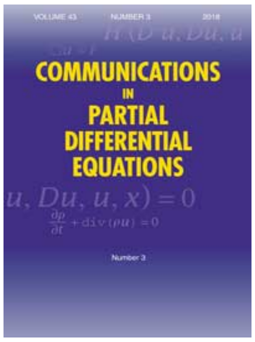


Communications in Partial Differential Equations

ISSN: 0360-5302 (Print) 1532-4133 (Online) Journal homepage: https://www.tandfonline.com/loi/lpde20

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**To cite this article:** David Fajman (2018) The nonvacuum Einstein flow on surfaces of nonnegative curvature, Communications in Partial Differential Equations, 43:3, 364-402, DOI: 10.1080/03605302.2018.1446159

**To link to this article:** https://doi.org/10.1080/03605302.2018.1446159



Published online: 19 Mar 2018.

COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
2018, VOL. 43, NO. 3, 364–402
https://doi.org/10.1080/03605302.2018.1446159

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# The nonvacuum Einstein flow on surfaces of nonnegative curvature

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**ABSTRACT**
We prove future nonlinear stability of homogeneous solutions to the Einstein–Vlasov system with massive particles on manifolds with topology $M = \mathbb{R} \times \Sigma$, where $\Sigma$ is either $\mathbb{S}^2$ or $\mathbb{T}^2$. For the sphere this implies the existence of an open subset of the initial data manifold with elements of strictly positive scalar curvature, whose developments are future geodesically complete. In combination with an earlier result for hyperbolic surfaces we conclude future completeness for the Einstein–Vlasov system in 2+1 dimensions independent of the compact spatial topology for an open set of initial data.

**ARTICLE HISTORY**
Received 28 November 2016
Accepted 28 October 2017

**KEYWORDS**
Closed universe recollapse; Einstein equations; Einstein–Vlasov system; nonlinear stability

**2010 MATHEMATICS SUBJECT CLASSIFICATION**
53Z05; 83C05; 35Q75

## 1. Introduction

An intriguing open problem in general relativity concerns the effect of the spatial topology on the long–time behavior of the Einstein flow on spacetimes with topology $I \times \Sigma$ with $I \subset \mathbb{R}$, where $\Sigma$ is a closed manifold. The main competing scenarios are spacetimes which are complete in one direction and incomplete in the other—we refer to those as *past incomplete* and *future complete*—and those which are incomplete in both directions, i.e., recollapse. In 2+1 dimensions, the uniformization theorem implies that $\Sigma$ is either the sphere, the torus or a higher genus surface. It is believed that for the case of vanishing cosmological constant positive spatial curvature causes recollapse of spacetime while negative spatial curvature leads to future completeness (cf. [31]). The former behavior is been referred to as the *closed universe recollapse conjecture*, stated by Barrow et al. [9]. It concerns the case of 3+1-dimensional solutions to the Einstein-matter equations with vanishing cosmological constant and may be reformulated following [31]:

*Let $(M, g)$ be a maximal globally hyperbolic cosmological solution to the Einstein-matter equations with a compact hypersurface $\Sigma$. If $\Sigma$ is of positive Yamabe type, the spacetime admits a foliation of constant mean curvature (CMC) hypersurfaces with mean curvature taking all real values. This foliation covers the entire spacetime.* In particular, validity of this conjecture determines that the existence of a CMC—hypersurface with an induced Riemannian metric of positive scalar curvature implies the recollapse of this spacetime, i.e., the geodesic incompleteness in both time directions. The conjecture has been proven under simplifying assumptions for the vacuum and nonvacuum case. Some results are given in the following. For the case of Bianchi type IX solutions the conjecture was proved by Lin and Wald [23] (cf. also [28] for a

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detailed presentation of the proof), for spherically symmetric spacetimes by Burnett [11]. For spatially homogeneous spacetimes with a perfect fluid or collisionless matter it was proved by Rendall [32]. For spherically symmetric spacetimes on $\mathbb{S}^2 \times \mathbb{S}^1$ with a massless scalar field or collisionless matter the conjecture was proven by Rendall [33] and Burnett and Rendall [12] under the assumption of the existence of a CMC surface. This assumption was then later removed by Henkel [22]. For additional references and the relation to geometrization we refer to the article by Anderson [1]. Complementary to the recollapse of positively curved initial data, there are results proving that negative spatial curvature implies future completeness such as the case of the Milne model and generalizations in different dimensions [3].

The first main result of the present work is the nonlinear stability of a counterexample to the recollapse conjecture, which has been constructed in [19]. We consider 2+1-dimensional spacetimes with the spatial topology of the sphere. This class is delicate in the sense that there are no vacuum solutions to the system with that topology, as discussed in the next section. A key observation presented in [19] is the fact that an energy density created by massive particles allows for the construction of homogeneous future complete solutions with this topology—despite the positive spatial curvature. Here, we upgrade this result to general initial data close to the homogeneous models of [19], where closeness is defined in terms of suitable Sobolev spaces. This is made precise below. The reason for future completeness results from the sign of the energy density, which acts as a negative correction to the rescaled scalar curvature in the Hamiltonian constraint. Moreover, for massive particles this effect is uniform in time as we describe below. It allows for a solution of the Hamiltonian constraint despite the spherical topology and avoids recollapse of spacetime. We used this fact to construct a class of explicit, future complete solutions with spherical topology in [19]. The first theorem of this work states their nonlinear stability.

In the second part of this paper we study the Einstein–Vlasov flow on the 2-torus. Here, similar to the sphere, the energy density is exploited to construct stable, future complete solutions. We comment on details of the construction in the following.

We remark that the corresponding system for massless particles leads to different classes of spacetimes, which show a relation between the long-time behavior and the spatial topology. These spacetimes are constructed and analyzed in [20].

### 1.1. Spacetimes with spherical spacelike topology

A necessary condition for the existence of solutions to the vacuum CMC-Einstein equations on manifolds of the form

$$M = [\mathrm{T}_0, \infty) \times \mathbb{S}^2, \tag{1}$$

where $\mathrm{T}_0 \in \mathbb{R}$, is positivity of the cosmological constant, $\Lambda > 0$. This is a direct consequence of the constraint equations for the induced metric $g$ and the second fundamental form $k$ with mean curvature $\tau = \mathrm{tr}_g k$,

$$\begin{aligned} R(g) - |k|_g^2 + \tau^2 &= 2\rho + 2\Lambda \\ \nabla^i k_{ia} - \nabla_a(\mathrm{tr}_g k) &= \jmath_a, \end{aligned} \tag{2}$$

as the vacuum-CMC momentum constraint ($\partial_{x^i}\tau = 0$ for $i = 1, 2$ and $\jmath = 0$) implies that the trace-free part of the second fundamental form $k$ vanishes, which is due to the fact that the 2-sphere admits no nontrivial transverse-traceless tensors. Then the left-hand side of the vacuum Hamiltonian constraint ($\rho = 0$) is strictly positive, which implies that $\Lambda$ needs to be strictly positive to allow for the existence of solutions [4]. This follows from rewriting the Hamiltonian constraint as an elliptic equation for the conformal factor (cf. (16)), [26].

In the general case, a CMC surface always exists [5, 8, 25] so $\mathbb{S}^2$ topology is ruled out for vacuum solutions with $\Lambda = 0$. However, the presence of a non-vanishing energy-momentum tensor may allow for solutions in the case of vanishing cosmological constant. It has been demonstrated in [14] that a non-vanishing energy density allows for (local in time) solutions of the Einstein equations with $\Lambda = 0$ for the case of a massless scalar field coupled to the Einstein equations (which in this specific case originates from a $U(1)$-symmetry in $3+1$ dimensions). The global structure of these solutions is however not investigated in [14]. We consider from now on the case $\Lambda = 0$ for the remainder of the paper. A key observation of [19] is discussed in the following. If one couples massive Vlasov matter to the Einstein equations the asymptotic behavior is more accessible in the following sense. The $L^1$-norm of the energy density $\rho$, taken with respect to the Riemannian metric induced on $\mathbb{S}^2$, is bounded from below by the total mass of the slice

$$\int \rho(f)\mu_g > \mathbf{m}_\infty \equiv \int_{T\mathbb{S}^2} f\mu_{T\mathbb{S}^2}, \tag{3}$$

where $f$ denotes the representation of the distribution function with domain $T\mathbb{S}^2$ (cf. Section 2.4 for details). $\mathbf{m}_\infty$ is a conserved quantity in the homogeneous case, which is an immediate consequence of the transport equation for $f$ (cf. Section 2.5). For initial data satisfying the condition

$$4\pi < \mathbf{m}_\infty, \tag{4}$$

this lower bound on the mass is preserved during the evolution and assures the solvability of the Hamiltonian constraint and in particular a sharp lower bound on the conformal factor, in terms of the time function. We conclude the existence of future complete homogeneous solutions, which are future-asymptotically of the form

$$g_\infty = -4dt^2 + \frac{\mathbf{m}_\infty - 4\pi}{2\pi} t^2 \cdot \sigma_{\mathbb{S}^2}. \tag{5}$$

Here $\sigma_{\mathbb{S}^2}$ denotes the round metric on the 2-sphere with volume $8\pi$ and constant scalar curvature $R(\sigma_{\mathbb{S}^2}) = 1$. The exact solution is given in (42). Note in particular that the asymptotic behavior of these solutions is uniquely determined by the total mass $\mathbf{m}_\infty$. We prove future global-in-time existence for solutions with initial data close to these homogeneous solutions. The asymptotics imply future completeness of the future-development of any initial data close to these solutions. The main result is

**Theorem 1.1.** *Every homogeneous solution to the Einstein–Vlasov system on* $[\mathrm{T}_0, \infty) \times \mathbb{S}^2$ *of type* (42) *is future nonlinearly stable. In particular, for any* $\mathrm{T}_0 > 0$, *there is an open set of initial data at* $\mathrm{T}_0$ *containing the initial data induced by the homogeneous solution, such that the future development of each of its elements is globally hyperbolic and future timelike- and null geodesically complete and remains in a fixed neighborhood of the homogeneous solution.*

**Remark 1.2.** We make the specifications of the above theorem precise. By homogeneous solution we refer to solutions of the type (42). By initial data we refer to data $(\lambda, h, N, X, f_0) \in H^5 \times H^4 \times H^6 \times H^5 \times H_{\mathrm{Vl,c,4}}$ solving (15)–(18), where $H^k$ are standard Sobolev spaces (cf. Section 2.1.1) and $H_{\mathrm{Vl,c},k}$, are Sobolev spaces for distribution functions (cf. Section 2.4.2).

The considered set of initial data is open in the sense of the aforementioned function space topology.

## 1.2. Spacetimes with toroidal spacelike topology

The spatial topology of the 2-torus, $\mathbb{T}^2$, allows for solutions to the vacuum Einstein equations [4, 13, 27]. However, the problem of their nonlinear stability is a priori difficult as the conformal geometry degenerates asymptotically (cf. the introduction of [15] and Chapter 3.3 of [13]). The reason for this behavior is the vanishing scalar curvature of the conformal metric in the Hamiltonian constraint. If the conformal factor is spatially constant it is essentially equal to the square of the trace-free part of the second fundamental form, $h = k - \tau/2g$. If one considers nonvacuum Einstein–Vlasov initial data, the energy-density in the Hamiltonian constraint lifts this sensitive dependence on $h$ and yields a behavior which resembles the case of negative spatial curvature. In the homogeneous case this yields solutions of the following form. We consider the homogeneous equations (44) for initial data with $\mathbf{m}_\infty > 0$. The homogeneous Hamiltonian constraint takes the form

$$0 = e^{2\lambda}\tau^2/2 - e^{2\lambda}\rho. \tag{6}$$

In particular,

$$e^{2\lambda}\tau^2 = 2e^{2\lambda}\rho \overset{t\to\infty}{\longrightarrow} \frac{2\mathbf{m}_\infty}{\mathrm{vol}_{\sigma_{\mathbb{T}^2}}(\mathbb{T}^2)}. \tag{7}$$

This yields future complete spacetimes on $\mathbb{T}^2$, with the future-asymptotic form

$$g_\infty = -4dt^2 + \frac{2\mathbf{m}_\infty}{\mathrm{vol}_{\sigma_{\mathbb{T}^2}}(\mathbb{T}^2)} t^2 \cdot \sigma_{\mathbb{T}^2}, \tag{8}$$

where here $\sigma_{\mathbb{T}^2}$ denotes a fixed Riemannian metric on $\mathbb{T}^2$ with vanishing scalar curvature. The exact form of the solutions is (155). These have first been constructed in [19]. The following is the second main theorem.

**Theorem 1.3.** *Every homogeneous solution to the Einstein–Vlasov system on* $[\mathrm{T}_0, \infty) \times \mathbb{T}^2$ *of the form* (155) *is future nonlinearly stable. In particular, for any* $\mathrm{T}_0 > 0$, *there is an open set of initial data at* $\mathrm{T}_0$ *containing the initial data induced by the homogeneous solution, such that the future development of each of its elements is globally hyperbolic and future timelike- and null geodesically complete and remains in a uniformly bounded neighborhood of the homogeneous solution.*

**Remark 1.4.** The same specifications as in Remark 1.2 apply to Theorem 1.3.

## 1.3. Massive particles—oblivion to topology

In a more general sense, the connection between the spatial topology and the asymptotics of the Einstein flow has been analyzed for the case of a positive cosmological constant by Ringström in [30] and with Vlasov matter in [29]. For a certain class of initial data the Einstein flow localizes and the spatial topology has no influence on its asymptotic behavior. The latter has been referred to as *oblivion to topology*.

The results in the present work, in combination with the main result of [17], where it is shown that the Einstein–Vlasov flow on hyperbolic surfaces yields future complete and stable solutions, directly imply that a variation of *oblivion to topology* holds for the Einstein–Vlasov flow in 2+1 dimensions with massive particles but vanishing cosmological constant. In particular, for an open set of initial data, the future behavior of the Einstein–Vlasov flow is independent of the spatial topology—all solutions are future complete and expand at a quadratic rate in inverse mean curvature time $t = -\tau^{-1}$. This is concluded by the following corollary, which is an immediate consequence of Theorems 1.1, 1.3, and the main result of [17]. For a closed surface $\Sigma$ we denote its genus by $\mathsf{gen}(\Sigma)$.

**Corollary 1.5.** *Let $\Sigma$ be a closed surface and $\sigma_\Sigma$ a fixed Riemannian metric of constant scalar curvature on $\Sigma$, $R(\sigma) \in \{-1, 0, 1\}$. Then there exists initial data to the massive Einstein–Vlasov flow on $\Sigma$ such that its maximal development is future geodesically complete and the spacetime is future-asymptotically of the form*

$$g_\infty = -4dt^2 + \mathbf{c} \cdot t^2 \sigma_\Sigma, \tag{9}$$

*where $\mathbf{c} = \mathbf{c}(\mathsf{gen}(\Sigma), \mathbf{m}_\infty)$ is a positive constant depending only on the genus of $\Sigma$ and its total mass. These solutions are future stable.*

### 1.4. Remarks

The existence of an open set of future complete spacetimes with topology $[\mathrm{T}_0, \infty) \times \mathbb{S}^2$ is in fact desirable as it provides the possibility to study the Einstein-flow with vanishing cosmological constant on one of the most accessible topological models—the 2-sphere. This provides a basis to consider large data perturbations of the homogeneous background geometry by studying for instance the future development of surfaces of revolution "far away" from the geometry of the round sphere. Similarly, future complete models with torus topology may be investigated with respect to large perturbations under an additional symmetry assumption. In the symmetric case, both models provide effectively 1+1-dimensional systems, which can serve as examples to study large data perturbations.

In the context of quantum gravity, where 2+1-dimensional spacetimes are a well-studied system (cf. [13]), models with these topologies and vanishing cosmological constant provide a new class of explicit solutions.

