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Approximate Similarity Reduction

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APPROXIMATE SIMILARITY REDUCTION

by

RUI ZHANG

(Under the Direction of Chunshan Zhao)

ABSTRACT

The nonlinear $K(n, 1)$ equation with damping is investigated via the approximate homotopy symmetry method and approximate homotopy direct method. The approximate homotopy symmetry and homotopy similarity reduction equations of different orders are derived and the corresponding homotopy series reduction solutions are obtained. As a result, the formal coincidence for both methods is displayed.

Index Words: symmetry method, direct method, homotopy model

APPROXIMATE SIMILARITY REDUCTION

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CHAPTER 1

INTRODUCTION – PRELIMINARIES

With progress and development of the modern science and technology, nonlinear science has gradually become a cross-discipline. Modern technology has played a pivotal role in providing a theoretical support to advance the technological innovation for industrial production. Physics and many other disciplines are now usually in the state of describing the issues in their field via the models of the nonlinear partial differential equations. Therefore, it becomes increasingly important to investigate the various nonlinear problems involved in these disciplines.

There are numerous partial differential equations (PDEs) with small parameters or weak perturbations involved in the fields of applied mathematics, nonlinear physics and engineering, etc. In order to further discuss such equations mentioned above, one can apply perturbation theory [1]-[4] to these problems via diverse approaches, among which approximate symmetry perturbation method [5] and approximate direct method [6] are two optimal choices. Approximate symmetry perturbation method and approximate direct method are both efficient ways in investigating weakly perturbed PDEs, especially in finding whose similarity reductions and approximate solutions [7],[8], [9]. Nonetheless, in more general situations, the parameters may not be small at all. For investigating these PDEs with strong perturbations, one can consult to some other methods such as homotopy analysis method [11] , the linear [12] and nonlinear nonsensitive homotopy approaches [13], etc.

In this thesis, we try to combine homotopy analysis method with symmetry reduction method and direct method respectively to form approximate homotopy symmetry method [14] and the approximate homotopy direct method [15].

This thesis is organized in the following ways:

- In the first chapter, we introduce the development of approximate symmetry method, which leads to the perturbed problems based on the perturbation theory. For the partial differential equations with strong perturbation, we then introduce the homotopy mode which leads to the K(n,1) equation with strong damping.

- In the second chapter, we describe the approximate symmetry method. Via the approximate symmetry method, we can further obtain the approximate series solutions of the K(n,1) equation with damping.

- In the third chapter, we describe the approximate direct method. Via the approximate direct method, we can further obtain the approximate series solutions of the K(n,1) equation with damping.

- In the fouth chapter, the formal coincidence of results for both methods is displayed.

- In the fifth chapter, we give the concluding remarks.

1.1 Homotopy analysis method

The homotopy analysis method (HAM) targetss to solve nonlinear differential equations.The homotopy analysis method derives the concept of the homotopy from topology to generate a convergent series solution for nonlinear systems. We could use homotopy-Mclaurin series to handle with the nonlinearities involved in the given system. The homotopy analysis method (HAM) was devised by Shijun Liao of Shanghai Jiaotong University in 1992. The method is different from other analytical methods in the following aspects. Firstly, it is a series expansion method but it is entirely independent of small embedding parameters. Thus, it is applicable for not only weakly but also strongly nonlinear problems, going beyond some of the limitations well known in perturbation methods. Secondly, the HAM is an unified method. This method allows for strong convergence of the solution over larger spacial and parameter domains. Thirdly, the homotopy analysis method (HAM) shows great freedom in the expression of the solution and how the solution is gained. It also provides a simplier path to enable the convergence of the solution, flexibiity to choose the basis functions of the desired solution and flexibility in determining the linear operator of the homotopy. Fourthly, combined with symbolic computation, the homotopy analysis method (HAM) ccould be connected with many other efficient PDE methods such as reduction methods, series expansion methods and numerical methods.

Definition 1.1.1. *Homotopy bewteen two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0,1] \to Y$ from the product of the space X with unit interval $[0,1]$ to Y such that, if $x \in X$ then $H(x,0) = f(x)$ and $H(x,1) = g(x)$.*

When perturbations is not weak, the homotopy analysis method can be successfully applied to the nonlinear PDEs

$$A(u) = A(x, t, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0 \tag{1.1}$$

where A is a nonlinear operator and $u = u(x, t)$ is an undetermined function of the independent variables $\{x, t\}$. The homotopy model $H(u, q) = 0$ with an embedding homotopy parameter $q \in [0, 1]$ has the following properties

$$H(u, 0) = H_0(u), H(u, 1) = A(u) \tag{1.2}$$

where $H_0(u)$ is a general form of a differential equation for certain type. Usually, the homotopy model $H(u, q)$ can be selected freely. For the briefness later, we introduce the following linear homotopy model in this study

$$(1 - q)H_0(u) + qA(u) = 0, \tag{1.3}$$

1.2 Perturbation theory

Perturbation theory are used to find an approximate solution to a given problem which is too complex to solve exactly, by starting from the exact solution of a related problem.

Perturbation theory is derived from early celestial mechanics, where the theory was majorly utilized to make modifications to the predicted paths of planets. Curiously, it was increasingly needed for epicycles that eventually led to the 16th century Copernican revolution in the understanding of planetary orbits. The development of fundamental perturbation theory for nonlinear problems was fairly completed in the middle of the 19th century. At that time Charles-Eugéne Delaunay was investigating the perturbative expansion for the Earth-Moon-Sun system and discovered the problem of small denominators where the denominator appearing in the n'th term of the perturbative expansion could become arbitrarily small, causing the n'th correction to be as large or larger than the first-order correction. In the early 20th century, this problem led Henri Poincaré to make one of the first deductions of the existence of chaos, or known as the butterfly effect: that even a very small perturbation can have a very large impact on a system.

Perturbation theory shows dramatic extension and evolution with the arrival of quantum mechanics. Although perturbation theory was pratical, the calculations were astonishingly complicated, and subject to some extent ambiguous explanation. The discovery of Heisenberg's matrix mechanics allowed a vast simplification of the application of perturbation theory. Other applications of perturbation theory include the fine structure and the hyperfine structure in the hydrogen atom.

In recent times, perturbation theory was largely used in quantum chemistry and quantum field theory. In chemistry, perturbation theory was utilized to obtain the first solutions for the helium atom.

In the middle of the 20th century, Richard Feynman realized that the perturbation theory could be applied to give an exact graphical representation in terms of Feynman diagrams. Although originally those applications limited in quantum field theory, such diagrams now are increasingly used in any area where perturbative expansions are investigated.

In 1954, the KAM theorem Developed by Andrey Kolmogorov, Vladimir Arnold and Jrgen Moser stated the conditions under which a system of partial differential equations will have only mildly chaotic behaviour under small perturbations.

