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Opial's Inequality on Time Scales and an Application

Tamara Vernee Gray

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**OPIAL'S INEQUALITY ON TIME SCALES
AND
AN APPLICATION**

by
TAMARA V. GRAY

(Under the Direction of Billûr Kaymakçalan)

ABSTRACT

In view of the recently developed theory of calculus for dynamic equations on time scales (which unifies discrete and continuous systems), in this project we give some basic Opial type dynamic Inequalities. We discuss the background developed for Opial Inequalities which have many important applications both in the continuous and discrete cases in many areas of applicable analysis. Finally, having the above developed tools at our disposal, we give an example that concerns upper bound estimates of Dynamic Initial Value Problems and illustrate the usage of the developed Dynamic Opial Inequality.

INDEX WORDS: Discrete, Continuous, Time Scales, Dynamic Equations, Opial's inequality, Upper bounds of solutions of Initial Value Problems.

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B.S., Savannah State University, 2000

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TAMARA V. GRAY

Major Professor: Billûr Kaymakçalan

Committee: Martha L. Abell
Goran Lesaja

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CHAPTER 1

INTRODUCTION

Difference equations appear as natural descriptions of observed evolution phenomena, because most measurements of time evolving variables are discrete and as such, these equations are in their own right important mathematical models. More importantly, since nonlinear problems cannot be solved, for such equations discretization methods which make use of difference equations are employed. Several results in difference equations have been obtained as more or less natural discrete analogs of corresponding results of differential equations. Furthermore, the application of the theory of difference equations to various fields such as numerical analysis, control theory, finite mathematics, computer science, probability theory, queuing problems, statistical problems, stochastic time series, etc. is rapidly increasing. Therefore, difference equations are not merely the discrete analogs of differential equations, in fact they have led the way for the development of the latter. In [1] several examples from the diverse fields have been illustrated which are sufficient to convey the importance of the serious qualitative as well as quantitative study of difference equations.

In recent years discrete time dynamical systems have experienced rapidly growing popularity thereby paving its way into an independent discipline [1, 38]. They are no longer only auxiliary systems for continuous time dynamical systems through discretization but rather they have developed their own independent right to exist. Despite this tendency of independence; however, there is a well-known striking similarity or even duality between the two concepts and therefore the discrete systems are commonly treated "along the lines" of continuous time theory. Nevertheless, the two kinds of systems have a number of significant differences mainly due to the topological fact that in one case the time scale and the corresponding trajectories are

connected while in the other case they are not. All the investigations on the two time scales show that much of the analysis is analogous but, at the same time, usually additional assumptions are needed in the discrete case in order to overcome the topological deficiency of lacking connectedness.

In view of many well-known analogies in the concepts of difference calculus with the difference operator

$$(\Delta_h f)(t) = \frac{f(t+h) - f(t)}{h}$$

on one hand and differential calculus with the differential operator

$$\left(\frac{df}{dt}\right)(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$$

on the other hand, it is an obvious desire to have a better way of dealing with problems treated in a parallel manner in both continuous and discrete time scales by establishing a method for theory which allows us to systematically handle both time scales simultaneously. In the context of such a theory it should also be possible to get some insight and better understanding of the sometimes subtle differences between the two types of systems. To create the desired theory first of all requires to set up a certain structure \mathbb{T} which is to play the role of the time scale generalizing \mathbb{R} and \mathbb{Z} . Furthermore, an operation on the space of functions from \mathbb{T} to the state space has to be defined generalizing the differential and difference operations.

Perhaps from a modeling point of view, it is more realistic to model a phenomenon by a dynamic system which incorporates both continuous and discrete times, namely, time as an arbitrary closed set of reals. It is therefore natural to ask whether it is possible to provide a framework which allows us to handle both dynamic systems simultaneously so that we can get some insight and a better understanding of the subtle differences of these two systems. The answer is affirmative and the recently

developed theory of "dynamic systems on time scales" "or dynamic systems on measure chains", (by a measure chain we mean the union of disjoint closed intervals of \mathbb{R}) offers the desired unified approach.

In [9, 10, 11, 25, 26, 27], Aulbach and Hilger have initiated the development of this theory with the aim of treating dynamic problems from a qualitative point of view. A calculus on measure chains meeting the requirements described above is developed and in this general framework some basic results for linear dynamic systems are given.

Later in [30, 32, 34, 35, 36] Kaymakçalan and others have extended this theory to a unified analysis of nonlinear systems from the point of view of investigating qualitative and quantitative behaviors of such systems. Most of these results are contained in the monograph [33], which is the first book containing extensive coverage of up-to-date results in the area of "Time-Scales" or "Measure Chains". Recently, many new results in the area have been obtained, and the two monographs [15, 16] by Bohner and Peterson, give very thorough insight into some of the more contemporary developments in the area.

Having the basics of the Theory of Time-Scales at our disposal, in this thesis we want to focus our attention to obtaining unified results in the context of Inequalities, in particular some kinds which involve integrals of functions and derivatives, namely Opial Type inequalities. It has been shown that these type of inequalities are of great importance in Mathematics with applications in the theory of differential equations, approximations and probability. For detailed investigations of Opial's type inequalities , their several generalizations, extensions and discretizations along with applications such as their role in establishing the existence and uniqueness of

solutions to initial and boundary value problems for ordinary and partial differential equations as well as difference equations, the recently published monograph [7], which is the first book dedicated to the theory of Opial type inequalities, serves as an excellent reference. In addition to the above mentioned applications, many qualitative behaviours such as oscillation, non-oscillation, boundedness, have also been discussed [7] in the light of Opial's inequality .

We begin this work, with giving the basics of Time-Scales in Chapter 2. Throughout \mathbb{T} denotes a time-scale (any closed subset of \mathbb{R} with order and topological structure defined in a canonical way) with $t_0 \geq 0$ as a minimal element. In Section 2.1, some of the essential features of the order and topological structure of time-scales are introduced, with the Induction Principle which is well suited for Time-Scales and constituting an efficient tool for unified results for both continuous and discrete systems, being given in Section 2.2. The concepts and results given in sections 2.3, 2.4, 2.5, 2.6 and 2.7 form the basics of the calculus developed and includes those features necessary for our Opial inequality unification purpose, concepts such as rd-continuity, differentiation, integral and the exponential function.

In Chapter 3, along with the basic Opial inequality given by Zdzidlaw Opial in 1960, we state some basic analogous discrete and continuous inequalities and results pertaining to them. Then in Section 3.2, the introduced inequalities are given in the time scale set-up along with generalizations. In these unifications the calculus on time-scales plays a vital role, the theorems are modified in a manner to reflect the differences of the discrete and continuous results at the same time.

Chapter 4 is devoted to establishing upper bound estimates for the solutions of I.V.P.'s by making use of the Opial type inequalities obtained in Chapter 3. As

auxiliary tools, generalized Gronwall type inequalities which were obtained earlier in time-scales are also employed for our purpose. Examples which realize the usage of the obtained upper-bound estimates are given.

CHAPTER 2

BASIC CONCEPTS OF TIME SCALES

2.1 Order and Topological Structure

In this chapter we give the main definitions and characteristics of the calculus on time scales initiated by Aulbach and Hilger [9, 10, 11, 25, 26, 27] which comprise those features of the differential and difference calculus as they are relevant for the development of a qualitative theory of dynamical systems. We note that the contents of such a development of some higher ranging calculus is quite extensive. So we suffice only with giving the essentials necessary for the further aims of this work and refer to [9, 10, 15, 33] for more details.

Throughout this chapter we remark the many similarities and differences in considering the Time Scale as in the \mathbb{R} or \mathbb{Z} set-up.

We begin this chapter with highlighting the basics of the order and topological structure of Time Scale. In this context the special features of openness brings additional considerations into account. Next, we continue by giving the special type of Induction Principle that is used as a main tool in the arguments. We proceed by paying special attention to concepts such as Continuity, Rd-Continuity, Differentiability which possess important roles in the analysis of discrete and continuous scales in a unified manner. As a result of these concepts, we end this chapter by considering Integrals and Exponential Functions for Time Scales, and give some so-called Useful Time Scales Formulas that will be employed throughout the thesis.

As subsets of \mathbb{R} , time scales carry an order structure in a canonical way. A time scale \mathbb{T} may be bounded above or below. As a consequence of the embedding of \mathbb{T} in \mathbb{R} , all order theoretical notions such as bounds, least upper bounds, greatest lower bounds and intervals are available in \mathbb{T} as they are in \mathbb{R} .

We note that the order structure of \mathbb{R} induces an order structure on each time-scale \mathbb{T} . On time scales, there exist primarily two order structures that should be distinguished from each other. But this distinction is easily seen to be only figurative; since it can be shown that due to the closedness assumption of time-scales, the \mathbb{R} or \mathbb{T} suffixes need not be mentioned. Hence, when order theoretical concepts are concerned the \mathbb{R} or \mathbb{T} specifications can be dropped and for instance; bound, boundedness or supremum concepts can be used, instead of \mathbb{R} -boundedness, \mathbb{T} -boundedness, \mathbb{R} -supremum, \mathbb{T} -supremum, etc.

