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Approximation and Integration on Compact Subsets of Euclidean Space

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APPROXIMATION AND INTEGRATION ON COMPACT SUBSETS OF EUCLIDEAN SPACE

by

ROCHELLE RANDALL

(Under the Direction of Professor Steven Damelin)

ABSTRACT

This thesis deals with approximation of real valued functions. It considers interpolation and numerical integration of functions. It also looks at error and precision, the Weierstrass Theorem and Taylor's Theorem. In addition, spherical harmonics, the Laplacian, Hilbert spaces and linear projections are considered with respect to the unit sphere. An example of distributing points equally on a sphere is illustrated and a covering theorem for a unit sphere is proved.

INDEX WORDS: Approximation, Interpolation, Numerical integration, Unit sphere covering theorem

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DEDICATION

I would like to dedicate this to Professor Mulatu Lemma whose encouragement prompted my return to full time studies.

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I wish to tender my most sincere gratitude to my advisor, Professor Steven Damelin for his unfailing patience and support and his special gift for encouragement. I also wish to thank him for showing me where I fit in the field of mathematics.

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Introduction

Interpolation is a technique for fitting a given function that passes through known points or essentially "connecting the dots". There are various types of interpolation from linear interpolation of functions to interpolation of operators. We will be looking at interpolation of functions, specifically polynomial interpolation. First with reference to Cheney [1] we will consider the existence and uniqueness theorem for a polynomial of degree $\leq n$ for both Lagrange and Hermite interpolation. We also look at a theorem involving Tchebycheff norms which gives us a way of measuring error. We then prove the Weierstrass Approximation Theorem which is significant in polynomial interpolation as it tells us that any continuous function on a finite interval can be approximated as closely as desired by a polynomial.
Next, we take a brief look at numerical integration, considering some of the quadrature rules and error and precision as well as Taylor's theorem. We look at an example of using Taylor's theorem to approximate e^x and then determine the error. Finally, we consider the unit sphere and spherical harmonics, the Laplacian, Hilbert spaces and linear projections. As part of Hilbert spaces and linear projections, we also look at Reimer's [14] definition and proof of the existence and uniqueness of a reproducing kernel function. We look at an example of distributing points equally about a sphere and finally we look at a covering theorem for a unit sphere.

Chapter 2

Approximation of real valued functions

2.1 Interpolation

Interpolation is a technique for fitting a given function that passes through known points. For example, given two points, $x_1 \neq x_2$, there exists a unique straight line of the form $y = mx + b$ that passes through those two points. In a similar fashion, given three points, $x_1 \neq x_2 \neq x_3$, there exists a unique parabola of the form $y = ax^2 + bx + c$ passing through those points. Interpolation is a generalization of this idea. [1]

Theorem 1 *Let $n \geq 1$. There exists a unique polynomial of degree $\leq n$ which assumes prescribed values at $n + 1$ distinct points.*

We present three different proofs for the above theorem.

Proof. 1. Let there exist points $x_0, x_1, \ldots, x_n$ and let $y_0, y_1, \ldots, y_n$ be the prescribed values. We seek a polynomial P such that $P(x_i) = y_i \ (i = 0, \ldots, n)$. This polynomial may be expressed as

$$P(x) = \sum_{j=0}^{n} c_j x^j, \text{ for some constant } c_0, \ldots, c_n. \tag{2.1}$$

It is of degree $\leq n$. Combining these statements allows us to write

$$\sum_{j=0}^{n} c_j x^j = y_i \, (i = 0, \ldots, n) \tag{2.2}$$

and to express equation (2.2) in matrix form as

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ \ldots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \ldots \\ y_n \end{bmatrix}.$$

This is a square Vandermonde matrix and its determinant is

$$\prod_{0 \leq j < i \leq n} (x_i - x_j)\,. \tag{2.3}$$

From equation (2.3) it can be seen that the determinant does not equal zero if and only if the points are distinct. ■

Proof. 2. In the second proof, we construct polynomials l_i with the property that

$$l_i\,(x_j) = \delta_{ij}. \tag{2.4}$$

Note that δ_{ij} is known as the Kronecker delta which denotes a function of two variables satisfying the following equation

$$\delta_{ij} = \left\{ \begin{array}{c} 1,\ i = j \\ 0,\ i \neq j \end{array} \right\}. \tag{2.5}$$

We can then write $P\,(x) = \sum_j y_i l_j\,(x)\,,\ x \in \mathbb{R}$ and
$P\,(x_i) = \sum_j y_i l_j\,(x_i) = \sum_j y_i \delta_{ij} = y_i$. It can then be seen that

$$l_i\,(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_i}{x_i - x_j},\ i = 0, \ldots, n,\ x \in \mathbb{R}. \tag{2.6}$$

It is still necessary to show that P is unique. Consider two polynomials, P and Q, both of degree $\leq n$ which have the property $Q\,(x_i) = P\,(x_i) = y_i$. Then $P - Q$ is of degree $\leq n$ and vanishes at the $n+1$ distinct points x_i. Thus $P - Q \equiv 0$. ■

Proof. 3. Our third proof begins by attempting to determine a polynomial of the form

$$P\,(x) = a_0 + a_1\,(x - x_0) + a_2\,(x - x_0)\,(x - x_1) + \cdots + a_n\,(x - x_0) \cdots (x - x_{n-1})\,. \tag{2.7}$$

If we set x equal to x_0, $P(x_0) = a_0$. Since $P(x_0) = y_0$, $a_0 = y_0$. It then is possible to solve for each succeeding coefficient with the following results.

$$
\begin{aligned}
a_0 &= y_0 \\
a_1 &= \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - a_0}{x_1 - x_0} \\
a_2 &= \frac{y_2 - a_0 - a_1 (x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \\
a_3 &= \frac{y_3 - a_0 - a_1 (x_3 - x_0) - a_2 (x_3 - x_0)(x_3 - x_1)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \\
a_4 &= \frac{y_4 - a_0 - a_1 (x_4 - x_0) - a_2 (x_4 - x_0)(x_4 - x_1) - a_3 (x_4 - x_0)(x_4 - x_1)(x_4 - x_2)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} \\
& \ldots \\
a_n &= \frac{y_n - a_0 - \sum\limits_{j=1}^{n-1} a_j \prod\limits_{j=0}^{n-2} (x_n - x_j)}{\prod\limits_{j=0}^{n-1} (x_n - x_j)}
\end{aligned}
$$

We can see that the denominators do not vanish, thus P exists. ■

Hermite interpolation generalizes Lagrange interpolation given that it also uses values of derivatives. The existence theorem follows.

Theorem 2 *Let $n \geq 1$. There exists a unique polynomial P of degree $\leq 2n - 1$ such that P and its derivative P' take on prescribed values at n points.*

Proof. Let $P(x_i) = y_i$ and $P'(x_i) = y_i'$ for $i = 1, \ldots, n$. We can express this polynomial explicitly as

$$P(x) = \sum_{i=1}^{n} \left[y_i A_i(x) + y_i' B_i(x) \right] \tag{2.8}$$

noting that A_i and B_i are polynomials of degree $\leq 2n - 1$ with the properties $A_i(x_j) = \delta_{ij}$, $B_i(x_j) = 0$, $A_i'(x_j) = 0$ and $B_i'(x_j) = \delta_{ij}$. Expressing A_i and B_i in terms of $l_i(x)$, we have

$$A_i(x) = [1 - 2(x - x_i)\, l_i'(x_i)]\, l_i^2(x) \tag{2.9}$$

$$B_i(x) = (x - x_i)\, l_i^2(x) \tag{2.10}$$

Then it is clear that $l_i(x_j) = \delta_{ij}$ and that A_i and B_i satisfy equation (2.8) ■

Before beginning the next theorem, the Tchebycheff norm, also called the supremum norm needs to be defined. Generally we define this norm as
$\|f\|_\infty = \sup\{|f(x)| : x \text{ is in domain of } f\}$. When f is continuous and on a compact set, it is bounded and the supremum can be replaced by the maximum.

