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Compact Filters and the Arzelá-Ascoli Theorem

Nathan Denton Farmer

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COMPACT FILTERS AND THE ARZELÀ-ASCOLI THEOREM

by

NATHAN FARMER

(Under the Direction of Frédéric Mynard)

ABSTRACT

The Arzelà-Ascoli Theorem gives conditions for a set to be compact in a function space, and is a useful theorem for existence proofs in many different mathematical contexts. The most natural structure for Arzelà-Ascoli type theorems is the continuous convergence - which is not necessarily topological even if the underlying base spaces are.

Compact filters, on the other hand, are the logical extension of compact sets to filters with similar compactness-type properties. Compact filters have proved to be a useful tool in optimization and analysis.

This thesis will show conditions for a filter to be compact on a function space, thus establishing an Arzelà-Ascoli type of result for filters.

INDEX WORDS: Arzelà-Ascoli, Compact Filters, Convergence Spaces, Function Spaces, Filters

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CHAPTER 1

INTRODUCTION

The ambition of this thesis is to explore the Arzelà-Ascoli theorem in a very general context.

The Arzelà-Ascoli theorem gives conditions for a subset of a function space to be (relatively) compact. Variants of the Arzelà-Ascoli theorem show up in a variety of fields with conditions and nomenclature suited to their particular applications. Broader generalizations of the theorem allow one to see the underlying structure that make the theorem work in different settings.

We'll start by developing filters, as we'll need to consider spaces that can not be characterized by sequences alone. We'll then look at convergence spaces, structures where convergence of filters is primal. Using filters, we'll characterize compact sets and filters in convergence spaces and topologies.

With the preliminaries out of the way, we'll look at function space structure as convergences, and in particular how they relate to the continuity of the evaluation map.

Finally, we'll characterize compact filters in a general convergence space setting.

1.1 Typographical Conventions

$f \in X$ An element, written in lowercase.

$F \subseteq X$ A set containing elements, written in uppercase.

$\mathcal{F} \subseteq 2^X$ A set of sets, written in uppercase script.

$\mathbb{F} \subseteq 2^{2^X}$ A set of families, written in a double struck script.

CHAPTER 2

GENERAL PRELIMINARIES

In a sequential topological space, knowing just the convergent sequences gives you enough information to reconstruct the topological structure of the space. One cannot do the same in a non-sequential topological space (for example, the cocountable topology has *no* non-stabilizing convergent sequences).

Generalized sequences, in the form of filters, provide a remedy. Knowing which *filters* converge in a topological space is sufficient to reconstruct the topology.

The idea of a convergence space comes from considering convergence (of filters) as the primary notion. Rather than starting with open sets, general convergence spaces simply specify which filters converge where with only a little additional structure. It is interesting to see, then, that topological spaces (viewed as convergence spaces) are rather rigidly structured in terms of which filters can converge and how.

As a further incentive to consider convergence spaces, there are natural nearly-topological spaces (such as pseudotopologies and pretopologies) that come from weakening standard topological axioms to “complete” certain constructions.

For further information see [3].

2.1 Filters

To begin with, consider a set X without any particular structure.

Definition 2.1.1. A family \mathcal{F} is *isotone* when

$$A \in \mathcal{F} \text{ and } A \subseteq B \Rightarrow B \in \mathcal{F}.$$

Further, if \mathcal{G} is a family,

$$\mathcal{G}^\uparrow := \{ B \subseteq X : \exists A \in \mathcal{G}, A \subseteq B \}$$

is isotone.

Definition 2.1.2. A family $\mathcal{F} \subseteq 2^X$ is a **filter** on X when

$$\begin{aligned} \emptyset &\notin \mathcal{F} && \text{(Non-degenerate)} \\ A, B \in \mathcal{F} &\Rightarrow A \cap B \in \mathcal{F} && \text{(Closed under finite intersections)} \\ \mathcal{F} &= \mathcal{F}^\uparrow. && \text{(Isotone)} \end{aligned}$$

Definition 2.1.3. If $(a_n)_{n \in \mathbb{N}}$ is a sequence into a set X , then

$$\{a_n\}_n^\uparrow := \{ \{a_k : k \geq n\} \subseteq X : k \in \mathbb{N} \}^\uparrow$$

is the **filter generated by tails of the sequence** $(a_n)_{n \in \mathbb{N}}$.

Definition 2.1.4. For a fixed set $A \subseteq X$,

$$A^\uparrow := \{A\}^\uparrow$$

is the **principal filter** of the set A .

In contexts where a filter is expected, it suffices to just write “ A ”.

Definition 2.1.5. Two families \mathcal{A} and \mathcal{B} **mesh** when

for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$A \cap B \neq \emptyset.$$

This situation is denoted $\mathcal{A} \# \mathcal{B}$.

If for the principal filter of A we have $\{A\}^\uparrow \# \mathcal{B}$, then by definition 2.1.4 we can simply write $A \# \mathcal{B}$.

Definition 2.1.6. The **set of all filters** on an underlying set X is

$$\mathbb{F}X := \{ \mathcal{F} \subseteq 2^X : \mathcal{F} \text{ filter} \}.$$

Definition 2.1.7. If $\mathcal{F} \subseteq \mathcal{G}$ for two filters \mathcal{F} and \mathcal{G} on the same underlying set, then we say that

\mathcal{F} is **coarser** than \mathcal{G}

or equivalently

\mathcal{G} is **finer** than \mathcal{F} .

We denote this situation $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \geq \mathcal{F}$, respectively.

The relation \geq on $\mathbb{F}X$ forms a partial order.

Definition 2.1.8. The **set of all filters finer than the filter \mathcal{F}** on an underlying set X is

$$\mathbb{F}(\mathcal{F}) := \{ \mathcal{G} \in \mathbb{F}X : \mathcal{G} \geq \mathcal{F} \}.$$

Proposition 2.1.9. The **infimum filter** (greatest lower bound) of a collection of filters $\{ \mathcal{F}_\alpha : \alpha \in I \}$ on the same underlying set is

$$\bigwedge_{\alpha \in I} \mathcal{F}_\alpha := \bigcap_{\alpha \in I} \mathcal{F}_\alpha.$$

Proof.

- It is relatively straightforward to verify that $\bigwedge_{\alpha \in I} \mathcal{F}_\alpha$ is a filter.
- The filter $\bigwedge_{\alpha \in I} \mathcal{F}_\alpha$ is a lower bound for each \mathcal{F}_α .
 \equiv The filter $\bigwedge_{\alpha \in I} \mathcal{F}_\alpha \leq \mathcal{F}_\alpha$ for all $\alpha \in I$.

The is true since $\bigcap_{\alpha \in I} \mathcal{F}_\alpha \subseteq \mathcal{F}_\alpha$ for each $\alpha \in I$.

- The filter $\bigwedge_{\alpha \in I} \mathcal{F}_\alpha$ is the greatest lower bound of $\{\mathcal{F}_\alpha : \alpha \in I\}$.
 \equiv For any filter \mathcal{G} that is a lower bound of $\{\mathcal{F}_\alpha : \alpha \in I\}$ we have $\mathcal{G} \leq \bigwedge_{\alpha \in I} \mathcal{F}_\alpha$.

Let \mathcal{G} be a lower bound of $\mathcal{G} \leq \bigwedge_{\alpha \in I} \mathcal{F}_\alpha$.

Then $\mathcal{G} \leq \mathcal{F}_\alpha$ for any $\alpha \in I$

So $G \in \mathcal{G}$ implies $G \in \mathcal{F}_\alpha$ for each $\alpha \in I$.

Thus $G \in \bigwedge_{\alpha \in I} \mathcal{F}_\alpha$.

□

The intersection of any collection of filters is a filter itself and the greatest lower bound for the partial order on $\mathbb{F}X$. The supremum, in contrast, does not generally exist for arbitrary sets of filters.

Proposition 2.1.10. *If $\mathcal{F} \# \mathcal{G}$ for two filters \mathcal{F} and \mathcal{G} on the same underlying set, then the **supremum** filter (least upper bound) is*

$$\mathcal{F} \vee \mathcal{G} := \{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}^\uparrow.$$

Proof.

- The proof that $\mathcal{F} \vee \mathcal{G}$ is a filter when $\mathcal{F} \# \mathcal{G}$ is straightforward.
- The filter $\mathcal{F} \vee \mathcal{G}$ is an upper bound for \mathcal{F} and \mathcal{G} .
 $\equiv \mathcal{F} \leq \mathcal{F} \vee \mathcal{G}$ and $\mathcal{G} \leq \mathcal{F} \vee \mathcal{G}$.

Let $F \in \mathcal{F}$.

Then for an arbitrary $G \in \mathcal{G}$ we have that $F \cap G \in \mathcal{F} \vee \mathcal{G}$ and $F \cap G \subseteq F$.

So $F \supseteq F \cap G$ is in $\mathcal{F} \vee \mathcal{G}$ by the isotone property of the filter $\mathcal{F} \vee \mathcal{G}$.

Thus $\mathcal{F} \subseteq \mathcal{F} \vee \mathcal{G}$.

Similarly, $\mathcal{G} \subseteq \mathcal{F} \vee \mathcal{G}$.

- The filter $\mathcal{F} \vee \mathcal{G}$ is the least upper bound for \mathcal{F} and \mathcal{G} .
 \equiv For any filter \mathcal{H} where $\mathcal{F} \leq \mathcal{H}$ and $\mathcal{G} \leq \mathcal{H}$ we have $\mathcal{F} \vee \mathcal{G} \leq \mathcal{H}$.

