Fall 2011

On Gorenstein Projective and Gorenstein Flat Modules

Chasen Grady Smith

ON GORENSTEIN PROJECTIVE AND GORENSTEIN FLAT MODULES

by

CHASEN GRADY SMITH

(Under the Direction of Alina C. Iacob)

ABSTRACT

H. Holm's metatheorem states,"Every result in classical homological algebra has a counterpart in Gorenstein homological algebra". We support this statement by showing over commutative Noetherian rings of finite Krull dimension, every Gorenstein flat module has finite Gorenstein projective dimension. This statement is the Gorenstein counterpart of a famous theorem of Gruson, Jensen, and Raynaud. Using this result we prove that over such rings, a module M having finite Gorenstein flat dimension is equivalent to M having finite Gorenstein projective dimension.

Key Words: module, projective, injective, flat, Krull dimension, Noetherian, Gorenstein

2009 *Mathematics Subject Classification*:

ON GORENSTEIN PROJECTIVE AND GORENSTEIN FLAT MODULES

by

CHASEN GRADY SMITH

B.S. in Mathematics

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial Fulfillment

of the Requirement for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

2011

ON GORENSTEIN PROJECTIVE AND GORENSTEIN FLAT MODULES

by

CHASEN GRADY SMITH

Major Professor: Alina C. Iacob

Committee: David R. Stone

Andrew V. Sills

Electronic Version Approved:

December 2011

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

Homological Algebra is at the root of modern techniques in many areas, including commutative algebra and algebraic geometry. While classical homological algebra can be viewed as based on projective, injective, and flat modules, Gorenstein homological algebra is its relative version that uses Gorenstein projective, Gorenstein injective, and Gorenstein flat modules.

The methods of Gorenstein homological algebra play a part in commutative and non-commutative algebra, in algebraic geometry, and in triangulated category theory. It also has applications to mathematical physics and to knot theory.

There is an active program in Gorenstein homological algebra. It is partly motivated by H. Holm's metatheorem, that states "Every result in classical homological algebra has a counterpart in Gorenstein homological algebra".

We support this statement by proving the Gorenstein counterpart of a famous result of Gruson, Jensen, and Raynaud. They showed that over commutative Noetherian rings of finite Krull dimension, every flat module has finite projective dimension. We show that over commutative Noetherian rings of finite Krull dimension, every Gorenstein flat module has finite Gorenstein projective dimension.

Using this result, we prove the following: Let R be a commutative Noetherian ring of finite Krull dimension. If M is a module over R, then M having finite Gorenstein flat dimension is equivalent to M having finite Gorenstein projective dimension.

Previously the result was only known for a more restrictive class of rings, that of commutative, noetherian rings with dualizing complexes. We note that throughout this thesis we adopt and state many well-known definitions and propositions from [3].

CHAPTER 2

PRELIMINARIES

2.1 Projective, Injective, and Flat Modules

The idea of a module over a ring is a generalization of the notion of a vector space. When the ring is a field the axioms for a module are precisely the same as those for a vector space. Since we will mainly consider different types of modules, we recall the following definition for a module M over a ring R.

Definition 2.1.1

Let R be a ring (not necessarily commutative nor with 1). A left R-module or a left module over R is a set M together with

1. *a binary operation $+$ on M under which M is an abelian group, and*
2. *an action of R on M (that is, a map $R \times M \to M$) denoted by rm, such that for all $m, n \in M$ and for all $r, s \in R$*
   (a) $(r+s)m = rm + sm$,
   (b) $(rs)m = r(sm)$, *and*
   (c) $r(m+n) = rm + rn$.

   If the ring R has a 1 we impose the additional axiom,

   (d) $1m = m$.

If $(rs)m = r(sm)$ is replaced by $(sr)m = r(sm)$, then M is said the be a *right* R-module, and we denote the image of (r, x) by xr and so $(sr)m = r(sm)$ becomes $(sr)m = (ms)r$. For the sake of notation, we will denote a left R-module M by ${}_RM$

and similarly denote a right R-module N by N_R. If the ring R is *commutative* and M is a left R-module, then M is also a right R-module. If R is *not* commutative, Axiom $2(b)$ in Definition 2.1.1 will not hold in general. So not every left R-module is also a right R-module. For the most part we will assume the ring R to be commutative and with 1, unless otherwise stated.

Definition 2.1.2

If M and N are R-modules, the set of all the R-homomorphisms from M to N will be denoted by $\mathrm{Hom}_R(M, N)$.

Note that $\mathrm{Hom}_R(M, N)$ is an abelian group under addition. Any R-module homomorphism is a homomorphism of the additive groups, but not every group homomorphism need be a module homomorphism.

Proposition 2.1.3

If M is an R-module, then the map $\varphi : \mathrm{Hom}_R(R, M) \to M$ defined by $\varphi(f) = f(1)$ is an R-isomorphism.

PROOF:

Let $f \in \mathrm{Hom}_R(R, M)$. If $f(1) = 0$,

$$f(r) = f(r \cdot 1) = r \cdot f(1) = 0, \ \forall r \in R.$$

If f is not the zero function, the $\varphi(f) = f(1) \neq 0$.

Now we claim φ is a bijective R-module homomorphism. For all $f, g \in \mathrm{Hom}_R(R, M)$

and all $r \in R$, φ is well-defined and

$$\begin{aligned}\varphi(f+rg) &= (f+rg)(1) \\ &= f(1)+rg(1) \\ &= f(1)+(rg)(1) \\ &= f(1)+r(g(1)) \\ &= \varphi(f)+r\varphi(g)\end{aligned}$$

So φ is an R-module homomorphism. Next suppose that $\varphi(f) = \varphi(g)$ for any $f, g \in \mathrm{Hom}_R(R, M)$. Then

$$\begin{aligned}\varphi(f) = \varphi(g) &\Longrightarrow f(1) = g(1) \\ &\Longrightarrow f(1) - g(1) = 0 \\ &\Longrightarrow (f-g)(1) = 0 \\ &\Longrightarrow (f-g)(r) = r(f-g) = 0\end{aligned}$$

Hence $f - g$ is the zero function. Thus $f = g$, implying that φ is injective. Now let $m \in M$. Let $f : R \to M$ be defined by $f(r) = rm$. Then

$$f \in \mathrm{Hom}_R(R, M) \text{ and } m = f(1).$$

□

From the previous proposition, we see that $\mathrm{Hom}_R(R, M) \simeq M$. In particular, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$. We also have that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$.

Definition 2.1.4

A left R-module M is finitely generated if M is generated by a finite set.

For example, a vector space V over a field K is a finitely generated K-module if and only if V is finite-dimensional.

For any ring R with 1, the (left) R-module R is finitely generated; it is generated by 1.