As a consequence of the definition of a time-scale \mathbb{T} being a closed subset of \mathbb{R} , topological structure of \mathbb{T} , especially from the openness point of view has various features. Obviously any subset A of \mathbb{T} which is open in \mathbb{R} , is also open in \mathbb{T} . The reverse is generally not true, though, as the simple example $\mathbb{T} := \mathbb{Z}$ shows, where any subset in the induced topology is open in \mathbb{T} but not open in \mathbb{R} . This is taken care of by distinguishing between \mathbb{R} -openness and \mathbb{T} -openness. In order to investigate the details of the notion of openness in time-scales, we must define the concept of neighborhood in this set-up. We give two different versions of neighborhood definitions, distinguishing between the concepts of \mathbb{R} -neighborhood and \mathbb{T} -neighborhood, giving way to the distinction between \mathbb{R} -openness and \mathbb{T} -openness.

Given a time-scale \mathbb{T} , $t \in \mathbb{T}$ and an $\epsilon > 0$, we denote by

$$\mathbb{R}_\epsilon(t) := \{x \in \mathbb{R} : t - \epsilon < x < t + \epsilon\}$$

$$\mathbb{T}_\epsilon(t) := \{x \in \mathbb{T} : t - \epsilon < x < t + \epsilon\}$$

the ϵ -neighborhoods of t in \mathbb{R} and \mathbb{T} , respectively.

An interval, in the time-scale context, is always understood as the intersection of a real interval with a given time-scale.

The following definition will lead the way to the concept of \mathbb{T} -openness.

Definition 2.1.1. Let \mathbb{T} be a time-scale and $t \in \mathbb{T}$. The set $U \subseteq \mathbb{R}$ is called an \mathbb{R} -neighborhood of t provided that there is $\epsilon > 0$ with $\mathbb{R}_\epsilon(t) \subseteq U$. The set $V \subseteq \mathbb{T}$ is called a \mathbb{T} -neighborhood of t , provided that there is $\epsilon > 0$ with $\mathbb{T}_\epsilon(t) \subseteq V$.

Neighborhood concepts give rise to further topological notions.

Definition 2.1.2. A subset A of a time-scale \mathbb{T} is open in \mathbb{T} if for each $t \in A$ there is an $\epsilon > 0$ such that $\mathbb{T}_\epsilon(t) \subseteq A$.

Remark 2.1.1. : For any time-scale \mathbb{T} , \emptyset and \mathbb{T} are open in \mathbb{T} .

Example 2.1.1:

For the time-scale $\mathbb{T} := \{x \in \mathbb{R} : -1 \leq x \leq 0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, any $a \in \mathbb{T}, a > 0$ is of the form $a = \frac{1}{n}$, for some $n \in \mathbb{N}$ and since

$$\mathbb{T}_\epsilon(a) = \{x \in \mathbb{T} : a - \epsilon < x < a + \epsilon\} = \{a\} \subseteq \{a\},$$

$\{a\}, a > 0$, is a \mathbb{T} -open set. Similarly; for $-1 \leq b \leq c \leq 1; (b, c)$ is a \mathbb{T} -open set, and since

$$\mathbb{T}_\epsilon(-1) = \{x \in \mathbb{T} : -1 - \epsilon < x < -1 + \epsilon\} \subseteq [-1, d], \quad d > 0$$

and

$$\mathbb{T}_\epsilon(d) = \{x \in \mathbb{T} : d - \epsilon < x < d + \epsilon\} = \{d\} \subseteq [-1, d],$$

we observe that $[-1, d]$ is a \mathbb{T} -open set for $d > 0$.

Likewise, $[-1, d)$ is \mathbb{T} -open for all $d < 0$.

For all $e \in \mathbb{T}, e > 0$, $\mathbb{T} \setminus \{e\}$ is \mathbb{T} -open, since for any $t \in \mathbb{T} \setminus \{e\}, (t \neq e)$,

$$\mathbb{T}_\epsilon(t) = \{x \in \mathbb{T} : t - \epsilon < x < t + \epsilon\} \subseteq \mathbb{T} \setminus \{e\}.$$

On the other hand, for $a \in \mathbb{T}$, $a \leq 0$, $\{a\}$ is not \mathbb{T} -open, since

$$\mathbb{T}_\epsilon(a) = \{x \in \mathbb{T} : a - \epsilon < x < a + \epsilon\} \not\subseteq \{a\}.$$

Example 2.1.2: For the time-scale

$$\mathbb{T} := \{x \in \mathbb{R} : -1 \leq x \leq 0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\},$$

the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is open in \mathbb{T} . This can easily be seen by observing that A is an arbitrary union of sets of the form $\{\frac{1}{n}\}$, which are shown to be \mathbb{T} -open for each $n \in \mathbb{N}$ by Example 2.1.1 Hence A is also \mathbb{T} -open.

The next result enables us to observe a connection between the concepts of \mathbb{T} -openness and \mathbb{R} -openness.

Theorem 2.1.1. *A set $A \subseteq \mathbb{T}$ is open in \mathbb{T} if and only if there is a set $B \subseteq \mathbb{R}$, open in \mathbb{R} , with $A = B \cap \mathbb{T}$.*

Example 2.1.3: For the time-scale

$$\mathbb{T} := \{x \in \mathbb{R} : -1 \leq x \leq 0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\},$$

we have seen in Example 2.1.2 that the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is \mathbb{T} -open. Hence by Theorem 2.1.1 there should be a set B , open in \mathbb{R} , such that $A = B \cap \mathbb{T}$. Clearly, the set $B = \bigcup_{n \in \mathbb{N}} I_n$, where

$$I_n = \left\{ x \in \mathbb{R} : \frac{1}{2n} \leq x < \frac{3}{2n} \right\} \quad \text{for each } n \in \mathbb{N}$$

being an \mathbb{R} -open interval, has this property. Namely; $B \cap \mathbb{T} = \{\frac{1}{n} : n \in \mathbb{N}\} = A$ and B is \mathbb{R} -open.

Next, we give the following result which was mentioned earlier and remark once again that the converse is not true, referring to the Example 2.1.3.

Theorem 2.1.2. *If A is a subset of \mathbb{T} , then A is open in \mathbb{R} implies that A is open in \mathbb{T} .*

Although, due to the closedness of \mathbb{T} , the notion of closedness in \mathbb{T} coincides with that of \mathbb{R} , also the definition of \mathbb{T} -openness gives way in a natural manner to the following concept of closedness.

Definition 2.1.3. A subset A of \mathbb{T} is called closed in \mathbb{T} provided that $\mathbb{T} \setminus A$ is open in \mathbb{T} .

Having mentioned the concepts of openness and closedness for time-scales; we can also remark about other topological concepts such as compactness and connectedness and their special features in time-scales.

Definition 2.1.4. A subset A of a time-scale \mathbb{T} is called compact in \mathbb{T} provided that A is bounded and closed in \mathbb{T} .

Next, in order to consider the concept of connectedness for time-scales we first concentrate on the following example.

Example 2.1.4: Considering the time-scale

$$\mathbb{T} := \{x \in \mathbb{R} : -1 \leq x \leq 0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\},$$

as a result of Example 2.1.1 and Example 2.1.3, it is clear that $\{1\}$ is both open and closed in \mathbb{T} . Therefore, \mathbb{T} can be written as a disjoint union of non-empty \mathbb{T} -open sets, giving way to the disconnectedness of this particular time-scale \mathbb{T} .

On the other hand, obviously there exist connected time-scales such as $\mathbb{T} := \mathbb{R}$.

Consequently; from the connectedness point of view, there can not be one single notion that applies to all time scales, and we conclude that a time-scale \mathbb{T} may or may not be connected. In order to overcome this topological deficiency, we introduce the concept of jump operators.

Definition 2.1.5. The mappings $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$, such that

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

are called jump operators. In the case \mathbb{T} is bounded above, we supplement the definition by $\sigma(\max \mathbb{T}) := \max \mathbb{T}$ and accordingly $\rho(\min \mathbb{T}) := \min \mathbb{T}$ if \mathbb{T} is bounded below.

These jump operators enable us to classify the points $\{t\}$ of a time-scale as right-dense, right-scattered, left-dense and left-scattered depending on whether $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively, for any $t \in \mathbb{T}$.

Example 2.1.5: For the time-scale;

$$\mathbb{T} := \{x \in \mathbb{R} : -2 \leq x \leq 0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{x \in \mathbb{R} : 2 \leq x \leq 3\} \cup \{4\}$$

points which are;

- right-dense and left-dense: all $t \in [-2, 0] \cup (2, 3)$,
- right-dense and left-scattered: 2, 4,
- right-scattered and left-dense: 3,
- right-scattered and left-scattered: all $\frac{1}{n}, n \in \mathbb{N}$.

Here, we take into account the facts that -2 is a minimal and 4 is a maximal point, respectively. Hence $\rho(-2) = -2$, thereby implying -2 to be a left-dense point and $\sigma(4) = 4$, also implying 4 to be a right-dense point.

Remark 2.1.2. :

(1) So far, we have not specified any direction in a time-scale and used both (positive and negative) directions in a symmetric manner. But from now on we have to cease this symmetric consideration, since the time-scale calculus will naturally include the classical difference operation as a special case and needless to mention, symmetry will automatically be destroyed in the development of this “difference” process. Hence, we will consider the direction for a time-scale \mathbb{T} to be in the sense of increasing values of t , for $t \in \mathbb{T}$.

