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Gorenstein Injective Modules

Emily McLean

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GORENSTEIN INJECTIVE MODULES

by

EMILY MCLEAN

(Under the Direction of Alina Iacob)

ABSTRACT

One of the open problems in Gorenstein homological algebra is: when is the class of Gorenstein injective modules closed under arbitrary direct sums? Our main result gives a sufficient condition for this to happen. We prove that when the ring R is noetherian and such that every R -module has finite Gorenstein injective dimension, every direct sum of Gorenstein injective modules is still Gorenstein injective.

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EMILY MCLEAN

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by

EMILY MCLEAN

Major Professor: Alina Iacob

Committee: Scott Kersey
Goran Lesaja

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CHAPTER 1

INTRODUCTION

Homological Algebra is at the root of modern techniques in many areas, including commutative algebra and algebraic geometry.

While classical homological algebra can be viewed as based on injective and projective modules, Gorenstein homological algebra is its relative version that uses Gorenstein injective and Gorenstein projective modules.

The methods of Gorenstein homological algebra play a part in commutative and non commutative algebra as well as in algebraic geometry and in triangulated category theory.

There is now an active program in Gorenstein homological algebra. It is partly motivated by H. Holm's metatheorem that states: "Every result in classical homological algebra has a counterpart in Gorenstein homological algebra".

In classical homological algebra, the question: "When is the class of injective modules closed under arbitrary direct sums?" was solved by Bass. He proved that the class of injective modules is closed under arbitrary direct sums if and only if the ring is noetherian.

The open question that we consider concerns the Gorenstein counterpart of Bass's theorem. More precisely, we consider the following open question: "For which rings it is true that every direct sum of Gorenstein injective modules is still injective?"

It is known that the noetherianity of the ring is necessary in order for the class of Gorenstein injectives be closed under arbitrary direct sums. It is not known whether or not the condition is sufficient.

We do not give a complete solution to the problem. But our main result gives a sufficient condition in order that every direct sum of Gorenstein injective modules be still such a module. We prove that if the ring R is noetherian and such that every R -module has finite Gorenstein injective dimension the the class of Gorenstein injective modules is closed under arbitrary direct sums.

CHAPTER 2

PRELIMINARIES

2.1 Module

The idea of a module over a ring is a generalization of that of a vector space over a field. We start by recalling a few basic facts about modules. The definitions in this section are from [3].

Definition 2.1.1. *Let R be a ring with 1, and let $(M, +)$ be an abelian group. Suppose we have a function $R \times M \rightarrow M$ where the image of (r, x) is denoted rx . M is a left R -module if the following axioms are satisfied:*

1. $1 \cdot x = x$
2. $r(x + y) = rx + ry \quad \forall r \in R, \forall x, y \in M$
3. $(r + s)x = rx + sx \quad \forall r, s \in R, \forall x \in M$
4. $(rs)x = r(sx) \quad \forall r, s \in R, \forall x \in M$

Throughout this thesis R is a ring with 1. Unless otherwise specified, by module we mean left R -module.

Remark 2.1.2. *If R is a field then the definition above becomes that of a vector space.*

Example 2.1.3. *If $R = \mathbb{Z}$ then any abelian group M is a left \mathbb{Z} -module with*

$$n \cdot x = \underbrace{x + \cdots + x}_{n \text{ times}}, \forall n \in \mathbb{Z}^+, \forall x \in M, \text{ and with}$$

$$(-n) \cdot x = -(nx) \text{ for any } n \in \mathbb{Z}^+ \text{ and } 0 \cdot x = 0.$$

Example 2.1.4. Let R be a ring with 1, and let $n \in \mathbb{Z}^+$.

Let $R^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in R, \text{ for all } i\}$.

Then R^n is an R -module with componentwise addition and multiplication by elements of R :

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

$$r(a_1, a_2, \dots, a_n) = (ra_1, ra_2, \dots, ra_n)$$

Linear Maps (R -homomorphisms)

Definition 2.1.5. Let R be a ring and M, N two left R -modules. A function $f : M \rightarrow N$ is a R -homomorphism if

$$f(x + y) = f(x) + f(y) \quad \forall x, y \in M$$

and

$$f(rx) = rf(x) \quad \forall r \in R, \forall x \in M$$

The set of all homomorphisms from M to N is denoted $\text{Hom}_R(M, N)$.

Remark 2.1.6. $\text{Hom}_R(M, N)$ is an abelian group with the usual addition of maps $((f + g)(x) = f(x) + g(x))$

Example 2.1.7. For any R -module M , the identity map $1_M : M \rightarrow M$ is an R -homomorphism.

Example 2.1.8. The only \mathbb{Z} -homomorphism from $\mathbb{Z}/5\mathbb{Z}$ to \mathbb{Z} is the zero homomorphism.

Proof. Let $f : \mathbb{Z}/5\mathbb{Z} \rightarrow \mathbb{Z}$ be a homomorphism, and let $f(\bar{1}) = x$. Then $f(\bar{2}) = f(\bar{1} + \bar{1}) = f(\bar{1}) + f(\bar{1}) = 2x$, $f(\bar{3}) = 3x$, $f(\bar{4}) = 4x$ and $f(\bar{5}) = 5x$.

But $\bar{5} = \bar{0}$ in $\mathbb{Z}/5\mathbb{Z}$, and $f(\bar{0}) = 0$. Since $5x = 0$ in \mathbb{Z} , it follows that $x = 0$. \square

A similar argument gives:

Example 2.1.9. For any $n \in \mathbb{Z}^+$ we have that the only homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to \mathbb{Z} is the zero homomorphism, that is, $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$.