Theorem 3 *If f possesses n continuous derivatives on $[a, b]$, P is the polynomial of degree $<n$ which interpolates to f at n nodes x_i in $[a, b]$, and if $W(x) = \prod (x - x_i)$, then in terms of the Tchebycheff norm,*

$$\|f - P\| \leq \frac{1}{n!} \left\|f^{(n)}\right\| \|W\|. \tag{2.11}$$

Proof. We seek to prove that for each x in $[a, b]$ there is a corresponding $\xi \in (a, b)$ such that

$$f(x) - P(x) = \frac{1}{n!} f^{(n)}(\xi)\, W(x). \tag{2.12}$$

If x is one of the nodes, the formula follows from expanding $f - P$ in Taylor series and noting that the nth derivative of $P \equiv 0$. If x is not one of the nodes, we let $\phi = f - P - \lambda W$, Choose λ so that $\phi(x) = 0$, thus $\lambda = \frac{f-P}{W}$. We can see that ϕ also vanishes at the nodes x_i; thus, ϕ vanishes in at least $n+1$ points of $[a, b]$. We know by Rolle's theorem that ϕ' vanishes at least once between any two zeros of ϕ, and therefore vanishes in at least n points. We can thus see that $\phi^{(n)}$ has at least one root on the interval, we can say at the point ξ. However, since P is a polynomial of degree $< n$ and $W(z) = z^n + \ldots$, $\phi^{(n)} = f^{(n)} - \lambda n!$. Hence, $f^{(n)}(\xi) = \lambda n!$. But the value of $\lambda = \frac{f(x) - P(x)}{W(x)}$. ■

This gives us a way of measuring the error.

The following theorem presents where to locate the nodes in order to minimize the norm of W.

Theorem 4 *The uniform norm of* $W(x) = \prod_{i=1}^{n} (x - x_i)$ *is minimized on* $[-1, 1]$ *when* $x_i = \cos[(2i-1)\pi/2n]$.

Proof. We know that $\cos n\theta$ can be expressed in the form $\sum_{k=0}^{n} a_k \cos^k \theta$. Letting $T_n(x) = \sum_{k=0}^{n} a_k x^k$ gives us $T(\cos\theta) = \cos n\theta$. Thus the n roots of T_n are the points x_i as given in the theorem. Consider the polynomial $W = 2^{1-n} T_n$ which has a leading coefficient of unity. The maximum of $|W(x)|$ on $[-1, 1]$ occurs at the points $y_i = \cos(\frac{i\pi}{n})$ since $T_n(y_i) = \cos i\pi = (-1)^i$. Now let V be another polynomial of the same form as W for which $\|V\| < \|W\|$. Then $V(y_0) < W(y_0)$, $V(y_1) > W(y_1)$, and so on, from which it follows that $w - v$ must vanish at least once in each interval $(y_{1,} y_0)$, (y_2, y_1), ... for a total of n times. But this is not possible because both V and W have leading coefficient unity, and their difference is therefore of degree $< n$. ■

2.2 Weierstrass Theorem

We now consider some general results of approximation of functions. The main aim of this section is to prove the Weierstrass Theorem.

Theorem 5 (Weierstrass) *Lef f be a continuous function defined on* $[a, b]$. *Then for each* $\epsilon > 0$ *there is a corresponding polynomial P such that* $\|f - P\| < \epsilon$. *Thus* $|f(x) - P(x)| < \epsilon$ *for all* $x \in [a, b]$.

This theorem tells us that any continuous function on a finite interval can be approximated arbitrarily closely by polynomials, i.e. polynomials are dense in the space of continuous functions on $[a, b]$. Various extensions of the Weierstrass Theorem are true on other sets.

In order to prove the Weierstrass Theorem, we want to consider Bernstein polynomials and the Theorem on Monotone Operators. First, here and throughout $C[0,1]$ denotes the class of real valued continuous functions on $[a, b]$. For a given $f \in C[0,1]$, Bernstein constructed a sequence of polynomials (which today are called *Bernstein* polynomials) using the formula

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}. \tag{2.13}$$

Note that $\binom{n}{k}$ is the binomial coefficient $\frac{n!}{(n-k)!k!}$. Additionally, the formula defines for each n a linear operator, B_n so that to each element f in $C[0,1]$ there is a corresponding element $B_n f$ of $C[0,1]$ such that the condition of linearity is met. This is expressed as

$$B_n(af + bg) = aB_n f + bB_n g. \tag{2.14}$$

The operators B_n also have the property espressed below as

$$f \geq g \Longrightarrow B_n f \geq B_n g. \tag{2.15}$$

An operator for which the above is true is called a monotone operator. The $f \geq g$ means that $f(x) \geq g(x)$ for all x in the domain of f.

Theorem 6 (Korovkin's) *For a sequence of monotone linear operators L_n on $C[a, b]$ the following conditions are equivalent:*

(i) $L_n f \longrightarrow f$ *(uniformly) for all* $f \in C[a, b]$

(ii) $L_n f \longrightarrow f$ *for the three functions* $f(x) = 1, x, x^2$

(iii) $L_n 1 \longrightarrow 1$ *and* $(L_n \phi_t)(t) \longrightarrow 0$ *uniformly in* f *where* $\phi_t(x) \equiv (t-x)^2$.

Proof. Since $f(x) = 1, x, x^2$ are all $\in C[a,b]$, the implication that $(i) \Longrightarrow (ii)$ is obvious. For $(ii) \Longrightarrow (iii)$, let $f_i(x) = x^i$. Since $\phi_t(x) = (t-x)^2 = t^2 - 2tx + x^2$, we can see that $\phi_t = t^2 f_0 - 2t f_1 + f_2$ and $L_n \phi_t = t^2 L_n f_0 - 2t L_n f_1 + L_n f_2$. Then

$$\begin{aligned}(L_n\phi_t)(t) &= t^2\left[(L_n f_0)(t) - 1\right] - 2t\left[(L_n f_1)(t) - t\right] + \left[(L_n f_2)(t) - t^2\right] \\ &\le t^2 \|L_n f_0 - f_0\| + |2t| \|L_n f_1 - f_1\| + \|L_n f_2 - f_2\|.\end{aligned}$$

As t^2 and $|2t|$ are bounded on $[a,b]$, we can see that $(L_n\phi_t)(t)$ converges uniformly to zero, thus proving (iii).

To prove that *(iii)* $\Longrightarrow$ *(i)*, we let f be an arbitrary element of $C[a,b]$. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$|x_1 - x_2| < \delta \Longrightarrow |f(x_1) - f(x_2)| < \varepsilon.$

Now let $\alpha = \frac{2\|f\|}{\delta^2}$, and let t be an arbitrary fixed point of $[a,b]$. If $|t-x| < \delta$, then $|f(t) - f(x)| < \varepsilon$, but if $|t-x| \ge \delta$, then

$|f(t) - f(x)| \le 2\|f\| \le \frac{2\|f\|(t-x)^2}{\delta^2} = \alpha\phi_t(x)$. So for all x, the following is satisfied

$$-\varepsilon - \alpha\phi_t(x) \le f(t) - f(x) \le \varepsilon + \alpha\phi_t(x). \tag{2.16}$$

Now if we want to express the above inequality on the functions, we let $f_0 = 1$ with the resulting expression

$$-\varepsilon f_0 - \alpha\phi_t \le f(t) f_0 - f \le \varepsilon f_0 + \alpha\phi_t. \tag{2.17}$$

Now because L_n is linear and monotone, we have

$$\begin{aligned}-\varepsilon(L_n f_0)(t) - \alpha(L_n\phi_t)(t) &\le f(t)(L_n f_0)(t) - (Lnf)(t) \\ &\le \varepsilon(L_n f_0)(t) + \alpha(L_n\phi_t)(t)\end{aligned} \tag{2.18}$$

which gives

$$|f(t)(L_n f_0)(t) - (Lnf)(t)| \leq \varepsilon \|L_n f_0\| + \alpha (L_n \phi_t)(t). \tag{2.19}$$

