a special case and needless to mention, symmetry will automatically be destroyed in the development of this "difference" process. Hence, we will consider the direction for a time-scale $\mathbb{T}$ to be in the sense of increasing values of t, for $t \in \mathbb{T}$.

(2) If a time-scale $\mathbb{T}$ has a maximal element, which is moreover left-scattered, then this point plays a particular role in several respects and therefore we call it degenerate. All other elements of $\mathbb{T}$ are called non-degenerate and the subset of non-degenerate points of $\mathbb{T}$ is denoted by $\mathbb{T}^k$. Since each closed subset A of a time-scale $\mathbb{T}$ is also a time-scale, it is possible that A^k is formed. Naturally $A = A^k$ is possible as long as A does not have a left-scattered maximum. Likewise $\mathbb{T}_k$ is defined as the set $\mathbb{T}_k = \mathbb{T}\,[\inf \mathbb{T}, \sigma(\inf \mathbb{T})]$ if $|\inf \mathbb{T}| < \infty$, and $\mathbb{T}_k = \mathbb{T}$ if $\inf \mathbb{T} = -\infty$.

Having characterized the points of a time-scale by use of jump operators we can remark about the topology on $\mathbb{T}$ and show that the so-called order topology (interval topology) on $\mathbb{T}$ is compatible with the metric topology on $\mathbb{T}$, which is induced on $\mathbb{T}$ as a consequence of the embedding of $\mathbb{T}$ in $\mathbb{R}$.

To show this, let $t_0 \in \mathbb{T}$ and consider the neighborhood systems in both topologies. In the open interval topology (order topology), the neighborhood system is generated by open intervals of the form (t_1, t_2) such that $t_1 < t_0 < t_2$. In the metric topology, the neighborhood system is generated by open balls of the form $\{t \in \mathbb{T} : |t - t_0| < \epsilon; \epsilon \in \mathbb{R}_+\}$. It is easy to observe that corresponding to each member of the first system, we can find a member of the second contained in it and vice versa.

Let $t_1, t_2 \in \mathbb{T} \cup \{-\infty, +\infty\}$ such that $t_1 < t_0 < t_2$. Clearly, there exists $\epsilon > 0$

such that $t_1 \leq t_0 - \epsilon < t_0 < t_0 + \epsilon \leq t_2$. Thus $\{t \in \mathbb{T} : |t - t_0| < \epsilon\} \subseteq (t_1, t_2)$.

Conversely, if $\delta > 0$ is given, then $\{t \in \mathbb{T} : |t - t_0| < \delta\}$ is a neighborhood in the metric topology on $\mathbb{T}$. We consider the interval $(\sigma(t_0 - \delta), \rho(t_0 + \delta))$, where σ, ρ are the jump operators given by Definition 2.1.5 Then $s \in (\sigma(t_0 - \delta), \rho(t_0 + \delta)) \cap \mathbb{T}$ implies

$$t_0 - \delta \leq \sigma(t_0 - \delta) < s < \rho(t_0 + \delta) \leq t_0 + \delta.$$

Hence any interval in the order topology is contained in a neighborhood in the metric topology. So the (interval) topology on $\mathbb{T}$ is metrizable. In fact we can give the following theorem.

Theorem 2.1.3. A time-scale $\mathbb{T}$ equipped with the order topology is metrizable and is a K_σ-space; i.e. it is a union of at most countably many compact sets. The metric on $\mathbb{T}$ which generates the order topology is given by

$$d(r, s) := |\mu(r, s)|$$

where $\mu(\cdot) = \mu(\cdot, \tau)$ for a fixed $\tau \in \mathbb{T}$ is defined as follows:

Definition 2.1.6. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$ such that $\mu(t) = \sigma(t) - t$ is called *graininess*.

When $\mathbb{T} = \mathbb{R}$, $\mu(t) \equiv 0$ and for $\mathbb{T} = \mathbb{Z}$, $\mu(t) \equiv 1$, if $\mathbb{T} = h\mathbb{Z}$, $\mu(t) = h$.

Definition 2.1.7. The mapping $\nu : \mathbb{T}_\kappa \to \mathbb{R}_0^+$ such that $\nu(t) = t - \rho(t)$ is called *backwards graininess*.

2.2 Continuity, Rd-Continuity and Ld-Continuity

In order to introduce the concept of integration on time-scales, we need a notion which is related to the approximation of continuous functions by step-functions.

Definition 2.2.1. If a function is defined on a compact interval $[t_a, t_b]$ of a time-scale $\mathbb{T}$ and if there are finite number of elements $t_0, \cdots, t_n$ of $\mathbb{T}$ with $t_a = t_0 < t_1 < \cdots < t_n = t_b$ and such that $f : [t_a, t_b] \to \mathbb{R}$ is constant on $[t_i, t_{i+1}]$, for $i = 1, 2, \cdots, n-1$, then f is called a *step-function.*

There is no difficulty in defining the continuity concept for $\mathbb{R}^n$-valued functions on Time Scales. The continuity definition can be taken from Real Analysis without any alteration.

Definition 2.2.2. The function $f : \mathbb{T} \to \mathbb{R}$ is said to be *continuous* at $t_0 \in \mathbb{T}$ if for all $\epsilon > 0$, there exists a neighborhood $U(t_0)$ such that

$$|f(t) - f(t_0)| < \epsilon \quad \text{for all} \quad t \in U(t_0).$$

In analysis on $\mathbb{R}$ discontinuity points are usually given graphically by jump points. But since Time Scales are not connected in general, a similar result need not be necessary.

In order to pave our way to the concept of integration, we first have to obtain an appropriate class of functions having anti-derivatives. For this purpose we define the following notions.

Definition 2.2.3. Let X be an arbitrary topological space and $\mathbb{T}$ a time-scale. The mapping $g : \mathbb{T} \to X$ is said to be regulated if at each left-dense $t \in \mathbb{T}, g(t^-) = \lim_{s \to t^-} g(s)$ exists and at each right-dense point $t \in \mathbb{T}, g(t^+) = \lim_{s \to t^+} g(s)$ exists.

Definition 2.2.4. The mapping $g : \mathbb{T} \to X$ is called *rd-continuous* if

(i) it is continuous at each right-dense or maximal $t \in \mathbb{T}$,

(ii) at each left-dense point, left sided limit $g(t^-)$ exists.

We denote by $C_{rd}[\mathbb{T}, X]$ the set of rd-continuous mappings from $\mathbb{T}$ to X. The class of rd-continuous functions turns out to be a "natural" class within the context of the Time Scale calculus. The function $\mu : \mathbb{T} \to \mathbb{R}$ in case $\mathbb{T} := [0, 1] \cup \mathbb{N}$, for example, is rd-continuous but not continuous at 1.

The following implications are immediate:

$$\text{continuous} \Rightarrow \text{rd- continuous} \Rightarrow \text{regulated.}$$

If $\mathbb{T}$ contains *ldrs*-points (left-dense and right-scattered) then the first implication is not invertible. However, on a discrete time scale all three notions coincide.

As a generalization of Definition 2.2.4, we give the following.

Definition 2.2.5. The mapping $f : \mathbb{T}^k \times X \to X$ is called *rd-continuous* if it

(i) is continuous at each (t, x) with right-dense or maximal t,

and

(ii) the limits $f(t^-, x) : \lim\limits_{\substack{(s,y) \to (t,x) \\ s < t}} f(s, y)$ and $\lim\limits_{y \to x} f(t, y)$ exist at each (t, x) with left-dense t.

Hence, in general for left-dense t, the function $f(t, \cdot) : \mathbb{T}^k \times X \to X$ is in no way a continuous continuation of the mapping $f : (-\infty, t) \times X \to X$ to the point t.

Example 2.2.1. Given an rd-continuous function $g : \mathbb{T} \to X_1$, which is in the sense of Definition 2.3.4, a continuous function $h : X_2 \to X_3$ and another continuous function $f : X_1 \times X_2 \to X_3$, the composite function $f(g(\cdot), h(\cdot))$ is rd-continuous in the sense of Definition 2.3.5.

We conclude this section by introducing a tool which is useful in some qualitative properties and give a relevant result to this section.

Definition 2.2.6. Consider the mapping $f^{\tau]} : (-\infty, \tau] \times X \to X$ which is defined for a fixed $\tau \in \mathbb{T}^k$ as:

$$f^{\tau]}(t, x) = \begin{cases} f(t, x), & \text{if} \quad (t, x) \in (-\infty, \tau) \times X \\ f(\tau^-, x), & \text{if} \quad (t, x) \in \{\tau\} \times X. \end{cases} \tag{2.3.1}$$

Here f is assumed to be rd-continuous on $\mathbb{T} \times X$. $f^{\tau]}$ does not necessarily coincide with f on $(-\infty, \tau]$ if τ is a ldrs-point, otherwise it does.

Using Definition 2.2.6, we can give the following result which is related to rd-continuous functions.

Lemma 2.2.1. For each continuous function $x : \mathbb{T} \to X$, the mapping $f^{\tau]}(\cdot, x(\cdot))$ is rd-continuous with respect to the interval $(-\infty, \tau]$, where $f \in C_{rd}[\mathbb{T} \times X, X]$.

Definition 2.2.7. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. We say that f is ld-continuous if it is continuous at each left-dense point in T and $\lim_{s \to t^+} f(s)$ exists as a finite number for all right-dense points $t \in T$.

2.3   Delta Derivative

When we consider functions which are defined on a time-scale $\mathbb{T}$ and taking their values in a topological space X, due to the embedding of $\mathbb{T}$ in $\mathbb{R}$, the concept of continuity arises in a straightforward manner. For the concept of differentiation, however, the topological structure of $\mathbb{T}$ plays a vital role. In fact the generally lacking openness of $\mathbb{T}$ requires a particular proceeding when one aims at a concept of differentiability which contains as special cases the differential calculus on one hand and the difference calculus on the other.

Definition 2.3.1. Let $\mathbb{T}$ be a time-scale and $f : \mathbb{T} \to \mathbb{R}$. f is called *delta differentiable* at $t_0 \in \mathbb{T}^k$, if there exists an $a \in \mathbb{R}$ with the following property:

For any $\epsilon > 0$, there exists a neighborhood U of t_0, such that

$$\left| f(\sigma(t_0)) - f(t) - (\sigma(t_0) - t)a \right| \leq \epsilon |\sigma(t_0) - t| \tag{2.4.1}$$

for all $t \in U$. We denote a as $f^\Delta(t_0)$.

$f : \mathbb{T} \to X$, where X is any Banach Space, is called delta differentiable if f is differentiable for each $t \in \mathbb{T}$.

Remark 2.3.1. In the two special cases $\mathbb{R}$ and $\mathbb{Z}$ the delta derivative is uniquely determined, in fact one gets $a = \frac{df}{dt}(t_0)$ and $a = f(t_0 + 1) - f(t_0)$, respectively.

Theorem 2.3.1. If $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at a non-degenerate point t_0, then from the Definition 2.4.1, $a \in \mathbb{R}$ is uniquely determined. It is denoted by $f^\Delta(t_0)$ and called the delta derivative of f at t_0.

Remark 2.3.2. Sometimes we may need the generalized delta derivatives correspond-

ing to Dini derivatives from the right, in which case we write

$$f(\sigma(t)) - f(s) - (\sigma(t) - s)a < \epsilon(\sigma(t) - s)$$

for all $s \in W$; W being a right-neighborhood of $t \in \mathbb{T}$. We denote in this case $a = D^+ f^\Delta(t)$.

Note that if t is right-scattered, then $D^+ f^\Delta(t)$ is the same as the $f^\Delta(t)$ given above. In this case we write

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{(\sigma(t) - t)}$$

The relationship between differentiability and continuity follows from Real Analysis.

Theorem 2.3.2. If $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t_0 \in \mathbb{T}$, then it is continuous at t_0.

Theorem 2.3.3. If f is continuous at t_0 and t_0 is right-scattered, then f is delta differentiable at t_0 with derivative

$$f^\Delta(t_0) = \frac{f(\sigma(t_0)) - f(t_0)}{\sigma(t_0) - t_0}$$

as indicated in Remark 2.4.2.

Example 2.3.1. Let $\mathbb{T} := \{x \in \mathbb{R} : -2 \leq x \leq 0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and $f(x) := |x|$, $f_n(x) = |x - \frac{1}{n}|$, $n \in \mathbb{N}$, for all $x \in \mathbb{T}$.

Then all functions f_n's are delta differentiable for all points of $\mathbb{T}$, but the limit function f has no delta derivative at 0, hence it is not delta differentiable.

Example 2.3.2. For $\mathbb{T} := \{0\} \cup \bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R} : \frac{1}{2n} \leq x \leq \frac{1}{2n-1}\}$ and

$$f(x) := \begin{cases} 0, & \text{if} \quad x = 0 \\ \frac{1}{2n}, & \text{if} \quad \frac{1}{2n} \leq x \leq \frac{1}{2n-1} \end{cases}$$

which gives, $f^\Delta(0) = 1$.

