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Nonlocal Electrostatics in Spherical Geometries

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NONLOCAL ELECTROSTATICS IN SPHERICAL GEOMETRIES

by

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ABSTRACT

NONLOCAL ELECTROSTATICS IN SPHERICAL GEOMETRIES

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Nonlocal continuum electrostatic models have been used numerically in protein simulations, but analytic solutions have been absent. In this paper, two modified nonlocal continuum electrostatic models, the Lorentzian Model and a Linear Poisson-Boltzmann Model, are presented for a monatomic ion treated as a dielectric continuum ball. These models are then solved analytically using a system of differential equations for the charge distributed within the ion ball. This is done in more detail for a point charge and a charge distributed within a smaller ball. As the solutions are a series, their convergence is verified and criteria for improved convergence is given.

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1 Introduction

Electostatic models are used in the simulation of protein. In particular, they have been used to determine the electrostatic potential for protein embedded in a solvent domain. In the literature, when such models are discussed, they are derived into integral of differential equations. Depending on the specific model, numerical methods have been designed and a series of numerical tests carried out, such as in [8]. It is also know that differential equations can be decomposed into orthogonal components and the individual parts solved as a means to solve the original equations. This idea of orthogonal decomposition has been used to produce some numerical methods [9]. However, while it produces solutions, they are only as a limit of convergent steps in the numerical algorithm. Thus, the main purpose of this paper is to derive solutions analytically under a particular model, the Lorentzian Model, to be given in detail in Section 2, as well as a Nonlocal Modified Linear Poisson-Boltzmann Model, given in detail in Section 6.

One of the features of electrostatic model equations is a ρ function, which gives the source charges of the system. These typically are made to represent point charges, which are represented with Dirac delta functions. Numerical methods are then either designed to exploit this or work around the singularity. [4]. Analytic techniques do not require such restrictions to ρ to create solutions.

The trade off is that while numerical techniques may not require our domains to be well-behaved, the techniques of this paper require the domains to be balls.

As is frequent in the discussion of electrostatic models, we begin in Section 1.1 discussing the derivation of the Lorentzian Model. We solve the differential equations by splitting it into a system and decomposing into orthogonal components, the details of which begin in Section 3. These analytic solutions are verified component-wise in Section 4. Finally convergence is verified in Section 5. Much of this work is repeated for the second model, the Nonlocal Modified Linear Poisson-Boltzmanm Model from Section 6. The solution is derived in Section 6.1, verified component-wise in Section6.2, and shown to convergence in Section 6.3. Finally, we comment in Section 7 about the necessity of studying an electrostatic model wherein the protein region is only a ball.

1.1 Model Origin

Let $\mathbb{R}^3 = D_s \cup D_p \cup \Gamma$ be the decomposition of the space into a water solvent region, open set D_s, that surrounds a the region holding the protein, open set D_p. Γ will denote the interface between them. Since D_p is to represent a protein, or any finite substance we might wish to study, D_p will be bounded and connected. There are options for the domain to change over time, but the models we will study will be independent of time. The developed formulas also assume that the domains are well-behaved, so let us assume that Γ is smooth.

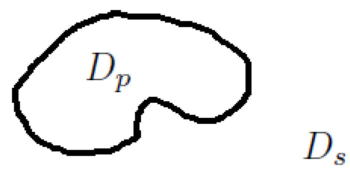


Figure 1: General appearance of a protein in water

Given a dielectric function $\epsilon(\mathbf{r}, \mathbf{r}')$, displacement field $\mathbf{d}$, electric field $\mathbf{e}$, charge density function $\rho(\mathbf{r})$, and electrostatic potential function $\Phi(\mathbf{r})$, it is known that they will satisfy the following relations [2]

$$\mathbf{e}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) \tag{1}$$

$$\nabla \cdot \mathbf{d}(\mathbf{r}) = \rho(\mathbf{r}) \tag{2}$$

$$\mathbf{d}(\mathbf{r}) = \epsilon_0 \int_{\mathbb{R}^3} \epsilon(\mathbf{r}, \mathbf{r}')\mathbf{e}(\mathbf{r}')d\mathbf{r}' \tag{3}$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

Before we introduce a specific dielectric function, it is necessary to define some of the constants that will appear in many such functions: the relative permittivities. We will denote the linear relative permittivities of the two mediums by ϵ_s and ϵ_p respectively. ϵ_∞ will represent the relative permittivity of water under high frequencies. These permittivities are relative to the permittivity of a vacuum, given by ϵ_0, measured in Farad per meter. These measure the ability of the substance to store an electrical charge.

While presented as a constant, these relative permittivities do vary in terms of the frequency of the waves applied to it as well as the temperature of the surface. At room temperature, possible permittivity values for water are $\epsilon_s = 80, \epsilon_\infty = 1.8$ [9]. For any temperature, the high frequencies are less able of interacting with the material, and thus $\epsilon_s > \epsilon_\infty$.

If we assume that the protein substance is uniform in D_p, it is reasonable for $\epsilon(\mathbf{r}, \mathbf{r}') = \epsilon_p \delta(\mathbf{r} - \mathbf{r}')$ for $\mathbf{r} \in D_p$. While the analogous definition for the water region is reasonable, as water has been studied, more accurate models have been developed [1, 4]. We shall focus on that dielectric function:

$$\epsilon(\mathbf{r}, \mathbf{r}') = \epsilon_p \delta(\mathbf{r} - \mathbf{r}') \quad \mathbf{r} \in D_p \tag{4}$$

$$\epsilon(\mathbf{r}, \mathbf{r}') = (\epsilon_s - \epsilon_\infty) \frac{1}{4\pi\lambda^2 |\mathbf{r}' - \mathbf{r}|} e^{-|\mathbf{r}' - \mathbf{r}|/\lambda} \quad \mathbf{r} \in D_s, \mathbf{r} \neq \mathbf{r}' \tag{5}$$

$$\epsilon(\mathbf{r}, \mathbf{r}') = \epsilon_\infty \quad \mathbf{r} \in D_s, \mathbf{r} = \mathbf{r}' \tag{6}$$

.

This introduces a new parameter, λ, representing the polarization correlations in water molecules. There is disagreement as to what values for λ are appropriate as they are often chosen by comparing experiments and simulations. Values between 3 and 30 Angstroms have been used where $|\mathbf{r}|$ is also measured in Angstroms. [9, 1].

