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Weighted Inverse Weibull and Beta-Inverse Weibull Distribution

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WEIGHTED INVERSE WEIBULL AND BETA-INVERSE WEIBULL DISTRIBUTION

by

JING XIONG KERSEY

(Under the Direction of Dr. Broderick O. Oluyede)

ABSTRACT

The weighted inverse Weibull distribution and the beta-inverse Weibull distribution are considered. Theoretical properties of the inverse Weibull model, weighted inverse Weibull distribution including the hazard function, reverse hazard function, moments, moment generating function, coefficient of variation, coefficient of skewness, coefficient of kurtosis, Fisher information and Shanon entropy are studied. The estimation for the parameters of the length-biased inverse Weibull distribution via maximum likelihood estimation and method of moment estimation techniques are presented, as well as a test for the detection of length-biasedness in the inverse Weibull model. Furthermore, the beta-inverse Weibull distribution which is a weighted distribution is presented, including the cumulative distribution function (cdf), probability density function (pdf), density plots, moments, and the moment generating function. Also, some useful transformations that lead to the generation of observations from the beta-inverse Weibull distribution are derived.

Key Words: Weighted distribution, Weighted inverse Weibull distribution,
Beta-inverse Weibull distribution

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TABLE OF CONTENTS

LIST OF TABLES

Table Page

LIST OF FIGURES

Figure Page

CHAPTER 1

INTRODUCTION

1.1 Background of Weighted Distribution

The use and application of weighted distributions in research related to reliability, bio-medicine, ecology and several other areas are of tremendous practical importance in mathematics, probability and statistics. These distributions arise naturally as a result of observations generated from a stochastic process and recorded with some weight function. Several authors have presented important results on length-biased distributions and on weighted distributions in general. Rao [23] unified the concept of weighted distributions. Bhattacharyya and Roussas [1] studied and compared nonparametric unweighted and length-biased density estimates of fibers. Vardi [26] derived the nonparametric maximum likelihood estimate (NPMLE) of a lifetime distribution in the presence of length bias and established convergence to a pinned Gaussian process with a simple covariance function under mild conditions. For additional and important results on weighted distributions see Patil and Rao [20], Patil and Ord [21], Gupta and Keating [12], Oluyede [19] among others.

Traditional environmetric theory and practice have been occupied largely with randomization and replication, however, in environmental and ecological work, observations also fall in the non experimental, non replicated, and nonrandom categories. The problems of model specification and data interpretation then acquire special importance and great concern. The theory of weighted distributions provides a unifying approach for these problems. Weighted distributions take into account the method of ascertainment by adjusting the probabilities of actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded. Failure to make such adjustments can lead to incorrect conclusions.

The concept of weighted distributions can be traced to the study of the effect of methods of ascertainment upon estimation of frequencies by Fisher [10]. In extending the basic ideas of Fisher, Rao [23] saw the need for a unifying concept and identified various sampling situations that can be modeled by what he called weighted distributions. Within the context of cell kinetics and the early detection of disease, Zelen and Feinleib [27] introduced weighted distributions to represent what he broadly perceived as length-biased sampling. In a series of papers with his colleages, Patil and Ord [21] pursued weighted distributions for the purpose of encountered data analysis, equilibrium population analysis subject to harvesting and predation, meta-analysis incorporating publication bias and heterogeneity, modeling clustering and extraneous variation, to mention just a few areas.

To introduce the concept of a weighted distribution, suppose X is a non-negative random variable (rv) with its natural probability density function (pdf) $f(x;\theta)$, where the natural parameter is $\theta \in \Omega$ (Ω is the parameter space). Suppose a realization x of X under $f(x;\theta)$ enters the investigator's record with probability proportional to $w(x,\beta)$, so that the recording (weight) function $w(x,\beta)$ is a non-negative function with the parameter β representing the recording (sighting) mechanism. Clearly, the recorded x is not an observation on X, but on the rv X^w, say, having a pdf

$$f^w(x;\theta,\beta) = \frac{w(x,\beta)f(x;\theta)}{\omega}, \tag{1.1}$$

where ω is the normalizing factor obtained to make the total probability equal to unity by choosing $\omega = E[w(X,\beta)] < \infty$. The rv X^w is called the weighted version of X, and its distribution is related to that of X and is called the weighted distribution with weight function w. Note that the weight function $w(x,\beta)$ need not lie between zero and one, and actually may exceed unity, as, for example, when $w(x, \beta) = x$, in which case $X^* = X^w$ is called the *size-biased version* of X. The distribution of X^* is called the *size-biased distribution* with pdf

$$f^*(x;\theta) = \frac{xf(x;\theta)}{\mu}, \tag{1.2}$$

where $\mu = E[X] < \infty$. The pdf f^* is called the length-biased or size-biased version of f, and the corresponding observational mechanism is called *length-biased* or *size-biased sampling*. Weighted distributions have seen much use as a tool in the selection of appropriate models for observed data drawn without a proper frame. In many situations the model given above is appropriate, and the statistical problems that arise are the determination of a suitable weight function, $w(x, \beta)$, and drawing inferences on θ. Appropriate statistical modeling helps accomplish unbiased inference in spite of the biased data and, at times, even provides a more informative and economic setup. See Rao [22], [23] for a comprehensive review and additional details on weighted distributions.

1.2 Outline of Results

This paper is organized as follows: In chapter 2, some basic utility notions, the Weibull model, the inverse Weibull model, weighted Weibull distribution and weighted inverse Weibull distribution are introduced. This chapter also includes some properties (such as reverse hazard function, mean, variance, coefficient of variation, coefficient of skewness, coefficient of kurtosis, and application of Glaser's [11] results) of those distributions. Chapter 3 presents the moments and moment generating functions, Fisher information and Shanon entropy of the weighted inverse Weibull distribution introduced in chapter 2. Chapter 4 contains the estimation of parameters in the weighted inverse Weibull distribution including the maximum likelihood estimators and method of moments estimation technique. Chapter 5 contains the derivation and properties as well as important results on the beta-inverse Weibull distribution.

CHAPTER 2

WEIGHTED INVERSE WEIBULL DISTRIBUTION

2.1 Introduction and Basic Utility Notions

The distribution of a random variable X can essentially be characterized by four functions, namely, the survival function which is the probability that an individual survives beyond a time x, the probability density (or mass) function which is the approximate unconditional probability of the event occurring at time x, the failure rate function or hazard rate, which is approximately the probability or chance an individual of age x experiences the event in the next instant in time, and the mean residual life at time x which is the mean time to the event of interest, given the event has not occurred at x. See Ross [24]; Shaked and Shanthikumar [25] for additional details.

The survival function of a continuous random variable X, denoted by $\overline{F}(x)$ or $S(x)$ is a continuous monotone, decreasing function, given by

$$\overline{F}(x) = 1 - F(x) = \int_x^{\infty} f(t)dt, \tag{2.1}$$

where $f(t)$ is the probability density function (pdf) and $F(t)$ is the cumulative distribution(cdf). The hazard function also known as the conditional failure rate in reliability is given by

$$\lambda_F(x) = \lim_{\Delta x \to 0} \frac{P(x \leq X < x + \Delta x | X \geq x)}{\Delta x}. \tag{2.2}$$

Note that $\lambda_F(x)\Delta x$ is the approximate probability that an individual who has survived to age x will experience the event in the interval $(x, x+\Delta x)$. The mean residual life function is given by

$$\delta_F(x) = E_F(X - x | X \geq x) = \frac{\int_x^{\infty} \overline{F}(u)du}{\overline{F}(x)} = \frac{\int_x^{\infty} (u-x)f(u)du}{\overline{F}(x)}. \tag{2.3}$$

This is a parameter that measures for an individual at age x, their expected remaining lifetime.

In general, if X is a continuous random variable with distribution function F, and probability density function (pdf) f, then the hazard function, reverse hazard function and mean residual life functions are given by $\lambda_F(x) = \frac{f(x)}{\overline{F}(x)}$, $\tau_F(x) = \frac{f(x)}{F(x)}$, and $\delta_F(x) = \frac{\int_x^\infty \overline{F}(u)du}{\overline{F}(x)}$ respectively. The functions $\lambda_F(x)$, $\delta_F(x)$, and $\tau_F(x)$ are equivalent. See Shaked [24] and Shanthikumar [25].

The concept of reverse hazard rate was introduced as the hazard rate in the negative direction and received minimal attention, if any, in the literature. Keilson and Sumita [15] demonstrated the importance of the reverse hazard rate and reverse hazard orderings. Shaked and Shanthikumar [25] presented results on reverse hazard rate. See Chandra and Roy [6], Block and Savits [2] for additional details. We present formal definitions of the reverse hazard function of a distribution function F and stochastic order of two distributions F and G respectively.

Definition 2.1.1. *Let $(a,\ b)$, $-\infty \leq a < b < \infty$, be an interval of support for F. Then the reverse hazard function of X (or F) at $t > a$ is denoted by $\tau_F(t)$ and is defined as*

$$\tau_F(t) = \frac{d}{dt}\log F(t) = \frac{f(t)}{F(t)}. \tag{2.4}$$

Definition 2.1.2. *Let X and Y be two random variables with distribution functions F and G respectively. We say $F <_{st} G$, stochastically ordered, if $\overline{F}(x) \leq \overline{G}(x)$, for $x \geq 0$ or equivalently, for any increasing function $\Phi(x)$,*

$$E(\Phi(X)) \leq E(\Phi(Y)). \tag{2.5}$$

2.2 Weibull Distribution

Consider the Weibull distribution used to model degradation of mechanical components such as pistons, crankshafts of diesel engines, as well as breakdown of insulating fluid to mention just a few areas. If a random variable X has a Weibull distribution with shape parameter α and scale parameter β, we write $X \sim Weibull(\alpha, \beta)$. The probability density function (pdf), cumulative distribution function (cdf), and non central moments are given by

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, \quad x \geq 0,\ \alpha > 0,\ \beta > 0, \tag{2.6}$$

$$F(x;\alpha,\beta) = \int_0^x \alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}}\,dt = 1 - e^{-\alpha x^{\beta}}, \quad x \geq 0,\ \alpha > 0,\ \beta > 0, \tag{2.7}$$

and

$$E(X^k) = \int_0^\infty x^k \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx = \frac{k\Gamma(k/\beta)}{\beta\alpha^{k/\beta}}. \tag{2.8}$$

respectively.

When $k = 1$, the mean μ is given by

$$E(X) = \int_0^\infty x\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx = \frac{\Gamma(1/\beta)}{\beta\alpha^{1/\beta}}. \tag{2.9}$$

The Fisher information (FI) in X are denoted by

$$\begin{aligned}
I(\alpha,x) &= E\Big[\frac{\partial \log\big(f(x;\alpha,\beta)\big)}{\partial\alpha}\Big]^2, \\
I(\beta,x) &= E\Big[\frac{\partial \log\big(f(x;\alpha,\beta)\big)}{\partial\beta}\Big]^2, \\
I(\alpha,\beta,x) &= \begin{bmatrix} E\Big[\frac{\partial \log\big(f(x;\alpha,\beta)\big)}{\partial\alpha}\Big]^2 & E\Big[\frac{\partial^2 \log\big(f(x;\alpha,\beta)\big)}{\partial\alpha\partial\beta}\Big] \\ E\Big[\frac{\partial^2 \log\big(f(x;\alpha,\beta)\big)}{\partial\beta\partial\alpha}\Big] & E\Big[\frac{\partial \log\big(f(x;\alpha,\beta)\big)}{\partial\beta}\Big]^2 \end{bmatrix}
\end{aligned} \tag{2.10}$$

and can be readily obtained.

2.3 Inverse Weibull Distribution

Consider the inverse Weibull distribution which can be readily applied to a wide range of situations including applications in medicine, reliability and ecology. Keller [16] obtained the inverse Weibull model by investigating failures of mechanical components subject to degradation. Calabria and Pulcini [4] obtained the maximum likelihood and least squares estimates of the parameters of the inverse Weibull distribution. They also obtained the Bayes estimator of the parameters. See Johnson [14] for details. The inverse Weibull (IW) cumulative distribution function (cdf) is given by

$$F(x, \alpha, \beta) = exp\Big[- (\alpha(x - x_0))^{-\beta}\Big], \qquad x \geq 0,\ \alpha > 0,\ \beta > 0, \tag{2.11}$$

where α, x_0 and β are the scale, location and shape parameters respectively. Often the parameter x_0 is called the minimum life or guarantee time. When $\alpha = 1$ and $x = x_0 + \alpha$, then $F(\alpha + x_0; 1; \beta) = F(\alpha + x_0; 1) = e^{-1} = 0.3679$. This value is in fact the characteristic life of the distribution. In what follows, we assume that $x_0 = 0$, and the inverse cumulative Weibull distribution function becomes

$$F(x, \alpha, \beta) = exp[-(\alpha x)^{-\beta}], \qquad x \geq 0,\ \alpha > 0,\ \beta > 0. \tag{2.12}$$

Note that when $\alpha = 1$, we have the Frechet distribution function. The inverse Weibull probability density function (pdf) is given by

$$f(x, \alpha, \beta) = \beta\alpha^{-\beta}x^{-\beta-1}exp[-(\alpha x)^{-\beta}], \qquad x \geq 0,\ \alpha > 0,\ \beta > 0. \tag{2.13}$$

When $\beta = 1$ and $\beta = 2$, the inverse Weibull distribution pdfs are referred to as the inverse exponential and inverse Raleigh pdfs respectively.