2.4   Nabla Derivative

Following the delta dynamic equations' development, in [11] the corresponding theory for nabla derivatives was studied extensively. Most of the following are in view of [11] and the subsequent works.

Definition 2.4.1. Let $\mathbb{T}$ be a time-scale and $f : \mathbb{T} \to \mathbb{R}$. f is called *nabla differentiable* at $t_0 \in \mathbb{T}$, if there exists an $a \in \mathbb{R}$ with the following property: For any $\epsilon > 0$, there exists a neighborhood U of t_0, such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \leq \epsilon|\rho(t) - s|$$

for all $t \in U$.

Theorem 2.4.1. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_\kappa$. Then we have the following:

(i) If f is nabla differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is left scattered, then f is nabla differentiable at t with

$$f^\nabla(t) = \frac{f(t) - f(\rho)}{\nu(t)}$$

(iii) If f is left-dense, then f is nabla differentiable at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is nabla differentiable at t, then

$$f(\rho(t)) = f(t) - \nu(t) f^{\nabla}(t).$$

Theorem 2.4.2. Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_{\kappa}$, Then:

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with

$$(f + g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t).$$

(ii) The product $fg : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t, and we get the product
rules

$$(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f^{\rho}(t)g^{\nabla}(t) = f(t)g^{\nabla}(t) + f^{\nabla}(t)g^{\rho}(t).$$

(iii) If $g(t)g^{\rho}(t) \neq 0$, then f/g is nabla differentiable at t, and we get the quotient
rule

$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)g(t) - f(t)g^{\nabla}(t)}{g(t)g^{\rho}(t)}$$

2.5 Antiderivative and Integral

After the development of the Time-Scale Analysis in the previous sections up to
the concept of delta and nabla differentiability, now the concepts of delta and nabla

antiderivative (primitive) and integration are presented. For this purpose, restricting ourselves to the class of differentiable functions, we consider the definition of the immediate concepts of antidifferentiation.

Once the main theorem, which guarantees the existence of a primitive (antiderivative) function for rd-continuous (respectively ld-continuous for nabla derivative) functions, has been given, we must introduce the concept of Cauchy-Integral which is an essential tool for further purposes.

In the following, $\mathbb{T}$ is a Time-Scale and $\mathbb{T}'$ is a subinterval of $\mathbb{T}$.

Definition 2.5.1. Let $f : \mathbb{T}' \to \mathbb{R}$ be a delta differentiable function. The function

$$f^{\Delta} : \begin{cases} (\mathbb{T})^k \to \mathbb{R} \\ t \to f^{\Delta}(t) \end{cases}$$

is called the *delta derivative* of f on $\mathbb{T}'$. In the case of $\mathbb{T}'= \mathbb{T}$, the statement "on $\mathbb{T}'$" disappears.

Remark 2.5.1. (i) From the Theorem 2.4.2, it is obvious that a mapping which is delta differentiable on $\mathbb{T}'$ is continuous.

(ii) If $\mathbb{T}' \subset \mathbb{R}$ is an interval, which is open in $\mathbb{R}$; then the above concept coincides with differential calculation.

Definition 2.5.2. A function $f : \mathbb{T}^k \to \mathbb{R}$ is called a delta antiderivative (primitive) of g on $\mathbb{T}'$, if it is differentiable on $\mathbb{T}'$ and for all $t \in \mathbb{T}'^k$, the condition $f^{\Delta}(t) = g(t)$ is satisfied.

As shown by the following theorem, just like in Real Analysis, there corresponds a delta anti-derivative for each rd-continuous function on Time-Scale.

Theorem 2.5.1. For any rd-continuous mapping $g : \mathbb{T}^k \to \mathbb{R}$, there exists a delta antiderivative function $f : \mathbb{T} \to \mathbb{R}$; $f : t \to \int_s^t g(s)\Delta s$, $s, t \in \mathbb{T}^k$.

Since, as a result of Definition 2.5.2 we can deduce that the difference of two delta antiderivative functions is constant, the following definition can be used similar to that in Real Analysis.

Definition 2.5.3. If the function $g : \mathbb{T}^k \to \mathbb{R}$ has a delta antiderivative function f on $[r, s] \subset \mathbb{T}$, then

$$\int_r^s g(t)\Delta t := f(s) - f(r)$$

is called the *Cauchy-Integral* from r to s of the function g.

For $\mathbb{T} = \mathbb{R}$ the Cauchy-integral coincides with the Riemann integral.
For $\mathbb{T} = h\mathbb{Z}$ where $h > 0$, the identity

$$\int_r^s g(t)\,\Delta t = \begin{cases} \displaystyle\sum_{i=\frac{r}{h}}^{\frac{s}{h}-1} g(ih)h & \text{if } s > r \\[2em] 0 & \text{if } s = r \\[2em] -\displaystyle\sum_{i=\frac{s}{h}}^{\frac{r}{h}-1} g(ih)h & \text{if } s < r \end{cases}$$

can be shown. For further properties of integrals on $\mathbb{T}$ we refer to [14].

Definition 2.5.4. A function $F : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\nabla(t) = f(t)$ holds for all $t \in \mathbb{T}_\kappa$. We then define the integral of f by

$$\int_a^t f(\tau)\nabla\tau = F(t) - F(a) \quad \text{for all} \quad t \in \mathbb{T}$$

Theorem 2.5.2. If f and f^∇ are continuous, then

$$\left(\int_a^t f(t,s)\nabla s\right)^\nabla = f(\rho(t),t) + \int_a^t f^\nabla(t,s)\nabla s$$

Theorem 2.5.3. (Existence of a nabla antiderivative) Every ld-continuous function has a nabla antiderivative.

Theorem 2.5.4. If $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous and $t \in \mathbb{T}_\kappa$, then

$$\int_{\rho(t)}^t f(\tau)\nabla\tau = f(t)\nu(t)$$

Theorem 2.5.5. If $a,b,c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f,g : \mathbb{T} \to \mathbb{R}$ are ld-continuous; then

(i) $\int_a^b [f(t) + g(t)]\nabla t = \int_a^b f(t)\nabla t + \int_a^b g(t)\nabla t$

(ii) $\int_a^b \alpha f(t)\nabla t = \alpha \int_a^b f(t)\nabla t$

(iii) $\int_a^b f(t)\nabla t = - \int_b^a f(t)\nabla t$

(iv) $\int_a^b f(t)\nabla t = \int_a^c f(t)\nabla t + \int_c^b f(t)\nabla t$

(v) $\int_a^b f(\rho(t))g^\nabla(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^\nabla(t)g(t)\nabla t$

(vi) $\int_a^b f(t)g^\nabla(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^\nabla(t)g(\rho(t))\nabla t$

(vii) $\int_a^a f(t)\nabla t = 0$

The following Theorem gives some relations between delta and nabla derivatives.

Theorem 2.5.6.

(i) Assume that $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$. Then f is nabla differentiable at t and

$$f^\nabla(t) = f^\Delta(\rho(t))$$

for $t \in \mathbb{T}^\kappa$ such that $\sigma(\rho(t)) = t$. If, in addition, f^Δ is continuous on $\mathbb{T}^\kappa$, then f is nabla differentiable at t and $f^\nabla(t) = f^\Delta(\rho(t))$ holds for any $t \in \mathbb{T}_\kappa$.

(ii) Assume that $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}_\kappa$. Then f is delta differentiable at t and

$$f^\Delta(t) = f^\nabla(\sigma(t))$$

for $t \in \mathbb{T}^\kappa$ such that $\rho(\sigma(t)) = t$. If , in addition, f^∇ is continuous on $\mathbb{T}_\kappa$, then f is delta differentiable at t and $f^\Delta(t) = f^\nabla(\sigma(t))$ holds for any $t \in \mathbb{T}^\kappa$.

2.6   The Exponential Function

We start this section by introducing the so-called Hilger complex plane, $\mathbb{C}_h$ and refer to [20, 36, 37] for many results concerning this concept.

Let $h > 0$ be fixed and define the Hilger complex numbers by

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}.$$

Also if $h = 0$, let $\mathbb{C}_0 := \mathbb{C}$.

Now recalling that the unique solution of the initial value problem

$$x' = \alpha x, \quad x(0) = 1$$

is the exponential function $x(t) = e^{\alpha t}$, we are motivated to the following definition

Definition 2.6.1. For $\alpha \in \mathbb{C}_h$, we define the delta exponential function $e_\alpha(t)$ on $h\mathbb{Z}$ to be the unique solution of the initial value problem

$$x^\Delta(t) = \alpha x(t), \quad t \in \mathbb{T} := h\mathbb{Z}, \qquad x(0) = 1. \tag{2.6.1}$$

The following theorem, whose proof can be found in the [36], gives the explicit formulation of the defined delta exponential function $e_\alpha(t)$, i.e. the unique solution of the I.V.P. (2.6.1)

Theorem 2.6.1. For $\alpha \in \mathbb{C}_h$, the delta exponential function $e_\alpha(t)$ is given by

$$e_\alpha(t) = (1 + \alpha h)^{\frac{t}{h}} \tag{2.6.2}$$

for $t \in \mathbb{T} := h\mathbb{Z}$.

We can define a more general exponential function as follows;

Definition 2.6.2. Assume $\alpha : \mathbb{T} \to \mathbb{C}$ is rd-continuous and $\alpha(t)\mu(t) + 1 \neq 0$ for $t \in \mathbb{T}^k$. Then for $t \in \mathbb{T}$, $s \in \mathbb{T}^k$, $e_\alpha(t, s)$ is defined to be the function such that for each fixed $s \in \mathbb{T}^k$, $e_\alpha(t, s)$ is the solution of the initial value problem

$$x^\Delta(t) = \alpha(t)x(t), \quad t \in \mathbb{T}^k, \quad x(s) = 1.$$

For each fixed s, we say that $\alpha(t)$ is the growth rate of the delta exponential function $e_\alpha(t, s)$

The above defined generalized delta exponential function $e_\alpha(t, s)$ can be explicitly given by;

Theorem 2.6.2. The explicit formulation of the delta exponential function $e_\alpha(t, s)$ is

$$e_\alpha(t, s) = \exp\left(\int_s^t \xi_{\mu(r)}(\alpha(r))\Delta r\right) \tag{2.6.3}$$

for $t, s \in \mathbb{T}$, where for $h > 0$, the so-called cylindrical transformation $\xi_h : \mathbb{C}_h \to Z_h$ is defined by

$$\xi_h(z) = \frac{1}{h}Log(1 + zh) \tag{2.6.4}$$

with Log being the principle logarithm function.

For $h = 0$, $\xi_0(z)$ is defined to be equal z for all $z \in \mathbb{C}$.

For further details we refer to [20] , [36] and [37].

Theorem 2.6.3. For $\alpha \in \mathbb{C}_h$, the nabla exponential function $\hat{e}_\alpha(t)$ is given by

$$\hat{e}_\alpha(t) = (1 - \alpha h)^{\frac{-t}{h}} \qquad (2.6.5)$$

for $t \in \mathbb{T} := h\mathbb{Z}$.

Definition 2.6.3. Assume $\alpha : \mathbb{T} \to \mathbb{C}$ is ld-continuous and $1 - \alpha(t)\nu(t) \neq 0$ for $t \in \mathbb{T}_k$. Then for $t \in \mathbb{T}$, $s \in \mathbb{T}_k$, $\hat{e}_\alpha(t, s)$ is defined to be the function such that for each fixed $s \in \mathbb{T}_k$, $\hat{e}_\alpha(t, s)$ is the solution of the initial value problem

$$x^\nabla(t) = \alpha(t)x(t), \quad t \in \mathbb{T}_k, \quad x(s) = 1.$$

For each fixed s, we say that $\alpha(t)$ is the growth rate of the nabla exponential function $\hat{e}_\alpha(t, s)$.

The above defined generalized nabla exponential function $\hat{e}_\alpha(t, s)$ can be explicitly given by;

Theorem 2.6.4. The explicit formulation of the nabla exponential function $\hat{e}_\alpha(t, s)$ is

$$\hat{e}_\alpha(t, s) = \exp\left(\int_s^t \hat{\xi}_{\nu(r)}(\alpha(r))\nabla r\right) \qquad (2.6.6)$$

for $t, s \in \mathbb{T}$, where for $h > 0$, the so-called cylindrical transformation $\hat{\xi}_h : \mathbb{C}_h \to Z_h$ is defined by

$$\hat{\xi}_h(z) = -\frac{1}{h}Log(1 - zh) \qquad (2.6.7)$$

with Log being the principle logarithm function.

For $h = 0$, $\hat{\xi}_0(z)$ is defined to be equal z for all $z \in \mathbb{C}$.