(2) If a time-scale \mathbb{T} has a maximal element, which is moreover left-scattered, then this point plays a particular role in several respects and therefore we call it degenerate. All other elements of \mathbb{T} are called non-degenerate and the subset of non-degenerate points of \mathbb{T} is denoted by \mathbb{T}^k . Since each closed subset A of a time-scale \mathbb{T} is also a time-scale, it is possible that A^k is formed. Naturally $A = A^k$ is possible as long as A does not have a left-scattered maximum.

Having characterized the points of a time-scale by use of jump operators we can remark about the topology on \mathbb{T} and show that the so-called order topology (interval topology) on \mathbb{T} is compatible with the metric topology on \mathbb{T} , which is induced on \mathbb{T} as a consequence of the embedding of \mathbb{T} in \mathbb{R} .

To show this, let $t_0 \in \mathbb{T}$ and consider the neighborhood systems in both topologies. In the open interval topology (order topology), the neighborhood system is generated by open intervals of the form (t_1, t_2) such that $t_1 < t_0 < t_2$. In the metric topology, the neighborhood system is generated by open balls of the form $\{t \in \mathbb{T} : |t - t_0| < \epsilon; \epsilon \in \mathbb{R}_+\}$. It is easy to observe that corresponding to each member of the first system, we can find a member of the second contained in it and vice versa.

Let $t_1, t_2 \in \mathbb{T} \cup \{-\infty, +\infty\}$ such that $t_1 < t_0 < t_2$. Clearly, there exists $\epsilon > 0$ such that $t_1 \leq t_0 - \epsilon < t_0 < t_0 + \epsilon \leq t_2$. Thus $\{t \in \mathbb{T} : |t - t_0| < \epsilon\} \subseteq (t_1, t_2)$. Conversely, if $\delta > 0$ is given, then $\{t \in \mathbb{T} : |t - t_0| < \delta\}$ is a neighborhood in the metric topology on \mathbb{T} . We consider the interval $(\sigma(t_0 - \delta), \rho(t_0 + \delta))$, where σ, ρ are the jump operators given by Definition 2.1.5. Then $s \in (\sigma(t_0 - \delta), \rho(t_0 + \delta)) \cap \mathbb{T}$ implies

$$t_0 - \delta \leq \sigma(t_0 - \delta) < s < \rho(t_0 + \delta) \leq t_0 + \delta.$$

Hence any interval in the order topology is contained in a neighborhood in the metric topology. So the (interval) topology on \mathbb{T} is metrizable. In fact we can give the following theorem.

Theorem 2.1.3. *A time-scale \mathbb{T} equipped with the order topology is metrizable and is a K_σ -space; i.e. it is a union of at most countably many compact sets. The metric on \mathbb{T} which generates the order topology is given by*

$$d(r, s) := |\mu(r, s)|$$

where $\mu^*(\cdot) = \mu(\cdot, \tau)$ for a fixed $\tau \in \mathbb{T}$ is defined as follows:

Definition 2.1.6. The mapping $\mu^* : \mathbb{T} \rightarrow \mathbb{R}^+$ such that $\mu^*(t) = \sigma(t) - t$ is called *graininess*.

When $\mathbb{T} = \mathbb{R}$, $\mu^*(t) \equiv 0$ and for $\mathbb{T} = \mathbb{Z}$, $\mu^*(t) \equiv 1$, if $\mathbb{T} = h\mathbb{Z}$, $\mu^*(t) = h$.

2.2 Induction Principle

A basic tool which is employed in the proofs is the following Induction Principle, well suited for time scales.

Theorem 2.2.1. *Let \mathbb{T} be a time-scale with minimal element $t_0 \geq 0$. Suppose for any $t \in \mathbb{T}$, there is a statement $A(t)$ such that the following conditions are verified:*

- (I) $A(t_0)$ is true;
- (II) If t is right-scattered and $A(t)$ is true, then $A(\sigma(t))$ is also true;
- (III) For each right-dense t , there exists a neighborhood U such that whenever $A(t)$ is true, $A(s)$ is also true for all $s \in U, s \geq t$;
- (IV) For left-dense t , $A(s)$ is true for all $s \in [t_0, t)$ implies $A(t)$ is true.

Then the statement $A(t)$ is true for all $t \in \mathbb{T}$.

Remark 2.2.1. :

For $\mathbb{T} = \mathbb{N}$, conditions (III) and (IV) are redundant and the theorem reduces to the usual principle of mathematical induction for the natural numbers.

On the other hand, if \mathbb{T} is a real interval closed at its left end, condition (II) becomes superfluous.

This induction principle then contains the (topological) principle that a nonvoid subset of an interval which is at the same time open and closed coincides with the interval under consideration.

The continuous type of Induction Argument given by the above theorem is being used by Dieudonné [21] for proving basic results in differential and integral calculus. Therefore, his presentation serves as a good guideline for the endeavor to generalize fundamental results such as the intermediate value theorem, the mean value theorem and existence theorems for differential equations. The use of the above induction principle makes particularly clear the difference between real intervals on the one hand and arbitrary time-scales on the other hand. This is due to the possible absence or presence of jumps. As it turns out, the verification of (II) is always algebraic. In

contrast, the proofs in which (III) and (IV) are valid usually require topological or analytical tools such as inequalities or fixed point theorems.

Since time-scales are not-connected in general, the Intermediate Value Theorem must be modified. An application of the Induction Principle, Theorem 2.2.1, gives the following form of the Intermediate Value Theorem on time-scales.

Corollary 2.2.1. *Suppose \mathbb{T} is a time-scale and f is a real-valued continuous function on the closed interval $[t_1, t_2] = \{t \in \mathbb{T} : t_1 \leq t \leq t_2\}$, $t_1, t_2 \in \mathbb{T}$ with $f(t_2) < 0 < f(t_1)$. Then there exists a $t \in [t_1, t_2]$ such that*

$$f(t)f(\sigma(t)) \leq 0,$$

Such a point t with the property (2.2.1) is called “*node point*” and in that case f is said to “*vanish between*” t and $\sigma(t)$.

Remark 2.2.2. :

If τ is right-dense, then from the statement of Corollary 2.2.1, we necessarily have that $f(\tau) \equiv 0$. In all other cases $f : \tau \rightarrow \sigma(\tau)$.

Remark 2.2.3. :

In some situations we might encounter points of a time-scale which are both left-dense and right-scattered at the same time. For instance, in the consideration of impulsive systems such points arise naturally. In order to consider dynamic systems containing such points we must guarantee the applicability of the Induction Principle at these points. It can be observed that assuming conditions (II) and (IV) of the Induction Principle to hold gives rise intrinsically to the validity of a statement at such left dense-right scattered points. To show this, let U be a left-neighborhood of the point t_0 which is both left-dense and right-scattered. Then $\rho(t_0) = t_0$ and by (IV) assuming

$A(s)$ to be true for all $s \in U, s < t_0 = \rho(t_0)$, we have that $A(\rho(t_0))$ is true. Now by (II), validity of $A(\rho(t_0))$ implies $A(\sigma(\rho(t_0)))$ to be true. But $A(\sigma(\rho(t_0))) = A(t_0)$ and hence the statement $A(t)$ is true at t_0 .

2.3 Continuity and Rd-Continuity

In order to introduce the concept of integration on time-scales, we need a notion which is related to the approximation of continuous functions by step-functions.

Definition 2.3.1. If a function is defined on a compact interval $[t_a, t_b]$ of a time-scale \mathbb{T} and if there are finite number of elements t_0, \dots, t_n of \mathbb{T} with $t_a = t_0 < t_1 < \dots < t_n = t_b$ and such that $f : [t_a, t_b] \rightarrow \mathbb{R}$ is constant on $[t_i, t_{i+1}]$, for $i = 1, 2, \dots, n - 1$, then f is called a *step-function*.

There is no difficulty in defining the continuity concept for \mathbb{R}^n -valued functions on Time Scales. The continuity definition can be taken from Real Analysis without any alteration.

Definition 2.3.2. The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be *continuous* at $t_0 \in \mathbb{T}$ if for all $\epsilon > 0$, there exists a neighborhood $U(t_0)$ such that

$$|f(t) - f(t_0)| < \epsilon \quad \text{for all } t \in U(t_0).$$

In analysis on \mathbb{R} discontinuity points are usually given graphically by jump points. But since Time Scales are not connected in general, a similar result need not be necessary.

Example 2.3.1: Consider the function $f : \mathbb{T} \rightarrow \mathbb{R}$ given by

$$f(t) = \begin{cases} |t|, & t < 0 \\ t + 1, & t \geq 0. \end{cases}$$

- (i) If $\mathbb{T} := \mathbb{R}$, f is not continuous at $t = 0$.
- (ii) If $\mathbb{T} := \mathbb{Z}$, f is continuous for all $t \in \mathbb{Z}$. Here attention should be paid to various topological properties. For instance, $\{0\}$ can be chosen for the neighborhood of “0”.

Remark 2.3.1. :

- (i) The last example shows that at right and left-scattered points every function is continuous. This may be possible by choosing $U = \{t_0\}$ as the neighborhood of t_0 , so that at every point of this neighborhood, the difference between the value of the function at that point and the value of the function at t_0 remains less than ϵ .
- (ii) Properties of continuous functions may be adapted similarly from Real Analysis by taking the topological properties into account.

In order to pave our way to the concept of integration, we first have to obtain an appropriate class of functions having anti-derivatives. For this purpose we define the following notions.

