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MONITORING FOR A SHIFT IN A PROCESS COVARIANCE MATRIX USING THE GENERALIZED VARIANCE

by

KELLEN M. PARHAM

(Under the Direction of Dr. Charles W. Champ)

ABSTRACT

The commonly recommended charts for monitoring the mean vector are affected by a shift in the covariance matrix. As in the univariate case, a chart for monitoring for a change in the covariance matrix should be examined first before examining the chart used to monitor for a change in the mean vector. One such chart is the one that plots the generalized sample variance $|\mathbf{S}|$ verses the sample number t. We propose to study charts based on the statistics $V = \left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}$ and $U = \ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right)$, where n is the sample size and $\boldsymbol{\Sigma}_0$ is the in-control value of the process covariance matrix $\boldsymbol{\Sigma}$. In particular, we will study the Shewhart V and U charts supplemented with runs rules. Also, we examine the methods that are useful in studying the run length properties of the cumulative sum (CUSUM) U charts. Further, we will study the effect that estimating $\boldsymbol{\Sigma}_0$ has on the performance of these charts. Guidance will be given for designing the Shewhart charts with runs rules with illustrative examples.

Key Words: Average run length, Cumulative sum charts, Independent samples, Integral equations, Markov chain, Multivariate normal distribution, Shewhart charts.

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by

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CHAPTER 1

INTRODUCTION

Monitoring for a change in the mean of a multivariate quality measurement is an important problem in various industrial settings. Control charts that have been designed for this purpose are affected by a change in the mean vector of the distribution of a multivariate quality measurement, but also they are affected by a change in the covariance structure. Consequently, the practitioner is faced with the problem of monitoring for a change in the covariance matrix. A variety of charts have been proposed for this purpose.

One of the most commonly used charts for monitoring for a change in covariance matrix $\mathbf{\Sigma}$ of a multivariate quality measurement $\mathbf{X}$ is a Shewhart chart based on the sample generalized variance $|\mathbf{S}|$, where $\mathbf{S}$ is the sample covariance matrix. One method for selecting the lower (LCL) and upper (UCL) control limits for this chart that is commonly recommended (see Montgomery (2001))is

$$LCL = |\mathbf{\Sigma}_0| \left(b_1 - 3b_2^{1/2} \right) \text{ and } UCL = |\mathbf{\Sigma}_0| \left(b_1 + 3b_2^{1/2} \right),$$

where $\mathbf{\Sigma}_0$ is the in-control value of $\mathbf{\Sigma}$. The values b_1 and b_2 are discussed in the next chapter. Montgomery and Wadsworth (1972) give a method for determining the control limits using an asymptotic normal approximation of the distribution of $|\mathbf{S}|$. These control limits have the general form

$$LCL = |\mathbf{\Sigma}_0| \left(b_1 - z_\tau b_2^{1/2} \right) \text{ and } UCL = |\mathbf{\Sigma}_0| \left(b_1 + z_{\alpha-\tau} b_2^{1/2} \right),$$

where $0 \leq \tau \leq \alpha$ with τ commonly selected to be $\alpha/2$.

Alt (1985) proposed a Shewhart type chart based on the statistic

$$W_t = -pn + pn \ln(n) - n \ln\left(\left| (n-1)\, \mathbf{\Sigma}_0^{-1} \mathbf{S}_t \right| \right) + tr\left((n-1)\, \mathbf{\Sigma}_0^{-1} \mathbf{S}_t \right)$$

for monitoring for a change in $\mathbf{\Sigma}$. The chart signals a potential out-of-control process if $W_t \leq h_L$ or $W_t \geq h_U$ with $h_L < h_U$. He recommended using the asymptotic upper control limit of $\chi^2_{p(p+1)/2,\alpha}$. We point out here that there exist no proof in the literature that any Phase II chart is equivalent to a (sequential) test of hypothesis. The plotted statistic W_t is viewed here only as a chart statistic.

Healy (1987) proposed a multivariate CUSUM chart for detecting a change in the covariance matrix from $\mathbf{\Sigma}_0$ to $c\mathbf{\Sigma}_0$ for $c > 0$. The multivariate CUSUM statistic C_t is for this chart is

$$C_t = \max\left\{0, Q_{t-1} + \ln \frac{f\left(\mathbf{x}_{t,1}, \ldots, \mathbf{x}_{t,n} \left|\mu_0, c\mathbf{\Sigma}_0\right.\right)}{f\left(\mathbf{x}_{t,1}, \ldots, \mathbf{x}_{t,n} \left|\mu_0, \mathbf{\Sigma}_0\right.\right)}\right\}$$

for $c > 0$, where $f\left(\mathbf{x}_{t,1}, \ldots, \mathbf{x}_{t,n} \left|\mu, \mathbf{\Sigma}\right.\right)$ is the joint density of a random sample $\mathbf{X}_{t,1}, \ldots, \mathbf{X}_{t,n}$ from a multivariate normal distribution with mean vector μ and covariance matrix $\mathbf{\Sigma}$. Using "rescaling," one can express C_t as

$$C_t = \max\left\{0, C_{t-1} + \sum\nolimits_{j=1}^{n} \left(\mathbf{X}_{t,j} - \mu_0\right)^{\mathbf{T}} \mathbf{\Sigma}_0^{-1} \left(\mathbf{X}_{t,j,} - \mu_0\right) - k\right\},$$

where $k = pnc\ln(c) / (c-1)$ and $0 \leq Q_0 < h$. The chart signals at the first sampling stage in which $Q_t \geq h \geq 0$ (h is the control limit). Crosier (1986) introduced a multivariate CUSUM chart based on the statistics

$$C_t = \max\left\{0, C_{t-1} + \sqrt{\sum\nolimits_{j=1}^{n} \left(\mathbf{X}_{t,j} - \mu_0\right)^{\mathbf{T}} \mathbf{\Sigma}_0^{-1} \left(\mathbf{X}_{t,j,} - \mu_0\right)} - k\right\}$$

with $0 \leq C_0 < h$ and without the restriction Healy (1987) placed on the chart parameter k. The chart signals a potential change in $\mathbf{\Sigma}$ from $\mathbf{\Sigma}_0$ if $C_t > h \geq 0$ (h is the control limit). Note that one can obtain Shewhart versions of the charts proposed by Healy (1987) and Crosier (1986) by selecting the control limits to be zero.

A multivariate exponentially weighted moving average (EWMA) chart based on the sequence of generalized sample variances would have the form

$$E_0 = E\left(Y\,|\boldsymbol{\Sigma}_0\right) \text{ and } E_t = rY_t + (1-r)\,E_{t-1}$$

where $Y = |\mathbf{S}|$. The chart signals if $E_t \leq h_L$ or $E_t \geq h_U$. Bernard (2001) studied a multivariate EWMA chart with

$$Y_t = \begin{cases} \ln\left(\sqrt{|\mathbf{S}_i|}\right), & \text{if } p = 2; \\ \ln\left(|\mathbf{S}_i|\right), & \text{if } p > 2. \end{cases}$$

Yeh, Lin, Zhou, and Venkataramani (2003) and Yeh, Huwang, and Wu (2004,2005) proposed multivariate EWMA charts for monitoring process variability based on the EWMA sequence of matrices

$$\mathbf{E}_0 = \mathbf{X}_1\mathbf{X}_1^{\mathbf{T}} \text{ and } \mathbf{E}_i = r\mathbf{X}_i\mathbf{X}_i^{\mathbf{T}} + (1-r)\,\mathbf{E}_{i-1}.$$

Observing that the expectation of $\mathbf{E}_i$ is the matrix of parameters $\boldsymbol{\Sigma}+\mu\mu^{\mathbf{T}}$. Hence the chart not only depends on a change in the covariance matrix $\boldsymbol{\Sigma}$ but also a change in the mean vector. Champ and Jones-Farmer (2005) show that commonly recommended charts for monitoring the mean vector are affected by shifts in both μ and $\boldsymbol{\Sigma}$. This suggest that only one of these charts is needed.

We begin by looking at our data model, the meaning of statistical process control, sampling assumptions, and some distributional results. These concepts are discussed in Chapter 2. In Chapter 3, the Shewhart chart based on a function of the $|\mathbf{S}|$ supplemented with runs rules is discussed. A method is given for selecting the warning and control limits for these charts. These limits depend on the number of quality measurements taken on an item. Methods are discussed for determining the run length distribution both when the process is in-control and when it is out-of-control as discussed in Chapter 2. In the fourth chapter, we give the details of analytical methods that are useful for analyzing a cumulative sum (CUSUM) charts based on a function of the sample generalized matrix. An example is given in Chapter 5.

CHAPTER 2

MODEL AND RELATED DISTRIBUTIONAL RESULTS

2.1 Introduction

A multivariate quality measurement $\mathbf{X} = [X_1, \ldots, X_p]^{\mathbf{T}}$ is to be taken on an item from the output of a production process. These variables have been identified by the practitioner as important quality measurements in the sense that parameters of their joint distribution describe important quality measures of the process. Control charts were introduced by Shewhart (1931) as statistical methods for aiding the practitioner (1) in bringing a process into a state of statistical in-control, (2) defining what is meant by an in-control process, and (3) monitoring for a change in a process. Charts used to achieve (1) and (2) are often referred to as retrospective or Phase I charts and to accomplish (3) as prospective or Phase II charts.

A Phase I or II chart is selected depending on the collection of process parameters the practitioner has an interest in controlling. It is often the case the practitioner is mainly interested in controlling the mean vector of the distribution of the quality vector $\mathbf{X}$. Although this may be the primary interest of the practitioner, it will be seen as in the univariate case that the changes in the collection of parameters that describe dispersion in the distribution of $\mathbf{X}$ affect the performance of charts for controlling the mean vector. Consequently, the practitioner must also be interested in controlling measures of dispersion.

As with most statistical methods, control charts are designed assuming some model for the vector of quality measurements $\mathbf{X}$ and a method as to how information is to be collected from the process. The model is commonly the multivariate normal distribution and the sampling method is to sample items periodically from the process assuming the quality measurements on these items are independent and identically distributed random vectors. We will refer to this model as the independent multivariate normal model. Under this model, the distribution of the plotted statistic is determined and a performance analysis of the chart can be determined.

In the next section, we discuss the independent multivariate normal model. This section is followed by a section giving some distributional results that will be used to study the performance of various multivariate control charts for monitoring the process dispersion. In some cases, exact distributions of the plotted statistics can be determined; in others, approximated distributional methods are discussed. We make some concluding remarks in the final section.

2.2 Model

It is typical in developing and analyzing control charts to assume that the distribution of $\mathbf{X}$ is a multivariate normal distribution with mean vector μ and covariance matrix $\mathbf{\Sigma}$. Further, it is assumed that the $\mathbf{\Sigma}$ is a positive definite matrix. The joint probability density function of the distribution of $\mathbf{X}$ has the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{p/2} |\mathbf{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^{\mathbf{T}}\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}.$$

Under the assumption that $\mathbf{\Sigma}$ is a positive definite matrix, the eigenvalues $\xi_1, \ldots, \xi_p$ of $\mathbf{\Sigma}$ are positive real numbers. Letting $\mathbf{v}_i$ be the normalized eigenvector associated with the eigenvalue ξ_i, we can write

$$\mathbf{\Sigma} = \mathbf{VCV}^{\mathbf{T}},$$

where $\mathbf{C} = \mathit{Diagonal}(\xi_1, \ldots, \xi_p)$ and $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_p]$. Note that $\mathbf{v}_i^{\mathbf{T}}\mathbf{v}_i = 1$ and

$\mathbf{v}_i^{\mathbf{T}}\mathbf{v}_j = 0$ for $i \neq j = 1, \ldots, p$. It is convenient to let $\mathbf{P} = \mathbf{V}\mathbf{C}^{1/2}$, where $\mathbf{C}^{1/2} = \mathit{Diagonal}\left(\xi_1^{1/2}, \ldots, \xi_p^{1/2}\right)$.

Three scalar transformations of $\mathbf{\Sigma}$ often found in the literature that provide scalar measures of dispersion in the distribution of $\mathbf{X}$ are

$$\omega_1 = \sum\nolimits_{i=1}^{p} \xi_i,\ \omega_2 = \sum\nolimits_{i=1}^{p} \xi_i^2, \text{ and } \omega_3 = \prod\nolimits_{i=1}^{p} \xi_i^2.$$

The parameter ω_2 is found in Mechanics and is a measure of inertia. The ω_3 is the determinant $|\mathbf{\Sigma}|$ of $\mathbf{\Sigma}$ known as the generalized variance. In the case in which $p = 1$, we see that

$$\omega_1 = \sigma \text{ and } \omega_2 = \omega_3 = \sigma^2.$$

For $p = 2$, we have that

$$\xi_1 = \frac{(\sigma_1^2 + \sigma_2^2) - \sqrt{(\sigma_1^2 - \sigma_2^2)^2 + 4\rho^2\sigma_1^2\sigma_2^2}}{2} \text{ and}$$
$$\xi_2 = \frac{(\sigma_1^2 + \sigma_2^2) + \sqrt{(\sigma_1^2 - \sigma_2^2)^2 + 4\rho^2\sigma_1^2\sigma_2^2}}{2}.$$

Hence,

$$\omega_1 = \sigma_1^2 + \sigma_2^2,\ \omega_2 = \sigma_1^4 + 2\rho^2\sigma_1^2\sigma_2^2 + \sigma_2^4, \text{ and } \omega_3 = \sigma_1^2\sigma_2^2\left(1 - \rho^2\right).$$