2.7   Some Useful Time Scale Formulas

The following formulas are useful and will be employed in the later chapters:

- $f^\sigma = f + \mu f^\Delta$;

- $f^\rho = f - \nu f^\nabla$;

- $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta$ ("Product Rule");

- $(fg)^\nabla = f^\nabla g + f^\rho g^\nabla$ ("Product Rule");

- $(f/g)^\Delta = (f^\Delta g - f g^\Delta)/(gg^\sigma)$ ("Quotient Rule");

- $(f/g)^\nabla = (f^\nabla g - f g^\nabla)/(gg^\rho)$ ("Quotient Rule").

We have that (see e.g. [14, Theorem 7])

$$f(t) \geq 0, \quad a \leq t < b \quad \text{implies} \quad \int_a^b f(t)\Delta t \geq 0. \tag{2.7.1}$$

Throughout this we assume that $0 \in \mathbb{T}$ and let $h > 0$ with $h \in \mathbb{T}$. Hence, if $\mathbb{T} = \mathbb{R}$, then $\int_0^h f(t)\Delta t = \int_0^h f(t)\mathrm{d}t$, and if $\mathbb{T} = \mathbb{Z}$, then $\int_0^h f(t)\Delta t = \sum_{t=0}^{h-1} f(t)$.

Other examples of time scales (to which our inequalities apply as well) are e.g.

$$h\mathbb{Z} = \{hk : k \in \mathbb{Z}\} \quad \text{for some} \quad h > 0,$$

$$q^{\mathbb{Z}} = \{q^k : k \in \mathbb{Z}\} \cup \{0\} \quad \text{for some} \quad q > 1$$

(which produces so-called *q-difference equations*),

$$\mathbb{N}^2 = \{k^2 : k \in \mathbb{N}\}, \quad \left\{\sum_{k=1}^n \frac{1}{k} : n \in \mathbb{N}\right\}, \quad \bigcup_{k\in\mathbb{Z}} [2k, 2k+1],$$

and the Cantor set.

We conclude this section with giving two simple consequences of the product rule as well as Hölder's inequality for time scales. Those results we shall need in Chapter 3 to come.

First,

$$(f^2)^\Delta = (f \cdot f)^\Delta = f^\Delta f + f^\sigma f^\Delta = (f + f^\sigma)f^\Delta, \qquad (2.7.2)$$

$$(f^2)^\nabla = (f \cdot f)^\nabla = f^\nabla f + f^\rho f^\nabla = (f + f^\rho)f^\nabla,$$

and in general, one can use mathematical induction to prove the formula

$$(f^{l+1})^\Delta = \left\{ \sum_{k=0}^{l} f^k (f^\sigma)^{l-k} \right\} f^\Delta, \quad l \in \mathbb{N}. \qquad (2.7.3)$$

Finally, Hölder's inequalities for time scales (see [21, Lemma 2.2 (iv)]) reads

Theorem 2.7.1. (Delta Hölder's inequality)

$$\int_0^h |f(t)g(t)|\Delta t \le \left\{ \int_0^h |f(t)|^p \Delta t \right\}^{\frac{1}{p}} \left\{ \int_0^h |g(t)|^q \Delta t \right\}^{\frac{1}{q}}, \qquad (2.7.4)$$

where $p > 1$ and $q = p/(p-1)$.

Correspondingly for nabla integrals the following holds:

Theorem 2.7.2. (Nabla Hölder's inequality) Let $f, g, h \in C_{ld}([a,b], \mathbb{R})$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$; then

$$\left\{ \int_a^b |h(x)||f(x)|^p \nabla x \right\}^{\frac{1}{p}} \left\{ \int_a^b |h(x)||g(x)|^q \nabla x \right\}^{\frac{1}{q}} \ge \int_a^b |h(x)||f(x)g(x)|\nabla x.$$

CHAPTER 3

DIAMOND-ALPHA DYNAMIC EQUATIONS

3.1   Differentiation

Definition 3.1.1. Let $\mathbb{T}$ be a time scale, $\alpha \in [0,1]$, $\mu_{ts} = \sigma(t) - s$, $\nu_{ts} = \rho(t) - s$, and $f : \mathbb{T} \to \mathbb{R}$. The $\Diamond_\alpha$-derivative of f at t is defined to be the value $f^{\Diamond_\alpha}(t)$, if it is exist, such that for all $\epsilon > 0$ there is a neighborhood $U \subseteq \mathbb{T}$ of t such that for all $s \in U$,

$$|\alpha[f^\sigma(t) - f(s)]\nu_{ts} + (1-\alpha)[f^\rho(t) - f(s)]\mu_{ts} - f^{\Diamond_\alpha}(t)\mu_{ts}\nu_{ts}| \le \epsilon|\mu_{ts}\nu_{ts}|$$

A function f is said $\Diamond_\alpha$-differentiable on $\mathbb{T}_\kappa^\kappa$, provided $f^{\Diamond_\alpha}(t)$ exists for all $t \in \mathbb{T}_\kappa^\kappa$.

Let $\mathbb{T}$ be a time scale then f is diamond-α differentiable if and only if f is Δ and ∇ differentiable. The diamond-α derivative reduces to the standard Δ derivative for $\alpha = 1$, and to the standard ∇ derivative for $\alpha = 0$. On the other hand, it represents a "weighted dynamic derivative" for $\alpha \in (0,1)$. Furthermore, the combined dynamic derivative offers a centralized derivative formula on any uniformly discrete time scale $\mathbb{T}$ when $\alpha = 1/2$.

It has been shown that from application point of view the diamond-alpha derivative provides a better approximation of the exact derivative in comparison with delta and nabla derivatives.

Theorem 3.1.1. Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$. Then,

(i) $f + g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$ with

$$(f+g)^{\Diamond_\alpha}(t) = f^{\Diamond_\alpha}(t) + g^{\Diamond_\alpha}(t)$$

(ii) For any constant c, $cf : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$ with

$$(cf)^{\Diamond_\alpha}(t) = cf^{\Diamond_\alpha}(t)$$

(iii) $f, g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$ with

$$(fg)^{\Diamond_\alpha}(t) = f^{\Diamond_\alpha}(t)g(t) + \alpha f^\sigma(t)g^\Delta(t) + (1-\alpha)f^\rho(t)g^\nabla(t)$$

(iv) $\left(\dfrac{1}{g}\right)^{\Diamond_\alpha}(t) = -\dfrac{(g^\sigma(t) + g^\rho(t))g^{\Diamond_\alpha}(t) - \alpha g^\Delta(t)g^\sigma(t) - (1-\alpha)g^\nabla(t)g^\rho(t)}{g(t)g^\sigma(t)g^\rho(t)}.$

(vi) For $g(t)g^\sigma(t)g^\rho(t) \neq 0$, $f/g : \mathbb{T} \to \mathbb{R}$ is diamond-α is differentiable at $t \in \mathbb{T}$ with

$$\left(\dfrac{f}{g}\right)^{\Diamond_\alpha}(t) = \dfrac{f^{\Diamond_\alpha}(t)g^\sigma(t)g^\rho(t) - \alpha f^\sigma(t)g^\rho(t)g^\Delta(t) - (1-\alpha)f^\rho(t)g^\sigma(t)g^\nabla(t)}{g(t)g^\sigma(t)g^\rho(t)}.$$

Theorem 3.1.2. Let $f : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$. Then the following hold:

(i) $f^{\Diamond_\alpha \Delta}(t) = \alpha f^{\Delta\Delta}(t) + (1-\alpha)f^{\nabla\Delta}(t)$.

(ii) $f^{\Diamond_\alpha \nabla}(t) = \alpha f^{\Delta\nabla}(t) + (1-\alpha)f^{\nabla\nabla}(t)$.

(iii) $f^{\Delta \Diamond_\alpha}(t) = \alpha f^{\Delta\Delta}(t) + (1-\alpha)f^{\Delta\nabla}(t) \neq f^{\Diamond_\alpha \Delta}(t)$.

(iv) $f^{\nabla \Diamond_\alpha}(t) = \alpha f^{\nabla\Delta}(t) + (1-\alpha)f^{\nabla\nabla}(t) \neq f^{\Diamond_\alpha \nabla}(t)$.

(v) $f^{\Diamond_\alpha \Diamond_\alpha}(t) = \alpha^2 f^{\Delta\Delta}(t) + \alpha(1-\alpha)(f^{\Delta\nabla}(t) + f^{\nabla\Delta}(t)) + (1-\alpha)^2 f^{\nabla\nabla}(t) \neq \alpha^2 f^{\Delta\Delta}(t)$
$+ (1-\alpha)^2 f^{\nabla\nabla}(t)$.

Theorem 3.1.3. Suppose that f is a continuous function on $[a, b]$ and has a diamond-α derivative at each point of $[a, b)$. If $f(a) = f(b)$, then there exist a points $\xi, \xi' \in [a, b)$ such that $f^{\Diamond_\alpha}(\xi') \leq 0 \leq f^{\Diamond_\alpha}(\xi)$.

Theorem 3.1.4. (Mean Value Theorem) Suppose that f is a continuous function on $[a, b]$ and has a diamond-α derivative at each point of $[a, b)$. Then there exist points $\xi, \xi' \in [a, b)$ such that

$$f^{\Diamond_\alpha}(\xi')(b - a) \leq f(b) - f(a) \leq f^{\Diamond_\alpha}(\xi)(b - a).$$

Corollary 3.1.1. Let f be a continuous function on $[a, b]$ and has a diamond-α derivative at each point of $[a, b)$. Then f is increasing, nonincreasing, nondecreasing on $[a, b]$ if $f^{\Diamond_\alpha}(t) > 0$, $f^{\Diamond_\alpha}(t) < 0$, $f^{\Diamond_\alpha}(t) \leq 0$, and $f^{\Diamond_\alpha}(t) \geq 0$ for all $t \in [a, b)$, respectively.

We can remove that $a < b$ in (3.1.3). Thus we state the following results.

Theorem 3.1.5. Let a and b be two arbitrary points in $\mathbb{T}$ and let us set $\alpha = \min\{a, b\}$ and $\beta = \max\{a, b\}$. Let, further, f be a continuous function on $[\alpha, \beta]$ that has a diamond-α derivative at each point of $[\alpha, \beta)$. Then there exist points $\xi, \xi' \in [\alpha, \beta)$ such that

$$f^{\Diamond_\alpha}(\xi')(b - a) \leq f(b) - f(a) \leq f^{\Diamond_\alpha}(\xi)(b - a).$$

Passing now to the two-variable case, let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. As given in [47], for $i = 1, 2$ let σ_i, ρ_i and $\Diamond_{\alpha_i}$ denote the forward jump operator, the backward jump operator, and the diamond-α dynamic differentiation operator on $\mathbb{T}_i$, respectively.

Definition 3.1.2. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. At $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we say that f has a "partial $\Diamond_{\alpha_1}$ derivative" $\frac{\partial f(t_1, t_2)}{\Diamond_{\alpha_1} t_1}$ (with respect to t_1) if for each $\epsilon > 0$ there exists a neighbourhood U_{t_1}, (open in the relative topology of $\mathbb{T}_1$), of t_1 such that

$$|\alpha_1[f(\sigma_1(t_1), t_2) - f(s, t_2)]\eta_{t_1s} + (1 - \alpha_1)[f(\rho_1(t_1), t_2) - f(s, t_2)]\mu_{t_1s}$$

$$- f^{\Diamond_{\alpha_1}}(t_1, t_2)\eta_{t_1s}\mu_{t_1s}| \leq \epsilon|\eta_{t_1s}\mu_{t_1s}|$$

for all $s \in U_{t_1}$, where $\eta_{t_1s} = \sigma_1(t_1) - s, \quad \mu_{t_1s} = \rho(t_1) - s.$

Definition 3.1.3. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. At $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we say that f has a "partial $\Diamond_{\alpha_2}$ derivative" $\frac{\partial f(t_1, t_2)}{\Diamond_{\alpha_2} t_1}$ (with respect to t_2)if for each $\epsilon > 0$ there exists a neighbourhood U_{t_2} of t_2 such that

$$|\alpha_2[f(t_1, \sigma_2(t_2)) - f(t_1, t)]\eta_{t_2t} + (1 - \alpha_2)[f(t_1, \rho_2(t_2)) - f(t_1, t)]\mu_{t_2t}$$

$$- f^{\Diamond_{\alpha_2}}(t_1, t_2)\eta_{t_2t}\mu_{t_2t}| \leq \epsilon|\eta_{t_2t}\mu_{t_2t}|$$

for all $s \in U_{t_2}$, where $\eta_{t_2s} = \sigma_1(t_2) - s, \quad \mu_{t_2s} = \rho(t_2) - s.$

These derivatives will be denoted also by $f^{\Diamond_{\alpha_1}}(t_1, t_2)$ and $f^{\Diamond_{\alpha_2}}(t_1, t_2)$, respectively. If f has partial derivatives $\frac{\partial f(t_1, t_2)}{\Diamond_{\alpha_1} t_1}$ and $\frac{\partial f(t_1, t_2)}{\Diamond_{\alpha_2} t_2}$, then we can also consider their partial derivatives. These are called second-order partial derivatives. We write $\frac{\partial^2 f(t_1, t_2)}{\Diamond_{\alpha_1} t_1^2}$ and $\frac{\partial^2 f(t_1, t_2)}{\Diamond_{\alpha_2} t_2 \Diamond_{\alpha_1} t_1}$ or $f^{\Diamond_{\alpha_1}\Diamond_{\alpha_1}}(t_1, t_2)$ and $f^{\Diamond_{\alpha_1}\Diamond_{\alpha_2}}(t_1, t_2)$ for the partial diamond-α dynamic derivatives of $\frac{\partial f(t_1, t_2)}{\Diamond_{\alpha_1} t_1}$ with respect to t_1 and with respect to t_2, respectively. Thus $\frac{\partial^2 f(t_1, t_2)}{\Diamond_{\alpha_1} t_1^2} = \frac{\partial}{\Diamond_{\alpha_1} t_1}\left(\frac{\partial f(t_1, t_2)}{\Diamond_{\alpha_1} t_1}\right)$ and $\frac{\partial^2 f(t_1, t_2)}{\Diamond_{\alpha_1} t_2 \Diamond_{\alpha_1} t_1} = \frac{\partial}{\Diamond_{\alpha_2} t_2}\left(\frac{\partial f(t_1, t_2)}{\Diamond_{\alpha_1} t_1}\right).$ Higher-order partial diamond-α dynamic derivatives are defined similarly.

