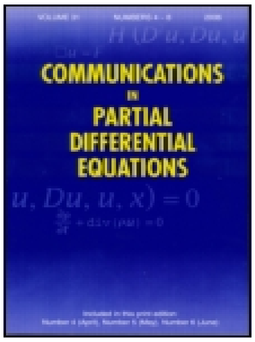

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Communications in Partial Differential Equations

ISSN: 0360-5302 (Print) 1532-4133 (Online) Journal homepage: https://www.tandfonline.com/loi/lpde20

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To cite this article: Jonathan Bennett , Neal Bez , Susana Gutiérrez & Sanghyuk Lee (2014) On the Strichartz Estimates for the Kinetic Transport Equation, Communications in Partial Differential Equations, 39:10, 1821-1826, DOI: 10.1080/03605302.2013.850880

To link to this article: https://doi.org/10.1080/03605302.2013.850880

Communications in Partial Differential Equations, 39: 1821–1826, 2014
Published with license by Taylor & Francis
ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605302.2013.850880

On the Strichartz Estimates for the Kinetic Transport Equation

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We show that the endpoint Strichartz estimate for the kinetic transport equation is false in all dimensions. We also present an alternative approach to proving the non-endpoint cases using multilinear analysis.

Keywords Endpoint estimates; Kinetic transport equations; Multilinear analysis; Strichartz estimates.

Mathematics Subject Classification 42B37.

1. Introduction

The solution of the kinetic transport equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = 0, \quad f(0,x,v) = f^0(x,v)$$

for $(t,x,v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, satisfies the Strichartz estimates

$$\|f\|_{L^q_t L^p_x L^r_v} \lesssim \|f^0\|_{L^a_{x,v}}, \tag{1.1}$$

where

$$\frac{2}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right), \quad \frac{1}{a} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{p}\right), \quad q > a, \ \ p \geq a. \tag{1.2}$$

Received August 28, 2013; Accepted September 23, 2013

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(We write $X \lesssim Y$ and $Y \gtrsim X$ if $X \leq CY$ for some finite constant C depending at most on the parameters (q, p, r, a, d), and $X \sim Y$ if $X \lesssim Y$ and $X \gtrsim Y$.) With (q, p, r, a) satisfying (1.2), but with the further condition $q > 2 \geq a$, this was proved by Castella and Perthame [1], and it was observed by Keel and Tao [5] that this latter condition can be relaxed to $q > a$ and hence (1.2) suffices. In [5] it is tentatively conjectured that the Strichartz estimate (1.1) holds at the endpoint $q = a$, at least for $d > 1$. Using the invariance under the transformations

$$f^0 \leftrightarrow (f^0)^\lambda, \quad f \leftrightarrow f^\lambda, \quad (q, p, r, a) \leftrightarrow \left(\frac{q}{\lambda}, \frac{p}{\lambda}, \frac{r}{\lambda}, \frac{a}{\lambda}\right) \tag{1.3}$$

this conjectured endpoint can be (and usually is) stated for initial data in $L^2_{x,v}$ as

$$\|f\|_{L^2_t L_x^{\frac{2d}{d-1}} L_v^{\frac{2d}{d+1}}} \lesssim \|f^0\|_{L^2_{x,v}}. \tag{1.4}$$

The main purpose of this paper is to disprove this conjecture.

Theorem 1. *The endpoint Strichartz estimate* (1.4) *for the kinetic transport equation fails for all* $d \geq 1$.

The case $d = 1$ of Theorem 1 was proved by Guo and Peng [4] and later by Ovcharov [7] using different arguments (where the norm in x on the left-hand side is L^∞_x). The argument we present uses key ideas from recent work of Frank et al. [3].

We remark that for the free Schrödinger propagator, the endpoint Strichartz estimate fails in the case $d = 2$ (see [6]) but is true for all $d > 2$, as shown in the landmark paper of Keel and Tao [5]. Thus, Theorem 1 highlights a fundamental difference in the Strichartz estimates for these related equations.

In the next section, we prove Theorem 1. In the final section, we provide an alternative proof of the Strichartz estimates (1.1) in all non-endpoint cases using multilinear analysis.