Again we see that ω_3 is the determinant $|\mathbf{\Sigma}|$ of $\mathbf{\Sigma}$. We also note that if $\mathbf{\Gamma} = \mathit{Diagonal}\,(\sigma_1, \ldots, \sigma_p)$ and $\mathbf{\Psi}$ is the corresponding correlation matrix associated with $\mathbf{\Sigma}$, then

$$|\mathbf{\Sigma}| = |\mathbf{\Psi}|\left(\prod\nolimits_{i=1}^{p} \sigma_i^2\right) = \prod\nolimits_{i=1}^{p} \psi_i\sigma_i^2,$$

where $\psi_1, \ldots, \psi_p$ are the eigenvalues of $\mathbf{\Psi}$. The eigenvalues of $\mathbf{\Psi}$ are positive real numbers since $\mathbf{\Psi}$ is a positive definite. This follows by observing that for all $\mathbf{x} \neq \mathbf{0}$, we have that $\mathbf{y} = \mathbf{\Gamma}^{-1}\mathbf{x} \neq \mathbf{0}$ and

$$\mathbf{x}^{\mathbf{T}}\mathbf{\Psi}\mathbf{x} = \left(\mathbf{\Gamma}^{-1}\mathbf{x}\right)^{\mathbf{T}}(\mathbf{\Gamma}\mathbf{\Psi}\mathbf{\Gamma})\left(\mathbf{\Gamma}^{-1}\mathbf{x}\right) = \mathbf{y}^{\mathbf{T}}\mathbf{\Sigma}\mathbf{y} > \mathbf{0}.$$

The collection of parameters that characterize the dispersion of a multivariate quality measurement $\mathbf{X}$ of p dimension is the $p \times p$ covariance matrix $\boldsymbol{\Sigma}$. The typical reason a practitioner has an interest in a change in $\boldsymbol{\Sigma}$ is that changes in the covariance structure affects the commonly recommended charts for monitoring the mean vector μ. For example, suppose the practitioner is interested in monitoring for a change in μ and the in-control mean vector μ_0 and covariance matrix $\boldsymbol{\Sigma}_0$ are known. We will assume that $\boldsymbol{\Sigma}$ is a positive definite matrix. A plotted statistic often recommended for monitoring μ is the Hotelling's T^2 statistic defined by

$$T_k^2 = n\left(\overline{\mathbf{X}}_k - \mu_0\right)^{\mathbf{T}} \boldsymbol{\Sigma}_0^{-1}\left(\overline{\mathbf{X}}_k - \mu_0\right),$$

where $\overline{\mathbf{X}}_k$ is the vector of means of the kth sample taken in Phase II. We observe that when $\mu = \mu_0$ and $\boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0$ the statistic T^2 can be expressed as

$$T^2 = (\boldsymbol{\Lambda}\mathbf{Z})^{\mathbf{T}}(\boldsymbol{\Lambda}\mathbf{Z}) = \mathbf{Z}^{\mathbf{T}}\left(\boldsymbol{\Lambda}^{\mathbf{T}}\boldsymbol{\Lambda}\right)\mathbf{Z}$$

where

$$\mathbf{Z} = \sqrt{n}\mathbf{P}^{-1}\left(\overline{\mathbf{X}} - \mu_0\right) \text{ and } \boldsymbol{\Lambda} = \mathbf{P}_0^{-1}\mathbf{P}.$$

The matrix $\mathbf{P}$ is the product of the diagonal matrix of the square roots of the eigenvectors and corresponding matrix of normalized eigenvectors of $\boldsymbol{\Sigma}$.

For any non-zero vector $\mathbf{x}$, we have that

$$0 < \mathbf{x}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{x} = (\mathbf{P}\mathbf{x})^{\mathbf{T}}(\mathbf{P}\mathbf{x}),$$

since $\boldsymbol{\Sigma}$ is assumed to be a positive definite matrix. Thus, the vector $\mathbf{P}\mathbf{x}$ is a non-zero vector. Next observe that

$$\mathbf{x}^{\mathbf{T}}\left(\boldsymbol{\Lambda}^{\mathbf{T}}\boldsymbol{\Lambda}\right)\mathbf{x} = (\mathbf{P}\mathbf{x})^{\mathbf{T}}\boldsymbol{\Sigma}_0^{-1}(\mathbf{P}\mathbf{x}) = \left((\mathbf{P}\mathbf{x})^{\mathbf{T}}\boldsymbol{\Sigma}_0(\mathbf{P}\mathbf{x})\right)^{-1} > 0,$$

since $\mathbf{\Sigma}_0$ is a positive definite matrix. Hence, the matrix $\mathbf{\Lambda}^{\mathbf{T}}\mathbf{\Lambda}$ is positive definite. It then follows that the eigenvalues of $\mathbf{\Lambda}^{\mathbf{T}}\mathbf{\Lambda}$ are positive real numbers and we can write

$$\mathbf{\Lambda}^{\mathbf{T}}\mathbf{\Lambda} = \mathbf{W}\mathbf{D}\mathbf{W}^{\mathbf{T}} = \left(\mathbf{W}\mathbf{D}^{1/2}\right)\left(\mathbf{W}\mathbf{D}^{1/2}\right)^{\mathbf{T}},$$

where $\mathbf{D}$ is the diagonal matrix of eigenvalues and $\mathbf{W}$ the corresponding matrix whose columns are the normalized eigenvalues of $\mathbf{\Lambda}^{\mathbf{T}}\mathbf{\Lambda}$. Using these results, we can express T^2 as

$$T^2 = \left(\mathbf{W}^{1/2}\mathbf{W}^{\mathbf{T}}\mathbf{Z}\right)^{\mathbf{T}}\left(\mathbf{D}^{1/2}\mathbf{W}^{\mathbf{T}}\mathbf{Z}\right) = \left(\mathbf{D}^{1/2}\mathbf{Y}\right)^{\mathbf{T}}\left(\mathbf{D}^{1/2}\mathbf{Y}\right) = \sum\nolimits_{i=1}^{p} \varsigma_i^2 Y_i^2,$$

where $\varsigma_1, \ldots, \varsigma_p$ are the eigenvalues of $\mathbf{\Lambda}^{\mathbf{T}}\mathbf{\Lambda}$ and $Y_1, \ldots, Y_p$ are independent and identically distributed as standard normal random variables. If $\mathbf{\Sigma} = \mathbf{\Sigma}_0$ (process is in-control), then $\varsigma_1 = \ldots = \varsigma_p = 1$ and

$$T^2 = \sum\nolimits_{i=1}^{p} Y_i^2 \sim \chi_p^2,$$

where $\chi_{p,0}^2$ is a random variable having a chi square distribution with p degrees of freedom. If $\mathbf{\Sigma} \neq \mathbf{\Sigma}_0$ (process is out-of-control), then not ς_j's are equal and T^2 is a linear combination of independent chi square random variables each with 1 degree of freedom. Hence, the distribution of T^2 depends on the change in the covariance structure through the eigenvalues of $\mathbf{\Lambda}^{\mathbf{T}}\mathbf{\Lambda}$.

2.3 Distributional Results

In this thesis, we are interested in the distribution of various functions of the generalized sample variance $|\mathbf{S}|$. Under the assumption of random sampling and the data following a multivariate normal distribution, it is shown in Anderson (2003) that

$$|\mathbf{S}| \sim \frac{|\mathbf{\Sigma}|}{(n-1)^p} \prod\nolimits_{i=1}^{p} \chi_{n-i}^2$$

with $\chi^2_{n-1,0},\ldots,\chi^2_{n-p,0}$ independent chi square random variables. It easily follows that

$$\left|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}\right| \sim \prod\nolimits_{i=1}^{p} \chi^2_{n-i}. \tag{2.1}$$

We are interested in plotting functions of the statistics

$$V = \left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right| \text{ and } U = \ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right).$$

Observing that we can express V as

$$V = \left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right|\left|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}\right|,$$

then it follows from (2.1) that

$$V \sim \left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right| \prod\nolimits_{i=1}^{p} \chi^2_{n-i}. \tag{2.2}$$

Using results from the previous section, we can write the parameter $\left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right|$ a variety of ways. These include the following.

$$\begin{aligned}&(1)\ \ \left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right| = |\boldsymbol{\Sigma}|\,/\,|\boldsymbol{\Sigma}_0| = \prod\nolimits_{i=1}^{p} \frac{\xi_i}{\xi_{0,i}} \text{ and}\\ &(2)\ \ \left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right| = \prod\nolimits_{i=1}^{p} \frac{\psi_i}{\psi_{0,i}}\frac{\sigma_i^2}{\sigma_{0,i}^2} = \prod\nolimits_{i=1}^{p} \frac{\psi_i}{\psi_{0,i}}\lambda_i^2.\end{aligned}$$

In what follows, it will be convenient to let $\lambda^2 = \left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right|$ and we then have

$$V \sim \lambda^2 \prod\nolimits_{i=1}^{p} \chi^2_{n-i}. \tag{2.3}$$

Next, we observe that we can write

$$\begin{aligned}U &= \ln\left(\lambda^{2/p}\right) + \ln\left(\left|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}\right|^{1/p}\right)\\ &\sim \ln\left(\lambda^{2/p}\right) + \ln\left(\left(\prod\nolimits_{i=1}^{p} \chi^2_{n-i}\right)^{1/p}\right).\end{aligned}$$

It was shown in Anderson (2003) that

$$\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|^{1/p} \sim \begin{cases} \chi^2_{n-1}, & \text{for } p=1; \text{ and} \\ \chi^2_{2n-4}/2, & \text{for } p=2. \end{cases}$$

It follows then that for these two cases that

$$V \sim \begin{cases} \lambda^2\chi^2_{n-1}, & \text{for } p=1; \text{ and} \\ \lambda^2\left(\chi^2_{2n-4}/2\right)^2, & \text{for } p=2; \end{cases}$$

and

$$U \sim \begin{cases} \ln\left(\lambda^2\right)+\ln\left(\chi^2_{n-1}\right), & \text{for } p=1; \text{ and} \\ \ln\left(\lambda\right)+\ln\left(\chi^2_{2n-4}/2\right), & \text{for } p=2. \end{cases}$$

Unfortunately, convenient expressions do not exist for describing the distributions of V and U for $p \geq 3$. In the next section, we will discuss two approximation methods given in the literature for the distribution of the generalized sample covariance matrix.

It is not typical for the in-control parameters μ_0 and $\mathbf{\Sigma}_0$ to be given. When they are not, we will assume the practitioner will have available m independent random samples $\mathbf{X}_{i,1},\ldots,\mathbf{X}_{i,n}$, $i=1,\ldots,m$, each of size n from an in-control process to estimate μ_0 and $\mathbf{\Sigma}_0$. These data typically come from a Phase I analysis of the process. The most commonly used estimators for these parameters are

$$\widehat{\mu}_0 = \overline{\overline{\mathbf{X}}} = \frac{1}{m}\sum\nolimits_{i=1}^{m}\overline{\mathbf{X}}_i \text{ and } \widehat{\mathbf{\Sigma}}_0 = \overline{\mathbf{S}} = \frac{1}{m}\sum\nolimits_{i=1}^{m}\mathbf{S}_i,$$

where

$$\overline{\mathbf{X}}_i = \frac{1}{n}\sum\nolimits_{j=1}^{n}\mathbf{X}_{i,j} \text{ and } \mathbf{S}_i = \frac{1}{n-1}\sum\nolimits_{j=1}^{n}\left(\mathbf{X}_{i,j}-\overline{\mathbf{X}}_i\right)\left(\mathbf{X}_{i,j}-\overline{\mathbf{X}}_i\right)^{\mathbf{T}}.$$

Under our model, we have that

$$\overline{\overline{\mathbf{X}}} \sim N_p\left(\mu_0, \frac{1}{mn}\mathbf{\Sigma}_0\right) \text{ and } \overline{\mathbf{S}} \sim Wishart\left(\mathbf{\Sigma}_0, m\left(n-1\right)\right).$$

It follows from results given in Anderson (2003) that

$$\left|\overline{\mathbf{S}}\right| \sim \frac{|\mathbf{\Sigma}_0|}{m^p\,(n-1)^p} \prod\nolimits_{i=1}^{p} \chi^2_{m(n-1)-(i-1)},$$

where $\chi^2_{m(n-1)}, \ldots, \chi^2_{m(n-1)-(p-1)}$ are independent chi square random variables with degrees of freedom $m\,(n-1)\,, \ldots, m\,(n-1)-(p-1)$, respectively. It then follows that

$$\begin{aligned} V_0 &= \left|m\,(n-1)\,\mathbf{\Sigma}_0^{-1}\overline{\mathbf{S}}\right| \sim \prod\nolimits_{i=1}^{p} \chi^2_{m(n-1)-(i-1)} \text{ and} \\ U_0 &= \ln\left(\left|m\,(n-1)\,\mathbf{\Sigma}_0^{-1}\overline{\mathbf{S}}\right|^{1/p}\right) \sim \ln\left(\left(\prod\nolimits_{i=1}^{p} \chi^2_{m(n-1)-(i-1)}\right)^{1/p}\right). \end{aligned}$$

Further from the results given in Anderson (2003), we have

$$V_0 \sim \begin{cases} \lambda^2\chi^2_{m(n-1)}, & \text{for } p=1\text{; and} \\ \lambda^2\chi^2_{2m(n-1)-2}/\left(2n\,(n-1)\right), & \text{for } p=2; \end{cases}$$

and

$$U_0 \sim \begin{cases} \ln\left(\lambda^2\right) + \ln\left(\chi^2_{n-1}\right), & \text{for } p=1\text{; and} \\ \ln\left(\lambda\right) + \ln\left(\chi^2_{2n-4}/\left(2n-2\right)\right), & \text{for } p=2. \end{cases}$$

Often control limits for various charts are expressed in terms of the in-control mean and standard deviation of the plotted statistic. Since we have that

$$V = \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right| \sim \lambda^2 \prod\nolimits_{i=1}^{p} \chi^2_{n-i},$$

then

$$\begin{aligned} \mu_V &= \lambda^2 \prod\nolimits_{i=1}^{p} E\left(\chi^2_{n-i}\right) = \lambda^2 \prod\nolimits_{i=1}^{p} (n-i) \text{ and} \\ \mu_{V^2} &= \lambda^4 \prod\nolimits_{i=1}^{p} E\left[\left(\chi^2_{n-i}\right)^2\right] = \lambda^4 \prod\nolimits_{i=1}^{p} (n-i)\,(n-i+2)\,. \end{aligned}$$

It follows that

$$\sigma_V^2 = \lambda^4 \left(\prod_{i=1}^p (n-i)\right) \left(\prod_{i=1}^p (n-i+2) - \prod_{i=1}^p (n-i)\right) \text{ and}$$
$$\sigma_V = \lambda^2 \sqrt{\left(\prod_{i=1}^p (n-i)\right) \left(\prod_{i=1}^p (n-i+2) - \prod_{i=1}^p (n-i)\right)}.$$