Proposition 2.1.10. For any ring R with 1, and any left R -module M , we have $\text{Hom}_R(R, M) \simeq M$.

Proof. Let $\theta : \text{Hom}_R(R, M) \rightarrow M$, $\theta(f) = f(1)$. We show that θ is a group isomorphism from $(\text{Hom}(R, M), +)$ to $(M, +)$.

- If $\theta(f) = \theta(g)$ then $f(1) = g(1)$ and therefore $f(r) = rf(1) = rg(1) = g(r)$ for any $r \in R$. This shows that θ is injective.

- For any $x \in M$, let $f : R \rightarrow M$ be defined by $f(r) = rx$. Then f is an R -homomorphism and $\theta(f) = f(1) = x$. So θ is surjective. \square

Remark 2.1.11. By example 2.1.9 and Proposition 2.1.10 above, for any $n \in \mathbb{Z}^+$ we have that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$, but $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$.

We also recall the definition of a submodule, and that of quotient module:

Definition 2.1.12. Let M be an R -module. N is a submodule of M if $N \subseteq M$ and N is a module.

We note that this means that $N \subseteq M$ is closed under the addition on M and under scalar multiplication.

For example, any ideal (left) I of a ring R is a submodule of R regarded as a left R -module.

Definition 2.1.13. *Let R be a ring (with 1) and let N be a submodule of M . The additive abelian quotient group M/N can be made into an R -module by defining scalar multiplication by $r(x + N) = (rx) + N$ for all $r \in R$, $x + N \in M/N$.*

This module is the quotient module M/N .

2.2 Complexes of modules

The definition of Gorenstein injective modules uses two notions: one is the notion of injective module, the other one is the notion of exact complex of modules. We start by recalling the definition of a complex of R -modules.

Definition 2.2.1. *A complex of R -modules is a sequence of R -modules and homomorphisms*

$$C = \dots \rightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \rightarrow \dots$$

such that $d_n \circ d_{n+1} = 0$, $\forall n \in \mathbb{Z}$ (equivalent to $\text{Im } d_{n+1} \subseteq \text{Ker } d_n$, $\forall n \in \mathbb{Z}$).

Definition 2.2.2. *The complex C is said to be exact if $\text{Ker } d_n = \text{Im } d_{n+1}$ for each $n \in \mathbb{Z}$.*

For example, a sequence $0 \rightarrow A \xrightarrow{f} B$ is exact if and only if f is injective, and a sequence $B \xrightarrow{g} C \rightarrow 0$ is exact if and only if g is surjective.

An exact sequence of the form $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is called a *short exact sequence* (we note that this means that f is injective, that g is a surjective map, and such that $\text{Im}(f) = \text{Ker}(g)$).

A particular case of short exact sequence is a *split short exact sequence*.

Definition 2.2.3. The short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ of R -modules is said to be *split exact* if A is isomorphic to a direct summand of B .

The following result gives a characterization of the split short exact sequences:

Proposition 2.2.4. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an exact sequence of R -modules. Then the following are equivalent:

1. The sequence is split exact.
2. There exists an R -homomorphism $f' : B \rightarrow A$ such that $f' \circ f = 1_A$.
3. There exists an R -homomorphism $g'' : C \rightarrow B$ such that $g \circ g'' = 1_C$.

Proof. This is Proposition 1.2.15 in [3]. □

Definition 2.2.5. Let $f : M' \rightarrow M$ be an R -homomorphism. For any R -module A we can define a homomorphism $\text{Hom}(A, f) : \text{Hom}_R(A, M') \rightarrow \text{Hom}_R(A, M)$ by setting $\text{Hom}(A, f)(h) = f \circ h$ for every $h \in \text{Hom}_R(A, M')$

When no confusion is possible, we will simply denote $\text{Hom}(A, f) = \overline{f}$.

Similarly, for a Module R , we can define $\text{Hom}(f, B) : \text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M', B)$, by $\text{Hom}(f, B)(v) = v \circ f$ for every $v \in \text{Hom}(M, B)$.

The behavior of short exact sequences when $\text{Hom}(A, -)$ or $\text{Hom}(-, B)$ is applied is described by the following:

Proposition 2.2.6. *The following statements hold:*

1. *If $0 \rightarrow N' \rightarrow N \rightarrow N''$ is exact, then for any R -module A , the sequence $0 \rightarrow \text{Hom}(A, N') \rightarrow \text{Hom}(A, N) \rightarrow \text{Hom}(A, N'')$ is still exact.*
2. *If $N' \rightarrow N \rightarrow N'' \rightarrow 0$ is an exact sequence, then for any R -module B the sequence $0 \rightarrow \text{Hom}(N'', B) \rightarrow \text{Hom}(N, B) \rightarrow \text{Hom}(N', B)$ is an exact sequence.*

Proof. This is Proposition 1.2.12 in [3]. □

2.3 Injective Modules

Together with the projective and flat modules, the injective modules are essential in classical homological algebra. In fact, classical homological algebra can be viewed as based on injective, projective and flat resolutions.

Most definitions and results in this section are from [3].

Definition 2.3.1. *Given a ring R , and a left R -module E , we say that E is injective if for any left R -module M and any submodule $S \subset M$, every homomorphism $f : S \rightarrow E$ can be extended to a homomorphism $g : M \rightarrow E$ (i.e. there is $g \in \text{Hom}_R(M, E)$ such that $g \circ i = f$ where $i : S \rightarrow M$ is the inclusion map).*

$$\begin{array}{ccc}
 S & \xhookrightarrow{i} & M \\
 f \downarrow & \nearrow g & \\
 & E &
 \end{array}$$

Example 2.3.2. $E = 0$ is an injective module.