2. Proof of Theorem 1

Using the invariance under the transformation (1.3) with $\lambda = \frac{2d}{d+1}$, estimate (1.4) is equivalent to

$$\|f\|_{L_t^{\frac{d+1}{d}} L_x^{\frac{d+1}{d-1}} L^1_v} \lesssim \|f^0\|_{L_{x,v}^{\frac{d+1}{d}}}. \tag{2.1}$$

Since $f(t, x, v) = f^0(x - tv, v)$, it is clear that (2.1) implies

$$\|\rho(f^0)\|_{L_t^{\frac{d+1}{d}} L_x^{\frac{d+1}{d-1}}} \lesssim \|f^0\|_{L_{x,v}^{\frac{d+1}{d}}}, \tag{2.2}$$

where $\rho(f^0)$ is the macroscopic density defined by the linear mapping

$$\rho(f^0)(t, x) = \int_{\mathbb{R}^d} f^0(x - tv, v)\, \mathrm{d}v.$$

Hence, by duality, (2.2) implies

$$\|\rho^* g\|_{L^{d+1}_{x,v}} \lesssim \|g\|_{L_t^{d+1} L_x^{\frac{d+1}{2}}}, \tag{2.3}$$

where the adjoint ρ^* is given by

$$\rho^* g(x, v) = \int_{\mathbb{R}} g(t, x + tv)\, \mathrm{d}t.$$

From here, the argument strongly uses ideas from the paper of Frank et al. [3] concerning refined Strichartz estimates for the free Schrödinger propagator associated with orthonormal initial data.

Suppose $g \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d) \setminus \{0\}$ is nonnegative and such that $\widehat{g} \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ is also nonnegative. Here, we use $\widehat{g}$ to denote the space-time Fourier transform of g given by

$$\widehat{g}(\tau, \xi) = \int_{\mathbb{R} \times \mathbb{R}^d} g(t, x) e^{-i(t\tau + x\cdot\xi)}\, \mathrm{d}t\mathrm{d}x.$$

In this proof, we shall also use c to denote a constant depending on at most d, which may change from line to line.

Proceeding formally, using Fourier inversion we get

$$\begin{aligned}
\|\rho^* g\|_{L^{d+1}_{x,v}}^{d+1} &= \int \prod_{j=1}^{d+1} g(t_j, x + t_j v)\, \mathrm{d}\vec{t}\mathrm{d}x\mathrm{d}v \\
&= c \int \prod_{j=1}^{d+1} \widehat{g}(\tau_j, \xi_j) \prod_{k=1}^{d+1} e^{it_k(\tau_k + v\cdot\xi_k)} e^{ix\cdot\sum_{\ell=1}^{d+1}\xi_\ell}\, \mathrm{d}\vec{\tau}\mathrm{d}\vec{\xi}\mathrm{d}\vec{t}\mathrm{d}x\mathrm{d}v \\
&= c \int \prod_{j=1}^{d+1} \widehat{g}(\tau_j, \xi_j) \prod_{k=1}^{d+1} \delta(\tau_k + v\cdot\xi_k)\ \delta\left(\sum_{\ell=1}^{d+1} \xi_\ell\right) \mathrm{d}\vec{\tau}\mathrm{d}\vec{\xi}\mathrm{d}v \\
&= c \int \prod_{j=1}^{d+1} \widehat{g}(-v\cdot\xi_j, \xi_j)\, \delta\left(\sum_{k=1}^{d+1} \xi_k\right) \mathrm{d}\vec{\xi}\mathrm{d}v
\end{aligned}$$

and hence

$$\|\rho^* g\|_{L^{d+1}_{x,v}}^{d+1} = c \int \prod_{j=1}^{d} \widehat{g}(-v\cdot\xi_j, \xi_j) \widehat{g}\left(v \cdot \sum_{k=1}^{d} \xi_k, -\sum_{\ell=1}^{d} \xi_\ell\right) \mathrm{d}\vec{\xi}\mathrm{d}v. \tag{2.4}$$

We remark that by appropriately truncating the integrals in the above identities and limiting arguments, (2.4) makes sense in $[0, \infty]$ for the class of g under consideration.

Define K to be the d by d matrix whose consecutive rows are $-\xi_1, \ldots, -\xi_d$. Using the change of variables $w = Kv$, so that $w_j = -\xi_j \cdot v$ for each $1 \leq j \leq d$, we obtain

$$\|\rho^* g\|_{L^{d+1}_{x,v}}^{d+1} = c \int \prod_{j=1}^{d} \widehat{g}(w_j, \xi_j) \widehat{g}\left(-\sum_{k=1}^{d} w_k, -\sum_{\ell=1}^{d} \xi_\ell\right) \frac{1}{|\det K|} \mathrm{d}\vec{w}\mathrm{d}\vec{\xi}.$$

Writing each $\xi_j = r_j \theta_j$ in polar coordinates, we have

$$|\det K| = \left(\prod_{j=1}^{d} r_j\right) |\det(\theta_1 \cdots \theta_d)|.$$