But $L_n f_0 \longrightarrow f_0$ and $(L_n \phi_t)(t) \longrightarrow 0$. One last thing to do is consider just how large n must be. To do this we show

$$\begin{aligned} |f(t) - (L_n f)(t)| &\leq |f(t) - f(t)(L_n f_0)(t)| + |f(t)(L_n f_0)(t) - (L_n f)(t)| \\ &\leq |f(t)| |1 - (L_n f_0)(t)| + \varepsilon \|L_n f_0\| + \alpha (L_n \phi_t)(t) \\ &\leq \|f\| \|f_0 - L_n f_0\| + \varepsilon (1 + \|f_0 - L_n f_0\|) + \alpha (L_n \phi_t)(t) \end{aligned}$$

We can see that we should choose N so that when $n \geq N$ we get $(\|f\| + \varepsilon) \|f_0 - L_n f_0\| < \varepsilon$ and $\alpha (L_n \phi_t)(t) < \varepsilon$. ■

We now want to prove the Weierstrass Approximation Theorem.[2]

Proof. Without loss of generality we prove the theorem for the interval $[0, 1]$. Indeed, once we have the theorem for the interval $[0, 1]$, we can extend to an arbitrary interval $[a, b]$ by making the change of variable $x = a + t(b - a)$. By theorem (6), it will be sufficient to show that $B_n h_k \rightarrow h_k$ for $k = 0, 1, 2$ where $h_k(x) = x^k$. We use the Binomial Theorem to write

$$(B_n h_0)(x) = \sum_{k=0}^{n} g_{nk}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1 \tag{2.20}$$

For h_1 we can write

$$\begin{aligned} (B_n h_1)(x) &= \sum_{k=0}^{n} \left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} \\ &= x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} = x \end{aligned} \tag{2.21}$$

and for h_2, we have

$$\begin{aligned}
(B_n h_2)(x) &= \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=1}^{n} \left(\frac{k}{n}\right) \binom{n-1}{k-1} x^k (1-x)^{n-k} \\
&= \sum_{k=1}^{n} \left(\frac{n-1}{n}\frac{k-1}{n-1} + \frac{1}{n}\right) \binom{n-1}{k-1} x^k (1-x)^{n-k} \\
&= \frac{n-1}{n} x^2 \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2} (1-x)^{n-2} + \frac{x}{n} \\
&= \frac{n-1}{n} x^2 + \frac{x}{n} \to x^2. \qquad (2.22)
\end{aligned}$$

■

Chapter 3

Numerical Integration of real valued functions

3.1 Some Quadrature Rules

There are various numerical (quadrature) rules of integration. Newton-Cotes formulas use values at equally spaced points. There are two types of Newton-Cotes formulas. The open formulas do not use the end points of the interval, but the closed formulas do use the endpoints. An open formula is the midpoint rule. An example of closed Newton-Cotes is the trapezoid rule. The idea for the trapezoid rule is to draw a straight line between the end points of the interval and then use the formula for the area of a trapezoid. This formula can be derived by using the Lagrange interpolating polynomial to approximate the function and then integrating.

Let $f(x)$ be the function and a and b the endpoints of our interval. Then $a = x_1$ and $b = x_2$ and $h = b - a = x_2 - x_1$. Using the Lagrange interpolating polynomial we have:

$$P(x) = \frac{(x - x_2)}{(x_1 - x_2)} f(x_1) + \frac{(x - x_1)}{(x_2 - x_1)} f(x_2). \tag{3.1}$$

Then,

$$
\begin{aligned}
\int_a^b f(x)\,dx &\approx \int_a^b \left[\frac{(x-x_2)}{(x_1-x_2)} f(x_1) + \frac{(x-x_1)}{(x_2-x_1)} f(x_2)\right] dx \\
&= \frac{1}{h}\int_a^b \left[-(x-x_2) f(x_1) + (x-x_1) f(x_2)\right] dx \\
&= \frac{1}{h}\int_a^b \left[(x_2-x) f(x_1) + (x-x_1) f(x_2)\right] dx \\
&= \frac{1}{h}\int_a^b x_2 f(x_1) - x f(x_1) + x f(x_2) - x_1 f(x_2)\, dx \\
&= \frac{1}{h}\left[x x_2 f(x_1) - \frac{x^2}{2} f(x_1) + \frac{x^2}{2} f(x_2) - x x_1 f(x_2)\right]_a^b \\
&= \frac{1}{h}\left[x b f(x_1) - \frac{x^2}{2} f(x_1) + \frac{x^2}{2} f(x_2) - x a f(x_2)\right]_a^b \\
&= \frac{1}{h}\left[b^2 f(x_1) - \frac{b^2}{2} f(x_1) + \frac{b^2}{2} f(x_2) - ba f(x_2) - ab f(x_1) + \frac{a^2}{2} f(x_1) - \frac{a^2}{2} f(x_2) + a^2 \right. \\
&= \frac{1}{h}\ \left[f(x_1)\left(\frac{2b^2}{2} - \frac{b^2}{2} + \frac{a^2}{2} - \frac{2ab}{2}\right) + f(x_2)\left(\frac{b^2}{2} - \frac{2ab}{2} - \frac{a^2}{2} + \frac{2a^2}{2}\right)\right] \\
&= \frac{1}{2h}\left[f(x_1)\left(b^2 + a^2 - 2ab\right) + f(x_2)\left(b^2 + a^2 - 2ab\right)\right] \\
&= \frac{1}{2(b-a)}\left[(b-a)^2 \left(f(x_1) + f(x_2)\right)\right] \\
&= \frac{1}{2}\ (b-a)\left(f(x_1) + f(x_2)\right).
\end{aligned}
$$

Looking at Figure 1, it can be seen that $(b-a)$ is the height of the trapezoid and $f(x_1)$ and $f(x_2)$ are the two bases.

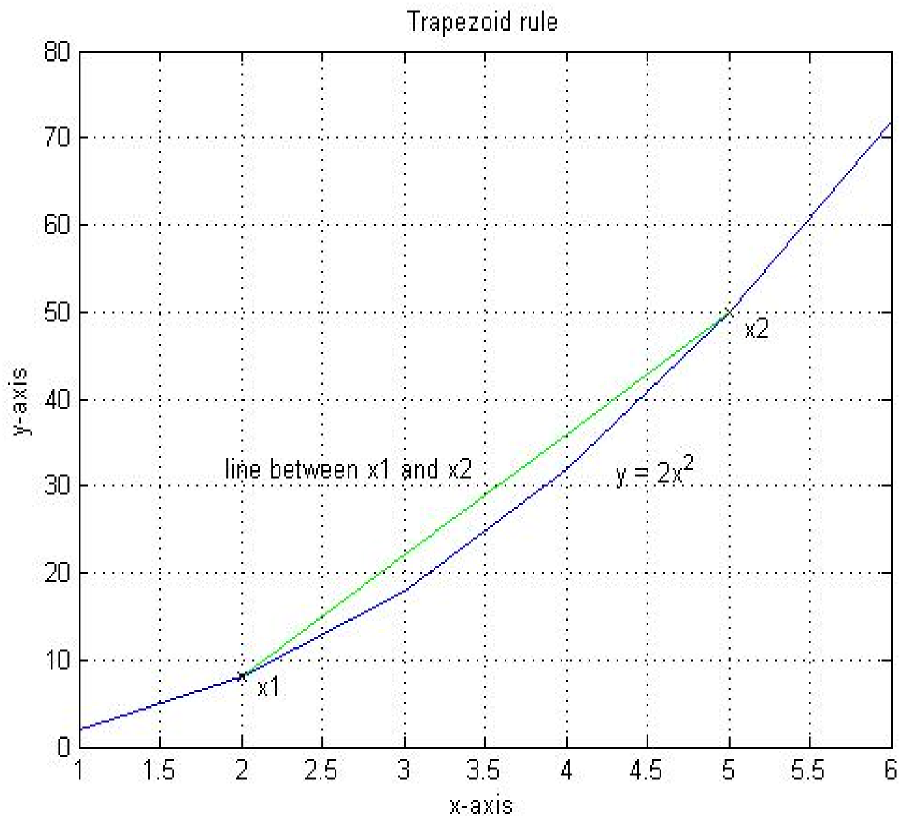

Figure 1 Trapezoid Rule

Composite rules can be obtained by partitioning the interval into n subintervals and then applying the simple rule (trapezoid, et al.) to each subinterval. For example, with equal subintervals, the composite trapezoid rule can be expressed as

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f\left(a + ih\right) + f\left(b\right)\right] \tag{3.2}$$

where $h = \frac{b-a}{n}$ and $x_i = a + ih$; see [8].