Finally, we remark that the results presented in this paper and the corresponding result in [17] relate, in a broader sense, to a recent series of results on nonlinear stability for the Einstein–Vlasov system [2, 21, 24, 35].

### 1.5. Organization of this paper

The paper is divided into two parts. Sections 2 to 4 are concerned with the case of the sphere, i.e., the proof of Theorem 1.1. Section 5 treats the case of the torus and contains the proof of Theorem 1.3.

We remark that the Einstein–Vlasov system in 2+1 dimensions has been discussed for the case of hypersurfaces of genus $\mathrm{gen}(\Sigma) > 1$ in [17]. Some fundamentals are similar for the topologies considered here. But several details differ in the present case, in particular, the matter sources are not small as in [17]. For the sphere the energy density is necessarily large (cf. (4)). This requires a number of additional steps in the proof compared to the case considered in [17].

For further background on the Einstein equations in CMC-gauge we refer to [3, 4, 15, 17]. Some relevant geometric facts for the sphere are taken from [14]. These fundamentals on the geometry are given in Section 2. Main facts about the energy-momentum tensor and the distribution function for Vlasov matter as well as the transport equation are revisited in Section 2.4. For a thorough introduction to the Einstein–Vlasov system we refer to [6, 29, 34].

In Section 2.5, we review the class of explicit homogeneous solutions with spherical topology introduced in [19]. Starting from there, we begin with the preparations of the proof of Theorem 1.1. Section 3 contains the proof of the main energy estimates, which are used in conjunction with a bootstrap argument in the proof of Theorem 1.1. To allow for a concise deduction and presentation of the relevant estimates, we initially make a number of bootstrap assumptions (cf. (53)), which reduce the estimates to a compact form. Namely, we present the conditional decay for the perturbation of geometry and matter directly in terms of the time function. Section 3 is organized as follows. In Section 3.1 we make the bootstrap assumptions which are the basis for all following estimates in the remainder concerning the spherical case. Sections 3.2 and 3.3 treat the elliptic estimates for the trace-free part of the second fundamental form, the conformal factor, the lapse function and the shift vector. Also, we derive estimates for the time-derivative of the lapse function and the shift vector, which appear in the Vlasov equation. In Section 3.4 we use the evolution equation for the conformal factor and the trace-free part of the second fundamental form to derive energy estimates for Sobolev norms of both quantities. These are necessary in addition to the elliptic estimates, since they provide a smallness factor coming from the initial data or the inverse of the initial time, $T_0$, which, by means of Cauchy stability, can then also be turned into a smallness factor. We distinguish between an energy for the rescaled conformal factor and one for its gradient, as the latter is required to be small, while the former is not. This is discussed in more detail in Section 3.4. We then summarize all estimates on the perturbation of the geometry in Section 3.5. Section 3.6 presents all estimates on the relevant matter quantities, the total mass, the bound on the momentum support, the $L^2$-energies for the distribution function and the resulting estimates on the Sobolev norms of the matter quantities which appear in the Einstein equations. The $L^2$-energies for distribution functions have been introduced in [17] and a mechanism of correction has been set up therein. This idea is briefly revisited in Section 3.6 for the sake of completeness, but the computations are essentially similar to [17] and not repeated. These estimates require certain decay properties of the perturbations of the geometry, which are assured by the bootstrap assumptions. Eventually, Section 4 presents the proof of Theorem 1.1.

The proof of Theorem 1.3 is given in Section 5. The main approach is similar to the case of spherical topology, except that a non-trivial evolution of the conformal metric occurs, which requires additional control. For the sake of completeness, we give all relevant estimates and prove those explicitly, which deviate from the case of the sphere.

## 2. The Einstein-flow on the 2-sphere

### *2.1. Notations and elliptic estimates*

We begin with some general notations and collect some standard tools from elliptic theory and Sobolev spaces.

#### 2.1.1. General setup

We consider the manifold $M = [\mathrm{T}_0, \infty) \times \mathbb{S}^2$ for $\mathrm{T}_0 > 0$ and with the standard ADM ansatz for the Lorentzian metric on $M$,

$$^{(3)}g = -N^2 dt \otimes dt + g_{ab}(dx^a + X^a dt) \otimes (dx^b + X^b dt), \tag{10}$$

where $N$ is the lapse function, $X$ the shift vector field and $g$ the physical metric on $\mathbb{S}^2$. We denote by $k$ the second fundamental form. A useful notation is $\hat{X} \equiv X/N$. Furthermore, $\sigma_{\mathbb{S}^2}$ denotes the round metric on $\mathbb{S}^2$ with scalar curvature $R(\sigma_{\mathbb{S}^2}) = 1$ and $D$ be its covariant derivative. The corresponding Laplacian is $\Delta_\sigma$. This implies that the volume of $\mathbb{S}^2$ with respect to $\sigma_{\mathbb{S}^2}$ is $\mathrm{vol}_{\sigma_{\mathbb{S}^2}}(\mathbb{S}^2) = 8\pi$. We define all Sobolev spaces $W^{s,p}$ and $H^k$ and standard Sobolev norms $\|.\|_{H^s}$ on $\mathbb{S}^2$ with respect to the metric $\sigma_{\mathbb{S}^2}$ and its covariant derivative $D$ (cf. [7]). We use $\|.\| = \|.\|_{L^2}$. As $\sigma_{\mathbb{S}^2}$ is fixed, all relevant constants for elliptic regularity estimates and Sobolev embedding up to a chosen order are bounded by a uniform constant denoted by $C$. Further notations and definitions will be introduced throughout the work.

#### 2.1.2. Standard elliptic estimate

The following generalization of the standard elliptic regularity estimate is relevant for elliptic operators with nontrivial kernel as occurring below in particular for the shift vector.

**Lemma 2.1** ([10], p. 463). *Consider a linear, second order elliptic, operator $L$ on a closed manifold $\Sigma$. Let $\ell \geq 0$ be an integer, then there are positive constants $C_1$, $C_2$ s.t.*

$$\|u\|_{H^{\ell+2}} \leq C_1 \|Lu\|_{H^\ell} + C_2 \|u\|_{L^1} \quad \forall u \in H^{\ell+2}(\Sigma). \tag{11}$$

*If in addition one restricts to the set of functions*

$$\{u \,|\, u \text{ is } L^2\text{-orthogonal to } \ker L\}, \tag{12}$$

*then the estimate holds with $C_2 = 0$.*

### 2.2. Geometry of the 2-sphere

In the following some relevant geometric properties of the 2-sphere are discussed, which are used to bring the Einstein equations into a more concise form.

**Lemma 2.2** (cf. [14]).

(i) *$\mathbb{S}^2$ admits no non-trivial TT-tensors, i.e., for a Riemannian metric $g$ on $\mathbb{S}^2$ and some vector field $Y$ the conditions $\mathrm{tr}_g Y = 0$ and $\nabla^i Y_{ij} = 0$ imply $Y \equiv 0$.*

(ii) *Every Riemannian metric $g_t$ on $\mathbb{S}^2$ is conformally equivalent to the canonical metric $\sigma_{\mathbb{S}^2}$ with scalar curvature $R(\sigma_{\mathbb{S}^2}) = 1$, i.e.,*

$$g_t = e^{2\lambda_t} \varphi_t^* \sigma_{\mathbb{S}^2} \tag{13}$$

*for a conformal factor $\lambda_t$ and an arbitrary diffeomorphism $\varphi_t : \mathbb{S}^2 \to \mathbb{S}^2$.*

(iii) *$\mathbb{S}^2$ admits 6 linearly independent conformal Killing vectors $Z^{(A)}$, $A \in \{1, \ldots, 6\}$, i.e.,*

$$D_a Z_b^{(A)} + D_b Z_b^{(A)} - \sigma_{\mathbb{S}^2\, ab} D_c Z^{(A),c} = 0. \tag{14}$$

### 2.3. Nonvacuum Einstein flow on the 2-sphere

We discuss in the following the choice of gauge and the corresponding reduced Einstein equations.

#### 2.3.1. Choice of gauge

The gauge freedom is fixed by additional choices following [14, 26]. The family of diffeomorphisms $(\varphi_t)$ is chosen to be the identity $(\mathrm{id}_{\mathbb{S}^2})$. The second fundamental form is decomposed into $k = h + \tau/2g$, where $h$ is the trace-free part of $k$ and $\tau$ the mean curvature. The mean curvature is chosen to be constant on every spatial slice $\mathbb{S}^2_t$ and the time coordinate is chosen to be $t = -\tau^{-1}$. Finally, the shift vector is chosen orthogonal to the space of conformal Killing vector fields $\{Z^{(A)}\}$.

#### 2.3.2. The reduced Einstein equations

The reduced Einstein equations read

$$D_b h^b_a = -e^{2\lambda} J_a \tag{15}$$

$$2\Delta_\sigma \lambda = e^{2\lambda}/2\tau^2 + 1 - 2e^{2\lambda}\rho - e^{-2\lambda}|h|^2_\sigma \tag{16}$$

$$\Delta_\sigma N = Ne^{2\lambda}\left(e^{-4\lambda}|h|^2_\sigma + \tau^2/2 + \eta\right) - \partial_t \tau e^{2\lambda} \tag{17}$$

$$[L_\sigma n]_{ab} = 2Ne^{-2\lambda} h_{ab}, \tag{18}$$

$$\partial_t \lambda = -\frac{1}{2}\left[N\tau -^g \nabla_c X^c\right] \tag{19}$$

$$\begin{aligned}\partial_t h_{ab} = (N-1)\tau^2/n g_{ab} - \nabla_a\nabla_b N - N(T_{ab} - g_{ab}T)\\ + \mathcal{L}_X h_{ab} + N(Ric_{ab} - 2h_{ai}h^i_b).\end{aligned} \tag{20}$$

tr $T$ denotes the ${}^{(3)}g$-trace of the energy-momentum tensor and $R(g)$ and $Ric$ are scalar and Ricci curvature of $g$, respectively. We use the notation $n_a \equiv X_a e^{-2\lambda}$, where the index is lowered by $g$ and $L_\sigma$ denotes the conformal Killing operator, $[L_\sigma Y]_{ab} \equiv D_a Y_b + D_b Y_a - \sigma_{\mathbb{S}^2\,ab} D_c Y^c$. The matter quantities are the energy density $\rho = N^2 T^{00}$, the matter current $J_a = NT^0_a$ and the pressure $\eta = g^{ab}T_{ab}$. The individual equations are momentum constraint, Hamiltonian constraint, lapse and shift equation as well as the evolution equation for the conformal factor and the trace free part of the second fundamental form, respectively. A solution to (15)–(18) with divergence-free energy momentum tensor solves the original Einstein equations [14]. To allow for a more concise treatment of the evolution, we define the *rescaled conformal factor*

$$\phi^2 \equiv \tau^2 e^{2\lambda}. \tag{21}$$

To obtain a preview on the approximate values of the metric variables in the small data setting considered in the nonlinear stability analysis we refer to the exact background solution (39), (42) (whose perturbations we consider) and the eventual decay rates given in Proposision 3.24.

### 2.4. Vlasov matter

For a detailed introduction to the geometric fundamentals of the Vlasov model as discussed here the reader is referred to [6, 29, 34]. We recall the relevant notations. Moreover, the presentation in this section includes the spatial topology of the torus as there is no substantial difference concerning the setup of the structure for the transport equation. In this section $\Sigma$ denotes either the sphere or the torus.

We consider the tangent bundle of the spacetime $M = [\mathrm{T}_0, \infty) \times \Sigma$, $TM$ and its subset, the mass-shell

$$TM \supset \mathcal{P} \equiv \left\{(x,p) \in TM \,\Big|\, -m^2 = |p|^2_{{}^{(3)}g(x)},\, p^0 > 0\right\}. \tag{22}$$

We set $m = 1$. Let $\mathcal{P}_x$ denote the corresponding fibre over the point $x \in M$ and $\mu_{\mathcal{P}_x}$ the induced volume for on $\mathcal{P}_x$. A distribution function is of the type $\hat{f} \in C^1(\mathcal{P})$ with $\mathrm{im}(\hat{f}) \subset \mathbb{R}_+$. The energy-momentum tensor of a distribution function $\hat{f}$ is given by

$$T_{\alpha\beta}(x) = \int_{\mathcal{P}_x} \hat{f} p_\alpha p_\beta \mu_{\mathcal{P}_x}. \tag{23}$$

Let $\mathfrak{X}$ denote the geodesic flow field on $\mathcal{P}$. In the frame corresponding to horizontal and vertical directions on $TM$, $\{\mathbf{A}_\alpha = \partial_\alpha - p^\mu \Gamma^\nu_{\mu\alpha}\partial_{p^\nu}, \mathbf{B}_a = \partial_{p^a}\}$ the geodesic flow field reads $\mathfrak{X} = p^\alpha \mathbf{A}_\alpha$. The Vlasov equation is then given by $\mathfrak{X}\hat{f} = 0$. For the projection $\pi : TM \to I \times T\Sigma$, we consider the pushed forward Vlasov equation for a function $f : \mathbb{R} \times T\Sigma \to [0,\infty)$ of the form $[\pi_* \mathfrak{X}]f = 0$. In coordinates this equation takes the form

$$p^\alpha \partial_\alpha f - \Gamma^e_{\alpha\beta} p^\alpha p^\beta \partial_{p^e} f = 0, \tag{24}$$

where the Christoffel symbols have the form

$$\begin{aligned} \Gamma^a_{00} &= \partial_t X^a - 2\tau X^a + \tilde{\Gamma}^a_1, \\ 2\Gamma^a_{(b}0) &= -2\tau\delta^a_b + 2\tilde{\Gamma}_{2,b}^{\ a}, \\ \Gamma^a_{bc} &= \Gamma^a_{bc} + \tilde{\Gamma}_{3,bc}^{\ a}. \end{aligned} \tag{25}$$

We made the abbreviations

$$\begin{aligned} \tilde{\Gamma}^a_1 &= (2-N)\tau X^a + X^b\nabla_b X^a - 2Nh^a_c X^c + N\nabla^a N \\ &\quad - \frac{1}{N} X^a(\partial_t N + X^b \nabla_b N - |X|^2_h - \tau/2|X|^2_g) \\ 2\tilde{\Gamma}_{2,b}^{\ a} &= (2-N)\tau\delta^a_b - 2Nh^a_b + 2\nabla_b X^a \\ &\quad - \frac{2}{N} X^a \nabla_b N + \frac{2}{N}(h_{bc} + \tau/2 g_{bc}) X^c X^a \\ \tilde{\Gamma}_{3,bc}^{\ a} &= \frac{1}{N}(h_{bc} + \tau/2 g_{bc}) X^a, \end{aligned} \tag{26}$$

which are all tensors on $\Sigma$. The final form of the Vlasov equation reads

$$\begin{aligned} \partial_t f = &-p^e/p^0 \mathbf{A}_e f - 2\tau p^e \mathbf{B}_e f + p^0(\partial_t X^e - 2\tau X^e)\mathbf{B}_e f \\ &+ p^0 \tilde{\Gamma}^e \mathbf{B}_e f + p^u 2\tilde{\Gamma}^e_u \mathbf{B}_e f + p^a p^b \tilde{\Gamma}^e_{ab} \mathbf{B}_e f. \end{aligned} \tag{27}$$

The form of the equation is adapted to the expected behavior of the terms on the right-hand side. The terms in the first line are leading order, i.e., they are present in the background geometry or in case of the shift vector terms, have too slow decay to be estimated with the perturbative terms. The second line contains all fast decaying terms, which are treated as perturbative terms.