We observe that when the process is in-control the statistic U can be expressed as

$$\begin{aligned} U &= \ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right) = \ln\left(\left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right|^{1/p}\left|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}\right|^{1/p}\right) \\ &= \frac{1}{p}\ln\left(\lambda^2\right) + \frac{1}{p}\ln\left(\prod_{i=1}^p \chi_{n-i}^2\right) \\ &= \frac{1}{p}\sum_{i=1}^p \ln\left(\chi_{n-i}^2\right). \end{aligned}$$

Hence, the $E(U)$ and $E(U^2)$ can be expressed as

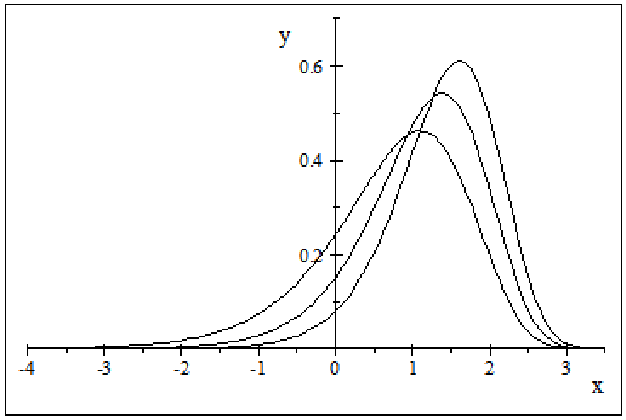
$$E(U) = \frac{1}{p}\sum_{i=1}^p E\left(\ln\left(\chi_{n-i}^2\right)\right) \text{ and}$$
$$E\left(U^2\right) = \frac{1}{p^2}\sum_{i=1}^p V\left(\ln^2\left(\chi_{n-i}^2\right)\right).$$

It is not difficult to show that the probability density function describing the distribution of $W = \ln\left(\chi_{n-i}^2\right)$ is

$$f_W(w) = e^w f_{\chi_{n-i}^2}(e^w) = \frac{1}{\Gamma\left(\frac{n-i}{2}\right)2^{(n-i)/2}} e^{-(e^w-(n-i)w)/2}.$$

A graph of these densities for $n = 6$ and $p = 3$ are given in Figure 2.1 for $i = 1, 2, 3$.

Figure 2.1: $f_W(w) = \frac{1}{\Gamma\left(\frac{n-i}{2}\right)2^{(n-i)/2}e^{-(e^w-(n-i)w)/2}}$



The distribution of Y is skewed in the negative direction with the heaviest tails corresponding with smaller degrees of freedom. We have that

$$E\left[\ln\left(\chi^2_{6-1}\right)\right] = \int_{-\infty}^{\infty} y \frac{1}{\Gamma\left(\frac{6-1}{2}\right)2^{(6-1)/2}} e^{-(e^y-(6-1)y)/2} dy = 1.396303821;$$
$$E\left[\ln\left(\chi^2_{6-2}\right)\right] = \int_{-\infty}^{\infty} y \frac{1}{\Gamma\left(\frac{6-2}{2}\right)2^{(6-2)/2}} e^{-(e^y-(6-2)y)/2} dy = 1.115931516; \text{ and}$$
$$E\left[\ln\left(\chi^2_{6-3}\right)\right] = \int_{-\infty}^{\infty} y \frac{1}{\Gamma\left(\frac{6-3}{2}\right)2^{(6-3)/2}} e^{-(e^y-(6-3)y)/2} dy = 0.7296371545.$$

These values were obtained numerically. Hence, the mean of the distribution of U for an in-control process is

$$\begin{aligned} \mu_U &= E(U) = \frac{1}{p}\sum_{i=1}^{p} E\left(\ln\left(\chi^2_{n-i}\right)\right) \\ &= \frac{1}{3}(1.396303821 + 1.115931516 + 0.7296371545) \\ &= 1.080624164. \end{aligned}$$

Further, we have that

$$
\begin{aligned}
E\left[\ln^2\left(\chi^2_{6-1}\right)\right] &= \int_{-\infty}^{\infty} y^2 \frac{1}{\Gamma\left(\frac{6-1}{2}\right)2^{(6-1)/2}} e^{-(e^y-(6-1)y)/2} dy \\
&= 2.440022117; \\
E\left[\ln^2\left(\chi^2_{6-2}\right)\right] &= \int_{-\infty}^{\infty} y^2 \frac{1}{\Gamma\left(\frac{6-2}{2}\right)2^{(6-2)/2}} e^{-(e^y-(6-2)y)/2} dy \\
&= 1.890237214; \text{ and} \\
E\left[\ln^2\left(\chi^2_{6-3}\right)\right] &= \int_{-\infty}^{\infty} y^2 \frac{1}{\Gamma\left(\frac{6-3}{2}\right)2^{(6-3)/2}} e^{-(e^y-(6-3)y)/2} dy \\
&= 1.467172578.
\end{aligned}
$$

The variances are

$$
\begin{aligned}
V\left[\ln\left(\chi^2_{6-1}\right)\right] &= 2.440022117 - (1.396303821)^2 \\
&= 0.49036; \\
V\left[\ln\left(\chi^2_{6-2}\right)\right] &= 1.890237214 - (1.115931516)^2 \\
&= 0.64493 \\
&; \text{ and} \\
V\left[\ln\left(\chi^2_{6-3}\right)\right] &= 1.467172578 - (0.7296371545)^2 \\
&= 0.9348.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sigma_U^2 &= V(U) = \frac{1}{3^2}(0.49036 + 0.64493 + 0.9348) \\
&= 0.23001 \\
&\text{and} \\
\sigma_U &= \sqrt{0.23001} = 0.47959.
\end{aligned}
$$

In Phase II (monitoring phase), the practitioner is interested in monitoring the process for a change from in-control to out-of-control. Available to the practitioner for this purpose will be samples taken periodically from the process. We assume the sample data in this phase are independent random samples with the tth sample denoted by $\{\mathbf{X}_{t,1},\ldots,\mathbf{X}_{t,n}\}$ for $t=1,2,3,\ldots$. One statistic that is commonly recommended for monitoring for a change in the covariance matrix is the sample generalize variance $|\mathbf{S}|$. It shown in Anderson that

$$|\mathbf{S}| \sim \frac{|\mathbf{\Sigma}|}{(n-1)^p}\prod\nolimits_{i=1}^{p}\chi^2_{n-i} \text{ or}$$
$$\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right| \sim \prod\nolimits_{i=1}^{p}\chi^2_{n-i},$$

where $\chi^2_{n-1},\ldots,\chi^2_{n-p}$ are independent chi square random variables with degrees of freedom $n-1,\ldots,n-p$, respectively. Under our model assumptions, we have that

$$\mu_{\left|\overline{\mathbf{S}}\right|} = b_1\left|\mathbf{\Sigma}_0\right| \text{ and } \sigma_{\left|\overline{\mathbf{S}}\right|} = \sqrt{b_2}\left|\mathbf{\Sigma}_0\right|,$$

where

$$b_1 = b_{1,m,n,p} = \prod\nolimits_{i=1}^{p}\frac{m\,(n-1)-(i-1)}{m\,(n-1)} \text{ and}$$

$$\begin{aligned}
b_2 &= b_{2,m,n,p}\\
&= \left(\prod\nolimits_{i=1}^{p}\frac{m\,(n-1)-(i-1)}{m\,(n-1)}\right)\\
&\quad\times\left(\prod\nolimits_{i=1}^{p}\frac{(m\,(n-1)-(i-1)+2)}{m\,(n-1)}-\prod\nolimits_{i=1}^{p}\frac{(m\,(n-1)-(i-1))}{m\,(n-1)}\right).
\end{aligned}$$

Further, we have that

$$\mu_{|\mathbf{S}|} = b_{1,1,n,p}\left|\mathbf{\Sigma}\right| \text{ and } \sigma_{|\mathbf{S}|} = \sqrt{b_{2,1,n,p}}\left|\mathbf{\Sigma}\right|.$$

It is now easy to see that $\left|\overline{\mathbf{S}}\right|/b_{1,m,n,p}$ is an unbiased estimator of $\left|\mathbf{\Sigma}_0\right|$ and $\left|\mathbf{S}\right|/b_{1,1,n,p}$ is an unbiased estimator of $|\mathbf{\Sigma}|$.

It is our interest to study charts based on the plotted statistic

$$U = \ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right).$$

We can see that

$$\begin{aligned} U &= \ln\left(\left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right|^{1/p}\right) + \ln\left(\left|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}\right|^{1/p}\right) \\ &\sim \ln\left(\left|\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}\right|^{1/p}\right) + \ln\left(\left(\prod_{i=1}^{p}\chi^2_{n-i}\right)^{1/p}\right). \end{aligned}$$

It is given in Anderson (2003) that

$$\left|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}\right| \sim \begin{cases} \chi^2_{n-1}, & \text{for } p=1; \\ \left(\chi^2_{2n-4}/2\right)^2, & \text{for } p=2. \end{cases}$$

Using these results, one can show that

$$\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}\right|^{1/p}\right) \sim \begin{cases} \ln\left(\chi^2_{n-1}\right), & \text{for } p=1; \\ \ln\left(\chi^2_{2n-4}/2\right), & \text{for } p=2. \end{cases}$$

For $p \geq 3$, a simple expression does not exist to describe the distribution of $|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}|$. We must then turn to approximations methods. Two methods found in the literature for approximating the distribution of (a function of) $|\mathbf{S}|$ are found in Hoel (1937) and Steyn (1978). We begin by discussing the method given in Steyn (1978).

The method given in Steyn (1978) provides an approximation to the distribution of the statistic

$$Y = \frac{p}{2}\left|(n-1)\,\boldsymbol{\Sigma}^{-1}\mathbf{S}\right|^{1/p}$$

using the function

$$\begin{aligned} h\left(y\,|n,p\right) = &\left(1+\frac{(p-1)\,(p-2)}{4}\right) g\left(y\,\middle|\,1,\frac{p\,(n-p)}{2}\right) \\ &-\frac{(p-1)\,(p-2)}{4} g\left(y\,\middle|\,1,\frac{p\,(n-p)-2}{2}\right) \end{aligned}$$

where $g(y|\theta,\kappa)$ is the density function of a gamma distribution with location parameter θ and scale parameter κ. The support for the probability density function is

$$\left\{ y \left| y \geq \frac{(p-1)(p-2)(p(n-p)/2-1)}{4+(p-1)(p-2)} \right. \right\}.$$

The function $h(y|n,p)$ can be expressed as

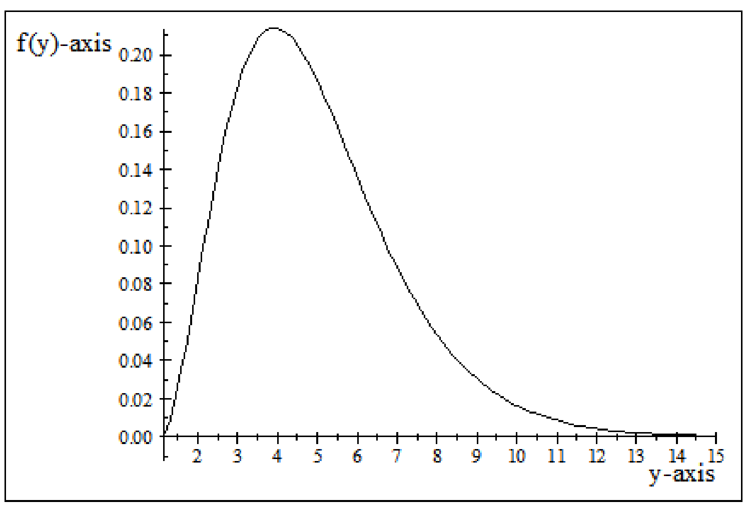
$$h(y|n,p) = \left(1+\frac{(p-1)(p-2)}{4}\right)\frac{1}{\Gamma\left(\frac{p(n-p)}{2}\right)}y^{p(n-p)/2-1}e^{-y} - \frac{(p-1)(p-2)}{4}\frac{1}{\Gamma\left(\frac{p(n-p)-2}{2}\right)}y^{(p(n-p)-2)/2-1}e^{-y}.$$

As an example, consider the case in which $n=6$ and $p=3$. It follows that

$$y \geq \frac{(3-1)(3-2)(3(6-3)/2-1)}{4+(3-1)(3-2)} = 1.1\overline{6}.$$

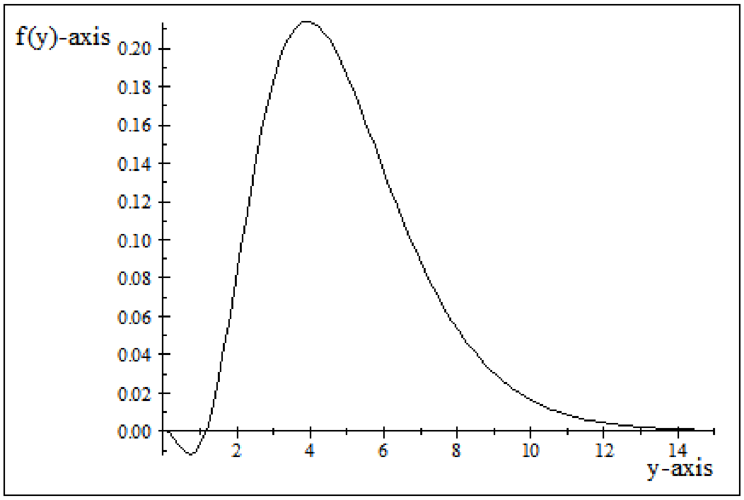
The graph of $g(y|n,p)$ is shown is Figure 2.2 for $y \geq 1.1\overline{6}$.

Figure 2.2: $g(y|n,p)$ versus y



It interesting to observe the graph of the function $g(y|6,3)$ versus y over the positive reals (see Figure 2.3). Since the generalized sample variance is a positive real number with probability 1 under the independent, multivariate normal model, the probability density function describing its distribution should be greater than or equal to zero over the positive reals. This is not the case for the function $g\left(y\,|n,p\right)$ over the positive reals.