Gaussian quadrature rules can be expressed as $\displaystyle\int_a^b f\left(x\right) w\left(x\right) dx \approx \sum_{i=0}^{n} A_i f\left(x\right)$ where $w\left(x\right)$ is a positive weight function.

3.2 Error and Precision

The precision of an integration rule is the highest degree of polynomial for which the method exactly integrates that polynomial as noted by Fausett in *Numerical Methods* [7]. When considering the error involved in numerical integration, we can use the error term in polynomial interpolation. There is a theorem which states that given an interval I which contains $n+1$ interpolating points $x_0, x_1, ...x_n$ and $f(x)$ which is continuous and has continuous derivatives of order $n+1$ for all x in I and if $p(x)$ is the polynomial which interpolates at the points $x_0, x_1, ...x_n$, then at any point x on I, the following equation holds. [9]

$$f(x) - p(x) = \frac{\psi(x) f^{(n+1)}(\xi)}{(n+1)!} \tag{3.3}$$

where $\psi(x) = \prod_{i=0}^{n} (x - x_i)$ and ξ is some point on the interval I.

3.3 Taylor's Theorem

Let f be $n+1$ times continuously differentiable on [a,b]. Then for each $x, c \in [a, b]$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-c)^{n+1} \tag{3.4}$$

where ξ is in the interval (x, c) or (c, x).

The Taylor polynomial can be expressed as

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k \tag{3.5}$$

Two ways we can use Taylor's theorem to obtain an error estimate are (1) increasing the polynomial degree and (2) decreasing the width of the interval.[10]

$$
\begin{aligned}
e_n &= |f(x) - T_n(x)| \\
&= \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-c)^{n+1} \right| \\
&\leq \frac{M}{(n+1)!} (b-a)^{n+1}
\end{aligned}
$$

where $x, \xi \in [a,b]$ and $M = \left\| f^{(n+1)} \right\|_{\infty}, [a,b]$.

Instead of increasing the polynomial degree, the interval width can be decreased and the error estimate would appear as:

$$
\begin{aligned}
e_n &= |f(x) - T_n(x)| \\
&= \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-c)^{n+1} \right| \\
&\leq \frac{M}{(n+1)!} h^{n+1},
\end{aligned}
$$

which we can say because $|x - c| \leq (b-a) = h$.

Let us look at an example using e^x about 0. Then

$$
T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{3.6}
$$

Letting $n = 5$ we will look at $T_n(x)$ and e_n.

$$\begin{aligned}
T_0(x) &= \frac{x^0}{0!} = 1 \\
T_1(x) &= \frac{x^0}{0!} + \frac{x}{1!} = 1 + x \\
T_2(x) &= \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} = 1 + x + \frac{x^2}{2} \\
T_3(x) &= \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \\
T_4(x) &= \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \\
T_5(x) &= \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}.
\end{aligned} \tag{3.7}$$

Now let $x = 1$ and $e^x = e = 2.71828$. We can see below that as n increases e_n decreases.

$$\begin{aligned}
e_0 &= |2.71828 - 1| = 1.71828 \\
e_1 &= |2.71828 - 2| = 0.71828 \\
e_2 &= |2.71828 - 2.5| = 0.21828 \\
e_3 &= |2.71828 - 2.66667| = 0.051615 \\
e_4 &= |2.71828 - 2.70833| = 0.00995 \\
e_5 &= |2.71828 - 2.71666| = 0.00162
\end{aligned} \tag{3.8}$$

At $T_8(x)$, $e_8 = 0.00000$ with five decimal places. If we carry e to fifteen decimal places (2.718281828459046), $T_{17}(x)$ will give us an e_{17} that equals zero to fifteen decimal places.

Chapter 4

Unit Sphere

4.1 Spherical Harmonics and the Laplacian

Primarily we want to look at interpolation on the unit sphere. In Euclidean space $\mathbb{R}^r$, the unit sphere $\mathbb{S}^{r-1}$ and the ball $\mathbb{B}^r$ are defined in Reimer [3] as:

$$\mathbb{S}^{r-1} := \{x \in \mathbb{R}^r\colon\ |x| = 1\}$$

$$\mathbb{B}^r := \{x \in \mathbb{R}^r : |x| \leq 1\}$$

It can be seen that $\mathbb{S}^{r-1}$ is essentially the boundary of $\mathbb{B}^r$.

The Laplacian is the divergence of the gradient of a function. In rectangular coordinates it is expressed as

$$\nabla \cdot \nabla f = \nabla^2 f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}. \tag{4.1}$$

In spherical coordinates we express the laplacian as follows.

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 f}{\partial \phi^2}\right). \tag{4.2}$$

As MacRobert [4] states, "Any solution V_n of Laplace's Equation

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{4.3}$$

which is homogeneous, of degree n, in x, y, z is called a *Solid Spherical Harmonic of degree n.* The degree n may be any number and the function need not be rational." Some examples of spherical harmonics are $x^2 - y^2 + yz$ and $(z + ix)^n$.

Yuan Xu [5] looks at the problem of interpolation at $(n+1)^2$ points on the unit sphere $S^2 = \{x : \|x\| = 1\}$ in $\mathbb{R}^3$, where $\|x\|$ is the Euclidean norm in $\mathbb{R}^3$, by spherical polynomials of degree at most n. Specifically, let $X = \left\{\mathbf{a}_i : 1 \le i \le (n+1)^2\right\}$ be a set of distinct points on S^2. Then what are the conditions on X such that there is a unique polynomial $T \in \prod_n (S^2)$ (where $\prod_n (S^2)$ denotes the space of spherical polynomials of 3 variables) that satisfies

$$T(\mathbf{a}_i) = f_i, \quad \mathbf{a}_i \in X, \quad 1 \le i \le (n+1)^2, \tag{4.4}$$

for any given data $\{f_i\}$. We say the problem is poised and that X solves this problem when there is a unique solution.

Now if we consider the space of all polynomials of the form $\sum c_n x^n$, c is some constant, and denote this space by $\mathbb{P}^r$, we will denote the space of all polynomials from $\mathbb{P}^r$ with total degree at most n to be $\mathbb{P}^r_n$ and the space of all polynomials from $\mathbb{P}^r$ with total degree exactly equal to n will be denoted by $\mathbb{PO}^r_n$. We can denote the space of all polynomials restricted to the unit sphere with total degree at most n as $\mathbb{P}^r_n(S^{r-1})$ and the space of all homogeneous polynomials restricted to the unit sphere with total degree of exactly n to be $\mathbb{PO}^r_n(S^{r-1})$. There are certain theorems about the dimensions of these polynomial spaces which follow.