2.2 Categories and Functors

Definition 2.2.1

A category $\mathbf{C}$ *consists of the following.*

1. *A class of objects, denoted* $Ob(\mathbf{C})$.
2. *For any pair* $A, B \in Ob(\mathbf{C})$*, a set denoted* $\mathrm{Hom}_{\mathbf{C}}(A, B)$ *with the property that* $\mathrm{Hom}_{\mathbf{C}}(A, B) \cap \mathrm{Hom}_{\mathbf{C}}(A', B') = \varnothing$ *whenever* $(A, B) \neq (A', B')$. $\mathrm{Hom}_{\mathbf{C}}(A, B)$ *is called the set of* morphisms *from* A *to* B.
3. *A* composition $\mathrm{Hom}_{\mathbf{C}}(B, C) \times \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{C}}(A, C)$ *for all objects* $A, B, C \in Ob(\mathbf{C})$*, denoted* $(g, f) \mapsto gf$ *(or* $g \circ f$*), satisfying the following properties:*
   - *(i) for each* $A \in Ob(\mathbf{C})$*, there is an* identity morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$ *such that* $f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f$ *for all* $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$,
   - *(ii)* $h(gf) = (hg)f$ *for all* $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathbf{C}}(B, C)$*, and* $h \in \mathrm{Hom}_{\mathbf{C}}(C, D)$.

Example of categories include sets, abelian groups, topological spaces, and left R-modules. Their morphisms are functions, group homomorphisms, continuous maps, and R-homomorphisms, respectively, with usual composition. We will denote the category of all left R-modules and the category of abelian groups ${}_{\mathbf{R}}\mathbf{Mod}$ and $\mathbf{Ab}$, respectively.

An important notion is that of a functor, which is defined in terms of categories.

Definition 2.2.2

If $\mathbf{C}$ and $\mathbf{D}$ are categories, then we say that we have a functor $F : \mathbf{C} \to \mathbf{D}$ if we have

1. *a function $F : Ob(\mathbf{C}) \to Ob(\mathbf{D})$*
2. *functions $F : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(F(A), F(B))$ such that*
   - *(i) if $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathbf{C}}(B, C)$, then $F(gf) = F(g)F(f)$, and*
   - *(ii) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for each $A \in Ob(\mathbf{C})$.*

If $\mathbf{C}$ is a category and $A \in Ob(\mathbf{C})$, then the Hom functor $T_A : \mathbf{C} \to \mathbf{Sets}$, usually denoted $\mathrm{Hom}(A, -)$, is defined by

$$T_A(B) = \mathrm{Hom}(A, B) \text{ for all } B \in Ob(\mathbf{C}),$$

and if $f : B \to B'$ in $\mathbf{C}$, then $T_A(f) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$ is defined by

$$T_A(f) = h \mapsto fh.$$

We call $T_A(f) = \mathrm{Hom}(A, f)$ the *induced map.*

We are mainly interested in the category ${}_{\mathbf{R}}\mathbf{Mod}$. We will consider the Hom functors

$$\mathrm{Hom}(A,-) : {}_{\mathbf{R}}\mathbf{Mod} \to \mathbf{Ab} \qquad B \mapsto \mathrm{Hom}(A,B),$$

which associates an R-homomorphism $f : B \to B'$ with $\mathrm{Hom}(A,f) : \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B')$ defined by $\mathrm{Hom}(A,f)(h) = fh$, and

$$\mathrm{Hom}(-,M) : {}_{\mathbf{R}}\mathbf{Mod} \to \mathbf{Ab} \qquad B \mapsto \mathrm{Hom}(B,M),$$

which associates an R-homomorphism $g : B \to B'$ with $\mathrm{Hom}(g,M) : \mathrm{Hom}(B',M) \to \mathrm{Hom}(B,M)$ defined by $\mathrm{Hom}(g,M)(h) = hg$.

Since, in a way, we think of modules as generalized vector spaces, we give the following example to view functors in this context. Recall that a linear functional on a vector space V over a field K is a linear transformation $T : V \to K$. The dual space of V is $V^* = \mathrm{Hom}_K(V,K)$. Now V^* is a K-module if we define $af : V \to K$ by

$$af : v \mapsto a[f(v)]$$

for any $f \in V^*$ and any $a \in K$. That is, V^* is a vector space over K. The dual space functor is $\mathrm{Hom}_K(-,K)$.

Besides the Hom functors, another key ingredient in the definition of Gorenstein projective modules is that of an exact complex. We recall first that a *complex* of R-modules is a sequence of R-modules and R-homomorphisms

$$\cdots \longrightarrow M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} \cdots$$

such that $f_i f_{i+1} = 0$ for each integer i. That is, $\mathrm{Im}\, f_{i+1} \subseteq \mathrm{Ker}\, f_i$.

Definition 2.2.3

A sequence

$$\cdots \longrightarrow M_2 \longrightarrow M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} M_{-1} \xrightarrow{f_{-1}} M_{-2} \longrightarrow \cdots,$$

with each M_i being an R-module and each f_i an R-homomorphism, is said to be exact at M_i if $\operatorname{Im} f_{i+1} = \operatorname{Ker} f_i$. The sequence is said to be exact if it is exact at each M_i. An exact sequence of the the form $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is said to be a short exact sequence.

For example, A sequence $0 \longrightarrow A \xrightarrow{f} B$ of R-modules is exact if and only if f in injective, and a sequence $B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if g is surjective.

A particular case of a short exact sequence is that of a split exact sequence. Since we work with projective and injective modules, and both classes can be characterized in terms of split exact sequences, we also recall the following definition.

Definition 2.2.4

The short exact sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ of R-modules is said to be split exact if $\operatorname{Im} f$ is a direct summand of M.

Remark. In the short exact sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$, $M' \simeq \operatorname{Im} f$. So $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is split exact if and only if $M \simeq M' \oplus M''$.

The exact sequence notation is a convenient way to analyze the extent to which the properties of M' and M'' determine the properties of M.

Remark. To say $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact at M is the same as saying that the sequence $0 \longrightarrow \operatorname{Im} f \longrightarrow M \longrightarrow M/\operatorname{Ker} g \longrightarrow 0$ is a short exact sequence.

For example, let A and C be R-modules. Then we can form their direct sum $B = A \oplus C$. Then the sequence

$$0 \longrightarrow A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \longrightarrow 0$$

where $\iota(a) = (a, 0)$ and $\pi(a, c) = c$ is a split exact sequence.

The following result characterizes short exact sequences that are split exact.

Proposition 2.2.5

Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence of R-modules. Then the following are equivalent.

1. *The sequence is split exact.*
2. *There exists an R-homomorphism $f' : M \longrightarrow M'$ such that $f' \circ f = \mathrm{id}_{M'}$.*
3. *There exists an R-homomorphism $g'' : M'' \longrightarrow M'$ such that $g \circ g'' = \mathrm{id}_{M''}$.*

PROOF:

We will only show that (1.) is equivalent to (2.) and note that (1.) and (3.) are equivalent by a similar argument.

(1.) $\Longrightarrow$ (2.)