Let $A \in \mathcal{F} \vee \mathcal{G}$.

Then there is $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G \subseteq A$ and $F \cap G \in \mathcal{F} \vee \mathcal{G}$.

But then $F \cap G \in \mathcal{H}$ since \mathcal{H} is a filter containing \mathcal{F} and \mathcal{G} and closed under finite intersections.

Thus $A \in \mathcal{H}$ by isotony and therefore $\mathcal{F} \vee \mathcal{G} \subseteq \mathcal{H}$.

□

2.2 Ultrafilters

Recall that a **chain** is a subset of a partially ordered set whose members are pairwise comparable.

Proposition 2.2.1.

$$\{\mathcal{F}_\alpha : \alpha \in I\} \text{ is a chain} \Rightarrow \bigcup_{\alpha \in I} \mathcal{F}_\alpha \text{ is a filter.}$$

The proof is a straightforward verification.

Proposition 2.2.2. *Every filter is contained in a maximal filter.*

Proof.

Every filter is contained in a maximal filter.

\equiv For any $\mathcal{F} \in \mathbb{F}X$ there is a $\mathcal{G} \in \mathbb{F}(\mathcal{F})$ such that for any $\mathcal{H} \in \mathbb{F}(\mathcal{F})$ we have $\mathcal{H} \leq \mathcal{G}$.

Consider a chain $\mathbb{C} \subseteq \mathbb{F}(\mathcal{F})$.

We wish to assert that every chain \mathbb{C} in the partially ordered set $\mathbb{F}(\mathcal{F})$ has an upper bound in $\mathbb{F}(\mathcal{F})$ so that we may invoke Zorn's Lemma.

We claim the filter $\mathcal{G} := \bigcup \mathbb{C}$ to be that upper bound.

- The filter $\bigcup \mathbb{C} \in \mathbb{F}(\mathcal{F})$.

By proposition 2.2.1 the union of filters in a chain is itself a filter.

Further, each $\mathcal{C} \in \mathbb{C}$ has $\mathcal{C} \geq \mathcal{F}$ by definition of $\mathbb{F}(\mathcal{F})$.

Therefore $\bigcup_{\mathcal{C} \in \mathbb{C}} \mathcal{C} \geq \mathcal{F}$.

- The filter $\bigcup \mathbb{C}$ is an upper bound.

$\equiv \bigcup \mathbb{C} \geq \mathcal{C}$ for any $\mathcal{C} \in \mathbb{C}$.

This is due to the fact that $\bigcup \mathbb{C} \supseteq \mathcal{C}$ for every $\mathcal{C} \in \mathbb{C}$.

Since every chain in $\mathbb{F}(\mathcal{F})$ has an upper bound somewhere in $\mathbb{F}(\mathcal{F})$, by Zorn's Lemma, $\mathbb{F}(\mathcal{F})$ must contain at least one maximal element.

□

Definition 2.2.3. *A filter which is maximal in $\mathbb{F}X$ is called an **ultrafilter**.*

Definition 2.2.4. *The set of all ultrafilters on X is denoted*

$$\mathbb{U}X := \{ \mathcal{U} \subseteq 2^X : \mathcal{U} \text{ ultrafilter} \}$$

The set of all ultrafilters finer than \mathcal{F} is denoted $\mathbb{U}(\mathcal{F})$.

Proposition 2.2.5. *If $\mathcal{U} \in \mathbb{U}X$, then*

for any set $A \subseteq X$, either

$$A \in \mathcal{U} \text{ and } A^c \notin \mathcal{U}$$

or

$$A^c \in \mathcal{U} \text{ and } A \notin \mathcal{U}.$$

Proof.

Let $A \subseteq X$ and $\mathcal{U} \in \mathbb{U}X$.

If $A \# \mathcal{U}$ then $A \vee \mathcal{U} \geq \mathcal{U}$ and $A \in A \vee \mathcal{U}$ (by isotony).

But \mathcal{U} is maximal, so $A \vee \mathcal{U} = \mathcal{U}$.

Therefore $A \in \mathcal{U}$ and $A^c \notin \mathcal{U}$ (for otherwise $A \cap A^c = \emptyset$).

If, on the other hand, $A^c \# \mathcal{U}$, we have $A^c \in \mathcal{U}$ by a similar process.

□

2.3 Calculus of Relations

For a relation $R \subseteq X \times Y$.

Definition 2.3.1. The **image** of an element $x \in X$ under a relation $R \subseteq X \times Y$ is

$$R(x) := \{y \in Y : (x, y) \in R\}$$

and the **preimage** of an element $y \in Y$ under a relation $R \subseteq X \times Y$ is

$$R^-(y) := \{x \in X : (x, y) \in R\}.$$

Definition 2.3.2. The **image** of a set $A \subseteq X$ under a relation $R \subseteq X \times Y$ is

$$R[A] := \bigcup_{x \in A} R(x)$$

and the **preimage** of a set $B \subseteq Y$ under a relation $R \subseteq X \times Y$ is

$$R^-[B] := \bigcup_{y \in B} R^-(y).$$

Proposition 2.3.3. For $F \subseteq X$ and $G \subseteq Y$ under a relation $R \subseteq X \times Y$,

$$(F \times G) \# R \Leftrightarrow R[F] \# G \Leftrightarrow F \# R^{-}[G]$$

Proof.

$$\begin{aligned} (F \times G) \# R &\Leftrightarrow \exists(x, y) \in X \times Y \text{ such that } (x, y) \in F \times G \text{ and } (x, y) \in R \\ &\Leftrightarrow \exists x \in F \text{ and } \exists y \in G \text{ such that } y \in R(x) \\ &\Leftrightarrow R[F] \# G \\ &\Leftrightarrow \exists y \in G \text{ and } \exists x \in F \text{ such that } x \in R^{-}(y) \\ &\Leftrightarrow F \# R^{-}[G] \end{aligned}$$

□

Definition 2.3.4. *The (potentially degenerate) **image filter** of $\mathcal{F} \in \mathbb{F}X$ under the relation R is*

$$R[\mathcal{F}] := \{ R[F] \subseteq Y : F \in \mathcal{F} \}^\uparrow$$

*and the (potentially degenerate) **preimage filter** $\mathcal{G} \in \mathbb{F}Y$ under a relation R is*

$$R^{-}[\mathcal{G}] := \{ R^{-}[G] \subseteq X : G \in \mathcal{G} \}^\uparrow.$$

Note that $R[\mathcal{F}]$ and $R^{-}[\mathcal{G}]$ are not guaranteed to be filters since $R[F]$ or $R^{-}[G]$ could be the empty set. If R is a function, however, $R[\mathcal{F}]$ will always be a filter since each $R[F] \neq \emptyset$.

Proposition 2.3.5. *If $\mathcal{U} \in \mathbb{U}X$ and $f : X \rightarrow Y$, then $f[\mathcal{U}] \in \mathbb{U}Y$*

Proof.

Let $\mathcal{U} \in \mathbb{U}X$ and $f : X \rightarrow Y$.

Then $f[\mathcal{U}] \in \mathbb{F}Y$.

Suppose $f[\mathcal{U}] \notin \mathbb{U}Y$.

Then there would be $\mathcal{W} \in \mathbb{U}Y$ such that $\mathcal{W} \geq f[\mathcal{U}]$, and further a $W \in \mathcal{W}$ such that $W \notin \mathcal{U}$.

The set $f^{-}[W] \notin \mathcal{U}$ (for otherwise $W \in f[\mathcal{U}]$).

But then $(f^{-}[W])^c \in \mathcal{U}$ by proposition 2.2.5.

Consequently, $f[(f^{-}[W])^c] \in f[f\mathcal{U}]$ and therefore $W^c \in f[\mathcal{U}]$ by isotony.

But $\mathcal{W} \geq \mathcal{U}$, so $W \in \mathcal{W}$ and $W^c \in \mathcal{W}$, contradicting the non-degeneracy of \mathcal{W} .

□

Corollary 2.3.6. *For a function $f : X \rightarrow Y$ and*

for every $A \subseteq X$ and $B \subseteq Y$,

$$B \# f[A] \Leftrightarrow f^{-}[B] \# A.$$

Proposition 2.3.7. *If $\mathcal{F} \in \mathbb{F}X$, $\mathcal{G} \in \mathbb{F}Y$ and $R \subseteq X \times Y$, then*

$$(\mathcal{F} \times \mathcal{G}) \# R \Leftrightarrow R[\mathcal{F}] \# \mathcal{G} \Leftrightarrow \mathcal{F} \# R^{-}[\mathcal{G}].$$

Proof.

$$(\mathcal{F} \times \mathcal{G}) \# R \Leftrightarrow \text{For every } F \in \mathcal{F} \text{ and } G \in \mathcal{G} \text{ we have } (F \times G) \# R$$

$$\Leftrightarrow \text{For every } F \in \mathcal{F} \text{ and } G \in \mathcal{G} \text{ we have } R[F] \# G$$

$$\Leftrightarrow R[\mathcal{F}] \# \mathcal{G}$$

$$\Leftrightarrow \text{For every } F \in \mathcal{F} \text{ and } G \in \mathcal{G} \text{ we have } R[F] \# G$$

$$\Leftrightarrow \text{For every } F \in \mathcal{F} \text{ and } G \in \mathcal{G} \text{ we have } F \# R^{-}[G]$$

$$\Leftrightarrow \mathcal{F} \# R^{-}[\mathcal{G}]$$

□

Definition 2.3.8. *If $\mathcal{R} \in \mathbb{F}(X \times Y)$ is a filter on the product $X \times Y$, then*

$$\mathcal{R}[\mathcal{F}] := \{ R[F] \subseteq Y : R \in \mathcal{R}, F \in \mathcal{F} \}^{\uparrow}$$

*is the **image** filter of a filter $\mathcal{F} \in \mathbb{F}X$ and*

$$\mathcal{R}^{-}[\mathcal{G}] := \{ R^{-}[G] \subseteq X : G \in \mathcal{G} \}^{\uparrow}$$

is the **preimage filter** of a filter $\mathcal{G} \in \mathbb{F}Y$.