We may apply (3) to obtain

$$\mathbf{d}(\mathbf{r}) = \epsilon_0 \epsilon_p \nabla \Phi(\mathbf{r}) \quad \mathbf{r} \in D_p \tag{7}$$

$$\mathbf{d}(\mathbf{r}) = \epsilon_0 (\epsilon_\infty \nabla \Phi(\mathbf{r}) + (\epsilon_s - \epsilon_\infty) \int_{\mathbb{R}^3} Q_\lambda(\mathbf{r} - \mathbf{r}') \nabla \Phi(\mathbf{r}') d\mathbf{r}') \quad \mathbf{r} \in D_s \tag{8}$$

where

$$Q_\lambda(\mathbf{r}) = \frac{1}{4\pi\lambda^2 |\mathbf{r}|} e^{-|\mathbf{r}|/\lambda} \quad \mathbf{r} \neq 0 \tag{9}$$

.

Applying $\nabla \cdot \mathbf{d}(\mathbf{r}) = \rho(\mathbf{r})$ gives differential equations to solve for Φ, given as (15) and (16).

Since we are given distinct functions for regions D_p and D_s, it is convenient to define Φ_p and Φ_s to be Φ restricted to those spaces. For that matter, we shall use subscripts of p and s to denote that the variable in question relates to D_p and D_s respectively.

We also shall need some conditions on the values along the boundary, Γ. We know from linear dielectric theory that we have continuity across the interface [2]. That is,

$$\Phi_p(\mathbf{r}) = \Phi_s(\mathbf{r}) \quad \mathbf{r} \in \Gamma \tag{10}$$

where $\Phi_p(\mathbf{r})$ and $\Phi_s(\mathbf{r})$ are understood to be the continuous extensions of the functions onto the boundary. It is also known that the displacement field has continuity in the normal vector direction of Γ [2]. That is,

$$\frac{\partial \mathbf{d_p}(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} = \frac{\partial \mathbf{d_s}(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} \quad \mathbf{r} \in \Gamma \tag{11}$$

which gives us

$$\epsilon_p \nabla\Phi(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) = \epsilon_\infty \nabla\Phi(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) + (\epsilon_s - \epsilon_\infty) \int_{\mathbb{R}^3} Q_\lambda(\mathbf{r} - \mathbf{r}')\nabla\Phi(\mathbf{r}')d\mathbf{r}') \cdot \mathbf{n}(\mathbf{r}) \quad \mathbf{r} \in \Gamma \tag{12}$$

where $\mathbf{n}(\mathbf{r})$ is the normal vector to Γ. As D_s is unbounded, for the purposes of having Φ represent the potential, we should also note that $\Phi((r)) \to 0$ as $|\mathbf{r}| \to \infty$.

As we would expect the charges to come from the protein region rather than from water, we will assume that ρ is 0 in D_s. If one wanted to include a charge on Γ, we would develop a new set of equations from taking the limit as the charge approaches the boundary. Care needs to be made for modifying the boundary conditions in that case. As defined, these boundary conditions, (10) and (12), give us (18) and (19), completing the formulation of the Lorentzian Model. In other models and their derivations, the integrals are made over D_p rather than over all of $\mathbb{R}^3$. Working on those models will require care concerning the swapping of integration and differentiation if Γ is not sufficiently well-behaved.

Before presenting the Lorentzian Model, let us also discuss ρ_p. Depending on what is known about the physical system it is supposed to represent, different ρ are used. For instance, if the protein is known to have atoms at locations $\{\mathbf{r_j}\}$ with respective charge numbers c_j, and elementary charge e_c, ρ_p can be estimated by

$$\rho_p(\mathbf{r}) = e_c \sum_j c_j \delta(\mathbf{r} - \mathbf{r_j}) \tag{13}$$

where δ is the Dirac delta distribution. Another interpretation is to view the charges as evenly distributed over balls of radius b_j.

$$\rho_p(\mathbf{r}) = e_c \sum_j c_j \frac{3}{4\pi b_j^3} I_{B(\mathbf{r_j}, b_j)}(\mathbf{r}) \tag{14}$$

.

In later sections, we will explore a point charge model (13) and the ball charge model (14), but we do not restrict ourselves to only those cases during our calculations. Naturally, since ρ_p may not be continuous, we may only be able to guarantee a weak solution for what we desire, Φ, the electrostatic potential function.

2 Lorentzian Model

Explicitly written, the Lorentzian Model, a nonlocal Poisson dielectric model is given by [7]

$$-\epsilon_p \Delta \Phi_p(\mathbf{r}) = \frac{1}{\epsilon_0}\rho_p(\mathbf{r}) \quad \mathbf{r} \in D_p \tag{15}$$

$$-\epsilon_\infty \Delta \Phi_s(\mathbf{r}) - (\epsilon_s - \epsilon_\infty)\nabla \cdot \mathbf{v}(\mathbf{r}) = 0 \quad \mathbf{r} \in D_s \tag{16}$$

$$\Phi_s(\mathbf{r}) \to 0 \qquad \text{as } |\mathbf{r}| \to \infty \tag{17}$$

subject to the interface equations

$$\Phi_s(\mathbf{r}) = \Phi_p(\mathbf{r}) \quad \mathbf{r} \in \Gamma \tag{18}$$

$$\epsilon_\infty \frac{\partial \Phi_s(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} + (\epsilon_s - \epsilon_\infty)\mathbf{v}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) - \epsilon_p \frac{\partial \Phi_p(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} = 0 \quad \mathbf{r} \in \Gamma \tag{19}$$

,

where $\mathbf{n}(\mathbf{r})$ is the outward normal, and

$$Q_\lambda(\mathbf{r}) = \frac{1}{4\pi\lambda^2|\mathbf{r}|}e^{-|\mathbf{r}|/\lambda} \quad \mathbf{r} \neq 0$$

$$\mathbf{v}(\mathbf{r}) = \int_{\mathbb{R}^3} Q_\lambda(\mathbf{r}-\mathbf{r}')\nabla\Phi(\mathbf{r}')d\mathbf{r}'$$

.

As we shall desire solutions, we present Theorem 2.1.

Theorem 2.1. *Let $\epsilon_p, \epsilon_\infty, \epsilon_s, a > 0$, be constants with $\epsilon_s > \epsilon_\infty$. Also let $\epsilon_0 = 1$.*

$D_p = \{\mathbf{r}|r < a\}$, $\Gamma = \{\mathbf{r}|r = a\}$, $D_s = \{\mathbf{r}|r > a\}$

Let ρ_p be a distribution defined on D_p, with support inside some closed set X within D_p. Let $\int_{D_p} |\rho_p(\mathbf{r})|d\mathbf{r} < \infty$. Denote $\frac{1}{\lambda}\sqrt{\frac{\epsilon_s}{\epsilon_\infty}}$ by ω.