Suppose the sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is split exact. Then $M = \mathrm{Im}\, f \oplus G$ for some R-module G. Consider the sequence

$$0 \longrightarrow \mathrm{Im}\, f \xrightarrow{\iota} \mathrm{Im}\, f \oplus G \longrightarrow M'' \longrightarrow 0$$

with $\iota(x) = (x, 0)$. Then there exists an R-homomorphism $f' : M \to \mathrm{Im}\, f$ defined by

$f'(x,y) = x$. So for any $x \in \operatorname{Im} f$,

$$(f' \circ \iota)(x) = f'(\iota(x)) = f'((x,0)) = x.$$

Hence $f' \circ \iota = \mathrm{id}_{\operatorname{Im} f}$. Since $\operatorname{Im} f \simeq M'$, $f' \circ f = \mathrm{id}_{M'}$ as desired.

(2.) $\Longrightarrow$ (1.)

Suppose there exists an R-homomorphism $f' : M \longrightarrow M'$ such that $f' \circ f = \mathrm{id}_{M'}$ and define a map $\varphi : M \to M' \oplus M''$ by $\varphi(m) = (f'(m), g(m))$. Note that φ is an R-homomorphism. Now suppose $\varphi(m) = (0,0)$ for any $m \in M$. Then $f'(m) = 0$ and $g(m) = 0$. By having exactness at M,

$$g(m) = 0 \Longrightarrow m = f(m')$$

for some $m' \in M'$. Thus $0 = f'(m) = f'(f(m')) = m'$ by assumption. Hence $m = f(m') = f(0) = 0$. Therefore $\operatorname{Ker} \varphi = \{0\}$ implying that φ is injective.

To show that φ is surjective, let $(m', m'') \in M' \oplus M''$. Since g is surjective, $m'' = g(m)$ for some $m \in M$. So,

$$m'' = g(m) = g(m + f(x))$$

for any $x \in M'$. To have $\varphi(m + f(x)) = (m', m'')$, we need $x \in M'$ such that

$$\begin{aligned} m' &= f'(m + f(x)) \\ &= f'(m) + f'(f(x)) \\ &= f'(m) + x \end{aligned}$$

So choose $x = m' - f'(m)$. Then,

$$\begin{aligned}\varphi(m + f(x)) &= (f'(m + f(x)), g(m + f(x))) \\ &= (f'(m) + f'(f(x)), g(m) + g(f(x))) \\ &= (f'(m) + x, g(m) + 0) \\ &= (m', m'')\end{aligned}$$

Thus φ is bijective, and hence $\operatorname{Im}\varphi = M' \oplus M''$, making the sequnce $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ split exact.

□

We now look at what happens to an exact sequence after the Hom functor is applied to a sequence of R-modules.

Proposition 2.2.6

The following statements hold.

1. *If* $0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N''$ *is an exact sequence of* R*-modules, then for any module* A *the sequence*

$$0 \longrightarrow \operatorname{Hom}_R(A, N') \xrightarrow{\operatorname{Hom}(A,f)} \operatorname{Hom}_R(A, N) \xrightarrow{\operatorname{Hom}(A,g)} \operatorname{Hom}_R(A, N'')$$

   is exact.

2. *If* $M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *is an exact sequence of* R*-modules, then for any module* A *the sequence*

$$0 \longrightarrow \operatorname{Hom}_R(M'', A) \xrightarrow{\operatorname{Hom}(g,A)} \operatorname{Hom}_R(M, A) \xrightarrow{\operatorname{Hom}(f,A)} \operatorname{Hom}_R(M', A)$$

   is also exact.

Proof:

We only show the proof for (1.); a similar argument follows to show (2.) to be true. Let $h \in \operatorname{Hom}_R(A, N')$ such that $\operatorname{Hom}(A, f)(h) = 0$. Then $fh = 0$, and since f is injective, $h = 0$. That is, $\operatorname{Ker}(\operatorname{Hom}(A, f)) = \{0\}$. Hence $\operatorname{Hom}(A, f)$ is injective.

Now let $\sigma \in \operatorname{Hom}_R(A, N')$. Since $gf = 0$,

$$(\operatorname{Hom}(A, g) \circ \operatorname{Hom}(A, f))(\sigma) = gf\sigma = 0.$$

Hence $\operatorname{Im}(\operatorname{Hom}(A, f)) \subset \operatorname{Ker}(\operatorname{Hom}(A, g))$.

Let $\tau \in \operatorname{Ker}(\operatorname{Hom}(A, g))$. Then $\operatorname{Hom}(A, g)(\tau) = g\tau = 0$. So, $\operatorname{Im}\tau \subset \operatorname{Ker} g = \operatorname{Im} f$. Note that $f : N' \to \operatorname{Im} f$ is a bijective map. So there is a function $f^{-1} : \operatorname{Im} f \to N'$. Thus let $\sigma : A \to N'$ be defined by $\sigma = f^{-1}\tau$. Then

$$\operatorname{Hom}(A, f)(\sigma) = f\sigma = \tau.$$

That is, $\tau \in \operatorname{Im}(\operatorname{Hom}(A, f))$. Therefore $\operatorname{Im}(\operatorname{Hom}(A, f)) = \operatorname{Ker}(\operatorname{Hom}(A, g))$, which implies exactness at $\operatorname{Hom}_R(A, N)$.

□

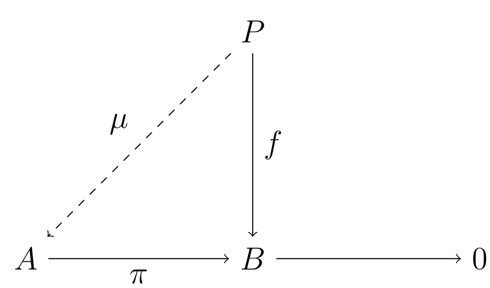
Notice that if $M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is an exact sequence then $\operatorname{Hom}_R(A, M'') \longrightarrow \operatorname{Hom}_R(A, M) \longrightarrow \operatorname{Hom}_R(A, M') \longrightarrow 0$ is *not necessarily* exact. For example, consider the exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$, with $n \in \mathbb{Z}$, $n \geq 2$ and π being the canonical surjection. Applying the functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ to this exact sequence, we obtain the sequence $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$, which is not exact at $\mathbb{Z}/n\mathbb{Z}$.

2.3 Projective, Injective, and Flat Modules

We now introduce the class of projective modules. As already noted, together with the injective and flat modules, projective modules are fundamental in classical homological algebra.

Definition 2.3.1

An R-module P is said to be projective if given an exact sequence $A \xrightarrow{\pi} B \longrightarrow 0$ of R-modules and an R-homomorphism $f : P \to B$, then there exists an R-homomorphism $\mu : P \to A$ such that $f = \pi \circ \mu$.



This definition is equivalent to saying that P is a projective module if given any exact sequence $A \longrightarrow B \longrightarrow 0$, then the sequence $\mathrm{Hom}(P, A) \longrightarrow \mathrm{Hom}(P, B) \longrightarrow 0$ is also exact.

For the category of modules there is another description of projective modules involving free modules.

Definition 2.3.2

An R-module F is said to be free if it is a direct sum of copies of R.

The connection between free modules and projective modules is given in the following theorem.

Proposition 2.3.3

The following are equivalent for an R-module P.