Theorem 3.1.6. (Mean Value Theorem) Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$ and $(a_1, a_2), (b_1, b_2)$ be any two points in $\mathbb{T}_1 \times \mathbb{T}_2$ and let us set

$$\alpha_i = \min\{a_i, b_i\} \quad \beta = \max\{a_i, b_i\} \quad \text{for} \quad i \in 1, 2.$$

Let further, f be a continuous function on $[\alpha_1, \beta_1] \times [\alpha_1, \beta_1] \subset \mathbb{T}_1 \times \mathbb{T}_2$ that has first order partial $\Diamond_\alpha$-dynamic derivatives $\frac{\partial f(t_1, a_2)}{\Diamond_{\alpha_1} t_1}$ for each $t_1 \in [\alpha_1, \beta_1)$ and $\frac{\partial f(b_1, t_2)}{\Diamond_{\alpha_2} t_2}$ for each $t_1 \in [\alpha_2, \beta_2)$. Then there exist $\xi_1, \eta_1 \in [\alpha_1, \beta_1)$ and $\xi_2, \eta_2 \in [\alpha_2, \beta_2)$ such that

$$\frac{\partial f(\eta_1, \alpha_2)}{\Diamond_{\alpha_1} t_1}(a_1, b_1) + \frac{\partial f(b_1, \eta_2)}{\Diamond_{\alpha_2} t_2}(a_2, b_2) \leq f(a_1, a_2) - f(b_1, b_2)$$

$$\leq \frac{\partial f(\xi_1, \alpha_2)}{\Diamond_{\alpha_1} t_1}(a_1 - b_1) + \frac{\partial f(b_1, \xi_2)}{\Diamond_{\alpha_2} t_2}(a_2 - b_2).$$

Also, if f has first order partial $\Diamond_\alpha$-dynamic derivatives $\frac{\partial f(t_1, b_2)}{\Diamond_{\alpha_1} t_1}$ for each $t_1 \in [\alpha_1, \beta_1)$ and $\frac{\partial f(a_1, t_2)}{\Diamond_{\alpha_2} t_2}$ for each $t_2 \in [\alpha_2, \beta_2)$. Then there exist $\tau_1, \theta_1 \in [\alpha_1, \beta_1)$ and $\tau_2, \theta_2 \in [\alpha_2, \beta_2)$ such that

$$\frac{\partial f(\theta_1, b_2)}{\Diamond_{\alpha_1} t_1}(a_1, b_1) + \frac{\partial f(a_1, \theta_2)}{\Diamond_{\alpha_2} t_2}(a_2, b_2) \leq f(a_1, a_2) - f(b_1, b_2)$$

$$\leq \frac{\partial f(\tau_1, b_2)}{\Diamond_{\alpha_1} t_1}(a_1 - b_1) + \frac{\partial f(a_1, \tau_2)}{\Diamond_{\alpha_2} t_2}(a_2 - b_2).$$

Theorem 3.1.7. Let a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ have the mixed partial $\Diamond_\alpha$-dynamic derivatives $\frac{\partial^2 f(t_1, t_2)}{\Diamond_{\alpha_2} t_2 \Diamond_{\alpha_1} t_1}$ and $\frac{\partial^2 f(t_1, t_2)}{\Diamond_{\alpha_1} t_1 \Diamond_{\alpha_2} t_2}$ in some neighbourhood of point $(t_1^0, t_2^0) \in \mathbb{T}_{1_\kappa}^\kappa \times \mathbb{T}_{2_\kappa}^\kappa$. If these derivatives are continuous at the point (t_1^0, t_2^0), then

$$\frac{\partial^2 f(t_1^0, t_2^0)}{\Diamond_{\alpha_1} t_1 \Diamond_{\alpha_2} t_2} = \frac{\partial^2 f(t_1^0, t_2^0)}{\Diamond_{\alpha_2} t_2 \Diamond_{\alpha_1} t_1}.$$

3.2 Antiderivative and Integral

Definition 3.2.1. Let $a, t \in \mathbb{T}$, and $h : \mathbb{T} \to \mathbb{R}$. Then, the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(\tau) \Diamond_\alpha \tau = \alpha \int_a^t h(\tau) \Delta\tau + (1 - \alpha) \int_a^t h(\tau) \nabla\tau, \quad 0 \leq \alpha \leq 1,$$

provided that there exists delta and nabla integrals of h on $\mathbb{T}$. It is clear that the diamond-α integral of h exists when h is a continuous function.

We may notice that the $\Diamond_\alpha$-combined derivative is not a dynamic for the absence of its antiderivative. Moreover, in general, we do not have

$$\left(\int_a^t h(\tau) \Diamond_\alpha \tau \right)^{\Diamond_\alpha} = h(t), \quad t \in \mathbb{T}.$$

Example 3.2.1. Let $\mathbb{T} = 0, 1, 2, a = 0$, and $h(\tau) = \tau^2$, $\tau \in \mathbb{T}$. It is a simple exercise to see that

$$\left(\int_a^t h(\tau) \Diamond_\alpha \tau \right)^{\Diamond_\alpha} \bigg|_{t=1} = h(1) + 2\alpha(1 - \alpha),$$

so that the equality above holds only when $\Diamond_\alpha = \Delta$ or $\Diamond_\alpha = \nabla$.

Theorem 3.2.1. Let $a, b, t \in \mathbb{T}, c \in \mathbb{R}$, and f and g be continuous functions on $[a, b] \cup \mathbb{T}$. Then the following properties hold.

(i) $\int_a^t (f(\tau) + g(\tau)) \Diamond_\alpha \tau = \int_a^t f(\tau) \Diamond_\alpha \tau + \int_a^t g(\tau) \Diamond_\alpha \tau$.

(ii) $\int_a^t c f(\tau) \Diamond_\alpha \tau = c \int_a^t f(\tau) \Diamond_\alpha \tau$.

(iii) $\int_a^t f(\tau) \Diamond_\alpha \tau = \int_a^b f(\tau) \Diamond_\alpha \tau + \int_b^t f(\tau) \Diamond_\alpha \tau$.

(iv) If $f(t) \geq 0$ for all $t \in [a, b]_\mathbb{T}$, then $\int_a^b f(t) \Diamond_\alpha t \geq 0$.

(v) If $f(t) \geq g(t)$ for all $t \in [a, b]_\mathbb{T}$, then $\int_a^b f(t) \Diamond_\alpha t \geq \int_a^b g(t) \Diamond_\alpha t$.

(vi) If $f(t) \geq g(t)$ for all $t \in [a, b]_\mathbb{T}$, then $f(t) = 0$ if and only if $\int_a^b f(t) \Diamond_\alpha t = 0$

Corollary 3.2.1. Let $t \in \mathbb{T}^\kappa_\kappa$, and $f, g : \mathbb{T} \to \mathbb{R}$, then

$$\int_t^{\sigma(t)} f(\tau) \Diamond_\alpha \tau = \mu(t)(\alpha f(t) + (1 - \alpha) f^\sigma(t))$$

$$\int_{\rho(t)}^t f(\tau) \Diamond_\alpha \tau = \nu(t)(\alpha f^\rho(t) + (1 - \alpha) f(t)).$$

3.3 Diamond-alpha Exponential function

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. By $\mathcal{R}$ it is denoted the set of all regressive and rd-continuous functions on $\mathbb{T}$. Similarly, a function $q : \mathbb{T} \to \mathbb{R}$ is called ν-regressive provided $1 - \nu(t)q(t) \neq 0$ for all $t \in \mathbb{T}_\kappa$. By $\mathcal{R}_v$ we denote the set of all ν-regressive and ld-continuous functions on $\mathbb{T}$.

Roughly speaking, the $\Diamond_\alpha$-calculus is the convex combination of Δ and ∇ calculuses. We may be then tempted to define the $\Diamond_\alpha$-exponential function by a simple combination of Δ and ∇ exponentials. We consider here two such functions: $_\alpha E_p$ and $_\alpha e_p$, where $p \in \mathcal{R} \cup \mathcal{R}_\nu$ and $\alpha \in 0, 1$.

Definition 3.3.1. (Combined-Exponentials) We consider here two such functions $_\alpha E_p$ and $_\alpha e_p$, where $p \in \mathcal{R} \cap \mathcal{R}_\nu$ and $\alpha \in [0, 1]$

$$_\alpha E_p(., t_0) = \alpha e_p(., t_0) + (1 - \alpha)\hat{e}_p(., t_0), \quad t_0 \in \mathbb{T},$$

and

$$_\alpha e_p(., t_0) = \exp\left(\alpha \int_{t_0}^t \xi_{\mu(t)}(p(\tau))\Delta(\tau) + (1 - \alpha) \int_{t_0}^t \hat{\xi}_{\nu(t)}(p(\tau))\nabla(\tau)\right), \quad t, t_0 \in \mathbb{T}$$

where e_p and $\hat{e}_p$ are delta and nabla exponential functions, respectively, as given in Chapter 2.

Example 3.3.1. Consider the time scale $\mathbb{T} = \mathbb{Z}$ and the constant function $p(t) = 1/2$. Take $t_0 = 0$. Then, $e_p(t, 0) = (3/2)^t$ is the solution of the initial value problem $y^\Delta(t) = \frac{1}{2}y(t), y(t_0) = 1$. Moreover, $\hat{e}_p(t, 0) = 2^t$ is the unique solution of $y^\nabla(t) = \frac{1}{2}y(t), y(t_0) = 1$. $_\alpha E_p(t, 0) = \alpha(3/2)^t + (1-\alpha)2^t, t \in \mathbb{Z}$. Combined-exponentials can not be really called an exponential function. Indeed, they seem to fail the most important

property of an exponential function: they are not a solution of an appropriate initial value problem.

Definition 3.3.2. A time scale $\mathbb{T}$ is said to be regular if the following two conditions are satisfied simultaneously:

(a) for all $t \in \mathbb{T}$ $\sigma(\rho(t)) = t$

(b) for all $t \in \mathbb{T}$ $\rho(\sigma(t)) = t$

Remark 3.3.2. If $\mathbb{T}$ is a regular time scale, then both operators ρ and σ are invertible with $\sigma^{-1} = \rho$ and $\rho^{-1} = \sigma$.

Next proposition gives direct formulas for the $\Diamond_\alpha$-derivative of the exponential functions $e_p(., t_0)$ and $\hat{e}_p(., t_0)$.

Proposition 3.3.3. Let $\mathbb{T}$ be a regular time scale. Assume that $t, t_0 \in \mathbb{T}$ and $p \in \mathcal{R} \cap \mathcal{R}_\nu$. Then

$$e_p^{\Diamond_\alpha}(t, t_0) = \left[\alpha p(t) + \frac{(1 - \alpha)p^\rho(t)}{1 + \nu(t)p^\rho(t)} \right] e_p(t, t_0)$$

$$\hat{e}_p^{\Diamond_\alpha}(t, t_0) = \left[\alpha p(t) + \frac{(1 - \alpha)p^\sigma(t)}{1 + \mu(t)p^\sigma(t)} \right] \hat{e}_p(t, t_0)$$

$e_p(., t_0)$ is a solution of the initial value problem $y^{\Diamond_\alpha}(t) = q(t)y(t), y(t_0) = 1$, where

$$q(t) = \alpha p(t) + \frac{(1 - \alpha)p^\rho(t)}{1 + \nu(t)p^\rho(t)}$$

Example 3.3.4. Let $\mathbb{T} = c\mathbb{Z}$, $c > 0$, and consider the $\Diamond_\alpha$-dynamic equation $y^{\Diamond_\alpha}(t) = 0$, $y(t_0) = 1$ where $\alpha \in (0, 1)$.