Let P_n^m denote the Associated Legendre polynomial, $i_n(r), k_n(r)$ denote the modified spherical Bessel functions as defined in (185), (182), (183)

Define Φ_p and Φ_s by

$$\begin{gathered}\Phi_p(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\frac{C_{p,n,m}}{\epsilon_p}r^n + \frac{r^n G_{n,m}(a)}{\epsilon_p a^n})P_n^m(\cos\phi)e^{im\theta} \\ + \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\frac{G_{n,m}(r)}{\epsilon_p} - \frac{r^n G_{n,m}(a)}{\epsilon_p a^n})P_n^m(\cos\phi)e^{im\theta}\end{gathered} \tag{20}$$

$$\begin{gathered}\Phi_s(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s}B_{s,n,m}k_n(\omega r)P_n^m(\cos\phi)e^{im\theta} \\ + \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{D_{s,n,m}}{\epsilon_s}r^{-n-1}P_n^m(\cos\phi)e^{im\theta}\end{gathered} \tag{21}$$

where the spherical coordinates of $\mathbf{r}$ are given by (r, ϕ, θ).

Let u_p and u_s defined on D_p and D_s be defined by

$$\begin{gathered}u_p(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-A_{p,n,m} - \frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})\epsilon_p\lambda})i_n(\frac{r}{\lambda})P_n^m(\cos\phi)e^{im\theta} \\ + \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\frac{C_{p,n,m}}{\epsilon_p}r^n + \frac{r^n G_{n,m}(a)}{\epsilon_p a^n})P_n^m(\cos\phi)e^{im\theta} \\ + \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{\epsilon_p\lambda}(\lambda G_{n,m}(r) - \lambda\frac{r^n G_{n,m}(a)}{a^n} - H_{n,m}(r) + \frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})}i_n(\frac{r}{\lambda}))P_n^m(\cos\phi)e^{im\theta}\end{gathered} \quad . \tag{22}$$

$$u_s(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{-\epsilon_\infty}{\epsilon_s} B_{s,n,m} k_n(\omega r) P_n^m(\cos\phi) e^{im\theta} + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{D_{s,n,m}}{\epsilon_s} r^{-n-1} P_n^m(\cos\phi) e^{im\theta} \tag{23}$$

satisfy

$$u(\mathbf{r}) = (Q_\lambda * \Phi)(\mathbf{r}) = \int_{\mathbb{R}^3} Q_\lambda(\mathbf{r} - \mathbf{r}')\Phi(\mathbf{r}')d\mathbf{r}'. \tag{24}$$

.

The coefficients within Φ_s*,* Φ_p*,* u_s*, and* u_p *are given by* (74), (75), (76), (77)*. These in turn are defined by* (68), (69), (70), (71), (48), (49)*.*

Then outside of X*,* Φ_s *and* Φ_p *weakly solve* (15), (16), (17) *subject to* (18), (19)*. The convergence of the series is geometric with the ratio dependent on* ρ*.*

To prove Theorem 2.1, we shall need to derive a solution. This will produce function u. We will need to prove Lemma 4.1 to show this u behaves. Then, as the solution is a series, we shall need to prove converge in Lemma 5.1.

It is worth mentioning that while the model is still valid when D_p is not a ball, we restrict ourselves to this case for the purposes of having analytical solutions. While more complicated protein structures will have more interesting domains, the monatomic ion is naturally represented by a ball. Thus, the $D_p = \{\mathbf{r} | r < a\}$, $\Gamma = \{\mathbf{r} | r = a\}$, $D_s = \{\mathbf{r} | r > a\}$ case is worth studying in its own right.

All of our equations discussed earlier are linear in terms of ρ. Thus, as e_c and ϵ_0 occur only with the presence of ρ, we may solve the differential equations with $e_c = \epsilon_0 = 1$ and then scale our result by the true e_c and ϵ_0.

One might wonder how essential it is for Γ to be a sphere centered at the origin. We shall mention the difficulties that arise if Γ is not a sphere when we use the properties of the sphere. Such observations will also be summarized in the remarks at the end.

We solve the differential equations in section 3.

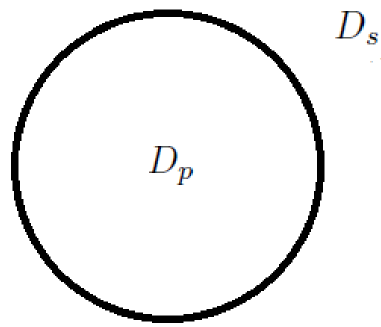


Figure 2: Spherical appearance of a protein in water

3 Deriving the Solution

3.1 Split into a system

We can create an equivalent version of (16) by recognizing that differentiation of a convolution is the convolution of the derivative in the second function, and that all functions applying to this set of differential equations go to 0 at infinity. Define u by

$$u(\mathbf{r}) = (Q_\lambda * \Phi)(\mathbf{r}) = \int_{\mathbb{R}^3} Q_\lambda(\mathbf{r} - \mathbf{r}')\Phi(\mathbf{r}')d\mathbf{r}' \tag{25}$$

.

Observing that Q_λ solves $-\lambda^2 \Delta Q_\lambda(\mathbf{r}) + Q_\lambda(\mathbf{r}) = \delta(\mathbf{r})$, we have that u solves

$$-\lambda^2 \Delta u(\mathbf{r}) + u(\mathbf{r}) - \Phi(\mathbf{r}) = 0$$

.

When divided by λ^2, and inserted into (16), we get

$$-\epsilon_\infty \Delta \Phi_s(\mathbf{r}) + \frac{(\epsilon_s - \epsilon_\infty)}{\lambda^2}(\Phi_s(\mathbf{r}) - u(\mathbf{r})) = 0 \quad \mathbf{r} \in D_s \tag{26}$$

.

With $u_p(\mathbf{r}) = u(\mathbf{r})$ for $\mathbf{r} \in D_p$ and $u_s(\mathbf{r}) = u(\mathbf{r})$ for $\mathbf{r} \in D_s$, we can reformulate the equations as

$$-\epsilon_p \Delta \Phi_p(\mathbf{r}) = \rho_p(r) \quad \mathbf{r} \in D_p \tag{27}$$

$$-\epsilon_\infty \Delta \Phi_s(\mathbf{r}) + \frac{(\epsilon_s - \epsilon_\infty)}{\lambda^2}(\Phi_s(\mathbf{r}) - u_s(\mathbf{r})) = 0 \quad \mathbf{r} \in D_s \tag{28}$$

$$-\lambda^2 \Delta u_p(\mathbf{r}) + u_p(\mathbf{r}) - \Phi_p(\mathbf{r}) = 0 \quad \mathbf{r} \in D_p \tag{29}$$

$$-\lambda^2 \Delta u_s(\mathbf{r}) + u_s(\mathbf{r}) - \Phi_s(\mathbf{r}) = 0 \quad \mathbf{r} \in D_s \tag{30}$$

.