1. *P is projective.*
2. *$\mathrm{Hom}(P, -)$ is right exact.*
3. *Every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ is split exact.*
4. *P is a direct summand of a free R-module.*

PROOF:

See the proof of Theorem 2.1.2 on page 40 of [3].

□

Proposition 2.3.4

Every free R-module is projective.

PROOF:

Immediate from the proof of Proposition 2.3.3 □

For example R and R^n are projective R-modules because they can be written as a direct sum of copies of R. However not every projective module is free. For example, $\mathbb{Z}$ regarded as a module over $R = \mathbb{Z} \oplus \mathbb{Z}$ is projective (as a direct summand of R), but it is too "small" to be a free R-module.

By [3], Proposition 1.2.2, for any R-module M there exists a surjective R-homomorphism $P_0 \longrightarrow M$ with P_0 a free module.

Using this we can construct for each ${}_RM$ an exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each P_i projective. Such a sequence is called a *projective resolution* of M.

Using projective resolutions, one can find the Ext modules. Let $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a projective resolution of an R-module M. Consider the *deleted projective resolution*

$$\cdots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \longrightarrow 0.$$

By applying the functor $\mathrm{Hom}(-, A)$ to this deleted resolution, we obtain the following complex:

$$0 \longrightarrow \mathrm{Hom}_R(P_0, A) \xrightarrow{\alpha_1} \mathrm{Hom}_R(P_1, A) \xrightarrow{\alpha_2} \mathrm{Hom}_R(P_2, A) \xrightarrow{\alpha_3} \cdots .$$

We note that this complex is not, in general, an exact one. The module $\mathrm{Ext}^i_R(M, A)$ is by definition the i^{th} homology module of this complex; that is,

$$\mathrm{Ext}^i_R(M, A) = {}^{\mathrm{Ker}\,\alpha_{i+1}} \Big/ {}_{\mathrm{Im}\,\alpha_i}$$

for any R-module A. Note that $\mathrm{Ext}^0_R(M, A) = \mathrm{Ker}\,\alpha_1 \simeq \mathrm{Hom}_R(M, A)$ (by Proposition 2.2.6). That is, $\mathrm{Ext}^i_R(M, A)$ is a measure of how close this complex is to being exact.

Theorem 2.3.5

Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of R-modules. Then there is a long exact sequence of abelian groups

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(M'', A) \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M', A) \\ \longrightarrow \mathrm{Ext}^1(M'', A) \longrightarrow \mathrm{Ext}^1(M, A) \longrightarrow \mathrm{Ext}^1(M', A) \\ \longrightarrow \mathrm{Ext}^2(M'', A) \longrightarrow \mathrm{Ext}^2(M, A) \longrightarrow \mathrm{Ext}^2(M', A) \\ \longrightarrow \mathrm{Ext}^3(M'', A) \longrightarrow \cdots \end{aligned}$$

for any R-module A.

PROOF:

See the proof of Theorem 8 on page 784 of [2].

□

Proposition 2.3.6

If P is a projective module, then $\mathrm{Ext}^i_R(P, A) = 0$ for all R-modules A and for all $i \geq 1$.

PROOF:

In general a projective resolution is infinite in length but if P is projective , then P has a simple projective resoultion:

$$0 \longrightarrow P \xrightarrow{\mathrm{id}_P} P \longrightarrow 0$$

Then the deleted projective resolution would just be $0 \longrightarrow P \longrightarrow 0$.

□

The converse is also true (by [3], Proposition 8.4.3). So we have the following characterization of projective modules.

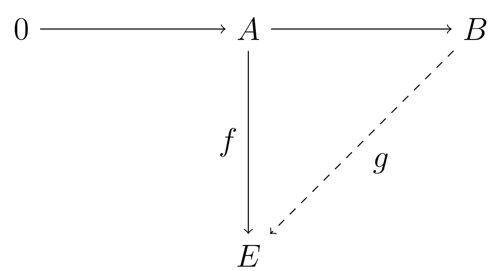
Proposition 2.3.7

An R-module P is projective if and only if $\operatorname{Ext}^i_R(P, A) = 0$ *for all R-modules A and for all $i \geq 1$.*

The dual notion of a projective module is the injective module.

Definition 2.3.8

An R-module E is said to be injective *if given R-modules $A \subset B$ and a homomorphism $f : A \to E$, then there exits a homomorphism $g : B \to E$ such that $g\big|_A = f$.*



Remark. By [3], Theorem 3.1.14, every R-module can be embedded in an injective module. Consequently every R-module N has an exact sequence $0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$ with each E^i injective. This sequence is called an *injective resolution* of N.

Over Principal Ideal Domains, there is another description of injective modules. The following Theorem gives a necessary and sufficient condition for when an R-module is injective. Since the result uses Baer's Criterion, we first recall the following theorem.

Theorem 2.3.9 (*Baer's Criterion*)

An R-module E is injective if and only if for all ideals I of R, every homomorphism $f : I \to E$ can be extended to R.

Proof:

See the proof of Theorem 3.1.3 on page 69 of [3].

□

Definition 2.3.10

A left R-module M is said to be divisible if $rM = M$ for all nonzero $r \in R$.

Proposition 2.3.11

Let R be a Principal Ideal Domain. Then an R-module M is injective if and only if it is divisible.

Proof:

($\Longrightarrow$)

Let $m \in M$ and $r \in R$ be a nonzero divisor. Define a map $f : \langle r \rangle \to M$ by $f(sr) = sm$. Extend the map f to $g : R \to M$ such that

$$m = f(r) = g(r) = rg(1).$$

Thus M is divisible.

($\Longleftarrow$)

Let I be an ideal of R and $f : I \to M$ be an R-homomorphism. Extend f to R for $I \neq 0$. But R is a Principal Ideal Domain, and so $I = \langle s \rangle$ for some $s \in R - \{0\}$. If M is divisible, then there exists $m \in M$ such that $f(s) = sm$. Define $g : R \to M$ by $g(r) = rm$. Then $g\big|_I = f$ for if $r' \in R$ then

$$g(r's) = r'sm = r'f(s) = f(r's).$$

□

As an example, $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module. Thus by the previous Theorem, $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. On the other hand since $\mathbb{Z}$ is not divisible, $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module.

There is another important class of modules — flat modules. These are defined not in terms of Hom as projective and injective modules are, but defined using the tensor product.

Definition 2.3.12

Let M be a right R-module, N a right R-module, and G an abelian group. Then a map $\sigma : M \times N \to G$ is said to be bilinear if it is biadditive, that is,

$$\sigma(x + x', y) = \sigma(x, y) + \sigma(x', y),$$

$$\sigma(x, y + y') = \sigma(x, y) + \sigma(x, y'),$$

$$\sigma(xr, y) = \sigma(x, ry)$$

for all $x, x' \in M,\ y, y' \in N,\ r \in R$.