2.3.1 Moments

In this subsection, we present the raw moments, mean, variance, coefficients of variation, skewness and kurtosis for the inverse Weibull distribution. See Johnson [14].

The k^{th} raw or non central moments are given by

$$E(X^k) = \int_0^\infty x^k f(x;\alpha,\beta)\,dx = \int_0^\infty \beta(\alpha x)^{-\beta} x^{k-1} e^{-(\alpha x)^{-\beta}}\,dx.$$

Let $(\alpha x)^{-\beta} = t$, then $x = \alpha^{-1}t^{-1/\beta}$ and $dx = -\alpha^{-1}\beta^{-1}t^{-1/\beta-1}\,dt$

so that $t = (\alpha x)^{-\beta}|_{x=0} = \infty$, and $t = (\alpha x)^{-\beta}|_{x=\infty} = 0$.

Consequently,

$$\begin{aligned} E(X^k) &= \int_0^\infty \beta t e^{-t}(-\alpha^{-k}\beta^{-1}t^{-k/\beta-1})\,dt \\ &= \alpha^{-k}\int_0^\infty t^{-k/\beta}e^{-t}\,dt = \frac{\Gamma(1-k/\beta)}{\alpha^k}, \quad \text{for} \quad \beta > k. \end{aligned} \tag{2.14}$$

Note that $E(X^k)$ does not exist, when $\beta \le k$.

The mean is given by

$$\mu_F = E(X) = \frac{\Gamma(1-1/\beta)}{\alpha}.$$

The variance σ^2, coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \frac{\Gamma(1-\frac{2}{\beta}) - \Gamma^2(1-\frac{1}{\beta})}{\alpha^2}, \tag{2.15}$$

$$CV = \frac{\sigma}{\mu} = \frac{[\Gamma(1-\frac{2}{\beta}) - \Gamma^2(1-\frac{1}{\beta})]^{1/2}}{\Gamma(1-\frac{1}{\beta})} = \frac{(\delta_2 - \delta_1^2)^{1/2}}{\delta_1}, \tag{2.16}$$

$$\begin{aligned} CS &= \frac{E(X-\mu)^3}{[E(X-\mu)^2]^{3/2}} \\ &= \frac{\Gamma(1-\frac{3}{\beta}) - 3\Gamma(1-\frac{1}{\beta})\Gamma(1-\frac{2}{\beta}) + 2\Gamma^3(1-\frac{1}{\beta})}{[\Gamma(1-\frac{2}{\beta}) - \Gamma(1-\frac{1}{\beta})^2]^{3/2}} \\ &= \frac{\delta_3 - 3\delta_1\delta_2 + 2\delta_1^3}{(\delta_2-\delta_1^2)^{3/2}}, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} CK &= \frac{E(X-\mu)^4}{[E(X-\mu)^2]^2} \\ &= \frac{\Gamma(1-\frac{4}{\beta}) - 4\Gamma(1-\frac{1}{\beta})\Gamma(1-\frac{3}{\beta}) + 6\Gamma^2(1-\frac{1}{\beta})\Gamma(1-\frac{2}{\beta}) - 3\Gamma^4(1-\frac{1}{\beta})}{[\Gamma(1-\frac{2}{\beta}) - \Gamma(1-\frac{1}{\beta})^2]^2} \\ &= \frac{\delta_4 - 4\delta_1\delta_3 + 6\delta_1^2\delta_2 - 3\delta_1^4}{(\delta_2-\delta_1^2)^2}, \end{aligned} \tag{2.18}$$

where $\delta_k = \Gamma(1-k/\beta)$.

2.3.2 Some Basic Properties

In this subsection, some basic properties involving the reverse hazard function as well as stochastic order relations in the inverse Weibull distribution are established. Stochastic and reverse order relations between two inverse Weibull distributions F and G are presented. The first result gives the length-biased pdf in terms of the corresponding parent cumulative distribution function. The second result gives the reverse hazards function and the third result deals with stochastic and reverse order relations.

1. $xf(x;\alpha,\beta) = \beta F(x;\alpha,\beta)(-ln[F(x;\alpha,\beta)])$

   Proof. With $F(x;\alpha,\beta) = e^{-(\alpha x)^{-\beta}}$, we have

$$f(x;\alpha,\beta) = F'(x;\alpha,\beta) = e^{-(\alpha x)^{-\beta}}\beta\alpha(\alpha x)^{-\beta-1},$$

so that

$$xf(x;\alpha,\beta) = e^{-(-\alpha x)^{-\beta}}\beta(\alpha x)^{-\beta} = \beta e^{-(\alpha x)^{-\beta}}(\alpha x)^{-\beta}.$$

Also,

$$lnF(x;\alpha,\beta) = ln(e^{-(\alpha x)^{-\beta}}) = -(\alpha x)^{-\beta} \Rightarrow -lnF(x;\alpha,\beta) = (\alpha x)^{-\beta}.$$

Therefore,

$$xf(x;\alpha,\beta) = \beta F(x;\alpha,\beta)(-lnF(x;\alpha,\beta)),$$

and the length-biased pdf is

$$\frac{xf(x;\alpha,\beta)}{\mu} = \frac{\alpha\beta F(x;\alpha,\beta)(-ln[F(x;\alpha,\beta)])}{\Gamma(1-1/\beta)}.$$

□

2. **Reverse Hazard Function.**

The reverse hazard function is given by:

$$\tau_F(x;\alpha,\beta) = \frac{f(x;\alpha,\beta)}{F(x;\alpha,\beta)} = -\frac{\beta}{x}lnF(x;\alpha,\beta)$$

Proof.

$$\begin{aligned} \tau_F(x;\alpha,\beta) &= \frac{xf(x;\alpha,\beta)}{xF(x;\alpha,\beta)} \\ &= \frac{\beta F(x;\alpha,\beta)(-lnF(x;\alpha,\beta))}{xF(x;\alpha,\beta)} \\ &= -\frac{\beta}{x}lnF(x;\alpha,\beta). \end{aligned} \tag{2.19}$$

□

3. **Stochastic and Reverse Hazard Order Relations.**

Consider the inverse Weibull cumulative distribution functions given by $F(x;\alpha_1,\beta_1) = e^{-(\alpha_1 x)^{-\beta_1}}$ and $G(x;\alpha_2,\beta_2) = e^{-(\alpha_2 x)^{-\beta_2}}$ respectively.

Note that by stochastic order relations we have $F >_{st} G$, if $\bar{F}(x) \geq \bar{G}(x)$, for any x.

(a) Now, let $\beta_1 = \beta_2$, then $\bar{F}(x;\alpha_1,\beta_1) \leq \bar{G}((x;\alpha_2,\beta_2)$ if $\alpha_1 \geq \alpha_2$.

(b) $\tau_{_F}(x;\alpha_1,\beta_1) \leq \tau_{_G}(x;\alpha_2,\beta_2)$, if and only if $F(x;\alpha_1,\beta_1) \geq G(x;\alpha_2,\beta_2)$, and $\beta_1 \geq \beta_2$.

Proof. Note that $\tau_{_F}(x;\alpha_1,\beta_1) = \frac{f(x;\alpha_1,\beta_1)}{F(x;\alpha_1,\beta_1)} = \alpha_{_1}\beta_{_1}(\alpha_{_1}x)^{-(\beta_1+1)}$ and $\tau_{_G}(x;\alpha_2,\beta_2) = \frac{g(x;\alpha_2,\beta_2)}{G(x;\alpha_2,\beta_2)} = \alpha_{_2}\beta_{_2}(\alpha_{_2}x)^{-(\beta_2+1)}$, so that for fixed $\beta_1 = \beta_2 > 0$, $\tau_{_F}(x;\alpha_1,\beta_1 \leq \tau_{_G}(x;\alpha_2,\beta_2)$ if $\alpha_{_1} \geq \alpha_{_2}$.
Also, note that $xf(x;\alpha,\beta) = \beta F(x;\alpha,\beta)(-lnF(x;\alpha,\beta))$,
so that $\tau_{_F}(x;\alpha_1,\beta_1) \leq \tau_{_G}(x;\alpha_2,\beta_2)$, if and only if
$F(x;\alpha_1,\beta_1) \geq G(x;\alpha_2,\beta_2)$, and $\beta_1 \geq \beta_2$. □

2.4 Weighted Distributions

Weighted distributions occur in various areas including medicine, ecology, reliability and branching processes. Results on applications in these and other areas can be seen in Patil and Rao [20], Gupta and Keating [12], Oluyede [19] and in other references therein.

Recall that in a weighted distribution problem, a realization x of X enters into the investigators record with probability proportional to a weight function $w(x)$. The recorded x is not an observation of X, but rather an observation on a weighted random variable X_w.

Let X be a nonnegative random variable with distribution function $F(x)$ and probability density function (pdf) $f(x)$. Let $w(x)$ be a positive weight function such that $0 < E(w(X)) < \infty$. The pdf of the weighted random variable X_w is given by

$$f_w(x) = \frac{w(x)f(x)}{E(w(X))}, \qquad x \geq 0, \text{ and } 0 < E(w(X)) < \infty. \tag{2.20}$$

The corresponding weighted survival or reliability function of X_w is given by

$$\overline{F}_w(x) = \frac{E_F[w(X)|X > x]}{E_F[w(X)]}\overline{F}(x), \qquad x \geq 0, \text{ and } 0 < E(W(X)) < \infty. \tag{2.21}$$

If the weight function is monotone increasing and concave, then the weighted distribution of an increasing failure rate (IFR) distribution is an IFR distribution. Also, the size-biased distribution of a decreasing mean residual (DMRL) distribution has decreasing mean residual life. The residual life at age t, is a weighted distribution, with survival function given by

$$\overline{F}_t(x) = \frac{\overline{F}(x+t)}{\overline{F}(t)}, \tag{2.22}$$

for $x \geq 0$. The weight function is $w(x) = \frac{f(x+t)}{f(x)}$, where $f(u) = \frac{dF(u)}{du}$, the hazard function and mean residual life functions are $\lambda_{F_t}(x) = \lambda_F(x+t)$ and $\delta_{F_t}(x) = \delta_F(x+t)$. It is clear that if F is IFR (DMRL) distribution, then F_t is IFR (DMRL) distribution, where the hazard function $\lambda_F(x)$ and mean residual life function $\delta_F(x)$ of the distribution function F are given by $\lambda_F(x) = f(x)/\overline{F}(x)$, and $\delta_F(x) = \int_x^\infty \overline{F}(u)du/\overline{F}(x)$ respectively.

2.5 Weighted Inverse Weibull Distribution

In this section, we present the weighted inverse Weibull distribution, the corresponding reliability function, hazard and reverse hazard functions, and apply Glaser [11] result to study the behavior of the hazard function.

Consider the weight function $w(x) = x$ and inverse Weibull distribution given by

$$f(x;\alpha,\beta) = \beta\alpha^{-\beta}x^{-\beta-1}e^{-(\alpha x)^{-\beta}}, \quad x \geq 0,\ \alpha > 0,\ \beta > 0.$$

The length-biased inverse Weibull (LBIW) distribution is given by

$$\begin{aligned} g_w(x;\alpha,\beta) &= \frac{xf(x;\alpha,\beta)}{\mu_F} = \frac{x\beta\alpha^{-\beta}x^{-(\beta+1)}e^{-(\alpha x)^{-\beta}}}{\frac{\Gamma(1-1/\beta)}{\alpha}} \\ &= \frac{\beta\alpha^{-\beta+1}}{\Gamma(1-1/\beta)}x^{-\beta}e^{-(\alpha x)^{-\beta}}, \quad x \geq 0,\ \alpha > 0,\ \beta > 1. \end{aligned} \tag{2.23}$$

Note that

$$\lim_{x\to 0} g_w(x;\alpha,\beta) = \frac{\beta\alpha^{-\beta+1}x^{-\beta}}{\Gamma(1-\frac{1}{\beta})}exp(-(\alpha x)^{-\beta}) = 0, \tag{2.24}$$

and by letting $u = \alpha x$, we have

$$\lim_{x\to\infty} g_w(x) = \lim_{u\to\infty} \frac{\beta\alpha u^{-\beta}}{\Gamma(1-\frac{1}{\beta})}exp(-(u)^{-\beta}) = 0. \tag{2.25}$$

Note that $\frac{\partial(ln(g_w(x)))}{\partial x} = 0$ implies $\beta(\frac{(\alpha x)^{-\beta}-1}{x}) = 0$, so that $g_w(x)$ has a maximum at $x_0 = \alpha_0^{-1}$. Hence $g_w(x)$ increases to maximum at x_0 and then decreases. Also

$$\frac{\partial^2(ln(g_w(x)))}{\partial x^2} = -\beta\Big((\beta+1)\alpha^2(\alpha x)^{-\beta-2} - x^{-2}\Big) < 0, \tag{2.26}$$

all values of x.