To formulate the interface equations, since u is continuously differentiable,

$$u_p(\mathbf{r}) = u_s(\mathbf{r}) \quad \mathbf{r} \in \Gamma \tag{31}$$

$$\frac{\partial u_p(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} = \frac{\partial u_s(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} \quad \mathbf{r} \in \Gamma \tag{32}$$

.

Since $\Phi_s(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$, the same applies to u_s. That is, $u_s(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$.

The first original interface equation remains:

$$\Phi_s(\mathbf{r}) = \Phi_p(\mathbf{r}) \quad \mathbf{r} \in \Gamma \tag{33}$$

By substituting u into (19), we get

$$\epsilon_\infty \frac{\partial \Phi_s(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} + (\epsilon_s - \epsilon_\infty) \frac{\partial u_s(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} - \epsilon_p \frac{\partial \Phi_p(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} = 0 \quad \mathbf{r} \in \Gamma \tag{34}$$

.

So, our new problem to solve is (27), (28), (29), (30) subject to (31), (32), (33), (34). For the solution to this system to be consistent with our original problem, we shall also need to verify the other definition for u given by (25).

To solve (27), (29) and (28), (30), for different choices of A, B, C, we have the following system of equations

$$-A\Delta f(\mathbf{r}) + Bf(\mathbf{r}) - Bg(\mathbf{r}) = h(\mathbf{r}) \tag{35}$$

$$-C\Delta g(\mathbf{r}) + g(\mathbf{r}) - f(\mathbf{r}) = 0 \tag{36}$$

C(35)-A(36) gives a single equation to solve for $f - g$

$$-AC\Delta(f-g)(\mathbf{r}) + (BC + A)(f-g)(\mathbf{r}) = Ch(\mathbf{r}) \tag{37}$$

-(35)-B(36) gives a single equation to solve for $Af + CBg$

$$\Delta(Af + CBg)(\mathbf{r}) = -h(\mathbf{r}) \tag{38}$$

From here, we can derive solutions for f and g.

$$f = \frac{CB(f-g) + (Af + CBg)}{CB + A}$$

$$g = \frac{-A(f-g) + (Af + CBg)}{CB + A}$$

These will be used to solve (35) and (36)

with $A = \epsilon_p$, $B = 0$, $C = \lambda^2, h = \rho_p$ for (27) and (29) and with $A = \epsilon_\infty$, $B = \frac{\epsilon_s - \epsilon_\infty}{\lambda^2}$, $C = \lambda^2, h = 0$ for (28) and (30). In both cases, $f = \Phi$, $g = u$

3.2 Solving the system (homogeneous part)

As with any differential equation, we desire a particular solution and homogeneous solutions and then we will later use the boundary conditions to determine the correct homogeneous solution to pair with the particular. Occasionally, such as in this case, we are also able to decompose the problem into orthogonal components and solve them separately. The completeness and orthogonality of spherical harmonics, $P_n^m(\cos\phi)e^{im\theta}$, defined in the appendix at (185) , on the sphere implies that it is enough to solve the homogeneous equation with solutions of the form $f(\mathbf{r}) = f_{n,m}(r)P_n^m(\cos\phi)e^{im\theta}$ and then sum the solutions over $n = 0, 1, 2, 3...$ and $m = -n, -n+1, -n+2, ..., n$ [5]

Then, since

$$\Delta f = \frac{1}{r^2}[2rf_r + r^2 f_{rr} + \frac{\cos\phi}{\sin\phi} f_\phi + f_{\phi\phi} + \frac{1}{\sin^2\phi} f_{\theta\theta}], \tag{39}$$

we get

$$\Delta f_{n,m}(r)P_n^m(\cos\phi)e^{im\theta} = \frac{1}{r^2}\begin{bmatrix} 2rf'_{n,m}(r)P_n^m(\cos\phi)e^{im\theta} \\ +r^2f''_{n,m}(r)P_n^m(\cos\phi)e^{im\theta} \\ -2f_{n,m}(r)P_n^{m'}(\cos\phi)\cos\phi e^{im\theta} \\ +(1-\cos^2\phi)2f_{n,m}(r)P_n^{m''}(\cos\phi)e^{im\theta} \\ +\frac{-m^2}{1-\cos^2\phi}2f_{n,m}(r)P_n^m(\cos\phi)e^{im\theta} \end{bmatrix}.$$

To simplify this, we appeal to the properties of the Associated Legendre polynomials given in (189), and we get

$$\Delta f_{n,m}(r)P_n^m(\cos\phi)e^{im\theta} = \frac{1}{r^2}\left[-n(n+1)f_{n,m}(r) + 2rf'_{n,m}(r) + r^2f''_{n,m}(r)\right]P_n^m(\cos\phi)e^{im\theta}. \quad (40)$$

As an ordinary differential equation, $\Delta f_{n,m}(r) = 0$ has independent solutions r^n and r^{-n-1}. Thus, $\Delta f = 0$ has solution

$$f(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_{n,m}r^nP_n^m(\cos\phi)e^{im\theta} + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} D_{n,m}r^{-n-1}P_n^m(\cos\phi)e^{im\theta}. \quad (41)$$

As an ordinary differential equation, $\Delta f_{n,m}(r) - \kappa^2 f_{n,m}(r) = 0$ with $\kappa > 0$ has independent solutions $i_n(\kappa r)$ and $k_n(\kappa r)$ from (186) and (187) Thus, $\Delta f - \kappa^2 f = 0$, with $\kappa > 0$ has solution

$$f(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{n,m}i_n(\kappa r)P_n^m(\cos\phi)e^{im\theta} + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} B_{n,m}k_n(\kappa r)P_n^m(\cos\phi)e^{im\theta}. \quad (42)$$

The next step is to apply the required constants of A, B, C to solve for $f = \Phi$, $g = u$ in 37 and 38. Before giving the results, recall how we desire solutions that are bounded as $r \to \infty$ in D_s and as $r \to 0$ in D_p. Therefore from the limits of i_n and k_n, $B_{p,n,m} = D_{p,n,m} = 0$, and $A_{s,n,m} = C_{s,n,m} = 0$. It is worth mentioning that depending on the choice of ρ, Φ_p and u_p may be unbounded. Therefore, all that we can enforce is that the homogeneous solution is bounded.