Figure 2.3: $g\left(y\,|6,3\right)$ versus y



What is important in our work is that

$$G\left(y\,|n,p\right) = \int_0^y g\left(t\,|n,p\right)dt$$

provides a good approximation to the cumulative distribution function $F_Y\left(y\right)$ describing the distribution of Y. More specifically, we are interested in approximating the distribution of U for $p \geq 3$.

Observing that

$$U = \ln\left(\frac{2\lambda^{2/p}}{p}Y\right).$$

The cumulative distribution function of U can be expressed in terms of the cumulative distribution function of Y as

$$F_U\left(u\,|\lambda,p,n\right) = P\left(Y \leq \frac{pe^u}{2\lambda^{2/p}}\right) = F_Y\left(\frac{pe^u}{2\lambda^{2/p}}\right).$$

Hence, the probability densities are related by

$$f_U\left(u\,|\lambda,p,n\right)=\frac{pe^u}{2\lambda^{2/p}}f_Y\left(\frac{pe^u}{2\lambda^{2/p}}\right).$$

We also note that the 100γ percentile $(0<\gamma<1)$ of the distributions of U and Y are related by

$$u_{p,n,1-\gamma}=\ln\left(\frac{2\lambda^{2/p}}{p}y_{p,n,1-\gamma}\right).$$

For $p>2$, the approximation $f\left(u\,|\lambda,p,n\right)$ to the probability density function $f_U\left(u\,|\lambda,p,n\right)$ of U based on Steyn (1978) approximation method is

$$f\left(u\,|\lambda,p,n\right)=\frac{\left(1+\frac{(p-1)(p-2)}{4}\right)e^{-\left(pe^x/\lambda^{2/p}-p(n-p)x\right)/2}}{\Gamma\left(\frac{p(n-p)}{2}\right)\left(2\lambda^{2/p}/p\right)^{p(n-p)/2}}-\frac{\frac{(p-1)(p-2)}{4}e^{-\left(pe^x/\lambda^{2/p}-(p(n-p)-2)x\right)/2}}{\Gamma\left(\frac{p(n-p)-2}{2}\right)\left(2\lambda^{2/p}/p\right)^{(p(n-p)-2)/2}}$$

with support

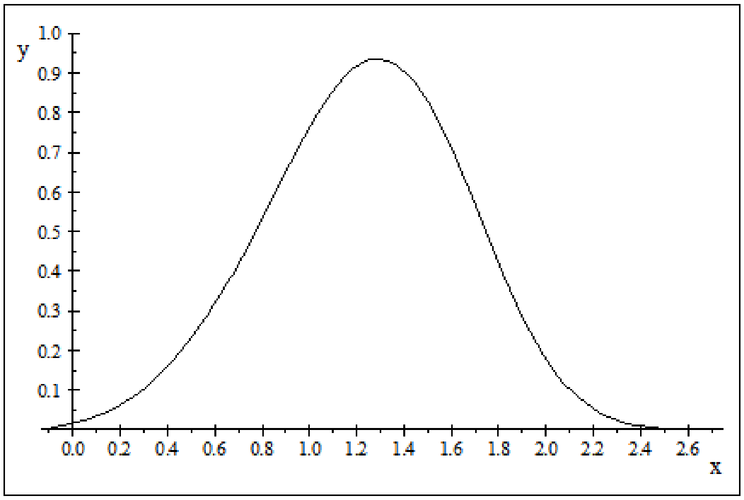
$$\left\{u\left|u\geq\ln\left(\frac{2\lambda^{2/p}\left(p-1\right)\left(p-2\right)\left(p\left(n-p\right)/2-1\right)}{p\left(4+\left(p-1\right)\left(p-2\right)\right)}\right)\right.\right\}.$$

For the case in which $n=6$, $p=1$, and $\lambda=1.2$, a graph of the function $f\left(u\,|1,6,3\right)$ is given in Figure 2.4 for

$$u\geq\ln\left(\frac{2\left(1.2\right)^{2/3}\left(3-1\right)\left(3-2\right)\left(3\left(6-3\right)/2-1\right)}{3\left(4+\left(3-1\right)\left(3-2\right)\right)}\right)\ \text{or}$$

$$u\geq-0.1297667238.$$

Figure 2.4: $f\,(u\,|1,6,3)$



We are also interested in approximations for the distributions of V_0 and U_0. The approximation for the distribution of U_0 will be given here. It is easy to see using Steyn (1978) that the approximation to the probability density function, $f_{U_0}\,(u\,|p,n\,)$, describing the distribution of U_0 is

$$f\,(u\,|p,n\,) = \frac{\left(1+\frac{(p-1)(p-2)}{4}\right)e^{-\left(pe^x/\lambda^{2/p}-p(m(n-1)+1-p)x\right)/2}}{\Gamma\left(\frac{p(m(n-1)+1-p)}{2}\right)(2/p)^{p(m(n-1)+1-p)/2}} - \frac{\frac{(p-1)(p-2)}{4}e^{-\left(pe^x/\lambda^{2/p}-(p(m(n-1)+1-p)-2)x\right)/2}}{\Gamma\left(\frac{p(m(n-1)+1-p)-2}{2}\right)(2/p)^{(p(m(n-1)+1-p)-2)/2}}$$

with support

$$\left\{u\,\middle|\,u \geq \ln\left(\frac{2\,(p-1)\,(p-2)\,(p\,(m\,(n-1)+1-p)\,/2-1)}{p\,(4+(p-1)\,(p-2))}\right)\right\}.$$

The approximation given in Hoel (1938) is for the random variable $Z = |\mathbf{S}|^{1/p}$. The statistics U is related to Z by

$$U = \ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\right|^{1/p} Z\right)$$

We have that

$$F_U\left(u\,|\lambda,p,n\right)=F_Z\left(\frac{\left|\boldsymbol{\Sigma}_0\right|^{1/p}e^u}{n-1}\,|\left|\boldsymbol{\Sigma}\right|,n,p\right).$$

It then follows that

$$f_U\left(u\,|\lambda,p,n\right)=\frac{\left|\boldsymbol{\Sigma}_0\right|^{1/p}e^u}{n-1}f_Z\left(\frac{\left|\boldsymbol{\Sigma}_0\right|^{1/p}e^u}{n-1}\,|\left|\boldsymbol{\Sigma}\right|,n,p\right).$$

2.4 Conclusion

The design of a statistical method typically depends on a model for the distribution of the data. The performance of a statistical methods depends on the distribution of the data. Studying the performance of a method usually requires one to also model the distribution of the measurement(s) of interest. Even the so called distribution free methods usually requires a parametric model for the data under the alternative hypothesis. Control charting procedures are statistical methods. Most are designed under the assumption the distribution of the quality measurement(s) is a (multivariate) normal distribution.

In this thesis, we are interested in processes in which the quality measurement is a multivariate measurement on the output of the process. We assume that the multivariate quality measurement X has a p-variate normal distribution with mean vector μ and positive definite covariance matrix $\boldsymbol{\Sigma}$. Further, we assume that the vectors of quality measurements taken on a sample of items from the output of the process constitute a random sample (independent and identically distributed) as well as that our samples are independent. We refer to this model as the independent multivariate normal model.

The process being in a state of in- or out-of-control is modeled in terms of the two groups of parameters, μ and $\mathbf{\Sigma}$. The simplest model is to assume there are value μ_0 and $\mathbf{\Sigma}_0$ such that if $\mu = \mu_0$ and $\mathbf{\Sigma} = \mathbf{\Sigma}_0$ the process is in a state of statistical in-control as discussed in Shewhart (1931). Our interest is examining the use of control charts in detecting a change in $\mathbf{\Sigma}$ from $\mathbf{\Sigma}_0$ by studying changes in the ratio $\lambda^2 = |\mathbf{\Sigma}| \,/\, |\mathbf{\Sigma}_0|$ where the process is in-control if $\lambda^2 = 1$.

CHAPTER 3

SHEWHART $\ln\left(\left|(N-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/P}\right)$ CHART WITH RUNS RULES

3.1 Introduction

Champ and Woodall (1987) studied the performance of the Shewhart $\overline{X}$ chart when supplemented with runs rules. The runs rules considered by them each had the general form of causing the chart to signal if j out of the last i plotted statistics fall in the standardized interval (a, b) with $a < b$. The actual interval associated with the interval (a, b) has the form

$$(\mu_{U,0} + a\sigma_{U,0}, \mu_{U,0} + b\sigma_{U,0}),$$

where $\mu_{U,0}$ and $\sigma_{U,0}$ are the in-control mean and standard deviation, respectively, of the plotted statistics U. They used the notation $T(j, i, a, b)$ as a compact way to state a runs rule. In this article, we describe a runs rule as a rule that causes a chart to signal if j out of the last i plotted statistics fall in the interval $(u_{p,n,1-a}, u_{p,n,1-b})$ for $0 < a < b < 1$. We will represent this runs rule by

$$T(j, i, u_{p,n,1-a}, u_{p,n,1-b}),$$

where $u_{p,n,1-\gamma}$ is the 100γth percentile $(0 < \gamma < 1)$ of the distribution of U.

Lowry, Champ, and Woodall (1995) examined the selection of runs rules for the R and S charts so the charts had the same in-control performance as the $\overline{X}$ chart supplemented with similar runs rules. Their method can be used to select runs rules to supplement the Shewhart type chart based on the plotted statistics U. The runs rules we will consider are the same eight considered by Lowry, Champ, and Woodall (1995). These are listed in Table 3.1.1 using the notation $T(j, i, a, b)$ of Champ and Woodall (1987).

Table 3.1: Runs Rules

1.	$\mathrm{T}(1, 1, -\infty, -3)$
2.	$\mathrm{T}(2, 3, -3, -2)$
3.	$\mathrm{T}(4, 5, -3, -1)$
4.	$\mathrm{T}(8, 8, -3, 0)$
5.	$\mathrm{T}(8, 8, 0, 3)$
6.	$\mathrm{T}(4, 5, 1, 3)$
7.	$\mathrm{T}(2, 3, 2, 3)$
8.	$\mathrm{T}(1, 1, 3, \infty)$

Rules 1 and 8 define the basic Shewhart chart with the values -3 and 3 referred to as control limits. For the real number a or b that is not a control limit, these values are referred to as warning limits (or lines) (see Page (1954)). For the U chart, these rules have the form $T\left(j, i, u_{p,n,1-a}, u_{p,n,1-b}\right)$. Their descriptions are given in Table 3.2

Table 3.2: Runs Rules

1.	$\mathrm{T}\big(1, 1, 0, u_{p,n,1-\Phi(-3)}\big)$
2.	$\mathrm{T}\big(2, 3, u_{p,n,1-\Phi(-3)}, u_{p,n,1-\Phi(-2)}\big)$
3.	$\mathrm{T}\big(4, 5, u_{p,n,1-\Phi(-3)}, u_{p,n,1-\Phi(-1)}\big)$
4.	$\mathrm{T}\big(8, 8, u_{p,n,1-\Phi(-3)}, u_{p,n,1-\Phi(0)}\big)$
5.	$\mathrm{T}\big(8, 8, u_{p,n,1-\Phi(0)}, u_{p,n,1-\Phi(3)}\big)$
6.	$\mathrm{T}\big(4, 5, u_{p,n,1-\Phi(1)}, u_{p,n,1-\Phi(3)}\big)$
7.	$\mathrm{T}\big(2, 3, u_{p,n,1-\Phi(2)}, u_{p,n,1-\Phi(3)}\big)$
8.	$\mathrm{T}\big(1, 1, u_{p,n,1-\Phi(3)}, \infty\big)$

We now need only to determine the values of $u_{p,n,1-c}$ for $c = -3, -2, -1, 0, 1, 2, 3$ and given values of p and n.

In the previous section, the distribution of U for given values of p and n is discussed. For $p = 1, 2$, the distribution is known exactly under the independent normal model. For $p > 2$, relatively simple expression for the distribution of U is not known. Hence, we considered two approximation methods, one by Hoel (1937) and the other by Steyn (1978). It was found that the method of Steyn (1978) approximation method gave the best approximations. These values are given in Tables 5-6 for selected values of p and n.

As stated in the previous section for $p = 1$, we have that

$$U = \ln\left(\frac{(n-1)\,S^2}{\sigma_0^2}\right) \sim \ln\left(\lambda^2\right) + \ln\left(\chi^2_{n-1}\right)$$

where $\lambda^2 = \sigma^2/\sigma_0^2$. Hence, the 100γth percentile $u_{1,n,1-\gamma}$ of the distribution of U can be expressed in terms of the 100γth percentile $\chi^2_{n-1,1-\gamma}$ of a chi square distribution with $n-1$ degrees of freedom as

$$u_{1,n,1-\gamma} = \ln\left(\lambda^2\right) + \ln\left(\chi^2_{n-1,1-\gamma}\right).$$

Table 3.3 list the percentiles of the distribution of U for $p = 1$ rounded to five decimal places and values of $n = 2\,(1)\,6$ associated with the eight runs rules given in Table 3.2. The values in Table 3.3 were calculated using the Scientific WorkPlace functions ChiSquareInv $(t; \nu)$ and NormalDist $(x; \mu, \sigma)$ with

$$\ln\left(\chi^2_{n-1,1-\Phi(c)}\right) = \ln\left(\text{ChiSquareInv}\left(\text{NormalDist}\left(c; 0, 1\right); n-1\right)\right)$$

Table 3.3: Percentage Points $u_{1,n,1-\Phi(c)}$, λ=*1*

c	n=2	n=3	n=4	n=5	n=6
-3	-12.76387	-5.91390	-3.51628	-2.24655	-1.43569
-2	-7.11451	-3.07855	-1.59900	-0.77590	-0.22741
-1	-3.21714	-1.06274	-0.18165	0.34810	0.72076
0	-0.78760	0.32663	0.86119	1.21096	1.47051
1	0.68662	1.30347	1.64601	1.88693	2.07398
2	1.64625	2.02371	2.25710	2.43057	2.57016
3	2.32952	2.58139	2.74923	2.87923	2.98676

For the case in which $p = 2$, it is shown in Anderson (2003) that

$$U = \ln\left(\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/2}\right) \sim \ln\left(\left(\lambda^2\right)^{1/2}\right) + \ln\left(\chi^2_{2n-4}/2\right),$$

where $\lambda^2 = \left|\mathbf{\Sigma}_0^{-1}\mathbf{\Sigma}\right|$. It follows that

$$u_{2,n,1-\gamma} = \ln\left(\left(\lambda^2\right)^{1/2}\right) + \ln\left(\chi^2_{2n-4,1-\gamma}/2\right),$$

with

$$\ln\left(\chi^2_{2n-4,1-\Phi(c)}/2\right) = \ln\left(\text{ChiSquareInv}\left(\text{NormalDist}\left(c;0,1\right);2\left(n\right)-4\right)/2\right).$$

Table 3.4 list the percentiles of the distribution of U for $p = 2$ rounded to five decimal places and values of $n = 3\,(1)\,7$ associated with the eight runs rules given in Table 3.2.