Proposition 2.3.9. *If $\mathcal{F} \in \mathbb{F}X$, $\mathcal{G} \in \mathbb{F}Y$, and $\mathcal{R} \in \mathbb{F}(X \times Y)$, then*

$$(\mathcal{F} \times \mathcal{G}) \# \mathcal{R} \Leftrightarrow \mathcal{R}[\mathcal{F}] \# \mathcal{G} \Leftrightarrow \mathcal{F} \# \mathcal{R}^{-}[\mathcal{G}].$$

The proof is a relatively straightforward extension of proposition 2.3.7.

2.4 Convergence

Definition 2.4.1. *If (x, \mathcal{F}) is contained in a relation $\xi \subseteq X \times \mathbb{F}X$, we say*

\mathcal{F} converges to x in ξ .

This situation is denoted $x \in \lim_{\xi} \mathcal{F}$.

Alternatively, $x \in \lim_X \mathcal{F}$ or $x \in \lim \mathcal{F}$ (where unambiguous).

Definition 2.4.2. *A pair (X, ξ) consisting of a set X and a relation ξ between X and $\mathbb{F}X$ is a **convergence space** when*

for every $x \in X$, (centered)

$$x \in \lim_{\xi} \{x\}^{\uparrow}$$

$\mathcal{G} \geq \mathcal{F}$ and $x \in \lim_{\xi} \mathcal{F} \Rightarrow x \in \lim_{\xi} \mathcal{G}$. (isotone)

The definition of a convergence space requires that we specify which filters go where (the relation ξ), making certain that at least the principal filters of a point converge ($x \in \lim_{\xi} \{x\}^{\uparrow}$), and ensuring that if a filter converges then each finer filter converges (isotone).

Convergence spaces are often defined with a single statement, e.g. $x \in \lim_{\xi} \mathcal{F} \Leftrightarrow \mathcal{F} = \{x\}^{\uparrow}$ defines the discrete convergence.

Definition 2.4.3. *If $\xi \subseteq \tau$ for two convergence spaces ξ and τ , then we say that*

ξ is **finer** than τ

or equivalently

τ is **coarser** than ξ .

We denote this situation $\xi \geq \tau$ or $\tau \leq \xi$.

Equivalently, one may say:

$$x \in \lim_{\xi} \mathcal{F} \Rightarrow x \in \lim_{\tau} \mathcal{F}.$$

Definition 2.4.4. A convergence is **Hausdorff** if each filter has a most one limit point.

Definition 2.4.5. If $\mathcal{G} \in \mathbb{F}X$ and $f : X \rightarrow Y$, then the **image filter** is $f[\mathcal{G}] \in \mathbb{F}Y$ as in definition 2.3.4. More explicitly,

$$f[\mathcal{G}] := \{ f[G] \subseteq Y : G \in \mathcal{G} \}^{\uparrow}.$$

Alternatively,

$$A \in f[\mathcal{G}] \Leftrightarrow f^{-}[A] \in \mathcal{G}.$$

Definition 2.4.6. A function $f : X \rightarrow Y$ between two convergence spaces is **continuous** when

for every $x \in X$

and every $\mathcal{G} \in \mathbb{F}X$,

$$x \in \lim_X \mathcal{G} \Rightarrow f(x) \in \lim_Y f[\mathcal{G}].$$

Definition 2.4.7. The **adherence** of a set F is the set

$$\text{adh } F := \bigcup_{\mathcal{G} \geq \{F\}} \lim \mathcal{G}.$$

The adherence of a set is a preclosure operator: it satisfies all of the closure axioms with the exception of idempotence. Restricted to a topological space, the adherence is in fact the closure.

Definition 2.4.8. *The **adherence** of a filter \mathcal{F} is the set*

$$\text{adh } \mathcal{F} := \bigcup_{\mathcal{G} \geq \mathcal{F}} \lim \mathcal{G}.$$

Proposition 2.4.9. *For every filter \mathcal{F} ,*

$$\text{adh } \mathcal{F} = \bigcup_{\mathcal{G} \# \mathcal{F}} \lim \mathcal{G} = \bigcup_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim \mathcal{U}.$$

Proof.

Number the left-most expression as (1), the right-most as (3), and middle as (2).

(1) \subseteq (2) Let $x \in \text{adh } \mathcal{F}$.

Then there must be a $\mathcal{H} \geq \mathcal{F}$ such that $x \in \lim \mathcal{H}$.

Since $\mathcal{H} \geq \mathcal{F}$, we have $\mathcal{H} \# \mathcal{F}$.

So $x \in \bigcup_{\mathcal{G} \# \mathcal{F}} \lim \mathcal{G}$.

(2) \subseteq (3) Let $x \in \bigcup_{\mathcal{G} \# \mathcal{F}} \lim \mathcal{G}$.

Then there is a $\mathcal{H} \# \mathcal{F}$ such that $x \in \lim \mathcal{H}$.

Then $\mathcal{H} \vee \mathcal{F}$ exists as a filter and $\mathcal{H} \vee \mathcal{F} \geq \mathcal{F}$.

By proposition 2.2.2, there is an ultrafilter $\mathcal{U} \geq \mathcal{H} \vee \mathcal{F}$.

The ultrafilter $\mathcal{U} \geq \mathcal{F}$ so $\mathcal{U} \in \mathbb{U}(\mathcal{F})$.

But also, $\mathcal{U} \geq \mathcal{H}$, so $x \in \lim \mathcal{U}$.

Thus $x \in \bigcup_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim \mathcal{U}$

(3) \subseteq (1) Let $x \in \bigcup_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim \mathcal{U}$.

Then there exists an ultrafilter $\mathcal{H} \in \mathbb{U}(\mathcal{F})$ such that $x \in \lim \mathcal{H}$.

Since $\mathcal{H} \in \mathbb{U}(\mathcal{F})$, we have $\mathcal{H} \geq \mathcal{F}$.

So $x \in \bigcup_{\mathcal{G} \geq \mathcal{F}} \lim \mathcal{G}$.

□

Corollary 2.4.10. *If \mathcal{U} is an ultrafilter in a convergence space, then*

$$\text{adh} \mathcal{U} = \lim \mathcal{U}.$$

Definition 2.4.11. *A set C is **closed** in a convergence space when*

$$\text{adh} C = C.$$

A closed set contains its limit points.

Definition 2.4.12. *A set O in a convergence space is **open** when for every filter \mathcal{F} ,*

$$O \# \lim \mathcal{F} \Rightarrow O \in \mathcal{F}.$$

Any filter on an open set eventually gets “smaller” than that open set.

Definition 2.4.13. *A convergence is a **pseudotopology** when for every filter \mathcal{F} ,*

$$\lim \mathcal{F} = \bigcap_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim \mathcal{U}.$$

Proposition 2.4.14. *In a pseudotopology,*

$$\lim \mathcal{F} = \bigcap_{\mathcal{G} \# \mathcal{F}} \text{adh} \mathcal{G}$$

for every filter \mathcal{F} .

Proof.

In a pseudotopology,

\subseteq Let $x \in \lim \mathcal{F}$.

Then $x \in \bigcap_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim \mathcal{U}$.

That is, $x \in \lim \mathcal{U}$ for every $\mathcal{U} \in \mathbb{U}(\mathcal{F})$.

Consider a filter \mathcal{G} such that $\mathcal{G} \# \mathcal{F}$.

Then there exists an ultrafilter $\mathcal{W} \geq \mathcal{G} \vee \mathcal{F}$.

The ultrafilter $\mathcal{W} \geq \mathcal{G}$ and $\mathcal{W} \geq \mathcal{F}$.

So $x \in \lim \mathcal{W}$ and $\mathcal{W} \in \mathbb{U}(\mathcal{F})$.

Since this works for every filter \mathcal{G} such that $\mathcal{G} \# \mathcal{F}$, we have $x \in \bigcap_{\mathcal{G} \# \mathcal{F}} \text{adh } \mathcal{G}$.

\supseteq Suppose contrapositively, that $x \notin \bigcap_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim \mathcal{U}$

Then there is some ultrafilter $\mathcal{U} \in \mathbb{U}(\mathcal{F})$ such that $x \notin \lim \mathcal{U}$.

But $\mathcal{U} \# \mathcal{F}$ and $\text{adh } \mathcal{U} = \lim \mathcal{U}$ by corollary 2.4.10.

Which means $x \notin \text{adh } \mathcal{U}$.

So $x \notin \bigcap_{\mathcal{G} \# \mathcal{F}} \text{adh } \mathcal{G}$.

□

Definition 2.4.15. In a topological space X , the **neighborhood filter** of a point $x \in X$ is

$$\mathcal{N}(x) := \{O \subseteq X : O \text{ open set, } x \in O\}^\uparrow.$$

If $N \in \mathcal{N}(x)$, then N is a **neighborhood** of x .