Merely substituting will produce the homogeneous solutions

$$\Phi_{p,h}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{C_{p,n,m}}{\epsilon_p} r^n P_n^m(\cos\phi) e^{im\theta}$$

$$u_{p,h}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} -A_{p,n,m} i_n(r\tfrac{1}{\lambda}) P_n^m(\cos\phi) e^{im\theta} + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tfrac{C_{p,n,m}}{\epsilon_p} r^n P_n^m(\cos\phi) e^{im\theta}$$

$$\Phi_{s,h}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tfrac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} B_{s,n,m} k_n(\tfrac{r}{\lambda}\sqrt{\tfrac{\epsilon_s}{\epsilon_\infty}}) P_n^m(\cos\phi) e^{im\theta} + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tfrac{D_{s,n,m}}{\epsilon_s} r^{-n-1} P_n^m(\cos\phi) e^{im\theta}$$

$$u_{s,h}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tfrac{-\epsilon_\infty}{\epsilon_s} B_{s,n,m} k_n(\tfrac{r}{\lambda}\sqrt{\tfrac{\epsilon_s}{\epsilon_\infty}}) P_n^m(\cos\phi) e^{im\theta} + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tfrac{D_{s,n,m}}{\epsilon_s} r^{-n-1} P_n^m(\cos\phi) e^{im\theta}$$

We will abbreviate the frequently occurring $\frac{1}{\lambda}\sqrt{\frac{\epsilon_s}{\epsilon_\infty}}$ by ω.

3.3 Finding the particular solution

Again, we note that since $P_n^m(\cos\phi)e^{im\theta}$ is complete and orthogonal on the sphere, we can decompose $f(\mathbf{r})$ on $\mathbb{R}^3$

$$f(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{m,n}(r) P_n^m(\cos\phi) e^{im\theta}$$

Define the normalization factor

$$c_{n,m} = \int_0^{\pi} \int_0^{2\pi} P_n^m(\cos\phi) e^{im\theta} \overline{P_n^m(\cos\phi) e^{im\theta}} \sin\phi \, d\theta \, d\phi = \frac{4\pi(n+m)!}{(2n+1)(n-m)!}$$

Then by orthogonality,

$$\int_0^{\pi}\int_0^{2\pi} f(\mathbf{r})\overline{P_n^m(\cos\phi)e^{im\theta}}\sin\phi d\theta d\phi = f_{n,m}(r)c_{n,m}$$

$$f_{n,m}(r) = \frac{1}{c_{n,m}}\int_0^{\pi}\int_0^{2\pi} f(\mathbf{r})\overline{P_n^m(\cos\phi)e^{im\theta}}\sin\phi d\theta d\phi \tag{43}$$

In the literature, the spherical harmonics are often normalized as $\frac{P_n^m(\cos\phi)e^{im\theta}}{\sqrt{c_{n,m}}}$ to create an orthonormal sequence. Similar to the 2π factor in the Fourier transforms, the modifications do not fundamentally change any results. For instance, the orthogonality of $P_n^m(\cos\phi)e^{im\theta}$ respect to the $\sin(\phi)d\phi$ integral on a sphere tells us that $f = g$ if $f_{n,m} = g_{n,m}, \forall n, m$

Now we work to solve the inhomogeneous problems (37) and (38) with $A = \epsilon_p$, $B = 0$, $C = \lambda^2, h = \rho_p, f = \Phi, g = u$, we proceed with those substitutions using (40).

Then, by equating coefficients of $P_n^m(\cos\phi)e^{im\theta}$, we get

$$\frac{1}{r^2}(-n(n+1)(\epsilon_p\Phi)_{n,m}(r) + 2r(\epsilon_p\Phi)'_{n,m}(r) + r^2(\epsilon_p\Phi)''_{n,m}(r)) = -\rho_{p,n,m}(r) \tag{44}$$

$$\frac{1}{r^2}(-n(n+1) - \frac{1}{\lambda^2}r^2(\Phi - u)_{n,m}(r) + 2r(\Phi - u)'_{n,m}(r) + r^2(\Phi - u)''_{n,m}(r)) = \frac{-\rho_{p,n,m}(r)}{\epsilon_p}. \tag{45}$$

Then (44) will solve (38), and (45) will solve (37).

So, to solve (44) , we identify independent solutions to the homogeneous equation

$$\frac{1}{r^2}(-n(n+1)(\epsilon_p\Phi)_{n,m}(r) + 2r(\epsilon_p\Phi)'_{n,m}(r) + r^2(\epsilon_p\Phi)''_{n,m}(r)) = 0$$

As mentioned before, they are $y_1(r) = r^n$ and $y_2(r) = r^{-n-1}$. Thus, using (209), a particular solution to (44) is

$$(\epsilon_p\Phi)_{n,m} = -r^n \int \frac{r^{-n-1}\rho_{p,n,m}(r)r^2}{2n+1} + r^{-n-1} \int \frac{r^n \rho_{p,n,m}(r)r^2}{2n+1}$$

$$(\epsilon_p\Phi)_{n,m}(r) = \frac{1}{2n+1}\left(-r^n \int r^{1-n}\rho_{p,n,m}(r)dr + r^{-1-n}\int r^{n+2}\rho_{p,n,m}(r)dr\right). \tag{46}$$

To solve (45) , we identify the solutions to the homogeneous equation

$$\frac{1}{r^2}(-n(n+1) - \frac{1}{\lambda^2}r^2(\Phi-u)_n(r) + 2r(\Phi-u)'_n(r) + r^2(\Phi-u)''_n(r)) = 0.$$

They are $y_1(r) = i_n(r/\lambda)$ and $y_2(r) = k_n(r/\lambda)$. Thus, using (209), a particular solution to (45) is

$$(\Phi-u)_{n,m} = -i_n(\frac{r}{\lambda})\int \frac{k_n(\frac{r}{\lambda})\rho_{p,n,m}(r)r^2}{\epsilon_p\lambda}dr + k_n(\frac{r}{\lambda})\int \frac{i_n(\frac{r}{\lambda})\rho_{p,n,m}(r)r^2}{\epsilon_p\lambda}dr$$

$$(\Phi-u)_{n,m}(r) = \frac{1}{\epsilon_p\lambda}\left(-i_n(\frac{r}{\lambda})\int k_n(\frac{r}{\lambda})\rho_{p,n,m}(r)r^2dr + k_n(\frac{r}{\lambda})\int i_n(\frac{r}{\lambda})\rho_{p,n,m}(r)r^2dr\right). \tag{47}$$

The choice of bounds of integration will change the particular solution by a homogeneous solution. To produce specific solutions, we will choose solutions

$$(\epsilon_p\Phi)_{n,m}(r) = \frac{1}{2n+1}\left(-r^n\int_0^r t^{1-n}\rho_{p,n,m}(t)dt + r^{-1-n}\int_0^r t^{n+2}\rho_{p,n,m}(t)dt\right) \tag{48}$$

$$(\Phi-u)_{n,m}(r) = \frac{1}{\epsilon_p\lambda}\left(-i_n(\frac{r}{\lambda})\int_0^r k_n(\frac{t}{\lambda})\rho_{p,n,m}(t)t^2dt + k_n(\frac{r}{\lambda})\int_0^r i_n(\frac{t}{\lambda})\rho_{p,n,m}(t)t^2dt\right). \tag{49}$$