Table 3.4: Percentage Points $u_{2,n,1-\Phi(c)}$, λ=*1*

c	***n*=3**	***n*=4**	***n*=5**	***n*=6**	***n*=7**
-3	-6.60705	-2.93970	-1.55269	-0.76510	-0.23337
-2	-3.77170	-1.46905	-0.51703	0.05635	0.45924
-1	-1.75589	-0.34505	0.31283	0.73509	1.04391
0	-0.36651	0.51781	0.98360	1.30075	1.54135
1	0.61032	1.19378	1.53425	1.77803	1.96889
2	1.33057	1.73743	1.99449	2.18624	2.34040
3	1.88824	2.18608	2.38597	2.54007	2.66671

$$\mathrm{u}_{2,n,1-\Phi(c)} = \ln\left(\text{ChiSquareInv}\left(\text{NormalDist}\left(c;0,1\right);2\left(n\right)-4\right)/2\right)$$

As discussed in the previous section, exact expressions for the distribution of U when $p > 2$ are not available. Two approximations found in the literature due to Hoel (1937) and Steyn (1978), the one by Steyn (1978) appears to provide the best approximation to the distribution of U. For $n > p > 2$, the cumulative distribution function $F_U(u)$ of U can be expressed approximately using approximation method of Steyn (1978) as

$$\begin{aligned} F_U\left(u\left|p,n\right.\right) \approx & \left(1+\frac{(p-1)(p-2)}{4}\right) F_Y\left(pe^u/2\left|\frac{p(n-p)}{2},1\right.\right) \\ & -\frac{(p-1)(p-2)}{4} F_Y\left(pe^u/2\left|\frac{p(n-p)-2}{2},1\right.\right), \end{aligned}$$

where $F_Y(y\,|\kappa,\theta)$ is the cumulative distribution function of a gamma distribution with parameters κ and θ. The approximation to the 100γth percentile $u_{p,n,1-\gamma}$ of the distribution of U is the value of u that satisfies the equation

$$F_U\left(u\left|p,n\right.\right) = \gamma.$$

Table 3.5 list the percentiles of the distribution of U for $p = 3$ rounded to five decimal places and values of $n = 4\,(1)\,8$ associated with the eight runs rules given in Table 3.2.

Table 3.5: Percentage Points $u_{3,n,1-\Phi(c)}$, λ=*1*

c	n=4	n=5	n=6	n=7	n=8
-3	-0.93513	-0.29194	0.09805	0.38900	0.63059
-2	-0.86779	-0.16668	0.28756	0.63240	0.90903
-1	-0.52935	0.23108	0.69070	1.01893	1.27319
0	0.07404	0.73782	1.13051	1.41109	1.62971
1	0.69704	1.21899	1.53923	1.77336	1.95881
2	1.24135	1.64532	1.90546	2.10080	2.25831
3	1.70175	2.01833	2.23156	2.39572	2.53034

Table 3.6 list the percentiles of the distribution of U for $p = 4$ rounded to five decimal places and values of $n = 5\,(1)\,9$ associated with the eight runs rules given in Table 3.2.

Table 3.6: Percentage Points $u_{4,n,1-\Phi(c)}$, λ=*1*

c	n=5	n=6	n=7	n=8	n=9
-3	-0.20982	0.38193	0.74085	1.00136	1.20677
-2	-0.18398	0.41436	0.78117	1.05052	1.26544
-1	-0.03118	0.57998	0.95732	1.23979	1.45324
0	0.32209	0.89464	1.24673	1.50377	1.70692
1	0.76691	1.25469	1.56105	1.78753	1.96813
2	1.20191	1.60271	1.86332	2.05972	2.21838
3	1.59447	1.92248	2.14360	2.31362	2.45284

More extensive tables are given in Appendix.

3.2 Run Length Distribution

Champ and Woodall (1987) developed a Markov chain representation of a Shewhart chart supplemented with runs rules that is useful in determining the run length distribution. Their method can be applied to the Shewhart ln $\left(\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right)$ chart supplemented with runs rules. The Markov chain representation of a control chart has one absorbing state. In this case, the transition matrix $\mathbf{P}$ with q non-absorbing states and one absorbing state has the form

$$\mathbf{P}^{(q+1)\times(q+1)} = \begin{bmatrix} \mathbf{Q}^{q\times q} & \left(\mathbf{I}^{q\times q} - \mathbf{Q}^{q\times q}\right)\mathbf{1}^{q\times 1} \\ \left(\mathbf{0}^{q\times 1}\right)^{\mathbf{T}} & 1 \end{bmatrix},$$

where $\mathbf{I}^{q\times q}$ is the identity matrix, $\mathbf{1}^{q\times 1}$ is a vector of ones. The last row and column of $\mathbf{P}$ are associated with the absorbing state.

The transition matrix is determined from a transition by regions matrix $\mathbf{H}$. The regions for a given chart depend on the set of runs rules defining the chart. As an example, consider the Shewhart $\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right)$ chart supplemented with rules 1, 2, 7, and 8. The regions of this chart are

$$R_1 = \left(0, u_{p,n,1-\Phi(-3)}\right]$$
$$R_2 = \left(u_{p,n,1-\Phi(-3)}, u_{p,n,1-\Phi(-2)}\right]$$
$$R_3 = \left(u_{p,n,1-\Phi(-2)}, u_{p,n,1-\Phi(2)}\right]$$
$$R_4 = \left[u_{p,n,1-\Phi(2)}, u_{p,n,1-\Phi(3)}\right)$$
$$R_5 = \left[u_{p,n,1-\Phi(3)}, \infty\right)$$

The matrix $\mathbf{H}$ is a $(q+1) \times 8$ matrix with row i corresponding to state i and column j corresponding to the jth region. The (i,j)th component $h_{i,j}$ of $\mathbf{H}$ is the state to which the Markov chain transitions if the $t+1$ plotted statistic falls in region R_j given the present state of the chart is i. It follows that the $(i, h_{i,j})$th component of $\mathbf{P}$ is

$$P\left(\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right) \in R_j\right)$$

if $h_{i,j}$ is the number of a non-absorbing state. The matrix $\mathbf{H}$ as determined by Champ and Woodall (1987) is

$$\mathbf{H} = \begin{bmatrix} 8 & 2 & 1 & 4 & 8 \\ 8 & 8 & 3 & 5 & 8 \\ 8 & 8 & 1 & 4 & 8 \\ 8 & 7 & 6 & 8 & 8 \\ 8 & 8 & 6 & 8 & 8 \\ 8 & 2 & 1 & 8 & 8 \\ 8 & 8 & 3 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \end{bmatrix}.$$

We see that if the Markov chain is in state 4 it can only transition to the non-absorbing states 6 and 7 with respective transition probabilities

$$\begin{aligned} p_{4,6} &= p_3 = P\left(u_{p,n,1-\Phi(-2)} < U < u_{p,n,1-\Phi(2)}\right) \\ &= F_U\left(u_{p,n,1-\Phi(2)}\,|\lambda, p, n\right) - F_U\left(u_{p,n,1-\Phi(-2)}\,|\lambda, p, n\right) \text{ and} \\ p_{4,7} &= p_2 = P\left(u_{p,n,1-\Phi(-3)} < U < u_{p,n,1-\Phi(-2)}\right) \\ &= F_U\left(u_{p,n,1-\Phi(-2)}\,|\lambda, p, n\right) - F_U\left(u_{p,n,1-\Phi(-3)}\,|\lambda, p, n\right), \end{aligned}$$

with $p_{4,1} = p_{4,2} = p_{4,3} = p_{4,4} = p_{4,5} = 0$ and $p_{4,8} = 1 - p_{4,6} - p_{4,7}$. Note that

$$\begin{aligned} p_4 &= P\left(u_{p,n,1-\Phi(2)} < U < u_{p,n,1-\Phi(3)}\right) \\ &= F_U\left(u_{p,n,1-\Phi(3)}\,|\lambda, p, n\right) - F_U\left(u_{p,n,1-\Phi(2)}\,|\lambda, p, n\right). \end{aligned}$$

For our example, the matrix $\mathbf{Q}$ can be expressed as

$$\mathbf{Q} = \begin{bmatrix} p_3 & p_2 & 0 & p_4 & 0 & 0 & 0 \\ 0 & 0 & p_3 & 0 & p_4 & 0 & 0 \\ p_3 & 0 & 0 & p_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_3 & p_2 \\ 0 & 0 & 0 & 0 & 0 & p_3 & 0 \\ p_3 & p_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_3 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If the process is in-control then

$$p_2 = \Phi(-2) - \Phi(-3), \; p_3 = \Phi(2) - \Phi(-2), \text{ and } p_4 = \Phi(3) - \Phi(2).$$

These values are

$$p_2 = \text{NormalDist}(-2; 0, 1) - \text{NormalDist}(-3; 0, 1) = 0.02140023392$$

$$p_3 = \text{NormalDist}(2; 0, 1) - \text{NormalDist}(-2; 0, 1) = 0.9544997361$$

$$p_4 = \text{NormalDist}(3; 0, 1) - \text{NormalDist}(2; 0, 1) = 0.02140023392$$

The function $\text{NormalDist}(x; \mu, \sigma)$ is the cumulative distribution function evaluated at the value x for a normal distribution with mean μ and standard deviation σ.

The run length T_i, given that the chart begins in non-absorbing state i, is the number of the sample in which the chart first signals, $i = 1, \ldots, q$. It is shown in Brook and Evans (1972) that the joint probability mass function describing the run length distribution is determined by

$$[P(T_1 = 1), \ldots, P(T_q = 1)]^{\mathbf{T}} = (\mathbf{I} - \mathbf{Q})\,\mathbf{1} \text{ and}$$

$$[P(T_1 = t), \ldots, P(T_q = t)]^{\mathbf{T}} = \mathbf{Q}^{t-1}(\mathbf{I} - \mathbf{Q})\,\mathbf{1},$$

for $t = 2, 3, 4, \ldots$. It follows that the joint cumulative distribution function of the distribution of the vector of run lengths is given by

$$[P(T_1 \leq t), \ldots, P(T_q \leq t)]^{\mathbf{T}} = \left(\mathbf{I} - \mathbf{Q}^t\right)\mathbf{1}.$$

The vector of expected run lengths and expected squared run lengths can be determined as follows.

$$\begin{aligned}[E(T_1), \ldots, E(T_q)]^{\mathbf{T}} &= (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{1} \text{ and} \\ \left[E\left(T_1^2\right), \ldots, E\left(T_q^2\right)\right]^{\mathbf{T}} &= (\mathbf{I} - \mathbf{Q})^{-1}\left[\mathbf{I} + 2(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{Q}\right]\mathbf{1}.\end{aligned}$$

For the case in which $n = 6$, and the chart begins in state 1, the ARL, $SDRL$, and various percentiles are presented in Table 4.1. When the process is in-control, the vector of ARL's are

$$\begin{bmatrix} E(T_1) \\ E(T_2) \\ E(T_3) \\ E(T_4) \\ E(T_5) \\ E(T_6) \\ E(T_7) \end{bmatrix} = (\mathbf{I} - \mathbf{Q})^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 225.4384069 \\ 216.2945284 \\ 220.8096534 \\ 216.2945284 \\ 211.7627559 \\ 220.8096534 \\ 211.7627559 \end{bmatrix}$$

Another method for obtaining the tail probabilities of the run length distribution is to use a method presented in Woodall (1983). For large values of $t = t^*$, the tail probabilities are approximated by

$$P(T_i = t^* + t) \approx \zeta_i^t P(T_i = t^*),$$

for $i = 1, \ldots, q$. The rule for selecting an approximate value of ζ_i is based on considering the value of t^* to be large if

$$\frac{P\left(T_i = t^*\right)}{P\left(T_i = t^* - 1\right)} \text{ and } \frac{1 - \sum_{t=1}^{t^*} P\left(T_i = t\right)}{1 - \sum_{t=1}^{t^*-1} P\left(T_i = t\right)}$$

are "close." The value of ζ_i is then approximated with the value

$$\widehat{\zeta}_i = \frac{P\left(T_i = t^*\right)}{P\left(T_i = t^* - 1\right)}.$$

This method gives the following approximation the expected value of the run length T_i by

$$E\left(T_i\right) \approx \sum_{t=1}^{t^*} tP\left(T_i = t\right) + \widehat{\zeta}_i P\left(T_i = t^*\right) \left(\frac{t^*}{1 - \widehat{\zeta}_i} + \frac{1}{\left(1 - \widehat{\zeta}_i\right)^2}\right).$$

Other parameters such as the standard deviation and various percentage points of the run length distribution can be approximated by this method. We have used this method the evaluating the run length distribution of the Shewhart chart with runs rules based on the statistic U. Champ and Woodall (1992) give a program that may be altered for this purpose. We have alternated an expanded version of this program that determines various percentage points of the run length distribution as well as the average (ARL) and standard deviation of the run length ($SDRL$) for various shifts in the parameter $\lambda = \left|\mathbf{\Sigma}_0^{-1}\mathbf{\Sigma}\right|^{1/p}$. Table 3.7 gives the ARL, $SDRL$, and various percentage points of the run length distributions for selected value of λ. Recalling that the process is in-control if $\lambda = 1$, we see from Table 3.7 the in-control ARL is 225.43, the in-control $SDRL$ is 224.37, and the 50th percentile of the in-control run length distribution is 157.