Theorem 7 *The dimension of* $\mathbb{P}^r_n$ *is*

$$\dim \mathbb{P}^r_n = \binom{n+r}{r}. \tag{4.5}$$

Proof. A basis for $\mathbb{P}^r_n$ can be constructed by selecting all monomials of the form $x_1^{m_1}, \ldots, x_r^{m_r}$ where

$$m_1 + m_2 + \ldots + m_r \le n. \tag{4.6}$$

For this inequality, the number of non-negative integer solutions is $\binom{n+r}{r}$. ■

Theorem 8 *The dimension of* $\mathbb{PO}_n^r$ *is*

$$\dim \mathbb{PO}_n^r = \binom{n+r-1}{r-1}. \tag{4.7}$$

Proof. We first want to prove that the two spaces $\mathbb{PO}_n^r$ and $\mathbb{P}_n^{r-1}$ are isomorphic. We will do this by defining the map

$$\mathcal{H} \quad : \quad \mathbb{P}_n^{r-1} \to \mathbb{PO}_n^r \tag{4.8}$$

$$p\left(x_1, \ldots, x_{r-1}\right) \quad \longmapsto \quad x_r^n p\left(\frac{x_1}{x_r}, \ldots, \frac{x_{r-1}}{x_r}\right). \tag{4.9}$$

Clearly, the map $\mathcal{H}$ is both linear and surjective. Moreover, if $\mathcal{H}_p = \mathcal{H}_q, \forall x \in \mathbb{R}^r$, then if we let $x_r = 1$, the result is $p = q$ for all $x \in \mathbb{R}^{r-1}$, i.e. $p \equiv q$. Hence, $\mathcal{H}$ is bijective and the lemma is proved. ■

A theorem which shows an important property about the decomposition of $\mathbb{P}_n^r\left(S^{r-1}\right)$ follows.

Theorem 9 *For* $n \in N^+, r \geq 2$

$$\mathbb{P}_n^r\left(S^{r-1}\right) = \mathbb{PO}_n^r\left(S^{r-1}\right) \oplus \mathbb{PO}_{n-1}^r\left(S^{r-1}\right). \tag{4.10}$$

Proof. First, define the subspace $\mathbb{Q}_n^r\left(S^{r-1}\right)$ of $\mathbb{P}_n^r\left(S^{r-1}\right)$ as

$$\mathbb{Q}_n^r\left(S^{r-1}\right) = \left\{Q \in \mathbb{P}_n^r\left(S^{r-1}\right) : Q\left(-x\right) = \left(-1\right)^n Q\left(x\right)\right\}. \tag{4.11}$$

Now, since a polynomial can be decomposed into even/odd or odd/even parts, we can say

$$\mathbb{P}_n^r\left(S^{r-1}\right) = \mathbb{Q}_n^r\left(S^{r-1}\right) \oplus \mathbb{Q}_{n-1}^r\left(S^{r-1}\right). \tag{4.12}$$

Clearly, $\mathbb{PO}_n^r\left(S^{r-1}\right) \subset \mathbb{Q}_n^r\left(S^{r-1}\right)$ and $\mathbb{PO}_{n-1}^r\left(S^{r-1}\right) \subset \mathbb{Q}_{n-1}^r\left(S^{r-1}\right)$. Also, each element of $\mathbb{Q}_n^r\left(S^{r-1}\right)$ is of the form

$$Q\left(x\right)=\sum_{i=0}^{|n/2|}\sum_{|m|=n-2i}c_m x^m. \tag{4.13}$$

Since $x\in S^{r-1}, |x|=1$ and

$$Q\left(x\right)=|x|^n\,Q\left(\frac{x}{|x|}\right)=\sum_{i=0}^{|n/2|}\sum_{|m|=n-2i}c_m\left(x_1^2+\ldots+x_r^2\right)x^m. \tag{4.14}$$

This is an element of $\mathbb{PO}_n^r\left(S^{r-1}\right)$; thus $\mathbb{Q}_n^r\left(S^{r-1}\right)\subset\mathbb{PO}_n^r\left(S^{r-1}\right)$. Hence we have

$$\mathbb{Q}_n^r\left(S^{r-1}\right)=\mathbb{PO}_n^r\left(S^{r-1}\right). \tag{4.15}$$

Using similar arguments we can also obtain $\mathbb{Q}_{n-1}^r\left(S^{r-1}\right)=\mathbb{PO}_{n-1}^r\left(S^{r-1}\right)$, thus proving the theorem. ■

Now we will discuss some related concepts including zonal polynomials and spherical harmonics per Reimer. [11]

Theorem 10 *A polynomial P is called zonal with axis $\mathbf{t}\in S^{r-1}$ if*

$$P\left(x\right)=f\left(\mathbf{t}\cdot x\right),\ \forall x\in S^{r-1},\ f\text{ is univariate function }[-1,1]\rightarrow\mathbb{R}. \tag{4.16}$$

Note: The notation $x\cdot y$ stands for the usual dot product in $\mathbb{R}^r$.

Before the next theorem about zonal polynomials we want to define a subgroup U_t^r of the rotation group $\mathbf{SO}\left(r\right)$ to be

$$U_t^r=\left\{\mathbf{A}\in\mathbf{SO}\left(r\right):\mathbf{At}=\mathbf{t}\right\}. \tag{4.17}$$

Theorem 11 ***Theorem 12*** *Let $r\geq 3$, then the polynomial $P\in\mathbb{P}^r$ is zonal with axis $\mathbf{t}\in S^{r-1}$ if and only if $P_A=P$ holds for all $\mathbf{A}\in U_t^r$.*

Proof. Let $P_A = P$ for all $\mathbf{A} \in U_t^r$. Then an element $\mathbf{u} \in S^{r-1}$ exists such that $\mathbf{t} \cdot \mathbf{u} = 0$. Fix $\mathbf{u}$ and $\mathbf{t}$, then we can express an arbitrary vector $\mathbf{x}$ as a linear combination of two orthogonal vectors $\mathbf{t}$ and $\mathbf{v}$ where $\mathbf{v}$ is in the hyperplane spanned by $\mathbf{t}$ and $\mathbf{x}$ as follows

$$\mathbf{x} = (\mathbf{t} \cdot \mathbf{x})\,\mathbf{t} + \sqrt{1 - (\mathbf{t} \cdot \mathbf{x})^2}\,\mathbf{v}. \tag{4.18}$$

Now we can find a rotation $A \in U_t^r$ so we have $\mathbf{Av} = \mathbf{u}$ to result in

$$\mathbf{Ax} = (\mathbf{t} \cdot \mathbf{x})\,\mathbf{t} + \sqrt{1 - (\mathbf{t} \cdot \mathbf{x})^2}\,\mathbf{u}. \tag{4.19}$$

Then we can define $f \in C[-1,1]$ as $f(\zeta) = P\left(\zeta\mathbf{t} + \sqrt{1-\zeta^2}\,\mathbf{u}\right)$ for $\zeta \in [-1,1]$ so that we obtain

$$P(x) = P_A(x) = P(\mathbf{A}x) = f(\mathbf{t} \cdot \mathbf{x}), \quad \mathbf{x} \in S^{r-1}. \tag{4.20}$$

■

We can see that the other direction will come because for every orthogonal matrix A we can say

$$\mathbf{At} \cdot \mathbf{Ax} = (\mathbf{At})^T \mathbf{Ax} = \mathbf{t}^T \mathbf{A}^T \mathbf{Ax} = \mathbf{t}^T x = \mathbf{t} \cdot \mathbf{x}. \tag{4.21}$$

Now we want to provide two theorems for zonal functions. [11]

Theorem 13 *A function $G \in \mathbb{P}^r\left(S^{r-1} \times S^{r-1}\right)$ for $r \geq 3$, is called a zonal function if for any two arbitrary points $x, y \in S^{r-1}$ the following is true for some univariate function $g : [-1,1] \to \mathbb{R}$*

$$G(x,y) = g(x \cdot y). \tag{4.22}$$

Theorem 14 *For any arbitrary $x, y \in S^{r-1}$, the function $G \in \mathbb{P}^r\left(S^{r-1} \times S^{r-1}\right)$ is* *zonal if and only if*

$$G(\mathbf{A}x, \mathbf{A}y) = G(x, y)\, \forall \mathbf{A} \in \mathbf{SO}(r). \tag{4.23}$$

