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Suboptimal variable structure control of uncertain nonlinear slowly varying systems

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ABSTRACT
In this paper, a new robust suboptimal controller is designed to stabilize a class of uncertain nonlinear time-varying systems with slowly varying parameters. In the design procedure of the proposed controller, first a suboptimal control law is designed for the nominal system based on considering a given cost function and an appropriate Slowly Varying Control Lyapunov Function (SVCLF). After that, a robustifying term is added to the nominal controller in order to vanish the effects of model uncertainties and/or external disturbances in a finite time. For this purpose, a special sliding surface, which is a combination of terminal and integral sliding surfaces, is used. This surface has the advantages of both of terminal and integral surfaces. Due to the structure of this surface, the actual trajectories track the desired one in the finite time. The other innovation of the proposed approach is accessing a chattering-free Controller. Finally, in order to confirm the applicability of the proposed controller and verify the theoretical results, it is applied on a practical benchmark system (a time-varying inertia pendulum). Computer simulations show the efficiency of the proposed controller.

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1. Introduction

The time-varying systems are an important class of dynamical systems, which have various applications in physics and control engineering. Since the time-varying systems include time-varying parameters, the control design methods reported in the literature for these systems are more complicated in comparison with the time-invariant ones.

One of the major categories of the time-varying systems is systems with slowly varying parameters (called slowly varying systems) (Khalil, 2002). The slowly varying systems have various applications such as air vehicles cranes, underwater vehicles, satellites and so on (Binazadeh & Shafiei, 2013a; Feintuch, 2012; Liu, Haraguchi, & Hu, 2009; Shafiei & Yazdanpanah, 2010). Using the control algorithms of time-invariant systems to control the slowly varying systems may cause instability and bad performance (Feintuch, 2012). On the other hand, the given methods in the literature for the stabilization of the time-varying systems lead to complicated and conservative control laws for slowly varying systems (Feintuch, 2012).

The controller design for linear slowly varying systems has been presented in many articles (Amato, Celentano, & Garofalo, 1993; Desoer, 1969; Feintuch, 2012; Fung, Chen, & Grimble, 1984; Ilchmann, Owens, & Prätzel-Wolters, 1987; Liu et al., 2009; Rosenbrock, 1963; Zhang, 1993). However, very few articles have been studied the problem of control design for the nonlinear slowly varying systems (Binazadeh & Shafiei, 2013b, 2014; Shafiei & Yazdanpanah, 2010).

Control Lyapunov Function (CLF) is one of the stabilizing control methods for nonlinear time-invariant systems. Shafiei and Yazdanpanah (2010) developed a CLF-based control technique called SVCLF controller for the nonlinear slowly varying systems. After that, Binazadeh and Shafiei (2013b) extended the SVCLF technique for systems with a vector of the slowly varying parameter. Moreover, they showed that the resulted control law is a suboptimal solution of Hamilton–Jacobi–Bellman (HJB) equation.

Despite the advantages of the SVCLF method, it has not considered the external disturbances and/or model uncertainties which are the integral parts in modelling of the practical systems. Therefore, the design of the robust controller is necessary to guarantee the asymptotical stability of the closed-loop slowly varying systems in the presence of model uncertainties and/or external disturbances. Binazadeh and Shafiei (2014) proposed a robust stabilizing controller for uncertain nonlinear slowly vary-

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ing systems by adding an additional term to the SVCLF controller. This additional term was designed based on the Lyapunov redesign method.

Variable structure controllers (like sliding mode controllers) are powerful robust control methods which have effective performance in practical engineering problems (Hu, Wang, Gao, & Stergioulas, 2012; Hu, Wang, & Gao, 2011). The basis of this control method is the design of appropriate sliding surfaces (to achieve the control objectives) and switching control laws so that the system trajectories reach to the desired sliding surface in finite time and stay on it for all future times (Boum, Djidjio Keubeng, & Bitjoka, 2017; Chenarani & Binazadeh, 2017; Su, Liu, Shi, & Song, 2018). Eksin, Tokat, Güzelkaya, and Söylemez (2003) proposed a time-varying sliding surfaces in order to minimize the settling time for second-order systems. Optimal linear sliding surfaces for underactuated nonlinear systems were studied by Nikkhah, Ashrafiuon, and Muske (2006). Designing a sliding surface which results in the optimality of the closed-loop system was investigated by Janardhanan and Kariwala (2008). The optimal sliding mode with fuzzy approach was studied by Li, Wang, Wu, Lam, and Gao (2018). This technique was also studied for the control of nonlinear vehicle active suspension system (Chen, Wang, Yao, & Kim, 2017) and spacecraft position and attitude manoeuvres (Pukdeboon & Kumam, 2015).

In this paper, a variable structure controller is designed for a class of uncertain nonlinear slowly varying systems with a new approach. The main advantage of the proposed robust method is in the elimination of the effects of the model uncertainties and/or external disturbances, in a finite time. This control law is designed in two phases. First, the nominal part of the proposed controller is designed for the nominal system (by ignoring the model uncertainties and/or external disturbances) based on the SVCLF method with an optimal approach related to a given cost function. Then, the robustifying term, which is the main contribution of this paper, is added to the nominal controller to overcome the model uncertainties and/or external disturbances. The additional control term is a variable structure controller with a special sliding surface. This surface is created by the combination of two surfaces included terminal and integral ones. Due to the integral structure of this surface, the actual trajectories track the desired one in the finite time. The other innovation is accessing a specific type of chattering-free second-order sliding mode controller by the combination of the integral and terminal structures in the sliding surface. In this regard, a theorem is given which guarantees vanishing the effect of model uncertainties and/or external disturbances in a finite time and also the asymptotic stability of the uncertain closed-loop system. Finally, this proposed controller is applied to the time-varying inertia pendulum system and its efficiency is affirmed by simulation results.