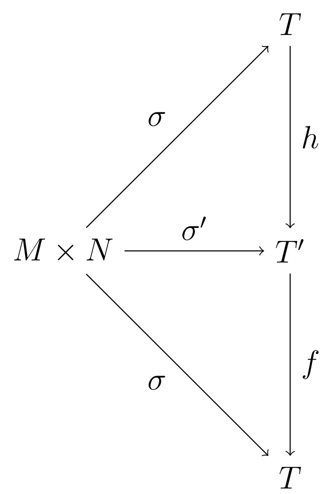
Definition 2.3.13

The map $\sigma : M \times N \to G$ is said to be a universal balanced map if for every abelian group G' and bilinear map $\sigma' : M \times N \to G'$, there exists a unique map $h : G \to G'$ such that $\sigma' = h\sigma$.

Definition 2.3.14

A tensor product of a right R-module M and left R-module N is an abelian group T together with a universal balanced map $\sigma : M \times N \to T$

If $\sigma : M \times N \to T$ and $\sigma' : M \times N \to T'$ are both universal balanced maps, then the following diagram commutes.



Notice that $fh = \mathrm{id}_T$ and $hf = \mathrm{id}_{T'}$, which implies h is an isomorphism. Thus tensor products are unique up to isomorphism. The tensor product of M_R and ${}_RN$ is denoted by $M \otimes_R N$, and it is known that the tensor product exists. (See for example [3], Theorem 1.2.19, for a construction of the tensor product.)

Definition 2.3.15

An R-module F is said to be flat if given any exact sequence $0 \longrightarrow A \longrightarrow B$ of right R-modules, the sequence $0 \longrightarrow A \otimes_R F \longrightarrow B \otimes_R F$ is exact.

Proposition 2.3.16

Every projective module is flat.

PROOF:

Suppose that P is a projective module. Then P is a direct summand of a free module F, say $F = P \oplus P'$. If the map $\psi : M' \to M$ of R-modules M' and M is injective then $1 \otimes \psi : F \otimes_R M' \to F \otimes_R M$ is also injective. Hence

$$1 \otimes \psi : (P \otimes_R M') \oplus (P' \otimes_R M') \to (P \otimes_R M) \oplus (P' \otimes_R M)$$

is injective. Thus $1 \otimes \psi : (P \otimes_R M') \to (P \otimes_R M)$ is injective.

□

The $\mathbb{Z}$-module $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. To see this is in fact true, suppose $\psi : L \to M$ is an injective map of $\mathbb{Z}$-modules. Then $(1/d) \otimes l \in \mathbb{Q} \otimes_{\mathbb{Z}} L$ for some nonzero integer d and some $l \in L$. If $(1/d) \otimes l \in \text{Ker}(1 \otimes \psi)$ then $(1/d) \otimes \psi(l)$ is 0 in $\mathbb{Q} \otimes_{\mathbb{Z}} M$. This means $c\psi(l) = 0$ in M for some nonzero integer c. Hence $\psi(cl) = 0$. By the injectivity of ψ, we have $cl = 0$ in L. This implies

$$(1/d) \otimes l = (1/cd) \otimes (cl) = 0$$

in L. Thus $1 \otimes \psi$ is injective.

As an example, $\mathbb{Z}$ is flat because it is a projective $\mathbb{Z}$-module. The arbitrary direct sum of flat modules is flat. In particular, $\mathbb{Q} \oplus \mathbb{Z}$ is flat. This module is neither projective nor injective because $\mathbb{Q}$ is not projective and $\mathbb{Z}$ is not injective.

An exact sequence $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ with each F_i flat is called a *flat resolution* of M.

Remark. Since every R-module M has a projective resolution and every projective module is flat (by Proposition 2.3.16), it follows that every module has a flat resolution.

Using flat resolutions, one can define the Tor modules:

Let $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ be a flat resolution of a left R-module M and

D be a right R-module. Then by applying the functor $D \otimes -$ to the deleted flat resolution, we obtain the complex

$$\cdots \longrightarrow D \otimes F_2 \xrightarrow{\beta_2} D \otimes F_1 \xrightarrow{\beta_1} D \otimes F_0 \longrightarrow 0.$$

The group $\operatorname{Tor}_i^R(D, M)$ is called the i^{th} homology group and is computed by

$$\operatorname{Tor}_i^R(D, M) = \operatorname{Ker} \beta_i \big/ \operatorname{Im} \beta_{i+1}$$

for all $i \geq 1$, and $\operatorname{Tor}_0^R(D, M) = (D \otimes F_0) \big/ \operatorname{Im} \beta_1$.

The homological dimensions — projective, injective, and flat — are defined in terms of resolutions (projective, injective, and flat, respectively).

Definition 2.3.17

The minimal length of a finite projective resolution $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ *of an* R*-module* M *is called the projective dimension of* M*, denoted* $\operatorname{pd}_R M$*. If* M *does not admit a finite projective resolution then the projective dimension is infinite.*

The *injective dimension* and *flat dimension* of an R-module are defined similarly using injective and flat resolutions, respectively. They are denoted as $\operatorname{inj\,dim} M$ and $\operatorname{flat\,dim} M$, respectively.

2.4 Noetherian Rings

Since our main result concerns Noetherian rings, we recall some basic facts about this class of rings.

Definition 2.4.1

1. *An R-module M is said to be Noetherian if every ascending chain of submodules of M terminates.*

2. *A ring R is said to be Noetherian if it is Noetherian as a left module over itself.*

When R is considered as a left module over itself, its R-submodules are precisely its ideals. Thus every Principal Ideal Domain is Noetherian.

Our main result concerns commutative Noetherian rings of finite Krull dimension. For this reason we also recall the following definition.

Definition 2.4.2

A prime ideal P of a ring R is an ideal such that $P \neq R$ and if $ab \in P$, then either $a \in P$ or $b \in P$ for all $a, b \in R$.

Definition 2.4.3

The Krull dimension of R, denoted $\dim R$, is the supremum of the number of strict inclusions in a chain of prime ideals.

A field has Krull dimension 0, and a Principal Ideal Domain that is not a field has Krull dimension 1. For example, $\mathbb{Z}$ and $k[x]$, where k is a field, both have Krull dimension 1 because they are both Principal Ideal Domains.

Now we introduce a theorem by Gruson, Jensen, and Raynaud. We will use this theorem as the foundation for our main result, which is the Gorenstein counterpart of this famous theorem.

Theorem 2.4.4 (*[7], Gruson-Jensen-Raynaud*)

Over commutative, Noetherian rings of finite Krull dimension every flat module has finite projective dimension.

2.5 Gorenstein Rings

The property of being Gorenstein imposes nice properties on modules over such rings.

Definition 2.5.1

A ring R is said to be an Iwanaga-Gorenstein ring *(or just a Gorenstein ring) if R is both left and right Noetherian and if R has finite self-injective dimension both as a left and as a right R-module.*

It is known (see [3], Proposition 9.1.8 for instance) that if the ring R is both left and right Noetherian and if R has finite injective dimension both as a left R-module and as a right R-module then the two injective dimensions coincide ($\operatorname{inj\,dim}{}_R R = \operatorname{inj\,dim} R_R$). Of course if R is a commutative ring then $\operatorname{inj\,dim}_R R = \operatorname{inj\,dim} R_R$ to begin with and Definition 2.5.1 above simply requires this dimension to be finite.