**Definition 2.3.** For a given distribution function $f$, we define the *total mass of a slice* $\Sigma_t$ *with respect to* $f$ by

$$\mathbf{m}_\infty \equiv \int_{T\Sigma} f \mu_{T\Sigma}, \tag{28}$$

where the volume form is given by $\mu_{T\Sigma} = |g| dpdx$.

**Remark 2.4.** In the case $X \equiv 0$, the total mass coincides with the number of particles and is therefore conserved.

#### *2.4.1. The characteristic system*

A simplified representation of the distribution function by the characteristic system for the Vlasov equation,

$$\begin{aligned} \frac{dx^a}{ds} &= P^a, \qquad \frac{dt}{ds} = P^0 \\ \frac{dP^a}{ds} &= -{}^{(3)}\Gamma^a_{\mu\nu} P^\mu P^\nu \end{aligned} \tag{29}$$

for curves $(x, P)(s)$ in $[\mathrm{T}_0, \infty) \times T\Sigma$ with initial data $x(t) = (t, x')$ and $P(t) = p$, is given by

$$f(t, x', p) = f_0((x, P)(0; t, x', p)) \tag{30}$$

where $f_0$ is the restriction of $f$ to the hypersurface given by $\{x^0(s = 0) = 0\}$ and $(x, P)(s; t, x', p)$ is the solution to (29).

#### *2.4.2. Sobolev norms*

We make use of the $L^2$-*energies* introduced in [17], which are partially weighted Sobolev norms of the distribution function with respect to the Sasaki metric on the tangent bundle $T\Sigma$. These norms are the essential tool to control higher Sobolev norms of the energy momentum tensor. The pointwise control and that of the $L^2$-norm result from estimates on the momentum-support and the constancy of the supremum of the distribution function. We recall the definition of the Sobolev norms.

**Definition 2.5.** Let

$$\|f\|_{\mathrm{Vl},s,\mu} \equiv \sqrt{\sum_{0 \le \ell \le s} \int_{T\Sigma} \overline{p}^{2\mu} |\overline{\nabla}^\ell f|^2_{\mathrm{g}} \mu_{T\Sigma}}. \tag{31}$$

We denote the corresponding weighted Sobolev space on $T\Sigma$ by $H_{\mathrm{Vl},s,\mu}(T\Sigma)$. Furthermore, let $H_{\mathrm{Vl,c},s}(T\Sigma)$ denote the subspace with distribution functions of compact momentum support.

Here, $\mathbf{g}$ denotes the Sasaki metric with respect to $g$ on $T\Sigma$, in coordinates $\mathbf{g}_{ab} = g_{ab}dx^a \otimes dx^b + g_{ab}Dp^a \otimes Dp^b$ for $Dp^i = dp^i + p^j\Gamma^i_{jk}dx^k$ with covariant derivative $\overline{\nabla}$ and $\overline{p} = \sqrt{1+e^{2\lambda}|p|_g^2}$, $\mu \in \mathbb{R}$. $\widehat{\mathbf{g}} = \tau^2 g_{ab}dx^a \otimes dx^b + (\omega t^{-4} + |p|_\sigma^2)^{-1}\tau^2 g_{ab}Dp^a \otimes Dp^b$ denotes the weighted Sasaki metric, with parameter $0 < \omega \leq \mathbb{R}$. The factors $\tau^2$ have been chosen so that the resulting metric $\tau^2 g_{ab}$ is essentially constant in time in case of the sphere or constant up to a perturbation in case of the torus. The volume form on $T\Sigma$ is $\mu_{T\Sigma} = |g|d^2x \wedge d^2p$.

### 2.4.3. Energy density, pressure, energy flux

We recall the explicit form of the energy momentum tensor as it appears in the Einstein equations.

$$\begin{aligned} \rho(f) &= N^2 \int_{T_x\Sigma} f\frac{(p^0)^2}{\hat{p}}\sqrt{g}dp, \quad J_a(f) = N\int_{T_x\Sigma} f\frac{p_a p^0}{\hat{p}}\sqrt{g}dp, \\ \eta(f) &= \int_{T_x\Sigma} f\frac{|p+p^0X|_g^2}{\hat{p}}\sqrt{g}dp, \quad T_{ab} = \int_{T_x\Sigma} f\frac{p_a p_b}{\hat{p}}\sqrt{g}dp. \end{aligned} \tag{32}$$

In the remainder, we denote the spatial components of the energy-momentum tensor by $T$ or $T_{ab}$. Let $\hat{X} = X/N$, then the following relations hold.

$$p^0 = N^{-1}(1-|\hat{X}|_g^2)^{-1}\left[\hat{X}_j p^j + \sqrt{(\hat{X}_j p^j)^2 + (1-|\hat{X}|_g^2)(1+|p|_g^2)}\right], \tag{33}$$

$$p^0 = \frac{1}{N}\frac{1}{\hat{p} - \langle \hat{X}, p\rangle_g}(1+|p|_g^2), \tag{34}$$

where

$$\hat{p} = \sqrt{(\hat{X}_j p^j)^2 + (1-|\hat{X}|_g^2)(1+|p|_g^2)}. \tag{35}$$

In addition, $p_0 = -N\hat{p}$. We define a bound on the rescaled momentum-support.

**Definition 2.6.** Consider a distribution function of compact momentum-support. Then, we define the rescaled volume of the momentum-support by

$$\mathbf{P}_\infty(t) \equiv \sup_{x\in\Sigma}\left\{\int_{T_x\Sigma\cap \operatorname{supp} f(t,x,.)} e^{2\lambda}\sqrt{g}dp\right\}. \tag{36}$$

**Remark 2.7.** Note, that the definition of $\mathbf{P}_\infty$ contains an overall factor of $e^{4\lambda}$, which is essentially $t^4$. This factor is absorbed by the expected decay of the volume of the momentum-support,

$$\sup_{x\in\Sigma} \operatorname{vol}(\operatorname{supp} f(t,x,.)) \approx t^{-4}. \tag{37}$$

In combination, this implies that $\mathbf{P}_\infty$ is of the order of a constant. This is rigorously inferred in Corollary 3.32.

In the following lemma we estimate the components of the energy-momentum tensor, as they appear in the Einstein equations, in terms of the $L^2$-energies of the distribution function and the bound on the momentum support.

**Lemma 2.8** (Estimates for matter quantities). *Let $\|\hat{X}\|_{H^4} \leq c < \infty$ for some $c$ sufficiently small, then*

$$\begin{aligned} \|e^{2\lambda}\rho(f)\|_{H^4} + \|e^{2\lambda}\jmath(f)\|_{H^4} &\leq C\|f\|_{\mathrm{Vl},4,1}, \\ \|e^{2\lambda}\eta(f)\|_{H^4} &\leq C(\tau^2\|\phi^{-2}\|_{H^4} + \|\hat{X}\|_{H^3})\|f\|_{\mathrm{Vl},4,2}, \\ \|T\|_{H^4} &\leq C\tau^2\|\phi^{-2}\|_{H^4}\|f\|_{\mathrm{Vl},4,2}, \end{aligned} \tag{38}$$

*where $C = C(c, \|\nabla\lambda\|_{H^3}, \mathbf{P}_\infty)$.*

*Proof.* These estimates are a straightforward consequence of the structure of $\rho$, $\jmath$, $\eta$ and $T$ and the energies $\|.\|_{\mathrm{Vl},k}$. We also use $\|u \cdot v\|_{H^k} \leq C\|u\|_{H^k}\|v\|_{H^k}$ for $k \geq 2$ for functions $u, v$. □

### 2.5. Homogeneous future complete solutions

We consider in the following the homogeneous Einstein–Vlasov system on $[\mathrm{T}_0, \infty) \times \mathbb{S}^2$ and begin by defining isotropic distribution functions.

**Definition 2.9.** Let $g_t$ be a Riemannian metric on $\mathbb{S}^2_t$, then a distribution function $f : [\mathrm{T}_0, \infty) \times T\mathbb{S}^2 \to [0, \infty)$ is $g_t$-**isotropic**, iff $f = f(t, |p|_{g_t})$.

The following lemma implies that all relevant matter quantities are spatially constant for isotropic distribution functions.

**Lemma 2.10.** *Let $X \equiv 0$ and $N = N(t)$. Then, if $f$ is a $g_t$-isotropic distribution function the following holds.*

(i) $\mathbf{A}_a f = 0$, *where $\mathbf{A}_a$ is the horizontal lift of $\partial_a$ with respect to $g_t$.*
(ii) $\partial_a \rho(f) = 0$,
(iii) $\jmath_a(f) = 0$,
(iv) $\partial_a \eta(f) = 0$.

*Proof.* This follows from direct computations. Note that the volume form, $\sqrt{g}dp$ yields a terms containing a Christoffel symbol, when the derivative is taken. This term is manipulated via integration by parts in $p$. □

**Lemma 2.11** (Homogeneous initial data). *Initial data of the form*

$$(h, e^{2\lambda}, N, X, f) = \left(0, \frac{2}{4\rho - \tau^2}, \frac{\tau^2}{\tau^2/2 + \eta}, 0, f_0(|p|_g)\right) \tag{39}$$

*on $\mathbb{S}^2$, where $f$ is a $g = e^{2\lambda}\sigma$-isotropic distribution function and $\rho(f_0) > \tau^2/4$, exists and solves the constraint equations (15), (16) and the elliptic system (17), (18).*

*Proof.* We first show that data of the form above solves equations (15)–(18). The $g$-isotropy of $f_0$ implies that the matter current vanishes, $\jmath(f_0) = 0$. In turn, the momentum constraint implies that $h$ is a $TT$-tensor and consequently vanishes, cf. Lemma 2.2. In particular, the momentum constraint holds.

With the vanishing shift vector, the shift equation holds as the right-hand side vanishes. The right-hand side of the lapse equation is constant on the slice $\mathbb{S}^2$, so the equation can be solved by the given constant for the lapse, where the conformal factor appears only as a common factor. Finally, the Hamiltonian constraint can be solved uniquely by the given term under the condition $\rho(f_0) > \tau^2/4$.

Concerning the existence of such initial data we have to note that the energy density and the pressure contain factors of the form $e^{2\lambda}$ in the density $\sqrt{g}$. In the previous construction we only used that $f_0$ is $g$-isotropic and that $\rho(f_0) > \tau^2/4$. It remains to show that there is in fact a $f_0$ with these features. We first fix a number $v^* > 0$ , and consider the positive solution $u_+$ to the equation

$$u^2 v^* - u\tau^2/2 - 1 = 0, \tag{40}$$

Then $u_+ \cdot v^* > \tau^2/4$, since $v^*$ is positive. We set now $e^{2\lambda} = u_+$. We choose now a $\sigma$-isotropic function $f_0$ such that

$$v^* = 2\int_{T_x\mathbb{S}^2} \sqrt{1 + u_+|p|^2_\sigma} f_0 \sqrt{\sigma}\, dp. \tag{41}$$

The choice of $f_0$ is not unique, but it suffices to pick one possible $f_0$ to obtain existence. As the conformal factor is spatially constant and as $f_0$ is $\sigma$-isotropic, it is automatically $g = e^{2\lambda}\sigma$-isotropic. In addition, with $\rho(f_0) = e^{2\lambda}v^* = u_+v^* > \tau^2/4$ by construction. □

**Proposition 2.12** (Future complete homogeneous solutions). *Homogeneous initial data of the form* (39) *with smooth isotropic $f_0$ has a future development of the form*

$$g_{\text{hom}} = -\left(\frac{2\tau^2}{\tau^2 + 2\eta}\right)^2 dt^2 + \frac{2\tau^2}{4\rho(f) - \tau^2} \cdot t^2 \sigma_{\mathbb{S}^2}, \tag{42}$$

*on* $[\mathrm{T}_0, \infty) \times \mathbb{S}^2$, *which is time- and null-geodesically complete in future direction. Asymptotically,*

$$\begin{aligned} \lim_{t\to\infty} \frac{2\tau^2}{\tau^2 + 2\eta} &= 2 \\ \lim_{t\to\infty} \frac{2\tau^2}{4\rho(f) - \tau^2} &= \frac{\mathbf{m}_\infty - 4\pi}{2\pi}. \end{aligned} \tag{43}$$

*Proof.* In the homogeneous and isotropic case, the set of equations reduces to

$$\begin{aligned} 0 &= e^{2\lambda}/2\tau^2 + 1 - 2e^{4\lambda}\tilde{\rho}(f) \\ 0 &= N\left(\tau^2/2 + \eta(f)\right) - \partial_t \tau \\ \partial_t f &= -N\tau p^e \mathbf{B}_e f, \end{aligned} \tag{44}$$

where $\tilde{\rho} = e^{-2\lambda}\rho$. Given functions $e^{2\lambda}$ and $N$ on a time interval the Vlasov equation can be rewritten for a function $f = f(t, |p|_\sigma)$ to

$$\partial_t f = -N\tau e^\lambda |p|_\sigma f', \tag{45}$$

where $f'$ is the derivative with respect to the second variable. The system (44), (45) is a simplified version of an elliptic-hyperbolic system coupled to a transport equation, which has a local-in-time unique solution given sufficiently regular initial data (cf. [18]). In particular, the system (44), (45) has a unique smooth solution on a short time interval $[\mathrm{T}_0, \mathrm{T}_1]$ given smooth initial data $f_0$ at $\mathrm{T}_0$. Global existence for these solutions follows straightforward, since lapse and the conformal factor are given explicitly and remain bounded with the asymptotics given above. For the distribution function a simplified version of the energy estimates in Corollary 3.35 holds, which proves that they remain uniformly bounded as well and have the desired decay properties. By the standard criterion [16] future completeness for these solutions holds.

The asymptotic behavior in (43) follows immediately for the first relation as $\eta$ decays in time. For the second relation we use the Hamiltonian constraint integrated over $\mathbb{S}^2$, which by the Gauss-Bonnet theorem implies

$$\tau^2/2\mathrm{vol}_g(\mathbb{S}^2) + 4\pi\chi(\mathbb{S}^2) - 2\int_{\mathbb{S}^2} \rho\mu_g = 0, \tag{46}$$

where $\chi(\mathbb{S}^2) = 2$ is the Euler characteristic. Asympotically for $t \to \infty$, this yields

$$\tau^2/2\mathrm{vol}_g(\mathbb{S}^2) + 8\pi - 2\mathbf{m}_\infty \to 0, \tag{47}$$

which implies the second relation in (43). □

**Remark 2.13.** Proposition 2.12 yields a large family of solutions—one for each sufficiently regular non-vanishing initial datum $f_0$, which realizes the bound $\mathbf{m}_\infty > 4\pi$. The asymptotics of the geometry of the spacetimes are however uniquely determined by the total mass $\mathbf{m}_\infty(\Sigma)$. For further details on this class of solutions we refer to [19].

## 2.6. Local well-posedness

The required lemma for the local existence theory is given in the following. We define a relevant notation before.

**Definition 2.14.** Let $\mathcal{B}_R^{(\ell)}((\lambda, h, N, X, f))$ denote the open ball of radius $R > 0$ in the function space $H^\ell \times H^{\ell-1} \times H^{\ell+1} \times H^\ell \times H_{\mathrm{Vl},\ell-1}$ centered at $(\lambda, h, N, X, f)$.

**Lemma 2.15 (Local well-posedness).** *Consider a homogeneous solution* $(g_{\mathrm{hom}} = (\lambda_{\mathrm{hom}}, 0, N_{\mathrm{hom}}, 0), f_{\mathrm{hom}})$ *of the type constructed in Proposition 5.1. Let* $\mathrm{T}_0 > 0$.