2. Problem definition

Consider the following time-varying system:

$$\dot{x} = f(x, \theta(t)) + h(x, \theta(t))(u + d(x, t)), \tag{1}$$

where $x \in R^n$ is the state vector, $u \in R$ is the control input, $\theta(t) \in \Omega \subset R^p$ is the vector of slowly varying parameters and the unknown nonlinear function $d(x, t)$ is due to the external disturbances and/or model uncertainties. Moreover, $f(0, \theta(t)) = 0, \ \ \forall t \geq 0$. The task is to design a robust control law as follows:

$$u(t) = u_1(t) + u_2(t), \tag{2}$$

where $u_1(t)$ is the nominal term and $u_2(t)$ is the robustifying term of the control law. The nominal model of the system (1) can be described as follows:

$$\dot{x} = f(x, \theta(t)) + h(x, \theta(t))u_1(t). \tag{3}$$

Furthermore, the cost function J is chosen as follows:

$$J = \int_0^\infty (l(x(\tau), \theta(\tau)) + u_1(\tau)^2)\mathrm{d}\tau, \tag{4}$$

where $l(x(t), \theta(t))$ is a positive-definite function which has influences on the transient responses of state variables and may be chosen by the designer after some trial and errors.

According to Binazadeh and Shafiei (2013b), if the changes of $\theta(t)$ be slow enough such that $||\dot{\theta}(t)||$ satisfies the following conditions (where $\alpha \in (0, 1)$ is selected such that the infimum is made positive)

$$\sup_t ||\dot{\theta}(t)|| \leq \inf_{\theta \in \Omega, x \in D} \frac{\sqrt{a^2(x, \theta(t)) + l(x, \theta(t))b^2(x, \theta(t))} - \alpha\gamma(\|x\|)}{\left\| \frac{\partial V}{\partial \theta} \right\|} \tag{5}$$

then the following suboptimal controller guarantees the asymptotic stability of the closed-loop system (3).

$$u_1(t) = k(x, \theta(t)) = \begin{cases} \dfrac{-b(x, \theta(t))}{\frac{a(x, \theta(t)) + \sqrt{a^2(x, \theta(t)) + l(x, \theta(t))b^2(x, \theta(t))}}{b^2(x, \theta(t))}} & \text{where} \quad b \neq 0 \\ 0 & \text{where} \quad b = 0 \end{cases} \tag{6}$$

where

$$a(x, \theta(t)) = (\partial V(x, \theta(t))/\partial x)f(x, \theta(t)),$$

$$b(x, \theta(t)) = (\partial V(x, \theta(t))/\partial x)h(x, \theta(t)).$$

Moreover, $l(x, \theta(t))$ is given in the cost function (4) and effects on the characteristics of the transient responses,

$\gamma(\|x\|)$ is a class K function and $V(x,\theta(t)) : D \times \Omega \to R$ is an SVCLF that satisfies the following conditions:

$$\alpha_1(\|x\|) \le V(x,\theta(t)) \le \alpha_2(\|x\|), \forall(x,\theta(t)) \in D \times \Omega \quad (7a)$$

$$a(x,\theta(t)) \le -\gamma(\|x\|), \forall(x,\theta(t)) \in D \times \Omega$$
$$\text{where} \quad b(x,\theta(t)) = 0 \quad (7b)$$

$$\left\|\frac{\partial V}{\partial \theta}\right\| < \infty \quad (x,\theta(t)) \in D \times \Omega \quad (7c)$$

where $\alpha_1(.)$ and $\alpha_2(.)$ are class K functions. The closed-loop system with the controller (6) is as follows:

$$\dot{x} = f(x,\theta(t)) + h(x,\theta(t))k(x,\theta(t)). \quad (8)$$

The time derivative of$V(x,\theta(t))$ along the system's trajectories of the above closed-loop system with considering the controller (6) is

$$\begin{aligned}\dot{V}(x,\theta(t)) &= \frac{\partial V}{\partial \theta}\dot{\theta}(t) + \frac{\partial V}{\partial x}(f + hk)\\ &= \frac{\partial V}{\partial \theta}\dot{\theta}(t) + a + bk\\ &= \begin{cases} -\sqrt{a^2 + lb^2} \\ +\frac{\partial V}{\partial \theta}\dot{\theta}(t), \quad \text{where } b(x,\theta(t)) \ne 0, \\ a + \frac{\partial V}{\partial \theta}\dot{\theta}(t), \quad \text{where } b(x,\theta(t)) = 0. \end{cases}\end{aligned} \quad (9)$$

In the region where $b(x,\theta(t)) \ne 0$, according to (9) and considering (5), one has

$$\begin{aligned}\dot{V}(x,\theta(t)) &= -\sqrt{a^2 + lb^2} + \frac{\partial V}{\partial \theta}\dot{\theta}\\ &\le -\sqrt{a^2 + lb^2} + \left\|\frac{\partial V}{\partial \theta}\right\| ||\dot{\theta}||\\ &\le -\sqrt{a^2 + lb^2} + \left\|\frac{\partial V}{\partial \theta}\right\| \frac{\sqrt{a^2 + lb^2} - \alpha\gamma(\|x\|)}{\left\|\frac{\partial V}{\partial \theta}\right\|}\\ &= -\alpha\gamma(\|x\|).\end{aligned}$$