Proof. Let $G(\mathbf{Ax}, \mathbf{Ay}) = G(x, y)$ for all matrices $\mathbf{A} \in \mathbf{SO}(r)$. Now fix $\mathbf{y}$ and let $\mathbf{A} \in \mathbf{U}_y^r$. We now have $G(\cdot, y)$ as a zonal polynomial with respect to the axis $\mathbf{y}$. Applying theorem 12 to $G(\cdot, y)$, we see that there exists a continuous univariate function $g_y : [-1, 1] \to \mathbb{R}$ such that

$$G(\mathbf{x}, \mathbf{y}) = g_y(\mathbf{x} \cdot \mathbf{y}), x \in S^{r-1}. \tag{4.24}$$

We can see that g_y does not depend on $\mathbf{y}$ since $\forall \mathbf{A} \in \mathbf{SO}(r)$ and $x, y \in S^{r-1}$

$$g_{Ay}(\mathbf{x} \cdot \mathbf{y}) = G(\mathbf{A}x, \mathbf{A}y) = G(\mathbf{x}, \mathbf{y}) = g_y(x \cdot y). \tag{4.25}$$

Thus $g_y = g$ for some $g : [-1, 1] \to \mathbb{R}$ independent of $\mathbf{y}$. Since the rotation preserves the dot product, the reverse direction is trivial. ■

At the beginning of this chapter we defined the unit sphere and the Laplacian operator in both rectangular and spherical coordinates. We will use the symbol $\triangle$ to denote the Laplacian operator in $\mathbb{R}^r$.

Now we can define a spherical harmonic of order l. We let $H_l(x)$ be an homogeneous polynomial of degree l in $\mathbb{R}^r$ that satisfies

$$\triangle H_l(x) = 0. \tag{4.26}$$

We can then say that $Y_l = H_l\, |_{S^{r-1}}$is a regular spherical harmonic of order l. Using this definition and Green's Theorem, we have the result that

$$0 = \int_{|x|\leq 1} (H_m \triangle H_n - H_n \triangle H_m)\, dV = \int_{S^{r-1}} H_m(x)\, H_n(x)\,(m - n)\, dS \tag{4.27}$$

noting that

$$mH_m(\mathbf{x}) = \left[\frac{\partial H_m(r\mathbf{x})}{\partial r}\right]_{r=1} \quad \text{and} \quad nH_n(\mathbf{x}) = \left[\frac{\partial H_m(r\mathbf{x})}{\partial r}\right]_{r=1}.$$

Hence, we have $\int_{S^{r-1}} Y_m(x)\, Y_n(x)\, dS = 0, \;\; m \neq n.$

Now we can denote the space of all spherical harmonics of order l by $HO_l^r(S^{r-1})$ and we will let $H_l^r(S^{r-1})$ denote the space of all spherical harmonics of degree $\leq n$. We can then express the relation between the spaces as follows. [11]

$$\mathbb{P}_n^r\left(S^{r-1}\right) = H_n^r\left(S^{r-1}\right), \tag{4.28}$$

$$H_n^r\left(S^{r-1}\right) = \bigoplus_{l=0}^{n} HO_l^r\left(S^{r-1}\right). \tag{4.29}$$

We know that $HO_l^r(S^{r-1})$ is rotation invariant because it is the eigenspace of the Laplace-Beltrami operator on S^{r-1}. Now let $\mathbf{A}$ be an orthogonal matrix , $Y_l(\mathbf{Ax}) \in HO_l^r(S^{r-1})$ and let the functions Y_{lj} be an orthonormal set, i.e.

$$\int_{S^{r-1}} Y_{lj}(\mathbf{x})\, Y_{lk}(\mathbf{x})\, dS = \delta_{jk.} \tag{4.30}$$

Now we can show $Y_{lj}(\mathbf{Ax})$ as a linear combination of $N(r,l)$ spherical harmonics of order l thus resulting in

$$Y_{lj}(\mathbf{Ax}) = \sum_{n=1}^{N(r,l)} c_{jn}^{l} Y_{ln}(\mathbf{x}). \tag{4.31}$$

The above two equations give the following result

$$\int_{S^{r-1}} Y_{lj}(\mathbf{Ax})\, Y_{lk}(\mathbf{Ax})\, dS = \sum_{n=1}^{N(r,l)} c_{jn}^{l} c_{kn}^{l}. \tag{4.32}$$

Now the orthogonal transformation $\mathbf{A}$ will leave the surface element dS unchanged so that we get

$$\sum_{n=1}^{N(r,l)} c_{jn}^{l} c_{kn}^{l} = \delta_{jk}. \tag{4.33}$$

Thus c^l_{jn} are elements of an orthogonal matrix such that

$$\sum_{n=1}^{N(r,l)} c^l_{nj} c^l_{nk} = \delta_{jk}. \tag{4.34}$$

Now for any two points x, y we can define the polynomial function

$$G_l(x,y) = \sum_{j=1}^{N(r,l)} Y_{lj}(x) Y_{lj}(y). \tag{4.35}$$

Now because of (4.34) for any orthogonal matrix $\mathbf{A}$ we have

$$\begin{aligned} G_l(\mathbf{Ax}, \mathbf{Ay}) &= \sum_{j=1}^{N(r,l)} Y_{lj}(\mathbf{A}x) Y_{lj}(\mathbf{A}y) \qquad (4.36) \\ &= \sum_{j=1}^{N(r,l)} \sum_{n=1}^{N(r,l)} c^l_{jn} Y_{ln}(x) \sum_{m=1}^{N(r,l)} c^l_{jm} Y_{lm}(\mathbf{y}) \\ &= \sum_{n=1}^{N(r,l)} \sum_{m=1}^{N(r,l)} \delta_{nm} Y_{ln}(x) Y_{lm}(\mathbf{y}) \\ &= G_l(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Applying theorems (13) and (14) we can say that G_l is a bizonal function.

Next, we want to look at the addition theorem.

Theorem 15 *Let $\{Y_{lk}\}$ be an orthonormal set of $N(r,l)$ spherical harmonics of order l on S^{r-1}, then*

$$\sum_{j=1}^{N(r,l)} Y_{lj}(x) Y_{lj}(y) = \frac{N(r,l)}{\varpi_{r-1}} P_l^{(r)}(\mathbf{x} \cdot \mathbf{y}), \tag{4.37}$$

where $\mathbf{x}, \mathbf{y} \in S^{r-1}$, ϖ_{r-1} is the total surface area of the sphere S^{r-1} and $P_l^{(r)}$ is the Legendre polynomial of degree l in $\mathbb{R}^r$.

Proof. Now we know from Muller [12] that the Legendre function is a homogeneous, harmonic polynomial that is zonal with respect to a fixed axis. Thus the univariate function $g : [-1, 1] \rightarrow \mathbb{R}$ in theorems (13) and (14) is clearly just a multiple of the Legendre function so we can see that

$$\sum_{j=1}^{N(r,l)} Y_{lj}(x) Y_{lj}(y) = cP_l^{(r)}(\mathbf{x} \cdot \mathbf{y}), \text{ where } c \text{ is some constant.} \tag{4.38}$$

Let $\mathbf{x} = \mathbf{y}$ and since $P_l(1) = 1$ we obtain

$$\sum_{j=1}^{N(r,l)} [Y_{lj}(x)]^2 = cP_l(1) = c. \tag{4.39}$$

If we integrate over S^{r-1} we get $N(r, l) = c\varpi_{r-1}$ which proves the theorem. ■

Delsarte, Goethals and Seidel [13] introduced the idea of a spherical t-design which is a set of points $\Upsilon = \{t_1, \ldots, t_m\}$ on the sphere S^{r-1} such that the equal-weight quadrature rule based on these points is exact for all polynomials of degree $\leq t$.