1. *There exists a* $\delta_{loc} > 0$ *such that for CMC-initial data* $(\lambda_0, h_0, N_0, X_0, f_0)$ *with*

$$(\lambda_0, h_0, N_0, X_0, f_0) \in \mathcal{B}^{(5)}_{\delta_{loc}}((g_{\mathrm{hom}}, f_{\mathrm{hom}})(\mathrm{T}_0)) \tag{48}$$

*there exists a* $\mathrm{T}_1 > \mathrm{T}_0$ *and a unique solution to the Einstein–Vlasov system* (15)–(20), (27), $(\lambda, h, N, X, f) \in H^5 \times H^4 \times H^6 \times H^5 \times H_{\mathrm{Vl},4}$ *on* $[\mathrm{T}_0, \mathrm{T}_1) \times \mathbb{S}^2$, *that coincides with the initial data in* $\mathrm{T}_0$, *such that the shift vector field is orthogonal to the space of Killing vector fields on* $\sigma_{\mathbb{S}^2}$.

2. *Let* $T_+$ *be the maximal time of existence of the solution. Then either* $T_+ = \infty$ *or*

$$\limsup_{t\to T_+} \|(\lambda, h, N, X)(t) - g_{\mathrm{hom}}(t)\|_5 + \|\eta(f) - \eta(f_{\mathrm{hom}}), \jmath(f) - \jmath(f_{\mathrm{hom}})\|_4 \geq 2\delta_{loc}. \tag{49}$$

3. *For every* $T_2 > T_0$ *and* $d > 0$ *there exists an* $\varepsilon = \varepsilon(T_2, d) > 0$ *such that every solution* $(\lambda, h, N, X, f)$ *with initial data* $(\lambda_0, h_0, N_0, X_0, f_0) \in \mathcal{B}_\varepsilon^{(5)}((g_{\rm hom}, f_{\rm hom})(T_0))$ *exists on the interval* $[T_0, T_2]$ *and*

$$(\lambda(T_2), h(T_2), N(T_2), X(T_2), f(T_2)) \in \mathcal{B}_d^{(5)}((g_{\rm hom}, f_{\rm hom})(T_2)). \quad (50)$$

**Remark 2.16.** The proof of the previous lemma is analogous to the case considered in [18]. Some adaptions are required, which result from the different background geometry considered in the present case. These issues only concern the geometry and have been discussed in [14], in particular that it is necessary to choose the shift vector field orthogonal to the space of conformal Killing fields of $\sigma_{\mathbb{S}^2}$. We note in particular that the smallness of the matter quantities discussed in [18] concerns the pressure $\eta$ and the energy flux $\jmath$ and does allow for a large energy density $\rho$ as considered here (cf. Remark 5.6 in [18]), by choosing the momentum support with a sufficiently small upper bound, which does not affect the energy density as $f$ can still be chosen sufficiently large to realize the lower bound (4).

## 3. Energy estimates

In view of the nonlinear stability problem for the homogeneous solutions of Proposition 2.12 we require a number of a priori estimates for general solutions to the system (16)–(20) with initial data close to that induced by a fixed homogeneous solution. These estimates will be derived in the following. To allow for a more concise presentation we impose a set of bootstrap assumptions on the data, which are consistent with the behavior of data close to the homogeneous solutions. We derive a number of estimates on the perturbation, which, due to the bootstrap assumptions, take a concise form. These conditional estimates will eventually improve the assumptions, which allows for a closure of the continuity argument in the proof of Theorem 1.1, which is presented in Section 4.

### *3.1. Assumptions*

Throughout this section we consider a fixed solution, $(h, \lambda, N, X, f)$ on $[T_0, T_1) \times \mathbb{S}^2$ for $T_1 > T_0$. Without loss of generality we assume $T_0 > 1$. Moreover, we define an indicator function for the mass, which measures the positivity, necessary for the solvability of the Hamiltonian constraint, by

$$\delta(t) \equiv \frac{\mathbf{m}_\infty(t)}{4\pi} - 1. \quad (51)$$

Recall that solutions of the type (42) require $\mathbf{m}_\infty - 4\pi > 0$, which is equivalent to (51). We also recall that $\mathrm{vol}_{\sigma_{\mathbb{S}^2}}(\mathbb{S}^2) = 8\pi$. In addition, we consider initial data with $\delta(T_0) \geq 1$.

**Remark 3.1.** We choose this lower bound on $\delta(T_0)$, however, every positive number would work similarly. Therefore this does not restrict the initial data we consider, but makes the presentation somewhat easier. We proceed with these assumptions without loss of generality.

We also define the energy of the rescaled conformal factor and the related energy without the $L^2$-component by

$$\begin{aligned} E_0(\phi) &\equiv \|\phi^2\|_{H^4}^2, \\ E(\phi) &\equiv \|D\phi^2\|_{H^3}^2. \end{aligned} \tag{52}$$

We formulate the bootstrap assumptions.

### 3.1.1. Bootstrap assumptions

We assume that the following estimates hold for $t \in [\mathrm{T}_0, \mathrm{T}_1]$.

$$\begin{aligned} E_0(\phi) &\le 10 \cdot (8\pi)^2 \delta(t)^2 \\ \|\phi^2 - 2\delta(t)\|_\infty &\le \delta(t) \\ \|\phi^{-2}\|_{H^4} &\le 10 \cdot (8\pi)^2 \\ \mathbf{P}_\infty(f)[t] &\le C_1 \\ \|e^{2\lambda}\rho(f)\|_{H^4} + \|e^{2\lambda}J(f)\|_{H^4} &\le C_{\mathrm{Vl}} \\ \|e^{2\lambda}\eta(f)\|_{H^4} &\le \tau^2 C_{\mathrm{Vl}} \\ \delta(t) &> \frac{1}{2} \\ |\mathbf{m}_\infty(t) - \mathbf{m}_\infty(\mathrm{T}_0)| &\le \frac{\mathbf{m}_\infty(\mathrm{T}_0)}{10} \\ \|\dot{N}\|_{H^4} &\le C_1 \cdot t^{-2} \\ \|\dot{X}\|_{H^4} &\le C_1 \cdot t^{-2} \end{aligned} \tag{53}$$

Here, $C_1$ is some fixed positive constant. Note, $C_{\mathrm{Vl}}$ is chosen large with respect to the initial data of the distribution function. We choose $C_{\mathrm{Vl}} \ge 10 \cdot \|f_0\|_{\mathrm{Vl},4}$. As an immediate consequence of the assumptions we obtain the bounds

$$\begin{aligned} &\phi^2 > \frac{1}{2}, \\ &\frac{1}{2} < \delta(t) \le \delta(\mathrm{T}_0) + \frac{1}{4\pi}\frac{\mathbf{m}_\infty(\mathrm{T}_0)}{10}. \end{aligned} \tag{54}$$

## 3.2. The constraints

In this subsection, we derive the relevant elliptic estimates from the constraint equations.

### 3.2.1. High derivatives of h

The momentum constraint yields an estimate for derivatives of $h$. Consider

$$D_b h^b_a = -e^{2\lambda} J_a. \tag{55}$$

In general, $h$ decomposes into a TT-tensor $q$ and a conformal Lie derivative. Since there are no TT-tensors on $\mathbb{S}^2$ except the zero tensor, we have

$$h_{ab} = D_a Y_b + D_b Y_a - \sigma_{\mathbb{S}^2\,ab} D_c Y^c \tag{56}$$

for some vector field $Y^c$ on $\mathbb{S}^2$. Taking the covariant derivative and the trace yields

$$\Delta_\sigma Y_b + \frac{1}{2}R(\sigma_{\mathbb{S}^2})Y_b = -e^{2\lambda} J_a. \tag{57}$$

Since there are no non-trivial TT tensors on $\mathbb{S}^2$ the tracefree solution to the momentum constraint is unique and Lemma 2.1 applied to (57), where $Y$ is chosen orthogonal to conformal Killing fields, which yields uniqueness, implies that $h$ fulfills an estimate of the following form.

**Lemma 3.2.** *For $k \in \mathbb{Z}_+$,*

$$\|h\|_{H^k} \le C_\sigma \|e^{2\lambda} J\|_{H^{k-1}}. \tag{58}$$

**Corollary 3.3.** *If the assumptions (53) hold, then*

$$\|h\|_{H^4} \le C_\sigma C_{\mathrm{Vl}}. \tag{59}$$

#### 3.2.2. Conformal factor

We derive estimates on the conformal factor. For derivatives up to fourth order in $L^2$ we use the evolution equation in the following section (cf. Corollary 3.19), while here we use the Hamiltonian constraint

$$2\Delta_\sigma \lambda = \phi^2/2 + 1 - 2e^{2\lambda}\rho - \tau^2\phi^{-2}|h|^2_\sigma. \tag{60}$$

Elliptic regularity (cf. [7]) implies the following lemma.

**Lemma 3.4.** *Let $\bar\lambda$ denote the mean value of $\lambda$, $\bar\lambda = \mathrm{vol}_\sigma(\mathbb{S}^2)^{-1} \cdot \int \lambda\mu_\sigma$. Then,*

$$\|\lambda - \bar\lambda\|_{H^5} \le C_\sigma\left(\|\phi^2/2 + 1 - 2e^{2\lambda}\rho - \tau^2\phi^{-2}|h|^2_\sigma\|_{H^3}\right) + C\|\lambda - \bar\lambda\|. \tag{61}$$

Using the Poincaré inequality for $\lambda - \bar\lambda$ (cf. [15], Section 8.2), yields

$$\|\lambda - \bar\lambda\| \le I_\sigma \|D\lambda\| = I_\sigma/2\|\phi^{-2}D\phi^2\|, \tag{62}$$

where $I_\sigma$ is the inverse of the first positive eigenvalue of $-\Delta_\sigma$, i.e., a constant of type $C_\sigma$, yields

**Corollary 3.5.** *Under the assumption (53),*

$$\|\lambda - \bar\lambda\|_{H^5} \le C(C_{\mathrm{Vl}}, C_\sigma)[1 + \tau^2]. \tag{63}$$

*Proof.* Beside assumptions (53) the estimate for $h$, (59) is used. □

### 3.3. The elliptic system

We prove the essential a priori estimates for the solutions to the elliptic system

$$\begin{aligned}\Delta_\sigma N &= N\left(\tau^2\phi^{-2}|h|^2_\sigma + \phi^2/2 + e^{2\lambda}\eta\right) - \phi^2\\ [L_\sigma n]_{ab} &= 2N\tau^2\phi^{-2}h_{ab}\end{aligned} \tag{64}$$

in the following.

### 3.3.1. Lapse equation

The maximum principle applied to the lapse equation implies $0 < N \leq 2$. Let $K > 0$ denote a fixed positive constant, which we choose explicitly further below. We then rewrite the lapse equation to

$$\Delta_\sigma(2-N) - K\cdot(2-N) = -N\left(\phi^{-2}\tau^2|h|^2_\sigma + e^{2\lambda}\eta\right) + (2-N)(\phi^2/2-K), \tag{65}$$

where $K$ as above. Then, at a minimum of $N$ we have

$$0 \geq K(2-N) - N\left(\phi^{-2}\tau^2|h|^2_\sigma + e^{2\lambda}\eta\right) + (2-N)(\phi^2/2-K). \tag{66}$$

This is

$$N\left(\phi^{-2}\tau^2|h|^2_\sigma + e^{2\lambda}\eta\right) \geq K\cdot(2-N) + (2-N)(\phi^2/2-K). \tag{67}$$

Under the condition $(\phi^2/2-K)\geq 0$ for all points on $\mathbb{S}^2$, this implies

$$N\left(\phi^{-2}\tau^2|h|^2_\sigma + e^{2\lambda}\eta\right) \geq K\cdot(2-N) \tag{68}$$

at a minimum of $N$ and thereby for all points on $\mathbb{S}^2$, when we replace the left-hand side by its supremum. In particular, in combination with the upper bound on $N$ we deduce

**Lemma 3.6.** *For $K > 0$ s.t. $(\phi^2/2-K)\geq 0$ for all points on $\mathbb{S}^2$,*

$$2-N \leq 2K^{-1}\left[\tau^2\|\phi^{-2}\|_\infty \||h|^2_\sigma\|_\infty + \|e^{2\lambda}\eta\|_\infty\right]. \tag{69}$$

In particular, in view of the first bound in (54) an admissible constant is $K = 1/8$ and we infer

**Corollary 3.7.** *Under the assumption* (53)

$$2-N \leq C(C_\sigma, C_{\mathrm{VI}})\cdot t^{-2}. \tag{70}$$

For higher derivatives of $N$ the following estimate holds.

**Lemma 3.8.**

$$\begin{aligned}\|2-N\|_{H^6} \leq C_\sigma\Bigg[&\tau^2\|\phi^{-2}\|_{H^4}\||h|^2_\sigma\|_{H^4} + \|e^{2\lambda}\eta\|_{H^4}\\ &+\left[1+\tau^2\|\phi^{-2}\|_{H^4}\||h|^2_\sigma\|_{H^4} + \|\phi^2/2-K\|_{H^4}\right]\\ &\cdot\Big[\tau^2\|\phi^{-2}\|_{H^4}\||h|^2_\sigma\|_{H^4} + \|e^{2\lambda}\eta\|_{H^4}\\ &+\Big(1+\tau^2\|\phi^{-2}\|_{H^4}\||h|^2_\sigma\|_{H^4} + \|e^{2\lambda}\eta\|_{H^4}\\ &+\|\phi^2/2-K\|_{H^4}\Big)\cdot\Big(\|2-N\| + \tau^2\|N\phi^{-2}|h|^2_\sigma\|\\ &+\|Ne^{2\lambda}\eta\| + \|(2-N)(\phi^2/2-K)\|\Big)\Big]\Bigg]\end{aligned} \tag{71}$$

*with $K$ as above.*

**Corollary 3.9.** *Under the assumptions* (53)

$$\|DN\|_{H^5} \leq C(C_1, C_\sigma, C_{\mathrm{Vl}}) \cdot t^{-2}. \tag{72}$$

### 3.3.2. Shift equation

We consider the equation for the rescaled shift vector,

$$[L_\sigma n]_{ab} = D_a n_b + D_b n_a - \sigma_{ab} D_c n^c = 2N\tau^2 \phi^{-2} h_{ab}, \tag{73}$$

where $n_a = X_a e^{-2\lambda}$. Taking the covariant derivative and the trace in the first index, this implies

$$\Delta_\sigma n_b + \frac{1}{2} n_b = 2D^a (N\tau^2 \phi^{-2} h_{ab}). \tag{74}$$

The kernel of the elliptic operator acting on $n$ is non-zero on the sphere, but the condition

$$0 = \int_{\mathbb{S}^2} \langle X, Z^{(A)} \rangle_\sigma \mu_\sigma, \quad A \in \{1, \ldots, 6\}, \tag{75}$$

assures uniqueness for $X$ and thereby Lemma 2.1 implies

**Lemma 3.10.**

$$\|n\|_{H^k} \leq C_\sigma \tau^2 \|N\|_{H^{k-1}} \|\phi^{-2}\|_{H^{k-1}} \|h\|_{H^{k-1}} \tag{76}$$

*for* $k \geq 3$.