In the region where $b(x,\theta(t)) = 0$, according to (9), one has

$$\begin{aligned}\dot{V}(x,\theta(t)) &= a(x,\theta) + \frac{\partial V}{\partial \theta}\dot{\theta}\\ &\le a(x,\theta) + \left\|\frac{\partial V}{\partial \theta}\right\| ||\dot{\theta}||.\end{aligned}$$

Since, where $b(x,\theta(t)) = 0$, then $a(x,\theta(t))$ is negative (refer to (7b)), thus $\sqrt{a^2 + lb^2}\Big|_{b=0}$ is equal to $-a$. Thus considering the relation (5) in this case one has

$$\begin{aligned}\dot{V}(x,\theta(t)) &\le a(x,\theta(t)) + \left\|\frac{\partial V}{\partial \theta}\right\| \frac{-a(x,\theta(t)) - \alpha\gamma(\|x\|)}{\left\|\frac{\partial V}{\partial \theta}\right\|}\\ &\le -\alpha\gamma(\|x\|).\end{aligned}$$

Thus $\dot{V}(x,\theta(t)) \le -\alpha\gamma(\|x\|)$ in both cases (i.e. for $b = 0$ and $b \ne 0$) and therefore, the closed-loop system (8) is asymptotically stable.

The nominal control law (6) is a suboptimal controller with respect to the cost function (4) and the following related Hamilton–Jacobi–Bellman equation:

$$\begin{aligned}&V_t^* + l(x,\theta(t)) + V_x^* f(x,\theta(t))\\ &\quad - \frac{1}{4}V_x^* h(x,\theta(t))h(x,\theta(t))^{\mathrm{T}} V_x^{*\mathrm{T}} = 0,\end{aligned}$$

where

$$\begin{aligned}&V^* = \min_{u(.)} \int_0^\infty (l(x(\tau),\theta(\tau)) + u_1^2(\tau))\mathrm{d}\tau\\ &s.t.\, \dot{x} = f(x,\theta(t)) + h(x,\theta(t))u(t)\\ &x(0) = x_0.\end{aligned}$$

Remark 1: It is worth noting that $\theta(t)$ is belonging to the known region for the considered problem and its *sup* estimation is related to physical aspects of the problem. Indeed, what is evaluated in (5) is the admissible upper bound of *sup* $||\dot{\theta}(t)||$ that depends on the selected SVCLF. If one can find the SVCLF that is independent of $\theta(t)$, there is not any limiting bound on $||\dot{\theta}(t)||$ (Binazadeh & Shafiei, 2013b).

Remark 2: In order to decrease the value of the cost function as far as possible, it is better to put some free parameters in the chosen SVCLF. Then, some acceptable ranges are obtained for these free parameters due to satisfying the conditions (7.a), (7.b) and (7.c). Finally, using a computer program, the best parameters that lead to the less cost function are obtained. In this way, the nominal control law (6) is a very good suboptimal controller and the nominal system response is very close to an optimal solution which is resulted from the related time-varying HJB equation (Binazadeh & Shafiei, 2015).

3. Design of the robustifying term of the control law

In this section, the additional state feedback term (i.e. $u_2(t)$) will be designed such that the whole controller (i.e. $u(t) = u_1(t) + u_2(t)$) guarantees the asymptotic stability of the uncertain closed-loop slowly varying system (1) and leads to vanishing the effect of model uncertainties and/or external disturbances in a finite time. Consider the following integral sliding surface:

$$s(t) = E\left[x(t) - \underbrace{\int_0^t (f + hu_1)\,\mathrm{d}t}_{x_{\mathrm{nom}}(t)}\right], \tag{10}$$

where $E = \begin{bmatrix} E_1 & \dots & E_n \end{bmatrix}$ is a row vector of the design parameter which is chosen such that $Eh(x, \theta(t)) \neq 0$. Moreover, $x_{\mathrm{nom}}(t)$ is the state vector which is resulted from the nominal closed-loop system (3) with the nominal controller (6) and $x(t)$ is the actual one related to system (1). The difference between $x(t)$ and $x_{\mathrm{nom}}(t)$ occurs due to external disturbances and/or model uncertainties. The main characteristics of the surface (10) are that sliding motion on this surface (i.e. $s(t) = 0$) leads to the nullification of the difference between the actual states of the uncertain system and the desired states of the nominal closed-loop system. This results in performance recovery in the uncertain system and forces the uncertain system to follow the nominal closed-loop system. Using an appropriate control action, this difference can be nullified and the actual trajectory tracks the required one (Castaños & Fridman, 2006).

The time derivative of $s(t)$ is as follows:

$$\dot{s}(t) = E[\dot{x}(t) - \dot{x}_{\mathrm{nom}}(t)]. \tag{11}$$

Using Equations (1), (2) and (3), the following relations are obtained as

$$\begin{aligned} \dot{s}(t) &= E[f(x, \theta(t)) + h(x, \theta(t))(u_1(t) + u_2(t)) \\ &\quad + h(x, \theta(t))d(x, t) - f(x, \theta(t)) - h(x, \theta(t))u_1(t)] \\ &= E[h(x, \theta(t))u_2(t) + h(x, \theta(t))d(x, t)]. \end{aligned} \tag{12}$$

In the conventional sliding mode, the switching control law $u_2(t)$ is designed such that the reaching law (i.e. $s(t)\dot{s}(t) < -\rho|s(t)|$ where $\rho > 0$) is satisfied where its satisfaction guarantees that the system trajectories reach to the surface in a finite time and stay on it for all future times (Khalil, 2002). This law results in appearing the sign function in the control law and therefore leads to chattering phenomena.