3.4 Solving the system

For ease of notation, we will denote

$$(\Phi - u)_{n,m}(r) = \frac{1}{\epsilon_p \lambda} H_{n,m}(r) \tag{50}$$

$$(\epsilon_p \Phi)_{n,m}(r) = G_{n,m}(r). \tag{51}$$

Thus for the components of the particular solution, we have

$$\Phi_{n,m}(r) = \frac{G_{n,m}(r)}{\epsilon_p} \tag{52}$$

$$u_{n,m}(r) = \frac{1}{\epsilon_p \lambda}(\lambda G_{n,m} - H_{n,m})(r). \tag{53}$$

Thus, the full general solutions become

$$\begin{aligned} \Phi_p(r) = \textstyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{C_{p,n,m}}{\epsilon_p} r^n P_n^m(\cos\phi) e^{im\theta} \\ + \textstyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{G_{n,m}(r)}{\epsilon_p} P_n^m(\cos\phi) e^{im\theta} \end{aligned} \tag{54}$$

$$\begin{aligned} u_p(\mathbf{r}) = \textstyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} -A_{p,n,m} i_n(\frac{r}{\lambda}) P_n^m(\cos\phi) e^{im\theta} \\ + \textstyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{C_{p,n,m}}{\epsilon_p} r^n P_n^m(\cos\phi) e^{im\theta} \\ + \textstyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{\epsilon_p \lambda}(\lambda G_{n,m} - H_{n,m})(r) P_n^m(\cos\phi) e^{im\theta} \end{aligned} \tag{55}$$

And as the equations defined in D_s had no non-homogeneous part, we repeat their general forms

$$\begin{aligned} \Phi_s(\mathbf{r}) = \textstyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} B_{s,n,m} k_n(\omega r) P_n^m(\cos\phi) e^{im\theta} \\ + \textstyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{D_{s,n,m}}{\epsilon_s} r^{-n-1} P_n^m(\cos\phi) e^{im\theta} \end{aligned} \tag{56}$$

$$u_s(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{-\epsilon_\infty}{\epsilon_s} B_{s,n,m} k_n(\omega r) P_n^m(\cos\phi) e^{im\theta} + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{D_{s,n,m}}{\epsilon_s} r^{-n-1} P_n^m(\cos\phi) e^{im\theta} \tag{57}$$

For the purposes of series convergence, to be shown later in Section 5, it will be more convenient to write the solutions as

$$\Phi_p(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \left(\frac{C_{p,n,m}}{\epsilon_p} r^n + \frac{r^n G_{n,m}(a)}{\epsilon_p a^n}\right) P_n^m(\cos\phi) e^{im\theta} + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \left(\frac{G_{n,m}(r)}{\epsilon_p} - \frac{r^n G_{n,m}(a)}{\epsilon_p a^n}\right) P_n^m(\cos\phi) e^{im\theta} \tag{58}$$

$$\begin{aligned} u_p(\mathbf{r}) &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \left(-A_{p,n,m} - \frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})\epsilon_p\lambda}\right) i_n(\tfrac{r}{\lambda}) P_n^m(\cos\phi) e^{im\theta} \\ &+ \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \left(\frac{C_{p,n,m}}{\epsilon_p} r^n + \frac{r^n G_{n,m}(a)}{\epsilon_p a^n}\right) P_n^m(\cos\phi) e^{im\theta} \\ &+ \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{1}{\epsilon_p\lambda}\left(\lambda G_{n,m}(r) - \lambda\frac{r^n G_{n,m}(a)}{a^n} - H_{n,m}(r) + \frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})} i_n(\tfrac{r}{\lambda})\right) P_n^m(\cos\phi) e^{im\theta} \end{aligned} \quad . \tag{59}$$

Observe that $(56), (57), (58), (59)$ match $(21), (23), (20), (22)$ of Theorem 2.1.

We wish to apply the boundary equations, $(31), (32), (33), (34)$, but as written, they are equating series over Γ. We may integrate these series over Γ respect to any $\overline{P_n^m(\cos\phi)e^{im\theta}}$ to eliminate all but one of the terms in the series. Then we need only equate the coefficients to $P_n^m(\cos\phi)e^{im\theta}$. This tells us that equating the coefficients to $P_n^m(\cos\phi)e^{im\theta}$ is not only a sufficient condition, but also a necessary one to achieve equality in the boundary equations.

This is where it is essential that D_p is a ball centered at the origin. Otherwise, integrating the boundary condition over Γ, with varying radii, will not eliminate orthogonal components. Asserting instead that the coefficients to $P_n^m(\cos\phi)e^{im\theta}$ must match for all of the radii on the boundary may be too constrictive. Additionally, the boundary conditions feature the derivative in the normal direction on Γ. For spheres, $\frac{\partial}{\partial \mathbf{n}(\mathbf{r})} = \frac{\partial}{\partial_r}$. To make use of the orthogonal components, we need the derivative to not involve ϕ or θ.

For each $n = 0, 1, 2, ...$ and $m = -n, -n+1, ...n$, the coefficients to the components in the boundary conditions are given by

$$\begin{aligned} -A_{p,n,m} i_n(\tfrac{a}{\lambda}) + \tfrac{C_{p,n,m}}{\epsilon_p} a^n + \tfrac{1}{\epsilon_p \lambda}(\lambda G_{n,m}(a) - H_{n,m}(a)) \\ = \tfrac{-\epsilon_\infty}{\epsilon_s} B_{s,n,m} k_n(\omega a) + \tfrac{D_{s,n,m}}{\epsilon_s} a^{-n-1} \end{aligned} \tag{60}$$

$$\begin{aligned} -A_{p,n,m} \tfrac{i'_n(\frac{a}{\lambda})}{\lambda} + \tfrac{C_{p,n,m}}{\epsilon_p} n a^{n-1} + \tfrac{1}{\epsilon_p \lambda}(\lambda G'_{n,m}(a) - H'_{n,m}(a)) \\ = \tfrac{-\epsilon_\infty}{\epsilon_s} B_{s,n,m} k'_n(\omega a)\omega + \tfrac{D_{s,n,m}}{\epsilon_s}(-n-1) a^{-n-2} \end{aligned} \tag{61}$$