Since $\widehat{g}(0,0) > 0$ and $\widehat{g}$ is continuous, it follows that

$$\|\rho^* g\|_{L^{d+1}_{x,v}}^{d+1} \gtrsim \int_{|r|\lesssim 1} \int_{(\mathbb{S}^{d-1})^d} \left(\prod_{j=1}^{d} r_j^{d-2}\right) \frac{1}{|\det(\theta_1 \cdots \theta_d)|} \mathrm{d}\vec{r}\mathrm{d}\vec{\theta}. \tag{2.5}$$

For $d = 1$ the radial integral is infinite, and for $d \geq 2$,

$$\int_{(\mathbb{S}^{d-1})^d} \frac{1}{|\det(\theta_1 \cdots \theta_d)|} \mathrm{d}\vec{\theta} = \infty,$$

so the angular integral is infinite. Hence, for all $d \geq 1$ we have shown that (2.3), and consequently (1.4), cannot hold.

Remark. The above argument shows that the endpoint estimate (2.3) fails rather generically. For example, the space-velocity norm $\|\rho^* g\|_{d+1}$ is infinite whenever $g \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^d) \setminus \{0\}$ is nonnegative and such that $\widehat{g} \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ is also nonnegative.

3. A Multilinear Approach to the Non-Endpoint Cases

Fix $\sigma > 1$. In this section, the notation $\lesssim$ allows, in addition, the implicit constant to depend on σ.

We shall prove

$$\|\rho^* g\|_{L^{\sigma(d+1)}_{x,v}} \lesssim \|g\|_{L_t^{q(\sigma)} L_x^{\frac{(d+1)\sigma}{2}}}, \tag{3.1}$$

for all $g \in L_t^{q(\sigma)} L_x^{\frac{(d+1)\sigma}{2}}$, where the exponent $q(\sigma)$ satisfies

$$\frac{1}{q(\sigma)} + \frac{d}{(d+1)\sigma} = 1.$$

Our argument is a multilinear variant of that in [1] and [9] (see also [8]), which deals with all non-endpoint cases directly. Similar multilinear arguments have been used recently in [3].

Using the invariance under transformations in (1.3), to prove the full range of non-endpoint Strichartz estimates, it suffices to consider (q, p, a) satisfying

$$q > a, \quad p \geq a, \quad \frac{2}{q} = d\left(1 - \frac{1}{p}\right), \quad \frac{1}{a} = \frac{1}{2}\left(1 + \frac{1}{p}\right) \tag{3.2}$$

and show that (1.1) holds with $r = 1$, or equivalently, that

$$\|\rho(f^0)\|_{L_t^q L_x^p} \lesssim \|f^0\|_{L^a_{x,v}}$$

holds for all $f^0 \in L^a_{x,v}$. By duality, this is equivalent to

$$\|\rho^* g\|_{L^{a'}_{x,v}} \lesssim \|g\|_{L_t^{q'} L_x^{p'}} \tag{3.3}$$

for all $g \in L_t^{q'} L_x^{p'}$. Note that (3.2) implies that $a' = 2p'$ and $\frac{1}{q'} + \frac{d}{a'} = 1$, in which case (3.3) reads

$$\|\rho^* g\|_{L^{a'}_{x,v}} \lesssim \|g\|_{L_t^{q'} L_x^{\frac{a'}{2}}},$$

and the condition $q > a$ is equivalent to $a' > d + 1$. Therefore, (3.1) with $\sigma = \frac{a'}{d+1} > 1$ implies the full range of non-endpoint Strichartz estimates (1.1).

Proof of (3.1). Without loss of generality, suppose g is nonnegative. By multiplying out and using Minkowski's integral inequality, we get

$$\begin{aligned}\|\rho^* g\|^{d+1}_{L^{\sigma(d+1)}_{x,v}} &= \left(\int \left(\int \prod_{j=1}^{d+1} g(t_j, x + t_j v) \mathrm{d}\vec{t} \right)^{\sigma} \mathrm{d}x \mathrm{d}v \right)^{1/\sigma} \\ &\le \int \left(\int \prod_{j=1}^{d+1} g(t_j, x + t_j v)^{\sigma} \mathrm{d}x \mathrm{d}v \right)^{1/\sigma} \mathrm{d}\vec{t}.\end{aligned}$$