Now consider the following sliding surface:

$$\sigma(t) = s(t) + \psi \dot{s}(t)^{n/m}, \tag{13}$$

where $\psi > 0$ is the switching gain and the odd numbers n and m are selected such that they satisfy the following condition:

$$1 < \frac{n}{m} < 2. \tag{14}$$

The advantages of choosing a fractional order sliding surface (13) are the combination of the structure of terminal and integral surfaces which leads to accessing a specific type of chattering-free second-order sliding mode controller which nullify the difference of $x(t)$ and $x_{\mathrm{nom}}(t)$in a finite time. More discussion in this regard is given in the proof of the following theorem.

Theorem 1: *Consider the uncertain nonlinear slowly varying system* (1). *The control law* $u(t) = u_1(t) + u_2(t)$ (*where* $u_1(t)$ *and* $u_2(t)$ *are proposed in* (6) *and* (15)) *guarantees vanishing the effect of model uncertainties and/or external disturbances in a finite time and also the asymptotic stability of the closed-loop system* (1) *in the presence of model uncertainties and/or external disturbances.*

$$u_2(t) = -\int_0^t \frac{1}{Eh}\left[\frac{m}{\psi n}\dot{s}(t)^{2-(n/m)} + E\dot{h}u_2 + \beta(x)\mathrm{sgn}(\sigma)\right]\mathrm{d}\tau, \tag{15}$$

where $\beta(x)$ is the design parameter function which is defined in the proof.

Proof: Considering (13), the time derivative of $\dot{\sigma}$ is as follows:

$$\begin{aligned} \dot{\sigma} &= \dot{s}(t) + \psi\frac{n}{m}(\dot{s}(t))^{(n/m)-1}\ddot{s}(t) \\ &= \psi\frac{n}{m}(\dot{s}(t))^{(n/m)-1}\left(\frac{m}{\psi n}(\dot{s}(t))^{2-(n/m)} + \ddot{s}(t)\right). \end{aligned} \tag{16}$$

Since, the design parameters n and m satisfy condition (14), it can be shown if $\dot{s}(t) \neq 0$, then $\dot{s}(t)^{(n/m)-1} > 0$ (it is because that $n - m$ is an even integer number). Consequently, the term $\psi\frac{n}{m}(\dot{s}(t))^{(n/m)-1}$ of Equation (16) can be substituted with the positive function $\eta(t) > 0$ and (16) can be written as follows:

$$\dot{\sigma} = \eta(t)\left(\frac{m}{\psi n}(\dot{s}(t))^{2-(n/m)} + \ddot{s}(t)\right). \tag{17}$$

Let define:

$$\bar{\eta} = \inf_{t \geq 0}\left(\psi\frac{n}{m}\dot{s}(t)^{(n/m)-1}\right). \tag{18}$$

Moreover, according to (12), one has

$$\ddot{s}(t) = E[h\dot{u}_2(t) + \dot{h}u_2(t) + h\dot{d} + \dot{h}d]. \tag{19}$$

Therefore, $\dot{\sigma}$ is

$$\dot{\sigma} = \eta(t)\left(\frac{m}{\psi n}\dot{s}(t)^{2-(n/m)} + E[h\dot{u}_2(t) + \dot{h}u_2(t) + h\dot{d} + \dot{h}d]\right). \tag{20}$$

Suppose

$$h\dot{u}_2(t) = -\frac{1}{E}\left[\frac{m}{\psi n}\dot{s}(t)^{2-(n/m)} + E\dot{h}(x, \theta(t))u_2(t) - \upsilon(t)\right], \tag{21}$$

where $\upsilon(t)$ is an additional discontinuous component which is added to cancel the effects of unknown terms.

Therefore, $u_2(t)$ is as follows:

$$u_2(t) = -\int_0^t \frac{1}{E h(x,\theta(t))} \left[\frac{m}{\psi n}\dot{s}(t)^{2-(n/m)} + E\dot{h}(x,\theta(t))u_2(t) - \upsilon(t)\right] d\tau. \quad (22)$$

Substituting (21) into (20) yields

$$\dot{\sigma} = \eta(t)(\upsilon(t) + E(\dot{h}d + h\dot{d})). \quad (23)$$

Now, consider the upper bound of the following unknown term is known and

$$|E(\dot{h}d + h\dot{d})| \leq \varphi(x), \quad (24)$$

where $\varphi(x) \geq 0$ is a known positive continuous function. Therefore, considering (23), the additional term $\upsilon(t)$ can be designed to force $\sigma(0)$ toward the surface$\sigma(t) = 0$. For this purpose, utilize $V(t) = 0.5\sigma^2(t)$ as a Lyapunov function candidate for (23), thus

$$\begin{aligned}\dot{V}(t) &= \sigma(t)\dot{\sigma}(t) \\ &= \sigma(t)\eta(t)\upsilon(t) + \sigma(t)\eta(t)(E(\dot{h}d + h\dot{d})) \\ &\leq \sigma(t)\eta(t)\upsilon(t) + |\sigma|\eta(t)\varphi(x).\end{aligned} \quad (25)$$