Proposition 2.5.2 (*[3], Proposition 9.1.2*)

If R is left (right) Noetherian and the left (right) self-injective dimension of R is $n < \infty$, then $\operatorname{inj\,dim} F \leq n$ for every flat left (right) R-module. And if $\operatorname{flat\,dim} M < \infty$ for a left (right) R-module M, then $\operatorname{proj\,dim} M \leq n$.

CHAPTER 3

GORENSTEIN PROJECTIVE AND GORENSTEIN FLAT MODULES

3.1 Gorenstein Projective Modules

In 1966 Auslander defined the notion of the G-dimension of a finitely generated module over a commutative Noetherian local ring. In 1969 Auslander and Bridger extended the definition to two-sided Noetherian rings. Calling the modules of G-dimension zero Gorenstein projective modules, in 1995 Enochs and Jenda defined the Gorenstein projective modules (whether finitely generated or not) and Gorenstein injective modules over arbitrary rings. These concepts are generalizations of projective and injective. Avramov, Buchweitz, Martsinkovsky, and Reiten proved that if the ring R is both right and left Noetherian and G is a finite Gorenstein projective module, then Enochs' and Jenda's definition agrees with that of Auslander and Bridger.

We start by recalling the definition of Gorenstein projective modules.

Definition 3.1.1

A module M is said to be Gorenstein projective if there is a $\operatorname{Hom}(-, \mathcal{P}roj)$ *exact exact sequence*

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective modules such that $M = \operatorname{Ker}(P^0 \to P^1)$.

Examples:

1. Every projective module is Gorenstein projective.

2. The converse however is not true: there exist Gorenstein projective modules

that are not projective. For example, over the ring $\mathbb{Z}/4\mathbb{Z}$ the module $\mathbb{Z}/2\mathbb{Z}$ is Gorenstein projective but it is not projective.

Remark. The complex above is a complete projective resolution of M. We note that if M is Gorenstein projective, then $\operatorname{Ext}^i(M, P) = 0$ for all $i \geq 1$ and all projective R-modules P and so by induction $\operatorname{Ext}^i(M, L) = 0$ for all $i \geq 1$ and all R-modules L of finite projective dimension. In particular, every left projective resolution of M is $\operatorname{Hom}(-, \mathcal{P}roj)$ exact.

Proposition 3.1.2 (*[3], Proposition 10.2.3*)

The projective dimension of a Gorenstein projective module is either zero or infinite.

Theorem 3.1.3

The following are equivalent for an R-module M.

1. *M is Gorenstein projective.*
2. *There is an exact and $\operatorname{Hom}(-, \mathcal{P}roj)$ exact sequence*

$$0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with each P^i projective, and $\operatorname{Ext}^i(M, P) = 0$ of all $i \geq 1$ and for any projective module P.

Proof:

(1.) $\Longrightarrow$ (2.)

By definition, there is an exact sequence $\cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{f_0} P^0 \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \cdots$ of projective R-modules with $M = \operatorname{Im} f_0$. In particular, this means that

M has an exact sequence $0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \cdots$ with each P^i projective. The sequence $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow P^2 \longrightarrow \cdots$ is also $\mathrm{Hom}(-, \mathcal{P}roj)$ exact. So for any projective module P, we have an exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(P^1, P) \longrightarrow \mathrm{Hom}(P^0, P) \xrightarrow{\alpha} \mathrm{Hom}(P_0, P) \xrightarrow{\beta} \mathrm{Hom}(P_1, P) \longrightarrow \cdots.$$

Thus $\mathrm{Im}\,\alpha = \mathrm{Ker}\,\beta$. But $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is exact, and $\mathrm{Hom}(-, \mathcal{P}roj)$ is left exact. So by Proposition 2.2.6 the sequence $0 \longrightarrow \mathrm{Hom}(M, P) \longrightarrow \mathrm{Hom}(P_0, P) \xrightarrow{\beta} \mathrm{Hom}(P_1, P)$ is exact. This means that $\mathrm{Ker}\,\beta \simeq \mathrm{Hom}(M, P)$. Thus

$$\cdots \longrightarrow \mathrm{Hom}(P^1, P) \longrightarrow \mathrm{Hom}(P^0, P) \longrightarrow \mathrm{Hom}(M, P) \longrightarrow 0$$

is exact for every projective module P.

(2.) $\Longrightarrow$ (1.)

Let $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be any projective resolution of M. Then $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ is an exact sequence of projective modules.

Let $K_0 = \mathrm{Ker}(P_0 \to M)$. The exact sequence $0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ gives that for every module P, we have a long exact sequence

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}(M, P) \longrightarrow \mathrm{Hom}(P_0, P) \longrightarrow \mathrm{Hom}(K_0, P)&\\ \longrightarrow \mathrm{Ext}^1(M, P) \longrightarrow \mathrm{Ext}^1(P_0, P) \longrightarrow \mathrm{Ext}^1(K_0, P)&\\ \longrightarrow \mathrm{Ext}^2(M, P) \longrightarrow \mathrm{Ext}^2(P_0, P) \longrightarrow \mathrm{Ext}^2(K_0, P)&\\ \longrightarrow \mathrm{Ext}^3(M, P) \longrightarrow \cdots&\end{aligned}$$

Since P_0 is a projective module, $\mathrm{Ext}^i(P_0, P) = 0$ for any $i \geq 1$ and for any module P. If P is a projective module, then by hypothesis, we also have that $\mathrm{Ext}^i(M, P) = 0$ for all $i \geq 1$. The exact sequence given above gives that $\mathrm{Ext}^i(K_0, P) = 0$ for all $i \geq 1$ and for any projective module P.

Similarly, $\text{Ext}^i(K_j, P) = 0$ for any $i \geq 1$ and any projective module P, where $K_j = \text{Ker}(P_j \to P_{j-1})$.

In particular, for each j there is an exact sequence $0 \longrightarrow K_j \longrightarrow P_j \longrightarrow K_{j-1} \longrightarrow 0$, and this gives a long exact sequence

$$0 \longrightarrow \text{Hom}(K_{j-1}, P) \longrightarrow \text{Hom}(P_j, P) \longrightarrow \text{Hom}(K_j, P) \longrightarrow \text{Ext}^1(K_{j-1}, P) = 0$$

provided that P is projective. So each sequence $0 \longrightarrow \text{Hom}(K_{j-1}, P) \longrightarrow \text{Hom}(P_j, P) \longrightarrow \text{Hom}(K_j, P) \longrightarrow 0$ is exact. Pasting them together, we obtain an exact sequence

$$0 \longrightarrow \text{Hom}(M, P) \longrightarrow \text{Hom}(P_0, P) \longrightarrow \text{Hom}(P_1, P) \longrightarrow \cdots .$$

By hypothesis, we also have the exact sequence

$$\cdots \longrightarrow \text{Hom}(P^1, P) \longrightarrow \text{Hom}(P^0, P) \longrightarrow \text{Hom}(M, P) \longrightarrow 0.$$

Splicing them together, we obtain the exact sequence

$$\cdots \longrightarrow \text{Hom}(P^1, P) \longrightarrow \text{Hom}(P^0, P) \longrightarrow \text{Hom}(P_0, P) \longrightarrow \text{Hom}(P_1, P) \longrightarrow \cdots .$$

Thus, M is Gorenstein projective.