**Corollary 3.11.** *If the assumptions* (53) *hold, then*

$$\|X\|_{H^5} \leq C(C_\sigma, C_1, C_{\mathrm{Vl}}) \cdot t^{-2}. \tag{77}$$

### 3.3.3. Time-derivatives of lapse and shift

The following estimates hold for the time derivatives of lapse and shift. The time derivatives of the lapse and shift are solutions to the following elliptic system.

$$\begin{aligned} \Delta_\sigma \dot{N} - (\phi^2/2)\dot{N} &= -\partial_t(\phi^2/2)(2 - N) + \dot{N}(\phi^{-2}\tau^2 |h|^2_\sigma + e^{2\lambda}\eta) \\ &\quad + N\partial_t(\phi^{-2}\tau^2 |h|^2_\sigma + e^{2\lambda}\eta) \\ &\equiv F_{\dot{N}} \end{aligned} \tag{78}$$

$$\begin{aligned} D_a \dot{n}_b + D_b \dot{n}_a - \sigma_{ab} D_c \dot{n}^c &= 2\Big[ -2N(\partial_t(\phi^{-2})\tau^2 + \phi^{-2}\partial_t \tau^2) h_{ab} \\ &\quad + \dot{N}\phi^{-2}\tau^2 h_{ab} + N\phi^{-2}\tau^2 \partial_t h_{ab} \Big] \\ &\equiv F_{\dot{n}} \end{aligned} \tag{79}$$

**Proposition 3.12.** *Consider a set of solutions* $(h, \lambda, N, X, \eta) \in H^4 \times H^5 \times H^5 \times H^5 \times H^3$. *Then the time derivative of the lapse* $\dot{N}$ *and of the rescaled shift vector* $\dot{n}$ *fulfill the following estimates.*

$$\begin{aligned} \|\dot{N}\|_\infty &\leq 2\|\phi^{-2}\|_\infty \|F_{\dot{N}}\|_\infty \\ \|\dot{N}\|_{H^4} &\leq C\left[\phi^2\right] C_\sigma \|F_{\dot{N}}\|_{H^2} \\ \|\dot{n}\|_{H^4} &\leq C_\sigma \|F_{\dot{n}}\|_{H^3} \end{aligned} \tag{80}$$

*where the constant $C$ depends on $\phi^2 = e^{2\lambda}\tau^2$ such that $0 < c_1 < e^{2\lambda}\tau^2 < c_2 < \infty$ implies $C$ is bounded.*

*Proof.* The first estimate results from an application of the maximum principle. The second estimate follows from standard elliptic regularity and the third estimate is analog to that for the shift equation, as the elliptic operators are identical. □

**Corollary 3.13.** *If the assumptions (53) hold, then*

$$\begin{aligned} \|\dot{N}\|_\infty &\leq C(C_1, C_\sigma, C_{\mathrm{Vl}}) \cdot t^{-3} \\ \|\dot{N}\|_{H^4} &\leq C(C_1, C_\sigma, C_{\mathrm{Vl}}) \cdot t^{-3} \\ \|\dot{n}\|_{H^4} &\leq C(C_1, C_\sigma, C_{\mathrm{Vl}}) \cdot t^{-3} \end{aligned} \tag{81}$$

*Proof.* The estimates are a straightforward consequence of the preceding proposition. The only important step is improving the decay for the shift vector before that of the lapse. As the lapse contains the time derivative of the distribution function, which in turn has to be replaced using the Vlasov equation—the resulting term containing the time derivative of the shift vector would prevent one from improving the bootstrap assumption for the lapse. With the optimal decay of the time derivative of the shift vector this does not occur. □

### 3.4. Energy estimates for the evolution equations

In this section we use the evolution equations (19) and (20) to derive a number of energy estimates for the energies $E(\phi)$ and $E_0(\phi)$ and deduce a pointwise estimate for $\phi^2$. Moreover we derive an energy estimate for the $H^4$-norm of $h$. These estimates are necessary, as they include smallness factors resulting from smallness at $\mathrm{T}_0$ or a sufficiently small $\mathrm{T}_0^{-1}$, which can be achieved by using the Cauchy stability argument in the eventual continuity argument.

#### 3.4.1. Pointwise estimates for the rescaled conformal factor

We consider the rescaled conformal factor, $\phi^2 = \tau^2 e^{2\lambda}$. The following adapted evolution equation holds.

$$\partial_t \phi^2 = \left[(2 - N)\tau + {}^g\nabla_c X^c\right]\phi^2 \tag{82}$$

Consider the $H^3$-norm of $D\phi^2$. We compute the time derivative

$$\begin{aligned} \partial_t \|D\phi^2\|^2_{H^3} &= \sum_{k=1}^4 2\int_{\mathbb{S}^2} \sigma_{\mathbb{S}^2}^{a_1 b_1}..\sigma_{\mathbb{S}^2}^{a_k b_k} D_{a_1}..D_{a_k}(\phi^2) D_{b_1}..D_{b_k}\partial_t(\phi^2)\mu_\sigma \\ &= \sum_{k=1}^4 2\int_{\mathbb{S}^2} \sigma_{\mathbb{S}^2}^{a_1 b_1}..\sigma_{\mathbb{S}^2}^{a_k b_k} D_{a_1}..D_{a_k}\phi^2 D_{b_1}..D_{b_k}(\left[(1 - N/2)\tau + {}^g\nabla_c X^c\right]\phi^2)\mu_\sigma \\ &\leq 2\|\phi^2\|^2_{H^4}\|(1 - N/2)\tau + {}^g\nabla_c X^c\|_{H^4}, \end{aligned} \tag{83}$$

where $a_i, b_i \in \{1, 2\}$ are spatial indices. Recall,

$$E(\phi) \equiv \|D\phi^2\|^2_{H^3} \text{ and } E_0(\phi) \equiv \|\phi^2\|^2_{H^4}. \tag{84}$$

**Lemma 3.14** (Energy estimate).

$$|\partial_t E(\phi)| \le 2\|(1-N/2)\tau + {}^g\nabla_c X^c\|_{H^4} E_0(\phi) \tag{85}$$

**Remark 3.15.** We note that the coefficient on the right-hand side of the previous energy estimate again contains third derivatives of $\lambda$ in $L^2$. These are however multiplied with eventually decaying shift-vector terms. In addition, we have the full energy $E_0$ on the right-hand side including the $L^2$ term. Also this term will be bootstrapped and eventually smallness of $E(\phi)$ can therefore be established under the present conditions by beginning at a sufficiently large initial time $T_0$.

**Corollary 3.16.** *Under the assumption* (53)

$$|\partial_t E(\phi)| \le C(C_1, C_\sigma, C_{\mathrm{V1}}) E_0(\phi) \cdot t^{-2}. \tag{86}$$

*Therefore*

$$E(\phi(t)) \le E(\phi(\mathrm{T}_0)) + C(C_1, C_\sigma, C_{\mathrm{V1}}) \cdot \mathrm{T}_0^{-1}. \tag{87}$$

We need to estimate the $L^2$-norm of $\phi^2$ in the following. We proceed by defining the mean value with respect to $\sigma_{\mathbb{S}^2}$, by

$$\overline{\phi^2} \equiv \frac{1}{8\pi}\int_{\mathbb{S}^2} \phi^2 \mu_{\sigma_{\mathbb{S}^2}}. \tag{88}$$

From the Hamiltonian constraint we obtain by integration,

$$0 = \overline{\phi^2} + 2 - \frac{4\varrho}{8\pi} - \frac{2\tau^2}{8\pi}\int_{\mathbb{S}^2} \phi^{-2}|h|^2_\sigma \mu_\sigma, \tag{89}$$

where $\varrho \equiv \int e^{2\lambda}\rho\mu_\sigma$. This, in turn, can be reformulated to

$$\overline{\phi^2} = 2\Big(\frac{\mathbf{m}_\infty}{4\pi} - 1\Big) + R_{\phi^2} = 2\delta(t) + R_{\phi^2} \tag{90}$$

where

$$R_{\phi^2} = \frac{4(\varrho - \mathbf{m}_\infty)}{8\pi} + \frac{2\tau^2}{8\pi}\int_{\mathbb{S}^2} \phi^{-2}|h|^2_\sigma \mu_\sigma \tag{91}$$

is a perturbation term. In addition, we quote the Poincaré inequality

$$\|\phi^2 - \overline{\phi^2}\| \le I_\sigma \|D\phi^2\|, \tag{92}$$

where $I_\sigma$ is a constant. In combination this yields

$$\|\phi^2 - \overline{\phi^2}\|_{H^2} \le (1 + I_\sigma)\|D\phi^2\| + \|D^2\phi^2\|. \tag{93}$$

With Sobolev embedding, we infer

$$\|\phi^2 - \overline{\phi^2}\|_\infty \le C_\sigma\Big((1 + I_\sigma)\|D\phi^2\| + \|D^2\phi^2\|\Big). \tag{94}$$

With the previous notation and in combination with the expression for the mean value this gives

**Lemma 3.17.**

$$\|\phi^2 - 2\delta(t)\|_\infty \leq C_\sigma(2 + I_\sigma)E(\phi) + |R_{\phi^2}| \tag{95}$$

**Corollary 3.18.** *Under the assumption* (53),

$$\|\phi^2 - 2\delta(t)\|_\infty \leq C(C_\sigma, C_1, C_{\mathrm{Vl}})\Big[E(\phi(\mathrm{T}_0)) + \mathrm{T}_0^{-1}\Big] + C(C_\sigma, C_1, C_{\mathrm{Vl}}) \cdot t^{-2}. \tag{96}$$

Note,

$$\begin{aligned}\|\phi^2\|_{L^2}^2 &\leq \|\phi^2 - 2\delta(t)\|_{L^2}^2 + (8\pi)^2\|2\delta(t)\|_\infty^2 \\ &\leq (8\pi)^2\Big[\|\phi^2 - 2\delta(t)\|_\infty^2 + \|2\delta(t)\|_\infty^2\Big].\end{aligned} \tag{97}$$

This implies

$$\begin{aligned}\|\phi^2\|_{L^2}^2 \leq\ & C(C_\sigma, C_1, C_{\mathrm{Vl}})\Big[E(\phi(\mathrm{T}_0)) + \mathrm{T}_0^{-1}\Big]^2 \\ &+ C(C_\sigma, C_1, C_{\mathrm{Vl}}) \cdot t^{-4} + 2(8\pi)^2\delta(t)^2.\end{aligned} \tag{98}$$

Finally, we combine the previous estimates to obtain an estimate for $E_0(\phi)$.

**Corollary 3.19.** *The assumptions* (53) *imply the following estimate.*

$$E_0(\phi) \leq 2(8\pi)^2\delta(t)^2 + E(\phi(\mathrm{T}_0)) + C(C_1, C_\sigma, C_{\mathrm{Vl}}) \cdot \Big[\mathrm{T}_0{}^{-1} + E(\phi(\mathrm{T}_0))^2\Big]. \tag{99}$$

To improve the bootstrap assumption on $\phi^{-2}$ we deduce the relevant estimate.

**Corollary 3.20.** *Under the assumptions* (53) *the following estimate holds*

$$\begin{aligned}\|\phi^{-2}\|_{H^4} \leq\ & \mathrm{vol}_\sigma^2 \cdot \left[\frac{1}{2\delta(t)} + \frac{\|\phi^2 - 2\delta(t)\|_\infty}{2\delta(t)\Big[2\delta(t) - \|\phi^2 - 2\delta(t)\|_\infty\Big]}\right] \\ &+ C_\phi(\kappa) \cdot \Big[\|D\phi^2\|_{H^2}^2\Big]\end{aligned} \tag{100}$$

*where*

$$\kappa = \frac{\|\phi^2 - 2\delta(t)\|_\infty}{2\delta(t)\Big[2\delta(t) - \|\phi^2 - 2\delta(t)\|_\infty\Big]} \tag{101}$$

*and $C_\phi(\kappa)$ is a function that is bounded if $\kappa$ is bounded.*

From the bootstrap assumptions we infer $0 \leq \kappa \leq 1$, which implies the following corollary.

**Corollary 3.21.** *Under the assumptions* (53) *the following estimate holds.*

$$\|\phi^{-2}\|_{H^4} \leq 2(8\pi)^2 + C(C_\sigma, C_1, C_{\mathrm{Vl}})(E(\phi(\mathrm{T}_0)) + \mathrm{T}_0^{-1}) \tag{102}$$

### 3.4.2. Energy estimates for the trace-free part of the second fundamental form

Defining the higher order energy of $h$ by $\mathbf{H}_k(t) \equiv \frac{1}{2}\|h\|^2_{H^k}$ we obtain the following energy estimate for $\mathbf{H}_k$. This estimate is based on the modified evolution equation for $h$, which follows from (20) in combination with the Hamiltonian constraint. This reads

$$\partial_t h = (N/2-1)\tau^2/2g_{ab} - \nabla_a\nabla_b N - N(T_{ab} - g_{ab}g^{ij}T_{ij}) + \mathcal{L}_X h_{ab} + N\Big[\frac{1}{2}|h|^2_g g_{ab} - 2h_{ai}h^i_b\Big]. \tag{103}$$

**Lemma 3.22.** *Let $h \in H^4$ be a solution to the evolution equation (20) with data $(\lambda, N, X, T) \in H^5 \times H^6 \times H^5 \times H^4$, then*

$$|\partial_t \mathbf{H}_4| \le C\Big[\|\mathrm{div}_\sigma X\|_\infty + C_\sigma \|D^{\le 5}X\|\Big]\mathbf{H}_4 + 2\|F_h\|_{H^4}\sqrt{\mathbf{H}_4}, \tag{104}$$

*where*

$$F_h = (N/2-1)\phi^2/2\sigma_{ab} - \nabla_a\nabla_b N - N(T_{ab} - g_{ab}g^{ij}T_{ij}) + N\left(\frac{1}{2}|h|^2_g g_{ab} - 2h_{ai}h^i_b\right) + 2h_{i(b}D_{a)}X^i. \tag{105}$$

*Proof.* The estimate is a direct consequence of Eq. (103). □

**Corollary 3.23.** *Under the assumption (53) the following estimate holds.*

$$|\partial_t \mathbf{H}_4| \le C(C_1, C_\sigma, C_{\mathrm{VI}}) \cdot t^{-2}\Big[\mathbf{H}_4 + \sqrt{\mathbf{H}_4}\Big]. \tag{106}$$

*In particular,*

$$\mathbf{H}_4(t) \le C(C_1, C_\sigma, C_{\mathrm{VI}}) \cdot \mathrm{T}_0^{-1} \exp\Big(C(C_1, C_\sigma, C_{\mathrm{VI}}) \cdot \mathrm{T}_0^{-1}\Big). \tag{107}$$

## 3.5. Total estimate for the perturbation of the metric

We give in the following a collection of all previously established estimates on the perturbation of the metric under the given assumptions (53).

**Proposition 3.24.** *Let $\mathrm{T}_0 > 1$. Under the assumptions (53) the following estimates hold for $t \in [\mathrm{T}_0, \mathrm{T}_1]$.*

$$\begin{aligned}
\|h\|_{H^4} &\le C_2 \\
\mathbf{H}_4(t) &\le C_2 \cdot \mathrm{T}_0{}^{-1} \exp\Big(C_2 \cdot \mathrm{T}_0{}^{-1}\Big) \\
\|\lambda - \bar{\lambda}\|_{H^5} &\le C_2 \\
2 - N &\le C_2 \cdot t^{-2} \\
\|DN\|_{H^4} &\le C_2 \cdot t^{-2} \\
\|X\|_{H^5} &\le C_2 \cdot t^{-2} \\
E(\phi(t)) &\le E(\phi(\mathrm{T}_0)) + C_2 \cdot \mathrm{T}_0{}^{-1}
\end{aligned}$$

$$
\begin{aligned}
\|\phi^2 - 2\delta(t)\|_\infty &\le C_2\Big[E(\phi(\mathrm{T}_0)) + \mathrm{T}_0{}^{-1}\Big] + C_2 \cdot t^{-2} \\
E_0(\phi) &\le 2(8\pi)^2\delta(t)^2 + E(\phi(\mathrm{T}_0)) + C_2\Big[E(\phi(\mathrm{T}_0))^2 + \mathrm{T}_0^{-1}\Big] \\
\|\phi^{-2}\|_{H^4} &\le 2(8\pi)^2 + C_2(E(\phi(\mathrm{T}_0)) + \mathrm{T}_0^{-1}) \\
\|\dot{X}\|_{H^4} &\le C_2 \cdot t^{-3} \\
\|\dot{N}\|_{H^4} &\le C_2 \cdot t^{-3},
\end{aligned}
\tag{108}
$$

*where* $C_2 = C_2(C_1, C_\sigma, C_{\mathrm{Vl}})$.