Table 3.7: Run Length Parameters, $p = 1$,$n = 6$

			Percentiles								
λ	ARL	SDRL	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
0.5	7.63	6.32	1	2	2	3	6	10	16	20	30
0.6	18.70	17.35	1	2	3	6	13	25	41	53	81
0.7	46.00	44.66	2	4	6	14	32	63	104	135	207
0.8	106.99	105.71	2	7	12	32	75	148	245	318	487
0.9	217.03	215.86	3	12	24	63	151	300	498	648	995
1.0	225.43	224.37	3	13	25	66	157	312	518	673	1034
1.1	87.47	86.40	2	6	10	26	61	121	200	260	399
1.2	32.55	31.50	1	3	4	10	23	45	74	95	146
1.3	15.35	14.35	1	2	3	5	11	21	34	44	67
1.4	8.83	7.88	1	1	2	3	6	12	19	25	37
1.5	5.85	4.95	1	1	1	2	4	8	12	16	24

3.3 Steady Run Length Distribution

A method was suggested by Crosier (1986) for determining a cyclic steady-state run length distribution. As shown in Champ (1992), this is done be replacing the transition matrix

$$\mathbf{P}^* = \begin{bmatrix} \mathbf{Q} & (\mathbf{I} - \mathbf{Q})\,\mathbf{1} \\ \mathbf{0^T} & 1 \end{bmatrix}$$

with the matrix

$$\mathbf{P}^* = \begin{bmatrix} \mathbf{Q} & (\mathbf{I} - \mathbf{Q})\,\mathbf{1} \\ \mathbf{e}_1^{\mathbf{T}} & 0 \end{bmatrix},$$

where $\mathbf{e}_1^{\mathbf{T}}$ is a $(q-1)\times 1$ vector in which the first component is 1 and the other components are 0. The matrix $\mathbf{P}^*$ is an erogodic transition matrix. A method is given in Lucas and Saccucci (1990) for calculating the $(q-1)\times 1$ steady-state probability vector π_{ss}. Champ (1992) shows that π_{ss} using Lucas and Saccucci (1990) method can be calculated as

$$\pi_{ss} = \left(\mathbf{G} - \mathbf{Q}^{\mathbf{T}}\right)^{-1} \mathbf{e}_1,$$

where

$$\mathbf{G} = \begin{bmatrix} 2 & \mathbf{1}^{\mathbf{T}} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

Using the example in the previous section, we have that the cyclic steady-state average run length is ARL_{ss} is

$$\begin{aligned} ARL_{ss} &= \pi_{ss}^{\mathbf{T}} \left(\mathbf{I} - \mathbf{Q}\right)^{-1} \mathbf{1} \\ &= \left(\left(\mathbf{G} - \mathbf{Q}^{\mathbf{T}}\right)^{-1} \mathbf{e}_1\right)^{\mathbf{T}} \left(\mathbf{I} - \mathbf{Q}\right)^{-1} \mathbf{1} \\ &= 223.8844836, \end{aligned}$$

with

$$\pi_{ss} = \begin{bmatrix} 0.9156777542 \\ 0.02001326795 \\ 0.01951146035 \\ 0.02001326795 \\ 0.0004282886157 \\ 0.01951146035 \\ 0.0004282886157 \end{bmatrix}.$$

Champ (1992) list this value as 224.88. This appears to be a typo. In general, we can determine the probability mass function, the cumulative distribution function,

$E(T_{ss})$, and $E(T_{ss}^2)$ by

$$P(T_{ss}=t)=\pi_{ss}^{\mathbf{T}}\mathbf{Q}^{t-1}(\mathbf{I}-\mathbf{Q})\,\mathbf{1},\ P(T_{ss}\leq t)=\pi_{ss}^{\mathbf{T}}\left(\mathbf{I}-\mathbf{Q}^{t}\right)\mathbf{1},$$
$$E(T_{ss})=\pi_{ss}^{\mathbf{T}}(\mathbf{I}-\mathbf{Q})^{-1}\,\mathbf{1},\text{ and}$$
$$E\left(T_{ss}^2\right)=\pi_{ss}^{\mathbf{T}}(\mathbf{I}-\mathbf{Q})^{-1}\left[\mathbf{I}+2(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{Q}\right]\mathbf{1}.$$

3.4 Parameters Estimated Chart

In the case when $\mathbf{\Sigma}_0$ is unknown and must be estimated, we define the statistics V^* and U^* by

$$V^*=\left|(n-1)\,\widehat{\mathbf{\Sigma}}_0^{-1}\mathbf{S}\right|^{1/p}=\left|(n-1)\,\overline{\mathbf{S}}^{-1}\mathbf{S}\right|^{1/p}\text{ and}$$
$$U^*=\ln\left(\left|(n-1)\,\overline{\mathbf{S}}^{-1}\mathbf{S}\right|^{1/p}\right).$$

We observe that V^* and U^* can be expressed as

$$V^*=m(n-1)\left(\left|m(n-1)\,\mathbf{\Sigma}_0^{-1}\overline{\mathbf{S}}\right|^{1/p}\right)^{-1}\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}$$
$$=m(n-1)\,V_0^{-1}V\text{ and}$$
$$U^*=\ln\left(m(n-1)\,V_0^{-1}V\right)=\ln(m(n-1))-U_0+U,$$

where

$$V_0=\left|m(n-1)\,\mathbf{\Sigma}_0^{-1}\overline{\mathbf{S}}\right|^{1/p}\text{ and }U_0=\ln\left(\left|m(n-1)\,\mathbf{\Sigma}_0^{-1}\overline{\mathbf{S}}\right|^{1/p}\right).$$

The conditional distribution of U^* given U_0 has cumulative distribution and probability density functions given respectively as

$$F_{U^*|U_0}(u\,|u_0)=F_U(u+u_0-\ln(m(n-1)))\text{ and}$$
$$f_{U^*|U_0}(u\,|u_0)=f_U(u+u_0-\ln(m(n-1))).$$

For the cases in which $p = 1$ and $p = 2$, we have

$$F_{U^*|U_0}\left(u\,|u_0\right) = \begin{cases} F_{\chi^2_{n-1}}\left(\frac{e^{u+u_0}}{m(n-1)\lambda^{2/p}}\right), & \text{if } p = 1; \\ F_{\chi^2_{2n-4}}\left(\frac{2e^{u+u_0}}{m(n-1)\lambda^{2/p}}\right), & \text{if } p = 2; \end{cases} \quad \text{and}$$

$$f_{U^*|U_0}\left(u\,|u_0\right) = \begin{cases} \frac{e^{u+u_0}}{m(n-1)\lambda^{2/p}} f_{\chi^2_{n-1}}\left(\frac{e^{u+u_0}}{m(n-1)\lambda^{2/p}}\right), & \text{if } p = 1; \\ \frac{2e^{u+u_0}}{m(n-1)\lambda^{2/p}} f_{\chi^2_{2n-4}}\left(\frac{2e^{u+u_0}}{m(n-1)\lambda^{2/p}}\right), & \text{if } p = 2. \end{cases}$$

Recalling that Steyn (1978) defined

$$Y = \frac{p}{2}\left|(n-1)\,\mathbf{\Sigma}^{-1}\mathbf{S}\right|^{1/p}$$

and gave a method for approximating the distribution of Y for $p \geq 3$. Observing that

$$U = \ln\left(\frac{2\lambda^{2/p}}{p}Y\right),$$

it follows that

$$F_U\left(u\right) = F_Y\left(\frac{pe^u}{2\lambda^{2/p}}\right).$$

Thus, we can write

$$F_{U^*|U_0}\left(u\,|u_0\right) = F_Y\left(\frac{pe^{u+u_0}}{2m\left(n-1\right)\lambda^{2/p}}\right) \text{ and}$$
$$f_{U^*|U_0}\left(u\,|u_0\right) = \frac{pe^{u+u_0}}{2m\left(n-1\right)\lambda^{2/p}} f_Y\left(\frac{pe^{u+u_0}}{2m\left(n-1\right)\lambda^{2/p}}\right).$$

Approximations for $F_{U^*|U_0}\left(u\,|u_0\right)$ and $f_{U^*|U_0}\left(u\,|u_0\right)$ are obtained by replacing $F_Y\left(y\right)$ and $f_Y\left(y\right)$ with the approximation given in Steyn (1978).

To determine the parameters estimated version of the chart, one first looks at the conditional run length distribution given U_0. We represent the joint probability mass function conditioned on $U_0 = u_0$ by

$$\left[P\left(T_1 = t\,|U_0 = u_0\right), \ldots, P\left(T_q = t\,|U_0 = u_0\right)\right]^{\mathbf{T}}.$$

For the region R_j, we denote the conditional probability that $U^* \in R_j$ given $U_0 = u_0$ by p_j^*. That is,

$$p_j^* = P\left(U^* \in R_j \left| U_0 = u_0 \right.\right).$$

Further, we represent the transition matrix conditioned on $U_0 = u_0$ of the Markov chain representation of the chart by $\mathbf{P}^*$. The matrix obtained by removing the row and column $\mathbf{P}^*$ associated with the absorbing state is represented by $\mathbf{Q}^*$. It then follows that

$$\begin{aligned}
\mathbf{p}_1^* &= \left[P\left(T_1 = 1 \left| U_0 = u_0 \right.\right), \ldots, P\left(T_q = 1 \left| U_0 = u_0 \right.\right)\right]^{\mathbf{T}} \\
&= \left(\mathbf{I} - \mathbf{Q}^*\right)\mathbf{1} \text{ and} \\
\mathbf{p}_t^* &= \left[P\left(T_1 = t \left| U_0 = u_0 \right.\right), \ldots, P\left(T_q = t \left| U_0 = u_0 \right.\right)\right]^{\mathbf{T}} \\
&= \left(\mathbf{Q}^*\right)^{t-1}\left(\mathbf{I} - \mathbf{Q}^*\right)\mathbf{1},
\end{aligned}$$

for $t = 2, 3, 4, \ldots$. In our example, we have

$$\begin{aligned}
&p_{4,6}^* = p_3^* = P\left(u_{p,n,1-\Phi(-2)} < U^* < u_{p,n,1-\Phi(2)} \left| U_0 = u_0 \right.\right) \\
&= P\left(u_{p,n,1-\Phi(-2)} + u_0 - \ln\left(m\left(n-1\right)\right) < U < u_{p,n,1-\Phi(2)} + u_0 - \ln\left(m\left(n-1\right)\right)\right) \\
&= F_U\left(u_{p,n,1-\Phi(2)} + u_0 - \ln\left(m\left(n-1\right)\right)\right) - F_U\left(u_{p,n,1-\Phi(-2)} + u_0 - \ln\left(m\left(n-1\right)\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&p_{4,7}^* = p_2^* \\
&= P\left(u_{p,n,1-\Phi(-3)} + u_0 - \ln\left(m\left(n-1\right)\right) < U < u_{p,n,1-\Phi(-2)} + u_0 - \ln\left(m\left(n-1\right)\right)\right) \\
&= F_U\left(u_{p,n,1-\Phi(-2)} + u_0 - \ln\left(m\left(n-1\right)\right)\right) - F_U\left(u_{p,n,1-\Phi(-3)} + u_0 - \ln\left(m\left(n-1\right)\right)\right).
\end{aligned}$$

To obtain the unconditional run length distribution of the chart, the conditional run length distributions are averaged over the values of u_0. This can be expressed by

$$\begin{aligned}\mathbf{p}_1 &= \left[P\left(T_1 = 1\right), \ldots, P\left(T_q = 1\right)\right]^{\mathbf{T}} \\ &= \int_{-\infty}^{\infty} \left(\mathbf{I} - \mathbf{Q}^*\right) \mathbf{1} f_{U_0}\left(u_0\right) du_0 \text{ and} \\ \mathbf{p}_t &= \left[P\left(T_1 = t\right), \ldots, P\left(T_q = t\right)\right]^{\mathbf{T}} \\ &= \int_{-\infty}^{\infty} \left(\mathbf{Q}^*\right)^{t-1} \left(\mathbf{I} - \mathbf{Q}^*\right) \mathbf{1} f_{U_0}\left(u_0\right) du_0.\end{aligned}$$

In particular, the vector of average run lengths are determined by

$$\begin{aligned}\mu_{\mathbf{T}} &= \left[E\left(T_1\right), \ldots, E\left(T_q\right)\right]^{\mathbf{T}} \\ &= \int_{-\infty}^{\infty} \left[E\left(T_1 \left| U_0 = u_0\right.\right), \ldots, E\left(T_q \left| U_0 = u_0\right.\right)\right]^{\mathbf{T}} f_{U_0}\left(u_0\right) du_0 \\ &= \int_{-\infty}^{\infty} \left(\mathbf{I} - \mathbf{Q}^*\right)^{-1} \mathbf{1} f_{U_0}\left(u_0\right) du_0\end{aligned}$$

It follows that the cyclic steady-state run length distribution with estimated parameters can be obtained in a similar way. In particular, we have

$$P\left(T_{ss} = t\right) = \int_{-\infty}^{\infty} \left(\pi_{ss}^*\right)^{\mathbf{T}} \left(\mathbf{Q}^*\right)^{t-1} \left(\mathbf{I} - \mathbf{Q}^*\right) \mathbf{1} f_{U_0}\left(u_0\right) du_0,$$

where

$$\pi_{ss}^* = \left(\mathbf{G} - \left(\mathbf{Q}^*\right)^{\mathbf{T}}\right)^{-1} \mathbf{e}_1.$$

3.5 Conclusion

A method is given for selecting the warning and control limits for Shewhart $\ln\left(\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right)$ chart supplemented with runs rules.Passing these limits through the exponential function gives the warning and control limits for the Shewhart $\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}$ and raising the results to the p power and multiplying by $\left|\mathbf{\Sigma}_0\right| / (n-1)^p$ gives the limits for the Shewhart $|\mathbf{S}|$ supplemented with runs rules. The Markov chain approach of Champ and Woodall (1987) is used to analyze the run length properties of the chart. A discussion about how to obtain the run length properties with estimated parameters was given.