Take

$$\upsilon(t) = -\beta(x(t))\mathrm{sgn}(\sigma(t)), \quad (26)$$

where $\beta(x) \geq \varphi(x) + \beta_0$ and $\beta_0 > 0$. Then

$$\begin{aligned}\dot{V}(t) &= \sigma(t)\dot{\sigma}(t) \\ &\leq (-\beta(x) + \varphi(x))\eta(t)|\sigma(t)| \\ &\leq (-\beta_0 - \varphi(x) + \varphi(x))\eta(t)|\sigma(t)| \\ &\leq -\beta_0\eta(t)|\sigma(t)| \\ &\leq -\beta_0\bar{\eta}|\sigma(t)|.\end{aligned} \quad (27)$$

For $\sigma(0) > 0$, the inequality (27) leads to $\dot{\sigma}(t) < -\beta_0\bar{\eta}$ and for $\sigma(0) < 0$ it leads to $\dot{\sigma}(t) > +\beta_0\bar{\eta}$ and consequently, in both cases, it is guaranteed $\sigma(t)$ becomes zero in the finite time t_r where $t_r \leq \frac{|\sigma(0)|}{\beta_0\bar{\eta}}$. Therefore, the trajectories starting off the sliding manifolds $\sigma(t) = 0$ reach them in a finite time. In this way, it is proved that $\sigma(0) \neq 0$ reaches $\sigma(t) = 0$ in the finite time and stays on it. Now, considering Equation (13), one has

$$\sigma(t) = 0 \Rightarrow \frac{1}{\psi^{m/n}}s^{m/n}(t) = -\dot{s}(t) \quad \Rightarrow \quad (28)$$
$$\frac{1}{\psi^{m/n}}dt = -\frac{ds}{s^{m/n}(t)}; \ \forall t \geq t_r.$$

Integrating (28), in the time interval $[t_r, \ t]$ results in

$$\begin{aligned}\int_{t_r}^{t} dt &= -\psi^{m/n}\int_{s(t_r)}^{s(t)} \frac{ds}{s^{m/n}} \Rightarrow \\ t - t_r &= -\frac{n}{n-m}\psi^{m/n}[s(t)^{(n-m)/n} - s(t_r)^{(n-m)/n}] \\ \Rightarrow s(t)^{(n-m)/n} &= -\frac{n-m}{n\psi^{m/n}}(t - t_r) + s(t_r)^{(n-m)/n}.\end{aligned} \quad (29)$$

According to (29), $s(t)$ will be decreasing and at $t = t_s$, $s(t_s)$ is zero where

$$t_s = t_r + \frac{n}{n-m}\psi^{m/n}s(t_r)^{(n-m)/n}. \quad (30)$$

Since n and m are odd numbers, $n-m$ is an even number and for all $s(t) \neq 0$, the term $s(t)^{(n-m)/n}$ is positive. Consequently, it is proved that the integral sliding surface $s(t)$ converges to zero in a finite time. Therefore, the difference between $x(t)$ and $x_{\mathrm{nom}}(t)$ is nullified and the actual trajectory tracks the required one in the finite time. In the other words, the effect of external disturbances and/or model uncertainties vanishes in the finite time and since $x_{\mathrm{nom}}(t)$ has the desirable behaviour (the asymptotical stable behaviour, Binazadeh & Shafiei, 2013b), therefore the control law $u(t) = u_1(t) + u_2(t)$ guarantees the asymptotic stability of the uncertain closed-loop slowly -varying system (1). ■

Remark 3: The controller (15) is a variable structure controller (because of existence the sign function), however, main advantages for the proposed robust control law in comparison with the existing literature are that the sign function is appeared inside the integral, thus the controller $u_2(t)$ is smooth and the resulted robust controller $u(t) = u_1(t) + u_2(t)$ is chattering free. On the other hand, the proposed controller leads to the closed-loop actual uncertain system tracks the nominal one (or the desired one) in the finite time. Consequently, the optimality approach in the design of the nominal control law (refer to Remark 2) leads to the suboptimal design of the robust control law, indirectly.

4. Design example (the time-varying inertia pendulum)

In order to show the applicability of the proposed controller, it is applied to a famous benchmark system: the nonlinear inertia pendulum. The nonlinear inertia pendulum that is used in this paper is time-varying one. It consists of a plate, a beam and a travelling mass (which moves along the beam) which is shown in Figure 1.

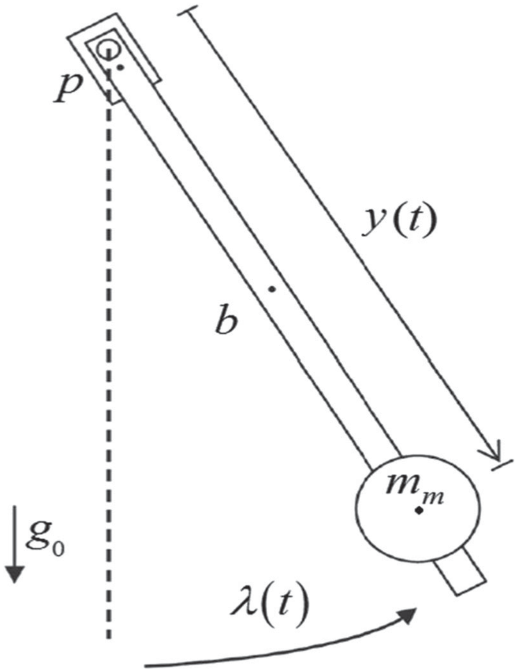

Figure 1. The time-varying inertia pendulum.