*Proof.* The estimates have all been proven in the foregoing part of the section. □

### 3.5.1. Implications for the perturbation terms

For the energy estimates for the distribution function in the following section we require the following estimate. Recall the definition of $\tilde{\Gamma}_i$ in (26) and that these are tensors.

**Lemma 3.25.** *Under assumptions (53) the following estimates hold.*

$$
\|t^2 \cdot \tilde{\Gamma}_1\|_{H^4} + \|\tilde{\Gamma}_2\|_{H^4} + \|t^{-2}\tilde{\Gamma}_3\|_{H^4} \le C_2 \cdot t^{-2} \tag{109}
$$

*Proof.* The estimate is a direct consequence of the explicit form of $\tilde{\Gamma}_i$ and Proposition 3.24. □

We state another improved estimate for the shift, which is a direct consequence of (107) and Lemma 3.10.

**Corollary 3.26.** *Under assumptions (53) we have*

$$
\|X\|_{H^5} \le C_2 \mathrm{T}_0^{-1} \cdot t^{-2}. \tag{110}
$$

## 3.6. Energy estimates for the distribution function

We have at hand now all necessary estimates concerning the perturbation of the metric. We proceed in this section by deriving the relevant estimates for the distribution function.

### 3.6.1. Evolution of $\mathrm{m}_\infty$

For the time derivative of the total mass, the following energy estimate holds.

**Lemma 3.27.** *The evolution equation for the total mass reads*

$$
\begin{aligned}
\frac{d}{dt}\mathbf{m}_\infty = \int_{T\mathbb{S}^2} \Big[&\langle p, \mathbf{A}(p^0)^{-1}\rangle_g - \Gamma^e_{00}\mathbf{B}_e p^0 - 2\tilde{\Gamma}_{2,e}{}^{e} - 2(N-2)\tau \\
&+ \tilde{\Gamma}_{3,ab}{}^{e}\mathbf{B}_e(p^a p^b (p^0)^{-1}) + g^{ab}\mathcal{L}_X g_{ab}\Big] \cdot f \mu_{T\mathbb{S}^2}.
\end{aligned}
\tag{111}
$$

*In particular, the energy estimate for the total mass takes the form*

$$\left|\frac{d}{dt}\mathbf{m}_\infty\right| \leq \mathbf{R}_{\mathbf{m}_\infty} \cdot \mathbf{m}_\infty, \tag{112}$$

*where* $\mathbf{R}_{\mathbf{m}_\infty}$ *is defined as the supremum of the term enclosed by the brackets* [.] *in* (111) *taken over the set* $\operatorname{supp} f \subset T\mathbb{S}^2$.

*Proof.* The equality is obtained by a straightforward computation using the Vlasov equation (27) and integration by parts. The energy estimate follows directly. □

**Corollary 3.28.** *Under the assumptions* (53) *the energy estimate takes the form*

$$\left|\frac{d}{dt}\mathbf{m}_\infty(t)\right| \leq C(C_1) \cdot t^{-2} \cdot \mathbf{m}_\infty(t). \tag{113}$$

*This implies, in particular,*

$$|\mathbf{m}_\infty(t) - \mathbf{m}_\infty(\mathrm{T}_0)| \leq \mathbf{m}_\infty(\mathrm{T}_0) C(C_1) \cdot \mathrm{T}_0^{-1} \exp\left[C(C_1) \cdot \mathrm{T}_0^{-1}\right]. \tag{114}$$

*For* $\delta(t)$ *(cf.* (51)*) this implies*

$$|\delta(t) - \delta(\mathrm{T}_0)| \leq \mathbf{m}_\infty(\mathrm{T}_0) C(C_1) \cdot \mathrm{T}_0^{-1} \exp\left[C(C_1) \cdot \mathrm{T}_0^{-1}\right]. \tag{115}$$

*This simplifies to*

$$|\delta(t) - \delta(\mathrm{T}_0)| \leq C(C_1, \mathbf{m}_\infty(\mathrm{T}_0)) \cdot \mathrm{T}_0^{-1}. \tag{116}$$

*Proof.* The energy estimate is a straightforward consequence of the decay assumptions (53). The estimate (114) follows by an application of Grönwall's lemma and the estimate for $\delta$ is an immediate consequence. □

### 3.6.2. *Evolution of the momentum-support*

The decay of the momentum-support of $f$ is established by use of a differential inequality for the auxilliary quantity

$$\mathbf{G}(t,x,p) \equiv \tau^{-2}|p + X/2|_g^2. \tag{117}$$

The derivative of $\mathbf{G} = \mathbf{G}(t,x,p)$ along geodesics is given in

**Proposition 3.29.** *Let* $(t(s), x(s), p(s))$ *be a solution to the characteristic system (29) and* $\mathbf{G}$ *as defined above, then the following estimate holds for the time derivative of* $\mathbf{G}$.

$$\begin{aligned}\left|\frac{d\mathbf{G}}{dt}\right| \leq& \left[|\tau||2-N| + 2|N||h|_g + 4|\tilde{\Gamma}_2|_g\right]\mathbf{G}\\ &+ |\tau^{-1}|\Big[2|\partial_t X|_g|\frac{1}{2} - p^0| + |p^0||\tilde{\Gamma}_1|_g + 2|X|_g|\tilde{\Gamma}_2|_g + 2|[p^0]^{-1}||\tilde{\Gamma}_3|_g|p|_g^2\\ &+ |[p^0]^{-1}||\nabla X|_g|p|_g + 4|X|_g|p^0 - \frac{1}{2}|\Big]\sqrt{\mathbf{G}}\end{aligned} \tag{118}$$

*Proof.* The estimate is a direct consequence of the characteristic system (29). □

**Corollary 3.30.** *Under the assumptions (53) the energy estimate takes the form*

$$\left|\frac{d\mathbf{G}}{dt}\right| \leq C(C_1, C_{\mathrm{Vl}}) \cdot t^{-2} \cdot \left[\mathbf{G} + \sqrt{\mathbf{G}}\right]. \tag{119}$$

*In particular,*

$$|\sqrt{\mathbf{G}} - \sqrt{\mathbf{G}_0}| \leq (\sqrt{\mathbf{G}_0} + 1) \cdot C(C_1, C_{\mathrm{Vl}}) \cdot \mathrm{T}_0^{-1} \cdot \exp(C(C_1, C_{\mathrm{Vl}}) \cdot \mathrm{T}_0^{-1}) \tag{120}$$

*where* $\mathbf{G}_0 = \mathbf{G}(\mathrm{T}_0, x(\mathrm{T}_0), p(\mathrm{T}_0))$.

Recall Definition 2.6 of $\mathbf{P}_\infty$ and let

$$\mathbf{G}_+(t) \equiv \sup\left\{\mathbf{G}(t,x,p)\Big|(x,p) \in \operatorname{supp} f(t,.,.)\right\}. \tag{121}$$

Define $\mathbf{G}_{0,+}$ analogously for $t = \mathrm{T}_0$.

**Lemma 3.31.** *Let f be continuous and of compact support. Then*

$$\mathbf{P}_\infty(t) \leq C_\sigma \|\phi^2\|_\infty \left[\mathbf{G}_+ + \||X|_g\|_\infty^2\right]. \tag{122}$$

*In combination, this yields the following estimate.*

**Corollary 3.32.** *Under the assumptions (53), the following estimate holds.*

$$\mathbf{P}_\infty(t) \leq C(C_1, C_{\mathrm{Vl}}) \cdot \left[\mathbf{G}_{0,+} + (\mathbf{G}_{0,+} + 1) \cdot \mathrm{T}_0^{-2} + C_2 \cdot t^{-2}\right]. \tag{123}$$

Finally, we define the supremum of the momentum-support,

$$\mathsf{p}_\infty(t) \equiv \sup\left\{|p|_\sigma \,\middle|\, p \in \operatorname{supp} f(t,.,.)\right\}. \tag{124}$$

### *3.6.3. Corrected energies and Energy estimates*

To estimate the Sobolev norms of the matter quantities via Lemma 2.8 we require energy estimates for the $L^2$-norms of the distribution function defined in (31). It is however not direct to obtain such estimates. A straightforward approach by taking the time derivative of the energy and integrating by parts yields perturbation terms, which have insufficient decay and yield a small polynomial $t$-growth of the energies. This is however incompatible with the perturbation analysis of the geometry, which in that case would yield terms growing in time, which cannot be compensated. Therefore, it is necessary to avoid these "bad" terms in the energy estimates. This can be achieved by a correction mechanism. These estimates, in combination with the bootstrap assumptions, then imply uniformly bounded Sobolev norms of the distribution function.

The problematic terms in the Vlasov equation appear in the $\Gamma^a_{00}$ component and read $\partial_t X^a + 2\tau X^a$ (cf. (25)). Their pointwise decay is of the order $\lesssim Ct^{-3}$, which is insufficient to compensate for the missing momentum variable $|p|_\sigma$ in these terms, which appears in front of $\mathbf{B}f$ in the energies. It is possible to avoid the appearance of the corresponding perturbation terms in the final estimate by considering a corrected energy which consists of the square of the $L^2$-norm (2.5) and a correction term, whose time derivative cancels the problematic terms and leaves only perturbation terms with sufficiently strong decay. The premise, for these corrected energies to be equivalent with the standard energies arising from (2.5), is quadratic decay in time of the shift vector, which holds in the present setting (cf. Proposition 3.24). A detailed construction of these corrected energies and of the related energy estimates has been worked out in [17] for low orders of regularity. As the construction does not depend on the spatial topology we shall not repeat the proof here. Rather we formulate a more general version of these estimates without giving the explicit expression of the correction terms—which is lengthy and not necessary for their application. What matters is only the fact that such correction terms exist, which follows identically to [17]. We formulate the according conditional estimates in the following lemma. The equivalence to the Vlasov energy is given under a smallness condition on the shift vector and bounds on the rescaled conformal factor $\phi^2$.

**Lemma 3.33.** *Let $c, C \in \mathbb{R}_+$ be such that*

$$\|X\|_{H^5} \leq Ct^{-2} \tag{125}$$

*and*

$$c^{-1} \leq \phi^2 \leq c, \tag{126}$$

*then there exists an $\omega_0 = \omega_0(c, C) > 0$, s.t. for $\omega_0 \leq \omega$ there exists a function*

$$\Phi = \Phi(\phi^2, X, \nabla X, \dots, \nabla^4 X, \overline{\nabla} f, \dots, \overline{\nabla}^4 f, p) \tag{127}$$

*with domain* $\mathrm{dom}\Phi \subset \mathrm{supp} f$ *and* $\mathrm{im}(\Phi) \subset \mathbb{R}$, *such that for the* corrected energy

$$\|f\|^2_{\mathrm{Vl},s,\mathbf{c}} \equiv \|f\|^2_{\mathrm{Vl},4} + \int_{T\mathbb{S}^2} \Phi \mu_{T\mathbb{S}^2} \tag{128}$$

*equivalency to the $L^2$-energy holds, i.e.,*

$$\frac{1}{2}\|f\|_{\mathrm{Vl},4,\mathbf{c}} \leq \|f\|_{\mathrm{Vl},4} \leq 2\|f\|_{\mathrm{Vl},4,\mathbf{c}}, \tag{129}$$

*where both energies are defined with respect to $\omega$ (cf. Definition 2.5 and $\omega$ is independent of $t$). Furthermore, an energy estimate of the form*

$$\left|\partial_t \|f\|_{\mathrm{Vl},4,\mathbf{c}}\right| \leq \mathbf{R}(t) \cdot \|f\|_{\mathrm{Vl},4,\mathbf{c}} \tag{130}$$

*holds, where*

$$\begin{aligned}\mathbf{R}(t) = C(c, C, \mathbf{G}_+, \omega_0, \|\mathbf{Rm}\|_{H^3})\Big[&\|t^2 \cdot \tilde{\Gamma}_1\|_{H^4} + \|\tilde{\Gamma}_2\|_{H^4} + \|t^{-2} \cdot \tilde{\Gamma}_3\|_{H^4} \\ &+ \|\left[(2-N)\tau + {}^g\nabla_c X^c\right]\|_\infty \\ &+ \mathit{pol}\Big(t^{-2}, \mathsf{p}_\infty, \|X\|_{H^5}, \|\partial_t X\|_{H^5}, \sup_{(x,p)\in \mathrm{supp} f} |\overline{\nabla}^{\leq 4} p^0|_{\hat{\mathbf{g}}}\Big)\Big]\end{aligned} \tag{131}$$

*and $\mathit{pol}(.)$ is a polynomial with vanishing constant term and $\nabla^{\leq 4}p^0$ represents at least the first derivative of $p^0$ (not the zeroth order).*

*Proof.* The proof is identical to the corresponding one in [17]. □

**Remark 3.34.** Note that in particular under assumptions (53) the conditions on the shift vector and on $\phi^2$ hold. Note also that all terms on the right-hand side of (131) decay at least like $t^{-2}$.

In combination with the decay estimates of Proposition 3.24, applied to estimate the coefficient **R**, this implies the following boundedness estimate for the corrected energy and in turn the corresponding estimate for the uncorrected energies.

**Corollary 3.35.** *Under the assumptions (53), we have*

$$\|f(t)\|_{\mathrm{Vl},4,\mathbf{c}} \leq \|f(\mathrm{T}_0)\|_{\mathrm{Vl},4,\mathbf{c}} \exp\left(C(C_1, C_{\mathrm{Vl}})\mathrm{T}_0^{-1}\right). \tag{132}$$

*For the $L^2$-energy this implies, by the equivalence of the energies,*

$$\|f(t)\|_{\mathrm{Vl},4} \leq 4 \cdot \|f(\mathrm{T}_0)\|_{\mathrm{Vl},4} \exp\left(C(C_1, C_{\mathrm{Vl}})\mathrm{T}_0^{-1}\right). \tag{133}$$

### *3.6.4. Total estimate for the perturbation of the matter*

The following proposition collects all estimates on the matter quantities from the foregoing section.

**Proposition 3.36.** *Under the assumptions (53) the following estimates hold for $t \in [\mathrm{T}_0, \mathrm{T}_1]$.*

$$\begin{aligned} \mathbf{P}_\infty(f)[t] &\leq C_2 \cdot \left[\mathbf{G}_{0,+} + \mathrm{T}_0^{-2}\right] \\ \|e^{2\lambda}\rho(f)\|_{H^4} + \|e^{2\lambda} J(f)\|_{H^4} &\leq 4\|f(\mathrm{T}_0)\|_{\mathrm{Vl},4} \exp\left(C_2\mathrm{T}_0^{-1}\right)\left[1 + C_2(\mathrm{T}_0^{-2} + \mathbf{G}_{0,+})\right] \\ \|e^{2\lambda}\eta(f)\|_{H^4} &\leq C_2 \cdot t^{-2}\|f(\mathrm{T}_0)\|_{\mathrm{Vl},4,2} \cdot \left[\mathbf{G}_{0,+}^2 + \mathrm{T}_0^{-2}\right] \\ \delta(t) &> \delta(\mathrm{T}_0) - C_2\mathrm{T}_0^{-1} \\ |\mathbf{m}_\infty(t) - \mathbf{m}_\infty(\mathrm{T}_0)| &\leq \mathbf{m}_\infty(\mathrm{T}_0)C_2\mathrm{T}_0^{-1} \exp\left[C_2\mathrm{T}_0^{-1}\right] \end{aligned} \tag{134}$$

*where $C_2 = C_2(C_1, C_\sigma, C_{\mathrm{Vl}})$.*

*Proof.* The estimates follow directly using $\mathrm{T}_0 > 1$, Corollary 3.26 and Corollary 3.35. □

## 4. Proof of Theorem 1.1

We are now able to prove the first theorem.