Later,it arises dissatisfaction with perturbation theory in the quantum physics community, including not only the difficulty of going beyond second order in the expansion, but also questions about whether the perturbative expansion is even convergent, has led to a strong interest in the area of the study of exactly solvable models. The typical example is the Kortewegde Vries equation cannot be reached by perturbation theory, even if the perturbations were carried out to infinite order. Much of the theoretical work in non-perturbative analysis goes under the name of quantum groups and non-commutative geometry.

Perturbation theory [1] is closely related to methods used in numerical analysis. The earliest use of what would now be called perturbation theory was to deal with the otherwise unsolvable nonlinear problems of celestial mechanics: Newton's solution for the orbit of the Moon, which moves differently from a simple Keplerian ellipse because of the balance of the gravitation between Earth and the Sun. Perturbation methods initiate with a simplified form of the original problem, which is simple enough to be solved exactly. In celestial mechanics, this is usually a Keplerian ellipse. Under non relativistic gravity, an ellipse is exactly correct when there are only two gravitating bodies (say, the Earth and the Moon) but not quite correct when there are three or more objects (say, the Earth, Moon, Sun, and the rest of the solar system).

Definition 1.2.1. *Perturbation series is an expression in terms of power series with small parameter writen as*

$$u = \sum_{k=0}^{\infty} \epsilon^k u_k \tag{1.4}$$

Theorem 1.2.2. *If perturbation series is represented as*

$$A_0 + A_1\epsilon + \ldots + A_n\epsilon^n + O(\epsilon^{n+1}) = 0 \tag{1.5}$$

for $\epsilon \to 0$ *and* $A_0, A_1, \ldots,$ *independent of* $\epsilon,$ *then* $A_0 = A_1 = \ldots = A_n = 0.$

1.3 Introduction of KdV Equation

KdV equation first arose as the modelling equation for solitary gravity waves in a shallow canal. Such waves are rare and not easy to produce, and they were apparently only first noticed in 1834 (by the naval architect, John Scott Russell). Early attempts by Stokes and Airy to model them mathematically seemed to indicate that such waves could not be stableand their very existence was at first a matter of debate. Later work by Boussinesqand Rayleigh corrected errors in this earlier theory, and finally a paper in 1894 by Korteweg and de Vries [KdV] settled the matter by giving a convincing mathematical argument that wave motion in a shallow canal is governed by KdV, and showing by explicit computation that their equation admitted travelling-wave solutions that had exactly the properties described by Russell, including the relation of height to speed that Russell had determined experimentally in a wave tank he had constructed.

But it was only much later that the truly remarkable properties of the KdV equation became evident. In 1954, Fermi, Pasta and Ulam (FPU) used one of the very first digital computers to perform numerical experiments on a one-dimensional, anharmonic lattice model, and their results contradicted the then current expectations of how energy should distribute itself among the normal modes of such a system. They showed that, in a certain continuum limit, the FPU lattice was approximated by the KdV equation. They then did their own computer experiments, solving the Cauchy Problem for KdV with initial conditions corresponding to those used in the FPU experiments. In the results of these simulations they observed the first example of a soliton, a term that they coined to describe a remarkable particle-like behavior (elastic scattering) exhibited by certain KdV solutions. Zabusky and Kruskal showed how the coherence of solitons explained the anomalous results observed by Fermi, Pasta, and Ulam. But in solving that mystery, they had uncovered a larger one; KdV solitons were unlike anything that had been seen before, and the search for an explanation of their remarkable behavior led to a series of discoveries that changed the course of applied mathematics for the next thirty years.

Deviation of KdV equation begin with a conservation equations for fluid motion

$$\partial_t \rho + \nabla(\rho \vec{v}) = 0, \tag{1.6}$$

$$\rho(\partial_t + \vec{v}\nabla)\vec{v} = -\nabla P + \vec{f} \tag{1.7}$$

With density ρ, velocity of the fluid $\vec{v}$, internal pressure P and external force density $\vec{f}$.

We assume the fluid is incompressible and irrotational which means

$$\nabla\rho = 0, \partial_t \rho = 0, \nabla \times \vec{v} = 0. \tag{1.8}$$

If we consider the external force $\vec{f}$ invovled in the equation caused by gravity, we caould rewrite it as $\vec{f} = -\rho g \tau$ .After inserting the gravity form of external force, we could transform our equation into

$$\rho(\partial_t + \vec{v}\nabla)\vec{v} = -\nabla P - \rho g \tau, \tag{1.9}$$

Then, we divide both side of the equation by ρ

$$(\partial_t + \vec{v}\nabla)\vec{v} + \nabla(\frac{P}{\rho}) + g\tau = 0, \tag{1.10}$$

Considering the relation, we have

$$\vec{v} \times (\nabla \times \vec{v}) = -(\vec{v}\nabla)\vec{v} + \frac{1}{2}\nabla(\vec{v}^2), \tag{1.11}$$

Under the condition of irrotation, we have

$$\vec{v} \times 0 = -(\vec{v}\nabla)\vec{v} + \frac{1}{2}\nabla(\vec{v}^2), \tag{1.12}$$

Then,

$$(\vec{v}\nabla)\vec{v} = \frac{1}{2}\nabla(\vec{v}^2), \tag{1.13}$$

Moreover, the potential satisfies the Laplace's equation:

$$\vec{v} = \nabla\phi = ui + vj, \nabla^2\phi = 0, \tag{1.14}$$

Hence, our original fluid equation could be rewritten as a gradient

$$\nabla(\partial_t\phi + \frac{1}{2}\vec{v}^2 + \frac{P}{\rho} + g\tau) = 0, \tag{1.15}$$

Under the notification that ∇ depend only on time, we lead to the following equation to determine the velocity:

$$\partial_t\phi + \frac{1}{2}(\nabla\phi)^2 + \frac{P}{\rho} + g\tau = \partial_t\phi + \frac{1}{2}(u^2 + v^2) + \frac{P}{\rho} + g\tau = 0, \tag{1.16}$$

Under the notification $u = \phi_x$ and $v = \phi_y$, we could differentiate both side of the equation with respect to x and get

$$u_t + uu_x + vv_x + g\tau_x = \phi_{xt} + \phi_x\phi_{xx} + \phi_y\phi_{xy} + g\tau_x = 0, \tag{1.17}$$

By introducing the series form of ϕ as $\sum_{n=0}^{\infty} \frac{(-1)^m y^{2m}}{(2m)!} f^{(2m)}$, where $f = \phi_0$ we have

$$u = \phi_x = f_x - \frac{1}{2}y^2 f_{xxx} + \cdots ; \tag{1.18}$$

$$v = \phi_y = -yf_{xx} + \frac{1}{6}y^3 f_{xxxx} + \cdots ; \tag{1.19}$$

Under these relations, we can derive the KdV equation from the fluid motion.