The constant function $y_1(t) = 1$ is a solution of such an initial value problem.

For a given $\alpha \in (0, 1)$ also $y_2(t) = e_q(t, t_0)$, $q = -\frac{1}{\alpha\mu}$, is a solution in view of the following:

$$\frac{q^\rho}{1 + q^\rho\nu} = \frac{1}{(1 - \alpha)c}, \quad \text{and}$$

$$\begin{aligned} y_2^{\Diamond_\alpha}(t) &= \left(\alpha q + (1 - \alpha)\frac{q^\rho}{1 + q^\rho\nu}\right)e_q(t, t_0) \\ &= \left(\frac{-\alpha}{\alpha c} + \frac{1 - \alpha}{(1 - \alpha)c}\right)e_q(t, t_0) \\ &= 0 \end{aligned}$$

with

$$y_2(t_0) = e_q(t_0, t_0) = 1$$

To have uniqueness of solution, a second boundary or initial condition is needed. We see that $e_q(\rho(t_0), t_0) = \alpha/(\alpha - 1) \neq 1$. Hence, with the second condition $y(\rho(t_0)) = 1$, $y(t) = 1$ is the unique solution.

We consider linear equations with $\Diamond_\alpha$ derivatives. Results are based on the fact that on a time scale of isolated points, a first order $\Diamond_\alpha$ equations becomes a linear equation of second order when written in terms of Δ and ∇ notions.

Example 3.3.5. Let $\mathbb{T} = \mathbb{N} \cup \{0\}$ and $\alpha = 0.5$. We have

$$\begin{aligned} 2y^{\Diamond_\alpha}(t) &= y^\Delta(t) + y^\nabla(t) \\ &= y(t + 1) - y(t) + y(t) - y(t - 1) \\ &= y(t + 1) - y(t - 1) \end{aligned}$$

Consider the $\Diamond_\alpha$ equation $y^{\Diamond_\alpha}(t) = 0.5y(t)$. Then

$$2y^{\Diamond_\alpha}(t) = y(t)$$

$$y(t+1) - y(t-1) = y(t)$$

$$y(t+1) = y(t) + y(t-1)$$

To have a unique solution, we need two initial or two boundary conditions. If we take $y(0) = 1$ and $y(1) = 1$ then the Fibonacci sequence is the unique solution to the boundary value problem $y^{\Diamond_{0.5}}(t) = 0.5y(t)$, $y(0) = 1$, $y(1) = 1$. For $t \in \mathbb{T}$, the solution is the Fibonacci sequence with the general formula

$$y(t) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{t+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{t+1}$$

The above two examples (Ex.3.3.4 and Ex.3.3.5) are taken from [55].

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CHAPTER 4

DIAMOND-ALPHA DYNAMIC INEQUALITIES

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In this Chapter we give a survey of some recently obtained Dynamic Inequalities in view of the newly developed theory of Diamond-alpha derivatives and integrals. Some of the results will be used in the subsequent chapters.

Theorem 4.0.1. (Hölder's inequality). Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and $f, g, h \in C([a, b]_\mathbb{T}, [0, \infty))$ with $\int_a^b h(x) g^q(x) \Diamond_\alpha x > 0$, where q is the Hölder conjugate number of p, that is, $1/p + 1/q = 1$ with $1 < p$. Then we have

$$\int_a^b h(x) f(x) g(x) \Diamond_\alpha x \leq \left(\int_a^b h(x) f^p(x) \Diamond_\alpha x \right)^{1/p} \left(\int_a^b h(x) g^q(x) \Diamond_\alpha x \right)^{1/q}.$$

Theorem 4.0.2. (Cauchy-Schwarz Inequality). For continuous functions $f, g : [a, b] \to \mathbb{R}$ we have

$$\int_a^b |f(t) g(t)| \Diamond_\alpha t \leq \sqrt{\left[\int_a^b |f(t)|^2 \Diamond_\alpha t \right] \left[\int_a^b |g(t)|^2 \Diamond_\alpha t \right]}$$

Theorem 4.0.3. (Minkowski's inequality) Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and $p > 1$. For continuous functions $f, g : [a, b]_\mathbb{T} \to \mathbb{R}$, we have

$$\left(\int_a^b |(f + g)(x)|^p \Diamond_\alpha x \right)^{1/p} \leq \left(\int_a^b |f(x)|^p \Diamond_\alpha x \right)^{1/p} + \left(\int_a^b |g(x)|^p \Diamond_\alpha x \right)^{1/p}.$$

Theorem 4.0.4. (Two Dimensional Diamond-alpha Hölder's Inequality, [10]). Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, $f, g, h : [a, b] \times [a, b] \to \mathbb{R}$ be $\Diamond_\alpha$-integrable functions, and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Then,

$$\int_a^b \int_a^b |h(x, y) f(x, y) g(x, y)| \Diamond_\alpha x \Diamond_\alpha y$$
$$\leq \left(\int_a^b \int_a^b |h(x, y)| |f(x, y)|^p \Diamond_\alpha x \Diamond_\alpha y \right)^{1/p} \left(\int_a^b \int_a^b |h(x, y)| |g(x, y)|^q \Diamond_\alpha x \Diamond_\alpha y \right)^{1/q}$$

Theorem 4.0.5. (Two Dimensional Diamond-alpha Cauchy-Schwartz's Inequality, [10]). Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$. For $\Diamond_\alpha$-integrable functions

$f, g, h : [a, b] \times [a, b] \to \mathbb{R}$, we have:

$$\int_a^b \int_a^b |h(x,y)f(x,y)g(x,y)| \Diamond_\alpha x \Diamond_\alpha y$$

$$\leq \sqrt{\left(\int_a^b \int_a^b |h(x,y)||f(x,y)|^2 \Diamond_\alpha x \Diamond_\alpha y \right) \left(\int_a^b \int_a^b |h(x,y)||g(x,y)|^2 \Diamond_\alpha x \Diamond_\alpha y \right)}.$$

Theorem 4.0.6. (Diamond-α Hardy-type inequalities, [10]). Let $\mathbb{T}$ be a time scale, $a, b \in$ with $a < b$, and $K(x,y)$, $f(x), g(y), \varphi(x)$, and $\psi(y)$ be nonnegative functions. Let

$$F(x) = \int_a^b K(x,y)\psi^{-p}(y) \Diamond_\alpha y$$

and

$$G(y) = \int_a^b K(x,y)\varphi^{-q}(x) \Diamond_\alpha x,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. Then, the two inequalities

$$\int_a^b \int_a^b K(x,y)f(x)g(y) \Diamond_\alpha x \Diamond_\alpha y \leq \left(\int_a^b \varphi^p(x)F(x)f^p(x) \Diamond_\alpha x \right)^{1/p} \times$$

$$\left(\int_a^b \psi^q(y)G(y)g^q(y) \Diamond_\alpha y \right)^{1/q}$$

and

$$\int_a^b G^{1-p}(y)\psi^{-p}(y) \left(\int_a^b K(x,y)f(x) \Diamond_\alpha x \right)^p \Diamond_\alpha y \leq \int_a^b \varphi^p(x)F(x)f^p(x) \Diamond_\alpha x$$

hold and are equivalent.

Theorem 4.0.7. (Jensen's Inequality, [9])Let $(t_1, t_2) \in R$ and $-\infty \leq m < n \leq \infty$ If $f : R \to (m, n)$ is continuous, $\Diamond_{\alpha_1}$ partial derivative of f and $\Diamond_{\alpha_2}$ partial derivative of f exist, are also continuous and $\Phi : (m, n) \to R$ is convex, then

$$\Phi \left(\frac{\int_c^d \int_a^b f(x_1, x_2) \Diamond_{\alpha_1} x_1 \Diamond_{\alpha_2} x_2}{\int_c^d \int_a^b \Diamond_{\alpha_1} x_1 \Diamond_{\alpha_2} x_2} \right) \leq \frac{\int_c^d \int_a^b \Phi(f(x_1, x_2)) \Diamond_{\alpha_1} x_1 \Diamond_{\alpha_2} x_2}{\int_c^d \int_a^b \Diamond_{\alpha_1} x_1 \Diamond_{\alpha_2} x_2}$$

Here; $\mathbb{T}_1$ is a time scale, $0 \leq a < b$ are points in $\mathbb{T}_1$;$\mathbb{T}_2$ is a time scale, $0 \leq c <$ d are points in $\mathbb{T}_2$; R is a rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$ defined by $R = [a,b) \times [c,d) =$ $(t,s) : t \in [a,b), s \in [c,d)$.

Theorem 4.0.8. (Hermite-Hadamard Inequality, [28]) Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function. Then,

$$f(x_\alpha) \leq \frac{1}{b-a} \int_a^b f(t)\Diamond_\alpha t \leq \frac{b-x_\alpha}{b-a} f(a) + \frac{x_\alpha - a}{b-a} f(b).$$

Theorem 4.0.9. (A general version of Hermite-Hadamard inequality, [28]) Let $\mathbb{T}$ be a time scale and $a, \lambda \in [0,1]$ and $a, b \in \mathbb{T}$. Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function Then,

(i) if f is nondecreasing on $[a,b]_\mathbb{T}$, then, for all $\alpha \in [0,\lambda]$ one has

$$f(x_\lambda) \leq \frac{1}{b-a} \int_a^b f(t)\Diamond_\alpha t \quad (4.1.1),$$

and for all $\alpha \in [\lambda, 1]$, one has

$$\frac{1}{b-a} \int_a^b f(t)\Diamond_\alpha t \leq \frac{b-x_\lambda}{b-a} f(a) + \frac{x_\lambda - a}{b-a} f(b) \quad (4.1.2).$$

(ii) If f is nonincreasing on $[a,b]_\mathbb{T}$, then, for all $\alpha \in [0,\lambda]$ one has the above inequality (4.1.2) and for all $\alpha \in [0,\lambda]$ one has the above inequality (4.1.1).

Theorem 4.0.10. (a weighted version of Hermite-Hadamard inequality, [28]) Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function and let $w : \mathbb{T} \to \mathbb{R}$ be a continuous function such that $w(t) \geq 0$ for all $t \in \mathbb{T}$ and $\int_a^b w(t)\Diamond_\alpha t > 0$. Then,

$$f(x_{w,\alpha}) \leq \frac{1}{\int_a^b w(t)\Diamond_\alpha t} \int_a^b f(t)w(t)\Diamond_\alpha t \leq \frac{b-x_{w,\alpha}}{b-a} f(a) + \frac{x_{w,\alpha}}{b-a} f(b),$$

where $x_{w,\alpha} = \int_a^b tw(t)\Diamond_\alpha t / \int_a^b w(t)\Diamond_\alpha t$.

Theorem 4.0.11. (the weighted diamond-alpha Grüss inequality, [25]) Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$ with $a < b$. If $f, g \in C(\mathbb{T}, \mathbb{R})$ and $p \in C(\mathbb{T}, [0, \infty))$ satisfy $\varphi \le f(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in [a, b] \cap \mathbb{T}$ and $\int_a^b p(x) \Diamond_\alpha x > 0$, then

$$\left| \int_a^b p(x) \Diamond_\alpha x \int_a^b p(x) f(x) g(x) \Diamond_\alpha x - \int_a^b p(x) f(x) \Diamond_\alpha x \int_a^b p(x) g(x) \Diamond_\alpha x \right|$$
$$\le \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma) \left(\int_a^b p(x) \Diamond_\alpha x \right)^2.$$

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CHAPTER 5

DYNAMIC OPIAL INEQUALITIES AND APPLICATIONS

5.1   Preliminaries

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In 1960, the Polish Mathematician Zdzidlaw Opial [68] published an inequality involving integrals of functions and their derivatives;

$$\int_0^h |x(t)x'(t)|\, dt \leq \frac{h}{4} \int_0^h |x'(t)|^2\, dt \tag{5.1.1}$$

where $x \in C^{(1)}[0,h]$, $x(0) = x(h) = 0$ and $x(t) > 0$ in $(0, h)$, and the constant $h/4$ is the best possible.

Inequalities which involve integrals of functions and their derivatives are of great importance in mathematics with applications in the theory of differential equations, approximations and probability. It has been shown that inequalities of the form (5.1.1) can be deduced from those of Wirtinger and Hardy type, but the importance of Opial's result is in the establishment of the best possible constant. A recently published monograph [7] is the first book dedicated to the theory of Opial type inequalities .

The positivity requirement of $x(t)$ in the original proof of Opial was shown to be unnecessary later by Olech in [67] where he proved that the inequality (5.1.1) holds even for functions $x(t)$ which are only absolutely continuous in [0,h]. Moreover, Olech's proof is simpler than that of Opial.

Theorem 5.1.1. (Olech): Let $x(t)$ be absolutely continuous in $[0, h]$ and $x(0) = x(h) = 0$. Then the following inequality holds;

$$\int_0^h |x(t)x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt \tag{5.1.2}$$

Works containing discrete analogues of Opial-Type inequalities started in 1967-69 with the papers of Lasota [52], Wong [69], Beesack [17] which provided discrete versions of inequality (5.1.1).