□

Proposition 3.1.4 (*[3], Theorem 10.2.8*)

Let R be a Noetherian ring and $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of finitely generated right R-modules. If M', M'' are Gorenstein projective, then so is M. If M, M'' are Gorenstein projective, then so is M'. If M, M' are Gorenstein projective, then M'' is Gorenstein projective if and only if $\text{Ext}^1(M'', P) = 0$ for all finitely generated projective R-modules P.

Definition 3.1.5

The minimal length of a finite exact sequence of an R-module M

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with each G_j being Gorenstein projective is called the Gorenstein projective dimension of M, denoted $\operatorname{Gpd}_R M$.

Proposition 3.1.6 (*[6], Theorem 2.24*)

If M has $\operatorname{Gpd}_R M = n < \infty$, then for any projective resolution of M

$$0 \longrightarrow G \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

G is Gorenstein projective.

Proposition 3.1.7 (*[6], Proposition 2.7*)

Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence. If two of the modules M', M, or M'' have finite Gorenstein projective dimension, then so does the third.

3.2 Gorenstein Flat Modules

The Gorenstein flat modules were introduced by Enochs, Jenda, and Torrecillas in [4] as a generalization of flat modules.

Definition 3.2.1

A module M is said to be Gorenstein flat if there exists an $\mathcal{I}nj \otimes -$ exact exact sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

of flat modules such that $M = \mathrm{Ker}(F^0 \to F^1)$.

It follows from the previous definition that for an injective R-module E, $\mathrm{Tor}_i(E, M) = 0$ for all $i \geq 1$.

Proposition 3.2.2 (*[3], Proposition 10.3.2*)

Let R be a Noetherian ring. Then every finitely generated Gorenstein projective R-module is Gorenstein flat.

Proposition 3.2.3 (*[3], Lemma 10.3.5*)

Let R be right Noetherian, M be an R-module, and $0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$ be a right $\mathcal{F}lat$-resolution. Then $\mathrm{Tor}_i(A, M) = 0$ for all $i \geq 1$ and all right R-modules A of finite injective dimension if and only if the sequence

$$\cdots \longrightarrow A \otimes F_1 \longrightarrow A \otimes F_0 \longrightarrow A \otimes F^0 \longrightarrow A \otimes F^1 \longrightarrow \cdots$$

is exact, where $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ is any flat resolution of M.

Proposition 3.2.4 (*[3], Theorem 10.3.14*)

Suppose R is Noetherian and $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of R-modules. If M' and M'' are Gorenstein flat, then so is M. If M and M'' are Gorenstein flat, then so is M'. If M' and M are Gorenstein flat, then M'' is Gorenstein flat if and only if $0 \longrightarrow E \otimes M' \longrightarrow E \otimes M$ is exact for any injective module E.

Definition 3.2.5

The minimal length of a finite exact sequence of an R-module M

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with each G_j being Gorenstein flat is called the Gorenstein flat dimension of M, denoted $\mathrm{Gfd}_R\, M$.

CHAPTER 4

MAIN RESULT

Let R be a commutative Noetherian ring of finite Krull dimension. It is known by Theorem 2.4.4 (Gruson-Jensen-Raynaud in [7]) that over such rings, every flat module has finite projective dimension.

Garcia-Rozas proved the corresponding result for complexes: Over a commutative Noetherian ring of finite Krull dimension, every flat complex F has finite projective dimension.

More precisely, if d is the Krull dimension of the ring, then for any flat R-module F and any exact sequence

$$0 \longrightarrow K \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow F \longrightarrow 0$$

with each P_i projective, then K is a projective module (so the projective dimension of any flat R-module is at most d).

We prove the Gorenstein counterpart of this result. Since our proof uses some notions of homological algebra in the category of complexes, we start by recalling some necessary definitions. They are from [5].

We recall that a complex of R-modules is a sequence of R-modules and R-homomorphisms

$$C = \cdots \longrightarrow M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} \cdots$$

such that $f_i f_{i+1} = 0$ for each integer i.

Given a complex C, we denote by $Z_j(C)$ its j^{th} cycle; that is, $Z_j(C) = \text{Ker}(f_j)$.

If $h : C' \to C$ and $g : C \to C''$ are maps of complexes now, we say that the sequence

$$C' \xrightarrow{h} C \xrightarrow{g} C''$$

is exact if for each n the sequence of modules

$$C'_n \xrightarrow{h_n} C_n \xrightarrow{g_n} C''_n$$

is exact.

The projective (injective, flat) complexes are defined in a similar manner with the projective (injective, flat) modules. For example a complex P is projective if for every exact sequence of complexes $A \longrightarrow B \longrightarrow 0$, the sequence (obtained by applying the functor $\mathrm{Hom}(P, -)$) $\mathrm{Hom}(P, A) \longrightarrow \mathrm{Hom}(P, B) \longrightarrow 0$ is still exact.

It is known (see [5] for instance) that a complex P is projective if and only if the complex is exact and $Z_n(P)$ is a projective module for each integer n. There is a similar result for flat complexes: a complex F is flat if and only if it is exact and for each integer n, the module $Z_n(F)$ is flat.

Theorem 4.0.1

Let R be a commutative Noetherian ring of finite Krull dimension. Then every Gorenstein flat R-module has finite Gorenstein projective dimension.

PROOF:

Let M be a Gorenstein flat module. Then there is an exact and $\mathcal{I}nj \otimes -$ exact sequence of flat modules

$$F = \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} F_{-1} \longrightarrow \cdots$$

such that $M = \mathrm{Ker}\, f_0$.

Let $0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow F \longrightarrow 0$ be an exact sequence of complexes such that P_0 is projective. Then K_0 is also an exact complex. And for each $j \in \mathbb{Z}$ we have an exact sequence of modules

$$0 \longrightarrow Z_j(K_0) \longrightarrow Z_j(P_0) \longrightarrow Z_j(F) \longrightarrow 0$$

where Z_j denotes the j^{th} cycle of F (*i.e.* the module $\operatorname{Ker} f_j$) and similarly $Z_j(K_0)$ and $Z_j(P_0)$ stand for the j^{th} cycle of K_0 and P_0, respectively.

Since $Z_j(P_0)$ is projective and $Z_j(F)$ is Gorenstein flat, and the ring is Noetherian, $Z_j(K_0)$ is Gorenstein flat for each j. Thus K_0 is also an exact and $\mathcal{I}nj \otimes -$ exact complex of flat modules.

We continue this way. Let $d = \dim R$ (the Krull dimension). After d steps we have an exact sequence

$$0 \longrightarrow K \longrightarrow P_{d-1} \longrightarrow P_{d-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow F \longrightarrow 0$$

with each P_j being a projective complex and with K an exact and $\mathcal{I}nj \otimes -$ exact complex of flat modules.