To understand the role of nonlinear dispersion in pattern formation, Rosenau and Hyman [16] introduced the $K(n,m)$ equation

$$u_t + (u^n)_x + (u^m)_{xxx} = 0, \tag{1.20}$$

where n and m are restricted as positive integers.The KdV equation has several connections to physical problems. In addition to being the governing equation of the string in the Fermi−Pasta−Ulam problem in the continuum limit, it approximately describes the evolution of long, one-dimensional waves in many physical settings Lou and Wu[17] characterized its Painlevé integrability under certain relations between n and m. The $K(2,1)$ and $K(3,1)$ models are just the KdV and mKdV equations respectively. Accordingly, the $K(2,1)$ equation with damping

$$u_t + auu_x + u_{xxx} = -\epsilon u, \tag{1.21}$$

was introduced and discussed in [18].

1.4 Application of both methods

The perturbation to the KdV equation is caused by an external force. In this thesis, we intend to investigate the nonlinear $K(n,1)$ equation with damping [19]

$$u_t + a(u^n)_x + u_{xxx} = -\epsilon u \tag{1.22}$$

via the approximate homotopy symmetry method and the approximate homotopy direct method, where u is a function of x and t. Hereafter, we put stress on the general case while $n > 2$, irrespective of the simple case of $n = 2$. Additionally, it should be underscored that ϵ is not a small parameter.

For the formal succinctness, we rewrite Eq. (1.22) as

$$A(u) = u_t + a(u^n)_x + u_{xxx} + \epsilon u = 0 \tag{1.23}$$

For the chosen linear homotopy model (1.3), $H_0(u)$ take the form

$$H_0(u) = u_t + a(u^n)_x + u_{xxx} \tag{1.24}$$

Accordingly, we change the Eq. (1.3) into

$$(1-q)(u_t + a(u^n)_x + u_{xxx}) + q(u_t + a(u^n)_x + u_{xxx} + \epsilon u) = 0 \tag{1.25}$$

It is obvious that Eq. (1.25) is $K(n,1)$ equation just when $q = 0$.

For Eq. (1.25), based on the perturbation theory, the solution can be represented as

$$u = \sum_{k=0}^{\infty} q^k u_k, \tag{1.26}$$

with u_k functions of x and t. Substituting Eq. (1.26) into Eq. (1.25) and vanishing the coefficients of all different powers of q, we obtain the following system

$$\begin{aligned} & u_{k,t} + na \sum_{i_1+i_2+\cdots+i_n=k} u_{i_1} u_{i_2} \ldots u_{i_{(n-1)}} u_{i_n,x} \\ & + u_{k,xxx} + \epsilon u_{k-1} = 0, \ (k = 0, 1, 2, \ldots) \end{aligned} \tag{1.27}$$

where $0 \leq i_m \leq k$ $(m = 1, \ldots, n)$ and u_{-1}=0.

CHAPTER 2

APPROXIMATE SYMMETRY TO EQUATION

2.1 Lie symmetry method

Towards the end of the nineteenth century, Sophus Lie introduced the notion of Lie group in order to study the solutions of ordinary differential equations (ODEs). He showed the following main property: the order of an ordinary differential equation can be reduced by one if it is invariant under one-parameter Lie group of point transformations. This observation unified and extended the available integration techniques. Lie devoted th e remainder of his mathematical career to developing these continuous groups that have now an impact on many areas of mathematically-based sciences. The applications of Lie groups to differential systems were mainly established by Lie and Emmy Noether.[2]

Definition 2.1.1. *A Lie group is a set G with two structures: G is a group and G is a (smooth, real) manifold. These structures agree in the following sense: multiplication and inversion are smooth maps.*

Lie symmetries were introduced by Lie in order to solve ordinary differential equations. Another application of symmetry methods is to reduce systems of differential equations, finding equivalent systems of differential equations of simpler form. This is called reduction.

Lie's fundamental theorems underline that Lie groups can be characterized by their infinitesimal generators. These mathematical objects form a Lie algebra of infinitesimal generators. Deduced "infinitesimal symmetry conditions" (defining equations of the symmetry group) can be explicitly solved in order to find the closed form of symmetry groups, and thus the associated infinitesimal generators.

Definition 2.1.2. *An infinitesimal generator V in the canonical basis of elementary* *derivations* $\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\}$ *can be written as*

$$V = \sum_{k=0}^{n} \zeta_{z_i} \frac{\partial}{\partial z_i}, \tag{2.1}$$

In order to study Lie symmetry reduction of Eq. (1.27), we construct the Lie point symmetry in the vector form

$$V = X \frac{\partial}{\partial x} + T \frac{\partial}{\partial t} + \sum_{k=0}^{\infty} U_k \frac{\partial}{\partial u_k}, \tag{2.2}$$

where X, T, and U_k are functions of x, t, and u_k, $(k = 0, 1, \ldots)$, equivalently, Eq. (1.27) is invariant under the transformation

$$\{x, t, u_k, k = 0, 1, \ldots\} \rightarrow \{x + \zeta X, t + \zeta T, u_k + \zeta U_k, k = 0, 1, \ldots\},$$

with infinitesimal parameter ζ.

2.2 Application of symmetry method

Since Eq. (1.22) is not explicitly dependent upon space-time x, t, the symmetry in the vector form (2.2) can be written as a function form

$$\sigma_k = U_k - XU_{k,x} - TU_{k,t}, \quad (k = 0, 1, \ldots), \tag{2.3}$$

Under notation (2.3), the symmetry equations for Eqs. (1.27)

$$\begin{aligned} &u_{k,t} + na \sum_{i_1+i_2+\cdots+i_n=k} u_{i_1} u_{i_2} \ldots u_{i_{(n-1)}} u_{i_n,x} \\ &+u_{k,xxx} + \epsilon u_{k-1} = 0, \ (k = 0, 1, 2, \ldots) \end{aligned} \tag{2.4}$$

read

$$\begin{aligned} &\sigma_{k,t} + na \sum_{i_1+i_2+\cdots+i_n=k} [\sigma_{i_1} u_{i_2} \ldots u_{i_{(n-2)}} u_{i_{(n-1)}} u_{i_n,x} + u_{i_1} \sigma_{i_2} \ldots \\ &u_{i_{(n-2)}} u_{i_{(n-1)}} u_{i_n,x} + \cdots + u_{i_1} u_{i_2} \ldots u_{i_{(n-2)}} u_{i_{(n-1)}} \sigma_{i_n,x}] + \sigma_{k,xxx} \\ &+\epsilon \sigma_{k-1} = 0, \ (k = 0, 1, 2, \ldots) \end{aligned} \tag{2.5}$$

which are the linearized equations for Eqs. (1.27), with $0 \leq i_m \leq k$ $(m = 1, \ldots, n)$ and $\sigma_{-1} = 0$.