Definition 2.3.3. Let X be an arbitrary topological space and \mathbb{T} a time-scale. The mapping $g : \mathbb{T} \rightarrow X$ is said to be **regulated** if at each left-dense $t \in \mathbb{T}$, $g(t^-) = \lim_{s \rightarrow t^-} g(s)$ exists and at each right-dense point $t \in \mathbb{T}$, $g(t^+) = \lim_{s \rightarrow t^+} g(s)$ exists.

Definition 2.3.4. The mapping $g : \mathbb{T} \rightarrow X$ is called *rd-continuous* if

- (i) it is continuous at each right-dense or maximal $t \in \mathbb{T}$,
- (ii) at each left-dense point, left sided limit $g(t^-)$ exists.

We denote by $C_{rd}[\mathbb{T}, X]$ the set of rd-continuous mappings from \mathbb{T} to X . The class of rd-continuous functions turns out to be a “natural” class within the context of the Time Scale calculus. The function $\mu^* : \mathbb{T} \rightarrow \mathbb{R}$ in case $\mathbb{T} := [0, 1] \cup \mathbb{N}$, for example, is rd-continuous but not continuous at 1.

The following implications are immediate:

$$\text{continuous} \Rightarrow \text{rd-continuous} \Rightarrow \text{regulated.}$$

If \mathbb{T} contains *ldrs*-points (left-dense and right-scattered) then the first implication is not invertible. However, on a discrete time scale all three notions coincide.

As a generalization of Definition 2.3.4, we give the following.

Definition 2.3.5. The mapping $f : \mathbb{T}^k \times X \rightarrow X$ is called *rd-continuous* if it

- (i) is continuous at each (t, x) with right-dense or maximal t ,
- and
- (ii) the limits $f(t^-, x) : \lim_{\substack{(s,y) \rightarrow (t,x) \\ s < t}} f(s, y)$ and $\lim_{y \rightarrow x} f(t, y)$ exist at each (t, x) with left-dense t .

Hence, in general for left-dense t , the function $f(t, \cdot) : \mathbb{T}^k \times X \rightarrow X$ is in no way a continuous continuation of the mapping $f : (-\infty, t) \times X \rightarrow X$ to the point t .

Example 2.3.2: Given an rd-continuous function $g : \mathbb{T} \rightarrow X_1$, which is in the sense of Definition 2.3.4, a continuous function $h : X_2 \rightarrow X_3$ and another continuous function

$f : X_1 \times X_2 \rightarrow X_3$, the composite function $f(g(\cdot), h(\cdot))$ is rd-continuous in the sense of Definition 2.3.5.

We conclude this section by introducing a tool which is useful in some qualitative properties and give a relevant result to this section.

Definition 2.3.6. Consider the mapping $f^{\tau] : (-\infty, \tau] \times X \rightarrow X$ which is defined for a fixed $\tau \in \mathbb{T}^k$ as:

$$f^{\tau]}(t, x) = \begin{cases} f(t, x), & \text{if } (t, x) \in (-\infty, \tau) \times X \\ f(\tau^-, x), & \text{if } (t, x) \in \{\tau\} \times X. \end{cases} \quad (2.3.1)$$

Here f is assumed to be rd-continuous on $\mathbb{T} \times X$. $f^{\tau]}$ does not necessarily coincide with f on $(-\infty, \tau]$ if τ is a ldrs-point, otherwise it does.

Using Definition 2.3.6, we can give the following result which is related to rd-continuous functions.

Lemma 2.3.1. For each continuous function $x : \mathbb{T} \rightarrow X$, the mapping $f^{\tau]}(\cdot, x(\cdot))$ is rd-continuous with respect to the interval $(-\infty, \tau]$, where $f \in C_{rd}[\mathbb{T} \times X, X]$.

2.4 Differentiation

When we consider functions which are defined on a time-scale \mathbb{T} and taking their values in a topological space X , due to the embedding of \mathbb{T} in \mathbb{R} , the concept of continuity arises in a straightforward manner. For the concept of differentiation, however, the topological structure of \mathbb{T} plays a vital role. In fact the generally lacking openness of \mathbb{T} requires a particular proceeding when one aims at a concept of differentiability which contains as special cases the differential calculus on one hand and the difference calculus on the other.

Definition 2.4.1. Let \mathbb{T} be a time-scale and $f : \mathbb{T} \rightarrow \mathbb{R}$. f is called *differentiable* at $t_0 \in \mathbb{T}$, if there exists an $a \in \mathbb{R}$ with the following property:

For any $\epsilon > 0$, there exists a neighborhood U of t_0 , such that

$$\left| f(\sigma(t_0)) - f(t) - (\sigma(t_0) - t)a \right| \leq \epsilon |\sigma(t_0) - t| \quad (2.4.1)$$

for all $t \in U$.

$f : \mathbb{T} \rightarrow X$, where X is any Banach Space, is called differentiable if f is differentiable for each $t \in \mathbb{T}$.

Remark 2.4.1. :

In the two special cases \mathbb{R} and \mathbb{Z} the derivative is uniquely determined, in fact one gets $a = \frac{df}{dt}(t_0)$ and $a = f(t_0 + 1) - f(t_0)$, respectively.

Theorem 2.4.1. *If $f : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at a non-degenerate point t_0 , then from the Definition 2.4.1, $a \in \mathbb{R}$ is uniquely determined. It is denoted by $f^\Delta(t_0)$ and called the **derivative** of f at t_0 .*

Remark 2.4.2. :

Sometimes we may need the generalized derivatives corresponding to Dini derivatives from the right, in which case we write

$$f(\sigma(t)) - f(s) - (\sigma(t) - s)a < \epsilon(\sigma(t) - s)$$

for all $s \in W$; W being a right-neighborhood of $t \in \mathbb{T}$. We denote in this case $a = D^+ f^\Delta(t)$.

Note that if t is right-scattered, then $D^+ f^\Delta(t)$ is the same as the $f^\Delta(t)$ given above. In this case we write

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{(\sigma(t) - t)}.$$

The relationship between differentiability and continuity follows from Real Analysis.

Theorem 2.4.2. *If $f : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at $t_0 \in \mathbb{T}$, then it is continuous at t_0 .*

Theorem 2.4.3. *If f is continuous at t_0 and t_0 is right-scattered, then f is differentiable at t_0 with derivative*

$$f^\Delta(t_0) = \frac{f(\sigma(t_0)) - f(t_0)}{\sigma(t_0) - t_0}$$

as indicated in Remark 2.4.2.

Example 2.4.1: Let $\mathbb{T} := \{x \in \mathbb{R} : -2 \leq x \leq 0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and

$f(x) := |x|$, $f_n(x) = |x - \frac{1}{n}|$, $n \in \mathbb{N}$, for all $x \in \mathbb{T}$.

Then all functions f_n 's are differentiable for all points of \mathbb{T} , but the limit function f has no derivative at 0, hence it is not differentiable.

Example 2.4.2: For $\mathbb{T} := \{0\} \cup \bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R} : \frac{1}{2n} \leq x \leq \frac{1}{2n-1}\}$ and

$$f(x) := \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{2n}, & \text{if } \frac{1}{2n} \leq x \leq \frac{1}{2n-1} \end{cases}$$

which gives, $f^\Delta(0) = 1$.

2.5 Antiderivative and Integral

After the development of the Time-Scale Analysis in the previous sections up to the concept of differentiability, now the concepts of antiderivative (primitive) and integration are to be presented. For this purpose, restricting ourselves to the class of differentiable functions, we consider the definition of the immediate concepts of antidifferentiation.

Once the main theorem, which guarantees the existence of a primitive (antiderivative) function for continuous functions, has been given, we must introduce the concept of Cauchy-Integral which is an essential tool for further purposes.

In the following, \mathbb{T} is a Time-Scale and \mathbb{T}' is a subinterval of \mathbb{T} .

Definition 2.5.1. Let $f : \mathbb{T}' \rightarrow \mathbb{R}$ be a differentiable function.

The function

$$f^\Delta : \begin{cases} (\mathbb{T})^k \rightarrow \mathbb{R} \\ t \rightarrow f^\Delta(t) \end{cases}$$

is called the *Derivative* of f on \mathbb{T}' . In the case of $\mathbb{T}' = \mathbb{T}$, the statement "on \mathbb{T}' " disappears.

Remark 2.5.1. :

- (i) From the Theorem 2.4.2, it is obvious that a mapping which is differentiable on \mathbb{T}' is continuous.
- (ii) If $\mathbb{T}' \subset \mathbb{R}$ is an interval, which is open in \mathbb{R} ; then the above concept coincides with differential calculation.

Definition 2.5.2. A function $f : \mathbb{T}^k \rightarrow \mathbb{R}$ is called an antiderivative (primitive) of g on \mathbb{T}' , if it is differentiable on \mathbb{T}' and for all $t \in \mathbb{T}'^k$, the condition $f^\Delta(t) = g(t)$ is satisfied.

As shown by the following theorem, just like in Real Analysis, there corresponds an anti-derivative for each rd-continuous function on Time-Scale.

Theorem 2.5.1. For any rd-continuous mapping $g : \mathbb{T}^k \rightarrow \mathbb{R}$, there exists an antiderivative function $f : \mathbb{T} \rightarrow \mathbb{R}$; $f : t \rightarrow \int_s^t g(s) \Delta s$, $s, t \in \mathbb{T}^k$.