Example 2.3.3. \mathbb{Z} (as a \mathbb{Z} -module) is not injective.

Proof. Suppose \mathbb{Z} is an injective \mathbb{Z} -module. Let $2\mathbb{Z}$ denote the ideal generated by 2, and let $f : 2\mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(2) = 1$ (then $f(2n) = n$ for any $n \in \mathbb{Z}$).

$$\begin{array}{ccc} 2\mathbb{Z} & \hookrightarrow & \mathbb{Z} \\ f \downarrow & \nearrow g & \\ \mathbb{Z} & & \end{array}$$

Since \mathbb{Z} is assumed to be injective there exists $g \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ such that $g|_{2\mathbb{Z}} = f$.

Let $g(1) = x$. Then $g(2) = g(1 + 1) = g(1) + g(1) = x + x = 2x$.

Thus $g(2) = f(2) = 1$.

It follows that $1 = 2x$ for some $x \in \mathbb{Z}$. Contradiction. □

It follows from the definition of an injective module that:

Proposition 2.3.4. Any direct summand of an injective module is injective.

Proof. Let E be an injective module. We prove that if $E = E' \oplus E''$ then E' is injective.

We have the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ f \downarrow & \nearrow h & \\ E' & & \\ e \uparrow \pi & \nearrow g & \\ E' \oplus E'' & & \end{array}$$

Let A be a submodule of B and let $f \in \text{Hom}_R(A, E')$. We need to prove that there exists a homomorphism $h : B \rightarrow E'$ such that $hi = f$.

Let $e \in \text{Hom}(E', E' \oplus E'')$, $e(x) = (x, 0)$. Since the module $E' \oplus E''$ is injective, there exists a homomorphism $g : B \rightarrow E' \oplus E''$ such that $gi = ef$.

Let $\pi : E' \oplus E'' \rightarrow E'$, $\pi(x, y) = x$. We have that $\pi e = 1_{E'}$.

Let $h = \pi g$ Then $h \in \text{Hom}(B, E')$ and $hi = \pi gi = \pi ef = 1_{E'}f = f$.

□

Theorem 2.3.5. *The following are equivalent for an R -module E :*

1. E is injective.
2. For every exact sequence $0 \rightarrow A \xrightarrow{f} B$, the sequence $\text{Hom}(B, E) \xrightarrow{\alpha} \text{Hom}(A, E) \rightarrow 0$ is still exact (with $\alpha(h) = hf$, for any $h \in \text{Hom}(B, E)$).
3. E is a direct summand of every R -module containing E .

Proof. This is Theorem 3.1.2 in [3].

□

Definition 2.3.6. *Let R be a ring, and let I be a subring of R . I is an ideal of R if $ra \in I$ and $ar \in I$ for all $a \in I$ and for all $r \in R$.*

Definition 2.3.7. *Let A be any subset of a ring R . Let (A) denote the smallest ideal of R containing A , called the ideal generated by A . An ideal generated by a single element is called a principal ideal.*

Definition 2.3.8. *A Principal Ideal Domain (P.I.D.) is an integral domain in which every ideal is principal.*

Over Principal Ideal Domains there is a nicer description of injective modules. Since its proof uses Baer's criterion, we start with this result:

Theorem 2.3.9. (*Baer's Criterion*) *An R -module E is injective if and only if for all ideals I of R , every homomorphism $f : I \rightarrow E$ can be extended to R .*

Proof. This is Theorem 3.1.3 in [3]. □

Theorem 2.3.10. *Let R be a Principal Ideal Domain. Then an R -module E is injective if and only if it is divisible (i.e. for every $x \in E$, for every nonzero divisor $r \in R$, there exists $y \in E$ such that $x = ry$).*

Proof. (\Rightarrow) Let $m \in E$ and let $r \in R$ be a nonzero divisor. Let $(r) = \{xr, x \in R\}$ denote the ideal generated by r and let $f : (r) \rightarrow E$ be defined by $f(xr) = sm$. Since the module E is injective, f can be extended to a homomorphism $g : R \rightarrow E$ such that $m = f(r) = g(r) = rg(1)$. Thus E is divisible.

(\Leftarrow) Let I be an ideal of R and let $f : I \rightarrow E$ be an R -homomorphism. By Baer's Criterion, it suffices to extend f to R for $I \neq 0$. But R is a principal ideal domain, so $I = (s)$ for some $s \in R$, $s \neq 0$. Since E is a divisible module, there exists $x \in E$ such that $f(s) = sx$. Now define an R -homomorphism $g : R \rightarrow E$ by $g(r) = rx$. Then the restriction of g to I , $g|_I$, is f because $g(r's) = r'sx = r'f(s) = f(r's)$, for any $r' \in R$. □

Example 2.3.11. ${}_Z\mathbb{Z}$ is not injective.

\mathbb{Z} is a Principal Ideal Domain, so \mathbb{Z} being injective is equivalent to \mathbb{Z} being divisible. But $2 \in \mathbb{Z}$, $3 \in \mathbb{Z} - \{0\}$ and $2 \neq 3x$ for any $x \in \mathbb{Z}$. So \mathbb{Z} is not divisible.

Example 2.3.12. \mathbb{Q} is an injective \mathbb{Z} -module.

$\forall x \in \mathbb{Q}$, $\forall n \in \mathbb{Z} - \{0\}$ there exists $\frac{x}{n} \in \mathbb{Q}$ such that $x = n \cdot \left(\frac{x}{n}\right)$. So \mathbb{Q} is divisible, hence injective.