*Proof of Theorem 1.1.* We begin by fixing positive constants

$$\begin{aligned} &C_1 < 1 \text{ and} \\ &C_{\mathrm{Vl}} \geq 10\|f_0\|_{\mathrm{Vl},4}. \end{aligned} \tag{135}$$

Then, let $C_2 = C_2(C_1, C_{\mathrm{Vl}})$ be the constant defined in Proposition 3.24. Choose $\mathrm{T}_0 > 1$ large enough to assure

$$C_2\mathrm{T}_0^{-1} < \min\left\{\frac{1}{20}, \frac{C_1}{2}, \frac{1}{20\exp(C_2 \cdot \mathrm{T}_0^{-1})}\right\}. \tag{136}$$

Note, that choosing $\mathrm{T}_0$ as above does not affect the lower bound on $C_{\mathrm{Vl}}$ as the norm of $f_0$ by local stability remains close to its original value, independent of the choice of $\mathrm{T}_0$.

So far these constants are not related to any interval of existence. By virtue of the Cauchy stability, Lemma 2.15, for the system, it is possible to choose $\varepsilon$ at the initial time sufficiently small to assure existence of the solution on an open interval containing $\mathrm{T}_0$ and moreover to deduce smallness with respect to the background solution at the time $\mathrm{T}_0$. We may therefore without loss of generality begin the evolution at a $\mathrm{T}_0$ such that (136) holds and choose the data at $\mathrm{T}_0$ small. We refer to $\varepsilon$ now as the smallness at $\mathrm{T}_0$. We may further increase $\mathrm{T}_0$ below if necessary.

Define

$$\mathrm{T}_* \equiv \sup\left\{T > \mathrm{T}_0 \,\middle|\, \text{the solution exists on } [\mathrm{T}_0, T) \text{ and (53) holds on } [\mathrm{T}_0, T)\right\}. \tag{137}$$

The local theory yields that choosing $\varepsilon$ sufficiently small implies the existence of $\mathrm{T}_* \in (\mathrm{T}_0, \infty) \cup \{\infty\}$. We consider from now on the solution on $[\mathrm{T}_0, \mathrm{T}_*)$. By construction, the bounds (53) hold on $[\mathrm{T}_0, \mathrm{T}_*)$. We use in the following Proposition 3.24 to improve the bootstrap assumptions one by one. Combining the smallness of $\mathrm{T}_0^{-1}$ as chosen in (136) with Proposition 3.24 we obtain the following estimates on $[\mathrm{T}_0, T_*)$, where for some we further decrease the smallness $\varepsilon$ at $\mathrm{T}_0$.

$$\begin{aligned} E_0(\phi) &\leq 4 \cdot (8\pi)^2 \delta(t)^2 \\ \|\phi^2 - 2\delta(t)\|_\infty &\leq \frac{\delta(t)}{2} \\ \|\phi^{-2}\|_{H^4} &\leq 4 \cdot (8\pi)^2 \\ \|\dot{N}\|_{H^4} &\leq \frac{C_1}{2} \cdot t^{-2} \\ \|\dot{X}\|_{H^4} &\leq \frac{C_1}{2} \cdot t^{-2} \end{aligned} \tag{138}$$

We obtain the complementary set of estimates for the matter quantities using Proposition 3.36 on the interval $[T_0, T_*)$. We possibly further decrease smallness at $\mathrm{T}_0$ and infer the following inequalities.

$$\begin{aligned} \mathbf{P}_\infty(f)[t] &\leq C_1/2 \\ \|e^{2\lambda}\rho(f)\|_{H^4} + \|e^{2\lambda}\jmath(f)\|_{H^4} &\leq C_{\mathrm{VI}}/2 \\ \|e^{2\lambda}\eta(f)\|_{H^4} &\leq C_{\mathrm{VI}}/2 \cdot t^{-2} \\ \delta(t) &> \frac{1}{2} + \frac{2}{5} \\ |\mathbf{m}_\infty(t) - \mathbf{m}_\infty(\mathrm{T}_0)| &\leq \frac{1}{20}\mathbf{m}_\infty(\mathrm{T}_0) \end{aligned} \tag{139}$$

Note that (138) and (139) improve (53) on $[\mathrm{T}_0, \mathrm{T}_*)$. As a result of (138) and (139) as well as the continuation criterion in Lemma 2.15 we intend to conclude existence of the solution on the interval $[\mathrm{T}_0, \mathrm{T}_*]$. If the existence of the solution is assured, the bounds (53) hold by the previous arguments. To show existence on $[\mathrm{T}_0, \mathrm{T}_*]$ it suffices to ensure that an upper bound on the norm in (49) of the form $3/2 \cdot \delta_{\mathrm{loc}}$ holds on $[\mathrm{T}_0, \mathrm{T}_*)$. This can be inferred straightforwardly from the estimates for $E(\phi)$, $\phi^2 - 2\delta(t)$, $N$ and $X$ in Proposition 3.24 as well as the estimates for $\eta$ and $\jmath$ in Proposition 3.36 by possibly increasing $\mathrm{T}_0$ again as discussed above and choosing $\varepsilon$ sufficiently small. By continuity, the bounds (53) hold on $[\mathrm{T}_0, \mathrm{T}_*]$. It is important to note that the smallness of $\varepsilon$ and the choice of $\mathrm{T}_0$ do not depend on $T_*$. This implies that for this fixed pair $(\varepsilon, \mathrm{T}_0)$, the solution defined on $[\mathrm{T}_0, T_*)$ for the corresponding $T_*$ as defined in (137) can be extended to $[\mathrm{T}_0, T_*]$. Then, a standard argument implies $\mathrm{T}_* = \infty$, which is future global existence of the solution. Using the estimates for the lapse, shift and trace-free part of the second fundamental form as given in Proposition 3.24, the main result of [16] implies future completeness of the solution. This finishes the proof of the theorem. □

## 5. Nonvacuum Einstein-flow on the 2-torus

In the remainder of this work we prove Theorem 1.3. The mechanism of proof here is similar to the case of the sphere, which is the fact that the energy density acts similar to a negative conformal curvature and therefore gives access to good estimates on the conformal factor. However, due to some differences between both systems, a detailed discussion of the case of the torus is necessary. The main difference is the non-trivial Teichmüller space of the torus, which yields an evolution of the conformal metric $\sigma_{\mathbb{T}^2}$ similar to the case of hyperbolic surfaces [17]. This changes the continuity argument for the proof of global existence. In particular, terms containing the time derivative of the conformal metric are non-vanishing and several constants and norms now depend on time due to the evolution of the conformal metric. In the continuity argument we assume a uniform bound on the point in Teichmüller space to overcome this complication analogous to [15]. We give a brief presentation of the Einstein-flow on the 2-torus and outline the proof of global existence. Details which follow identical arguments as in the case of the sphere are omitted to avoid redundancies in the presentation.

### *5.1. Gauge fixing*

The Einstein equations on the 2-torus have been addressed in different works including [14], [27]. We employ their essential techniques.

Every Riemannian metric $g$ on $\mathbb{T}^2$ is conformally equivalent to a flat metric $\sigma$ via $\sigma = e^{-2\lambda} g$, where $\lambda$ is the conformal factor determined by the Hamiltonian constraint. We denote the space of flat metrics on $\mathbb{T}^2$ by $\mathcal{M}_0(\mathbb{T}^2)$. The group of diffeomorphisms of $\mathbb{T}^2$ homotopic to the identity, $\mathcal{D}_0$, acts on $\mathcal{M}_0(\mathbb{T}^2)$. One may fix a gauge by passing to a slice of this group action in which the flat metrics are represented by spatially constant metrics on $\mathbb{T}^2$ of fixed volume. For an extensive discussion of Teichmüller spaces cf. [27] or the previous applications in [15, 17]. We set $\mathrm{vol}_{\sigma_{\mathbb{T}^2}}(\mathbb{T}^2) = 1$. In the following $\sigma_{\mathbb{T}^2}$ denotes a spatially constant Riemannian metric on $\mathbb{T}^2$ of unit volume and scalar curvature $R(\sigma_{\mathbb{T}^2}) = 0$. We refer to $\sigma_{\mathbb{T}^2}$ as the *conformal metric*. We denote the volume form of $\sigma_{\mathbb{T}^2}$ by $\mu_\sigma$. The remaining gauge freedom is fixed by the CMC condition $t = -\tau^{-1}$.

### *5.2. Evolution of the conformal metric*

The space of conformally inequivalent metrics, in which the equivalence class of $\sigma_{\mathbb{T}^2}$ evolves is two-dimensional. The evolution can be determined by the solvability condition of the shift equation as done in [15] for hyperbolic surfaces. The shift equation reads

$$L_\sigma n = 2Ne^{-2\lambda} h + \partial_t \sigma - \frac{1}{2} \sigma \,\mathrm{tr}_\sigma(\partial_t \sigma), \tag{140}$$

with $[L_\sigma n]_{ab} \equiv D_a n_b + D_b n_a - \sigma_{ab} D_c n^c$. The shift equation is solvable iff the right-hand side of the equation is orthogonal to the kernel of $L_\sigma$. The kernel is the space of $TT$-tensors on $(\mathbb{T}^2, \sigma_{\mathbb{T}^2})$. This space is 2-dimensional with a basis $\{X_1(\sigma_{\mathbb{T}^2}), X_2(\sigma_{\mathbb{T}^2})\}$. The orthogonality condition reads

$$\partial_t \sigma_{ab} X_I^{ab} = -2\mathrm{vol}_{\sigma_{\mathbb{T}^2}}(\mathbb{T}^2)^{-1} \int_{\mathbb{T}^2} N e^{2\lambda} h_{ab} X_I^{ab} \mu_\sigma, I \in \{1, 2\}, \tag{141}$$

where we used the constancy of $\sigma$ and $X_I$ in space. The metric $\sigma(t)$ may be parametrized by two parameters $(\mathsf{q}_1(t), \mathsf{q}_2(t)) = \mathsf{q}(t)$ as $\sigma$ is spatially constant and of fixed volume. The condition (141) can be modified into the form

$$\dot{\mathsf{q}}_I(t) = -2\mathrm{vol}_\sigma(\mathbb{T}^2)^{-1} \mathbf{A}_I^J(\mathsf{q}) \cdot \int_{\mathbb{T}^2} N e^{-2\lambda} h_{ab} X_J^{ab} \mu_\sigma, \tag{142}$$

where $\dot{\mathbf{q}} = (\dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2)$ and $\mathbf{A}$ is a matrix depending on $\mathsf{q}$ such that for $\mathsf{q}$ being uniformly bounded, the components of $\mathbf{A}$ are uniformly bounded. The symbol $\cdot$ denotes the matrix product. Equation (142) controls the evolution of $\sigma$. The time derivative of $\sigma$ in terms of $\dot{\mathsf{q}}$ is given by an equation of the form

$$\partial_t \sigma = \mathbf{B}(\mathsf{q}) \cdot \dot{\mathsf{q}}, \tag{143}$$

where $\mathbf{B}$ is of the same type as $\mathbf{A}$. These considerations are sufficient to control the evolution of $\sigma$ in terms of the evolution of perturbations.

### 5.3. Einstein equations on the torus

With the non-vanishing time derivative of the conformal metric the full set of equations determining the Einstein flow on the 2-torus reads

$$D_b h^b_a = -e^{2\lambda} j_a \tag{144}$$

$$2\Delta_\sigma \lambda = \phi^2/2 - e^{2\lambda}\rho - \phi^{-2}\tau^2 |h|^2_\sigma \tag{145}$$

$$\Delta_\sigma N = N\left[\phi^{-2}\tau^2 |h|^2_\sigma + \phi^2/2 + e^{2\lambda}\eta\right] - \phi^2 \tag{146}$$

$$[L_\sigma n]_{ab} = 2N\phi^{-2}\tau^2 h_{ab} + \partial_t \sigma_{ab} - \frac{1}{2}\sigma_{ab}\sigma^{cd}\partial_t \sigma_{cd} \tag{147}$$

$$\partial_t h_{ab} = (N-1)/n\phi^2 \sigma_{ab} - \nabla_a \nabla_b N - N(T_{ab} - g_{ab}T) \tag{148}$$

$$+ \mathcal{L}_X h_{ab} + N(Ric_{ab} - 2h_{ai}h^i_b) \tag{149}$$

$$\partial_t \lambda = -\frac{1}{2}\left[N\tau + \frac{1}{2}\mathrm{tr}_\sigma(\partial_t \sigma) - \mathrm{div}_g X\right] \tag{150}$$

$$\dot{\mathsf{q}}_I(t) = -2\mathrm{vol}_\sigma(\mathbb{T}^2)^{-1} \mathbf{A}_I^J(\mathsf{q}) \cdot \tau^2 \int_{\mathbb{T}^2} N\phi^{-2} h_{ab} X_J^{ab} \mu_\sigma \tag{151}$$

$$\partial_t \sigma = \mathbf{B}(\mathsf{q}) \cdot \dot{\mathsf{q}}, \tag{152}$$

where we rescaled using $\phi^2 = \tau^2 e^{2\lambda}$. The kernel of the conformal Killing operator $L_\sigma$ is the space of conformal Killing fields on $(\mathbb{T}^2, \sigma_{\mathbb{T}^2})$, which is non-trivial. As an additional restriction we require $X$ to be $L^2$-orthogonal to this space yielding a unique solution. We consider in the following a solution to the foregoing system with initial data $(\mathsf{q}_0, h_0, \lambda_0, N_0, X_0, f_0)$ sufficiently regular and $\varepsilon$-close to the fixed background solutions. Local stability for this system holds analogously to the case of the sphere. $\mathsf{q}$ is controlled with respect to the euclidian norm on $\mathbb{R}^2$ and the continuation criterion is generalized to include the case $|\mathsf{q} - \mathsf{q}_0| \to \infty$ as $t \to \mathrm{T}_+$ (cf. notations in Lemma 2.15). The analogous continuation criterion for the case of hyperbolic surfaces is formulated in [17] and contains the condition on $\mathsf{q}$.

## 5.4. Homogeneous solutions

Analogously to the sphere we obtain the following proposition on the existence of future complete homogeneous solutions. Prior, we define

$$\underline{\rho}(f) \equiv e^{-2\lambda} \rho(f). \tag{153}$$

**Proposition 5.1** (Future complete homogeneous solutions on $\mathbb{T}^2$). *Let $\sigma_{\mathbb{T}^2}$ be a Riemannian metric with constant coefficients on $\mathbb{T}^2$ of unit volume. Then, there exists homogeneous initial data of the form*

$$(\sigma, h, e^{2\lambda}, N, X, f) = \left(\sigma_0, 0, \frac{\tau^2}{4\underline{\rho}(f_0)}, \frac{\tau^2}{\tau^2/2 + \eta}, 0, f_0(|p|_g)\right) \tag{154}$$

*with smooth $f_0$. This initial data has the future complete development on $[\mathrm{T}_0, \infty) \times \mathbb{T}^2$ of the form*

$$g_{\text{hom}} = -\left(\frac{2\tau^2}{\tau^2 + 2\eta}\right)^2 dt^2 + \frac{\tau^4}{4\underline{\rho}} \cdot t^2 \sigma_{\mathbb{T}^2}. \tag{155}$$

*Asymptotically,*

$$\begin{aligned} \lim_{t\to\infty} \frac{2\tau^2}{\tau^2 + 2\eta} &= 2 \\ \lim_{t\to\infty} \frac{\tau^4}{4\underline{\rho}} &= \frac{4\mathbf{m}_\infty}{\mathrm{vol}_\sigma(\mathbb{T}^2)}. \end{aligned} \tag{156}$$

**Remark 5.2.** We refer to [19] for a detailed discussion of these background solutions.

## 5.5. Energy estimates

We establish in the following a number of estimates on the perturbation of the metric and the distribution function on an interval where we impose a set of bootstrap assumptions. The estimates are similar to those in the spherical case except for some adaptions, which are discussed explicitly. All Sobolev norms used in the following, i.e., $\|.\|_{H^k}$, are defined with respect to the evolving metric $\sigma_{\mathbb{T}^2}(t)$. The same holds for the related Sobolev constants. For definitions which are formally identical with the case of the sphere we keep the same notations.