Since, as a result of Definition 2.5.2 we can deduce that the difference of two antiderivative functions is constant, the following definition can be used similar to that in Real Analysis.

Definition 2.5.3. If the function $g : \mathbb{T}^k \rightarrow \mathbb{R}$ has an antiderivative function f on $[r, s] \subset \mathbb{T}$, then

$$\int_r^s g(t) \Delta t := f(s) - f(r)$$

is called the *Cauchy-Integral* from r to s of the function g .

For $\mathbb{T} = \mathbb{R}$ the Cauchy integral coincides with the Riemann integral.

For $\mathbb{T} = h\mathbb{Z}$ where $h > 0$, the identity

$$\int_r^s g(t) \Delta t = \begin{cases} \sum_{i=\frac{r}{h}}^{\frac{s}{h}-1} g(ih)h & \text{if } s > r \\ 0 & \text{if } s = r \\ -\sum_{i=\frac{s}{h}}^{\frac{r}{h}-1} g(ih)h & \text{if } s < r \end{cases}$$

can be shown. For further properties of integrals on \mathbb{T} we refer to [10].

2.6 The Exponential Function

We start this section by introducing the so-called Hilger complex plane, \mathbb{C}_h and refer to [27, 28, 15] for many results concerning this concept.

Let $h > 0$ be fixed and define the Hilger complex numbers by

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}.$$

Also if $h = 0$, let $\mathbb{C}_0 := \mathbb{C}$.

Now recalling that the unique solution of the initial value problem

$$x' = \alpha x, \quad x(0) = 1$$

is the exponential function $x(t) = e^{\alpha t}$, we are motivated to the following definition

Definition 2.6.1. For $\alpha \in \mathbb{C}_h$, we define the *exponential function* $e_\alpha(t)$ on $h\mathbb{Z}$ to be the *solution* of the initial value problem

$$x^\Delta(t) = \alpha x(t), \quad t \in \mathbb{T} := h\mathbb{Z}, \quad x(0) = 1. \quad (2.6.1)$$

The following theorem, whose proof can be found in the [27], gives the explicit formulation of the defined exponential function $e_\alpha(t)$, i.e. the unique solution of the I.V.P. (2.6.1)

Theorem 2.6.1. For $\alpha \in \mathbb{C}_h$, the exponential function $e_\alpha(t)$ is given by

$$e_\alpha(t) = (1 + \alpha h)^{\frac{t}{h}} \quad (2.6.2)$$

for $t \in \mathbb{T} := h\mathbb{Z}$.

We can define a more general exponential function as follows;

Definition 2.6.2. Assume $\alpha : \mathbb{T} \rightarrow \mathbb{C}$ is rd-continuous and $\alpha(t)\mu(t) + 1 \neq 0$ for $t \in \mathbb{T}^k$. Then for $t \in \mathbb{T}$, $s \in \mathbb{T}^k$, $e_\alpha(t, s)$ is defined to be the function such that for each fixed $s \in \mathbb{T}^k$, $e_\alpha(t, s)$ is the solution of the initial value problem

$$x^\Delta(t) = \alpha(t)x(t), \quad t \in \mathbb{T}^k, \quad x(s) = 1.$$

For each fixed s , we say that $\alpha(t)$ is the growth rate of the *exponential function* $e_\alpha(t, s)$

The above defined generalized exponential function $e_\alpha(t, s)$ can be explicitly given by;

Theorem 2.6.2. *The explicit formulation of the exponential function $e_\alpha(t, s)$ is*

$$e_\alpha(t, s) = \exp \left(\int_s^t \xi_{\mu(r)}(\alpha(r)) \Delta r \right) \quad (2.6.3)$$

for $t, s \in \mathbb{T}$, where for $h > 0$, the so-called cylindrical transformation

$\xi_h : \mathbb{C}_h \rightarrow Z_h$ is defined by

$$\xi_h(z) = \frac{1}{h} \text{Log}(1 + zh) \quad (2.6.4)$$

with Log being the principle logarithm function.

For $h = 0$, $\xi_0(z)$ is defined to be equal z for all $z \in \mathbb{C}$.

For further details we refer to [15], [27] and [28].

2.7 Some Useful Time Scale Formulas

The following formulas are useful and will be employed in the following chapters:

- $f^\sigma = f + \mu f^\Delta$;
- $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta$ (“Product Rule”);
- $(f/g)^\Delta = (f^\Delta g - fg^\Delta)/(gg^\sigma)$ (“Quotient Rule”).

We have that (see e.g. [10, Theorem 7])

$$f(t) \geq 0, \quad a \leq t < b \quad \text{implies} \quad \int_a^b f(t) \Delta t \geq 0. \quad (2.1)$$

Throughout this paper we assume that $0 \in \mathbb{T}$ and let $h > 0$ with $h \in \mathbb{T}$. Hence, if $\mathbb{T} = \mathbb{R}$, then $\int_0^h f(t) \Delta t = \int_0^h f(t) dt$, and if $\mathbb{T} = \mathbb{Z}$, then $\int_0^h f(t) \Delta t = \sum_{t=0}^{h-1} f(t)$.

Other examples of time scales (to which our inequalities apply as well) are e.g.

$$h\mathbb{Z} = \{hk : k \in \mathbb{Z}\} \quad \text{for some } h > 0,$$

$$q^{\mathbb{Z}} = \{q^k : k \in \mathbb{Z}\} \cup \{0\} \quad \text{for some } q > 1$$

(which produces so-called *q-difference equations*),

$$\mathbb{N}^2 = \{k^2 : k \in \mathbb{N}\}, \quad \left\{ \sum_{k=1}^n \frac{1}{k} : n \in \mathbb{N} \right\}, \quad \bigcup_{k \in \mathbb{Z}} [2k, 2k + 1],$$

and the Cantor set.

We conclude this section with giving two simple consequences of the product rule as well as Hölder's inequality for time scales. Those results we shall need in Sections 3.2 and 3.3 to come.

First,

$$(f^2)^\Delta = (f \cdot f)^\Delta = f^\Delta f + f^\sigma f^\Delta = (f + f^\sigma) f^\Delta, \quad (2.2)$$

and in general, one can use mathematical induction to prove the formula

$$(f^{l+1})^\Delta = \left\{ \sum_{k=0}^l f^k (f^\sigma)^{l-k} \right\} f^\Delta, \quad l \in \mathbb{N}. \quad (2.3)$$

Finally, Hölder's inequality for time scales (see [19, Lemma 2.2 (iv)]) reads

$$\int_0^h |f(t)g(t)| \Delta t \leq \left\{ \int_0^h |f(t)|^p \Delta t \right\}^{\frac{1}{p}} \left\{ \int_0^h |g(t)|^q \Delta t \right\}^{\frac{1}{q}}, \quad (2.4)$$

where $p > 1$ and $q = p/(p - 1)$.

CHAPTER 3
SOME OPIAL INEQUALITIES AND AN APPLICATION

3.1 Preliminaries

In 1960, the Polish Mathematician Zdzidlaw Opial [48] published an inequality involving integrals of functions and their derivatives;

$$\int_0^h |x(t)x'(t)| dt \leq \frac{h}{4} \int_0^h |x'(t)|^2 dt \quad (3.1.1)$$

where $x \in C^{(1)}[0, h]$, $x(0) = x(h) = 0$ and $x(t) > 0$ in $(0, h)$, and the constant $h/4$ is the best possible.

Inequalities which involve integrals of functions and their derivatives are of great importance in mathematics with applications in the theory of differential equations, approximations and probability. It has been shown that inequalities of the form (3.1.1) can be deduced from those of Wirtinger and Hardy type, but the importance of Opial's result is in the establishment of the best possible constant. A recently published monograph [7] is the first book dedicated to the theory of Opial type inequalities .

The positivity requirement of $x(t)$ in the original proof of Opial was shown to be unnecessary later by Olech in [47] where he proved that the inequality (3.1.1) holds even for functions $x(t)$ which are only absolutely continuous in $[0, h]$. Moreover, Olech's proof is simpler than that of Opial.

Theorem 3.1.1. *(Olech): Let $x(t)$ be absolutely continuous in $[0, h]$ and $x(0) = x(h) = 0$. Then the following inequality holds;*

$$\int_0^h |x(t)x'(t)| dt \leq \frac{h}{4} \int_0^h (x'(t))^2 dt \quad (3.1.2)$$

Works containing discrete analogues of Opial-Type inequalities started in 1967-69 with the papers of Lasota [39], Wong [49], Beesack [13] which provided discrete versions of inequality (3.1.1).

The next result is the discrete analogue of the above theorem.

Theorem 3.1.2. (*Lasota's Inequality*): Let $\{x_i\}_{i=0}^N$ be a sequence of numbers, and $x_0 = x_N = 0$. Then the following inequality holds

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \leq \frac{1}{2} \left[\left[\frac{N+1}{2} \right] \right] \sum_{i=0}^{N-1} |\Delta x_i|^2 \quad (3.1.3)$$

where Δ is the forward difference operator, and $[[\cdot]]$ is the greatest integer function.

Now, if we consider Olech's result under weaker conditions, we get the following estimate with a less sharper bound.

Theorem 3.1.3. Let $x(t)$ be absolutely continuous in $[0, a]$ and $x(0) = 0$. Then the following inequality holds.

$$\int_0^a |x(t)x'(t)| dt \leq \frac{a}{2} \int_0^a (x'(t))^2 dt \quad (3.1.4)$$

In (3.1.4) equality holds if and only if $x(t) = ct$

The following theorem is a non-trivial generalization of Theorem 3.1.3 and is given in Hua [29].