2.4 Gorenstein injective module

Gorenstein injective modules were introduced by Enochs and Jenda in 1995, as a generalization of the injective modules. While classical homological algebra can be viewed as based on injective, projective and flat modules, its relative version called Gorenstein homological algebra is based on Gorenstein injective, projective and flat modules and the resolutions built using these modules.

Before recalling Enochs' and Jenda's definition ([5]), we introduce one more notion, that of the complex $Hom(A, C)$ obtained by applying $Hom(A, -)$ to a complex C :

If A is a module and C is a complex as described above then $Hom(A, C)$ is the complex:

$$\dots \rightarrow Hom(A, C_{n+1}) \xrightarrow{\overline{d_{n+1}}} Hom(A, C_n) \xrightarrow{\overline{d_n}} Hom(A, C_{n-1}) \rightarrow \dots$$

with $\overline{d_n} : Hom(A, C_n) \rightarrow Hom(A, C_{n-1})$ defined by

$$\overline{d_n}(f) = d_n \circ f \quad \forall f \in Hom(A, C_n)$$

Remark 2.4.1. $Hom(A, -)$ does not, in general, preserve the exactness of a complex. More precisely, if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence, then the sequence $0 \rightarrow Hom(A, M') \rightarrow Hom(A, M) \rightarrow Hom(A, M'')$ is still exact, but the sequence $Hom(A, M) \rightarrow Hom(A, M'') \rightarrow 0$ **is not**, in general, exact.

Definition 2.4.2. A module G is Gorenstein injective if there is an exact complex of injective modules

$$C = \dots \rightarrow E_1 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \rightarrow \dots$$

such that the complex $\text{Hom}(A, C)$ is still exact for any injective module A and such that $G = \text{Ker } d_0$.

Remark 2.4.3. In fact $\text{Ker } d_n$ is Gorenstein injective for every $n \in \mathbb{Z}$.

Example 2.4.4. Any injective module E (over some ring R) is Gorenstein injective because there is an exact complex

$$C = \dots \rightarrow 0 \rightarrow E \xrightarrow{id} E \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

with E being injective and such that for every module M the complex

$$\text{Hom}(M, C) = 0 \rightarrow \text{Hom}(M, E) = \text{Hom}(M, E) \rightarrow 0 \text{ is exact.}$$

The converse however is not true; not every Gorenstein injective module is injective.

Example 2.4.5. Let $R = \mathbb{Z}/4\mathbb{Z}$ and let $G = \mathbb{Z}/2\mathbb{Z}$.

G is a Gorenstein injective R -module because we have the exact complex

$$\dots \rightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{d_1} \mathbb{Z}/4\mathbb{Z} \xrightarrow{d_0} \mathbb{Z}/4\mathbb{Z} \rightarrow \dots$$

of injective R -modules, with each map $d_n : \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z}$ being defined by multiplication by 2, such that $\mathbb{Z}/2\mathbb{Z} = \text{Ker } d_0$. But $\mathbb{Z}/2\mathbb{Z}$ is not an injective $\mathbb{Z}/4\mathbb{Z}$ module.

CHAPTER 3

INJECTIVE RESOLUTION. INJECTIVE DIMENSIONS. THE *EXT* MODULES

The injective resolutions are a main tool in classical homological algebra. Both the injective dimension and the Ext modules are defined by means of injective resolutions. The main result that allows showing that every module has an injective resolution is the following:

Theorem 3.0.6. *Every module can be embedded in an injective R -module.*

Proof. This follows from Theorem 3.1.14 in [3]. □

We explain below how to use the theorem above to show that every R -module N has an exact sequence $0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$ with E^i injective.

By Theorem 3.0.6 there exists an exact sequence

$$0 \rightarrow N \xrightarrow{i} E^0 \xrightarrow{\pi} X \rightarrow 0$$

with E^0 injective (we simply take $N \xrightarrow{i} E^0$ to be the inclusion map, and let $X = E^0/N$ and $\pi : E^0 \rightarrow E^0/N$, $\pi(x) = x + N$).

But X is also an N -module so it can be embedded in an injective R -module E^1 . So we have another exact sequence

$$0 \rightarrow X \xrightarrow{i_1} E^1 \xrightarrow{\pi_1} X_1 \rightarrow 0$$

Continuing, we obtain an exact complex:

$$0 \rightarrow N \rightarrow E^0 \xrightarrow{f_0} E^1 \xrightarrow{f_1} E^2 \rightarrow \dots$$

with $f_0 = i_1 \circ \pi$, $f_1 = i_2 \circ \pi_1$, \dots)

Definition 3.0.7. *An exact sequence $0 \rightarrow N \rightarrow E^0 \rightarrow E^{-1} \rightarrow \dots$ with each E^j injective is called an injective resolution of N .*

The injective dimension of a module is defined in terms of injective resolutions:

Definition 3.0.8. *N has finite injective dimension if it has a finite injective resolution $0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \dots \rightarrow E^d \rightarrow 0$*

The smallest such d is the injective dimension of N .

The injective dimension can be characterized in terms of the modules Ext .

First we explain how to obtain the modules $Ext_R^i(A, N)$ using an injective resolution of the module N :

Let N be an R -module and let $0 \rightarrow N \rightarrow E^0 \xrightarrow{f_0} E^1 \xrightarrow{f_1} E^2 \rightarrow \dots$ be an injective resolution. Consider the “deleted” injective resolution $0 \rightarrow E^0 \xrightarrow{f_0} E^1 \xrightarrow{f_1} E^2 \rightarrow \dots$

For any R -module A consider the complex obtained from the deleted resolution by applying $Hom(A, -)$:

$$0 \rightarrow Hom(A, E^0) \xrightarrow{\alpha_0} Hom(A, E^1) \xrightarrow{\alpha_1} Hom(A, E^2) \rightarrow \dots \quad (3.1)$$

with $\alpha_0 : Hom(A, E^0) \rightarrow Hom(A, E^1)$, $\alpha_0(u) = f_0 \circ u$, $A \xrightarrow{u} E^0 \xrightarrow{f_0} E^1$

$\alpha_1 : Hom(A, E^1) \rightarrow Hom(A, E^2)$, $\alpha_1(v) = f_1 \circ v$, etc.