$$\frac{\varpi_{r-1}}{M} \sum_{k=1}^{M} p(t_k) = \int_{S^{r-1}} p(x)\, dS, \quad \forall p \in \mathbb{P}_t^r\left(S^{r-1}\right). \tag{4.40}$$

Here ϖ_{r-1} is the surface area of S^{r-1}. Delsarte, Goethals and Seidel [13] give the following lower bounds for the cardinality of Υ in order for the quadrature rule to be exact.

$$M \geq \binom{r+n-1}{r-1} + \binom{r+n-2}{r-1}, \qquad M \geq 2\left(\frac{r+n-1}{r-1}\right) \tag{4.41}$$

for $t = 2n$ and $t = 2n + 1$ respectively. Note that a spherical t-design $\Upsilon = \{t_1, \ldots, t_m\}$ is called a tight spherical t-design if the cardinality of Υ attains its lower bound. [13]

4.2 Hilbert Spaces and Linear Projections

A Hilbert space is an inner product space that is complete, i.e. Cauchy sequences will converge to point that is in that space. Reimer [14] defines and proves the existence and uniqueness of a reproducing kernel function. Given a linear space of functions $\mathbf{V}$ with $D \rightarrow \mathbb{R}$ with the inner product $\langle \cdot, \cdot \rangle$, a function $G : D \times D \rightarrow \mathbb{R}$ is called a reproducing kernel of $\mathbf{V}$ if the following are satisfied.

(i) $G(x, \cdot) \in \mathbf{V}$ for all $x \in D$,

(ii) $G(x, y) = G(y, x)$ for all $(x, y) \in D^2$,

(iii) $\langle G(x, \cdot), F \rangle = F(x)$ for all $F \in \mathbf{V}$,$x \in D$.

Theorem 16 *If* $\mathbf{V}$ *is finite-dimensional, then a uniquely determined reproducing kernel exists.*

Proof. Let $S_1, \ldots, S_N$ be an arbitrary orthonormal basis in $\mathbf{V}$ and define $G : D \times D \rightarrow \mathbb{R}$ by

$$G(x, y) = \sum_{j=1}^{N} S_j(x) S_j(y) \qquad \text{for } x, y \in D. \tag{4.42}$$

It is clear that G satisfies (i) and (ii). Now let $F \in \mathbf{V}$ and $x \in D$. We then obtain

$$\langle G(x, \cdot), F \rangle = \sum_{j=1}^{N} \langle S_j, F \rangle S_j(x) = F(x) \tag{4.43}$$

which shows the validity of (iii). Thus, a reproducing kernel exists. Now assume that H is an arbitrary reproducing kernel of $\mathbf{V}$. Then we can say $H(x, \cdot) \in \mathbf{V}$,$x \in D$ is reproduced at $y \in D$ by $G(y, \cdot)$, and we get

$$H(x, y) = \langle G(y, \cdot), H(x, \cdot) \rangle = \langle H(x, \cdot), G(y, \cdot) \rangle. \tag{4.44}$$

But $H(x, \cdot)$ reproduces $G(y, \cdot) \in \mathbf{V}$ at x, and thus we see that

$$H(x, y) = G(y, x) = G(x, y) \tag{4.45}$$

Since x and y were arbitrary it implies that $H = G$ proving uniqueness. ■

Next we want to discuss linear projections.[15] If we let U be a normed linear space and V be a finite dimensional subspace of U, then for $u \in U$, we define the minimal deviation of u in V as

$$E(u, V) = \inf \{\|u - v\| : v \in V\}. \tag{4.46}$$

Then every element v in V with $\|v - u\| = E(u, V)$ is called a best approximation to u in V. The existence of v is given by a fundamental existence theorem, but first we need another theorem.

Theorem 17 *Every closed, bounded, finite-dimensional set in a normed linear space is compact.*

Proof. Let V be a closed, bounded, finite-dimensional subspace of U and let $\{v_1, \ldots, v_n\}$ be a basis for V. Then for each element v in V, there is a unique tuple $(a_1, \ldots, a_n)$ so that $v = \sum_{i=1}^{n} a_i v_i$. We can define the map $T : \mathbb{R}^n \to V$ that maps $a \longmapsto v$, where $a = (a_1, \ldots, a_n)$. Now if $\|a\| = \max_{i=1,\ldots,n} |a_i|$ then T is continuous. For example, if we let $a, b \in \mathbb{R}^n$, then

$$\|Ta - Tb\| = \left\| \sum_{i=1}^{n} a_i v_i - \sum_{i=1}^{n} b_i v_i \right\| \leq \sum_{i=1}^{n} |a_i - b_i| \, \|v_i\| \leq \|a - b\| \sum_{i=1}^{n} \|v_i\|. \tag{4.47}$$

To prove that V is compact, we can show that $A = \{a : Ta \in V\}$ is compact. First, we will show that A is closed. If $a^{(k)} \to a$, then $Ta = T(\lim_k) a^{(k)} = \lim_k T\left(a^{(k)}\right)$. Now since V is closed, $Ta \in V$, $a \in A$ and this shows that A is closed. Next, we will prove that A is bounded. Since the set $\{a : \|a\| = 1\}$ is compact and T is continuous, the infimum, α, of $\|Ta\|$ is obtained on that set. Now, since $\{v_1, \cdots, v_n\}$ is linearly independent, $\alpha > 0$. Thus for any $a \neq 0$, $\|Ta\| = \left\| T\left(\frac{a}{\|a\|}\right) \right\| \|a\| \geq \alpha \|a\|$. And since $\|Ta\|$ is bounded on A, $\|a\|$ is bounded on A. ■

Theorem 18 (Existence) *Let $\{v_1, \ldots, v_n\}$ be a basis for V and let u be an element in U. Then the problem of finding*

$$\min_{a_i} \|u - (a_1 v_1 + \ldots + a_n)\|$$

has a solution.

Proof. We can find the solution in the set $M = \{v \in V : \|u - v\| \leq \|u - w\|\}$, where w is an arbitrary fixed element in V. We know that M is compact by theorem (17). Now let $\delta = E(u, V)$ and using the definition of an infimum, we can find a sequence of points in M, $x_1, x_{2\ldots}$with the property that $\|u - x_n\| \to \delta$ as $n \to \infty$. Because M is compact, we can assume that the sequence converges to a point v of M. Now from the triangle inequality we can say

$$\|u - v\| \leq \|u - x_n\| + \|x_n - v\|.$$

We know that as $n \to \infty$, $\|u - v\| \leq \delta$ and since $v \in V$, $\|u - v\| \geq \delta$; hence $\|u - v\| = \delta$. ■

Now we can define a projection operator P if P is surjective and $P^2 = P$ where $P \in L(U, V)$ and $L(U, V)$ denotes the set of all bounded linear operators from U to V.

4.3 Polynomial Interpolation on Spheres

We want to briefly look at interpolation on spheres considering it as a linear projection from the space $C(S^{r-1})$ to a finite dimensional space $\mathbb{P} \subset \mathbb{P}^r(S^{r-1})$. First, we need to consider the concept of the fundamental system. The set $\{t_1, \ldots, t_N\}$ is called a fundamental system if the evaluation functionals $\{f \to f(t_j)\}$ for $j = 1, \ldots, N$ and $f \in \mathbb{P}$, is a linearly independent set. Note that N is the dimension of $\mathbb{P}$. Many methods of interpolation have been developed. We can define the Lagrange interpolating polynomial where $P(x)$ is of degree $\leq (n-1)$

to be:

$$P(x) = \sum_{j=1}^{N} P_j(x) \quad where\ P_j(x) = y_j \prod_{k \neq j} \frac{x - x_k}{x_j - x_k}. \tag{4.48}$$

Although Lagrangian interpolation in a multivariate setting has been considered problematic, Reimer [11] suggests using the theory of reproducing kernel Hilbert space and the Addition Theorem of spherical harmonics to bring down the dimension when studying multivariate interpolation on the sphere S^{r-1}. In order for the Addition Theorem of spherical harmonics to be applied in this way $\mathbb{P}$ must be a rotation-invariant subspace of $\mathbb{P}^r\left(S^{r-1}\right)$.