Theorem 3.1.4. (*Hua's generalization*)

Let $x(t)$ be absolutely continuous on $[0, a]$, and $x(0) = 0$. Further let ε be a positive integer. Then the following inequality holds

$$\int_0^a |x^\varepsilon(t)x'(t)| dt \leq \frac{a^\varepsilon}{\varepsilon + 1} \int_0^a |x'(t)|^{\varepsilon+1} dt \quad (3.1.5)$$

with equality being valid in (3.1.5) if and only if $x(t) = ct$.

Finally we give a discrete analogue of Theorem 3.1.4 due to Wong [49].

Theorem 3.1.5. (*Wong's inequality*)

Let $\{x_i\}_{i=0}^\tau$ be a non-decreasing sequence of nonnegative numbers, and $x_0 = 0$. Then for $\varepsilon \geq 1$, the following inequality holds

$$\sum_{i=1}^{\tau} x_i^\varepsilon \nabla x_i \leq \frac{(\tau + 1)^\varepsilon}{\varepsilon + 1} \sum_{i=1}^{\tau} (\nabla x_i)^{\varepsilon+1}. \quad (3.1.6)$$

where ∇ is the backward difference operator i.e., $\nabla x_i = x_i - x_{i-1}$.

Remark 3.1.1: In terms of the forward difference operator, Δx_i , the above Wong's inequality (3.1.6) can be restated as follows;

$\{x_i\}_{i=0}^\tau$ is a non-decreasing sequence of non-negative numbers with $x_0 = 0$, for $\ell \geq 1$, the inequality

$$\sum_{i=0}^{\tau-1} x_{i+1}^\varepsilon \Delta x_i \leq \frac{(\sigma(\tau))^\varepsilon}{\varepsilon + 1} \sum_{i=0}^{\tau-1} (\Delta x_i)^{\varepsilon+1}. \quad (3.1.7)$$

holds where Δ is the forward difference operator.

3.2 Time-Scale Set-up of Basic Opial Type Inequality

For convenience we now recall the following easiest versions of Opial's inequality.

Theorem 3.2.1 (Continuous Opial Inequality, [7, Theorem 1.4.1]). *For absolutely continuous $x : [0, h] \rightarrow \mathbb{R}$ with $x(0) = 0$ we have*

$$\int_0^h |x\dot{x}|(t)dt \leq \frac{h}{2} \int_0^h |\dot{x}|^2(t)dt,$$

with equality when $x(t) = ct$.

Theorem 3.2.2 (Discrete Opial Inequality, [7, Theorem 5.2.2]). *For $x_0 = 0$ and a sequence $\{x_i\}_{0 \leq i \leq h} \subset \mathbb{R}$, we have*

$$\sum_{i=1}^{h-1} |x_i(x_{i+1} - x_i)| \leq \frac{h-1}{2} \sum_{i=0}^{h-1} |x_{i+1} - x_i|^2,$$

with equality when $x_i = ci$.

In [18] a Dynamic Opial inequality is proven that contains both Theorem 3.2.1 and Theorem 3.2.2 as special cases. We now give this simplest version of Opial's inequality on time scales as presented in [18].

Theorem 3.2.3 (Dynamic Opial Inequality). *Let \mathbb{T} be a time scale. For delta differentiable $x : [0, h] \cap \mathbb{T} \rightarrow \mathbb{R}$ with $x(0) = 0$ we have*

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \leq h \int_0^h |x^\Delta|^2(t)\Delta t,$$

with equality when $x(t) = ct$.

Proof. Consider

$$y(t) = \int_0^t |x^\Delta(s)|\Delta s.$$

Then we have $y^\Delta = |x^\Delta|$ and $|x| \leq y$ so that

$$\begin{aligned} \int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t &\leq \int_0^h [(|x| + |x^\sigma|)|x^\Delta|](t)\Delta t \\ &\leq \int_0^h [(y + y^\sigma)|x^\Delta|](t)\Delta t \\ &= \int_0^h [(y + y^\sigma)y^\Delta](t)\Delta t \\ &= \int_0^h (y^2)^\Delta(t)\Delta t \\ &= y^2(h) - y^2(0) \\ &= \left\{ \int_0^h |x^\Delta(t)|\Delta t \right\}^2 \\ &\leq h \int_0^h |x^\Delta|^2(t)\Delta t, \end{aligned}$$

where we have used formulas (2.2) and (2.4) for $p = 2$.

Now, let $\tilde{x}(t) = ct$ for some $c \in \mathbb{R}$. Then $\tilde{x}^\Delta(t) \equiv c$, and it is easy to check that the equation

$$\int_0^h |(\tilde{x} + \tilde{x}^\sigma)\tilde{x}^\Delta|(t)\Delta t = h \int_0^h |\tilde{x}^\Delta|^2(t)\Delta t$$

holds. □

Next a generalization of Theorem 3.2.3 is offered where $x(0)$ does not need to be equal to 0. This result is not found in the book [7] (neither a continuous nor a discrete version of it), but both a weaker version of Theorem 3.2.3 (with $x(0) = 0$) and the subsequent Theorem 3.2.5 (with $x(0) = x(h) = 0$) are corollaries of Theorem 3.2.4, and continuous [7, Theorem 1.3.1] and discrete [7, Theorem 5.2.1, “Lasota’s inequality”] versions of Theorem 3.2.5 can be found in the book by Agarwal and Pang [7].

Theorem 3.2.4. *Let $x : [0, h] \cap \mathbb{T} \rightarrow \mathbb{R}$ be differentiable. Then*

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \leq \alpha \int_0^h |x^\Delta(t)|^2 \Delta t + 2\beta \int_0^h |x^\Delta(t)|\Delta t,$$

where

$$\alpha \in \mathbb{T} \quad \text{with} \quad \text{dist}(h/2, \alpha) = \text{dist}(h/2, \mathbb{T}) \tag{3.1}$$

and $\beta = \max\{|x(0)|, |x(h)|\}$.

Proof. We consider

$$y(t) = \int_0^t |x^\Delta(s)|\Delta s \quad \text{and} \quad z(t) = \int_t^h |x^\Delta(s)|\Delta s.$$

Then $y^\Delta = |x^\Delta|$, $z^\Delta = -|x^\Delta|$,

$$\begin{aligned} |x(t)| &\leq |x(t) - x(0)| + |x(0)| \\ &= \left| \int_0^t x^\Delta(s)\Delta s \right| + |x(0)| \\ &\leq \int_0^t |x^\Delta(s)|\Delta s + |x(0)| \\ &= y(t) + |x(0)|, \end{aligned}$$

and similarly $|x(t)| \leq z(t) + |x(h)|$. Let $u \in [0, h] \cap \mathbb{T}$. Then

$$\begin{aligned} \int_0^u |(x + x^\sigma)x^\Delta|(t)\Delta t &\leq \int_0^u [y(t) + |x(0)| + y^\sigma(t) + |x(0)|]y^\Delta(t)\Delta t \\ &= \int_0^u [(y + y^\sigma)y^\Delta](t)\Delta t + 2|x(0)| \int_0^u y^\Delta(t)\Delta t \\ &= y^2(u) + 2|x(0)|y(u) \\ &\leq u \int_0^u |x^\Delta(t)|^2\Delta t + 2|x(0)| \int_0^u |x^\Delta(t)|\Delta t, \end{aligned}$$

where we have used again (2.2) and (2.4) for $p = 2$. Similarly, one shows

$$\begin{aligned} \int_u^h |(x + x^\sigma)x^\Delta|(t)\Delta t &\leq z^2(u) + 2|x(h)|z(u) \\ &\leq (h - u) \int_u^h |x^\Delta(t)|^2\Delta t + 2|x(h)| \int_u^h |x^\Delta(t)|\Delta t. \end{aligned}$$

By putting $\nu(u) = \max\{u, h - u\}$ and adding the above two inequalities, we find

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \leq \nu(u) \int_0^h |x^\Delta(t)|^2\Delta t + 2\beta \int_0^h |x^\Delta(t)|\Delta t.$$

This is true for any $u \in [0, h] \cap \mathbb{T}$, so it is also true if $\nu(u)$ is replaced by $\min_{u \in [0, h] \cap \mathbb{T}} \nu(u)$.

However, this last quantity is easily seen to be equal to α . \square

Theorem 3.2.5. *Let $x : [0, h] \cap \mathbb{T} \rightarrow \mathbb{R}$ be differentiable with $x(0) = x(h) = 0$. Then*

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \leq \alpha \int_0^h |x^\Delta(t)|^2\Delta t,$$

where α is given in (3.1).

Proof. This follows easily from Theorem 3.2.4 since in this case we have $\beta = 0$. \square

3.3 Upper Bound Estimates of Solutions of Dynamic Initial Value Problems

We now proceed to give an application of Theorem 3.2.3.

Example 3.3.1. Let y be a solution of the initial value problem

$$y^\Delta = 1 - t + \frac{y^2}{t}, \quad 0 < t \leq 1, \quad y(0) = 0. \quad (3.2)$$

Then $y \leq \tilde{y}$ on $[0, 1] \cap \mathbb{T}$, where $\tilde{y}(t) = t$ solves (3.2).