The abelian group $Ext_R^n(A, N)$ is defined to be the quotient group $\frac{Ker \alpha_n}{Im \alpha_{n-1}}$ (the n th homology group of the complex in Equation 3.1).

Remark 3.0.9. *The Ext groups $Ext_R^n(A, N)$ are independent of the injective resolution of N that we use (in the sense that if we start with a different injective resolution we obtain isomorphic groups).*

This is the dual result of Theorem 6 in [4].

Proposition 3.0.10. *For any R -module A we have $Ext_R^0(A, N) \simeq Hom(A, N)$*

Proof. Let $0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \dots$ be an injective resolution of N . Since the sequence $0 \rightarrow N \rightarrow E^0 \rightarrow E^1$ is exact it follows (Proposition 2.2.6) that the sequence $0 \rightarrow Hom(A, N) \rightarrow Hom(A, E^0) \rightarrow Hom(A, E^1)$ is exact. Therefore $Ext_R^0(A, N) = Ker(Hom(A, E^0) \rightarrow Hom(A, E^1)) \simeq Hom(A, N)$. \square

When applying $Hom(A, -)$ to a short exact sequence one obtains an associated long exact sequence. The following result is Theorem 10 in [3]:

Theorem 3.0.11. *Let $0 \rightarrow L \rightarrow M \rightarrow N$ be a short exact sequence of R -modules. Then there is a long exact sequence of modules:*

$$0 \rightarrow Hom(A, L) \rightarrow Hom(A, M) \rightarrow Hom(A, N) \rightarrow Ext^1(A, L) \rightarrow Ext^1(A, M) \rightarrow Ext^1(A, N) \rightarrow \dots$$

Proposition 3.0.12. *A module M is injective if and only if $Ext_R^1(A, M) = 0$, for each R -module A (${}_RA$).*

Proof. - We prove first that if M is injective then $Ext_R^i(A, M) = 0$ for all ${}_RA$ and all $i \geq 1$.

Given that M is injective, an injective resolution of M is the exact complex

$$0 \rightarrow M \xrightarrow{1_M} M \rightarrow 0$$

with $1_M : M \rightarrow M$ the identity map. Then a deleted resolution of M is the complex $0 \rightarrow M \rightarrow 0$. After applying $\text{Hom}(A, -)$ we obtain the complex $0 \rightarrow \text{Hom}(A, M) \rightarrow 0$. Since except at the zeroth place, the complex has zero components everywhere, all the modules $\text{Ext}_R^i(A, M)$ with $i \geq 1$ are zero (by definition).

- Conversely, assume that M is a left R -module such that $\text{Ext}_R^i(A, M) = 0$ for any left R -module A and for any $i \geq 1$. By Theorem 2.0.5, there exists an exact sequence $0 \rightarrow M \xrightarrow{f} E \xrightarrow{g} X \rightarrow 0$ with E an injective module. This short exact sequence gives, an associated long exact sequence: $0 \rightarrow \text{Hom}(X, M) \rightarrow \text{Hom}(X, E) \rightarrow \text{Hom}(X, X) \rightarrow \text{Ext}_R^1(X, M) = 0$. Thus $\text{Hom}(X, E) \rightarrow \text{Hom}(X, X) \rightarrow 0$ is exact. This means that for every homomorphism $u : X \rightarrow X$ there exists a homomorphism $v : X \rightarrow E$ such that $u = gv$. In particular when u is the identity map 1_X of X , there exists $v \in \text{Hom}(X, E)$ such that $gv = 1_X$. This means that the short exact sequence

$$0 \rightarrow M \xrightarrow{f} E \xrightarrow{g} X \rightarrow 0$$

is split exact. Then M is a direct summand of E and therefore it is injective.

□

Remark 3.0.13. If $0 \rightarrow N \rightarrow E \rightarrow X \rightarrow 0$ is a short exact sequence with E an injective module then it follows from Theorem 3.0.11 that $\text{Ext}_R^i(A, X) \simeq \text{Ext}_R^{i+1}(A, N)$ for any R -module A and any $i \geq 1$.

More generally, if

$$0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{n-1} \rightarrow K \rightarrow 0$$

is exact with each E^i injective then for any R -module A we have that $\text{Ext}_R^i(A, K) \simeq \text{Ext}_R^{i+n}(A, N)$ for all $i \geq 1$.

Thus, Ext measures “how far” a module N is from being an injective module, in the sense that a module N has injective dimension $\leq n$ if and only if $\text{Ext}_R^{n+1}(A, N) = 0$ for every ${}_R A$.

More precisely, the injective dimension of N is the smallest $n \in \mathbb{Z}^+ \cup \{0\}$ with the property that $\text{Ext}_R^{n+i}(A, N) = 0$ for every $i \geq 1$, for every R -module A .

3.1 Noetherian rings

Our main result concerns a special class of rings, that of noetherian rings.

Definition 3.1.1. *A ring R (with 1) is said to be left Noetherian if every ascending chain of left ideals $I_1 \subseteq I_2 \subseteq \dots$ terminates (with $I_i \neq I_j$ for all $i \neq j$, $i, j \geq 1$).*

Example 3.1.2. \mathbb{Z} is noetherian.