Plots of the length-biased inverse Weibull (LBIW) probability density function are given below when either the parameter α or the parameter β is fixed.

- fix parameter β: see fig2.1
- fix parameter α: see fig2.2

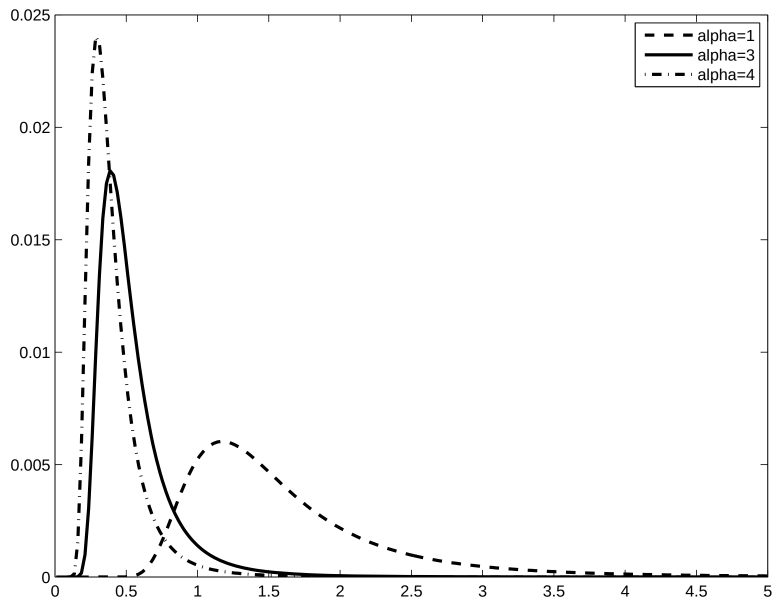

Figure 2.1: plot of $g_w(x; \alpha, \beta)$ with fixed β

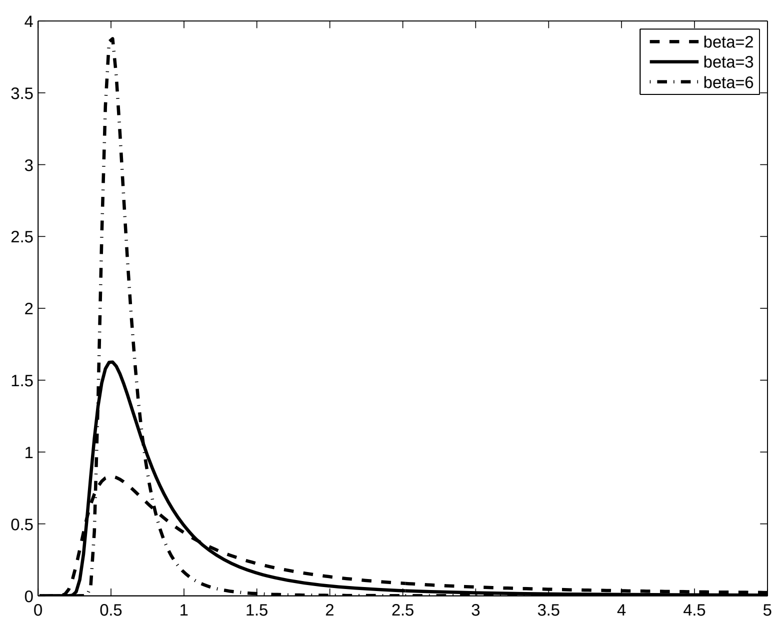

Figure 2.2: plot of $g_w(x; \alpha, \beta)$ with fixed α

If the weight function is $w(x;c) = x^c$, for $c \geq 1$, then the weighted inverse Weibull probability density function is given by

$$g_w(x;\alpha,\beta,c) = \frac{\beta\alpha^{c-\beta}}{\Gamma(1-c/\beta)} x^{c-\beta-1} e^{-(\alpha x)^{-\beta}}, \quad x \geq 0,\ \alpha > 0,\ \beta > c. \tag{2.27}$$

Now, the length-biased cdf is given by

$$G_w(x;\alpha,\beta) = \int_0^x g_w(t;\alpha,\beta)\,dt = \frac{\beta\alpha^{-\beta+1}}{\Gamma(1-1/\beta)} \int_0^x t^{-\beta} e^{-(\alpha x)^{-\beta}}\,dt$$

and the reliability or survival function is given by

$$\begin{aligned} \bar{G}_w(x;\alpha,\beta) &= 1 - G_w(x;\alpha,\beta) \\ &= 1 - \frac{\beta\alpha^{-\beta+1}}{\Gamma(1-1/\beta)} \int_0^x t^{-\beta} e^{-(\alpha t)^{-\beta}}\,dt \\ &= \frac{\beta\alpha^{-\beta+1}}{\Gamma(1-1/\beta)} \int_x^\infty t^{-\beta} e^{-(\alpha t)^{-\beta}}\,dt. \end{aligned} \tag{2.28}$$

The hazard function is given by

$$\begin{aligned} \lambda_{G_w}(x;\alpha,\beta) &= \frac{g_w(x;\alpha,\beta)}{\bar{G}_w(x;\alpha,\beta)} \\ &= \frac{x^{-\beta} e^{-(\alpha x)^{-\beta}}}{\int_x^\infty t^{-\beta} e^{-(\alpha t)^{-\beta}}\,dt}. \end{aligned} \tag{2.29}$$

The reverse hazard function is given by

$$\tau_{G_w}(x;\alpha,\beta) = \frac{g_w(x)}{G_w(x)} = \frac{\frac{\beta\alpha^{1-\beta}}{\Gamma(1-1/\beta)} x^{-\beta} e^{-(\alpha x)^{-\beta}}}{\frac{\beta\alpha^{1-\beta}}{\Gamma(1-1/\beta)} \int_0^x t^{-\beta} e^{-(\alpha t)^{-\beta}}\,dt} = \frac{x^{-\beta} e^{-(\alpha x)^{-\beta}}}{\int_0^x t^{-\beta} e^{-(\alpha t)^{-\beta}}\,dt}. \tag{2.30}$$

We study the behavior of the hazard function of the LBIW distribution via the following lemma, due to Glaser [11].

Lemma 2.5.1. *Let $f(x)$ be a twice differentiable probability density function of a continuous random variable X. Define $\eta(x) = \frac{-f'(x)}{f(x)}$, where $f'(x)$ is the first derivative of $f(x)$ with respect to x. Furthermore, suppose the first derivative of $\eta(x)$ exist.*

1. If $\eta'(x) < 0$, for all $x > 0$, then the hazard function is monotonically decreasing (DFR).

2. If $\eta'(x) > 0$, for all $x > 0$, then the hazard function is monotonically increasing (IFR).

3. If there exist x_0 such that $\eta'(x) > 0$, for all $0 < x < x_0$, $\eta'(x_0) = 0$ and $\eta'(x) < 0$ for all $x > x_0$. In addition, $\lim_{x\to 0} f(x) = 0$, then the hazard function is upside down bathtub shape (UBT).

4. If there exist x_0 such that $\eta'(x) < 0$, for all $0 < x < x_0$, $\eta'(x_0) = 0$ and $\eta'(x) > 0$ for all $x > x_0$. In addition, $\lim_{x\to 0} f(x) = \infty$, then the hazard function is bathtub shape (BT).

Now consider the weighted distribution discussed above. We compute the quantity $\eta(x) = \frac{-g'(x)}{g(x)}$, and apply Glaser [11] result. Note that

$$g'_w(x) = g_w(x)\{\alpha\beta(\alpha x)^{-\beta-1} - \beta x^{-1}\}, \tag{2.31}$$

so that

$$\eta(x) = \frac{-g'_w(x)}{g_w(x)} = \beta x^{-1} - \alpha\beta(\alpha x)^{-\beta-1}, \tag{2.32}$$

and

$$\eta'(x) = \frac{\beta}{x^2}\{(\alpha x)^{-\beta}(1+\beta) - 1\}. \tag{2.33}$$

Since $\alpha > 0$, $\beta > 1$ and $x > 0$, we have $\eta'(x) > 0$ if $(\alpha x)^{-\beta}(1+\beta) - 1 > 0$.

Theorem 2.5.2. *Let $\beta > 1$, and $x^* = \frac{(1+\beta)^{1/\beta}}{\alpha}$, then $\eta'(x) = 0$ if $x = x^*$, $\eta'(x) > 0$ if $x < x^*$, and $\eta'(x) < 0$ if $x > x^*$.*

Proof: The results in the theorem follows from the fact that $\eta'(x) = 0$ implies $x = \frac{(1+\beta)^{1/\beta}}{\alpha}$.

2.6 Concluding Remarks

In chapter 2, the weighted inverse Weibull distribution was presented. At first, we introduced four functions that characterize the distribution of a random variable X, namely the survival function, the probability density or mass function, the failure rate function or hazard function and the mean residual life. Then we gave a simple review on the Weibull distribution and the inverse Weibull distribution. We also presented the moments $E(X^k)$, the coefficient of variation CV, the coefficient of skewness CS and the coefficient of Kurtosis CK of the inverse Weibull distribution as well as some basic properties of this distribution. A brief discussion of weighted distributions was presented followed by an introduction of the weighted inverse Weibull distribution. We computed the cdf $G_w(x;\alpha,\beta)$, reliability or survival function $\bar{G}_w(x;\alpha,\beta)$, hazard function $\lambda_{G_w}(x;\alpha,\beta)$, reverse hazard function $\tau_{G_w}(x;\alpha,\beta)$ and found that the hazard function of $g_w(x;\alpha,\beta)$, (using the function $\eta(x) = \frac{-g'_w(x)}{g_w(x)}$,) is upside down bathtub shape on applying the Glaser's [11] result. In chapter 3, we will continue discussing more properties of the weighted inverse Weibull distribution including the moments, moment generating function, the variance, the coefficient of variation CV, the coefficient of skewness CS, and the coefficient of Kurtosis CK. We will also present results on Fisher information and Shanon entropy.

CHAPTER 3

MOMENTS AND MOMENT GENERATING FUNCTION, FISHER INFORMATION AND SHANON ENTROPY

3.1 Introduction

This chapter contains the moments, moment generating function as well as the mean, variance, coefficients of variation, skewness, and kurtosis for the weighted inverse Weibull distribution. Also, the Fisher information and the Shanon entropy for the weighted inverse Weibull distribution are presented.

3.2 Moments and Moment Generating Function

3.2.1 Moments

The moments of the length-biased random variable Y are related to those of the original or parent random variable X by

$$E_{G_w}(Y^k) = \frac{E_F(X^{k+1})}{E_F(X)}, \qquad k = 1, 2, ..., \text{ provided } E_F(X^{k+1}) \text{ exists.} \tag{3.1}$$

Noting that the moments of F are given by

$$E_F(X^k) = \gamma_k = \frac{\Gamma(1 - \frac{k}{\beta})}{\alpha^k}, \qquad k \geq 1,\ \beta > k, \tag{3.2}$$

we obtain the moments of Y as follows:

$$E_{G_w}(Y^k) = \frac{\Gamma(1 - \frac{(k+1)}{\beta})}{\alpha^k \Gamma(1 - \frac{1}{\beta})} = \frac{\gamma_{k+1}}{\gamma_1}, \qquad k \geq 1,\ \beta > k. \tag{3.3}$$

The mean and variance of Y are given by

$$\mu_{G_w} = E_{G_w}(Y) = \frac{\Gamma(1 - \frac{2}{\beta})}{\alpha \Gamma(1 - \frac{1}{\beta})} = \frac{\gamma_2}{\gamma_1}, \tag{3.4}$$

and

$$\sigma^2_{G_w} = EY^2_{G_w} - [E_{G_w}(Y)]^2 = \frac{\Gamma(1-\frac{3}{\beta})}{\alpha^2\Gamma(1-\frac{1}{\beta})} - \left\{\frac{\Gamma(1-\frac{2}{\beta})}{\Gamma(1-\frac{1}{\beta})}\right\}^2 = \frac{\gamma_1\gamma_3 - \gamma_2^2}{\gamma_1^2} \qquad (3.5)$$

respectively.

The coefficient of variation (CV) is given by

$$\begin{aligned} CV &= \frac{\sigma_{G_w}}{\mu_{G_w}} = \frac{\sqrt{\frac{\Gamma(1-\frac{3}{\beta})}{\alpha^2\Gamma(1-\frac{1}{\beta})} - \left\{\frac{\Gamma(1-\frac{2}{\beta})}{\alpha\Gamma(1-\frac{1}{\beta})}\right\}^2}}{\frac{\Gamma(1-\frac{2}{\beta})}{\alpha\Gamma(1-\frac{1}{\beta})}} \\ &= \sqrt{\frac{\gamma_3\gamma_1}{\gamma_2^2} - 1}, \qquad (3.6) \end{aligned}$$

where $\gamma_k = \frac{\Gamma(1-\frac{k}{\beta})}{\alpha^k}$.