It seems difficult to figure out X, T and $U_k, (k = 0, 1, \ldots)$ directly because there are infinite number of equations and arguments concerning or in X, T and $U_k, (k = 0, 1, \ldots)$. To make brief of it, we begin the discussion with finite number of equations.

Confining the range of k to $(k = 0 - 2)$ in Eqs. (1.27), (2.3) and (2.5), we see that X, T, U_0, U_1 and U_2 are functions of x, t, u_0, u_1 and u_2. In this case, the determining equations can be derived by substituting Eq. (2.3) into Eq. (2.5), eliminating $u_{0,t}$,

$u_{1,t}$ and $u_{2,t}$ in terms of Eq. (1.27). Some of the determining equations read

$$
\begin{aligned}
&T_x = T_{u_0} = T_{u_1} = T_{u_2} = 0, X_t = X_{u_0} = X_{u_1} = X_{u_2} = 0,\\
&U_{0,u_0u_0} = U_{0,u_0u_1} = U_{0,u_0u_2} = U_{0,u_1u_1} = U_{0,u_1u_2} = U_{0,u_2u_2} = 0,\\
&U_{1,u_0u_0} = U_{1,u_0u_1} = U_{1,u_0u_2} = U_{1,u_1u_1} = U_{1,u_1u_2} = U_{1,u_2u_2} = 0,\\
&U_{2,u_0u_0} = U_{2,u_0u_1} = U_{0,u_0u_2} = U_{2,u_1u_1} = U_{2,u_1u_2} = U_{2,u_2u_2} = 0.
\end{aligned}
\tag{2.6}
$$

The general solution to Eqs. (2.6) is

$$
\begin{aligned}
&X = X(x),\ T = T(t),\\
&U_0 = a_0(x,t)u_0 + a_1(x,t)u_1 + a_2(x,t)u_2 + a_3(x,t),\\
&U_1 = a_4(x,t)u_0 + a_5(x,t)u_1 + a_6(x,t)u_2 + a_7(x,t),\\
&U_2 = a_8(x,t)u_0 + a_9(x,t)u_1 + a_{10}(x,t)u_2 + a_{11}(x,t).
\end{aligned}
\tag{2.7}
$$

Using relations (2.7), the remaining determining equations are immediately simplified to

$$
\begin{aligned}
&a_0 = -\frac{2}{n-1}X_x = \frac{1}{n-1}\left(X_x - T_t\right),\\
&a_5 + (n-2)a_0 = X_x, a_{10} + (n-1)a_0 - a_5 = X_x,\\
&a_1 = a_2 = a_3 = a_4 = a_6 = a_7 = a_8 = a_9 = a_{11} = X_{xx} = 0.
\end{aligned}
$$

It is straightforward to find that

$$
\begin{aligned}
&X = \frac{c}{3}x + x_0,\ T = ct + t_0,\ U_0 = -\frac{2}{3(n-1)}cu_0,\\
&U_1 = \left(1 - \frac{2}{3(n-1)}\right)cu_1,\ U_2 = \left(2 - \frac{2}{3(n-1)}\right)cu_2.
\end{aligned}
$$

Likewise, restricting the range of k to $\{k \mid k = 0,1,2,3\}$ in Eqs. (1.27) (2.3) and (2.5), where X, T, U_0, U_1, U_2 and U_3 are functions of x, t, u_0, u_1, u_2 and u_3, repeating the calculation process as before, then we have

$$
\begin{aligned}
&X = \frac{c}{3}x + x_0,\ T = ct + t_0,\\
&U_0 = -\frac{2}{3(n-1)}cu_0,\ U_1 = \left(1 - \frac{2}{3(n-1)}\right)cu_1,\\
&U_2 = \left(2 - \frac{2}{3(n-1)}\right)cu_2,\ U_3 = \left(3 - \frac{2}{3(n-1)}\right)cu_3.
\end{aligned}
$$

With more similar computation considered, we find that X, T and U_k ($k = 0, 1, \ldots$) are formally coherent, i.e.,

$$
X = \frac{c}{3}x + x_0,\ T = ct + t_0,\ U_k = \left(k - \frac{2}{3(n-1)}\right)cu_k,\ (k = 0, 1, \ldots) \tag{2.8}
$$

where c, x_0 and t_0 are arbitrary constants.

Subsequently, solving the characteristic equations

$$
\frac{dx}{X} = \frac{dt}{T},\ \frac{du_0}{U_0} = \frac{dt}{T},\ \ldots,\ \frac{du_k}{U_k} = \frac{dt}{T},\ \ldots. \tag{2.9}
$$

leads to the similarity solutions to Eq. (1.27). Two subcases are distinguished as follows.

2.3 Results

Case 1: When $c \neq 0$, without loss of generality, making the transformation $x_0 \longrightarrow \frac{1}{3}cx_0$ and $t_0 \longrightarrow ct_0$, we rewrite Eq. (2.8) as

$$\begin{aligned} &X = \frac{1}{3}c\,(x+x_0)\,,\ T = c\,(t+t_0)\,,\ U_0 = -\frac{2}{3(n-1)}cu_0, \\ &U_1 = \left(1-\frac{2}{3(n-1)}\right)cu_1,\ \ldots,\ U_k = \left(k-\frac{2}{3(n-1)}\right)cu_k,\ (k=0,1,\ldots) \end{aligned} \tag{2.10}$$

in this case, solving Eq. (2.9) leads to the following invariants

$$I(x,t) = \xi = (x+x_0)(t+t_0)^{-\frac{1}{3}}, \tag{2.11}$$

$$I_0(x,t) = V_0 = (t+t_0)^{\frac{2}{3(n-1)}}u_0, \tag{2.12}$$

and

$$I_k(x,t) = V_k = (t+t_0)^{\frac{2}{3(n-1)}-k}u_k,\ (k=1,2,\ldots) \tag{2.13}$$

viewing V_k $(k = 0, 1, \ldots)$ as functions of ξ, we get the similarity solutions

$$u_k = V_k(\xi)(t+t_0)^{k-\frac{2}{3(n-1)}},\ (k=0,1,\ldots) \tag{2.14}$$

to Eqs. (1.27) with similarity variable

$$\xi = (x+x_0)(t+t_0)^{-\frac{1}{3}}. \tag{2.15}$$

From Eq. (1.26), the series reduction solution to Eq. (1.25) is given by

$$u = \sum_{k=0}^{\infty} q^k (t+t_0)^{k-\frac{2}{3(n-1)}} V_k(\xi), (k=0,1,\ldots) \tag{2.16}$$