Definition 2.4.16. A convergence is *topological* when

for every $x \in X$,

$$x \in \lim \mathcal{F} \Leftrightarrow \mathcal{F} \geq \mathcal{N}(x) := \bigwedge_{x \in \lim \mathcal{G}} \mathcal{G}$$

and for every set $C \subseteq X$,

$$\text{adh}(\text{adh } C) = \text{adh } C.$$

Proposition 2.4.17. For a filter \mathcal{F} in a topological space,

$$\text{adh } \mathcal{F} = \bigcap_{F \in \mathcal{F}} \text{cl } F$$

where cl is the standard topological closure.

Proof.

\subseteq Let $x \in \text{adh } \mathcal{F}$, then there is a $\mathcal{G} \geq \mathcal{F}$ where $x \in \lim \mathcal{G}$.

Since \mathcal{G} converges to x in a topology, $\mathcal{G} \geq \mathcal{N}(x)$.

So every element of \mathcal{G} intersects every neighborhood of x .

And since $\mathcal{F} \subseteq \mathcal{G}$, every element of \mathcal{F} also intersects every neighborhood of x .

So x is a closure point for every $F \in \mathcal{F}$.

\supseteq Let $x \in \bigcap_{F \in \mathcal{F}} \text{cl } F$

and consider $\mathcal{F} \vee \mathcal{N}(x)$ (as defined in 2.4.16)

then $\mathcal{F} \vee \mathcal{N}(x) \geq \mathcal{N}(x)$, so it converges to x .

Since it is finer than \mathcal{F} and converges to x , then $x \in \text{adh } \mathcal{F}$

□

Definition 2.4.18. The *adherence filter* of a filter \mathcal{F} is

$$\text{adh}^{\natural} \mathcal{F} := \{ \text{adh } F : F \in \mathcal{F} \}^{\uparrow}.$$

Definition 2.4.19. A convergence is *regular* when

for every filter \mathcal{F} ,

$$\lim \mathcal{F} = \lim (\text{adh}^{\natural} \mathcal{F}).$$

Definition 2.4.20. If X and Y are two convergences, then a filter $\mathcal{H} \in \mathbb{F}(X \times Y)$ converges to (x, y) in the *product convergence* when there are two filters $\mathcal{F} \in \mathbb{F}X$ and $\mathcal{G} \in \mathbb{F}Y$ such that

$$x \in \lim \mathcal{F}, y \in \lim \mathcal{G}, \text{ and } \mathcal{H} \geq \mathcal{F} \times \mathcal{G}.$$

2.5 Compact Sets

In a topological space, a set is **compact** if every open cover of it has a finite sub-cover. It is well known (again, within topological spaces) that this is equivalent to every ultrafilter on that space having a least one limit point. By the equivalence in proposition 2.4.9, then, this is equivalent to every filter having non-empty adherence – which is the definition we will use for convergence spaces.

Definition 2.5.1. A subset K of a convergence space is *compact* when

for every filter \mathcal{F} ,

$$K \# \mathcal{F} \Rightarrow K \# \text{adh } \mathcal{F}.$$

By the above definition, an entire space X is compact if for any filter $\mathcal{F} \in \mathbb{F}X$, $\text{adh } \mathcal{F} \neq \emptyset$.

Definition 2.5.2. A subset K of a convergence space is **relatively compact** when for every filter \mathcal{F} ,

$$K \# \mathcal{F} \Rightarrow \text{adh } \mathcal{F} \neq \emptyset.$$

Compactness and relative compactness for sets differ only by which *set* that they are compact at; compact sets are compact at themselves and relatively compact sets are compact at the space itself. This leads to the following generalization.

Definition 2.5.3. A set K is **compact at a set L** when for every filter \mathcal{F} ,

$$K \# \mathcal{F} \Rightarrow L \# \text{adh } \mathcal{F}.$$

Definition 2.5.4. A convergence is **locally compact** if every convergent filter contains a compact set.

In a convergence space, the open cover definition is not always the same as definition 2.5.1. This convergence definition of compactness more closely resembles the topological definition of sequential compactness. A topological space is **sequentially compact** if every sequence on it has a convergence subsequence. The convergence definition — analogously — requires a finer filter (2.4.8) that converges.

The rest of this section is devoted to showing the equivalence of the convergence and topological definitions.

Definition 2.5.5. An family \mathcal{O} is an **ideal-base** when

$$A, B \in \mathcal{O} \Rightarrow A \cup B \in \mathcal{O} \text{ (Closed under finite unions)}$$

Additionally, if \mathcal{A} is a family,

$$\mathcal{A}^{\cup} := \left\{ \bigcup_{S \in \mathcal{S}} S : \mathcal{S} \subseteq \mathcal{A}, |\mathcal{S}| < \infty \right\}$$

is an ideal-base.

Definition 2.5.6. An open cover \mathcal{O} is an **ideal open cover** when

$$\mathcal{O} = \mathcal{O}^{\cup}.$$

Proposition 2.5.7. The following are equivalent in a topological space:

1. Every open cover of a set K has a finite subcover.
2. Every ideal open cover of K has an element that contains K .

Proof.

$1 \Rightarrow 2$ Let \mathcal{O} be an open ideal cover of a set K

Then \mathcal{O} is, in particular, an open cover.

So by (1), a finite union of elements of \mathcal{O} covers K .

But since \mathcal{O} is also an ideal open cover, the union of finite elements is also an element of \mathcal{O}

So \mathcal{O} contains an element that by itself covers K .

$2 \Rightarrow 1$ Let \mathcal{O} be an open cover of K .

Then \mathcal{O}^{\cup} is an ideal open cover of K .

So there is $O \in \mathcal{O}^{\cup}$ such that $K \subseteq O$ by (2).

The set O is a finite union of elements in \mathcal{O} .

So K has open cover in \mathcal{O} .

□

Proposition 2.5.8. The following are equivalent in a topological space:

1. K is compact.
2. Every ideal open cover of K has an element that contains K .

Proof.

$\neg 2 \Rightarrow \neg 1$ Suppose there is an ideal open cover \mathcal{O} where $K \subseteq \bigcup_{O \in \mathcal{O}} O$,

but for all $O \in \mathcal{O}$, $K \not\subseteq O$.

Then $K \# \{O^c : O \in \mathcal{O}\}^\uparrow$.

If K was compact,

$K \# \{O^c : O \in \mathcal{O}\}^\uparrow$ would imply that

$$\text{adh} \left(K \vee \{O^c : O \in \mathcal{O}\}^\uparrow \right) \neq \emptyset$$

$\Rightarrow \bigcap_{O \in \mathcal{O}} \text{cl} O^c \neq \emptyset$ (we can consider just the filter base)

$\Rightarrow \bigcap_{O \in \mathcal{O}} O^c \neq \emptyset$ (O^c is closed since O is open)

So there is $x_0 \in K$ and $x_0 \in \bigcap_{O \in \mathcal{O}} O^c$

So $x_0 \in \left(\bigcup_{O \in \mathcal{O}} O \right)^c$.

Which contradicts $x_0 \in K \subseteq \left(\bigcup_{O \in \mathcal{O}} O \right)$.

$2 \Rightarrow 1$ Suppose K is not compact,

then for some $\mathcal{F} \in \mathbb{F}X$ with $K \# \mathcal{F}$,

$$K \cap \text{adh} \mathcal{F} = \emptyset$$

$$\equiv K \cap \left(\bigcap_{F \in \mathcal{F}} \text{cl} F \right) = \emptyset$$

Consider $\mathcal{F}' := \{ \text{cl} F : F \in \mathcal{O} \}$.

$(K \# \mathcal{F} \Rightarrow K \# \mathcal{F}'$ and

$\text{adh} \mathcal{F} = \text{adh} \mathcal{F}')$

$$\Rightarrow K \cap \left(\bigcap_{F \in \mathcal{F}'} F \right) = \emptyset \quad (*)$$

Let $\mathcal{O} := \{ F^c : F \in \mathcal{F} \}$

then \mathcal{O} is an open ideal cover

and $(*) \Rightarrow K \subseteq \bigcap_{F \in \mathcal{F}'} F$ (cover of K)

Suppose 2 was true,

then there is $O \in \mathcal{O}$ where $K \subseteq O$

where $K \subseteq O = F^c$ some $F \in \mathcal{F}'$

$$\equiv K \cap F = \emptyset.$$

But $K \# \mathcal{F}$. Contradiction.

□

Corollary 2.5.9. *The following are equivalent in a topological space:*

1. *Every open cover of K has a finite subcover.*
2. *Every ideal open cover of K has an element that contains K .*
3. *K is compact.*
4. *For all $\mathcal{U} \in \mathbb{U}(K)$, we have $K \# \lim \mathcal{U}$.*

2.6 Compact Filters and Families

While a function whose domain is compact has many useful properties, it is not always the case that the domain can or should be made compact. It is still useful, though, to have compactness-like properties on non-compact sets.

Consider a lower semicontinuous function $f : X \rightarrow \mathbb{R}$.

If X is compact, then f has a minimum (a property that lower semicontinuous functions share with continuous functions).

If X is a topological space (but not necessarily a compact space), it suffices that any countably-based filter that meshes with the family

$$\mathcal{A} := \{ \{ x : f(x) \leq r \} : r \geq \inf(f) \}$$

has non-empty adherence.

For a sequence converging in the codomain to $\inf(f)$ (possibly equal to $-\infty$), we can construct a sequence of domain elements whose image bounds the converging sequence below. By the above condition, the filter generated by the constructed sequence must mesh with \mathcal{A} and therefore has non-empty adherence. The adherence points (which need not exist without the condition) show the existence of a converging subsequence that converges to a minimum despite being “beneath” the infimum.