Proof. Clearly \tilde{y} as defined above solves (3.2). We let y be a solution of (3.2) and consider R defined by

$$R(t) = 1 - t + \int_0^t |y^\Delta|^2(s) \Delta s.$$

Let $t \in [0, 1] \cap \mathbb{T}$. Then

$$\begin{aligned} |y^\Delta(t)| &= \left| 1 - t + \frac{y^2(t)}{t} \right| \leq |1 - t| + \frac{1}{t} |y^2(t)| \\ &= 1 - t + \frac{1}{t} \left| \int_0^t (y^2)^\Delta(s) \Delta s \right| \\ &\leq 1 - t + \frac{1}{t} \int_0^t |(y^2)^\Delta(s)| \Delta s \\ &= 1 - t + \frac{1}{t} \int_0^t |(y + y^\sigma)y^\Delta|(s) \Delta s \\ &\leq 1 - t + \frac{1}{t} \int_0^t |y^\Delta|^2(s) \Delta s \\ &= R(t), \end{aligned}$$

where we have used our Opial's inequality, Theorem 3.2.3. Hence

$$R^\Delta(t) = -1 + |y^\Delta(t)|^2 \leq R^2(t) - 1 \quad \text{and} \quad R(0) = 1.$$

Let w be the unique solution of

$$w^\Delta = (1 + R(t))w, \quad w(0) = 1.$$

Note that this w exists since $1 + \mu(R + 1) > 0$ (see e.g. [10, Theorem 8]) because $R > |y^\Delta| \geq 0$; actually $w(t) > 0$ for all $t \in [0, 1] \cap \mathbb{T}$. Hence, because of $(R - 1)^\Delta =$

$R^\Delta \leq R^2 - 1$, we have

$$\begin{aligned} 0 &\geq \frac{(R-1)^\Delta w - (R^2-1)w}{ww^\sigma} \\ &= \frac{(R-1)^\Delta w - (R-1)w^\Delta}{ww^\sigma} \\ &= \left(\frac{R-1}{w}\right)^\Delta \end{aligned}$$

(we used the quotient rule from Section 2.7) so that

$$\left(\frac{R-1}{w}\right)^\Delta(t) = \left(\frac{R-1}{w}\right)^\Delta(0) + \int_0^t \left(\frac{R-1}{w}\right)^\Delta(s) \Delta s \leq 0$$

and hence $R(t) \leq 1$. Therefore

$$y^\Delta(t) \leq |y^\Delta(t)| \leq R(t) \leq 1$$

and $y(t) = \int_0^t y^\Delta(s) \Delta s \leq \int_0^t \Delta s = t$. □

CHAPTER 4

EXTENSIONS OF DYNAMIC OPIAL INEQUALITIES

4.1 Extensions

Since its discovery more than four decades ago, Opial's inequality has found many interesting applications. In fact, Opial's inequality and its several generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. In addition, many qualitative behaviours such as oscillation, non-oscillation, boundedness, have been discussed in the light of Opial's inequality. It may be asserted that this subject will continue to play a very important part in the future of applied mathematics.

In this chapter, we give some of the various generalizations of the inequalities presented in the previous chapter. The continuous and/or discrete versions of those inequalities may be found in [7]. We have not included all of those results in this work, but most of them may be proved using similar techniques as the ones presented here. The results given here are in view of those obtained in [18] by Bohner and Kaymakçalan.

Theorem 4.1.1 (see [7, Theorem 2.5.1]). *Let p, q be positive and continuous on $[0, h]$, $\int_0^h \Delta t/p(t) < \infty$, and q non-increasing. For differentiable $x : [0, h] \cap \mathbb{T} \rightarrow \mathbb{R}$ with $x(0) = 0$ we have*

$$\int_0^h [q^\sigma |(x + x^\sigma)x^\Delta|(t)\Delta t] \leq \left\{ \int_0^h \frac{\Delta t}{p(t)} \right\} \left\{ \int_0^h p(t)q(t)|x^\Delta(t)|^2 \Delta t \right\}.$$

Proof. We consider

$$y(t) = \int_0^t \sqrt{q^\sigma(s)} |x^\Delta(s)| \Delta s.$$

Then $y^\Delta = \sqrt{q^\sigma}|x^\Delta|$ and (note that $0 \leq s < t$ implies $\sigma(s) \leq t$ and hence $q(\sigma(s)) \geq q(t)$; apply (2.1))

$$|x(t)| \leq \int_0^t |x^\Delta(s)| \Delta s \leq \int_0^t \sqrt{\frac{q^\sigma(s)}{q(t)}} |x^\Delta(s)| \Delta s = \frac{y(t)}{\sqrt{q(t)}}$$

so that (note that $t \leq \sigma(t)$ implies $q(t) \geq q(\sigma(t))$)

$$\begin{aligned} \int_0^h [q^\sigma|(x + x^\sigma)x^\Delta|](t) \Delta t &\leq \int_0^h q^\sigma(t) \left(\frac{y(t)}{\sqrt{q(t)}} + \frac{y^\sigma(t)}{\sqrt{q^\sigma(t)}} \right) \frac{y^\Delta(t)}{\sqrt{q^\sigma(t)}} \Delta t \\ &\leq \int_0^h (y(t) + y^\sigma(t)) y^\Delta(t) \Delta t \\ &= y^2(h) \\ &= \left\{ \int_0^h \frac{1}{\sqrt{p(s)}} \sqrt{p(s)q^\sigma(s)} |x^\Delta(s)| \Delta s \right\}^2 \\ &\leq \left\{ \int_0^h \frac{\Delta s}{p(s)} \right\} \left\{ \int_0^h p(s)q^\sigma(s) |x^\Delta(s)|^2 \Delta s \right\}. \end{aligned}$$

Again we have used (2.2) and (2.4) for $p = 2$. □

The following result involves higher order derivatives. As usual, we write f^{Δ^n} for the n th (delta) derivative of f .

Theorem 4.1.2 (see [7, Chapter 3]). *Suppose $l, n \in \mathbb{N}$. For n -times differentiable $x : [0, h] \cap \mathbb{T} \rightarrow \mathbb{R}$ with $x(0) = x^\Delta(0) = \dots = x^{\Delta^{n-1}}(0) = 0$ we have*

$$\int_0^h \left| \left\{ \sum_{k=0}^l x^k (x^\sigma)^{l-k} \right\} x^{\Delta^n} \right| (t) \Delta t \leq h^{ln} \int_0^h |x^{\Delta^n}(t)|^{l+1} \Delta t.$$

Proof. We consider

$$y(t) = \int_0^t \int_0^{\tau_{n-1}} \dots \int_0^{\tau_2} \left\{ \int_0^{\tau_1} |x^{\Delta^n}(s)| \Delta s \right\} \Delta \tau_1 \Delta \tau_2 \dots \Delta \tau_{n-1}.$$

Hence we have

$$\begin{aligned} y^\Delta(t) &= \int_0^t \int_0^{\tau_{n-2}} \dots \int_0^{\tau_2} \left\{ \int_0^{\tau_1} |x^{\Delta^n}(s)| \Delta s \right\} \Delta \tau_1 \Delta \tau_2 \dots \Delta \tau_{n-2}, \\ \dots, \quad y^{\Delta^{n-1}}(t) &= \int_0^t |x^{\Delta^n}(s)| \Delta s, \quad y^{\Delta^n}(t) = |x^{\Delta^n}(t)|, \end{aligned}$$

and for $0 \leq t \leq h$

$$\begin{aligned}
|x(t)| &\leq \int_0^t |x^\Delta(t_1)| \Delta t_1 \leq \int_0^t \int_0^{t_1} |x^{\Delta\Delta}(t_2)| \Delta t_2 \Delta t_1 \leq \dots \leq y(t) \\
&= \int_0^t y^\Delta(s) \Delta s \leq \int_0^t y^\Delta(t) \Delta s \leq \int_0^h y^\Delta(t) \Delta s = h y^\Delta(t) \\
&\leq h^2 y^{\Delta\Delta}(t) \leq \dots \leq h^{n-1} y^{\Delta^{n-1}}(t) = h^{n-1} f(t),
\end{aligned}$$

where we put $f = y^{\Delta^{n-1}}$. Therefore

$$\begin{aligned}
&\int_0^h \left| \left\{ \sum_{k=0}^l x^k (x^\sigma)^{l-k} \right\} x^{\Delta^n} \right| (t) \Delta t \leq \int_0^h \left\{ \sum_{k=0}^l |x|^k |x^\sigma|^{l-k} |x^{\Delta^n}| \right\} (t) \Delta t \\
&\leq \int_0^h \left\{ \sum_{k=0}^l (h^{n-1} f)^k (h^{n-1} f^\sigma)^{l-k} |x^{\Delta^n}| \right\} (t) \Delta t \\
&= h^{(n-1)l} \int_0^h \left\{ \sum_{k=0}^l f^k (f^\sigma)^{l-k} f^\Delta \right\} (t) \Delta t \\
&= h^{(n-1)l} \int_0^h (f^{l+1})^\Delta(t) \Delta t \\
&= h^{(n-1)l} f^{l+1}(h) \\
&= h^{(n-1)l} \left\{ \int_0^h |x^{\Delta^n}(t)| \Delta t \right\}^{l+1} \\
&\leq h^{(n-1)l} h^l \int_0^h |x^{\Delta^n}(t)|^{l+1} \Delta t \\
&= h^{nl} \int_0^h |x^{\Delta^n}(t)|^{l+1} \Delta t,
\end{aligned}$$

where we have used (2.1), (2.3), and (2.4) with $p = (l+1)/l$ and $q = l+1$. \square

The following two results are easy corollaries of the above Theorem 4.1.2.