The coefficients of skewness (CS) and kurtosis (CK) are given by

$$CS = \frac{E(Y-\mu_{G_w})^3}{(E(Y-\mu_{G_w})^2)^{3/2}} = \frac{\gamma_1^2\gamma_4 - 3\gamma_1\gamma_2\gamma_3 + 2\gamma_2^3}{(\gamma_1\gamma_3 - \gamma_2^2)^{3/2}} \qquad (3.7)$$

and

$$CK = \frac{E(Y-\mu_{G_w})^4}{(E(Y-\mu_{G_w})^2)^2} = \frac{\gamma_1^3\gamma_5 - 4\gamma_1^2\gamma_2\gamma_4 + 6\gamma_1\gamma_2^2\gamma_3 - 3\gamma_2^4}{\gamma_1^2\gamma_3^2 - 2\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4} \qquad (3.8)$$

respectively.

The Table 3.1 shows the mean, standard deviation (STD), coefficient of variation (CV), coefficient of Skewness (CS) and coefficient of Kurtosis (CK) with some values of the parameters α and β.

Note that CV, CS and CK do not depend on the parameter α. Plots for CV, CS, CK as functions of β are presented below. (see fig3.1, fig3.2 and fig 3.3)

α	β	Mean	STD	CV	CS	CK
1	3	1.9184	Inf	Inf	NaN	NaN
	5	1.2791	0.5188	0.4056	5.8578	Inf
	8	1.1246	0.2276	0.2023	2.5156	18.5192
	10	1.0895	0.1666	0.1529	2.0949	12.7770
2	3	0.9892	Inf	Inf	NaN	NaN
	5	0.6396	0.2594	0.4056	5.8578	Inf
	8	0.5623	0.1138	0.2023	2.5156	18.5192
	10	0.5447	0.0833	0.1529	2.0949	12.7770
5	3	0.3957	Inf	Inf	NaN	NaN
	5	0.2558	0.1038	0.4056	5.8578	Inf
	8	0.2249	0.0455	0.2023	2.5156	18.5192
	10	0.2179	0.0333	0.1529	2.0949	12.7770
10	3	0.1978	Inf	Inf	NaN	NaN
	5	0.1279	0.0519	0.4056	5.8578	Inf
	8	0.1125	0.0228	0.2023	2.5156	18.5192
	10	0.1089	0.0167	0.1529	2.0949	12.7770

Inf: ∞,   NaN: Not Defined

Table 3.1: Table of Mean, STD, Coefficients of Variation, Skewness and Kurtosis

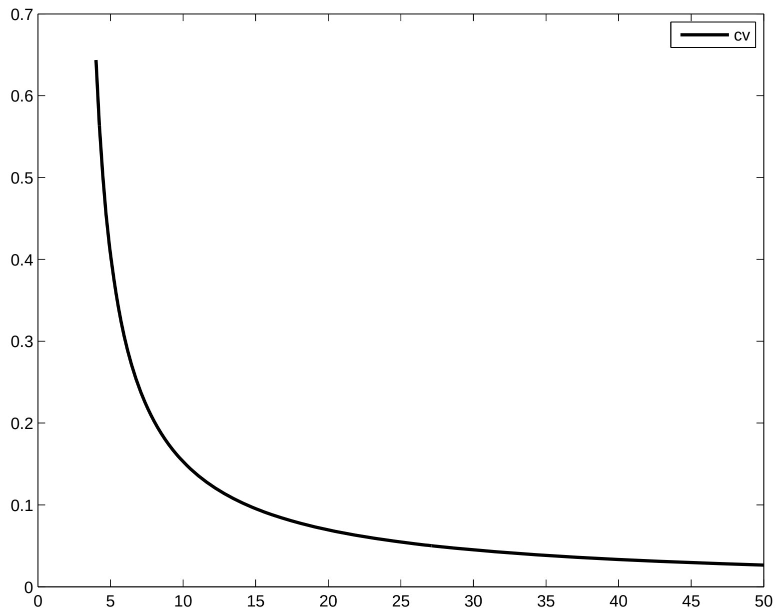

Figure 3.1: plot of CV against β

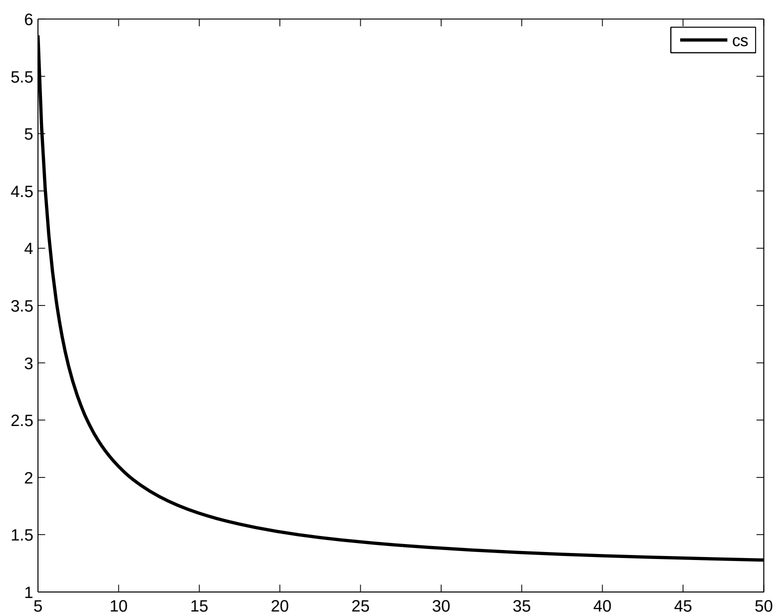

Figure 3.2: plot of CS against β

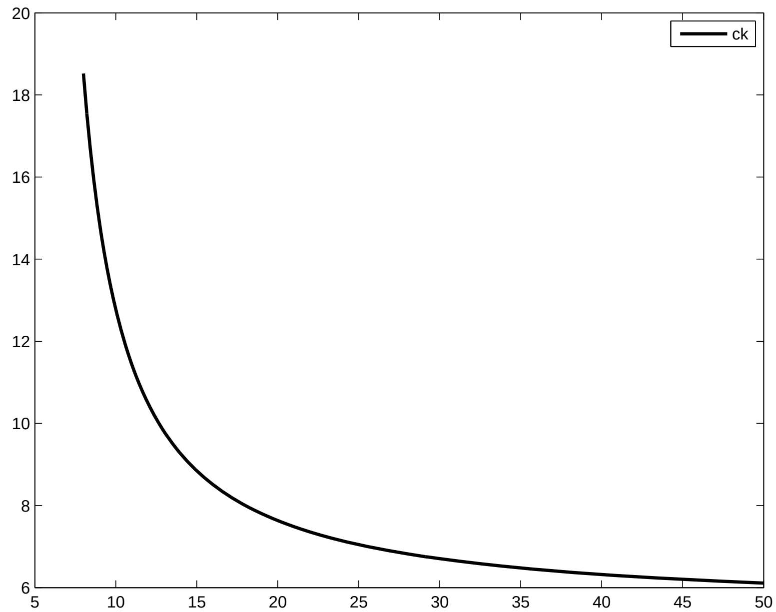

Figure 3.3: plot of CK against β

3.2.2 Moment Generating Function

The moment generating function of the length-biased inverse Weibull distribution is given by

$$\begin{aligned} M_Y(t) &= \frac{\beta\alpha^{-\beta+1}}{\Gamma(1-1/\beta)}\int_0^\infty e^{ty}y^{-\beta}e^{-(\alpha y)^{-\beta}}dy \\ &= \frac{\beta\alpha^{-\beta+1}}{\Gamma(1-1/\beta)}\sum_{j=0}^{\infty}\frac{t^j}{j!}\int_0^\infty y^{j-\beta}e^{-(\alpha y)^{-\beta}}dy \\ &= \frac{1}{\Gamma(1-1/\beta)}\sum_{j=0}^{\infty}\frac{t^j}{j!\alpha^j}\int_0^\infty t^{-(j+1)/\beta}e^{-t}dt \\ &= \frac{1}{\Gamma(1-1/\beta)}\sum_{j=0}^{\infty}\frac{t^j}{j!}\frac{\Gamma(1-\frac{j+1}{\beta})}{\alpha^j} \qquad (3.9) \\ &= \frac{\alpha}{\Gamma(1-1/\beta)}\sum_{j=0}^{\infty}\frac{t^j}{j!}\gamma_{j+1}, \qquad (3.10) \end{aligned}$$

for $\beta > j+1$.

3.3 Fisher Information and Shanon Entropy

The information (or Fisher information) that a random variable X contains about the parameter θ is given by

$$I(\theta) = E\left[\frac{\partial}{\partial\theta}log(f(X,\theta))\right]^2. \tag{3.11}$$

Now, if in addition, the second derivative with respect to θ of $f(x,\theta)$ exits for all x and θ and the second derivative with respect to θ of $\int f(x,\theta)dx = 1$ can be obtained by differentiating twice under the integral sign, then

$$I(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}log(f(X,\theta))\right]. \tag{3.12}$$

For the weighted inverse Weibull distribution, the Fisher information (FI) that X contains about the parameters $\theta = (\alpha, \beta)$ is obtained as follows:

$$\begin{aligned}
E\left\{\frac{\partial log(g_w(x;\alpha,\beta))}{\partial\alpha}\right\}^2 &= \int_0^\infty \left(\frac{1-\beta}{\alpha} + \beta\alpha^{-\beta-1}x^{-\beta}\right)^2 g_w(x;\alpha,\beta)\,dx \\
&= (1-\beta)^2\alpha^{-2}\int_0^\infty g_w(x;\alpha,\beta)\,dx \\
&+ \frac{2\beta^2(1-\beta)\alpha^{-2\beta-1}}{\Gamma(1-1/\beta)}\int_0^\infty x^{-2\beta}e^{-(\alpha x)^{-\beta}}\,dx \\
&+ \frac{\beta^3\alpha^{-3\beta-1}}{\Gamma(1-1/\beta)}\int_0^\infty x^{-3\beta}e^{-(\alpha x)^{-\beta}}\,dx \\
&= (1-\beta)^2\alpha^{-2} + \frac{2\beta(1-\beta)\alpha^{-2}}{\Gamma(1-1/\beta)}\Gamma(2-1/\beta) \\
&+ \frac{\beta^2\alpha^{-2}}{\Gamma(1-1/\beta)}\Gamma(3-1/\beta) \\
&= (1-\beta)^2\alpha^{-2} + 2(1-\beta)\beta\alpha^{-2}(1-1/\beta) \\
&+ \beta^2\alpha^{-2}(2-1/\beta)(1-1/\beta) \\
&= \beta(\beta-1)\alpha^{-2}.
\end{aligned} \tag{3.13}$$

Also,

$$
\begin{aligned}
E & \left\{\frac{\partial log(g_w(x;\alpha,\beta))}{\partial\beta}\right\}^2 \\
= & \int_0^\infty \{\frac{1}{\beta} - \frac{\Gamma'(1-1/\beta)}{\beta^2\Gamma(1-1/\beta)} + \log(\alpha x)[(\alpha x)^{-\beta}-1]\}^2 g_w(x;\alpha,\beta)\,dx \\
= & [\frac{1}{\beta} - \frac{\Gamma'(1-1/\beta)}{\beta^2\Gamma(1-1/\beta)}]^2 \int_0^\infty g_w(x;\alpha,\beta)\,dx \\
+ & 2[\frac{1}{\beta} - \frac{\Gamma'(1-1/\beta)}{\beta^2\Gamma(1-1/\beta)}] \int_0^\infty \log(\alpha x)[(\alpha x)^{-\beta}-1] g_w(x;\alpha,\beta)\,dx \\
+ & \int_0^\infty [\log(\alpha x)]^2[(\alpha x)^{-\beta}-1]^2 g_w(x;\alpha,\beta)\,dx \\
= & \left[\frac{1}{\beta} - \frac{\Gamma'(1-1/\beta)}{\beta^2\Gamma(1-1/\beta)}\right]^2 \\
- & 2\left[\frac{1}{\beta} - \frac{\Gamma'(1-1/\beta)}{\beta^2\Gamma(1-1/\beta)}\right]\frac{[\Gamma'(2-1/\beta)-\Gamma'(1-1/\beta)]}{\beta\Gamma(1-1/\beta)} \\
+ & \frac{\beta^2[\Gamma''(3-1/\beta)-2\Gamma''(2-1/\beta)+\Gamma''(1-1/\beta)]}{\Gamma(1-1/\beta)},
\end{aligned}
\tag{3.14}
$$

and,

$$
\begin{aligned}
E & \Big[\frac{\partial^2 \log\big(g_w(x;\alpha,\beta)\big)}{\partial\alpha\partial\beta}\Big] \\
= & E\Big[\frac{\partial^2 \log\big(g_w(x;\alpha,\beta)\big)}{\partial\beta\partial\alpha}\Big] \\
= & \int_0^\infty \Big[\alpha^{-\beta-1}x^{-\beta}(1-\beta\log\alpha-\beta\log x) - \frac{1}{\alpha}\Big] g_w(x;\alpha,\beta)\,dx \\
= & \alpha^{-\beta-1}(1-\beta\log\alpha)\int_0^\infty x^{-\beta} g_w(x;\alpha,\beta)\,dx \\
- & \frac{\alpha^{-2\beta}\beta^2}{\Gamma(1-1/\beta)}\int_0^\infty x^{-2\beta}\log x e^{-(\alpha x)^{-\beta}}\,dx - \frac{1}{\alpha}\int_0^\infty g_w(x;\alpha,\beta)\,dx \\
= & \alpha^{-\beta-1}(1-\beta\log\alpha)[(1-1/\beta)\alpha^{\beta}] - \frac{1}{\alpha} \\
+ & \frac{\alpha^{-1}\beta\log\alpha}{\Gamma(1-1/\beta)}\Gamma(2-1/\beta) + \frac{\alpha^{-1}}{\Gamma(1-1/\beta)}\Gamma'(2-1/\beta) \\
= & \alpha^{-1}\beta^{-2}(1-\beta) + \alpha^{-1}\beta^{-3}(\beta-1)\frac{\Gamma'(1-1/\beta)}{\Gamma(1-1/\beta)} \\
= & \frac{\beta-1}{\alpha\beta^3}\left[\frac{\Gamma'(1-1/\beta)}{\Gamma(1-1/\beta)} - \beta\right].
\end{aligned}
\tag{3.15}
$$