4.1. Time-varying inertia pendulum modelling

The following mathematical equation describes the motions of the time-varying inertia pendulum:

$$(I_0 + m_m y^2(t))\ddot{\lambda}(t) + (c_v + 2m_m y(t)\dot{y}(t))\dot{\lambda}(t) + (P_0 + m_m g_0 y(t))\sin(\lambda(t)) = \tau(t), \tag{31}$$

where

$$I_0 = \frac{1}{12}m_P(a_P^2 + b_P^2) + m_P d_P^2 + \frac{1}{12}m_b L_b^2 + m_b d_b^2 + \frac{1}{2}m_m r_m^2, \quad P_0 = m_P g_0 d_P + m_b g_0 d_b, \tag{32}$$

and m_p, m_b, m_m, c_v and $\tau(t)$ are the plate mass, the beam mass, the travelling mass, the viscous damping coefficient and the applied torque to the pendulum, respectively (and the gravity acceleration is considered as $g_0 = 9.8$). $y(t)$ is the slowly varying parameter which shows the distance of the travelling mass from the point O (Figure 1) and it slowly changes in the range of $0.1 \leq y(t) \leq 1$. The other parameters are available in Table 1.

The goal is designing $\tau(t)$ in a way that $\lambda(t)$ regulates to π. Thus choosing the state variables as $x_1 = \lambda - \pi$(rad), $x_2 = \dot{\lambda}$(rad/s) and the control input $u(t) = \tau(t)$, the state–space equations can be written as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -\xi_1(y)\sin(x_1 + \pi) - \xi_2(y,\dot{y})x_2\end{bmatrix} + \begin{bmatrix}0\\ \xi_3(y)\end{bmatrix}u, \tag{33}$$

Table 1. The time-varying inertia pendulum components.

Component	Mass (kg)	Sizes (m)	Centre of mass distance from O (m)
Plate	$m_P = 0.0713$	$a_P = 0.044, b_P = 0.063$	$d_P = 0.01$
Beam	$m_b = 0.29$	$L_b = 1$	$d_b = 0.5$
Travelling mass	$m_m = 0.5025$	$r_m = 0.05$	$y(t)$

where ξ_1, ξ_2 and $\xi_3$are as follows:

$$\xi_1(y) = \frac{P_0 + m_m g_0 y(t)}{I_0 + m_m y^2(t)}, \quad \xi_2(y,\dot{y}) = \frac{c_v + 2m_m y(t)\dot{y}(t)}{I_0 + m_m y^2(t)}, \quad \xi_3(y) = \frac{1}{I_0 + m_m y^2(t)}. \tag{34}$$

Suppose ξ_2 has uncertainties due to the unknown parameter c_v and inaccurate measurement of $\dot{y}(t)$. Therefore, ξ_2 can be written as follows:

$$\xi_2(y,\dot{y}) = \xi_3(y)(c_v + 2m_m y\dot{y}) = \hat{\xi}_2(y,\dot{y}) + \xi_3(y)\delta, \tag{35}$$

where $\hat{\xi}_2$ is the known nominal part. Considering $0 \leq c_v \leq 1$ and the maximum error 0.5(rad/s)in the estimation of $\dot{y}$, one has $|\delta| \leq 1$.

Considering $\theta(t) = y(t)$ as the slowly varying parameter, Equation (33) has the similar structure of (1) and it can be rewritten as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ \xi_1(\theta)\sin(x_1) - \hat{\xi}_2(\theta,\dot{\theta})x_2\end{bmatrix}}_{f(x,\theta(t))} + \underbrace{\begin{bmatrix}0\\ \xi_3(\theta)\end{bmatrix}}_{h(x,\theta(t))}(u + d(x,\theta(t))), \tag{36}$$

where $d(x,\theta(t)) = -\delta x_2$. Therefore, one has $|d| \leq |x_2|$.

The goal is designing a robust stabilizing controller for the system (36) in the set of $D = \{x \in R^2 : |x_1| \leq \pi \& |x_2| \leq 1\}$ by considering the following given cost function:

$$J = \int_0^\infty (\underbrace{10(x_1^2 + x_2^2)}_{l(x,\theta(t))} + u_1(t)^2)dt. \tag{37}$$

4.2. Controller design for the nonlinear time-varying inertia pendulum

In this section, the robust control law is designed for the time-varying inertia pendulum according to the proposed method. First, assume the nominal form of

Equation (36) (i.e. consider $d = 0$) and consider the following parametric Lyapunov function as

$$V = \frac{1}{2}x^{\mathrm{T}} \underbrace{\begin{bmatrix} P_1 & P_3 \\ P_3 & P_2 \end{bmatrix}}_{P} x,$$

where P_is are free parameters. Since P should be a positive-definite matrix, the parameters P_1, P_2 and $P_1P_2 - (P_3)^2$ should be positive. The proposed function is independent of the slowly varying parameter (refer to Remark 1) and therefore, condition (9) is satisfied. Moreover, according to Remark 2, some acceptable ranges may be obtained for the free parameters P_is by satisfying the conditions on (7a), (7b) and (7c). In this way that, by defining $\alpha_1(\|x\|) = \lambda_{\min}(P)\|x\|^2$, $\alpha_2(\|x\|) = \lambda_{\max}(P)\|x\|^2$, condition (7a) is satisfied. Also,

$$\begin{aligned} a(x, \theta(t)) = V_x f &= (P_1 - P_2\hat{\xi}_2)x_1x_2 + (P_2 - P_3\hat{\xi}_2)x_2^2 \\ &\quad + P_3\xi_1 x_2 \sin x_1 + P_2\xi_1 x_1 \sin x_1 \\ b(x, \theta(t)) = V_x g &= \xi_3(P_3x_2 + P_2x_1). \end{aligned}$$