$$\begin{aligned} \tfrac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} B_{s,n,m} k_n(\omega a) + \tfrac{D_{s,n,m}}{\epsilon_s} a^{-n-1} \\ = \tfrac{C_{p,n,m}}{\epsilon_p} a^n + \tfrac{G_{n,m}(a)}{\epsilon_p} \end{aligned} \tag{62}$$

$$\begin{aligned} \epsilon_\infty \left(\tfrac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} B_{s,n,m} k'_n(\omega a)\omega + \tfrac{D_{s,n,m}}{\epsilon_s}(-n-1) a^{-n-2} \right) \\ +(\epsilon_s - \epsilon_\infty) \left(\tfrac{-\epsilon_\infty}{\epsilon_s} B_{s,n,m} k'_n(\omega a)\omega + \tfrac{D_{s,n,m}}{\epsilon_s}(-n-1) a^{-n-2} \right) \\ -\epsilon_p \left(\tfrac{C_{p,n,m}}{\epsilon_p} n a^{n-1} + \tfrac{G'_{n,m}(a)}{\epsilon_p} \right) = 0 \end{aligned} .$$

This last equation can easily be simplified into

$$D_{s,n,m}(-n-1)a^{-n-2} - C_{p,n,m} n a^{n-1} - G'_{n,m}(r) = 0. \tag{63}$$

Now we may solve the system of 4 equations and 4 unknowns.

(60) + (62) gives

$$-A_{p,n,m} i_n(\frac{a}{\lambda}) + \frac{-H_{n,m}(a)}{\epsilon_p \lambda} + B_{s,n,m} k_n(\omega a) = 0. \tag{64}$$

(61) + (63)/ϵ_p gives

$$\begin{aligned} -A_{p,n,m} \tfrac{i'_n(\frac{a}{\lambda})}{\lambda} + \tfrac{-H'_{n,m}(a)}{\epsilon_p \lambda} + \tfrac{D_{s,n,m}(-n-1)a^{-n-2}}{\epsilon_p} \\ = \tfrac{-\epsilon_\infty}{\epsilon_s} B_{s,n,m} k'_n(\omega a)\omega + \tfrac{D_{s,n,m}}{\epsilon_s}(-n-1) a^{-n-2} \end{aligned} . \tag{65}$$

(62)n − (63)a/ϵ_p gives

$$\begin{array}{c} \frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s} n B_{s,n,m} k_n(\omega a) + \frac{D_{s,n,m}}{\epsilon_s} n a^{-n-1} + \frac{D_{s,n,m}}{\epsilon_p}(n+1)a^{-n-1} \\ = n\frac{G_{n,m}(a)}{\epsilon_p} - \frac{G'_{n,m}(a)a}{\epsilon_p} \end{array} .$$

(64)$i'_n(\frac{a}{\lambda})$ − (65)$i_n(\frac{a}{\lambda})\lambda$ gives

$$\begin{array}{c} \frac{-H_{n,m}(r)}{\epsilon_p\lambda} i'_n(\frac{a}{\lambda}) + B_{s,n,m}k_n(\omega a)i'_n(\frac{a}{\lambda}) + \frac{H'_{n,m}(r)}{\epsilon_p} i_n(\frac{a}{\lambda}) + \frac{D_{s,n,m}(n+1)a^{-n-2}i_n(\frac{a}{\lambda})\lambda}{\epsilon_p} = \\ \frac{\epsilon_\infty}{\epsilon_s} B_{s,n,m}k'_n(\omega a)\omega\lambda i_n(\frac{a}{\lambda}) + \frac{D_{s,n,m}}{\epsilon_s}(n+1)a^{-n-2}\lambda i_n(\frac{a}{\lambda}) \end{array} .$$

These last 2 equations give us

$$\alpha_{n,m} D_{s,n,m} + \beta_{n,m} B_{s,n,m} = \frac{1}{\epsilon_p}(nG_{n,m}(a) - aG'_{n,m}(a)) \tag{66}$$

$$\gamma_{n,m} D_{s,n,m} + \delta_{n,m} B_{s,n,m} = \frac{1}{\epsilon_p}(\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})) \tag{67}$$

where

$$\alpha_{n,m} = \frac{1}{a^{n+1}}(\frac{n}{\epsilon_s} + \frac{n+1}{\epsilon_p}) \tag{68}$$

$$\beta_{n,m} = \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} n k_n(\omega a) \tag{69}$$

$$\gamma_{n,m} = \frac{\lambda i_n(\frac{a}{\lambda})(n+1)}{a^{n+2}}(\frac{1}{\epsilon_p} - \frac{1}{\epsilon_s}) \tag{70}$$

$$\delta_{n,m} = k_n(\omega a)i'_n(\frac{a}{\lambda}) - \frac{\epsilon_\infty}{\epsilon_s}k'_n(\omega a)\omega\lambda i_n(\frac{a}{\lambda}). \tag{71}$$

To verify that the determinant, $\alpha_{n,m}\delta_{n,m} - \gamma_{n,m}\beta_{n,m} \neq 0$, we shall first need to rewrite $\delta_{n,m}$. Using (195) and (196),

$$\delta_{n,m} = k_n(\omega a)\frac{n\lambda}{a}i_n(\frac{a}{\lambda}) + k_n(\omega a)i_{n+1}(\frac{a}{\lambda}) - \frac{\epsilon_\infty}{\epsilon_s}k_n(\omega a)\frac{n\lambda}{a}i_n(\frac{a}{\lambda}) + \frac{\epsilon_\infty}{\epsilon_s}k_{n+1}(\omega a)\omega\lambda i_n(\frac{a}{\lambda}).$$

$$\delta_{n,m} = \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s}(k_n(a\omega)\frac{n\lambda}{a}i_n(\frac{a}{\lambda})) + k_n(a\omega)i_{n+1}(\frac{a}{\lambda}) + \frac{\epsilon_\infty}{\epsilon_s}k_{n+1}(a\omega)\omega\lambda i_n(\frac{a}{\lambda}). \tag{72}$$

Since $\epsilon_s \geq \epsilon_\infty > 0$ and $i_n(r), k_n(r) > 0$ for $r > 0$, we have $\delta_{n,m} > 0$.