There is a similarity to the condition above and the definition of relative compactness. While every filter that meshes with a compact set has non-empty adherence, in this case every (countably based) filter that meshes with the family \mathcal{A} has non-empty adherence. With that idea in mind, we have the following definition.

Definition 2.6.1. *A family \mathcal{K} in a convergence space is a **relatively compact family** when*

for every filter \mathcal{F} ,

$$\mathcal{K} \# \mathcal{F} \Rightarrow \text{adh } \mathcal{F} \neq \emptyset.$$

Definition 2.6.2. *A family \mathcal{K} in a convergence space is **compact at a family \mathcal{L}** when*

for every filter \mathcal{F} ,

$$\mathcal{K} \# \mathcal{F} \Rightarrow \mathcal{L} \# \text{adh } \mathcal{F}.$$

A family is **compact** if it is compact at itself.

It is interesting to note that convergence in a pseudotopology is precisely compactness of the convergent filter at that point.

Let X be a pseudotopology. Then $x \in \lim_X \mathcal{F}$

$\Leftrightarrow x \in \bigcap_{\mathcal{H} \# \mathcal{F}} \text{adh } \mathcal{H}$ (by definition 2.4.13)

\Leftrightarrow for all filters \mathcal{H} , $(\mathcal{H} \# \mathcal{F} \Rightarrow \{x\} \# \text{adh } \mathcal{H})$

$\Leftrightarrow \mathcal{F}$ compact at $\{x\}$.

CHAPTER 3
FUNCTION SPACE PRELIMINARIES

Definition 3.0.3. *The set of all functions from X to Y is*

$$Y^X := \{ f \subseteq X \times Y : f \text{ is a function} \}.$$

Definition 3.0.4. *The **evaluation map** is*

$$\begin{aligned} \langle \cdot, \cdot \rangle : X \times Y^X &\rightarrow Y \\ \langle x, f \rangle &\mapsto f(x) \end{aligned}$$

When considered as a map on the product space $X \times Y^X$, the evaluation map is denoted $\text{ev} : X \times Y^X \rightarrow Y$.

Definition 3.0.5. *The **image filter** of two filters $\mathcal{F} \in \mathbb{F}Y^X$ and $\mathcal{G} \in \mathbb{F}X$ under the evaluation map is*

$$\langle \mathcal{G}, \mathcal{F} \rangle := \{ \langle G, F \rangle : G \in \mathcal{G}, F \in \mathcal{F} \}^\uparrow$$

or equivalently

$$\text{ev}[\mathcal{G} \times \mathcal{F}] := \{ \text{ev}[G \times F] : G \in \mathcal{G}, F \in \mathcal{F} \}^\uparrow.$$

Where $\langle G, F \rangle := \{ f(x) : x \in G, f \in F \} \subseteq Y$ and $\langle \mathcal{G}, \mathcal{F} \rangle = \text{ev}[\mathcal{G} \times \mathcal{F}] \in \mathbb{F}Y$.

3.1 Pointwise Convergence

Definition 3.1.1. *A filter \mathcal{F} converges to f in the **pointwise convergence** on Y^X when*

for every $x \in X$,

$$f(x) \in \lim_Y \langle x, \mathcal{F} \rangle.$$

This is also denoted $f \in \lim_p \mathcal{F}$.

3.2 Continuous Convergence

Definition 3.2.1. A filter \mathcal{F} converges to a function f in the *continuous convergence* $[X, Y]$ on $C(X, Y)$ when

for every $x \in X$

and every $\mathcal{G} \in \mathbb{F}X$,

$$x \in \lim_X \mathcal{G} \Rightarrow f(x) \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle.$$

The continuous convergence is easily verified to be a convergence.

Proposition 3.2.2. The continuous convergence $[X, Y]$ is the coarsest convergence making the evaluation map continuous.

Proof.

ev is continuous

Let $(x, f) \in \lim_{X \times [X, Y]} \mathcal{H}$.

Then there is $\mathcal{G} \in \mathbb{F}X$ and $\mathcal{F} \in \mathbb{F}C(X, Y)$ such that $x \in \lim_X \mathcal{G}$, $f \in \lim_{[X, Y]} \mathcal{F}$, and $\mathcal{G} \times \mathcal{F} \leq \mathcal{H}$.

By the properties of $[X, Y]$, $f(x) \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle = \text{ev}[\mathcal{G} \times \mathcal{F}]$

So, $f(x) \in \lim_Y \text{ev}[\mathcal{G} \times \mathcal{F}]$ and since $\mathcal{H} \geq \mathcal{G} \times \mathcal{F}$, $f(x) \in \lim_Y \text{ev}[\mathcal{H}]$

Therefore ev is continuous.

$[X, Y]$ coarsest making ev cont

Suppose α is a convergence on $C(X, Y)$ making ev continuous

Let $f \in \lim_\alpha \mathcal{F}$.

Then for every $x \in X$, $\mathcal{G} \in \mathbb{F}X$ where $x \in \lim_X \mathcal{G}$,

$f(x) \in \lim_Y \text{ev}[\mathcal{G} \times \mathcal{F}]$ by the continuity of the evaluation map.

By the definition of $[X, Y]$, $f \in \lim_{[X, Y]} \mathcal{F}$.

□

Lemma 3.2.3.

$$\left. \begin{array}{l} Y \text{ regular} \\ \mathcal{F} \in \mathbb{F}Y^X \\ f \in \lim_{[X, Y]} \mathcal{F} \end{array} \right\} \Rightarrow f \text{ continuous}$$

Proof.

Let $x_0 \in \lim_X \mathcal{G}$.

Since $f \in \lim_{[X, Y]} \mathcal{F}$, $f(x_0) \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle$.

Since Y is regular, $f(x_0) \in \lim_Y \text{adh}^{\natural} \langle \mathcal{G}, \mathcal{F} \rangle$.

So it suffices to show that $f[\mathcal{G}] \geq \text{adh}^{\natural} \langle \mathcal{G}, \mathcal{F} \rangle$,

thereby showing $f(x_0) \in \lim_Y f[\mathcal{G}]$.

Consider $\langle G, F \rangle \in \langle \mathcal{G}, \mathcal{F} \rangle$.

For every $x \in G$,

$f(x) \in \lim_Y \langle x, \mathcal{F} \rangle$ (continuous convergence \Rightarrow pointwise convergence)

and $\langle x, \mathcal{F} \rangle \geq \langle G, F \rangle$

and so $f(x) \in \text{adh} \langle G, F \rangle$

Since this is true for every $x \in G$,

we have $f[G] \subseteq \text{adh} \langle G, F \rangle$

and since the above is true for every $G \in \mathcal{G}$ and $F \in \mathcal{F}$,

we have $f[\mathcal{G}] \geq \text{adh}^{\natural} \langle \mathcal{G}, \mathcal{F} \rangle$.

□

3.3 The Exponential Laws

Consider a function

$$f : X \times Y \rightarrow Z.$$

It is possible for f to be separately continuous in one or both variables, but yet not be continuous. Consider $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ that $(x, y) \mapsto \frac{2xy}{x^2 + y^2}$ and $(0, 0) \mapsto 0$. For a fixed y (or x), f is continuous in x (or y) as a composition of continuous functions. Yet the image of f under the filter generated by the sequence $\left\{ \left(\frac{1}{n}, \frac{1}{n} \right) \right\}_n$ does not converge to the function value at 0. So separate continuity in each variable does not suffice to show continuity of the function on the product.

Ideally, then, we should be able to show conditions under which separate continuity in one or both variables implies joint continuity.

More formally, f will be separately continuous in Y if for each fixed $y_0 \in Y$, the one parameter function $f(\cdot, y_0) : X \rightarrow Z$ is continuous.

Parameterizing in Y , we can then consider the “transposed” map

$$\begin{aligned} {}^t f : Y &\rightarrow C(X, Z) \\ y &\mapsto \left(x \mapsto f(x, y) \right) \end{aligned}$$

which represents the separate continuity of f in Y .

An interesting strategy in topology and elsewhere is looking at the structure of the function space to try and understand things about the base spaces. In this case, we are looking for the relationship between joint continuity and separate continuity. As it stands, ${}^t f$ is a bare set-theoretic function with a convergence structure on Y but no particular structure on $C(X, Z)$. We might consider putting some structure on $C(X, Z)$ to better “read” information from the function space. Typical non-trivial candidates might include the compact-open topology, etc.

Rather than specifying a particular structure (and therefore inheriting the re-

strictions that it requires), it might be better to ask what it is we want from the structure.

Ideally, if α is a convergence on $C(X, Z)$, we would want

$${}^t f : Y \rightarrow C_\alpha(X, Z) \text{ continuous} \Leftrightarrow f : X \times Y \rightarrow Z \text{ continuous} (*)$$

for any convergence Y and any $f : X \times Y \rightarrow Z$.

Dually, we could also start from $g : Y \rightarrow C_\alpha(X, Z)$, and have

$$g : Y \rightarrow C_\alpha(X, Z) \text{ continuous} \Leftrightarrow \widehat{g} : X \times Y \rightarrow Z \text{ continuous} (**)$$

where \widehat{g} maps $(x, y) \mapsto g(y)(x)$.

Both (*) and (**) are equivalent, since the maps $f \rightarrow {}^t f$ and $g \rightarrow \widehat{g}$ are inverses.