Then, in the points where $b(x, \theta(t)) = 0$ or $x_2 = -P_2x_1/P_3$, one has

$$a(x, \theta(t))|_{x_2=-\frac{P_2}{P_3}x_1} = -\underbrace{\frac{P_2}{(P_3)^2}[P_1P_2 - (P_3)^2]}_{\varepsilon} x_1^2.$$

Since the matrix P is positive definite, the coefficient $P_1P_2 - (P_3)^2$ is positive. In the above equation, it is evident that if P_2 be also a positive value, the coefficient of $(-x_1^2)$ is positive. Therefore, there is a positive constant ε, such that $a(x, \theta(t))|_{x_2=-P_3x_1/P_2} \leq -\varepsilon x_1^2$. Now, if the class K function γ is chosen as $\gamma(\|x\|) = \lambda_0(x_1^2 + x_2^2)$, $\lambda_0 > 0$, then in the points where $b(x, \theta(t)) = 0$, it can be rewritten as $\gamma(\|x\|)|_{x_2=-P_3x_1/P_2} = \lambda_0 \left(1 + \left(\frac{P_3}{P_2}\right)^2\right) x_1^2$. Therefore, by satisfying the inequality $-\varepsilon x_1^2 \leq -\lambda_0(1 + (P_2/P_3)^2)x_1^2$ that leads to determine the admissible range of λ_0 ($\lambda_0 \leq \varepsilon/(1 + (P_2/P_3)^2)$), condition (8) is also satisfied.

Now, using simple computer programming, the best possible free parameter values are obtained to achieve the minimum feasible cost value as $P_1 = 11.9$, $P_2 = 0.3$, $P_3 = 1$. This value is substituted in $a(x, \theta(t))$ and

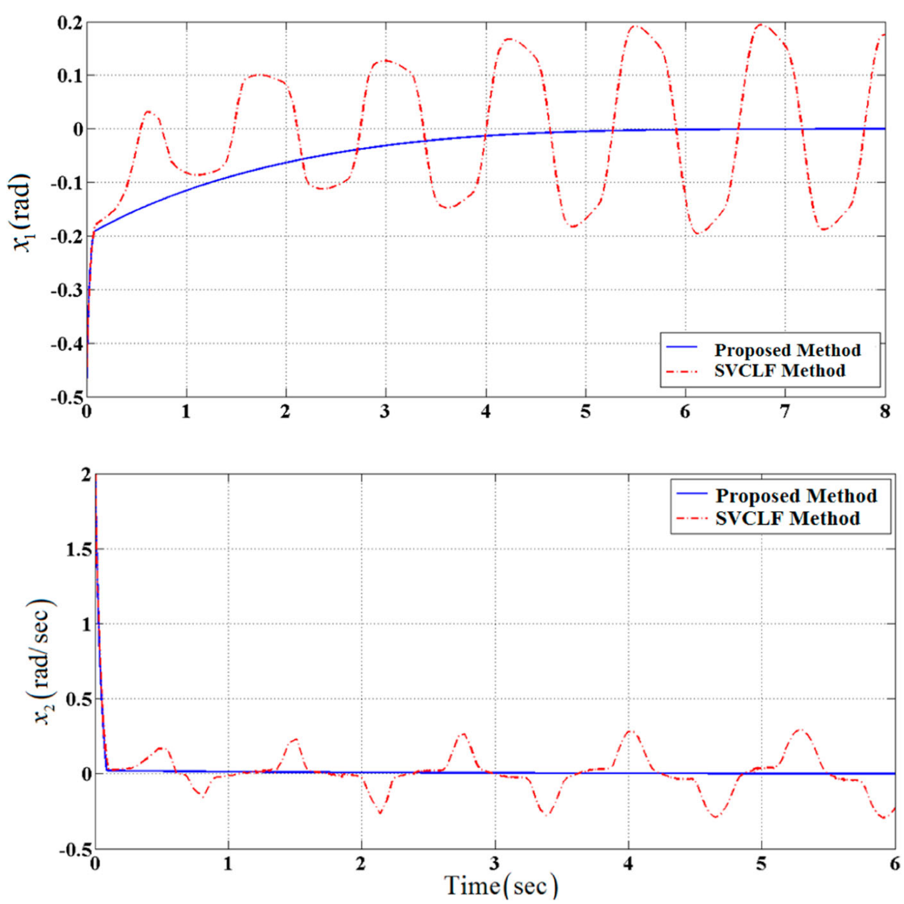

Figure 2. Time response of state variables of the uncertain closed-loop system (36).