Therefore,

$$
\begin{aligned}
I(\alpha,\beta) &= \begin{bmatrix} E\Big[\dfrac{\partial \log\big(g_w(x;\alpha,\beta)\big)}{\partial\alpha}\Big]^2 & E\Big[\dfrac{\partial^2 \log\big(g_w(x;\alpha,\beta)\big)}{\partial\alpha\partial\beta}\Big] \\ E\Big[\dfrac{\partial^2 \log\big(g_w(x;\alpha,\beta)\big)}{\partial\beta\partial\alpha}\Big] & E\Big[\dfrac{\partial \log\big(g_w(x;\alpha,\beta)\big)}{\partial\beta}\Big]^2 \end{bmatrix} \\
&= \begin{bmatrix} E\Big[\dfrac{\partial \log\big(g_w(x;\alpha,\beta)\big)}{\partial\alpha}\Big]^2 & \dfrac{\beta-1}{\alpha\beta^3}\Big[\dfrac{\Gamma'(1-1/\beta)}{\Gamma(1-1/\beta)}-\beta\Big] \\ \dfrac{\beta-1}{\alpha\beta^3}\Big[\dfrac{\Gamma'(1-1/\beta)}{\Gamma(1-1/\beta)}-\beta\Big] & E\Big[\dfrac{\partial \log\big(g_w(x;\alpha,\beta)\big)}{\partial\beta}\Big]^2 \end{bmatrix}, \qquad (3.16)
\end{aligned}
$$

where $E\Big\{\frac{\partial log(g_w(x;\alpha,\beta))}{\partial\alpha}\Big\}^2$ and $E\Big\{\frac{\partial log(g_w(x;\alpha,\beta))}{\partial\beta}\Big\}^2$ are given by equations (3.13) and (3.14) respectively.

Consider $g_w(x;\alpha_1,\beta_1)$ and $g_w(x;\alpha_2,\beta_2)$. Note that for $\beta_1=\beta_2$,

$$
\begin{aligned}
& E\Big\{\frac{\partial log(g_w(x;\alpha_1,\beta_1))}{\partial\alpha_1}\Big\}^2 \geq E\Big\{\frac{\partial log(g_w(x;\alpha_2,\beta_2))}{\partial\alpha_2}\Big\}^2 \\
\Longleftrightarrow\ & \frac{\beta_1(\beta_1-1)}{\alpha_1^2} \geq \frac{\beta_2(\beta_2-1)}{\alpha_2^2} \\
\Longleftrightarrow\ & \alpha_2^2 \geq \alpha_1^2 \\
\Longleftrightarrow\ & \alpha_2 \geq \alpha_1. \qquad (3.17)
\end{aligned}
$$

Similarly, for $\alpha_1=\alpha_2$,

$$
\begin{aligned}
& E\Big\{\frac{\partial log(g_w(x;\alpha_1,\beta_1))}{\partial\alpha_1}\Big\}^2 \geq E\Big\{\frac{\partial log(g_w(x;\alpha_2,\beta_2))}{\partial\alpha_2}\Big\}^2 \\
\Longleftrightarrow\ & \beta_1(\beta_1-1) \geq \beta_2(\beta_2-1) \\
\Longleftrightarrow\ & \beta_1^2-\beta_2^2-(\beta_1-\beta2) \geq 0 \\
\Longleftrightarrow\ & (\beta_1-\beta_2)(\beta_1+\beta_2-1) \geq 0. \qquad (3.18)
\end{aligned}
$$

Consequently,

$$
E\Big\{\frac{\partial log(g_w(x;\alpha_1,\beta_1))}{\partial\alpha_1}\Big\}^2 \geq E\Big\{\frac{\partial log(g_w(x;\alpha_2,\beta_2))}{\partial\alpha_2}\Big\}^2 \qquad (3.19)
$$

if and only if $\beta_1 \geq \beta_2$, since $\beta_j > 1$, $j = 1, 2$.

The Shanon entropy of a random variable X is a measure of the uncertainty and is given by $E_F(-log(f(X))$, where $f(x)$ is the pdf of the random variable X. Under the length-biased inverse Weibull distribution, the Shanon entropy is given by

$$\begin{aligned}
E_G(-log(g_w(x;\alpha;\beta)) &= \int_0^\infty \Big[-\log(\frac{\beta\alpha^{-\beta+1}}{\Gamma(1-1/\beta)}) + \beta \log x + (\alpha x)^{-\beta} \Big] g_w(x;\alpha,\beta)\ dx \\
&= -\log\Big(\frac{\beta\alpha^{-\beta+1}}{\Gamma(1-1/\beta)}\Big) + \beta \int_0^\infty \log x g_w(x;\alpha,\beta)\, dx \\
&+ \int_0^\infty (\alpha x)^{-\beta} g_w(x;\alpha,\beta)\, dx \\
&= -\log\Big(\frac{\beta\alpha^{-\beta+1}}{\Gamma(1-1/\beta)}\Big) + \beta\Big[-\log\alpha - \frac{\Gamma'(1-1/\beta)}{\beta\Gamma(1-1/\beta)}\Big] + \frac{\beta-1}{\beta} \\
&= \log\frac{\Gamma(1-1/\beta)}{\alpha\beta} - \frac{\Gamma'(1-1/\beta)}{\Gamma(1-1/\beta)} + \frac{\beta-1}{\beta}.
\end{aligned} \tag{3.20}$$

3.4 Concluding Remarks

This chapter includes some computation. Here are the main results :

- moments

$$E_{G_w}(Y^k) = \frac{\Gamma(1-\frac{(k+1)}{\beta})}{\alpha^{(k)}\Gamma(1-\frac{1}{\beta})} = \frac{\gamma_{k+1}}{\gamma_1}, \qquad k \geq 1,\ \beta > k, \tag{3.21}$$

where $\gamma_k = \frac{\Gamma(1-\frac{k}{\beta})}{\alpha^k}$.

- mean

$$\mu_{G_w} = E_{G_w}(Y) = \frac{\Gamma(1-\frac{2}{\beta})}{\alpha\Gamma(1-\frac{1}{\beta})} = \frac{\gamma_2}{\gamma_1}. \tag{3.22}$$

- variance

$$\sigma^2_{G_w} = EY^2_{G_w} - [E_{G_w}(Y)]^2 = \frac{\Gamma(1-\frac{3}{\beta})}{\alpha^2\Gamma(1-\frac{1}{\beta})} - \left\{\frac{\Gamma(1-\frac{2}{\beta})}{\Gamma(1-\frac{1}{\beta})}\right\}^2 = \frac{\gamma_1\gamma_3 - \gamma_2^2}{\gamma_1^2}. \tag{3.23}$$

- coefficient of variation

$$CV = \sqrt{\frac{\gamma_3\gamma_1}{\gamma_2^2} - 1}. \tag{3.24}$$

- coefficient of skewness

$$CS = \frac{E(Y-\mu_{G_w})^3}{(E(Y-\mu_{G_w})^2)^{3/2}} = \frac{\gamma_1^2\gamma_4 - 3\gamma_1\gamma_2\gamma_3 + 2\gamma_2^3}{(\gamma_1\gamma_3 - \gamma_2^2)^{3/2}}. \tag{3.25}$$

- coefficient of kurtosis

$$CK = \frac{E(Y-\mu_{G_w})^4}{(E(Y-\mu_{G_w})^2)^2} = \frac{\gamma_1^3\gamma_5 - 4\gamma_1^2\gamma_2\gamma_4 + 6\gamma_1\gamma_2^2\gamma_3 - 3\gamma_2^4}{\gamma_1^2\gamma_3^2 - 2\gamma_1\gamma_2^2\gamma_3 + \gamma_2^4}. \tag{3.26}$$

- the moment generating function

$$M_Y(t) = \frac{1}{\Gamma(1-1/\beta)}\sum_{j=0}^{\infty}\frac{t^j}{j!}\frac{\Gamma(1-\frac{j+1}{\beta})}{\alpha^j} = \frac{\alpha}{\Gamma(1-1/\beta)}\sum_{j=0}^{\infty}\frac{t^j}{j!}\gamma_{j+1}. \tag{3.27}$$

- Fisher information matrix

  see (3.16)

- some inequalities from Fisher information matrix

  For $\beta_1 = \beta_2$,

$$E\left\{\frac{\partial \log(g_w(x;\alpha_1,\beta_1))}{\partial \alpha_1}\right\} \geq E\left\{\frac{\partial \log(g_w(x;\alpha_2,\beta_2))}{\partial \alpha_2}\right\} \Leftrightarrow \alpha_2 \geq \alpha_1;$$

  For $\alpha_1 = \alpha_2$,

$$E\left\{\frac{\partial \log(g_w(x;\alpha_1,\beta_1))}{\partial \alpha_1}\right\} \geq E\left\{\frac{\partial \log(g_w(x;\alpha_2,\beta_2))}{\partial \alpha_2}\right\} \Leftrightarrow \beta_1 \geq \beta_2.$$

- Shanon Entropy

$$E_G(-log(g_w(x;\alpha;\beta)) = \log \frac{\Gamma(1-1/\beta)}{\alpha\beta} - \frac{\Gamma'(1-1/\beta)}{\Gamma(1-1/\beta)} + \frac{\beta-1}{\beta}. \tag{3.28}$$

CHAPTER 4

ESTIMATION OF PARAMETERS IN THE WEIGHTED INVERSE WEIBULL DISTRIBUTION

In this section we obtain estimates of the scale and shape parameters for the length-biased inverse Weibull (LBIW) distribution. Method of moment (MOM) and maximum likelihood (ML) estimators are presented.

4.1 Method of Moment Estimators

Let $Y_1, Y_2, \cdots, Y_n$ be an independent length biased sample, then the method of moments estimators are obtained by setting the moments $E(Y)$ and $E(Y^2)$ equal to the corresponding sample moments, that is

$$\frac{\Gamma(1-\frac{2}{\beta})}{\alpha\Gamma(1-\frac{1}{\beta})} = \frac{1}{n}\sum_{j=1}^{n} Y_j \qquad \text{and} \qquad \frac{\Gamma(1-\frac{3}{\beta})}{\alpha^2\Gamma(1-\frac{1}{\beta})} = \frac{1}{n}\sum_{j=1}^{n} Y_j^2. \tag{4.1}$$

For fixed $\beta > 1$, the method of moment estimate (MME) of α is

$$\hat{\alpha} = \frac{n}{\sum_{j=1}^{n} Y_j}\frac{\Gamma(1-\frac{2}{\beta})}{\Gamma(1-\frac{1}{\beta})}. \tag{4.2}$$

4.2 Maximum Likelihood Estimators

The log likelihood function for a single observation x of X is

$$\begin{aligned} l(\alpha,\beta) &= log\left\{\frac{\beta\alpha^{-\beta+1}}{\Gamma(1-(1/\beta))}x^{-(\beta)}exp(-(\alpha x)^{-\beta})\right\} \\ &= log(\beta) - (\beta-1)log(\alpha) - \beta log x - (\alpha x)^{-\beta} - log(\Gamma(1-1/\beta)). \end{aligned} \tag{4.3}$$

Now,

$$\frac{\partial l}{\partial \alpha} = -\frac{\beta-1}{\alpha} + \frac{\beta(\alpha x)^{-\beta}}{\alpha}, \tag{4.4}$$

and

$$\frac{\partial l}{\partial \beta} = \frac{1}{\beta} - log(\alpha) - log(x) + (\alpha x)^{-\beta} log(\alpha x) + \frac{\Gamma'(1-1/\beta)}{\beta^2 \Gamma(1-1/\beta)}. \tag{4.5}$$

From $E(\frac{\partial l}{\partial \alpha}) = 0$, we obtain

$$E(X^{-\beta}) = \frac{\alpha^{\beta}(\beta - 1)}{\beta}, \tag{4.6}$$

and from $E(\frac{\partial l}{\partial \beta}) = 0$, we have

$$E\{-logX + (\alpha X)^{-\beta} log(\alpha X)\} = log\alpha - \frac{1}{\beta} - \frac{\Gamma'(1-1/\beta)}{\beta^2 \Gamma(1-1/\beta)}. \tag{4.7}$$

Now the loglikelihood function for n observations of X is given by

$$L(\alpha, \beta) = nlog(\beta) - n(\beta - 1)log(\alpha) - \beta \sum_{j=1}^{n} log(X_j) - \sum_{j=1}^{n} (\alpha X_j)^{-\beta} - \sum_{j=1}^{n} log\Gamma(1 - 1/\beta). \tag{4.8}$$

The normal equations are

$$\frac{\partial L(\alpha, \beta)}{\partial \alpha} = \frac{-n(\hat{\beta} - 1)}{\hat{\alpha}} + \hat{\alpha}^{-\hat{\beta}-1} \sum_{j=1}^{n} X_j^{-\hat{\beta}} = 0, \tag{4.9}$$

and

$$\frac{\partial L(\alpha, \beta)}{\partial \beta} = \frac{n}{\hat{\beta}} - nlog\hat{\alpha} - \sum_{j=1}^{n} logX_j + \sum_{j=1}^{n} (\hat{\alpha} X_j)^{-\hat{\beta}} log(\hat{\alpha} X_j) + \frac{1}{\hat{\beta}^2} \sum_{j=1}^{n} \Psi(1 - 1/\hat{\beta}) = 0 \tag{4.10}$$

respectively.