More generally, any Principal Ideal Domain is noetherian.

Proposition 3.1.3. *A ring R is left noetherian if every left ideal of R is finitely generated.*

Proof. This is [4], Corollary 2.3.5 □

There is a well known characterization (due to Bass) of noetherian rings in terms of injective modules.

Theorem 3.1.4. *The following are equivalent for a ring R :*

1. R is left noetherian
2. Every direct sum of injective R -modules is injective.

Baer's Criterion together with Theorem 2.0.10 gives the following result:

Theorem 3.1.5. *The following are equivalent for an R -module E :*

1. E is injective
2. $\text{Ext}^i(M, E) = 0$ for all R -modules M and for all $i \geq 1$
3. $\text{Ext}^1(M, E) = 0$ for all R -modules M
4. $\text{Ext}^i(R/I, E) = 0$ for all ideals I of R and for all $i \geq 1$
5. $\text{Ext}^1(R/I, E) = 0$ for all ideals I of R .

Proposition 3.1.6. *Let $(N_i)_{i \in I}$ be a family of R -modules and let $\pi_j : \prod_{i \in I} N_i \rightarrow N_j$ be the projection map $(\pi_j((x_i)_i) = x_j)$. Then the map*

$$\text{Hom}_R(M, \prod_{i \in I} N_i) \xrightarrow{\varphi} \prod_{i \in I} \text{Hom}_R(M, N_i)$$

$$f \xrightarrow{\varphi} (\pi_i \circ f)_i$$

is an isomorphism.

(So $\text{Hom}_R(M, \prod_{i \in I} N_i) \simeq \prod_{i \in I} \text{Hom}_R(M, N_i)$)

The theorem and the proposition above can be used to prove that the class of injective modules is closed under direct products (over any ring).

First of all, using Proposition 3.1.6 and induction it can be proved that $\text{Ext}^n(M, \prod_{i \in I} N_i) \simeq \prod_{i \in I} \text{Ext}^n(M, N_i)$ for any $n \geq 1$.

Corollary 3.1.7. *A product of R -modules $\prod_{i \in I} E_i$ is injective if and only if each E_i is injective.*

Proof. This follows from the fact that $Ext^1(M, \Pi_i E_i) \simeq \Pi_i Ext^1(M, E_i)$ and the theorem above. \square

Not the same is true for direct sums of injective modules. It is still true that every **finite** direct sum of injective modules is an injective module (over an arbitrary ring). But for infinite direct sums this is not true in general. In fact, Bass' result shows that the class of injective modules is closed under arbitrary direct sums if and only if the ring is noetherian.

3.2 Open question: Direct sums of Gorenstein injective modules

It is known that:

- 1) Every *finite* direct sum of injective modules (over an arbitrary ring) is injective.
- 2) For any R -module A and any *finite* family of modules $(N_i)_i$ it is true that

$$Hom(A, \oplus N_i) \simeq \oplus Hom(A, N_i)$$

(this is ex. 2, page 16 in [4])

It follows from (1), (2), and the definition of Gorenstein injective modules that every *finite* direct sum of Gorenstein injective modules is still a Gorenstein injective module (over an arbitrary ring). However this is not the case for arbitrary direct sums of Gorenstein injective modules. In fact this is an open question in Gorenstein homological algebra, and this is the question we consider.

As we already mentioned, classical homological algebra is based on injective (and projective) resolutions. The Gorenstein counterpart of an injective resolution is a Gorenstein injective resolution. More precisely, a module M has a Gorenstein

injective resolution if there exists an exact complex

$$0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$$

with each G^j Gorenstein injective and such that the complex stays exact when applying $\text{Hom}(-, G)$ with G Gorenstein injective.

While for classical homological algebra the existence of the injective resolutions over arbitrary rings is well known, things are a little different for Gorenstein homological algebra. It is not known for which rings it is true that every module has a Gorenstein injective resolution. The best result up to date is one of Enochs and Lopez-Ramos: they proved the existence of the Gorenstein injective resolutions over noetherian rings.

It is natural to consider the following question:

Let $(N_i)_{i \in I}$ be a family of modules over a noetherian ring R . Then each N_i has a Gorenstein injective resolution $0 \rightarrow N_i \rightarrow G_i^0 \rightarrow G_i^1 \rightarrow \dots$. Also, over such a ring, the module $\bigoplus_{i \in I} N_i$ has a Gorenstein injective resolution $0 \rightarrow \bigoplus N_i \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$. *When is it the case that the Gorenstein injective resolution of $\bigoplus_{i \in I} N_i$ can simply be obtained by taking the direct sum of the Gorenstein injective resolutions of the N_i 's?*

The answer is not known, the problem is the open question that we mentioned above: it is not known when every direct sum of Gorenstein injective modules is still Gorenstein injective.

We do not come with a complete answer to this open question, but we give a sufficient condition to guarantee that every direct sum of Gorenstein injective modules is still such a module.

More precisely we prove that if R is noetherian and such that every R -module has finite Gorenstein injective dimension, then the class of Gorenstein injective modules is closed under arbitrary direct sums.

3.3 Gorenstein rings

One type of rings over which the class of Gorenstein injective modules behaves well (in the sense that it is closed under arbitrary direct sums) is that of Gorenstein rings. They play an important part in Gorenstein homological algebra and its applications in algebraic geometry.

Definition 3.3.1. *A ring R is called Iwanaga-Gorenstein (or simply a Gorenstein ring) if R is both left and right noetherian and if R has finite self-injective dimension both as a left and a right R -module.*

It is known that in fact, for such a ring the left and right injective of R agree.