4.4 Distribution of Points

One of the problems we encounter is distributing a large number of points equally on a sphere. Saff and Kuijlaars [6] suggest a construction of spiral points where the first point is at a pole and then the sphere is intersected by N horizontal planes which are spaced 2/(N-1) units apart. A point is placed on each latitude line. Using spherical coordinates (θ, ϕ), where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, they set:

$$\theta_k = \arccos(h_k), where\ h_k = -1 + \frac{2(k-1)}{(N-1)}, 1 \leq k \leq N,$$

$$\phi_k = \left(\phi_{k-1} + \frac{3.6}{\sqrt{N}} \frac{1}{\sqrt{1-h_k^2}}\right) \pmod{2\pi}, \text{ where } 2 \leq k \leq N-1,\ \phi_1 = \phi_N = 0.$$

Figure 1.shows the results of plotting 1,000 points on a sphere in MATLAB using the above construction. The view is of the pole of the sphere which shows the points as they begin spiral from the pole.

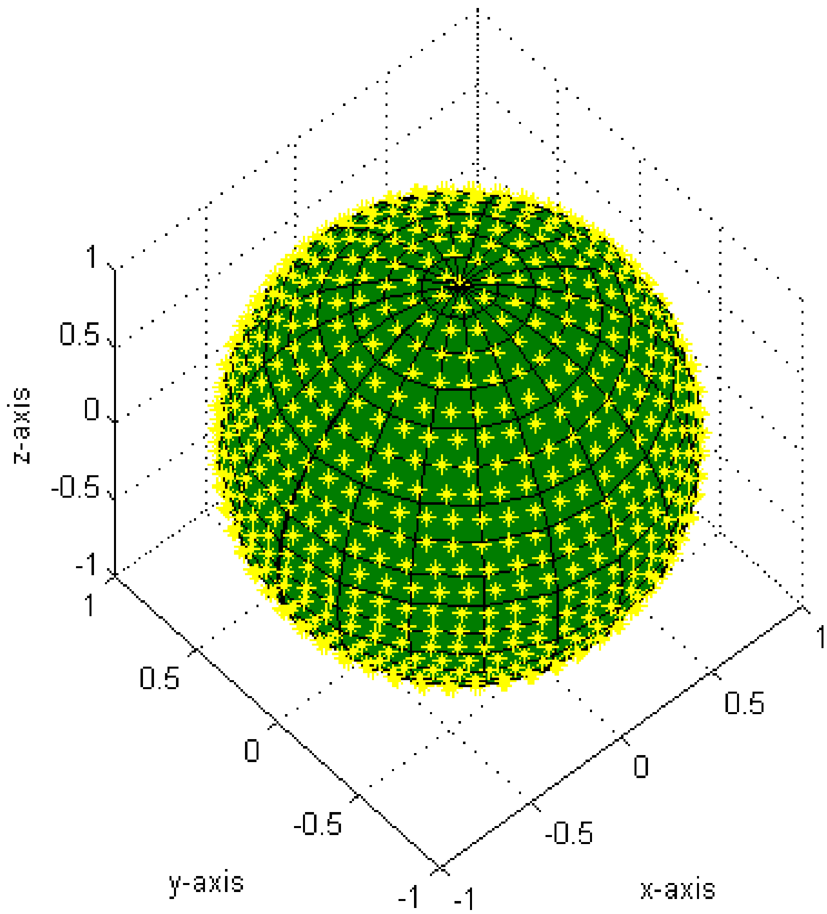

Figure 2 Spiral Points.

4.5 Covering Theorem

In the theorem and proof below, xy denotes the inner product in $\mathbb{R}^{d+1}$, μ denotes surface measure on S^d normalized to have total mass 1, P_n^λ is the Gegenbauer polynomial of degree n and parameter λ and $\prod_{2n}$ denotes the space of polynomials of degree $\leq 2n$, $n \geq 1$.

Theorem 19 *Let $n \geq 1$. Suppose we have a set of points $Y \subset S^d$, a set of numbers A_y where $y \in Y$, $A_y > 0$ and*

$$\sum_{y \in Y} A_y P(y) = \int_{S^d} P(y)\, d\mu(y) \quad \text{for } P \in \prod\nolimits_{2n}\left(S^d\right), \tag{4.49}$$

then

$$S^d \subseteq \bigcup_{y\in Y} B_{\rho(y)}, \quad \textit{where } \rho = \cos^{-1}\left(t_n^\lambda\right) \quad \textit{and} \;\; t_n^\lambda \textit{ is the largest zero of } P_n^\lambda.$$

Proof

Fix $x \in S^d$. Applying (1) we have the following

$$\sum_{y\in Y} A_y \frac{(P_n(xy))^2}{t_n^\lambda - xy} = \int_{S^d} \frac{P_n(xy)\cdot P_n(xy)}{t_n^\lambda - xy} d\mu(x) = 0. \tag{4.50}$$

Equation 2 follows because:

(1) Gegenbauer polynomials are orthogonal to each other, i.e.$\langle P_n, P_m\rangle = 0$, $n \neq m$

(2) As t_n^λ is the largest zero of P_n^λ, a polynomial of degree $\leq n$, note that t_n^λ is a simple zero of P_n^λ since P_n^λ is an orthogonal polynomial and hence has n simple zeros. $\frac{P_n(xy)}{t_n^\lambda - xy}$ is well defined for every x and y because a polynomial $\frac{P_n(t)}{(t-t_n)} = a(t-t_0)(t-t_1)\ldots(t-t_{n-1})$, a is some constant.

There are only two ways in which Equation 2 can equal zero. One is if $P_n(xy) = 0$, for all $y \in Y$; the other is if the numbers $\{t_n^\lambda - xy : y \in Y\}$ are not all the same sign. Let

$$d(x,y) = \cos^{-1}(xy),\;\; y \in Y. \tag{4.51}$$

Suppose $d(x,y) > 0$, $\forall y \in Y$. Then $d(x,y) = \cos^{-1}(xy) \Longleftrightarrow xy < t_n^\lambda$ so $\sum_{y\in Y} A_y \frac{(P_n(xy))^2}{t_n^\lambda - xy} > 0$. This is a contradiction. Now consider

$$\{t_y = xy : \;\; y \in Y,\; t_y \in [-1,1]\}, \tag{4.52}$$

then $d(x,y) = \cos^{-1}(t_y)$. But moving x will then produce an open set where $P_n^\lambda(x) = 0$ and so $P_n^\lambda \equiv 0$. So again we have a contradiction. ■

Chapter 5

Conclusion and Future Work

In conclusion, we have looked at some aspects of approximation including interpolation of real valued functions, specifically polynomial interpolation and a theorem that gives us a way of measuring error. We considered theorems and proofs for both the existence and uniqueness of Lagrange and Hermite interpolation. We proved the Weierstrass Approximation Theorem which tells us that any continuous function on a finite interval can be approximated as closely as desired by a polynomial. We considered some quadrature rules in numerical integration, error and precision and then we used Taylor's theorem to approximate e^x and determine the error. We moved on to consider the unit sphere and spherical harmonics. In our consideration of Hilbert spaces and linear projections we looked at Reimer's definition and proof of the existence and uniqueness of a reproducing kernel function. One of the problems encountered is how to distribute a large number of points equally on a sphere and we looked at the suggestion Saff and Kuijlaars [6] make of constructing spiral points. We then proved a covering theorem for a unit sphere.

Extending the proof of the covering theorem to two point space is a task for the future, also perhaps attempting to extend to more general manifolds. As well as looking at some hyperinterpolation problems, it is hoped that learning how to use Lie algebras will increase the sophistication of investigating approximation problems.

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