From equation (4.9), we have the MLE of α is

$$\hat{\alpha} = \left\{ \frac{n(\hat{\beta} - 1)}{\sum_{j=1}^{n} X_j^{-\hat{\beta}}} \right\}^{1/\hat{\beta}}. \tag{4.11}$$

Now replace $\hat{\alpha}$ in equation (4.10) to obtain

$$\begin{aligned} \left.\frac{\partial L(\alpha,\beta)}{\partial \beta}\right|_{\hat{\alpha},\hat{\beta}} &= \frac{n}{\hat{\beta}} - nlog\left(\frac{n(\hat{\beta}-1)}{\sum_{j=1}^{n} X_j^{-\hat{\beta}}}\right)^{1/\hat{\beta}} - \sum_{j=1}^{n} logX_j \\ &+ \sum_{j=1}^{n}\left[\left(\frac{n(\hat{\beta}-1)}{\sum_{j=1}^{n} X_j^{-\hat{\beta}}}\right)^{1/\hat{\beta}} X_j\right]^{-\hat{\beta}} log\left[\left(\frac{n(\hat{\beta}-1)}{\sum_{j=1}^{n} X_j^{-\hat{\beta}}}\right)^{1/\hat{\beta}} X_j\right] \\ &+ \frac{1}{\hat{\beta}^2}\sum_{j=1}^{n}\frac{\Gamma'(1-1/\hat{\beta})}{\Gamma(1-1/\hat{\beta})} \\ &= 0. \end{aligned} \tag{4.12}$$

This equation does not have a closed form solution and must be solved iteratively to obtain the MLE of the scale parameter β. When α is unknown and β is known, the MLE of α is obtained from equation (4.10) with the value of β in place of $\hat{\beta}$. When both α and β are unknown, the MLE of α and β are obtained by solving the normal equations in (4.9) and (4.10). The MLE of the reliability and hazard functions can be obtained by replacing α and β by their MLEs $\hat{\alpha}$ and $\hat{\beta}$.

4.3 Test for Length-Biasedness

Let $X_1, X_2, \cdots, X_n$, be a random sample from

$$f(x,\alpha,\beta) = \beta\alpha^{-\beta}x^{-\beta-1}exp[-(\alpha x)^{-\beta}], \qquad x \geq 0,\ \alpha > 0,\ \beta > 0. \tag{4.13}$$

Consider the weighted inverse Weibull pdf given by

$$g_w(x,\alpha,\beta,c) = \frac{\beta\alpha^{c-\beta}}{\Gamma(1-c/\beta)}x^{-\beta-1}exp[-(\alpha x)^{-\beta}], \qquad x \geq 0,\ \alpha > 0,\ \beta > 0,\ c > 0. \tag{4.14}$$

If $c = 1$, we have the length-biased inverse Weibull pdf.

We test the hypothesis

$$H_0 : f_0(x,\alpha,\beta) = f(x,\alpha,\beta),$$

against

$$H_A : f_0(x, \alpha, \beta) = g_w(x, \alpha, \beta).$$

To test whether a random sample of size n comes from the inverse Weibull distribution (parent distribution) or the weighted inverse Weibull distribution (weighted distribution), we use the following statistic:

$$\begin{aligned}\Lambda &= \prod_{i=1}^{n} \frac{g_w(x_i, \alpha, \beta, c)}{f(x_i, \alpha, \beta)} \\ &= \prod_{i=1}^{n} \left[\frac{\frac{\beta\alpha^{c-\beta}}{\Gamma(1-c/\beta)} x_i^{-\beta-1} exp[-(\alpha x_i)^{-\beta}]}{\beta\alpha^{-\beta} x_i^{-\beta-1} exp[-(\alpha x_i)^{-\beta}]} \right] \\ &= \prod_{i=1}^{n} \left[\frac{\alpha^c x_i^c}{\Gamma(1-1/\beta)} \right] \\ &= \frac{\alpha^{nc} \prod_{i=1}^{n} x_i^c}{(\Gamma(1-1/\beta))^n}. \end{aligned} \tag{4.15}$$

We reject H_0 when

$$\Lambda = \frac{\alpha^{nc} \prod_{i=1}^{n} x_i^c}{(\Gamma(1-1/\beta))^n} > K, \qquad \text{for some } K > 0. \tag{4.16}$$

Equivalently, we reject the null hypothesis when

$$\Lambda^* = \prod_{i=1}^{n} x_i^c > K^*, \qquad \text{where } K^* = \frac{K(\Gamma(1-c/\beta)^n}{\alpha^{nc}} > 0. \tag{4.17}$$

Note also that $-2\log\Lambda \sim \chi_1^2$, for large sample size n, we can get the p-value from the χ^2 table. Also, we can reject the null hypothesis when the probability value (p-value) given by

$$P(\Lambda^* > \lambda^*), \qquad \text{where } \lambda^* = \prod_{i=1}^{n} x_i^c, \tag{4.18}$$

is less than a specified level of significance, where λ^* is the observed value of the test statistic Λ^*. The p-value can be readily computed via Monte Carlo simulation. That is, simulate n samples from the distribution under H_0, and compute the test statistic Λ^* for each sample, then compute p-value for given λ^*.

$$p - value = \frac{\text{the number of simulated value in which is} \quad \Lambda^* > \lambda^*}{n}.$$

4.4 Concluding Remarks

Method of moment estimators and maximum likelihood estimators are used to estimate the scale and shape parameters of the length-biased inverse Weibull distribution in chapter 4. For fixed $\beta > 1$, the method of moment estimate (MME) of α is

$$\hat{\alpha} = \frac{n}{\sum_{j=1}^{n} Y_j} \frac{\Gamma(1 - \frac{2}{\beta})}{\Gamma(1 - \frac{1}{\beta})}. \tag{4.19}$$

The MLE of the parameters α and β does not have closed form. If the parameter β is known, then the maximum likelihood estimators (MLE) of α is given by

$$\hat{\alpha} = \left\{ \frac{n(\beta - 1)}{\beta \sum_{j=1}^{n} X_j^{-\beta-1}} \right\}^{1/(\beta+1)}. \tag{4.20}$$

A test procedure for the detection of length-biasedness in the inverse Weibull distribution was constructed.

CHAPTER 5

THE BETA-INVERSE WEIBULL DISTRIBUTION

In this section, we present results on the beta-inverse Weibull Distribution (BIW). In particular, we derive the probability density function (pdf), cumulative distribution function (cdf), moment generating function and some other useful distributional properties.

5.1 Introduction to Beta-Inverse Weibull Distribution

There are several generalizations of the beta distribution including those of Eugene [8] dealing with the beta-normal distribution, as well results on the moments of the beta-normal distribution given by Gupta and Nadarajah [13]. Famoye [9] discussed and presented results on the beta-Weibull distribution. Nadarajah [18] presented results on the exponentiated beta distribution. Kong and Sepanski [17] presented results on the beta-gamma distribution.

The pdf and cdf of beta distribution are given by:

$$f(x;a,b) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1, \tag{5.1}$$

and

$$F(x;a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}\, dt, \quad 0 < x < 1, \tag{5.2}$$

respectively, where $B(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$.

Recall that inverse Weibull distribution function is given by,

$$F(x) = \exp(-(\alpha x)^{-\beta}), \quad x \geq 0, \quad \alpha > 0, \quad \beta > 0. \tag{5.3}$$

In general the beta-inverse Weibull distribution is given by:

$$G(x) = \frac{B_{F(x)}(a,b)}{B(a,b)}, \tag{5.4}$$

where $B(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$, $B_{F(x)}(a,b) = \int_0^{F(x)} t^{a-1}(1-t)^{b-1}\,dt$, and

$$F(x) = \exp(-(\alpha x)^{-\beta}), \quad x \geq 0,\ \alpha > 0,\ \beta > 0. \tag{5.5}$$

Clearly, the beta-inverse Weibull distribution is a weighted distribution. The derivation of the cdf and pdf of the beta-inverse Weibull distributions are presented below.

- **Result 1:**

  The cdf of beta-inverse Weibull distribution is given by

  $$G(x;\alpha,\beta,a,b) = \frac{\beta\alpha^{-\beta}}{B(a,b)} \int_0^x t^{-(\beta+1)} \exp(-a(\alpha t)^{-\beta})(1-\exp(-(\alpha t)^{-\beta}))^{b-1}\,dt, \tag{5.6}$$

  for $x \geq 0$, $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$.

  Proof

  Let $u = F^{-1}(t)$, then $t = F(u)$, and
  $dt = d[\exp(-(\alpha u)^{-\beta})] = \dfrac{\beta}{\alpha^{\beta}u^{\beta+1}}\exp(-(\alpha u)^{-\beta})\,du$, so that the cumulative distribution function is

$$\begin{aligned}
G(x;\alpha,\beta,a,b) &= \frac{1}{B(a,b)}\int_0^{F(x)} t^{a-1}(1-t)^{b-1}\,dt \\
&= \frac{1}{B(a,b)}\int_0^{F(x)} [F(u)]^{a-1}[1-F(u)]^{b-1}\,dF(u) \\
&= \frac{\beta\alpha^{-\beta}}{B(a,b)}\int_0^{x} [\exp(-(\alpha u)^{-\beta})]^{a}[1-\exp(-(\alpha u)^{-\beta})]^{b-1}u^{-\beta-1}\,du \\
&= \frac{\beta\alpha^{-\beta}}{B(a,b)}\int_0^{x} [\exp(-a(\alpha u)^{-\beta})][1-\exp(-(\alpha u)^{-\beta})]^{b-1}u^{-(\beta+1)}\,du \\
&= \frac{\beta\alpha^{-\beta}}{B(a,b)}\int_0^{x} t^{-(\beta+1)}[\exp(-a(\alpha t)^{-\beta})][1-\exp(-(\alpha t)^{-\beta})]^{b-1}\,dt.
\end{aligned}$$

If we let $y = (\alpha t)^{-\beta}$, then $t = \alpha^{-1}y^{-1/\beta}$, and $dt = -\alpha^{-1}\beta^{-1}y^{-1/\beta-1}\,dy$

so that $(\alpha t)^{-\beta}|_{t=0} = \infty$, and $(\alpha t)^{-\beta}|_{t=x} = (\alpha x)^{-\beta}$.

The cdf can also be written as follows:

$$\begin{aligned}
G(x;\alpha,\beta,a,b) &= \frac{\beta\alpha^{-\beta}}{B(a,b)}\int_0^{x} t^{-(\beta+1)}\exp(-a(\alpha t)^{-\beta})[1-\exp(-(\alpha t)^{-\beta})]^{b-1}\,dt \\
&= \frac{\beta\alpha^{-\beta}}{B(a,b)}\int_{(\alpha x)^{-\beta}}^{\infty} \alpha^{\beta}\beta^{-1}\exp(-ay)[1-\exp(-y)]^{b-1}\,dy \\
&= \frac{1}{B(a,b)}\int_{(\alpha x)^{-\beta}}^{\infty} \exp(-ay)[1-\exp(-y)]^{b-1}\,dy.
\end{aligned}$$

- **Result 2:**

  The pdf of the beta-inverse Weibull distribution is given by

$$g(x;\alpha,\beta,a,b) = \frac{\beta\alpha^{-\beta}}{B(a,b)}x^{-(\beta+1)}\exp(-a(\alpha x)^{-\beta})(1-\exp(-(\alpha x)^{-\beta}))^{b-1}, \quad (5.7)$$

  for $x \geq 0$, $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$.