substituting Eqs. (2.14) into Eqs. (1.27), we get the following related similarity reduction equations

$$
\begin{aligned}
O(q^0): \quad & V_{0,\xi\xi\xi} + naV_0^{n-1}V_{0,\xi} - \frac{2}{3(n-1)}V_0 - \frac{1}{3}\xi V_{0,\xi} = 0, \\
O(q^1): \quad & V_{1,\xi\xi\xi} + naV_0^{n-1}V_{1,\xi} + n(n-1)aV_0^{n-2}V_1V_{0,\xi} \\
& + \left(1 - \frac{2}{3(n-1)}\right)V_1 - \frac{1}{3}\xi V_{1,\xi} + \epsilon V_0 = 0, \\
O(q^2): \quad & V_{2,\xi\xi\xi} + naV_0^{n-1}V_{2,\xi} + n(n-1)aV_0^{n-2}V_1V_{1,\xi} \\
& + n(n-1)aV_0^{n-2}V_2V_{0,\xi} + \frac{n(n-1)(n-2)}{2}aV_0^{n-3}V_1^2V_{0,\xi} \\
& + \left(2 - \frac{2}{3(n-1)}\right)V_2 - \frac{1}{3}\xi V_{2,\xi} + \epsilon V_1 = 0, \\
& \ldots, \\
O(q^k): \quad & V_{k,\xi\xi\xi} + na \sum_{i_1+i_2+\cdots+i_n=k} V_{i_1}V_{i_2}\ldots V_{i_{(n-1)}}V_{i_n,\xi} \\
& + \left(k - \frac{2}{3(n-1)}\right)V_k - \frac{1}{3}\xi V_{k,\xi} + \epsilon V_{k-1} = 0,
\end{aligned}
$$

with $0 \leq i_m \leq k$, $(m = 1, \ldots, n)$ and $V_{-1} = 0$. The kth $(k > 0)$ similarity reduction equation is in fact a third order linear ordinary differential equation (ODE) of V_k when the previous $V_0, V_1, \ldots, V_{k-1}$ are known, since it can be rewritten as

$$
\begin{aligned}
& V_{k,\xi\xi\xi} + na[{V_0}^{n-1}V_{k,\xi} + (n-1){V_0}^{n-2}V_kV_{0,\xi}] \\
& + \left(k - \frac{2}{3(n-1)}\right)V_k - \frac{1}{3}\xi V_{k,\xi} = G_k(\xi), \ (k = 0, 1, \ldots)
\end{aligned} \tag{2.17}
$$

where G_k is an only function of $\{V_0, V_1, \ldots V_{k-1}\}$

$$
G_k(\xi) = -\epsilon V_{k-1} - na \sum_{i_1+i_2+\cdots+i_n=k} V_{i_1}V_{i_2}\ldots V_{i_{(n-1)}}V_{i_n,\xi}, \ (k = 0, 1, \ldots) \tag{2.18}
$$

with $i_m \neq k$ $(m = 1, \ldots, n)$.

Case 2: When $c = 0$, we have

$$
X = x_0, \ T = t_0, \ U_k = \left(k - \frac{2}{3(n-2)}\right)cu_k = 0, \ (k = 1, 2, \ldots) \tag{2.19}
$$

the similarity solutions are

$$
u_k = V_k(\xi), \ \xi = t_0x - x_0t, \ (k = 1, 2, \ldots, n) \tag{2.20}
$$

thus the series reduction solution to Eq. (1.22) is

$$u = \sum_{k=0}^{\infty} q^k V_k(\xi), \tag{2.21}$$

where $V_k(\xi)$ $(k = 0, 1, 2 \ldots)$ yields

$$\begin{aligned}
O(q^0): \quad & (t_0)^3 V_{0,\xi\xi\xi} + nat_0 V_0^{n-1} V_{0,\xi} - x_0 V_{0,\xi} = 0, \\
O(q^1): \quad & (t_0)^3 V_{1,\xi\xi\xi} + nat_0 V_0^{n-1} V_{1,\xi} + n(n-1)at_0 \\
& V_0^{n-2} V_1 V_{0,\xi} - x_0 V_{1,\xi} + \epsilon V_0 = 0, \\
& \ldots, \\
O(q^k): \quad & (t_0)^3 V_{k,\xi\xi\xi} + nat_0 \sum_{i_1+i_2+\cdots+i_n=k} V_{i_1} V_{i_2} \ldots V_{i_{(n-1)}} V_{i_n,\xi} \\
& -x_0 V_{k,\xi} + \epsilon V_{k-1} = 0,
\end{aligned}$$

with $0 \leq i_m \leq k$, $(m = 1, \ldots, n)$ and $V_{-1} = 0$. The kth $(k > 0)$ similarity reduction equation can be rewritten as an ODE

$$(t_0)^3 V_{k,\xi\xi\xi} + nat_0[{V_0}^{n-1} V_{k,\xi} + (n-1){V_0}^{n-2} V_k V_{0,\xi}] - x_0 V_{k,\xi} = G_k(\xi), \ (k = 0, 1, \ldots) \tag{2.22}$$

of $V_k(\xi)$, where G_k is a function of $\{V_0, V_1, \ldots, V_{k-1}\}$ defined as

$$G_k(\xi) = -\epsilon V_{k-1} - nat_0 \sum_{i_1+i_2+\cdots+i_n=k} V_{i_1} V_{i_2} \ldots V_{i_{(n-1)}} V_{i_n,\xi}, \ (k = 0, 1, \ldots) \tag{2.23}$$

with $i_m \neq k$ $(m = 1, \ldots, n)$.

CHAPTER 3

APPROXIMATE DIRECT METHOD TO EQUATION

3.1 Direct method

Direct method or CK method was developed by P.A.Clarkson and M.D. Kruscal in 1989([6]). This method first is applied to investigate the approximate solution forboussinesq equation. CK method is different from Lie syemmetry method because CK method don't need any symmetry to achieve the solution and could get more general solution than Lie symmetry method.

In this section, we develop the direct method to investigate Eq.(1.27) for its similarity solutions of the form

$$u_k = f_k(x, t, P_k(z(x,t))),\ (k = 0, 1, \dots) \tag{3.1}$$

which satisfy a system of ODEs resulting from inserting Eq. (3.1) into Eq. (1.27).