The next result is the discrete analogue of the above theorem.

Theorem 5.1.2. (Lasota's Inequality): Let $\{x_i\}_{i=0}^N$ be a sequence of numbers, and $x_0 = x_N = 0$. Then the following inequality holds

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \leq \frac{1}{2}[[\frac{N+1}{2}]] \sum_{i=0}^{N-1} |\Delta x_i|^2 \tag{5.1.3}$$

where Δ is the forward difference operator, and $[[.]]$ is the greatest integer function.

Now, if we consider Olech's result under weaker conditions, we get the following estimate with a less sharper bound.

Theorem 5.1.3. Let $x(t)$ be absolutely continuous in $[0, a]$ and $x(0) = 0$. Then the following inequality holds.

$$\int_0^a |x(t)x'(t)| dt \leq \frac{a}{2} \int_0^a (x'(t))^2 dt \tag{5.1.4}$$

In (5.1.4) equality holds if and only if $x(t) = ct$

The following theorem is a non-trivial generalization of Theorem 5.1.3 and is given in Hua [38].

Theorem 5.1.4. (Hua's generalization)

Let $x(t)$ be absolutely continuous on $[0, a]$, and $x(0) = 0$. Further let ε be a positive integer. Then the following inequality holds

$$\int_0^a |x^\varepsilon(t)x'(t)| dt \leq \frac{a^\varepsilon}{\varepsilon + 1} \int_0^a |x'(t)|^{\varepsilon+1} dt \tag{5.1.5}$$

with equality being valid in (5.1.5) if and only if $x(t) = ct$.

Finally we give a discrete analogue of Theorem 3.1.4 due to Wong [69].

Theorem 5.1.5. (Wong's inequality)

Let $\left\{x_i\right\}_{i=0}^{\tau}$ be a non-decreasing sequence of nonnegative numbers, and $x_0 = 0$. Then for $\varepsilon \geq 1$, the following inequality holds

$$\sum_{i=1}^{\tau} x_i^{\varepsilon} \nabla x_i \leq \frac{(\tau+1)^{\varepsilon}}{\varepsilon+1} \sum_{i=1}^{\tau} (\nabla x_i)^{\varepsilon+1}. \tag{5.1.6}$$

where ∇ is the backward difference operator $i.e., \nabla x_i = x_i - x_{i-1}$.

Remark 5.1.1. In terms of the forward difference operator, Δx_i, the above Wong's inequality (5.1.6) can be restated as follows;

$\left\{x_i\right\}_{i=0}^{\tau}$ is a non-decreasing sequence of non-negative numbers with $x_0 = 0$, for $\varepsilon \geq 1$, the inequality

$$\sum_{i=0}^{\tau-1} x_{i+1}^{\varepsilon} \Delta x_i \leq \frac{(\sigma(\tau))^{\varepsilon}}{\varepsilon+1} \sum_{i=0}^{\tau-1} (\Delta x_i)^{\varepsilon+1}. \tag{5.1.7}$$

holds where Δ is the forward difference operator.

For convenience we now recall the following simple versions of Opial's inequality.

Theorem 5.1.1. [Continuous Opial Inequality, [7, Theorem 1.4.1]] For absolutely continuous function $x : [0, h] \to \mathbb{R}$ with $x(0) = 0$ we have

$$\int_0^h |x\dot{x}|(t)\mathrm{d}t \leq \frac{h}{2} \int_0^h |\dot{x}|^2(t)\mathrm{d}t,$$

with equality when $x(t) = ct$.

Theorem 5.1.2. [Discrete Opial Inequality, [7, Theorem 5.2.2]] For $x_0 = 0$ and a sequence $\{x_i\}_{0 \leq i \leq h} \subset \mathbb{R}$, we have

$$\sum_{i=1}^{h-1} |x_i(x_{i+1} - x_i)| \leq \frac{h-1}{2} \sum_{i=0}^{h-1} |x_{i+1} - x_i|^2,$$

with equality when $x_i = ci$.

5.2   Delta Dynamic Opial Inequality

In [23] a Dynamic Opial inequality is proven that contains both Theorem 5.1.1 and Theorem 5.1.2 as special cases. This result is presented in the theorem below.

Theorem 5.2.1. [Delta Dynamic Opial Inequality] Let $\mathbb{T}$ be a time scale. For delta differentiable function $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = 0$ we have

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \leq h \int_0^h |x^\Delta|^2(t)\Delta t,$$

with equality when $x(t) = ct$.

Proof. Consider

$$y(t) = \int_0^t |x^\Delta(s)|\Delta s.$$

Then we have $y^\Delta = |x^\Delta|$ and $|x| \leq y$ so that

$$
\begin{aligned}
\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \;&\leq\; \int_0^h [(|x| + |x^\sigma|)|x^\Delta|](t)\Delta t \\
&\leq\; \int_0^h [(y + y^\sigma)|x^\Delta|](t)\Delta t \\
&=\; \int_0^h [(y + y^\sigma)y^\Delta](t)\Delta t \\
&=\; \int_0^h (y^2)^\Delta(t)\Delta t \\
&=\; y^2(h) - y^2(0) \\
&=\; \left\{ \int_0^h |x^\Delta(t)|\Delta t \right\}^2 \\
&\leq\; h \int_0^h |x^\Delta|^2(t)\Delta t,
\end{aligned}
$$

where we have used formulas (2.7) and (2.7.4) for $p = 2$.

Now, let $\tilde{x}(t) = ct$ for some $c \in \mathbb{R}$. Then $\tilde{x}^\Delta(t) \equiv c$, and it is easy to check that the

equation

$$\int_0^h |(\tilde{x} + \tilde{x}^\sigma)\tilde{x}^\Delta|(t)\Delta t = h \int_0^h |\tilde{x}^\Delta|^2(t)\Delta t$$

holds. □

Next a generalization of Theorem 5.2.1 is offered where $x(0)$ does not need to be equal to 0. This result is not found in the book [7] (neither a continuous nor a discrete version of it), but both a weaker version of Theorem 5.2.1 (with $x(0) = 0$) and the subsequent Theorem 5.2.3 (with $x(0) = x(h) = 0$) are corollaries of Theorem 5.2.2, and continuous [7, Theorem 1.3.1] and discrete [7, Theorem 5.2.1, "Lasota's inequality"] versions of Theorem 5.2.3 can be found in the book by Agarwal and Pang [7].

Theorem 5.2.2. Let $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ be differentiable. Then

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \le \alpha \int_0^h |x^\Delta(t)|^2\Delta t + 2\beta \int_0^h |x^\Delta(t)|\Delta t,$$

where

$$\alpha \in \mathbb{T} \quad \text{with} \quad \text{dist}(h/2, \alpha) = \text{dist}(h/2, \mathbb{T}) \tag{5.2.1}$$

and $\beta = \max\{|x(0)|, |x(h)|\}$.

Proof. We consider

$$y(t) = \int_0^t |x^\Delta(s)|\Delta s \quad \text{and} \quad z(t) = \int_t^h |x^\Delta(s)|\Delta s.$$

Then $y^\Delta = |x^\Delta|$, $z^\Delta = -|x^\Delta|$.

Next, we have

$$
\begin{aligned}
|x(t)| &\leq |x(t) - x(0)| + |x(0)| \\
&= \left| \int_0^t x^\Delta(s)\Delta s \right| + |x(0)| \\
&\leq \int_0^t |x^\Delta(s)|\Delta s + |x(0)| \\
&= y(t) + |x(0)|,
\end{aligned}
$$

and similarly $|x(t)| \leq z(t) + |x(h)|$.

Let $u \in [0, h] \cap \mathbb{T}$. Then

$$
\begin{aligned}
\int_0^u |(x + x^\sigma)x^\Delta|(t)\Delta t &\leq \int_0^u [y(t) + |x(0)| + y^\sigma(t) + |x(0)|]y^\Delta(t)\Delta t \\
&= \int_0^u [(y + y^\sigma)y^\Delta](t)\Delta t + 2|x(0)| \int_0^u y^\Delta(t)\Delta t \\
&= y^2(u) + 2|x(0)|y(u) \\
&\leq u \int_0^u |x^\Delta(t)|^2 \Delta t + 2|x(0)| \int_0^u |x^\Delta(t)|\Delta t,
\end{aligned}
$$

where we have used again (2.7) and (2.7.4) for $p = 2$. Similarly, one shows

$$
\begin{aligned}
\int_u^h |(x + x^\sigma)x^\Delta|(t)\Delta t &\leq z^2(u) + 2|x(h)|z(u) \\
&\leq (h - u) \int_u^h |x^\Delta(t)|^2 \Delta t + 2|x(h)| \int_u^h |x^\Delta(t)|\Delta t.
\end{aligned}
$$

By putting $\nu(u) = \max\{u, h - u\}$ and adding the above two inequalities, we find

$$
\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \leq \nu(u) \int_0^h |x^\Delta(t)|^2 \Delta t + 2\beta \int_0^h |x^\Delta(t)|\Delta t.
$$

This is true for any $u \in [0, h] \cap \mathbb{T}$, so it is also true if $\nu(u)$ is replaced by $\min_{u \in [0,h] \cap \mathbb{T}} \nu(u)$.

However, this last quantity is easily seen to be equal to α. $\qquad \square$

Theorem 5.2.3. Let $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ be differentiable with $x(0) = x(h) = 0$. Then

$$
\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \leq \alpha \int_0^h |x^\Delta(t)|^2 \Delta t,
$$

where α is given in (5.2.1).

Proof. This follows easily from Theorem 5.2.2 since in this case we have $\beta = 0$. $\square$

5.3 Upper Bound Estimates of Solutions of Delta Dynamic Initial Value Problems

We now proceed to give an application of Theorem 5.2.1.

Example 5.3.1. Let y be a solution of the initial value problem

$$y^\Delta = 1 - t + \frac{y^2}{t}, \quad 0 < t \le 1, \quad y(0) = 0. \tag{5.3.1}$$

Then $y \le \tilde{y}$ on $[0, 1] \cap \mathbb{T}$, where $\tilde{y}(t) = t$ solves (5.3.1).

Proof. Clearly $\tilde{y}$, as defined above, solves (5.3.1). We let y be a solution of (5.3.1) and consider R defined by

$$R(t) = 1 - t + \int_0^t |y^\Delta|^2(s)\Delta s.$$

Let $t \in [0, 1] \cap \mathbb{T}$. Then

$$
\begin{aligned}
|y^\Delta(t)| &= \left|1 - t + \frac{y^2(t)}{t}\right| \le |1 - t| + \frac{1}{t}|y^2(t)| \\
&= 1 - t + \frac{1}{t}\left|\int_0^t (y^2)^\Delta(s)\Delta s\right| \\
&\le 1 - t + \frac{1}{t}\int_0^t |(y^2)^\Delta(s)|\Delta s \\
&= 1 - t + \frac{1}{t}\int_0^t |(y + y^\sigma)y^\Delta|(s)\Delta s \\
&\le 1 - t + \frac{1}{t}t\int_0^t |y^\Delta|^2(s)\Delta s \\
&= R(t),
\end{aligned}
$$

where we have used Opial's inequality in Theorem 5.2.1. Hence,

$$R^\Delta(t) = -1 + |y^\Delta(t)|^2 \le R^2(t) - 1 \quad \text{and} \quad R(0) = 1.$$

Let w be the unique solution of

$$w^\Delta = (1 + R(t))w, \quad w(0) = 1.$$

Note that this w exists since $1 + \mu(R + 1) > 0$ (see e.g. [14, Theorem 8]) because $R > |y^\Delta| \geq 0$; actually $w(t) > 0$ for all $t \in [0, 1] \cap \mathbb{T}$. Hence, because of $(R - 1)^\Delta = R^\Delta \leq R^2 - 1$, we have

$$
\begin{aligned}
0 &\geq \frac{(R - 1)^\Delta w - (R^2 - 1)w}{ww^\sigma} \\
&= \frac{(R - 1)^\Delta w - (R - 1)w^\Delta}{ww^\sigma} \\
&= \left(\frac{R - 1}{w}\right)^\Delta
\end{aligned}
$$

(we used the quotient rule from Section 2.7) so that

$$\left(\frac{R - 1}{w}\right)(t) = \left(\frac{R - 1}{w}\right)(0) + \int_0^t \left(\frac{R - 1}{w}\right)^\Delta (s)\Delta s \leq 0$$

and hence $R(t) \leq 1$. Therefore

$$y^\Delta(t) \leq |y^\Delta(t)| \leq R(t) \leq 1$$

and $y(t) = \int_0^t y^\Delta(s)\Delta s \leq \int_0^t \Delta s = t$. $\qquad \square$

5.4   Nabla Opial Inequalities

Theorem 5.4.1. [Nabla Dynamic Opial Inequality]

Let $\mathbb{T}$ be a time scale. For nabla differentiable function $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = 0$ we have

$$\int_0^h |(x + x^\rho)x^\nabla|(t)\nabla t \leq h \int_0^h |x^\nabla|^2(t)\nabla t,$$

with equality when $x(t) = ct$.