Proposition 3.3.2. *([4]) If R is left and right noetherian and $\text{inj dim}_R R = m < \infty$ and $\text{inj dim } R_R = n < \infty$, then $m = n$.*

Definition 3.3.3. *A Gorenstein ring with injective dimension at most n is called n -Gorenstein (by the proposition above, both $\text{inj dim}_R R$ and $\text{inj dim } R_R$ are at most n in this case.)*

3.4 Gorenstein injective modules. More Properties

Recall that a module N is said to be Gorenstein injective if there is an exact complex

$$E = \dots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \dots$$

of injective modules such that $\text{Hom}(\text{Inj}, E)$ is an exact sequence and $N = \text{Ker}(E_0 \rightarrow E_{-1})$.

$\dots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \dots$ is called a *complete resolution* of N for any such exact complex, E , where $\text{Hom}(\text{Inj}, E)$ also exact.

Some important properties of the Gorenstein injective modules are the following:

Proposition 3.4.1. *The injective dimension of a Gorenstein injective module is either zero or infinite.*

Proof. This is Theorem 10.1.2 in [4]. □

Theorem 3.4.2. *Let R be noetherian, and let $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ be an exact sequence of R -modules. If N' , N'' are Gorenstein injective then so is N . If N' and N are Gorenstein injective then so is N'' . If N and N'' are Gorenstein injective then N' is Gorenstein injective if and only if $\text{Ext}^1(E, N') = 0$ for any injective module E .*

Proof. This is Theorem 10.1.4 in [4]. □

The reason why the Gorenstein injective modules behave very well over Gorenstein rings is the following characterization of the Gorenstein injective modules over such rings:

Proposition 3.4.3. *Let R be an n -Gorenstein ring. A module M is Gorenstein injective if and only if there exists an exact sequence of injective modules $\dots \rightarrow E_2 \rightarrow E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} E_{-1} \xrightarrow{f_{-1}} \dots$ with $M = \text{Ker}(E_0 \rightarrow E_{-1})$.*

Proposition 3.4.3 allows us to prove:

Proposition 3.4.4. *Let R be an n -Gorenstein ring. The class of Gorenstein injective modules over R is closed under arbitrary direct sums.*

Proof. Let $(M_i)_{i \in I}$ be a family of Gorenstein injective modules.

By Proposition 3.4.3, for each $i \in I$ there is an exact sequence:

$$\dots \rightarrow E_1^i \rightarrow E_0^i \rightarrow E_{-1}^i \rightarrow E_{-2}^i \rightarrow \dots$$

with $M_i = \text{Ker}(E_{-1}^i \rightarrow E_{-2}^i)$.

Then we have an exact sequence:

$$\dots \rightarrow \oplus_i E_1^i \rightarrow \oplus_i E_0^i \rightarrow \oplus_i E_{-1}^i \rightarrow \oplus_i E_{-2}^i \rightarrow \dots$$

and $\oplus_i M_i = \text{Ker}(\oplus_i E_{-1}^i \rightarrow \oplus_i E_{-2}^i)$

By Bass' criterion each $\oplus_i E_j^i$ is an injective module.

By Proposition 3.4.3, $\oplus_{i \in I} M_i$ is a Gorenstein injective module. □

CHAPTER 4

MAIN RESULT

We give a sufficient condition for the classes of Gorenstein injective modules to be closed under arbitrary direct sums.

Since our proof uses the concept of strongly Gorenstein injective modules, we recall first the definition:

Definition 4.0.5. ([1]) *A module G is strongly Gorenstein injective if there exists an exact complex of injective modules*

$$\dots \rightarrow A \xrightarrow{a} A \xrightarrow{a} A \rightarrow \dots$$

such that:

- 1) *the complex remains exact when applying $\text{Hom}(E, -)$ to it, for any injective module E ;*
- 2) $G = \text{Ker}(a)$

It is known (from [1]) that a module is Gorenstein injective if and only if it is a direct summand of a strongly Gorenstein injective one.

We also use the notion of Gorenstein injective dimension in our proof. This is defined in a similar manner to the injective dimension.

Definition 4.0.6. *An R -module M has finite Gorenstein injective dimension if there exists an exact sequence*

$$0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \dots \rightarrow G^d \rightarrow 0$$

with each G^i a Gorenstein injective module.

If M has such an exact sequence then the smallest d with this property is the Gorenstein injective dimension.

It is known (see [2] for instance) that if M has Gorenstein injective dimension d then for any exact sequence $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{d-1} \rightarrow K \rightarrow 0$ with each E^i injective, the module K is Gorenstein injective.

Our main result is the following:

Theorem 4.0.7. *Let R be a noetherian ring.*

If every R -module M has finite Gorenstein injective dimension then the class of Gorenstein injective modules is closed under arbitrary direct sums.

Proof. - We start by proving that the class of strongly Gorenstein injective modules is closed under arbitrary direct sums.

Let $(G_i)_{i \in I}$ be a family of strongly Gorenstein injective modules. Then for each $i \in I$ there is an exact complex

$$\dots \rightarrow A_i \xrightarrow{a_i} A_i \xrightarrow{a_i} A_i \rightarrow \dots$$

with A_i injective R -module, and with $G_i = \text{Ker } a_i$.

Thus we have an exact complex:

$$\dots \rightarrow \oplus_{i \in I} A_i \xrightarrow{f} \oplus_{i \in I} A_i \xrightarrow{f} \oplus_{i \in I} A_i \rightarrow \dots$$

Since R is a noetherian ring, and each A_i is injective, we have that the module $\oplus_{i \in I} A_i$ is injective. We also have that $G = \oplus_{i \in I} G_i$ is the Kernel of $f = \oplus_{i \in I} a_i$.