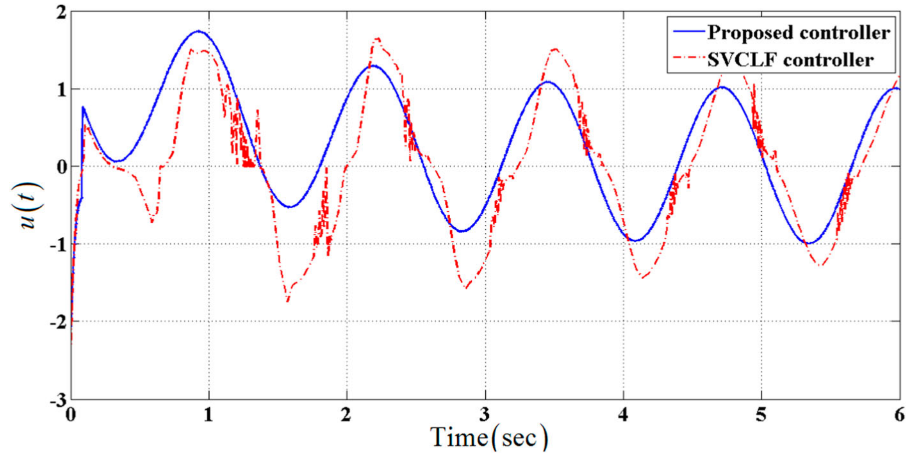

Figure 3. Time response of the nominal and robust control inputs.

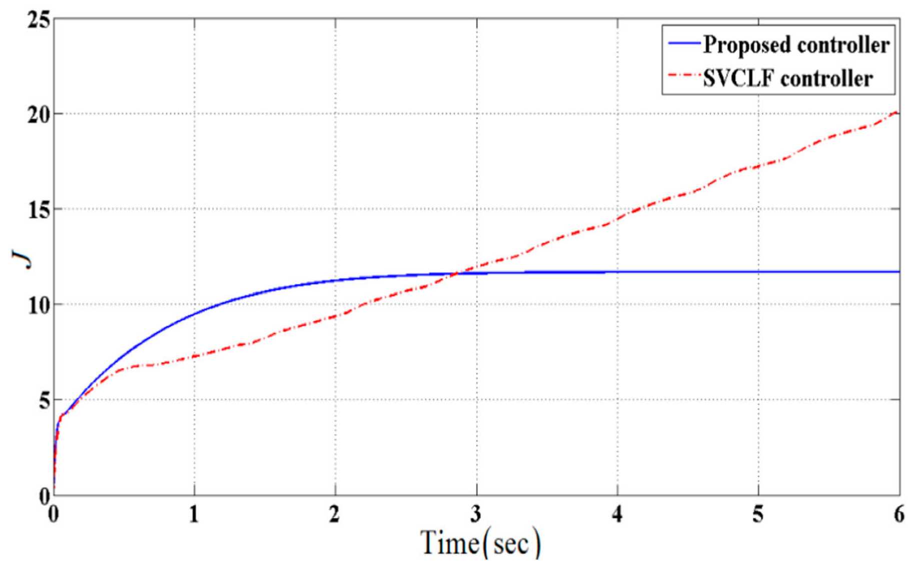

Figure 4. Time response of the cost function.

$b(x,\theta(t))$ and then, the nominal control law (i.e. $u_1(t)$) is calculated according to (6).

After that, consider the system in the presence of model uncertainties and/or external disturbances ($d(x,\theta(t)) \neq 0$). In a way that, the integral sliding surface (10) is designed, the design parameter $E = \begin{bmatrix} E_1 & E_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$ is chosen. Then, it is combined with the terminal sliding surface as in (13) with the switching gain $\psi = 0.005$ and the parameters $n = 9,\ m = 5$. Considering the upper bound (24) as $|E(\dot{h}d + h\dot{d})| \leq 4.088$ and choosing the design parameter $\beta(x) = 4.2$, the robust term ($u_2(t)$) is also designed.

4.3. Computer simulation

The computer simulation results are shown in Figures 2–4. As it is seen in Figure 2, by applying the proposed controller $u(t) = u_1(t) + u_2(t)$, the state variables converge to zero from their initial conditions and consequently $\lambda(t)$ is regulate to π; however, the nominal controller (i.e. $u_1(t)$) cannot tolerate the effects of model uncertainties and/or the external disturbances (the dotted lines). Also, some advantages of the proposed method like the desired characteristics of the transient responses are seen as well. Moreover, in Figure 3, the effective elimination of the chattering in the proposed control input is shown. The cost function value is also shown in Figure 4. Therefore, as seen from the simulation results, adding the robust term succeed in achieving the robust and desired performance for the uncertain closed-loop system.

5. Conclusion

In this paper, a new chattering-free stabilizing control law was presented for uncertain nonlinear slowly varying systems. The proposed controller had a suboptimal feature in addition to its robust performance in presence of model uncertainties and/or external disturbances. The design of the proposed controller had two steps. The first step was the design of the nominal part of the controller which was designed based on the SVCLF method for the nominal system by considering the given cost function. The second step was the design of a robustifying term to eliminate the effects of the model uncertainties and/or external disturbances such that the time response of the actual system converged to the nominal system in a finite time. For this purpose, a special sliding surface was provided by the combination of terminal and integral ones and a theorem was given and proved. At the end of the paper, the proposed controller was applied to a time-varying inertia pendulum system and its efficiency was confirmed. Further extension may be based on output feedback instead of state feedback and extended the proposed method for multi-input nonlinear slowly varying systems.

Disclosure statement

No potential conflict of interest was reported by the authors.

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