Thus we have the following bijection:

For all Y ,

$$C(Y, C_\alpha(X, Z)) \cong C(X \times Y, Z)$$

Further background information can be found in [7].

Proposition 3.3.1. *If $C_\alpha(X, Z)$ has property (*) (or equivalently (**)), then*

$$C_\alpha(X, Z) = [X, Z].$$

Proof.

\geq Any such structure α would work in particular for $Y = C_\alpha(X, Z)$ and for $g = \text{id} : C_\alpha(X, Z) \rightarrow C_\alpha(X, Z)$ (the identity map).

By the right implication of (**), this means that $\widehat{\text{id}} : X \times C_\alpha(X, Z) \rightarrow Z$ would be continuous.

The function $\widehat{\text{id}}$ is defined to be $(x, f) \mapsto \text{id}(f)(x) = f(x)$, thus $\widehat{\text{id}} = \text{ev}$ is the evaluation map.

So if we have (*), then we should also have the evaluation map continuous.

Thus $C_\alpha(X, Y) \geq [X, Z]$ by 3.2.2.

≤ The continuous convergence on $C(X, Z)$ makes $ev : X \times C(X, Z) \rightarrow Z$ continuous.

So by (*), ${}^t ev = id : [X, Z] \rightarrow C_\alpha(X, Z)$ is continuous.

So $[X, Z] \geq C_\alpha(X, Z)$.

□

Theorem 3.3.2. *For any converge spaces X , Y , and Z*

$$[Y, [X, Z]] \cong [X \times Y, Z]$$

where \cong is a homeomorphism.

Proof.

Consider ${}^t : [X \times Y, Z] \rightarrow [Y, [X, Z]]$.

The map t is a bijection, and $\hat{}$ is its inverse.

Let $f \in \lim_{[X \times Y, Z]} \mathcal{H}$,

then $(x, y) \in \lim_{X \times Y} \mathcal{F} \times \mathcal{G}$

implies $f(x, y) \in \lim_Z \langle \mathcal{F} \times \mathcal{G}, \mathcal{H} \rangle$

under $ev_{X \times Y} : (X \times Y) \times C(X \times Y, Z) \rightarrow Z$

To show continuity, we must have

$${}^t f(y) \in \lim_{[Y, [X, Z]]} {}^t [[\mathcal{H}]]$$

⇔ for all $\mathcal{G} \in \mathbb{F}Y$ where $y \in \lim_Y \mathcal{G}$,

$${}^t f(y) \in \lim_{[X, Z]} \langle \mathcal{G}, {}^t [\mathcal{H}] \rangle$$

under $\text{ev}_Y : Y \times C(Y, C(X, Z)) \rightarrow C(X, Z)$.

\Leftrightarrow for all $\mathcal{G} \in \mathbb{F}Y$ where $y \in \lim_Y \mathcal{G}$,

and for all $x \in X$, $\mathcal{F} \in \mathbb{F}X$ where $x \in \lim_X \mathcal{F}$,

${}^t f(y)(x) = f(x, y) \in \lim_Z \langle \mathcal{F}, \langle \mathcal{G}, {}^t[\mathcal{H}] \rangle \rangle$

under $\text{ev}_X : X \times C(X, Z) \rightarrow Z$.

Now,

$f(x, y) \in \lim_Z \langle \mathcal{F} \times \mathcal{G}, \mathcal{H} \rangle$

$\Leftrightarrow f(x, y) \in \lim_Z \{ \langle F \times G, H \rangle : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H} \}$

$\Leftrightarrow f(x, y) \in \lim_Z \{ \{ \langle (x, y), h \rangle : x \in F, y \in G, h \in H \} : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H} \}$

$\Leftrightarrow f(x, y) \in \lim_Z \{ \{ h(x, y) : x \in F, y \in G, h \in H \} : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H} \}$

$\Leftrightarrow f(x, y) \in \lim_Z \{ \{ \langle x, {}^t h(y) \rangle : x \in F, y \in G, {}^t h \in {}^t H \} : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H} \}$

$\Leftrightarrow f(x, y) \in \lim_Z \{ \{ \langle x, \langle y, {}^t h \rangle \rangle : x \in F, y \in G, {}^t h \in {}^t H \} : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H} \}$

$\Leftrightarrow f(x, y) \in \lim_Z \{ \langle F, \langle G, {}^t[H] \rangle \rangle : F \in \mathcal{F}, G \in \mathcal{G}, H \in \mathcal{H} \}$

$\Leftrightarrow f(x, y) \in \lim_Z \langle \mathcal{F}, \langle \mathcal{G}, {}^t[\mathcal{H}] \rangle \rangle$

The continuity of $\hat{}$ is proved similarly. □

3.4 Compact-Open Convergence

Definition 3.4.1. A filter \mathcal{F} converges to a function f in the **compact-open convergence** when

$$\mathcal{F} \geq \mathcal{N}_k(f) := \{ [K, U] : K \text{ compact, } U \text{ open, } f(K) \subseteq U \}^\dagger$$

where $[K, U] = \{ g \in C(X, Y) : g(K) \subseteq U \}$.

The set $C(X, Y)$ equipped with this convergence is denoted $C_k(X, Y)$.

Proposition 3.4.2.

X regular, Hausdorff, topological space

Y topological space

$$\text{ev}|_{K \times C_k(X,Y)} : K \times C_k(X,Y) \rightarrow Y \text{ is continuous}$$

for every compact set $K \subseteq X$.

Proof.

Let $x \in \lim_K \mathcal{G}$,

$$f \in \lim_{C_k(X,Y)} \mathcal{F}.$$

Then $\mathcal{G} \times \mathcal{F}$ is a convergent filter on $K \times C_k(X,Y)$.

To prove continuity, we must find for every $U \in \mathcal{N}(f(x))$ open
a $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $\langle G, F \rangle \subseteq U$.

Let $U \in \mathcal{N}_Y(f(x))$ be an arbitrary open set,

then $f^{-}(U) \in \mathcal{N}_X(x)$ is open

and $f^{-}(U) \cap K \in \mathcal{N}_K(x)$ is open.

By the regularity of X and therefore K , there is $V \in \mathcal{N}_K(x)$ closed,
such that $V \subseteq f^{-}(U) \cap K$.

As a consequence of $\mathcal{G} \geq \mathcal{N}_K(x)$, $V \in \mathcal{G}$.

The set V is a closed subset of a compact space and therefore compact itself.

Moreover $f(V) \subseteq f(f^{-}U \cap K) \subseteq U$.

Then since \mathcal{F} converges in $C_k(X,Y)$,

it must be the case that $[V, U] \in \mathcal{F}$.

So $\langle V, [V, U] \rangle \subseteq U$.

So $\text{ev}|_{K \times C_k(X,Y)}$ is continuous.

□

Proposition 3.4.3.*X locally compact topological space**Y topological space* $\text{ev} : X \times C_k(X, Y) \rightarrow Y$ is continuous.*Proof.*Let $x \in \lim_X \mathcal{G}$, $f \in \lim_{C_k(X, Y)} \mathcal{F}$.By local compactness, $x \in K \in \mathcal{G}$, for some compact set K .So $\text{ev}[\mathcal{G} \vee K \times \mathcal{F}] \geq \mathcal{N}(f(x))$ (by proposition 3.4.2)And since $\mathcal{G} \vee K = \mathcal{G}$ (\mathcal{G} contains all supersets of K), $\text{ev}[\mathcal{G} \times \mathcal{F}] \geq \mathcal{N}(f(x))$.

□

Proposition 3.4.4. *If X, Y topological and X regular (or Hausdorff) then* *$C_k(X, Y)$ is the coarsest convergence making*

$$\text{ev}|_{K \times C_k(X, Y)} : K \times C_k(X, Y) \rightarrow Y$$

continuous for every compact set $K \subseteq X$ *Proof.*Let $C_\alpha(X, Y) \leq C_k(X, Y)$ such that for any compact set $K \subseteq X$, $\text{ev} : K \times C_\alpha(X, Y) \rightarrow$ Y is continuous.Let $f \in \lim_{C_\alpha(X, Y)} \mathcal{F}$ and $K \subseteq X$ be compact.Let U be open in Y such that $f(K) \subseteq U$.Then for all $x \in K$ and $\mathcal{G} \in \mathbb{F}X$ such that $x \in \lim_X \mathcal{G}$,

$f(x) \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle$ by continuity of $\text{ev} : K \times C_\alpha(X, Y) \rightarrow Y$ and $f(x) \in U$ because $f(K) \subseteq U$.

Therefore $U \in \langle \mathcal{G}, \mathcal{F} \rangle$, i.e. there is $F_x \in \mathcal{F}$, $G_x \in \mathcal{G}$ where $\langle G_x, F_x \rangle \subseteq U$

By the above, then, there must be $F_x \in \mathcal{F}$ and $O_x \in \mathcal{G}$ open where

$$\langle F_x, O_x \rangle \subseteq U.$$

The union $\bigcup_{x \in K} O_x$ covers K ,

so we can find a finite subcover $\bigcup_{i=1}^n O_{x_i} \supseteq K$.

Using the associated F_x 's, $\bigcap_{i=1}^n F_{x_i} \in \mathcal{F}$ (\mathcal{F} is closed under finite intersections).

Further, $\bigcup_{i=1}^n O_{x_i} \subseteq U$.

So $\langle \bigcap_{i=1}^n F_{x_i}, \bigcup_{i=1}^n O_{x_i} \rangle \subseteq U$.

and therefore $\bigcap_{i=1}^n F_{x_i} \subseteq [K, U]$.