On substituting Eq. (3.1) into Eq. (1.27), since only one term $u_{k,xxx}$ in Eq. (1.27) generates the terms $P_{k,zzz}$ and $P_{k,z}P_{k,zz}$ during the substitution, it is easily seen that the coefficients of $P_{k,zzz}$ and $P_{k,z}P_{k,zz}$ are $f_{k,P_k}(z_x)^3$ and $3f_{k,P_kP_k}(z_x)^3$, respectively. We reserve uppercase Greek letters for undetermined functions of z hereafter. The ratios of the coefficients are functions of z, namely,

$$f_{k,P_k}(z_x)^3 = 3f_{k,P_kP_k}(z_x)^3\Gamma_k(z),\ (k = 0, 1, \dots) \tag{3.2}$$

Then, we have

$$f_{k,P_k} = 3f_{k,P_kP_k}\Gamma_k(z),\ (k = 0, 1, \dots) \tag{3.3}$$

Hence,

$$\frac{1}{3\Gamma(z)} = \frac{f_{k,P_kP_k}}{f_{k,P_k}}\Gamma_k(z),\ (k = 0, 1, \dots) \tag{3.4}$$

with the solution

$$f_k = \alpha_k(x,t) + \beta_k(x,t)e^{\frac{1}{3\Gamma(z)}P_k},\ (k = 0, 1, \ldots)$$

where $\alpha_k(x,t)$ and $\beta_k(x,t)$ are arbitrary functions. Hence, rewriting $e^{\frac{1}{3\Gamma(z)}P_k}$ as P_k, it is sufficient to seek the similarity reduction of Eq. (1.27) in the special form

$$u_k = \alpha_k(x,t) + \beta_k(x,t)P_k(z(x,t)),\ (k = 0, 1, \ldots) \tag{3.5}$$

instead of the general form Eq. (3.1).

Remark: Three freedoms in the determination of $\alpha_k(x,t)$, $\beta_k(x,t)$ and $z(x,t)$ should be notified:

(i) If $\alpha_k(x,t) = \alpha_k'(x,t) + \beta_k(x,t)\Omega(z)$, then one can take $\Omega(z) = 0$;

(ii) If $\beta_k(x,t) = \beta_k'(x,t)\Omega(z)$, then one can take $\Omega(z) = constant$;

(iii) If $z(x,t)$ is determined by $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then one can take $\Omega(z) = z$.

3.2 Application of Direct method

Substituting Eq. (3.5) into Eq. (1.27), we find that the coefficients for $P_{0,zzz}$, $P_0^{n-1}P_{0,z}$, $P_{0,zz}$ and $P_0^{n-2}P_{0,z}$ are $\beta_0(z_x)^3$, $na\beta_0^n z_x$, $3\beta_{0,x}(z_x)^2+3\beta_0 z_x z_{xx}$ and $n(n-1)a\alpha_0\beta_0^{n-1}z_x$, respectively. Since P_k is only a function of z, it requires that

$$na\beta_0^n z_x = \beta_0(z_x)^3\Phi_0(z), \tag{3.6}$$

$$3\beta_{0,x}(z_x)^2+3\beta_0 z_x z_{xx} = \beta_0(z_x)^3\Psi_0(z), \tag{3.7}$$

$$n(n-1)a\alpha_0\beta_0^{n-1}z_x = \beta_0(z_x)^3\Omega_0(z). \tag{3.8}$$

From Eq. (3.6) and remark (ii), we get

$$\beta_0^{n-1} = z_x^2, \tag{3.9}$$

$$\beta_0 = z_x^{\frac{2}{n-1}}. \tag{3.10}$$

From Eq. (3.8) and remark (i), we can see $\alpha_0 = 0$.

From Eqs. (3.7), (3.10) and remark (iii), we have

$$\frac{6}{n-1}z_x z_{xx} + 3z_x z_{xx} = z_x^3\Psi_0(z),$$

we have

$$z_{xx} = 0, \tag{3.11}$$

then

$$z = \theta(t)x + \sigma(t), \tag{3.12}$$

where $\theta(t)$ and $\sigma(t)$ are some functions to be settled.

Then Eq. (1.27) is degenerated into

$$\theta^4 P_{0,zzz} + na\theta^4 P_0^{n-1}P_{0,z} + \theta(x\theta_t+\sigma_t)P_{0,z} + \frac{2}{n-1}\theta_t P_0 = 0. \tag{3.13}$$

From the coefficients of $P_{0,zzz}$, $P_{0,z}$ and P_0 and the relations

$$x\theta_t + \sigma_t = \theta^3\Gamma_1(z), \quad \frac{2}{n-1}\theta_t = \theta^4\Gamma_2(z),$$

we have

$$\Gamma_1(z) = Az + B, \quad \Gamma_2(z) = \frac{2}{n-1}A, \quad \frac{d\theta}{dt} = A\theta^4, \quad \frac{d\sigma}{dt} = \theta^3(A\sigma + B), \tag{3.14}$$

where A and B are arbitrary constants.

Assume that $k \geq 1$, inserting Eq. (3.5) into Eq. (1.27), we know that the coefficients of P_{k-1}, $P_0^{n-2}P_{0,z}$ and $P_{k,zzz}$ are $\epsilon\beta_{k-1}$, $n(n-1)a\beta_0^{n-1}z_x\alpha_k$ and $\beta_k z_x^3$ respectively, which leads to

$$\epsilon\beta_{k-1} = \beta_k z_x^3\Phi_k(z), \quad n(n-1)a\beta_0^{n-1}z_x\alpha_k = \beta_k z_x^3\Psi_k(z), \quad (k \geq 1)$$

then using remark (i) and (ii), we have

$$\alpha_k = 0, \quad \beta_k = (z_x)^{\frac{2}{n-1}-3k} \; (k = 0, 1, 2, \ldots).$$

3.3 Results

We distinguish the following two subcases.

Case 1: When $A \neq 0$, Eq. (3.14) has solution

$$\theta = -(3A(t-t_0))^{-\frac{1}{3}}, \quad \sigma = -\frac{B}{A} + s_0(t-t_0)^{-\frac{1}{3}}, \tag{3.15}$$

where t_0 and s_0 are arbitrary constants.

In terms of Eqs. (3.5), (3.10), (3.12), (3.14) and (3.15), we get the following solution to Eq. (1.27)

$$u_k = (-1)^k (3A(t-t_0))^{k-\frac{2}{3(n-1)}} P_k(z), \ (k = 0, 1, 2, \ldots) \tag{3.16}$$

where the similarity variable $z = -(3A(t-t_0))^{-\frac{1}{3}} x + s_0(t-t_0)^{-\frac{1}{3}} - \frac{B}{A}$.