CHAPTER 4

CUSUM $\ln\left(\left|(N-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/P}\right)$ CHART

4.1 Introduction

Page (1986) first introduced the CUSUM chart for monitoring the mean of a quality measurement X. He studied this chart under the assumption the in-control mean and standard deviation of the distribution of X are known and that X follows a normal distribution. The CUSUM chart makes use of the information in the present sample but may use information in previous samples to make a decision about the state of the process. One version of the CUSUM chart plots on the same chart the statistic C_t^- and C_t^+ verses the sampling number t. The stochastic sequences $\left\{C_t^-\right\}$ and $\left\{C_t^+\right\}$ are defined by

$$C_t^- = \min\left\{0, C_{t-1}^- + U_t + k^-\right\} \text{ and } C_t^+ = \max\left\{0, C_{t-1}^+ + U_t - k^+\right\},$$

with $C_0^- = 0$, $C_0^+ = 0$, and

$$U_t = \ln\left(\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right).$$

The chart signals a potential out-of-control process if $C_t^- \leq h^- \leq 0$ or $C_t^+ \geq h^+ \geq 0$. We will design the chart such that the statistic C_t^- detects a change from Σ_0 to Σ with $\left|\Sigma_0^{-1}\Sigma\right| < 1$ and C_i^+ to detect for a change with $\left|\Sigma_0^{-1}\Sigma\right| > 1$. Lucas (1985) recommended setting $h^- < C_0^- < 0$ and/or $0 < C_0^+ < h^+$. This gives the chart a "head-start" in detecting a process that is initially out-of-control. The parameters of this chart include the initial values of C_0^- and C_0^+, the values k^- and k^+, the values h^- and h^+, and the sample size n. Often the chart parameters are selected such that $k^- = -k^+$ and $h^- = -h^+$. In this chapter, we outline a method for studying the run length properties of the CUSUM chart based on the statistic U.

4.2 Run Length Distribution

Most who study the run length properties of a quality control chart either use simulation, a Markov chain approach, or integral equations. Champ and Rigdon (1991) illustrated how the Markov chain approach for analyzing the run length properties of a CUSUM chart is equivalent to the integral equation approach. Both methods are approximation methods. A Markov chain can first be used to approximate the chart and then the run length properties are determined exactly for the approximation. Champ, Rigdon, and Scharnagl (2001) give several integral equations whose exact solutions are parameters of the run length distribution of the CUSUM chart of Page. Since exact expressions for the solutions to these integral equations cannot be obtained in a useful form, approximate solutions are obtained. Equating the approximation methods used to obtain the Markov chain and approximate the solutions to the integral equations equates the methods.

The distribution of the run length T of the upper one-sided CUSUM chart can be obtained iteratively from a sequence of integral equations. Define $pr^+(t|u^+)$ by

$$pr^+\left(t|u^+\right) = P\left(T = t|C_0^+ = u^+\right),$$

where T is the run length of the chart. From the results found in Champ, Rigdon, and Scharnagl (2001), we have that the function $pr^+(t|u^+)$ is the exact solution to the sequence of equations

$$pr^+\left(1|u^+\right) = 1 - F_U\left(h^+ - u^+ + k^+\right) \text{ and}$$
$$pr^+\left(t|u^+\right) = pr^+\left(t-1|0\right)F_U\left(k^+ - u^+\right) + \int_0^{h^+} pr^+\left(t-1|u\right)f_U\left(u - u^+ + k^+\right)du$$

for $t > 1$.

A convenient closed form express for the function $pr^+\left(t|u^+\right)$ does not exist, but it can be accurately approximated using Gaussian quadrature. To prepare the integral in the previous equation to be approximated numerically, we need to make the change of variable

$$u = \frac{h^+}{2}\left(v+1\right)$$

with $du = \left(h^+/2\right)dv$. It follows that

$$pr^+\left(t|u^+\right) = pr^+\left(t-1|0\right)F_U\left(k^+ - u^+\right) + \int_{-1}^{1} \frac{h^+}{2} pr^+\left(t-1|\frac{h^+}{2}\left(v+1\right)\right)f_U\left(\frac{h^+}{2}\left(v+1\right) - u^+ + k^+\right)dv.$$

Using $v_1, \ldots, v_\eta$ as the zeros and $\omega_1, \ldots, \omega_\eta$ the corresponding weights of the Legendre polynomials (see Abramowitz and Stegun (1972)), then we can approximate the integral with the sum

$$\sum_{j=1}^{\eta} \frac{h^+}{2} pr^+\left(t-1|\frac{h^+}{2}\left(v_j+1\right)\right)f_U\left(\frac{h^+}{2}\left(v_j+1\right) - u^+ + k^+\right)\omega_j.$$

Substituting u_j for $\left(h^+/2\right)\left(v_j+1\right)$, we can write the sum as

$$\sum_{j=1}^{\eta} \frac{h^+}{2} pr^+\left(t-1|u_j\right)f_U\left(u_j - u^+ + k^+\right)\omega_j.$$

It now follows that the function $pr^+\left(t|u^+\right)$ approximately satisfies the equation

$$pr^+\left(t|u^+\right) = pr^+\left(t-1|0\right)F_U\left(k^+ - u^+\right) + \frac{h^+}{2}\sum_{j=1}^{\eta} pr^+\left(t-1|u_j\right)f_U\left(u_j - u^+ + k^+\right)\omega_j.$$

Selecting u^+ to be the values $u_0, u_1, \ldots, u_\eta$ with $u_0 = 0$, we have the system of equations

$$\begin{aligned}
pr^+(t|u_0) &= pr^+(t-1|u_0)\,F_U\left(k^+ - u_0\right) \\
&\quad + \frac{h^+}{2}\sum\nolimits_{j=1}^{\eta} pr^+(t-1|u_j)\,f_U\left(u_j - u_0 + k^+\right)\omega_j \\
pr^+(t|u_1) &= pr^+(t-1|u_0)\,F_U\left(k^+ - u_1\right) \\
&\quad + \frac{h^+}{2}\sum\nolimits_{j=1}^{\eta} pr^+(t-1|u_j)\,f_U\left(u_j - u_1 + k^+\right)\omega_j \\
&\ \vdots \\
pr^+(t|u_i) &= pr^+(t-1|0)\,F_U\left(k^+ - u_i\right) \\
&\quad + \frac{h^+}{2}\sum\nolimits_{j=1}^{\eta} pr^+(t-1|u_j)\,f_U\left(u_j - u_i + k^+\right)\omega_j \\
&\ \vdots \\
pr^+(t|u_\eta) &= pr^+(t-1|0)\,F_U\left(k^+ - u_\eta\right) \\
&\quad + \frac{h^+}{2}\sum\nolimits_{j=1}^{\eta} pr^+(t-1|u_j)\,f_U\left(u_j - u_\eta + k^+\right)\omega_j.
\end{aligned}$$

It is easy to see that this system of equations can be expressed in the form of

$$\mathbf{p}_t = \mathbf{Q}\mathbf{p}_{t-1},$$

where the ith component of $\mathbf{p}_t$ is $pr^+(t|u_i)$, the ith component of $\mathbf{p}_{t-1}$ is $pr^+(t-1|u_i)$, and the (i,j)th element of $\mathbf{Q}$ is

$$\begin{cases} F_U\left(k^+ - u_i\right), & \text{for } i = 0, 1, \ldots, \eta,\ j = 0; \\ \frac{h^+}{2} f_U\left(u_j - u_i + k^+\right)\omega_j & \text{for } i = 0, 1, \ldots, \eta,\ j = 1, \ldots, \eta. \end{cases}$$

with

$$\mathbf{p}_1 = \begin{bmatrix} 1 - F_U\left(h^+ - u_0 + k^+\right) \\ 1 - F_U\left(h^+ - u_1 + k^+\right) \\ \vdots \\ 1 - F_U\left(h^+ - u_i + k^+\right) \\ \vdots \\ 1 - F_U\left(h^+ - u_\eta + k^+\right) \end{bmatrix}.$$

Note that

$$\begin{aligned}
&F_U\left(k^+ - u_i\right) + \sum\nolimits_{j=1}^{\eta} f_U\left(u_j - u_i + k^+\right)\omega_j + \left(1 - F_U\left(h^+ - u_i + k^+\right)\right) \\
&\approx F_U\left(k^+ - u_i\right) + \int_0^{h^+} f_U\left(u - u_i + k^+\right) du + \left(1 - F_U\left(h^+ - u_i + k^+\right)\right) \\
&= F_U\left(k^+ - u_i\right) + \int_{-u_i+k^+}^{h^+ - u_i + k^+} f_U\left(t\right) dt + \left(1 - F_U\left(h^+ - u_i + k^+\right)\right) \\
&= F_U\left(k^+ - u_i\right) + F_U\left(h^+ - u_i + k^+\right) - F_U\left(k^+ - u_i\right) + 1 - F_U\left(h^+ - u_i + k^+\right) \\
&= F_U\left(h^+ - u_i + k^+\right) + 1 - F_U\left(h^+ - u_i + k^+\right) \\
&= 1.
\end{aligned}$$

The average run length (ARL) of the chart is a function of the starting value $C_0^+ = u^+$. For convenience, we let

$$M\left(u^+\right) = E\left(T \left| C_0^+ = u^+\right.\right).$$

The integral equation that is useful in determining $M\left(u^+\right)$ is

$$M\left(u^+\right) = 1 + M\left(0\right) F_U\left(k^+ - u^+\right) + \int_0^{h^+} M\left(z\right) F_U\left(z - u^+ + k^+\right) dz,$$

where $C_0^+ = u^+$. The function $M\left(u^+\right)$ can be approximated at the values $u_0, u_1, \ldots, u_\eta$ as the solutions to the system of equations

$$\mathbf{M} = \mathbf{1} + \mathbf{QM},$$

where

$$\mathbf{M} = \left[M\left(u_0\right), M\left(u_1\right), \ldots, M\left(u_{\eta-1}\right)\right]^{\mathbf{T}}.$$

It is not difficult to see that the solution of this system of equation can be expressed as

$$\mathbf{M} = \left(\mathbf{I} - \mathbf{Q}\right)^{-1}\mathbf{1}.$$

For a Markov chain representation of the chart, we define

$$\mathbf{P} = \begin{bmatrix} \mathbf{Q} & \left(\mathbf{I}-\mathbf{Q}\right)\mathbf{1} \\ \mathbf{0}^{\mathbf{T}} & 1 \end{bmatrix}$$

with non-absorbing states $u_0, u_1, \ldots, u_\eta$. If the method proposed by Brook and Evans (1972) is used, then the non-absorbing states are the values

$$u_i = \frac{2h^+}{2\eta - 1} i$$

for $i = 0, 1, \ldots, \eta - 1$ and transition probabilities

$$P\left(u_i \to u_j\right) = \begin{cases} F_U\left(\frac{h^+}{2\eta-1} - u_i + k^+\right), & \text{for } i = 0, 1, \ldots, \eta,\ j = 0; \\ F_U\left(\frac{h^+(2j+1)}{2\eta-1} - u_i + k^+\right) & \\ -F_U\left(\frac{h^+(2j-1)}{2\eta-1} - u_i + k^+\right), & \text{for } i = 0, 1, \ldots, \eta,\ j = 1, \ldots, \eta. \end{cases}$$

For the case in which $p = 2$, recall that

$$U = \ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}_t\right|^{1/p}\right) = \ln\left(\lambda\chi^2_{2n-4}/2\right).$$

Thus, we have

$$F_U\left(u\right) = F_{\chi^2_{2n-4}}\left(\frac{2e^u}{\lambda}\right) \text{ and } f_U\left(u\right) = \frac{2e^u}{\lambda} f_{\chi^2_{2n-4}}\left(\frac{2e^u}{\lambda}\right).$$

Hence, the (i, j)th element of $\mathbf{Q}$ is

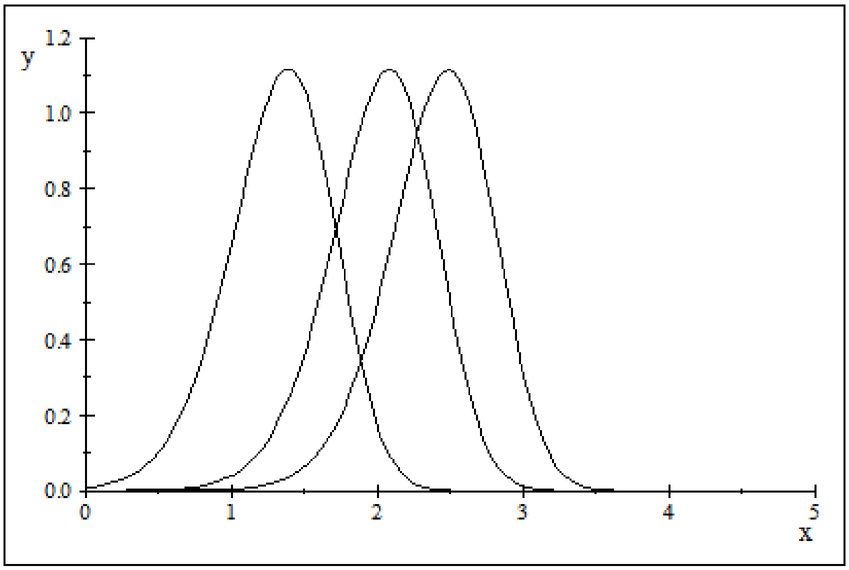
$$\begin{cases} F_{\chi^2_{2n-4}}\left(\frac{2e^{u_0-u_i+k^+}}{\lambda}\right), & \text{for } i = 0, 1, \ldots, \eta,\ j = 0; \\ \frac{2e^{u_j-u_i+k^+}}{\lambda} f_{\chi^2_{2n-4}}\left(\frac{2e^{u_j-u_i+k^+}}{\lambda}\right)\omega_j & \text{for } i = 0, 1, \ldots, \eta,\ j = 1, \ldots, \eta. \end{cases}$$

For the case in which $n = 10$, the density describing the distribution of U is given by

$$f_U(u) = \frac{1}{\Gamma(n-2)\,\lambda^{n-2}} e^{-(e^u/\lambda-(n-2)u)}.$$

A plot of the density of describing the distribution of U is given in Figure 4.1.