Corollary 4.1.1 (see [7, Theorem 3.2.1]). *Suppose $n \in \mathbb{N}$. For n -times differentiable*

$x : [0, h] \cap \mathbb{T} \rightarrow \mathbb{R}$ with $x(0) = x^\Delta(0) = \dots = x^{\Delta^{n-1}}(0) = 0$ we have

$$\int_0^h |(x + x^\sigma)x^{\Delta^n}|(t) \Delta t \leq h^n \int_0^h |x^{\Delta^n}(t)|^2 \Delta t.$$

Proof. This is Theorem 4.1.2 with $l = 1$. □

Corollary 4.1.2 (see [7, Theorem 2.3.1]). *Suppose $l \in \mathbb{N}$. For differentiable $x : [0, h] \cap \mathbb{T} \rightarrow \mathbb{R}$ with $x(0) = 0$ we have*

$$\int_0^h \left| \left\{ \sum_{k=0}^l x^k (x^\sigma)^{l-k} \right\} x^\Delta \right| (t) \Delta t \leq h^l \int_0^h |x^\Delta(t)|^{l+1} \Delta t.$$

Proof. This is Theorem 4.1.2 with $n = 1$. □

BIBLIOGRAPHY

- [1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, Inc. (New York) (1992).
- [2] R. P. Agarwal, *Difference Calculus with Applications to Difference Equations*, Int. Ser. Num. Math., 71, 95-110, (1984).
- [3] R. P. Agarwal, *Properties of Solutions of Higher Order Nonlinear Difference Equations II*, An. Sti. Univ. "Al. I. Cuza" Iasi, 85-96, (1985).
- [4] R.P. Agarwal and M. Bohner, *Basic Calculus on Time Scales and Some of its Applications*, Results of Mathematics, **35**(1-2), (1999), 3-22.
- [5] R.P. Agarwal and M. Bohner, *Inequalities on Time Scales: A survey*, Math. Inequal. Appl., **4**(4) (2001), 535-557.
- [6] R.P. Agarwal and R.C. Gupta, *Essentials of Ordinary Differential Equations*, McGraw-Hill (Singapore, New York) (1991).
- [7] R.P. Agarwal and P.Y.H. Pang, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer (Dordrecht, Netherlands) (1995).
- [8] R.P. Agarwal and P.Y.H. Pang, *Opial-type Inequalities Involving Higher Order Derivatives*, J. Math. Anal. Appl. , **189**(1), (1995), 85-103.
- [9] B. Aulbach, *Analysis auf Zeitmengen*, Lecture Notes, Universitat Augsburg, (1990).
- [10] B. Aulbach and S. Hilger, *Linear Dynamic Processes with Inhomogeneous Time Scales*, Nonlinear Dynamics and Quantum Dynamical Systems, Akademie-Verlag (Berlin) (1990).
- [11] B. Aulbach and S. Hilger, *A Unified Approach to Continuous and Discrete Dynamics*, In *Differential Equations: Qualitative Theory*, 37-56, Colloq. Math. Soc. Janos Bolyai, North-Holland, (1988).

- [12] D. Bainov and P. Simeonov, *Integral Inequalities and Applications*, Kluwer (Dordrecht) (1992).
- [13] R.P. Beesack, *On an Integral Inequality of Z. Opial*, Trans. Amer. Math. Soc., **104**, (1962), 470-475.
- [14] R.P. Beesack, *Elementary Proofs of Opial-Type Integral inequalities*, J. d'Analyse Mathématique, **36**, (1979), 1-14.
- [15] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser (Boston) (2001).
- [16] M. Bohner and A. Peterson, (eds.), *Advances in Dynamic Equations on Time Scales*, Birkhäuser (Boston) (2003).
- [17] M. Bohner, S. Clark, and J. Ridenhour, *Lyapunov inequalities on time scales*, J. Inequal. Appl., **7**(1), (2002), 61-77.
- [18] M. Bohner and B. Kaymakçalan, *Opial Inequalities on Time Scales*, Ann. Polon. Math., **77**(1), (2001), 11-20.
- [19] M. Bohner and D.A. Lutz, *Asymptotic Behavior of Dynamic Equations on Time Scales*, J. Differ. Equations Appl., **7**(1), (2001), 21-50. Special issue in memory of W. A. Harris, Jr.
- [20] C. Corduneanu, *Principles of Differential and Integral Equations*, Allyn and Bacon, Inc. (Boston) (1971).
- [21] J. Dieudonné, *Foundations of Modern Analysis*, Prentice Hall (New Jersey) (1966).
- [22] L. Erbe and S. Hilger, *Sturmian Theory on Measure Chains*, Differential Equations Dynam. Systems, **1**(3), (1993), 223-246.
- [23] L. Erbe and A. Peterson, *Green's Function and Comparison Theorems for Differential Equations on Measure Chains*, Dynam. Contin. Discrete Impuls. Systems, **6**(1), (1999), 121-137.
- [24] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press (Cambridge) (1952).

- [25] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, PhD thesis, Universität Würzburg, (1988).
- [26] S. Hilger, *Analysis on Measure Chains-a Unified Approach to Continuous and Discrete Calculus*, Res. Math., **18**, (1990), 18-56.
- [27] S. Hilger, *Differential and Difference Calculus*, Nonlinear Anal. Proceedings of Second World Congress of Nonlinear Analysts, **30**(5), (1997), 2683-2694.
- [28] S. Hilger, *Special Functions, Laplace and Fourier Transform on Measure Chain*, Dynam. Systems Appl., **8**, (1999), 471-488.
- [29] L.K. Hua, *On an Inequality of Opial*, Scientia Sinica, **14**, (1965), 789-790.
- [30] B. Kaymakçalan, *Existence and Comparison Results for Dynamic Systems on Time Scale*, J. Math. Anal. Appl., **172**, (1993), 243-255.
- [31] B. Kaymakçalan and E. Koralp, *Opial Type Inequalities on Time Scales and Some Applications* Proceedings of the International Conference on Dynamic Systems and Applications, III, Atlanta-U.S.A., May, 1999, **3**, Dynamic Publishers, (Atlanta), (2001), 335-342.
- [32] B. Kaymakçalan and V. Lakshmikantham, *Monotone Flows and Fixed Points for Dynamic Systems on Time Scales*, Comput. Math. Model., **28**, (1994), 185-189.
- [33] B. Kaymakçalan, V. Lakshmikantham, and S. Sivasundaram, *Dynamic Systems on Measure Chains*, Mathematics and its Applications **370**, Kluwer Academic Publishers (Dordrecht) (1996).
- [34] B. Kaymakçalan and S. Leela, *A Survey of Dynamic Systems on Time Scales*, Nonlinear Times Digest, **1**, (1994), 37-60.
- [35] B. Kaymakçalan, S. A. Özgün and A. Zafer, *Gronwall-Bihari type Inequalities on Time Scales*, Advances in Difference Equations, Proceeding of the Second International Conference on Difference Equations and Applications, Veszprem-Hungary, (1997), 481-490.
- [36] B. Kaymakçalan, S.A. Özgün and A. Zafer, *Asymptotic Behavior of Higher Order Equations on Time Scale*, Comput. Math. Model., Special Issue on Difference Equations, **36**(10-12), (1998), 299-306.

- [37] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Vol. I, Academic Press (New York) (1969).
- [38] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press (New York) (1988).
- [39] A. Lasota, *A Discrete Boundary Value Problem*, Ann. Polon. Math., **20**, (1968), 183-190.
- [40] C.M. Lee, *On a Discrete Analogue of Inequalities of Opial and Yang*, Canad. Math. Bull., **11**, (1968), 73-77.
- [41] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer (Dordrecht) (1991).
- [42] B.G. Patchpatte, *Finite-difference Inequalities and an Extension of Lyapunov's Method*, Michigan Math. J., **18**, (1971), 385-391.
- [43] B.G. Patchpatte, *Inequalities for Differential and Integral Equations*, **197** Mathematics in Science and Engineering, Academic Press Inc. (San Diego) (1998).
- [44] B.G. Patchpatte, *Inequalities for Finite Difference Equations*, **247** Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc. (New York) (2002).
- [45] R.N. Pederson, *On an Inequality of Opial, Beesack and Levinson*, Proc. Amer. Math. Soc. **16**, (1965), 174.
- [46] B. Stachurska, *On a Nonlinear Integral Inequality*, Zeszyty Nauk. Uniw. Jagiellonsk., 252, Prace Math., **15**, (1971), 151-157.
- [47] C. Olech, *A Simple Proof of a Certain Result of Z. Opial*, Ann. Polon. Math., **8**, (1960), 61-63.
- [48] Z. Opial, *Sur une inégalité*, Ann. Polon. Math. **8**, (1960), 29-32.
- [49] J.S.W. Wong, *A Discrete Analogue of Opial's Inequality*, Canad. Math. Bull. **10**, (1967), 115-118.

- [50] G.S. Yang, *On a Certain Result of Z. Opial*, Proc. Japan Acad., **42**, (1966), 78-83.