□

Corollary 3.4.5. *If X, Y topological,*

$$C_k(X, Y) \leq [X, Y]$$

Proof.

For any $f \in \lim_{[X, Y]} \mathcal{F}$,

the evaluation map is continuous.

So it is in particular continuous for every compact set $K \subseteq X$.

Therefore by prop 3.4.2, $f \in \lim_{C_k(X, Y)} \mathcal{F}$.

□

Theorem 3.4.6. *If X locally compact and Y topological,*

$$C_k(X, Y) = [X, Y]$$

Proof.

\leq By corollary 3.4.5

\geq For $f \in \lim_{C_k(X,Y)} \mathcal{F}$ and X locally compact,

then by proposition 3.4.3, the evaluation map is continuous.

Which means $f \in \lim_{[X,Y]} \mathcal{F}$.

□

CHAPTER 4

ARZELÀ-ASCOLI

The many forms of the Arzelà-Ascoli theorem all seek to give conditions for relative compactness in the space of continuous functions. Much like the pointwise limit of a sequence of continuous function may not be continuous, an adherence point of a sequence of continuous functions is likewise not necessarily continuous without having stronger conditions placed on it.

The original version of the Arzelà-Ascoli is strongly metric in nature, which does not comport well with the wide variety of fields it has found applications in. Most of these fields adapt the theorem's conditions by using area-specific restrictions and terminology. There is a strong sense in which all of these theorems are really the same thing, and convergence spaces provide a broad enough structure that most or all of them can be shown to be corollaries of a more general theorem.

4.1 Equicontinuous and Evenly Continuous

We start off by defining the traditional version of *equicontinuity*, a uniform or metric notion. We then define *even continuity* a similar, yet purely topological notion that is implied by equicontinuity. Finally, we end up with a filter-centric version of even continuity that we will take to be the definition.

Definition 4.1.1. *A sequence of (continuous) functions $(f_n)_n$ between two metric spaces X and Y is **equicontinuous** at x when*

for every $\varepsilon > 0$,

there exists a $\delta > 0$,

such that for every $n \in \mathbb{N}$,

$$f_n(B_\delta(x)) \subseteq B_\varepsilon(f_n(x)).$$

The set $B_\delta(x)$ and $B_\varepsilon(f_n(x))$ are open balls of radius δ and ε around x and $f_n(x)$ respectively.

The definition of equicontinuity at x says that we can force every function f_n to be within ε radius of its particular image of x using the same δ for every single one of them.

Definition 4.1.2. For topological spaces X and Y ,

a set $H \subseteq C(X, Y)$ is **evenly continuous** at x when

for every $y \in Y$ and every neighborhood U of y ,

there exists a neighborhood V of x and W of y where

for every $f \in H$,

$$f(x) \in W \Rightarrow f(V) \subseteq U.$$

Topological variants of equicontinuity exist, but the weaker (and more complicated) even continuity is more general and has a closer connection to joint continuity of the evaluation map.

We will now show that even continuity can be reinterpreted more straightforwardly in terms of filters.

Proposition 4.1.3. H is evenly continuous at x if and only if

for every filter $\mathcal{F} \geq H$,

$$x \in \lim_X \mathcal{G} \Rightarrow y \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle$$

$$y \in \lim_Y \langle x, \mathcal{F} \rangle$$

Proof.

\Rightarrow Assume $H \subseteq C(X, Y)$ is an evenly continuous set at x .

We must show $(x \in \lim_X \mathcal{G} \text{ and } y \in \lim_Y \langle x, \mathcal{F} \rangle \Rightarrow y \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle)$ for every filter $\mathcal{F} \geq H$.

Let $\mathcal{F} \geq H$, $x \in \lim_X \mathcal{G}$, and $y \in \lim_Y \langle x, \mathcal{F} \rangle$.

Since X and Y are topological, $\mathcal{G} \geq \mathcal{N}_X(x)$ and $\langle x, \mathcal{F} \rangle \geq \mathcal{N}_Y(y)$.

Let $U \in \mathcal{N}_Y(y)$.

By 4.1.2, there is a $V \in \mathcal{N}_X(x)$ and $W \in \mathcal{N}_Y(y)$ such that for all $f \in H$, $(f(x) \in W \Rightarrow f(V) \subseteq U)$.

Since $\langle x, \mathcal{F} \rangle \geq \mathcal{N}_Y(y)$, we can find a $F \in \mathcal{F}$, $F \subseteq H$ such that $\langle x, F \rangle \subseteq W$.

Then $\langle V, F \rangle \subseteq U$. Since we can do this for all $U \in \mathcal{N}_Y(y)$, we have $\langle \mathcal{N}_X(x), \mathcal{F} \rangle \geq \mathcal{N}_Y(y)$.

Further, since $\mathcal{G} \geq \mathcal{N}_X(x)$, $\langle \mathcal{G}, \mathcal{F} \rangle \geq \langle \mathcal{N}_X(x), \mathcal{F} \rangle \geq \mathcal{N}_Y(y)$.

So $y \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle$.

\Leftarrow Assume H is not evenly continuous at x .

Then there is $y_0 \in Y$ and $U_0 \in \mathcal{N}_Y(y_0)$ such that for all $V \in \mathcal{N}_X(x)$ and $W \in \mathcal{N}_Y(y_0)$ there is a $f_{V,W} \in H$ where $f_{V,W}(x) \in W$ and $f_{V,W}(V) \not\subseteq U_0$.

We will find a $\mathcal{F} \geq H$ such that $x \in \lim_X \mathcal{G}$ and $y \in \lim_Y \langle x, \mathcal{F} \rangle$ yet $y \notin \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle$.

Consider $y_0 \in Y$ and $U_0 \in \mathcal{N}_Y(y_0)$ as above.

Then for every $W \in \mathcal{N}_Y(y_0)$,

there is a set $F_W := \{f_{V,P} : V \in \mathcal{N}_X(x), P \in \mathcal{N}_Y(y_0), P \subseteq W\}$ with $f_{V,P}$ as above.

The collection $\mathcal{F} := \{F_W : W \in \mathcal{N}_Y(y_0)\}^\dagger$ is a filter.

– Claim: $y_0 \in \lim_Y \langle x, \mathcal{F} \rangle$

For every $W \in \mathcal{N}_Y(y_0)$ we can find $F_W \in \mathcal{F}$ such that $\langle x, F_W \rangle \subseteq W$.

So $\langle x, \mathcal{F} \rangle \geq \mathcal{N}_Y(y_0)$.

– Claim: for any $\mathcal{G} \geq \mathcal{N}_X(x)$, $y_0 \notin \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle$.

For otherwise, the set $U_0 \supseteq \langle V, F_W \rangle$ for some $V \in \mathcal{N}_X(x)$ and $W \in \mathcal{N}_Y(y_0)$.

But every F_W contains a $f_{V,W}$, which by assumption has $f_{V,W} \notin U_0$.

□

Motivated by the equivalence in the above proposition, we'll define an evenly continuous filter.

Definition 4.1.4. A filter \mathcal{H} on $C(X, Y)$ is *evenly continuous* at x when for every filter $\mathcal{F} \geq \mathcal{H}$,

$$\begin{aligned} x \in \lim_X \mathcal{G} &\quad \Rightarrow \quad y \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle \\ y \in \lim_Y \langle x, \mathcal{F} \rangle & \end{aligned}$$

Note that if $f \in \lim_{[X, Y]} \mathcal{F}$, then \mathcal{F} is automatically evenly continuous.

Lemma 4.1.5.

$$\left. \begin{array}{l} \mathcal{H} \text{ evenly continuous filter on } C(X, Y) \\ \mathcal{F} \geq \mathcal{H} \\ \mathcal{F} \text{ converges pointwise to } f \end{array} \right\} \Rightarrow f \in \lim_{[X, Y]} \mathcal{F}.$$

The proof is immediate from the definition.

Proposition 4.1.6. If X, Y metric, and $f_n : X \rightarrow Y$ for each $n \in \mathbb{N}$ then

$$(f_n)_n \text{ equicontinuous at } x \Rightarrow \{f_n\}_n^\uparrow \text{ evenly continuous at } x.$$

Proof.

Assume $(f_n)_n$ is equicontinuous, i.e. for every $\varepsilon > 0$ there is $\delta > 0$ such that $f_n(B_\delta(x)) \subseteq B_\varepsilon(f_n(x))$.

We will show that $\langle G, \mathcal{F} \rangle \geq \mathcal{N}_Y(y)$ for any \mathcal{G} such that $x \in \lim_X \mathcal{G}$ and any $\mathcal{F} \geq \{f_n\}_n^\uparrow$ where $\langle x, \mathcal{F} \rangle \geq \mathcal{N}_Y(y)$ (i.e. y is the pointwise limit of \mathcal{F}).

Let $B_\varepsilon(y) \in \mathcal{N}_Y(y)$.

Since $\langle x, \mathcal{F} \rangle \geq \mathcal{N}_Y(y)$, we can find an $F \in \mathcal{F}$ such that $\langle x, F \rangle \subseteq B_\varepsilon(y)$ and $F \subseteq (f_n)_k$ some $k \geq n$.

Each function in F is equicontinuous, so there is a $B_\delta(x)$ such that for each $f_n \in F$, $f_n(B_\delta(x)) \subseteq B_\varepsilon(y)$.

Which is equivalent to saying that $\langle B_\delta(x), F \rangle \subseteq B_\varepsilon(y)$.

Since we can do this for every $\varepsilon > 0$, we have $\langle \mathcal{N}_X(x), \mathcal{F} \rangle \geq \mathcal{N}_Y(y)$.