  This follows from the fact that
  $G(x;\alpha,\beta,a,b) = \dfrac{\beta\alpha^{-\beta}}{B(a,b)}\displaystyle\int_0^{x} t^{-(\beta+1)}\exp(-a(\alpha t)^{-\beta})[1-\exp(-(\alpha t)^{-\beta})]^{b-1}\,dt,$

so that,

$$g(t;\alpha,\beta,a,b) = \frac{\beta\alpha^{-\beta}}{B(a,b)}t^{-(\beta+1)}\exp(-a(\alpha t)^{-\beta})[1-\exp(-(\alpha t)^{-\beta})]^{b-1},$$

for $x \geq 0, \alpha > 0, \beta > 0, a > 0, b > 0$.

We can plot $g(x;\alpha,\beta,a,b)$ for fixed values of:

- β, a, b: see fig5.1
- α, a, b: see fig5.2
- α, β, b: see fig5.3
- α, β, a: see fig5.4

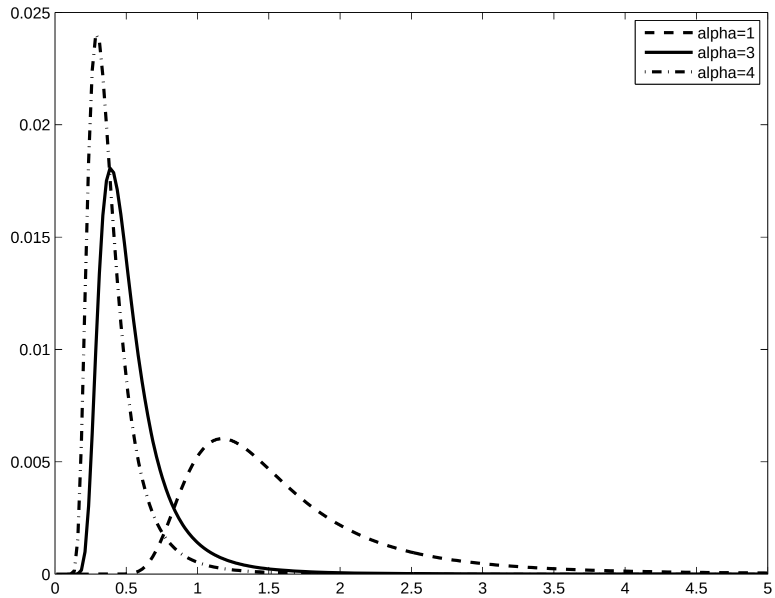

Figure 5.1: plot of $g(x;\alpha,\beta,a,b)$ with fixed values of β,a,b

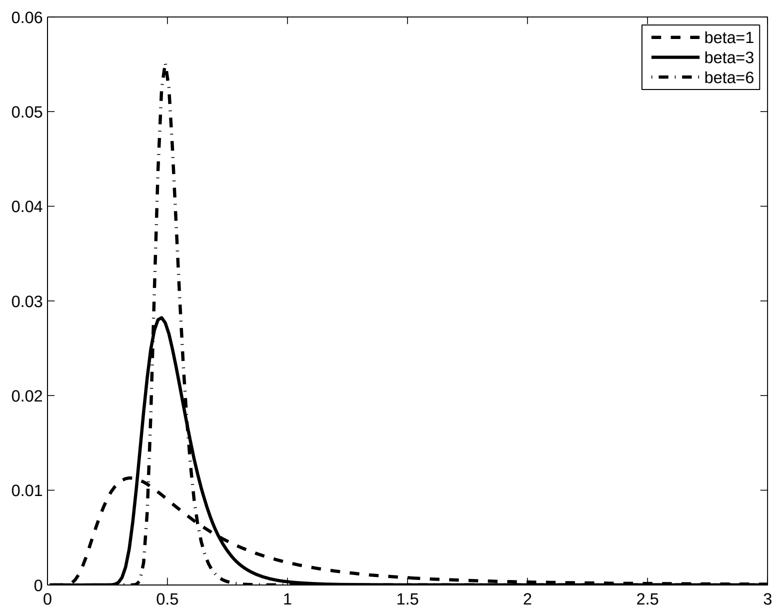

Figure 5.2: plot of $g(x;\alpha,\beta,a,b)$ with fixed values of α,a,b

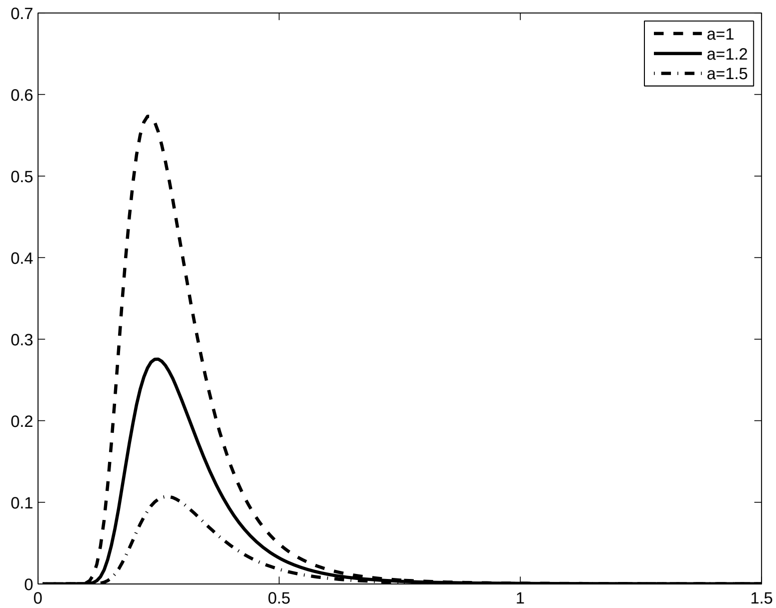

Figure 5.3: plot of $g(x;\alpha,\beta,a,b)$ with fixed values of α,β,b

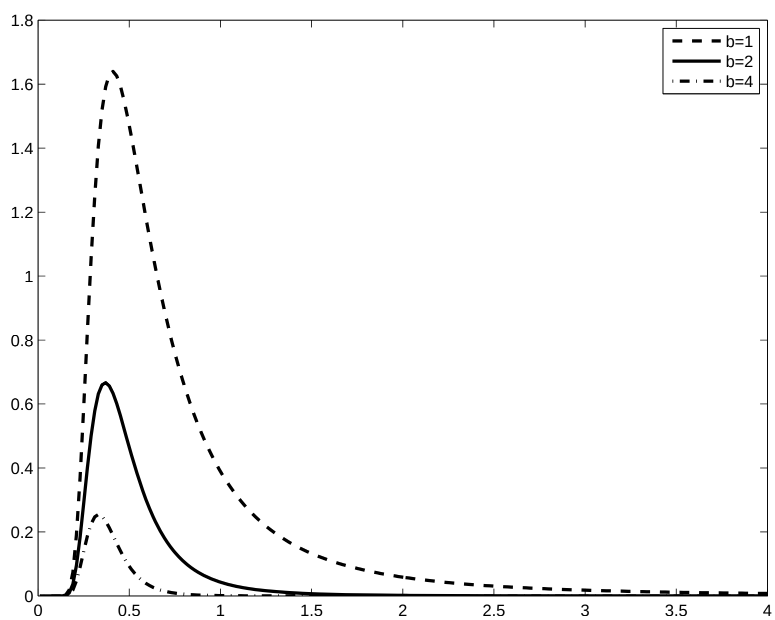

Figure 5.4: plot of $g(x;\alpha,\beta,a,b)$ with fixed values of α,β,a

If we use the formula $(1-w)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b) w^j}{\Gamma(b-j) j!}$, where $b > 0$ and b is non integer, real number, and let $y = (\alpha t)^{-\beta}$, then we have

$$\begin{aligned}
G(x;\alpha,\beta,a,b) &= \frac{\beta\alpha^{-\beta}}{B(a,b)} \int_0^x t^{-(\beta+1)} \exp(-a(\alpha t)^{-\beta})[1-\exp(-(\alpha t)^{-\beta})]^{b-1}\, dt \\
&= \frac{1}{B(a,b)} \int_{(\alpha x)^{-\beta}}^{\infty} \exp(-ay)[1-\exp(-y)]^{b-1}\, dy \\
&= \frac{1}{B(a,b)} \int_{(\alpha x)^{-\beta}}^{\infty} \exp(-ay) \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)[\exp(-y)]^j}{\Gamma(b-j) j!}\, dy \\
&= \frac{\Gamma(b)}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j) j!} \int_{(\alpha x)^{-\beta}}^{\infty} \exp(-ay)\exp(-jy)\, dy \\
&= \frac{\Gamma(b)}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j) j!} \int_{(\alpha x)^{-\beta}}^{\infty} \exp[-(a+j)y]\, dy \\
&= \frac{\Gamma(b)}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j) j!} \Big(-\frac{1}{a+j}\Big) \exp[-(a+j)y]\Big|_{(\alpha x)^{-\beta}}^{\infty} \\
&= \frac{\Gamma(b)}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{\Gamma(b-j) j!(a+j)} [0-\exp[-(a+j)(\alpha x)^{-\beta}]] \\
&= \frac{\Gamma(b)}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b-j) j!(a+j)} \exp[-(a+j)\alpha^{-\beta}x^{-\beta}]. \qquad (5.8)
\end{aligned}$$

This provides an alternative representation of the cdf of the beta-inverse Weibull distribution in terms of an infinite series.

5.2 Some Useful Transformations

In this section, some useful transformations and the resulting distribution are presented. We assume that the random variable Y has a beta distribution with parameters a and b. The following questions are addressed.

1. What is the distribution of X if $X = \dfrac{Y}{1-Y}$?
2. What is the distribution of X if $X = \dfrac{-[\log_e(1-Y)]^{-\frac{1}{\beta}}}{\alpha}$?
3. What is the distribution of X if $X = \dfrac{-[\log_e Y]^{-\frac{1}{\beta}}}{\alpha}$?

For 1, let $x = \dfrac{y}{1-y}$. Then $y = \dfrac{x}{1+x}$ and $y' = \dfrac{1}{(1+x)^2}$. Since the pdf of Y is given by

$$g_Y(y) = \frac{1}{B(a,b)} y^{a-1}(1-y)^{b-1},$$

the pdf of X is given by

$$\begin{aligned} g_X(x) &= g_Y\left(\frac{x}{1+x}\right)\frac{d\left(\frac{x}{1+x}\right)}{dx} \\ &= \frac{1}{B(a,b)}\left(\frac{x}{1+x}\right)^{a-1}\left(1-\frac{x}{1+x}\right)^{b-1}\frac{1}{(1+x)^2} \\ &= \frac{1}{B(a,b)} x^{a-1}(1+x)^{-a-b}, \end{aligned} \tag{5.9}$$

for $0 < x < 1$.

This is essentially the so called transformed beta distribution.

For 2, note that

$$x = \frac{-\left(\log_e(1-y)\right)^{-1/\beta}}{\alpha}$$

implies $-(\alpha x) = [\ln(1-y)]^{-1/\beta}$, so that $y = 1 - e^{(-(\alpha x))^{-\beta}}$, and

$$\frac{d\Big(1 - e^{(-\alpha x)^{-\beta}}\Big)}{dx} = (-\alpha)^{-\beta}\beta x^{-\beta-1}e^{(-\alpha x)^{-\beta}}.$$

Now,

$$\begin{aligned} g_X(x) &= g_Y(1 - e^{(-(\alpha x))^{-\beta}})\Big((-\alpha)^{-\beta}\beta x^{-\beta-1}e^{-((\alpha x))^{-\beta}}\Big) \\ &= \frac{1}{B(a,b)}\Big(1 - e^{(-(\alpha x))^{-\beta}}\Big)^{a-1}\Big(e^{(-(\alpha x))^{-\beta}}\Big)^{b-1}\alpha^{-\beta}\beta x^{-\beta-1}e^{(-(\alpha x))^{-\beta}} \\ &= \frac{1}{B(a,b)}\Big(1 - e^{(-(\alpha x))^{-\beta}}\Big)^{a-1}\Big(e^{(-(\alpha x))^{-\beta}}\Big)^{b}\alpha^{-\beta}\beta x^{-(\beta+1)} \\ &= \frac{\alpha^{-\beta}\beta x^{-(\beta+1)}}{B(a,b)}\Big(e^{(-(\alpha x))^{-\beta}}\Big)^{b}\Big(1 - e^{(-(\alpha x))^{-\beta}}\Big)^{a-1}, \end{aligned} \tag{5.10}$$

for $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$.

For 3, we have $x = \dfrac{[-\log_e(y)]^{-1/\beta}}{\alpha}$, so that,

$$(\alpha x)^{-\beta} = -\ln(y) \implies y = e^{-(\alpha x)^{-\beta}}.$$

Then,

$$\frac{dy}{dx} = e^{-(\alpha x)^{-\beta}}(-1)(-\beta)(\alpha x)^{-\beta-1}\alpha = \beta\alpha^{-\beta}x^{-(\beta+1)}e^{-(\alpha x)^{-\beta}},$$

and the pdf of $X = \dfrac{-[\log_e Y]^{-\frac{1}{\beta}}}{\alpha}$ is given by:

$$\begin{aligned} g_X(x) &= g_Y(e^{-(\alpha x)^{-\beta}})\beta\alpha^{-\beta}x^{-(\beta+1)}e^{-(\alpha x)^{-\beta}} \\ &= \frac{1}{B(a,b)}\Big(e^{-(\alpha x)^{-\beta}}\Big)^{a-1}\Big(1 - e^{-(\alpha x)^{-\beta}}\Big)^{b-1}\beta\alpha^{-\beta}x^{-(\beta+1)}e^{-(\alpha x)^{-\beta}} \\ &= \frac{\beta\alpha^{-\beta}}{B(a,b)}x^{-(\beta+1)}\Big(e^{-(\alpha x)^{-\beta}}\Big)^{a}\Big(1 - e^{-(\alpha x)^{-\beta}}\Big)^{b-1}, \end{aligned} \tag{5.11}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$.