From Eqs. (3.16) and (1.26), we obtain the series reduction solution

$$u = \sum_{k=0}^{\infty} (-1)^k q^k (3A(t-t_0))^{k-\frac{2}{3(n-1)}} P_k(z), \ (k = 0, 1, 2, \ldots) \tag{3.17}$$

to Eq. (1.25). Inserting Eq. (3.16) into Eq. (1.27), we get the similarity reduction equations

$$\begin{aligned} &P_{k,zzz} + na \sum_{i_1+i_2+\cdots+i_n=k} P_{i_1} P_{i_2} \ldots P_{i_{(n-1)}} P_{i_n,z} + (Az+B) P_{k,z} \\ &+ \left(\frac{2}{n-1} - 3k\right) A P_k + \epsilon P_{k-1} = 0, \ (k = 0, 1, 2, \ldots) \end{aligned} \tag{3.18}$$

with $P_{-1} = 0$.

Case 2: When $A = 0$, Eq. (3.14) has the solution

$$\theta = t_0, \quad \sigma = B t_0^3 t + s_0, \tag{3.19}$$

where t_0 and s_0 are arbitrary constants. By Eqs. (3.5), (3.10), (3.12), (3.14) and (3.19), we obtain the similarity solution

$$u_k = t_0^{\frac{2}{n-1}-3k} P_k(z), \ (k = 0, 1, 2, \ldots) \tag{3.20}$$

with the similarity variable $z = t_0 x + B t_0^3 t + s_0$. Based on this, the series reduction solution to Eq. (1.25) is

$$u = \sum_{k=0}^{\infty} q^k t_0^{\frac{2}{n-1} - 3k} P_k(z), \tag{3.21}$$

and the similarity reduction equation is boiled down to

$$\begin{aligned} & P_{k,zzz} + na \sum_{i_1 + i_2 + \cdots + i_n = k} P_{i_1} P_{i_2} \ldots P_{i_{(n-1)}} P_{i_n,z} \\ & + B P_{k,z} + \epsilon P_{k-1} = 0, \ (k = 0, 1, 2, \ldots) \end{aligned} \tag{3.22}$$

with $P_{-1} = 0$.

CHAPTER 4

ANALYSIS ON FORMAL COINCIDENCE FOR BOTH METHODS

In the following, we discuss the formal coincidence for both methods on the basis of the results obtained by both methods.

4.1 Formal Coincidence under Case 1

Case 1: We now compare Eqs. (3.16) and (3.18) with the results concerning similarity reduction equations and similarity solutions in Case 1 of Chapter 2 which are

$$\theta = -(3A(t-t_0))^{-\frac{1}{3}}, \ \ \sigma = -\frac{B}{A} + s_0(t-t_0)^{-\frac{1}{3}}, \tag{4.1}$$

where t_0 and s_0 are arbitrary constants. The series reduction solution written as

$$u = \sum_{k=0}^{\infty} (-1)^k q^k (3A(t-t_0))^{k-\frac{2}{3(n-1)}} P_k(z), \ (k = 0, 1, 2, \ldots) \tag{4.2}$$

we get the similarity reduction equations

$$\begin{aligned} &P_{k,zzz} + na \sum_{i_1+i_2+\cdots+i_n=k} P_{i_1} P_{i_2} \ldots P_{i_{(n-1)}} P_{i_n,z} + (Az+B)P_{k,z} \\ &+ \left(\frac{2}{n-1} - 3k\right) AP_k + \epsilon P_{k-1} = 0, \ (k = 0, 1, 2, \ldots) \end{aligned} \tag{4.3}$$

By the transformations$A \to -\frac{1}{3}$, $B \to 0$, $t_0 \to -t_0$ and $s_0 \to x_0$, we can get the similarity variable $z = (x+x_0)(t+t_0)^{-\frac{1}{3}}$, then Eqs. (3.16) and (3.18) are respectively changed into

$$u_k = (t+t_0)^{k-\frac{2}{3(n-1)}} P_k(z), \ (k = 0, 1, 2, \ldots) \tag{4.4}$$

and

$$\begin{aligned} &P_{k,zzz} + na \sum_{i_1+i_2+\cdots+i_n=k} P_{i_1} P_{i_2} \ldots P_{i_{(n-1)}} P_{i_n,z} - \frac{1}{3} z P_{k,z} \\ &+ \left(k - \frac{2}{3(n-1)}\right) P_k + \epsilon P_{k-1} = 0, \ (k = 0, 1, 2, \ldots), \end{aligned} \tag{4.5}$$

with $P_{-1} = 0$.

On the other hand, for Case 1 in Chapter 3,

$$I_k(x,t) = V_k = (t+t_0)^{\frac{2}{3(n-1)}-k} u_k,\ (k = 1, 2, \ldots) \tag{4.6}$$

$$u = \sum_{k=0}^{\infty} q^k (t+t_0)^{k-\frac{2}{3(n-1)}} V_k(\xi), (k = 0, 1, \ldots) \tag{4.7}$$

and

$$\begin{aligned}
O(q^0):\quad & V_{0,\xi\xi\xi} + naV_0^{n-1}V_{0,\xi} - \frac{2}{3(n-1)}V_0 - \frac{1}{3}\xi V_{0,\xi} = 0,\\
O(q^1):\quad & V_{1,\xi\xi\xi} + naV_0^{n-1}V_{1,\xi} + n(n-1)aV_0^{n-2}V_1V_{0,\xi}\\
& + \left(1 - \frac{2}{3(n-1)}\right)V_1 - \frac{1}{3}\xi V_{1,\xi} + \epsilon V_0 = 0,\\
O(q^2):\quad & V_{2,\xi\xi\xi} + naV_0^{n-1}V_{2,\xi} + n(n-1)aV_0^{n-2}V_1V_{1,\xi}\\
& + n(n-1)aV_0^{n-2}V_2V_{0,\xi} + \frac{n(n-1)(n-2)}{2}aV_0^{n-3}V_1^2V_{0,\xi}\\
& + \left(2 - \frac{2}{3(n-1)}\right)V_2 - \frac{1}{3}\xi V_{2,\xi} + \epsilon V_1 = 0,\\
& \ldots,\\
O(q^k):\quad & V_{k,\xi\xi\xi} + na\sum_{i_1+i_2+\cdots+i_n=k} V_{i_1}V_{i_2}\ldots V_{i_{(n-1)}}V_{i_n,\xi}\\
& + \left(k - \frac{2}{3(n-1)}\right)V_k - \frac{1}{3}\xi V_{k,\xi} + \epsilon V_{k-1} = 0,
\end{aligned}$$

with $0 \le i_m \le k$, $(m = 1, \ldots, n)$ and $V_{-1} = 0$. Making the transformation$V_k(\xi) \to P_k(\xi)$, Eqs. (2.14) and (2.17) are respectively converted into

$$u_k = (t+t_0)^{k-\frac{2}{3(n-1)}} P_k(\xi),\ (k = 0, 1, 2, \ldots) \tag{4.8}$$

and

$$\begin{aligned}
& P_{k,\xi\xi\xi} + na\sum_{i_1+i_2+\cdots+i_n=k} P_{i_1}P_{i_2}\ldots P_{i_{(n-1)}}P_{i_n,\xi} - \frac{1}{3}\xi P_{k,\xi}\\
& + \left(k - \frac{2}{3(n-1)}\right)P_k + \epsilon P_{k-1} = 0,\ (k = 0, 1, 2, \ldots)
\end{aligned} \tag{4.9}$$

where $P_{-1} = 0$, which are formally the same as Eqs. (4.4) and (4.5).