Figure 4.1: Density Plots of the Distribution of U



The value of λ for the "left-most" curve is 0.5; for the curve in the middle it is 1.0; and for the "right-most" curve its value is 1.5. Further we observe that the probability density function of the distribution of U is "mound-shaped" and only slightly skewed in the negative direction. The support of the distribution of U is the reals. While the CUSUM chart was designed assuming the sequence of statistics being "CUSUMed" has a normal distribution, we see that the statistic U has an approximate normal distribution both when the process is in- and out-of-control. This suggest that the CUSUM chart is well suited for monitoring for a change in expected value of U.

4.3 Estimated Parameters and Steady-State Run Length Distributions

The estimated parameters and steady-state run length distributions can be approximated by viewing the matrix $\mathbf{Q}$ as part of the transition matrix of a Markov chain representation of the chart after removing the row and column of the transition matrix association with the absorbing state. The results developed in the previous chapter can then be used to determine the unconditional and cyclic steady-state run length distributions. Similarly, the results in the previous chapter can be used in the estimated parameters case.

4.4 Conclusion

A well known property of the CUSUM chart is that it will detect a "small" to "moderate" shift in the process parameter(s) than a comparable Shewhart chart with runs rules. In this chapter, we have outlined how one can obtain the fixed state and cyclic steady-state run length properties both when parameters are known and estimated of the CUSUM chart based on the statistic $\ln\left(\left|(n-1)\,\boldsymbol{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right)$.

CHAPTER 5

AN EXAMPLE

Montgomery (2003) gives an example in which tensile (X_1) and diameter (X_2) of a textile fiber are the quality measurements of interest. At each sampling stage, random samples of size $n = 10$ are taken from the output of the process. The practitioner has available $m = 20$ samples each of size $n = 10$ for the process when the process was believed to be in-control. The summary statistics for these samples are given in Table 5.1 for completeness.

The estimates for the in-control mean vector and covariance matrix using these data are

$$\overline{\overline{\mathbf{x}}} = \begin{bmatrix} \overline{\overline{x}}_1 \\ \overline{\overline{x}}_2 \end{bmatrix} = \begin{bmatrix} 115.59 \\ 1.06 \end{bmatrix} \text{ and } \overline{\mathbf{S}} = \begin{bmatrix} 1.23 & 0.79 \\ 0.79 & 0.83 \end{bmatrix}.$$

Let us assume that $\overline{\overline{\mathbf{x}}}$ and $\overline{\mathbf{S}}$ are "very good" estimates of μ_0 and $\mathbf{\Sigma}_0$. Since Montgomery (2003) does not give any Phase II data for this process, we will simulate some Phase II data to illustrate the design of a Shewhart U chart supplemented with runs rules.

For illustrations purposes, we assume the process is in-control at sampling stages 1 through 5 and out-of-control in 6 through 10. The process changes from in-control to out-of-control by a sustained shift in the covariance matrix from the in-control value $\mathbf{\Sigma}_0 = \overline{\mathbf{S}}$ to the out-of-control value

$$\mathbf{\Sigma} = \begin{bmatrix} (1.2)^2\,(1.23) & (1.2)\,(0.79) \\ (1.2)\,(0.79) & 0.83 \end{bmatrix}.$$

That is, standard deviation $\sigma_{0,1} = \sqrt{1.23}$ of the distribution of X_1 made a sustained shift at sampling stage 6 to the value $\sigma_1 = (1.2)\sqrt{1.23}$. Under these assumptions

Table 5.1: Example Data

Sample i	$\bar{x}_1$	$\bar{x}_2$	s_1^2	s_2^2	s_{12}	$\lvert\mathbf{S}\rvert$	u
1	115.25	1.04	1.25	0.87	0.80	0.45	2.26
2	115.91	1.06	1.26	0.85	0.81	0.41	2.22
3	115.05	1.09	1.30	0.90	0.82	0.50	2.31
4	116.21	1.05	1.02	0.85	0.81	0.21	1.88
5	115.90	1.07	1.16	0.73	0.80	0.21	1.87
6	115.55	1.06	1.01	0.80	0.76	0.23	1.93
7	114.98	1.05	1.25	0.78	0.75	0.41	2.22
8	115.25	1.10	1.40	0.83	0.80	0.52	2.33
9	116.15	1.09	1.19	0.87	0.83	0.35	2.13
10	115.92	1.05	1.17	0.86	0.95	0.10	1.53
11	115.75	0.99	1.45	0.79	0.78	0.54	2.35
12	114.90	1.06	1.24	0.82	0.81	0.36	2.15
13	116.01	1.05	1.26	0.55	0.72	0.17	1.79
14	115.83	1.07	1.17	0.76	0.75	0.33	2.10
15	115.29	1.11	1.23	0.89	0.82	0.42	2.23
16	115.63	1.04	1.24	0.91	0.83	0.19	2.25
17	115.47	1.03	1.20	0.95	0.70	0.65	2.44
18	115.58	1.05	1.18	0.83	0.79	0.36	2.14
19	115.72	1.06	1.31	0.89	0.76	0.59	2.39
20	115.40	1.04	1.29	0.85	0.68	0.63	2.43

$$*u = \ln\left(\left|(n-1)\,\overline{\mathbf{S}}^{-1}\mathbf{S}\right|^{1/2}\right)$$

based on sample sizes of $n = 10$, ten values of the statistic U were generated. These are given in Table 5.2.

Table 5.2: Phase II Values of U

t	1	2	3	4	5	6	7	8	9	10
$\mathbf{U_t}$	2.43	1.58	1.55	1.65	1.89	1.87	2.52	2.03	2.37	2.87

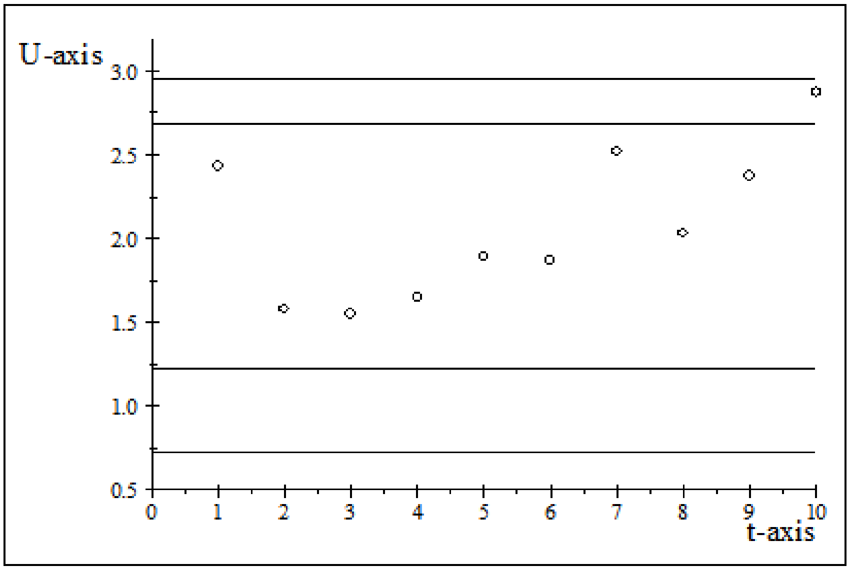
We consider a Phase II Shewhart U chart with the supplementary runs rules given in Table 5.3.

Table 5.3: Runs Rules

1.	$\mathrm{T}(1, 1, -\infty, 0.72644)$
2.	$\mathrm{T}(2, 3, 0.72644, 1.22102)$
3.	$\mathrm{T}(2, 3, 2.68034, 2.95354)$
4.	$\mathrm{T}(1, 1, 2.95354, \infty)$

A plot of the data given in Table 5.2 and the warning lines and control limits given in Table 5.3 is given in Figure 5.1.

Figure 5.1: Plot of U_t vs t



Under the modelling assumptions, an analysis of the run length distribution is given in Table 5.4. The values in the table were obtain using a modified version of the FORTRAN program given in Champ and Woodall (1992).

Table 5.4: Run Length Parameters, $p = 2$,$n = 10$

			Percentiles								
λ	ARL	SDRL	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
0.5	6.13	4.92	1	1	2	3	5	8	13	16	24
0.6	14.15	12.89	1	2	3	5	10	19	31	40	61
0.7	34.76	33.48	1	3	5	11	24	48	78	102	155
0.8	85.18	83.93	2	6	10	25	59	118	194	253	387
0.9	185.39	184.24	3	11	21	54	129	257	425	553	849
1.0	225.44	224.38	3	13	25	66	157	312	518	673	1034
1.1	117.16	116.07	2	7	13	34	82	162	268	349	535
1.2	51.10	49.99	1	4	6	15	36	70	116	151	231
1.3	25.37	24.28	1	2	4	8	18	35	57	74	113
1.4	14.58	13.52	1	2	3	5	10	20	32	42	63
1.5	9.43	8.41	1	1	2	3	7	13	20	26	40

CHAPTER 6

CONCLUSION

6.1 General Conclusions

The performance of commonly recommended control charts for monitoring the mean vector of the distribution of a multivariate quality measurement is not only affected by changes in the mean vector but also changes in the covariance matrix. This establishes the need for the practitioner to maintain a chart for monitoring for a change in the covariance matrix. One such chart that is commonly used for this purpose is the generalized variance chart. This chart is not affected by changes in the mean vector and as we have shown is quite useful in detecting certain types of shifts in the covariance matrix. One of the difficulties in studying this chart is that the distribution of the sample generalized variance is not known. We examined an approximation method by Steyn (1978) and found it to be useful in studying the run length performance of the chart.

Supplementary runs rules can be added to a Shewhart chart to enhance its performance. We have given a method for selecting runs rules for the generalized variance chart. This method selects the warning and action lines (control limits) so that the in-control ARL is the same as that of the Shewhart $\overline{X}$ supplemented with the same runs rules. This requires selecting percentage points of the plotted statistic, in our case, the statistics $V = \left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}$ and $U = \ln\left(\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right)$. The run length properties of the chart were studied using a Markov chain approach. We further discussed methods for designing these charts with estimated parameters.

The CUSUM chart is commonly recommended in the literature for monitoring for small to moderate shifts in the mean of a quality measurement. We examined its use for monitoring for a change in the distribution of U. An outline was given for studying the run length performance of the CUSUM U chart using the integral equation approach. It was shown that this method is equivalent to a Markov chain approach. This easily allows one to study the steady-state run length distribution properties. Further, we outlined how one would study the run length properties of the chart with estimated parameters. Our work was concluded with an example.

6.2 Areas for Further Research

This research has suggested a number of potential further research. (1) It would be useful for practitioners to have available a computer program that analyzes the fixed and steady-state run length properties of the Shewhart chart with run rules when parameters are to be estimated. (2) A similar program would be useful for the CUSUM chart. (3) Determining the run length properties of the CUSUM chart based on the statistic $\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}$, the statistic $\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|$, or the statistic $|\mathbf{S}|$ is far more difficult than the CUSUM chart based on the statistic $\ln\left(\left|(n-1)\,\mathbf{\Sigma}_0^{-1}\mathbf{S}\right|^{1/p}\right)$. The integral equations whose solutions are various run length parameters are not of the Fredholm type and are a bit more difficult to solve. Developing software to implement solutions to these equations would provide useful analyses of the run length properties of the chart for practitioner who may wish to use these charts. (4) Other charts appear in the literature that have been designed to monitor for a change in the process covariance matrix. It would be interesting to develop analytical methods to study the fixed and steady-state run length properties both when parameters are known and estimated for these charts and report a comparison of their run length performance.

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Appendix A
FIRST APPENDIX

Table A.1: Percentage Points (continued) $u_{1,n,1-\Phi(c)}$, λ=1

c	n=7	n=8	n=9	n=10
-3	-0.85954	-0.42128	-0.07195	0.21610
-2	0.17611	0.49187	0.74950	0.96611
-1	1.00598	1.23601	1.42823	1.59303
0	1.67675	1.84779	1.99390	2.12140
1	2.22740	2.35773	2.47118	2.57171
2	2.68763	2.78940	2.87939	2.96017
3	3.07912	3.16040	3.23322	3.29929

Table A.2: Percentage Points (continued) $u_{2,n,1-\Phi(c)}$, λ=1

c	n=8	n=9	n=10	n=11	n=12
-3	0.16123	0.47187	0.72644	0.94117	1.12630
-2	0.76697	1.01458	1.22102	1.39761	1.55165
-1	1.28649	1.48581	1.65472	1.80114	1.93026
0	1.73522	1.89757	2.03722	2.15975	2.26890
1	2.12614	2.26006	2.37679	2.48032	2.57339
2	2.46984	2.58170	2.68034	2.76867	2.84871
3	2.77477	2.86931	2.95354	3.029594	3.09901

Table A.3: Percentage Points (continued) $u_{3,n,1-\Phi(c)}$, λ=1

c	n=7	n=8	n=9	n=10
-3	0.38900	0.63059	0.84289	1.03339
-2	0.63240	0.90903	1.13809	1.33237
-1	1.01893	1.27319	1.48014	1.65434
0	1.41109	1.62971	1.80891	1.96077
1	1.77336	1.95881	2.11273	2.24448
2	2.10080	2.25831	2.39078	2.50533
3	2.39572	2.53034	2.64497	2.74506

Table A.4: Percentage Points (continued) $u_{4,n,1-\Phi(c)}$, λ=1

c	$\boldsymbol{n}$**=7**	$\boldsymbol{n}$**=8**	$\boldsymbol{n}$**=9**	$\boldsymbol{n}$**=10**
-3	0.74085	1.00136	1.20677	1.37685
-2	0.78117	1.05052	1.26544	1.44534
-1	0.95732	1.23398	1.45324	1.63508
0	1.24673	1.50377	1.70692	1.87512
1	1.56105	1.78753	1.96813	2.11870
2	1.86332	2.05972	2.21838	2.35194
3	2.14360	2.31362	2.45284	2.57123