This means that for each j there is an exact sequence

$$0 \longrightarrow K_j \longrightarrow P_{d-1,j} \longrightarrow \cdots \longrightarrow P_{0,j} \longrightarrow F_j \longrightarrow 0.$$

Using Gruson, Jensen, and Raynaud's result and [7], we have $\operatorname{pd}_R F_j \leq d$. Hence K_j must be projective for each j. But then K is an exact complex of *projective modules* that is $\mathcal{I}nj \otimes -$ exact. It is known that such a complex is totally acyclic (*i.e.* all cycles $Z_j(K)$ are Gorenstein projective modules).

Since $0 \longrightarrow K \longrightarrow P_{d-1} \longrightarrow P_{d-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow F \longrightarrow 0$ is an exact sequence of exact complexes, for each $j \in \mathbb{Z}$ we have an exact sequence of modules

$$0 \longrightarrow Z_j(K) \longrightarrow Z_j(P_{d-1}) \longrightarrow \cdots \longrightarrow Z_j(P_1) \longrightarrow Z_j(P_0) \longrightarrow Z_j(F) \longrightarrow 0$$

with each $Z_j(P_l)$ projective and with $Z_j(K)$ Gorenstein projective. This means that each $Z_j(F)$ has finite Gorenstein projective dimension less than or equal to d.

In particular, $\operatorname{Gpd}_R M \leq d$.

□

We use Theorem 4.0.1 to prove that the following result holds in a more general setting.

Theorem 4.0.2 (*[1], Theorem 1*)

If R is a commutative Noetherian ring with a dualizing complex, then the following are equivalent for an R-module M.

1. *M has finite Gorenstein projective dimension,* $\operatorname{Gpd}_R M < \infty$*;*
2. *M has finite Gorenstein flat dimension,* $\operatorname{Gfd}_R M < \infty$.

In our main result, we drop the dualizing complex condition on the ring in favor of having finite Krull dimension. As one can see, the definition of a dualizing complex is very technical.

Definition 4.0.3

Let S and R be rings. If S is left Noetherian and R is right Noetherian, we refer to the ordered pair $\langle S, R\rangle$ as a Noetherian pair of rings. A dualizing complex for a Noetherian pair of rings $\langle S, R\rangle$ is a complex ${}_S D_R$ of bimodules meeting the *requirements:*

1. *The homology of D is bounded and degreewise finitely generated over S and over R^{opp}.*

2. *There exists a quasi-isomorphism of complexes of bimodules, ${}_SP_R \xrightarrow{\simeq} {}_SD_R$, where ${}_SP_R$ is right-bounded and consists of modules projective over both S and R^{opp}.*

3. *There exists a quasi-isomorphism of complexes of bimodules, ${}_SD_R \xrightarrow{\simeq} {}_SI_R$, where ${}_SI_R$ is bounded and consists of modules injective over both S and R^{opp}.*

4. *The homothety morphisms $\grave{\chi}_D^{\langle S,R\rangle} : {}_SS_S \longrightarrow \mathbf{R}\operatorname{Hom}_{R^{\text{opp}}}({}_SD_R, {}_SD_R)$ and $\acute{\chi}_D^{\langle S,R\rangle} : {}_RR_R \longrightarrow \mathbf{R}\operatorname{Hom}_S({}_SD_R, {}_SD_R)$ are bijective in homology.*

We extend Christensen, Fraukild, and Holm's result to commutative Noetherian rings of finite Krull dimension.

Our proof uses Theorem 4.0.1, together with the following fact:

Proposition 4.0.4 (*[6], Proposition 3.4*)

If R is a commutative Noetherian ring of finite Krull dimension then every Gorenstein projective R-module is also Gorenstein flat.

We can now prove the following theorem.

Theorem 4.0.5 (*Main Theorem*)

Let R be a commutative noetherian ring of finite Krull dimension. The following are equivalent for a module M over R.

1. *M has finite Gorenstein projective dimension;*

2. *M has finite Gorenstein flat dimension.*

Proof:

(1.) $\Longrightarrow$ (2.)

Since $\operatorname{Gpd}_R M < \infty$, there is an exact sequence

$$0 \longrightarrow G_l \longrightarrow G_{l-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with G_j being Gorenstein projective for all $j \in \{0, 1, \ldots, l\}$. Since every Gorenstein projective module is Gorenstein flat, it follows that M has finite Gorenstein flat dimension.

(2.) $\Longrightarrow$ (1.)

There exists an exact sequence

$$0 \longrightarrow F_k \xrightarrow{f_k} F_{k-1} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0$$

with each F_j Gorenstein flat.

Let $V_k = \operatorname{Im} f_{k+1}$. We have an exact sequence

$$0 \longrightarrow F_k \longrightarrow F_{k-1} \xrightarrow{f_{k-1}} V_{k-2} \longrightarrow 0$$

with both F_k and F_{k-1} of finite Gorenstein projective dimension. It follows that V_{k-2} also has finite Gorenstein projective dimension.

Then we have an exact sequence

$$0 \longrightarrow V_{k-2} \longrightarrow F_{k-2} \xrightarrow{f_{k-2}} V_{k-3} \longrightarrow 0$$

with $\operatorname{Gpd}_R V_{k-2} < \infty$ and $\operatorname{Gpd}_R F_{k-2} < \infty$. It follows that $\operatorname{Gpd}_R V_{k-3} < \infty$.

Continuing, we obtain that $\operatorname{Gpd} V_0 < \infty$ with $V_0 = \operatorname{Im} f_1$. Then the exact sequence

$$0 \longrightarrow V_0 \longrightarrow F_0 \xrightarrow{f_0} M \longrightarrow 0$$

with $\operatorname{Gpd}_R V_0 < \infty$ and $\operatorname{Gpd}_R F_0 < \infty$ gives $\operatorname{Gpd}_R M < \infty$.

$\square$

REFERENCES

[1] L.W. Christensen, A. Fraukild, and H. Holm. On Gorenstein projective, injective and flat dimensions – a functional description with applications. *Journal of Algebra*, 302(1):231–279, 2006.

[2] David S. Dummit and Richard M. Foote. *Abstract Algebra*. John Wiley and Sons, Inc., 3^{rd} edition, 2004.

[3] Edgar E. Enochs and Overtoun M. G. Jenda. *Relative Homological Algebra*. De Gruyter Expositions in Mathematics, 2000.

[4] Edgar E. Enochs, Overtoun M. G. Jenda, and B. Torrecillas. Gorenstein flat modules. *Journal Nanjing University*, 10:1–9,19, 1998.

[5] J.R. Garcia-Rozas. *Covers and envelopes in the category of complexes*. CRC Press LLC, 1999.

[6] H. Holm. Gorenstein homological dimensions. *Journal of Pure and Applied Algebra*, 189:167–193, 2004.

[7] C. U. Jensen. *Les foncteurs derives de lim et leurs application en theorie des modules*, volume 254. Lecture Notes in Mathematics, Springer-Verlag, 1972.