So we have an exact complex of injective modules

$$\dots \rightarrow A \xrightarrow{f} A \xrightarrow{f} A \rightarrow \dots$$

with $G = \text{Ker } f$.

By hypothesis, G has finite Gorenstein injective dimension. Thus we have an exact sequence:

$$0 \rightarrow G \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^l \rightarrow G^l \rightarrow 0$$

with E^0, E^1, \dots, E^l injective modules, and with G^l Gorenstein injective.

We also have the (partial) injective resolution:

$$0 \rightarrow G \rightarrow A \rightarrow A \rightarrow \dots \rightarrow A \rightarrow G \rightarrow 0$$

This and the definition of an injective module give a commutative diagram:

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & G & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & \dots & \longrightarrow & E^{l-1} & \longrightarrow & G^l & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & A & \longrightarrow & A & \longrightarrow & \dots & \longrightarrow & A & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

with both rows being exact complexes.

Then the mapping cone of the map of complexes above is also an exact complex:

$$0 \rightarrow G \rightarrow G \oplus E^0 \rightarrow A \oplus E^1 \rightarrow A \oplus E^2 \rightarrow \dots \rightarrow A \oplus E^{l-1} \rightarrow A \oplus G^l \rightarrow G \rightarrow 0$$

After “factoring out” the exact subcomplex $0 \rightarrow G \xrightarrow{\sim} G \rightarrow 0$ we still have an exact complex:

$$0 \rightarrow E^0 \xrightarrow{\delta_0} A \oplus E^1 \xrightarrow{\delta_1} A \oplus E^2 \xrightarrow{\delta_2} \dots \rightarrow A \oplus E^{l-1} \xrightarrow{\delta_{l-1}} A \oplus G^l \xrightarrow{\delta_l} G \rightarrow 0$$

The exact sequence

$$0 \rightarrow E^0 \rightarrow A \oplus E^1 \xrightarrow{\delta_1} Im \delta_1 \rightarrow 0$$

with both E^0 and $A \oplus E^1$ injective gives that $Im \delta_1$ is also an injective module.

Then we have an exact sequence $0 \rightarrow Im \delta_1 \rightarrow A \oplus E^2 \rightarrow Im \delta_2 \rightarrow 0$ with both $Im \delta_1$ and $A \oplus E^2$ injective modules. It follows that $Im \delta_2$ is also injective.

Continuing in the same manner, we obtain that $Im \delta_j$ is injective for $j \in \{0, 1, \dots, l-1\}$.

Next we consider the exact sequence $0 \rightarrow Im \delta_{l-1} \rightarrow A \oplus G^l \rightarrow G \rightarrow 0$.

The module $A \oplus G^l$ is Gorenstein injective (as a finite direct sum of Gorenstein injective modules). And, by the above, the module $Im \delta_{l-1}$ is injective (hence Gorenstein injective).

By [1], the module G is also Gorenstein injective. Since it has an exact sequence $\dots \rightarrow A \rightarrow A \xrightarrow{f} A \rightarrow \dots$ with A injective, G is strongly Gorenstein injective.

Thus, the class of strongly Gorenstein injective modules is closed under arbitrary direct sums.

- We can prove now that every direct sum of Gorenstein injective modules is Gorenstein injective.

Let $(G_i)_{i \in I}$ be a family of Gorenstein injective modules.

By [1], each G_i is a direct summand of a strongly Gorenstein injective module H_i , that is, $\forall i \in I$, $H_i = G_i \oplus G'_i$, for some Gorenstein injective modules G'_i .

Then $\oplus_{i \in I} H_i = (\oplus_{i \in I} G_i) \oplus (\oplus_{i \in I} G'_i)$.

By the above, $\oplus_{i \in I} H_i$ is a strongly Gorenstein injective module. By [1], $\oplus_{i \in I} G_i$ is Gorenstein injective (as a direct summand of a strongly Gorenstein injective module).

□

Corollary 4.0.8. *If R is a Gorenstein ring, then the class of Gorenstein injective modules is closed under arbitrary direct sums.*

Proof. By Theorem 11.3.4 in [4], every R -module M has finite Gorenstein injective dimension. By our main result, the class of Gorenstein injective modules is closed under arbitrary direct sums.

□

Corollary 4.0.9 below follows from, Theorem 6.9 in [2].

However, our main result allows us to give a different, shorter proof.

Corollary 4.0.9. *If R is a commutative noetherian ring with a dualizing complex, the class of Gorenstein injective R -modules is closed under arbitrary direct sums.*

Proof. Let $(G_i)_i$ be a family of Gorenstein injective modules.

By Theorem 4.4 and Lemma 5.6 in [2], $G = \oplus_{i \in I} G_i$ has finite Gorenstein injective dimension.

Then by Theorem 4.0.7, G is Gorenstein injective.

□

REFERENCES

- [1] D. Bennis, N. Mahdou, *Strongly Gorenstein projective, injective and flat modules*, Journal of Pure and Applied Algebra, (210): 1709–1718, 2007. 10th ed., Dover Publ. (Yew York) (1974).
- [2] L. Christensen, A. Frankild, H. Holm, *On Gorenstein projective, injective and flat dimensions – a functorial description with applications*, Journal of Algebra, 302(1): 2311–279, 2006.
- [3] D. Dummit, R. Foote, *Abstract Algebra*, Third Edition, Wiley, 2004.
- [4] E. Enochs, O.M.G. Jenda, *Relative Homological Algebra*, DeGruyter Expositions on Mathematics, 2000.
- [5] E. Enochs, O.M.G. Jenda, *Gorenstein injective and projective modules*, Mathematische Zeitschrift, 220(1995): 611–633