Proof. Consider

$$y(t) = \int_0^t |x^\nabla(s)| \nabla s.$$

Then we have $y^\nabla = |x^\nabla|$ and $|x| \le y$ so that

$$
\begin{aligned}
\int_0^h |(x + x^\rho)x^\nabla|(t)\nabla t &\le \int_0^h [(|x| + |x^\rho|)|x^\nabla|](t)\nabla t \\
&\le \int_0^h [(y + y^\rho)|x^\nabla|](t)\nabla t \\
&= \int_0^h [(y + y^\rho)y^\nabla](t)\nabla t \\
&= \int_0^h (y^2)^\nabla(t)\nabla t \\
&= y^2(h) - y^2(0) \\
&= \left\{ \int_0^h |x^\nabla(t)|\nabla t \right\}^2 \\
&\le h \int_0^h |x^\nabla|^2(t)\nabla t,
\end{aligned}
$$

where we have used formulas (2.7) and (2.7.4) for $p = 2$.

Now, let $\tilde{x}(t) = ct$ for some $c \in \mathbb{R}$. Then $\tilde{x}^\nabla(t) \equiv c$, and it is easy to check that the equation

$$\int_0^h |(\tilde{x} + \tilde{x}^\rho)\tilde{x}^\nabla|(t)\nabla t = h \int_0^h |\tilde{x}^\nabla|^2(t)\nabla t$$

holds. $\qquad\square$

5.5 Extensions of Delta Dynamic Opial Inequalities

Since its discovery more than four decades ago, Opial's inequality has found many interesting applications. In fact, Opial's inequality and its several generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential

equations as well as difference equations. In addition, many qualitative behaviours such as oscillation, non-oscillation, boundedness, have been discussed in the light of Opial's inequality . It may be asserted that this subject will continue to play a very important part in the future of applied mathematics.

In this chapter, we give some of the various generalizations of the inequalities presented in the previous chapter. The continuous and/or discrete versions of those inequalities may be found in [7]. We have not included all of those results in this work, but most of them may be proved using similar techniques as the ones presented here. The results given here are in view of those obtained in [23] by Bohner and Kaymakçalan.

Theorem 5.5.1. [see [7, Theorem 2.5.1]] Let p, q be positive and continuous on $[0, h]$, $\int_0^h \Delta t / p(t) < \infty$, and q non-increasing. For differentiable $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = 0$ we have

$$\int_0^h [q^\sigma |(x + x^\sigma) x^\Delta|](t) \Delta t \leq \left\{ \int_0^h \frac{\Delta t}{p(t)} \right\} \left\{ \int_0^h p(t) q(t) |x^\Delta(t)|^2 \Delta t \right\}.$$

Proof. We consider

$$y(t) = \int_0^t \sqrt{q^\sigma(s)} |x^\Delta(s)| \Delta s.$$

Then $y^\Delta = \sqrt{q^\sigma} |x^\Delta|$. Note that $0 \leq s < t$ implies $\sigma(s) \leq t$ and hence, $q(\sigma(s)) \geq q(t)$; by applying (2.7.1)

$$|x(t)| \leq \int_0^t |x^\Delta(s)| \Delta s \leq \int_0^t \sqrt{\frac{q^\sigma(s)}{q(t)}} |x^\Delta(s)| \Delta s = \frac{y(t)}{\sqrt{q(t)}}$$

so that (note that $t \leq \sigma(t)$ implies $q(t) \geq q(\sigma(t))$)

$$\int_0^h [q^\sigma |(x + x^\sigma)x^\Delta|](t)\Delta t \leq \int_0^h q^\sigma(t)\left(\frac{y(t)}{\sqrt{q(t)}} + \frac{y^\sigma(t)}{\sqrt{q^\sigma(t)}}\right)\frac{y^\Delta(t)}{\sqrt{q^\sigma(t)}}\Delta t$$

$$\leq \int_0^h (y(t) + y^\sigma(t))y^\Delta(t)\Delta t$$

$$= y^2(h)$$

$$= \left\{\int_0^h \frac{1}{\sqrt{p(s)}}\sqrt{p(s)q^\sigma(s)}|x^\Delta(s)|\Delta s\right\}^2$$

$$\leq \left\{\int_0^h \frac{\Delta s}{p(s)}\right\}\left\{\int_0^h p(s)q^\sigma(s)|x^\Delta(s)|^2\Delta s\right\}.$$

Again we have used (2.7) and (2.7.4) for $p = 2$. $\qquad\qquad\qquad\qquad\qquad\square$

The following result involves higher order derivatives. As usual, we write f^{Δ^n} for the nth (delta) derivative of f.

Theorem 5.5.2. [see [7, Chapter 3]] Suppose $l, n \in \mathbb{N}$. For n-times differentiable function $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = x^\Delta(0) = \ldots = x^{\Delta^{n-1}}(0) = 0$ we have

$$\int_0^h \left|\left\{\sum_{k=0}^l x^k(x^\sigma)^{l-k}\right\}x^{\Delta^n}\right|(t)\Delta t \leq h^{ln}\int_0^h \left|x^{\Delta^n}(t)\right|^{l+1}\Delta t.$$

Proof. We consider

$$y(t) = \int_0^t \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2}\left\{\int_0^{\tau_1} \left|x^{\Delta^n}(s)\right|\Delta s\right\}\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{n-1}.$$

Hence, we have

$$y^\Delta(t) = \int_0^t \int_0^{\tau_{n-2}} \cdots \int_0^{\tau_2}\left\{\int_0^{\tau_1} \left|x^{\Delta^n}(s)\right|\Delta s\right\}\Delta\tau_1\Delta\tau_2\cdots\Delta\tau_{n-2},$$

$$\ldots, \quad y^{\Delta^{n-1}}(t) = \int_0^t \left|x^{\Delta^n}(s)\right|\Delta s, \quad y^{\Delta^n}(t) = \left|x^{\Delta^n}(t)\right|,$$

and for $0 \le t \le h$

$$
\begin{aligned}
|x(t)| &\le \int_0^t |x^\Delta(t_1)|\Delta t_1 \le \int_0^t \int_0^{t_1} |x^{\Delta\Delta}(t_2)|\Delta t_2 \Delta t_1 \le \ldots \le y(t) \\
&= \int_0^t y^\Delta(s)\Delta s \le \int_0^t y^\Delta(t)\Delta s \le \int_0^h y^\Delta(t)\Delta s = hy^\Delta(t) \\
&\le h^2 y^{\Delta\Delta}(t) \le \ldots \le h^{n-1}y^{\Delta^{n-1}}(t) = h^{n-1}f(t),
\end{aligned}
$$

where we put $f = y^{\Delta^{n-1}}$. Therefore,

$$
\begin{aligned}
\int_0^h \left| \left\{ \sum_{k=0}^l x^k (x^\sigma)^{l-k} \right\} x^{\Delta^n} \right| (t)\Delta t &\le \int_0^h \left\{ \sum_{k=0}^l |x|^k |x^\sigma|^{l-k} \left| x^{\Delta^n} \right| \right\} (t)\Delta t \\
&\le \int_0^h \left\{ \sum_{k=0}^l (h^{n-1}f)^k (h^{n-1}f^\sigma)^{l-k} \left| x^{\Delta^n} \right| \right\} (t)\Delta t \\
&= h^{(n-1)l} \int_0^h \left\{ \sum_{k=0}^l f^k (f^\sigma)^{l-k} f^\Delta \right\} (t)\Delta t \\
&= h^{(n-1)l} \int_0^h (f^{l+1})^\Delta (t)\Delta t \\
&= h^{(n-1)l} f^{l+1}(h) \\
&= h^{(n-1)l} \left\{ \int_0^h \left| x^{\Delta^n}(t) \right| \Delta t \right\}^{l+1} \\
&\le h^{(n-1)l} h^l \int_0^h \left| x^{\Delta^n}(t) \right|^{l+1} \Delta t \\
&= h^{nl} \int_0^h \left| x^{\Delta^n}(t) \right|^{l+1} \Delta t,
\end{aligned}
$$

where we have used (2.7.1), (2.7.3), and (2.7.4) with $p = (l+1)/l$ and $q = l+1$. $\quad\square$

The following two results are easy corollaries of the above Theorem 5.5.2.

Corollary 5.5.1. [see [7, Theorem 3.2.1]] Suppose $n \in \mathbb{N}$. For n-times differentiable function $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = x^\Delta(0) = \ldots = x^{\Delta^{n-1}}(0) = 0$ we have

$$
\int_0^h |(x + x^\sigma)x^{\Delta^n}|(t)\Delta t \le h^n \int_0^h \left| x^{\Delta^n}(t) \right|^2 \Delta t.
$$

Proof. This is Theorem 5.5.2 with $l = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Corollary 5.5.2. [see [7, Theorem 2.3.1]] Suppose $l \in \mathbb{N}$. For differentiable function $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = 0$ we have

$$\int_0^h \left| \left\{ \sum_{k=0}^l x^k (x^\sigma)^{l-k} \right\} x^\Delta \right| (t) \Delta t \leq h^l \int_0^h \left| x^\Delta(t) \right|^{l+1} \Delta t.$$

Proof. This is Theorem 5.5.2 with $n = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

5.6 Diamond-alpha Opial Inequalities

In this section, we give a sequence of Opial-type inequalities for first-order diamond alpha derivatives on time scales. Throughout, we say that a function $f : [0, h] \to \mathbb{R}$ is in the class $C_{\Diamond_\alpha}^1$ if f is $\Diamond_\alpha$-differentiable such that αf^Δ is rd-continuous, $(1 - \alpha)f^\nabla$ is ld-continuous, and $\alpha(1 - \alpha)f^{\Diamond_\alpha}$ is continuous. This ensures the existence of all occurring integrals. Note that $C_{\Diamond_\alpha}^1$ is, for $\alpha \in (0, 1)$, equal to the class of functions that are Δ-differentiable and ∇-differentiable such that f^Δ is rd-continuous, f^∇ is ld-continuous, and $f^{\Diamond_\alpha}$ is continuous. Moreover, $C_{\Diamond_0}^1$ is equal to the class of functions that are ∇-differentiable such that f^∇ is ld-continuous, and $C_{\Diamond_1}^1$ is equal to the class of functions that are Δ-differentiable such that f^Δ is rd-continuous. Firstly we introduce a set of Opial type Diamond-alpha Inequalities obtained by Bohner-Duman as given in [19].

Theorem 5.6.1. Let $\alpha \in [0, 1]$ and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C_{\Diamond_\alpha}^1$ with $f(0) = 0$ and $\alpha(1 - \alpha)f^\Delta f^\nabla \geq 0$, we have

$$\alpha^3 \int_0^h |(f + f^\sigma)f^\Delta|(t)\Delta t + (1 - \alpha)^3 \int_0^h |(f + f^\rho)f^\nabla|(t)\nabla t \leq h \int_0^h (f^\Delta)^2(t)\Diamond_\alpha t \quad (5.6.1).$$

Theorem 5.6.2. Let $\alpha \in [0,1]$ and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C^1_{\Diamond_\alpha}$ with $\alpha f^\Delta \geq 0$ and $(1-\alpha)f^\nabla \geq 0$, we have

$$\alpha^3 \int_0^h |(f+f^\sigma)f^\Delta|(t)\Delta t + (1-\alpha)^3 \int_0^h |(f+f^\rho)f^\nabla|(t)\nabla t$$
$$\leq h\beta \int_0^h (f^{\Diamond_\alpha})^2(t)\Diamond_\alpha t + 2\gamma(1-3\alpha+3\alpha^2)(f(h)-f(0)),$$

where $\beta := \min_{u \in [0,h] \cap \mathbb{T}} \max\{u, h-u\}$ and $\gamma := \max\{|f(0)|, |f(h)|\}$.

Corollary 5.6.1. Let $\alpha \in [0,1]$ and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C^1_{\Diamond_\alpha}$ with $\alpha f^\Delta \geq 0$, $(1-\alpha)f^\nabla \geq 0$, and $f(0) = f(h) = 0$, we have

$$\alpha^3 \int_0^h |(f+f^\sigma)f^\Delta|(t)\Delta t + (1-\alpha)^3 \int_0^h |(f+f^\rho)f^\nabla|(t)\nabla t \leq \beta \int_0^h (f^{\Diamond_\alpha})^2(t)\Diamond_\alpha t,$$

where β is as in Theorem 5.6.2.