This is the beta-inverse Weibull distribution. In essence, one can generate observations from beta-inverse Weibull distribution via the transformation $X = \frac{-[\log(Y)]^{-1/\beta}}{\alpha}$, where Y is a random variable that follows a beta distribution with parameters a and b.

5.3 Moments of the Beta-Inverse Weibull Distribution

The moments of the beta-inverse Weibull distribution are presented in this section. The r^{th} central moment is given by

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r \frac{\beta\alpha^{-\beta}}{B(a,b)} x^{-(\beta+1)} e^{-a(\alpha x)^{-\beta}} \left(1 - e^{-(\alpha x)^{-\beta}}\right)^{b-1} dx \\ &= \frac{\beta\alpha^{-\beta}}{B(a,b)} \int_0^\infty x^{r-\beta-1} e^{-a(\alpha x)^{-\beta}} \left(1 - e^{-(\alpha x)^{-\beta}}\right)^{b-1} dx \\ &= \frac{\beta\alpha^{-\beta}}{B(a,b)} \int_0^\infty x^{c-1} e^{-a(\alpha x)^{-\beta}} \left(1 - e^{-(\alpha x)^{-\beta}}\right)^{b-1} dx, \end{aligned} \tag{5.12}$$

where $c = r - \beta$. Let $y = (\alpha x)^{-\beta}$, then $y^{-1/\beta} = \alpha x$, and $x = \alpha^{-1} y^{-1/\beta}$.
This implies $dx = \alpha^{-1}(-1/\beta) y^{-1/\beta-1} dy = -\alpha^{-1}\beta^{-1} y^{-1/\beta-1} dy$, and $y = (\alpha x)^{-\beta}\Big|_{x=0} = \infty$, $y = (\alpha x)^{-\beta}\Big|_{x=\infty} = 0$.
Therefore,

$$\begin{aligned} E(X^r) &= \frac{\beta\alpha^{-\beta}}{B(a,b)} \int_\infty^0 (\alpha^{-1} y^{-1/\beta})^{c-1} e^{-ay} (1 - e^{-y})^{b-1} (-\alpha^{-1}\beta^{-1} y^{-1/\beta-1} dy) \\ &= \frac{\alpha^{-\beta-c}}{B(a,b)} \int_0^\infty y^{-c/\beta-1} e^{-ay} (1 - e^{-y})^{b-1} dy. \end{aligned}$$

Let $k = -\dfrac{c}{\beta}$, then,

$$E(X^r) = \frac{\alpha^{-(\beta+c)}}{B(a,b)} \int_0^\infty y^{k-1} e^{-ay} (1 - e^{-y})^{b-1} dy.$$

Define

$$\Delta_{k,a,b} = \int_0^\infty y^{k-1} e^{-ay} (1 - e^{-y})^{b-1} dy. \tag{5.13}$$

Then the moments of X can be written as

$$E(X^r) = \frac{\alpha^{-(\beta+c)}}{B(a,b)} \Delta_{k,a,b}, \tag{5.14}$$

where $c = r - \beta$ and $k = -\frac{c}{\beta} = \frac{\beta - r}{\beta}$. Since $c = r - \beta$, then $c + \beta = r$, and

$$E(X^r) = \frac{\alpha^{-r}}{B(a,b)} \Delta_{\frac{\beta-r}{\beta},a,b} = \frac{\Delta_{\frac{\beta-r}{\beta},a,b}}{\alpha^r B(a,b)}.$$

For positive integer b,

$$\begin{aligned}
\Delta_{c,a,b} &= \int_0^\infty y^{c-1} e^{-ay} (1 - e^{-y})^{b-1} dy \\
&= \int_0^\infty y^{c-1} e^{-ay} \sum_{j=0}^{b-1} \frac{(b-1)!(-1)^j (e^{-y})^j}{j!(b-1-j)!} dy \\
&= \int_0^\infty y^{c-1} \sum_{j=0}^{b-1} \frac{(b-1)!(-1)^j e^{-(a+j)y}}{j!(b-1-j)!} dy \\
&= \sum_{j=0}^{b-1} \frac{(b-1)!(-1)^j}{j!(b-1-j)!} \int_0^\infty y^{c-1} e^{-(a+j)y} dy.
\end{aligned} \tag{5.15}$$

Let $t = (a+j)y$, then $y = \frac{t}{a+j}$ and $dy = \frac{1}{a+j} dt$, so that

$$\begin{aligned}
\Delta_{c,a,b} &= \sum_{j=0}^{b-1} \frac{(b-1)!(-1)^j}{j!(b-j-1)!} \int_0^\infty \frac{t^{c-1}}{(a+j)^{c-1}} e^{-t} \frac{1}{a+j} dt \\
&= \sum_{j=0}^{b-1} \frac{(b-1)!(-1)^j}{j!(b-j-1)!} \int_0^\infty \frac{1}{(a+j)^c} t^{c-1} e^{-t} dt \\
&= \sum_{j=0}^{b-1} \frac{(b-1)!(-1)^j}{j!(b-j-1)!(a+j)^c} \int_0^\infty t^{c-1} e^{-t} dt \\
&= \Gamma(c) \sum_{j=0}^{b-1} \frac{(b-1)!(-1)^j}{j!(b-j-1)!(a+j)^c}.
\end{aligned} \tag{5.16}$$

For $c > 0$ and $b > 0$,

$$\begin{aligned}\Delta_{c,a,b} &= \int_0^\infty y^{c-1} e^{-ay} (1 - e^{-y})^{b-1} dy \\ &= \int_0^\infty y^{c-1} e^{-ay} \sum_{j=0}^\infty \frac{(-1)^j \Gamma(b) (e^{-y})^j}{j! \Gamma(b-j)} dy \\ &= \int_0^\infty y^{c-1} \sum_{j=0}^\infty \frac{(-1)^j \Gamma(b) (e^{-(a+j)y})}{j! \Gamma(b-j)} dy \\ &= \sum_{j=0}^\infty \frac{(-1)^j \Gamma(b)}{j! \Gamma(b-j)} \int_0^\infty y^{c-1} e^{-(a+j)y}) dy.\end{aligned} \tag{5.17}$$

With $t = (a+j)y$, we also have

$$\begin{aligned}\Delta_{c,a,b} &= \sum_{j=0}^\infty \frac{(-1)^j \Gamma(b)}{j! \Gamma(b-j)} \int_0^\infty \frac{t^{c-1}}{(a+j)^{c-1}} e^{-t} \frac{1}{a+j} dt \\ &= \sum_{j=0}^\infty \frac{(-1)^j \Gamma(b)}{j! \Gamma(b-j)} \int_0^\infty \frac{1}{(a+j)^c} t^{c-1} e^{-t} dt \\ &= \sum_{j=0}^\infty \frac{(-1)^j \Gamma(b)}{j! \Gamma(b-j)} \frac{1}{(a+j)^c} \int_0^\infty t^{c-1} e^{-t} dt \\ &= \Gamma(c) \sum_{j=0}^\infty \frac{(-1)^j \Gamma(b)}{j! \Gamma(b-j)(a+j)^c}.\end{aligned} \tag{5.18}$$

5.4 Moment Generating Function

In this section, we derive the moment generating function of the beta-inverse Weibull distribution. Recall that (by Taylor's series expansion of e^{tx} about zero)

$$e^{tx} = \sum_{j=0}^{\infty} \frac{(tx)^j}{j!},$$

so the moment-generating function (MGF) of the beta-inverse Weibull distribution is given by

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \int_0^{\infty} e^{tx} \frac{\beta\alpha^{-\beta}}{B(a,b)} x^{-(\beta+1)} e^{-a(\alpha x)^{-\beta}} (1 - e^{-(\alpha x)^{-\beta}})^{b-1} dx \\ &= \frac{\beta\alpha^{-\beta}}{B(a,b)} \int_0^{\infty} \sum_{j=0}^{\infty} \frac{(tx)^j}{j!} x^{-(\beta+1)} e^{-a(\alpha x)^{-\beta}} (1 - e^{-(\alpha x)^{-\beta}})^{b-1} dx \\ &= \frac{\beta\alpha^{-\beta}}{B(a,b)} \int_0^{\infty} \sum_{j=0}^{\infty} \frac{t^j}{j!} x^{-(\beta+1)+j} e^{-a(\alpha x)^{-\beta}} (1 - e^{-(\alpha x)^{-\beta}})^{b-1} dx. \end{aligned} \tag{5.19}$$

Let $y = (\alpha x)^{-\beta}$. Then $x = \alpha^{-1} y^{-1/\beta}$ and $dx = -\alpha^{-1}\beta^{-1}y^{-1/\beta-1}$,
so that $y = (\alpha x)^{-\beta}\Big|_{x=0} = \infty$ and $y = (\alpha x)^{-\beta}\Big|_{x=\infty} = 0$.
Consequently, the MGF of the beta-inverse Weibull distribution is given by

$$\begin{aligned} M_X(t) &= \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^{\infty} y^{-j/\beta} e^{-ay} (1 - e^{-y})^{b-1} dy \\ &= \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{t^j}{j!} \Delta_{-\frac{j}{\beta}+1,a,b} \\ &= \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{t^j}{j!} \Delta_{-\frac{\beta-j}{\beta},a,b}. \end{aligned} \tag{5.20}$$

5.5 Some Results

This section contains results that lead to further evaluation of the function $\Delta_{c,a,b}$ in terms of the moments of the beta random variable. The results are stated and proved below.

1. If $V \sim \text{Beta}(a,b)$, then $U = -\log(V) \sim 1 - F_V(e^{-y})$.

Proof.

$$\begin{aligned} F_U(y) &= P(U \leq y) = P(-\log V \leq y) \\ &= P(V \geq e^{-y}) \\ &= 1 - P(V < e^{-y}) \\ &= 1 - F_V(e^{-y}). \end{aligned}$$

□

2. For $c > 1$, c an integer, $E(U^{c-1}) = \frac{\Delta_{c,a,b}}{B(a,b)}$.

Proof.

$$F_U(y) = 1 - F_V(e^{-y}) \implies F_U'(y) = -F_V'(e^{-y})e^{-y}(-1) \implies F_U(y) = e^{-y}F_V(e^{-y}).$$

Since $V \sim \text{Beta}(a,b)$, then $f_V(v) = \frac{1}{B(a,b)}v^{a-1}(1-v)^{b-1}$. Therefore, the pdf of U is given by

$$f_U(y) = \frac{1}{B(a,b)}e^{-y}(e^{-y})^{a-1}(1-e^{-y})^{b-1} = \frac{1}{B(a,b)}e^{-ay}(1-e^{-y})^{b-1}.$$

Now,

$$
\begin{aligned}
E(U^{c-1}) &= \int_0^\infty u^{c-1} f_U(u)du \\
&= \int_0^\infty u^{c-1} \frac{1}{B(a,b)} e^{-au}(1-e^{-y})^{b-1}du \\
&= \frac{1}{B(a,b)} \int_0^\infty u^{c-1} e^{-au}(1-e^{-y})^{b-1}du \\
&= \frac{\Delta_{c,a,b}}{B(a,b)},
\end{aligned}
$$

which implies that

$$\Delta_{c,a,b} = E(U^{c-1})B(a,b).$$

□

3. Moment generating function of U:

$$
\begin{aligned}
M_U(t) = E(e^{tU}) = E(e^{-U(-t)}) = E(V^{-t}) = \int_0^1 v^{-t} \frac{1}{B(a,b)} v^{a-1}(1-v)^{b-1}dv \\
= \frac{1}{B(a,b)} \int_0^1 v^{a-t-1}(1-v)^{b-1}dv = \frac{B(a-t,b)}{B(a,b)}.
\end{aligned}
$$

5.6 Concluding Remarks

We presented the beta-inverse Weibull distribution in this chapter. The beta-inverse Weibull distribution is a weighted distribution and in fact contains a fairly large class of distributions with potential applications to a wide area of probability and statistics. Given the pdf or cdf of beta distribution, we computed the cdf and pdf of the beta-inverse Weibull distribution and analyzed the behavior of the probability density function by plotting the probability density function for some fixed values of the parameters. We also obtained very useful transformations which show us the relationships between beta distribution and inverse beta distribution, beta distribution and beta-inverse Weibull distribution in this chapter. The transformations provide a way to generate data from the beta-inverse Weibull distribution. Moments and moment generating function are derived in chapter 5 as well. There are still a lot more we could do on this topic. Future work include estimation of parameters, goodness of fit and tests of hypothesis in the beta-inverse Weibull distribution.

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