Now fix $t_1, \ldots, t_{d+1}$ and consider the multilinear form

$$\int \prod_{j=1}^{d+1} g_j(t_j, x + t_j v) \mathrm{d}x \mathrm{d}v.$$

A straightforward estimate via the change of variables $(x, v) \mapsto (x + t_i v, x + t_j v)$ gives

$$\int \prod_{j=1}^{d+1} g_j(t_j, x + t_j v) \mathrm{d}x \mathrm{d}v \lesssim \frac{1}{|t_i - t_j|^d} \|g_i(t_i, \cdot)\|_{L^1_x} \|g_j(t_j, \cdot)\|_{L^1_x} \prod_{k \neq i,j} \|g_k(t_k, \cdot)\|_{L^\infty_x}$$

for each $1 \le i < j \le d$. A multilinear interpolation argument yields

$$\int \prod_{j=1}^{d+1} g_j(t_j, x + t_j v) \mathrm{d}x \mathrm{d}v \lesssim \prod_{1 \le i < j \le d} |t_i - t_j|^{-\frac{2}{d+1}} \prod_{k=1}^{d+1} \|g_k(t_k, \cdot)\|_{L_x^{\frac{d+1}{2}}}.$$

Applying this with g^σ for each g_j we get

$$\begin{aligned}\|\rho^* g\|^{d+1}_{L^{\sigma(d+1)}_{x,v}} &\lesssim \int_{\mathbb{R}^{d+1}} \prod_{1 \le i < j \le d} |t_i - t_j|^{-\frac{2}{(d+1)\sigma}} \prod_{k=1}^{d+1} \|g(t_k, \cdot)\|_{L_x^{\frac{(d+1)\sigma}{2}}} \mathrm{d}\vec{t} \\ &\lesssim \|g\|^{d+1}_{L_t^{q(\sigma)} L_x^{\frac{(d+1)\sigma}{2}}}.\end{aligned}$$

The last inequality is a consequence of the multilinear Hardy–Littlewood–Sobolev inequality due to Christ [2]. □

Remark. It seems natural to conjecture that ρ^* satisfies the weak-type estimate

$$\|\rho^* g\|_{L^{d+1,\infty}_{x,v}} \lesssim \|g\|_{L_t^{d+1} L_x^{\frac{d+1}{2}}}.$$

A similar comment was made in an operator-theoretic context in [3].

Certain replacements for the endpoint are already known. For example, with $q = a = 2$, in [5], Keel and Tao obtain a substitute for (1.4) with the $L_x^p L_v^r$ norm (where $p = \frac{2d}{d-1}$ and $r = \frac{2d}{d+1}$) replaced by that of a certain real interpolation space which is between $L_x^{p,1} L_v^{r,1}$ and $L_x^{p,\infty} L_v^{r,\infty}$. See also work of Ovcharov [8] where a different substitute bound was given for velocities v belonging to a bounded subset of $\mathbb{R}^d$.

Acknowledgments

We would like to thank José Cañizo for his contributions during the early stages of this project.

Funding

This work was supported by the European Research Council (grant number 307617; Bennett); the Engineering and Physical Sciences Research Council (grant numbers EP/J021490/1, EP/J01155X/1; Bez, Gutiérrez); National Research Foundation of Korea (grant number 2012008373; Lee).

References

[1] Castella, F., Perthame, B. (1996). Estimations de Strichartz pour les èquations de transport cinétique [Strichartz estimates for kinetic transport equations]. *C. R. Acad. Sci. Paris Sér. I Math.* 332:535–540.

[2] Christ, M. (1985). On the restriction of the Fourier transform to curves: endpoint results and the degenerate case. *Trans. Amer. Math. Soc.* 287:223–238.

[3] Frank, R., Lewin, M., Lieb, E.H., Seiringer, R. A Strichartz inequality for orthonormal functions. *J. Euro. Math. Soc.*, to appear.

[4] Guo, Z., Peng, L. (2007). Endpoint Strichartz estimate for the kinetic transport equation in one dimension. *C. R. Math. Acad. Sci. Paris* 345:253–256.

[5] Keel, M., Tao, T. (1998). Endpoint Strichartz estimates. *Amer. J. Math.* 120:955–980.

[6] Montgomery-Smith, S.J. (1998). Time decay for the bounded mean oscillation of solutions of the Schrödinger and wave equations. *Duke Math. J.* 91:393–408.

[7] Ovcharov, E. (2011). Counterexamples to Strichartz estimates for the kinetic transport equation based on Besicovitch sets. *Nonlinear Anal.* 74:2515–2522.

[8] Ovcharov, E. (2011). Strichartz estimates for the kinetic transport equation. *SIAM J. Math. Anal.* 43:1282–1310.

[9] Perthame, B. (2004). Mathematical tools for kinetic equations. *Bull. Amer. Math. Soc.* 41:205–244.