### 5.5.1. Bootstrap assumptions

We define in this section

$$\delta(t) \equiv \frac{\mathbf{m}_\infty(t)}{\mathrm{vol}_{\sigma_{\mathbb{T}^2}}(\mathbb{T}^2)}. \tag{157}$$

We impose the following assumptions for $t \in [\mathrm{T}_0, \mathrm{T}_1]$.

$$\begin{aligned} \|h\| &\leq C_1 \\ E_0(\phi) &\leq 10 \cdot \mathbf{m}_\infty(\mathrm{T}_0) \\ \|\phi^2 - 2\delta(t)\|_\infty &\leq \delta(t) \\ \|\phi^{-2}\|_{H^4} &\leq 10 \cdot \mathrm{vol}_{\sigma_0}^2 \\ \mathbf{P}_\infty(f)[t] &\leq C_1 \\ \|e^{2\lambda}\rho(f)\|_{H^4} + \|e^{2\lambda}\jmath(f)\|_{H^4} &\leq C_{\mathrm{Vl}} \\ \|e^{2\lambda}\eta(f)\|_{H^4} &\leq C_1 \cdot t^{-2} \\ \delta(t) &> \frac{1}{2} \\ |\mathbf{m}_\infty(t) - \mathbf{m}_\infty(\mathrm{T}_0)| &\leq \frac{\mathbf{m}_\infty(\mathrm{T}_0)}{10} \\ \|\dot{N}\|_{H^4} &\leq C_1 \cdot t^{-2} \\ \|\dot{X}\|_{H^4} &\leq C_1 \cdot t^{-2} \\ |\mathsf{q} - \mathsf{q}_0| &\leq C_1 \end{aligned} \tag{158}$$

Note that $C_{\mathrm{Vl}}$ is large with respect to the initial data of the distribution function. We choose

$$C_{\mathrm{Vl}} \geq 10 \cdot \|f_0\|_{\mathrm{Vl},4}. \tag{159}$$

**Remark 5.3.** For simplicity we assume the lower bound on $\delta$ as above by $\frac{1}{2}$. Any positive constant would work similarly. So this choice does not restrict the validity of the proof.

### 5.5.2. Estimates on the perturbation of the geometry

The purpose of this section is to collect the relevant energy estimates for the solution to (144)–(152). Their proofs are mostly similar to those in the case of the sphere.

**Proposition 5.4.** *Let* $\mathrm{T}_0 > 1$. *Then, under the assumptions* (158) *the following estimates hold for* $t \in [\mathrm{T}_0, \mathrm{T}_1]$.

$$\begin{aligned} \|h\|_{H^4} &\leq C_2 \\ \mathbf{H}_4(t) &\leq C_2 \cdot \mathrm{T}_0^{-1} \exp(C_2/\mathrm{T}_0) \\ \|\lambda - \bar{\lambda}\|_{H^5} &\leq C_2 \\ 2 - N &\leq C_2 \cdot t^{-2} \\ \|DN\|_{H^5} &\leq C_2 \cdot t^{-2} \end{aligned}$$

$$
\begin{aligned}
\|X\|_{H^5} &\le C_2 \cdot t^{-2} \\
E(\phi(t)) &\le E(\phi(\mathrm{T}_0)) + C_2 \cdot \mathrm{T}_0^{-1} \\
\|\phi^2 - 2\delta(t)\|_\infty &\le C_2\Big[E(\phi(\mathrm{T}_0)) + \mathrm{T}_0^{-1}\Big] + C_2 t^{-2} \\
E_0(\phi) &\le 2\mathrm{vol}_\sigma^2 \delta(t)^2 + E(\phi(\mathrm{T}_0)) + C_2\mathrm{T}_0^{-1} + C_2\Big[E(\phi(\mathrm{T}_0)) + \mathrm{T}_0^{-1}\Big]^2 \\
\|\dot{X}\|_{H^4} &\le C_2 \cdot t^{-3} \\
\|\dot{N}\|_{H^4} &\le C_2 \cdot t^{-3} \\
|q - q_0| &\le C_2\mathrm{T}_0^{-1} \exp\left(C_2/\mathrm{T}_0\right) \\
\|\partial_t \sigma\|_{H^4} &\le C_2 \cdot t^{-2} \\
\|\partial_t^2 \sigma\|_{H^4} &\le C_2 \cdot t^{-3},
\end{aligned}
\tag{160}
$$

*with* $C_2 = C_2(C_1, C_{\mathrm{VI}})$.

*Proof.* We discuss the individual estimates in the following. We note that due to the uniform bound on the coordinates in Teichmüller space, all constants depending on $\sigma$ are uniformly bounded. We denote a bound on these constants by $C_\sigma$.

*Second fundamental form.* We decompose the trace-free part of the second fundamental form into $h = h^{TT} + h^\perp$, where $h^{TT}$ is a TT-tensor and $h^\perp$ is a conformal Lie derivative. For $h^\perp$ we obtain the following estimate analogous to section 3.2.1.

$$\|h^\perp\|_{H^4} \le C_\sigma \|e^{2\lambda} J\|_{H^3}. \tag{161}$$

For the TT-part, since all Sobolev norms are equivalent on this finite dimensional space (cf. [15]), we have

$$\|h^{TT}\|_{H^4} \le C_\sigma \|h^{TT}\| \le C_\sigma\Big[\|h\| + \|h^\perp\|\Big]. \tag{162}$$

In total, this yields

$$\|h\|_{H^4} \le C_\sigma\Big[\|h\| + \|e^{2\lambda} J\|_{H^3}\Big]. \tag{163}$$

With the bootstrap assumptions, this implies the estimate for $h$.

*Conformal metric.* With the uniform bound on the point in Teichmüller space, Eq. (151) implies an estimate of the form

$$|\dot{\mathsf{q}}| \le C_2 \tau^2 \|\phi^{-2}\| \|h\|, \tag{164}$$

which in turn yields

$$|\frac{d}{dt}|\mathsf{q} - \mathsf{q}_0| \le C_2(|\mathsf{q} - \mathsf{q}_0| + \mathsf{q}_0)\tau^2 \tag{165}$$

and by Grönwall's lemma

$$|\mathsf{q}-\mathsf{q}_0| \leq C_2 \cdot T_0^{-1} \exp\left(C_2/\mathrm{T}_0\right). \tag{166}$$

Equation (152) in combination with Eq. (151) immediately yields

$$\|\partial_t\sigma\|_{H^4} \leq C_2 \cdot t^{-2}. \tag{167}$$

*Conformal factor.* Analogously to Lemma 3.4 we have

$$\|\lambda-\overline{\lambda}\|_{H^5} \leq C_\sigma\Big[\|\phi^2/2 - e^{2\lambda}\rho - \tau^2\phi^{-2}|h|^2_\sigma\|_{H^3}\Big] + CI_\sigma/2\|\phi^{-2}D\phi^2\|. \tag{168}$$

In combination with the previous estimates and the bootstrap assumptions this implies the estimate for the conformal factor.

*Lapse.* The estimates for the lapse function follow identically to those for the sphere discussed in Section 3.3.1.

*Shift.* The estimate for the shift vector follows similarly to that one in Corollary 3.11. However, there are additional terms arising from the time derivative of the conformal metric, which can be estimated using (167).

*Time derivatives of lapse and shift.* The corresponding estimate for the lapse follows analogously as for the sphere in Corollary 3.13. For the time derivative of the shift there is an important difference with respect to the case of the sphere. The time derivative of the shift equation takes the form

$$L_\sigma \dot{n} = \partial_t F_n, \tag{169}$$

where $L_\sigma \dot{n} = D_a\dot{n}_b + D_b\dot{n}_a - \sigma_{ab}D_c\dot{n}^c$ and $F_n$ denotes the right-hand side of the shift equation. The term on the right-hand side can be computed using (151) and (152). As the conformal Killing operator $L_\sigma$ has a non-trivial kernel, which is the space of the conformal Killing fields with respect to $\sigma$. Equation (169) therefore only provides estimates for the part of $\partial_t X$ orthogonal to this space. We split,

$$\partial_t X = \partial_t X^\perp + \partial_t X^\parallel, \tag{170}$$

where

$$\int_{\mathbb{T}^2} \langle \partial_t X^\perp, Z^{(A)}\rangle_\sigma \mu_\sigma = 0. \tag{171}$$

Then, a representation of the following form exists,

$$\partial_t X^\parallel = \mathsf{r}_A Z^{(A)}, \tag{172}$$

where $\mathsf{r}_A = \mathsf{r}_A(t)$ and $\{\mathsf{Z}^{(A)}\}$ is a basis of the space of conformal Killing fields. The vector of coefficients can be determined by the time derivative of the orthogonality condition for the shift vector, which reads

$$0 = \int_{\mathbb{T}^2} \langle \partial_t X, Z^{(A)}\rangle_\sigma \mu_\sigma + \int_{\mathbb{T}^2} \langle X, \partial_t Z^{(A)}\rangle_\sigma \mu_\sigma + \int_{\mathbb{T}^2} \langle X, Z^{(A)}\rangle_{\partial_t\sigma} \mu_\sigma + \int_{\mathbb{T}^2} \langle X, Z^{(A)}\rangle_\sigma \partial_t\mu_\sigma$$

$$= \mathsf{r}_B \int_{\mathbb{T}^2} \langle Z^{(B)}, Z^{(A)} \rangle_\sigma \mu_\sigma + \int_{\mathbb{T}^2} \langle X, \partial_t Z^{(A)} \rangle_\sigma \mu_\sigma$$
$$+ \int_{\mathbb{T}^2} \langle X, Z^{(A)} \rangle_{\partial_t \sigma} \mu_\sigma + \int_{\mathbb{T}^2} \langle X, Z^{(A)} \rangle_\sigma \partial_t \mu_\sigma, \tag{173}$$

where the orthogonality condition for $X$ is used. Let $\mathsf{Z}$ be the invertible matrix with entries $\mathsf{Z}_{AB} = \int_{\mathbb{T}^2} \langle Z^{(B)}, Z^{(A)} \rangle_\sigma \mu_\sigma$ and $\mathsf{Y}$ the vector with components $\mathsf{Y}^A = \int_{\mathbb{T}^2} \langle X, \partial_t Z^{(A)} \rangle_\sigma \mu_\sigma + \int_{\mathbb{T}^2} \langle X, Z^{(A)} \rangle_{\partial_t \sigma} \mu_\sigma + \int_{\mathbb{T}^2} \langle X, Z^{(A)} \rangle_\sigma \partial_t \mu_\sigma$. Then, the following equation holds,

$$\mathsf{r} = \mathsf{Z}^{-1} \cdot \mathsf{Y} \tag{174}$$

and implies

$$|\mathsf{r}| \leq C |\mathsf{Z}| |\mathsf{Y}|. \tag{175}$$

By the bootstrap assumptions $|\mathsf{Z}|$ is uniformly bounded as it just depends on the conformal metric. For the components of $\mathsf{Y}$ we evaluate

$$\partial_t Z^{(A)} = \frac{\partial Z^{(A)}}{\partial \mathsf{q}_I} \frac{d\mathsf{q}_I}{dt}. \tag{176}$$

This implies, in combination with Eq. (164), (167) and the estimate on the $L^\infty$-norm of the shift vector,

$$|\mathsf{r}| \leq C_2 \cdot t^{-4}. \tag{177}$$

In combination with Lemma 2.1 applied to Eq. (169) for $\partial_t X^\perp$, this yields

$$\|\partial_t X\|_{H^4} \leq \|\partial_t X^\perp\|_{H^4} + \|\partial_t X^\parallel\|_{H^4} \leq C_2 \cdot t^{-3}, \tag{178}$$

which is the desired estimate for $\partial_t X$.

*Estimate for $\partial_t^2 \sigma$.* Taking the time derivative of the equation for $\partial_t \sigma$ in terms of $\dot{\mathsf{q}}$ we obtain

$$\partial_t^2 \sigma = \frac{\partial \mathbf{B}(\mathsf{q})}{\partial \mathsf{q}^I} \dot{\mathsf{q}}^I + \mathbf{B}(\mathsf{q}) \cdot \ddot{\mathsf{q}}. \tag{179}$$

Using in turn (151) in combination with the evolution equation for $h$ and the previous estimate for $\dot{N}$ we can infer the estimate for $\partial_t^2 \sigma$.

*Energy estimate for h.*

The energy estimate of the form derived in Lemma 3.22 holds similarly in the case of the torus. In combination with the previously established estimates it implies

$$|\partial_t \mathbf{H}_4| \leq C_2 t^{-2} \Big[ \mathbf{H}_4 + \sqrt{\mathbf{H}_4} \Big]. \tag{180}$$

An immediate consequence is

$$\mathbf{H}_4(t) \leq C_2 \cdot \mathrm{T}_0^{-1} \exp\left( C_2 \cdot \mathrm{T}_0^{-1} \right). \tag{181}$$

*Rescaled conformal factor.*

The evolution equation for the conformal factor (150) implies

$$\partial_t \phi^2 = \left[ (2-N)\tau - \frac{1}{2}\mathrm{tr}_\sigma(\partial_t \sigma) + {}^g\nabla_c X^c \right] \phi^2. \tag{182}$$

This differs from the analogous equation for the sphere by the term containing the time derivative of the conformal metric. However, this term obeys the same bound as the shift vector term and therefore can be handled identically. Following similarly to (87) we obtain

$$E(\phi(t)) \leq E(\phi(\mathrm{T}_0)) + C_2 \cdot \mathrm{T}_0^{-1}. \tag{183}$$

As in Corollary 3.18, noting the vanishing conformal curvature in the Hamiltonian constraint in the case of the torus, we obtain

$$\|\phi^2 - 2\delta(t)\|_\infty \leq C_2\left[E(\phi(\mathrm{T}_0)) + \mathrm{T}_0^{-1}\right] + C_2 t^{-2}. \tag{184}$$

Finally, analogous to Corollary 3.19 we obtain

$$E_0(\phi) \leq 2\mathrm{vol}_\sigma^2 \delta(t)^2 + E(\phi(\mathrm{T}_0)) + C_2\mathrm{T}_0^{-1} + C_2\left[E(\phi(\mathrm{T}_0)) + \mathrm{T}_0^{-1}\right]^2, \tag{185}$$

where here $\delta(t) = \mathbf{m}_\infty(t)/\mathrm{vol}_{\sigma_{\mathbb{T}^2}}(\mathbb{T}^2)$. This completes the list of estimates and the proof. □

### 5.5.3. *Estimates for the perturbation of the distribution function*

**Proposition 5.5.** *Under the assumptions* (158) *estimates of the same form as in Proposition 3.36 hold.*

*Proof.* The estimates follow analogous to the those in Proposition 3.36. □

## 5.6. *Proof of Theorem 1.3*

*Proof of Theorem 1.3.* The proof is analogous to the one of Theorem 1.1 in the foregoing section. The only difference concerns the non-trivial evolution in Teichmüller space for the torus. The initial bootstrap assumption on boundedness of $\mathsf{q}$ is improved as shown in Proposition 5.4. The other steps in the proof follow the foregoing one and are therefore not repeated. □


## Acknowledgment

The author is grateful to Lars Andersson, Vince Moncrief, Hans Ringström and Robert Beig for discussions and suggestions.


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