4.2 Formal Coincidence under Case 2

Case 2: For case 2 in Chapter 3, we have the similarity solution

$$u_k = t_0^{\frac{2}{n-1}-3k} P_k(z), \ (k = 0, 1, 2, \ldots) \tag{4.10}$$

with the similarity variable $z = t_0 x + B t_0^3 t + s_0$. Based on this, the series reduction solution to Eq. (1.25) is

$$u = \sum_{k=0}^{\infty} q^k t_0^{\frac{2}{n-1}-3k} P_k(z), \tag{4.11}$$

and the similarity reduction equation is boiled down to

$$\begin{aligned} &P_{k,zzz} + na \sum_{i_1+i_2+\cdots+i_n=k} P_{i_1} P_{i_2} \ldots P_{i_{(n-1)}} P_{i_n,z} \\ &+ B P_{k,z} + \epsilon P_{k-1} = 0, \ (k = 0, 1, 2, \ldots) \end{aligned} \tag{4.12}$$

with $P_{-1} = 0$. Suppose that $t_0 \neq 0$, by the transformations $B \to -\frac{x_0}{t_0^3}$, $t_0 \to t_0$ and $s_0 \to 0$, Eqs. (4.10) and (4.11) are respectively transformed into

$$u_k = t_0^{\frac{2}{n-1}-3k} P_k(z), \ (k = 0, 1, 2, \ldots) \tag{4.13}$$

and

$$u = \sum_{k=0}^{\infty} q^k t_0^{\frac{2}{n-1}-3k} P_k(z), \tag{4.14}$$

with similarity variable $z = t_0 x - x_0 t$, then Eq. (4.12) becomes

$$\begin{aligned} &P_{k,zzz} + na \sum_{i_1+i_2+\cdots+i_n=k} P_{i_1} P_{i_2} \ldots P_{i_{(n-1)}} P_{i_n,z} V \\ &- \frac{x_0}{t_0^3} P_{k,z} + \epsilon P_{k-1} = 0, \ (k = 0, 1, 2, \ldots) \end{aligned} \tag{4.15}$$

with $P_{-1} = 0$.

Meanwhile, for Case 2 in Chapter 3,we have the similarity solutions are

$$u_k = V_k(\xi), \ \xi = t_0 x - x_0 t, \ (k = 1, 2, \ldots, n) \tag{4.16}$$

thus the series reduction solution to Eq. (1.22) is

$$u = \sum_{k=0}^{\infty} q^k V_k(\xi), \tag{4.17}$$

where $V_k(\xi)$ $(k = 0, 1, 2 \ldots)$ yields

$$\begin{aligned}
O(q^0): \quad & (t_0)^3 V_{0,\xi\xi\xi} + nat_0 V_0^{n-1} V_{0,\xi} - x_0 V_{0,\xi} = 0, \\
O(q^1): \quad & (t_0)^3 V_{1,\xi\xi\xi} + nat_0 V_0^{n-1} V_{1,\xi} + n(n-1)at_0 \\
& V_0^{n-2} V_1 V_{0,\xi} - x_0 V_{1,\xi} + \epsilon V_0 = 0, \\
& \ldots, \\
O(q^k): \quad & (t_0)^3 V_{k,\xi\xi\xi} + nat_0 \sum_{i_1+i_2+\cdots+i_n=k} V_{i_1} V_{i_2} \ldots V_{i_{(n-1)}} V_{i_n,\xi} \\
& -x_0 V_{k,\xi} + \epsilon V_{k-1} = 0,
\end{aligned}$$

with $0 \le i_m \le k$, $(m = 1, \ldots, n)$ and $V_{-1} = 0$. The kth $(k > 0)$ similarity reduction equation can be rewritten as an ODE

$$(t_0)^3 V_{k,\xi\xi\xi} + nat_0 [{V_0}^{n-1} V_{k,\xi} \ + (n-1){V_0}^{n-2} V_k V_{0,\xi}] - x_0 V_{k,\xi} = G_k(\xi); (k = 0, 1, \ldots) \tag{4.18}$$

of $V_k(\xi)$, where G_k is a function of $\{V_0, V_1, \ldots, V_{k-1}\}$ defined as

$$G_k(\xi) = -\epsilon V_{k-1} - nat_0 \sum_{i_1+i_2+\cdots+i_n=k} V_{i_1} V_{i_2} \ldots V_{i_{(n-1)}} V_{i_n,\xi}, \ (k = 0, 1, \ldots) \tag{4.19}$$

with $i_m \ne k$ $(m = 1, \ldots, n)$.

We make $V_k(\xi) \to t_0^{\frac{2}{n-1}-3k} P_k(\xi)$ maps Eqs. (4.16) and (2.22) into

$$u_k = t_0^{\frac{2}{n-1}-3k} P_k(\xi), \ (k = 0, 1, 2, \ldots) \tag{4.20}$$

and

$$\begin{aligned}
& P_{k,\xi\xi\xi} + na \sum_{i_1+i_2+\cdots+i_n=k} P_{i_1} P_{i_2} \ldots P_{i_{(n-1)}} P_{i_n,\xi} \\
& -\frac{x_0}{t_0^3} P_{k,\xi} + \epsilon P_{k-1} = 0, \ (k = 0, 1, 2, \ldots)
\end{aligned} \tag{4.21}$$

with $P_{-1} = 0$, which are formally equivalent to Eqs. (4.13) and (4.15) respectively.

From the above analysis of the results from both methods, we can see that approximate homotopy direct method produces more general approximate homotopy similarity reduction than the approximate homotopy symmetry method does.

CHAPTER 5

CONCLUDING REMARKS

In sum, applying the approximate homotopy symmetry method and the approximate homotopy direct method to the nonlinear $K(n, 1)$ equation with damping, we have obtained the homotopy similarity reduction equations of different orders in general forms and gained the infinite homotopy series similarity reduction solutions in uniform formulas for Eq. (1.22). As a result, we have revealed the formal coincidence for both methods by relating both results. It is fascinating to take both methods into consideration while handling with other perturbed PDEs. Furthermore, the prolongations of approximate nonclassical symmetry ones is likely to improve this method.

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