□

4.2 Classic Arzelà-Ascoli and its Limitations

With the definitions of equicontinuity and even continuity in hand, we can now list some of the more common Arzelà-Ascoli variants.

Theorem 4.2.1 (First Arzelà-Ascoli). *A sequence of real-valued (continuous) functions $(f_n)_{n \in \mathbb{N}}$ on a closed and bounded set $[a, b]$ have a uniformly convergent subsequence if and only if the sequence is equicontinuous and uniformly bounded.*

By requiring that the sequence $(f_n)_{n \in \mathbb{N}}$ be equicontinuous, it is automatic that each of the functions $f_n : [a, b] \rightarrow \mathbb{R}$ are themselves individually continuous. So the underlying space is really $C([a, b], \mathbb{R})$. That $[a, b]$ is closed and bounded interval of the real line is another way of saying that it is compact.

Recall, also, that a (sub)sequence $(f_n)_{k \in \mathbb{N}}$ converges uniformly to a function f if

$\|(f_n)_k - f\|_\infty \xrightarrow{k \rightarrow \infty} 0$. In fact, the uniform/infinity norm makes $C([a, b], \mathbb{R})$ a metric space.

Putting all this together, we can see this theorem from a more topological perspective. In a metric space, sequential compactness is the same as compactness. Therefore this first Arzelà-Ascoli theorem gives criteria for a sequence to be relatively compact in the function space $C([a, b], \mathbb{R})$.

This particular version of the theorem is heavily dependent on the metric properties of the spaces involved. In particular, equicontinuity requires the range space to be metric, the domain space is a compact interval, and the theorem itself uses the equivalence of compactness and sequential compactness that is true in metric spaces.

Theorem 4.2.2 (Second Arzelà-Ascoli).

X compact, Hausdorff, topological space

Y metric space

C(X, Y) topology of uniform convergence

$$H \text{ relatively compact in } C(X, Y) \Leftrightarrow \begin{cases} H \text{ equicontinuous} \\ \text{for each } x \in X, \\ \langle x, H \rangle \text{ relatively compact in } Y \end{cases}$$

This version of the Arzelà-Ascoli theorem can be found in Munkres' Topology [6] as well as numerous other places. It is essentially a slightly abstracted version of the first Arzelà-Ascoli theorem. The equicontinuity of H is a variant for sets that simply replaces the $B_\varepsilon(x)$ in definition 4.1.1 with an open neighborhood of x . The theorem replaces uniform boundedness with pointwise relative compactness, requires Y be a metric space rather than \mathbb{R} , and explicitly rather than implicitly requires the topology of uniform convergence (sometimes called the topology of uniform convergence on compact sets) on $C(X, Y)$.

Theorem 4.2.3 (Third Arzelà-Ascoli).

X locally compact, topological space

Y regular, Hausdorff, topological space

$C_k(X, Y)$ compact-open topology

$$\begin{array}{l}
 H \text{ relatively compact for} \\
 C_k(X, Y)
 \end{array}
 \Leftrightarrow
 \left\{
 \begin{array}{l}
 H \text{ evenly continuous} \\
 \text{for each } x \in X, \\
 \langle x, H \rangle \text{ relatively compact in } Y
 \end{array}
 \right.$$

This version of the Arzelà-Ascoli can be found in Kelley's General Topology [4]. It is one of the most general forms of the Arzelà-Ascoli theorem that can be commonly found.

Dropping metricity in Y requires a new condition on H . Even continuity replaces equicontinuity and drops any dependence of the theorem on metric conditions, making this a purely topological version of the theorem. As we saw in proposition 4.1.3, even continuity is better understood as a statement about filters – and filters are a more appropriate replacement for sequences in nonsequential topological spaces.

4.3 Reformulated Arzelà-Ascoli

The most general form of the Arzelà-Ascoli theorem we will present here has its roots in convergence space variants. It requires practically nothing of the domain space, less of the range space, and characterizes compactness directly with respect to the continuous convergence.

The major addition of this thesis is in that we are characterizing compact *filters* rather than compact *sets*.

Theorem 4.3.1.

X convergence space

Y regular, Hausdorff, convergence space

$$\left. \begin{array}{l} \mathcal{H} \text{ evenly continuous filter} \\ \text{for each } x \in X, \\ \langle x, \mathcal{H} \rangle \text{ relatively compact in } Y \end{array} \right\} \Rightarrow \mathcal{H} \text{ relatively compact for } [X, Y]$$

Proof.

Let \mathcal{U} be an ultrafilter on $C(X, Y)$ finer than \mathcal{H} ,

then $\langle x, \mathcal{H} \rangle$ is relatively compact (by assumption)

and $\langle x, \mathcal{U} \rangle$ is an ultrafilter on Y

(image of an ultrafilter under $\langle x, \cdot \rangle$)

Moreover, $\langle x, \mathcal{H} \rangle \leq \langle x, \mathcal{U} \rangle$

So $\lim_Y \langle x, \mathcal{U} \rangle \neq \emptyset$.

So there is a $y_x \in \lim_Y \langle x, \mathcal{U} \rangle$

and Y is Hausdorff, so y_x is unique

Since we can do this process for every x ,

we actually have $f \in \lim_p \mathcal{U}$ (where $f(x) = y_x$).

By Lemma 4.1.5, $f \in \lim_c \mathcal{U}$.

Further, by 3.2.3, f is continuous.

□

A stronger condition on Y yields a converse, and the main theorem of this paper.

Theorem 4.3.2.

X convergence space

Y regular, Hausdorff, pseudotopological convergence space

$$\mathcal{H} \text{ relatively compact for } [X, Y] \Leftrightarrow \left\{ \begin{array}{l} \mathcal{H} \text{ evenly continuous filter} \\ \text{for each } x \in X, \\ \langle x, \mathcal{H} \rangle \text{ relatively compact in } Y \end{array} \right.$$

Proof.

\Leftarrow By theorem 4.3.1.

\Rightarrow Let $\mathcal{F} \geq \mathcal{H}$, $y \in \lim_Y \langle x, \mathcal{F} \rangle$, and $x \in \lim_X \mathcal{G}$ as in the definition of even continuity.

We will show that $y \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle$.

Since Y is pseudotopological, this is equivalent to saying that for any $\mathcal{W} \in \mathbb{U}(\langle \mathcal{G}, \mathcal{F} \rangle)$, $y \in \lim_Y \mathcal{W}$.

Let $\mathcal{W} \in \mathbb{U}(\langle \mathcal{G}, \mathcal{F} \rangle)$.

then $\mathcal{W} \# \langle \mathcal{G}, \mathcal{F} \rangle \Leftrightarrow \mathcal{W} \# \text{ev}[\mathcal{G} \times \mathcal{F}] \Leftrightarrow \text{ev}^-[\mathcal{W}] \# \mathcal{G} \times \mathcal{F} \Leftrightarrow (\text{ev}^-[\mathcal{W}])[\mathcal{G}] \# \mathcal{F}$.

The supremum $\mathcal{L} := (\text{ev}^-[\mathcal{W}])[\mathcal{G}] \vee \mathcal{F}$ must exist and $\mathcal{L} \geq \mathcal{F} \geq \mathcal{H}$.

Pick any $\mathcal{U} \in \mathbb{U}(\mathcal{L})$, then there is $f \in \lim_{[X, Y]} \mathcal{U}$ by the relative compactness of \mathcal{H} .

Because $f \in \lim_{[X, Y]} \mathcal{U}$, in particular $f(x) = \lim_Y \langle x, \mathcal{U} \rangle$.

Further, $\langle x, \mathcal{F} \rangle \leq \langle x, \mathcal{L} \rangle \leq \langle x, \mathcal{U} \rangle$ and $y \in \lim_Y \langle x, \mathcal{F} \rangle$ and Y Hausdorff implies $y = f(x)$.

By the definition of the continuous convergence $[X, Y]$, $f(x) \in \lim_Y \langle \mathcal{G}, \mathcal{U} \rangle$.

To show that $y \in \lim_Y \mathcal{W}$ recall that $\mathcal{U} \geq \mathcal{L} \geq (\text{ev}^-[\mathcal{W}])[\mathcal{G}]$.

$\mathcal{U} \# (\text{ev}^-[\mathcal{W}])[\mathcal{G}] \Leftrightarrow e[\mathcal{G} \times \mathcal{U}] \# \mathcal{W}$, which means $\mathcal{W} \geq \langle \mathcal{G}, \mathcal{U} \rangle$ and $y \in \lim_Y \mathcal{W}$

Thus proving that \mathcal{H} is an evenly continuous filter.

To show $\langle x, \mathcal{H} \rangle$ is relatively compact in Y for every $x \in X$, it suffices to note that $[X, Y]$ makes $\text{ev}_x : C(X, Y) \rightarrow Y$ continuous for every $x \in X$ and that

$\text{ev}_x[\mathcal{H}]$ is therefore relatively compact in Y as the image of a relatively compact set under a continuous function.

□

4.4 Conclusion

A fair amount of mathematical machinery is required to understand the final proof of this thesis: the characterization (in 4.3.2) of compact filters in function spaces. This thesis endeavored to show the role that filters and the continuous convergence play in making theorems of this type work in a more general context. Even so, it barely scratches the surface in explaining how and why these structures are natural choices in many different contexts. Interested readers are directed towards [1] for functional analysis applications, [2] for more information on the continuous convergence, and [5] and [8] for general Arzelà-Ascoli information.

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