Theorem 5.6.3. Let $\alpha \in [0,1]$ and $h \in \mathbb{T}$ with $h > 0$. Assume that $g : [0,h] \to \mathbb{R}^+$ is a nonincreasing continuous function. For any $f \in C^1_{\Diamond_\alpha}$ with $\alpha(1-\alpha)f^\Delta f^\nabla \geq 0$, we have

$$\alpha^3 \int_0^h [g^\sigma|(f+f^\sigma)f^\Delta|](t)\Delta t + (1-\alpha)^3 \int_0^h [g^\rho|(f+f^\rho)f^\nabla|](t)\nabla t$$
$$\leq h \int_0^h g(t)(f^{\Diamond_\alpha})^2(t)\Diamond_\alpha t.$$

5.7   Higher-Order Diamond-alpha Opial Inequalities

We continue to refer to [19] for the following higher-order Opial Inequalities of Diamond-alpha type. We say that a function $f : [0,h] \to R$ is in the class $C^n_{\Diamond_\alpha}$ if f is n times $\Diamond_\alpha$-differentiable such that αf^{Δ^n} is rd- continuous, $(1-\alpha)f^{\nabla^n}$ is ld-continuous, and $\alpha(1-\alpha)f^{\Diamond_\alpha^n}$ is continuous. This ensures the existence of all occuring integrals.

Lemma 5.7.1. Let $i \in \mathbb{N}$ and $i \in \mathbb{N}_0$. Assume f is $(i+j)$ times $\Diamond_\alpha$-differentiable. If $(1-\alpha)f^{\Diamond_\alpha^{i-1}\nabla\Delta^j} \geq 0$, then

$$f^{\Diamond_\alpha^i\Delta^j} \geq \alpha f^{\Diamond_\alpha^{i-1}\Delta^{j+1}},$$

and if $\alpha f^{\Diamond_\alpha^{i-1}\Delta\nabla^j} \geq 0$, then

$$f^{\Diamond_\alpha^i\nabla^j} \geq (1-\alpha)f^{\Diamond_\alpha^{i-1}\nabla^{j+1}}.$$

Lemma 5.7.2. Let $n \in \mathbb{N}$. Let f be n times $\Diamond_\alpha$-differentiable. If $(1-\alpha)f^{\Diamond_\alpha^{n-j-1}\nabla\Delta^j} \geq 0$ for all $j \in 0, ..., n-1$, then

$$f^{\Diamond_\alpha^n} \geq \alpha^n f^{\Delta^n},$$

and if $\alpha f^{\Diamond_\alpha^{n-j-1}\Delta\nabla^j} \geq 0$ for all $j \in 0, ..., n-1$, then

$$f^{\Diamond_\alpha^n} \geq (1-\alpha)^n f^{\nabla^n}.$$

Theorem 5.7.1. Let $m, n \in \mathbb{N}$, $\alpha \in [0,1]$, and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C_{\Diamond_\alpha}^n$ with $\alpha f^{\Delta^j}(0) = (1-\alpha)f^{\nabla^j}(0) = 0$, $(1-\alpha)f^{\Diamond_\alpha^{n-j-1}\nabla\Delta^j} \geq 0$, and $(1-\alpha)f^{\Diamond_\alpha^{n-j-1}\Delta\nabla^j} \geq 0$ for all $j \in 0, ..., n-1$ and $\alpha f^{\Delta^n} \geq 0$ and $(1-\alpha)f^{\nabla^n} \geq 0$, we have

$$\alpha^{n(m+1)+1} \int_0^h |If^{\Delta^n}|(t)\Delta t + (1-\alpha)^{n(m+1)+1} \int_0^h |Kf^{\nabla^n}|(t)\nabla t$$

$$\leq h^{nm} \int_0^h (f^{\Diamond_\alpha^n})^{(m+1)}(t)\Diamond_\alpha t.$$

where $I = \sum_{k=0}^m f^k(f^\sigma)^{m-k}$ and $K = \sum_{k=0}^m f^k(f^\rho)^{m-k}$.

Corollary 5.7.1. Let $m \in \mathbb{N}$, $\alpha \in [0,1]$, and $h \in \mathbb{T}$ with $h > 0$. For any $f \in C_{\Diamond_\alpha}^n$ $f(0) = 0$, $f^\nabla \geq 0$, and $f^\Delta \geq 0$, we have

$$\alpha^{m+2} \int_0^h |\left(\sum_{k=0}^m f^k(f^\sigma)^{m-k}\right) f^\Delta|(t)\Delta t + (1-\alpha)^{m+2} \int_0^h |\left(\sum_{k=0}^m f^k(f^\rho)^{m-k}\right) f^\nabla|(t)\nabla t$$

$$\leq h^m \int_0^h (f^{\Diamond_\alpha^n})^{(m+1)}(t)\Diamond_\alpha t.$$

5.8 Improved Diamond-alpha Opial Inequalities

In [12], the Theorem 5.6.1 is improved in the sense of removing the restriction given by the condition $\alpha(1-\alpha)f^\Delta f^\nabla \geq 0$ as well as the left hand side of (5.6.1) being refined to a compact form, including a single diamond-alpha integral. In this sense, the next theorem and its consequences extend and unify the previously obtained delta (5.2.1) and nabla (5.4.1) Opial dynamic inequalities in a much better way than that given by (5.6.1).

Theorem 5.8.1. Let $\mathbb{T}$ be a time scale. For $\Diamond_\alpha$ differentiable $x : [0,h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = 0$ we have

$$\int_0^h |(x^2)^{\Diamond_\alpha}|(t)\Diamond_\alpha t \leq h \int_0^h |x^{\Diamond_\alpha}|^2(t)\Diamond_\alpha t,$$

with equality when $x(t) = ct$.

Proof. Consider

$$y(t) = \int_0^t \left|x^{\Diamond_\alpha}(s)\right| \Diamond_\alpha s.$$

Then we have $y^\Delta = \left|x^\Delta\right|$, $y^\nabla = \left|x^\nabla\right|$ and $|x| \leq y$ so that

$$\int_0^h |(x^2)^{\Diamond_\alpha}|(t)\Diamond_\alpha t = \int_0^h \left|xx^{\Diamond_\alpha} + \alpha x^\sigma x^\Delta + (1-\alpha)x^\rho x^\nabla\right|(t)\Diamond_\alpha t$$

$$= \alpha \int_0^h \left|xx^{\Diamond_\alpha} + \alpha x^\sigma x^\Delta + (1-\alpha)x^\rho x^\nabla\right|(t)\Delta t$$

$$+ (1-\alpha) \int_0^h \left|xx^{\Diamond_\alpha} + \alpha x^\sigma x^\Delta + (1-\alpha)x^\rho x^\nabla\right|(t)\nabla t$$

$$\leq \alpha \int_0^h \left| xx^{\Diamond_\alpha} \right| (t) \Delta t + \alpha^2 \int_0^h \left| x^\sigma x^\Delta \right| (t) \Delta t$$

$$+ \alpha(1-\alpha) \int_0^h \left| x^\rho x^\nabla \right| (t) \Delta t + (1-\alpha) \int_0^h \left| xx^{\Diamond_\alpha} \right| (t) \nabla t$$

$$+ \alpha(1-\alpha) \int_0^h \left| x^\sigma x^\Delta \right| (t) \nabla t + (1-\alpha)^2 \int_0^h \left| x^\rho x^\nabla \right| (t) \nabla t$$

$$= \alpha^2 \int_0^h \left[(|x| + |x^\sigma|) \left| x^\Delta \right| \right] (t) \Delta t$$

$$+ (1-\alpha)^2 \int_0^h \left[(|x| + |x^\rho|) \left| x^\nabla \right| \right] (t) \nabla t$$

$$+ \alpha(1-\alpha) \int_0^h \left[(|x| + |x^\sigma|) \left| x^\Delta \right| \right] (t) \nabla t$$

$$+ \alpha(1-\alpha) \int_0^h \left[(|x| + |x^\rho|) \left| x^\nabla \right| \right] (t) \Delta t$$

$$\leq \alpha^2 \int_0^h \left[(y + y^\sigma) y^\Delta \right] (t) \Delta t + (1-\alpha)^2 \int_0^h \left[(y + y^\rho) y^\nabla \right] (t) \nabla t$$

$$+ \alpha(1-\alpha) \int_0^h \left[(y + y^\sigma) y^\Delta \right] (t) \nabla t$$

$$+ \alpha(1-\alpha) \int_0^h \left[(y + y^\rho) y^\nabla \right] (t) \Delta t$$

$$= \int_0^h (y^2)^{\Diamond_\alpha} (t) \Diamond_\alpha t$$

$$= \left[\int_0^h \left| x^{\Diamond_\alpha}(s) \right| \Diamond_\alpha s \right]^2$$

$$\leq h \int_0^h \left| x^{\Diamond_\alpha}(s) \right|^2 \Diamond_\alpha s.$$

where we have used the Fundamental theorem of calculus and Hölder's inequality in Chapter 4 for $h(x) = 1$ and $p = 2$. Let $x(t) = ct$ for some $c \in \mathbb{R}$. Then,

$$x^{\Diamond_\alpha}(t) = \alpha x^\Delta(t) + (1-\alpha) x^\nabla(t)$$

$$= \alpha c + (1-\alpha)c$$

$$= c$$

and it is easy to check that the equation holds. $\qquad \square$

Theorem 5.8.2. Let $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ be $\Diamond_\alpha$-differentiable function. Then

$$\int_0^h |(x^2)^{\Diamond_\alpha}|(t)\Diamond_\alpha t \leq \gamma \int_0^h |x^{\Diamond_\alpha}|^2(t)\Diamond_\alpha t + 2\beta \int_0^h |x^{\Diamond_\alpha}|(t)\Diamond_\alpha t,$$

where $\beta = \max\{|x(0), |x(h)|\}, \quad \gamma = \min_{u \in [0,h] \cap \mathbb{T}} \max\{u, h - u\}$ \hfill (5.8.1)

Proof. We consider $y(t) = \int_0^t |x^{\Diamond_\alpha}(s)|\Diamond_\alpha s$ and $z(t) = \int_t^h |x^{\Diamond_\alpha}(s)|\Diamond_\alpha s$

Then,

$$y^{\Diamond_\alpha} = |x^{\Diamond_\alpha}|, \quad z^{\Diamond_\alpha} = -|x^{\Diamond_\alpha}|, \quad y^\Delta = |x^\Delta|, \quad z^\Delta = -|x^\Delta|, \quad y^\nabla = |x^\nabla|, \quad z^\nabla = -|x^\nabla|$$

$$|x(t)| \leq |x(t) - x(0)| + |x(0)|$$

$$= |\int_0^t x^{\Diamond_\alpha}(s)\Diamond_\alpha s| + |x(0)|$$

$$\leq \int_0^t |x^{\Diamond_\alpha}(s)|\Diamond_\alpha s + |x(0)|$$

$$= y(t) + |x(0)|$$

and similarly $|x(t)| \leq z(t) + |x(h)|$.

Let $u \in [0, h] \cap \mathbb{T}$. Then

$$\int_0^h \left|(x^2)^{\Diamond_\alpha}\right|(t)\Diamond_\alpha t \leq \int_0^u \left[(y + |x(0)|)y^{\Diamond_\alpha} + \alpha(y^\sigma + |x(0)|)y^\Delta \right.$$

$$\left. + (1 - \alpha)(y^\rho + |x(0)|)y^\nabla\right](t)\Diamond_\alpha t$$

$$= y^2(u) - y^2(0) + 2|x(0)|y(u)$$

$$\leq u \int_0^u |x^{\Diamond_\alpha}(t)|^2 \Diamond_\alpha t + 2|x(0)| \int_0^u |x^{\Diamond_\alpha}(t)|\Diamond_\alpha t$$

where we have used the Fundamental theorem of calculus and Hölder inequality

(in Chapter 4) for $h(x) = 1$ and $p = 2$. Similarly we have

$$\int_u^h |(x^2)^{\Diamond_\alpha}|(t)\Diamond_\alpha t \le z^2(u) - z^2(h) + 2|x(h)|z(u)$$

$$\le (h - u) \int_u^h |x^{\Diamond_\alpha}(t)|^2 \Diamond_\alpha t + 2|x(h)| \int_u^h |x^{\Diamond_\alpha}(t)| \Diamond_\alpha t$$

Defining $\gamma = \min\limits_{u \in [0,h] \cup \mathbb{T}} \max\{u, h - u\}$, $\beta = \max\{|x(0)|, |x(h)|\}$ and adding the above two inequalities, we get

$$\int_0^h |(x^2)^{\Diamond_\alpha}|(t)\Diamond_\alpha t \le \gamma \int_0^h |x^{\Diamond_\alpha}(t)|^2 \Diamond_\alpha t + 2\beta \int_0^h |x^{\Diamond_\alpha}(t)| \Diamond_\alpha t.$$

$\square$

Theorem 5.8.3. Let $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ be $\Diamond_\alpha$-differentiable function with $x(0) = x(h) = 0$. Then

$$\int_0^h |(x^2)^{\Diamond_\alpha}(t)|\Diamond_\alpha t \le \gamma \int_0^h |x^{\Diamond_\alpha}(t)|^2 \Diamond_\alpha t,$$

where γ is given in (5.8.1).

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