Then

$$\begin{array}{c} \delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m} = \\ \frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s}(k_n(\omega a)\frac{n\lambda}{a}i_n(\frac{a}{\lambda}))\frac{1}{a^{n+1}}(\frac{n}{\epsilon_s} + \frac{n+1}{\epsilon_p}) \\ +(k_n(\omega a)i_{n+1}(\frac{a}{\lambda}) + \frac{\epsilon_\infty}{\epsilon_s}k_{n+1}(\omega a)\omega\lambda i_n(\frac{a}{\lambda}))\frac{1}{a^{n+1}}(\frac{n}{\epsilon_s} + \frac{n+1}{\epsilon_p}) \\ -\frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s}nk_n(\omega a)\frac{\lambda i_n(\frac{a}{\lambda})(n+1)}{a^{n+2}}(\frac{1}{\epsilon_p} - \frac{1}{\epsilon_s}) \end{array},$$

which simplifies to

$$\begin{array}{c} \delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m} = \\ \frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s}(k_n(a\omega)\frac{n\lambda}{a}i_n(\frac{a}{\lambda}))\frac{2n+1}{a^{n+1}\epsilon_s} \\ +(k_n(a\omega)i_{n+1}(\frac{a}{\lambda}) + \frac{\epsilon_\infty}{\epsilon_s}k_{n+1}(a\omega)\omega\lambda i_n(\frac{a}{\lambda}))(\frac{n}{\epsilon_s a^{n+1}} + \frac{n+1}{\epsilon_p a^{n+1}}) \end{array}. \tag{73}$$

Thus, since $\epsilon_s > \epsilon_\infty$ and $i_n(r), k_n(r) > 0$ for $r > 0$, we have $\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m} > 0$

Continuing with the solving, we get

$$D_{s,n,m} = \frac{\epsilon_p^{-1}}{\alpha_{n,m}\delta_{n,m} - \gamma_{n,m}\beta_{n,m}}\begin{pmatrix} \delta_{n,m}(nG_{n,m}(a) - aG'_{n,m}(a)) \\ -\beta_{n,m}(\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})) \end{pmatrix} \tag{74}$$

$$B_{s,n,m} = \frac{\epsilon_p^{-1}}{\alpha_{n,m}\delta_{n,m} - \gamma_{n,m}\beta_{n,m}} \left(\begin{array}{c} -\gamma_{n,m}(nG_{n,m}(a) - aG'_{n,m}(a)) \\ +\alpha_{n,m}(\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})) \end{array} \right). \tag{75}$$

From here, we further substitute back to get

$$C_{p,n,m} = \frac{\epsilon_p(\epsilon_s - \epsilon_\infty)}{\epsilon_s a^n} B_{s,n,m} k_n(\omega a) + \frac{D_{s,n,m}}{\epsilon_s} a^{-2n-1}\epsilon_p - \frac{G_{n,m}(a)}{a^n} \tag{76}$$

$$A_{p,n,m} = \frac{1}{i_n(\frac{a}{\lambda})}(B_{s,n,m}k_n(\omega a) - \frac{H_{n,m}(a)}{\epsilon_p \lambda}). \tag{77}$$

As certain forms appear, it becomes useful to find G' and H'.

(48) gives

$$G'_{n,m}(r) = \frac{1}{2n+1}\left(-nr^{n-1}\int_0^r t^{1-n}\rho_{p,n,m}(t)dt + (-1-n)r^{-2-n}\int_0^r t^{n+2}\rho_{p,n,m}(t)dt\right).$$

Combined, we get

$$nG_{n,m}(a) - aG'_{n,m}(a) = a^{-1-n}\int_0^a r^{n+2}\rho_{p,n,m}(r)dr. \tag{78}$$

(49) gives

$$H'_{n,m}(r) = -\frac{i'_n(\frac{r}{\lambda})}{\lambda}\int_0^r k_n(\frac{t}{\lambda})\rho_{p,n,m}(t)t^2dt + \frac{k'_n(\frac{r}{\lambda})}{\lambda}\int_0^r i_n(\frac{t}{\lambda})\rho_{p,n,m}(t)t^2dt.$$

Combined, we get

$$\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda}) = \frac{k_n(\frac{a}{\lambda})i'_n(\frac{a}{\lambda}) - k'_n(\frac{a}{\lambda})i_n(\frac{a}{\lambda})}{\lambda}\int_0^a i_n(\frac{r}{\lambda})\rho_{p,n,m}(r)r^2dr,$$

and applying (197), we have

$$\frac{H_{n,m}(a)i_n'(\frac{a}{\lambda})}{\lambda} - H_{n,m}'(a)i_n(\frac{a}{\lambda}) = \frac{\lambda}{a^2}\int_0^a i_n(\frac{r}{\lambda})\rho_{p,n,m}(r)r^2dr. \tag{79}$$

3.5 Specific example with point charge

As mentioned in the introduction, if there is a point charge at $\mathbf{r_0}$ of 1, where the radius of $\mathbf{r_0}$ is r_0 Then $\rho_p(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r_0})$ Where δ satisfies

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^2\sin(\phi)f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r_0})drd\phi d\theta = f(\mathbf{r_0})$$

While (13) included multiple, but finitely many, charges, linearity of the differential equations implies that we may decompose such a problem into unit charges at an arbitrary point.

Using (43), we get

$$\rho_{p,n,m}(r) = \frac{1}{c_{n,m}}\delta_r(r-r_0)r^{-2}\overline{P_n^m(\cos\phi_0)e^{im\theta_0}}$$

where

$$\int_0^{\infty} f(r)\delta_r(r-r_0)dr = f(r_0).$$

So, we have that (44) is solved by

$$(\epsilon_p\Phi)_{n,m,p}(r) = \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{(2n+1)c_{n,m}}(-r^n\int_0^r t^{-1-n}\delta_r(t-r_0)dt + r^{-1-n}\int_0^r t^n\delta_r(t-r_0)dt)$$

$$G_{n,m}(r) = (\epsilon_p\Phi)_{n,m,p}(r) = \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{(2n+1)c_{n,m}}\left(\begin{cases} -r^n r_0^{-1-n} + r^{-1-n}r_0^n & r \geq r_0 \\ 0 & r \leq r_0 \end{cases}\right) \tag{80}$$

The other inhomogeneous problem (45) is solved similarly,

$$(\Phi - u)_{n,m,p}(r) = \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\lambda\epsilon_p c_{n,m}}\left(-i_n(\frac{r}{\lambda})\int_0^r k_n(\frac{t}{\lambda})\delta_r(t-r_0)dt + k_n(\frac{r}{\lambda})\int_0^r i_n(\frac{t}{\lambda})\delta_r(t-r_0)dt\right)$$

$$H_{n,m}(r) = \lambda\epsilon_p(\Phi - u)_{n,m,p}(r) = \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{c_{n,m}}\left(\begin{cases} -i_n(\frac{r}{\lambda})k_n(\frac{r_0}{\lambda}) + k_n(\frac{r}{\lambda})i_n(\frac{r_0}{\lambda}) & r \geq r_0 \\ 0 & r \leq r_0 \end{cases}\right) \tag{81}$$

For the purposes of solving the system for the homogeneous solution, we have from (78) and (79) give us

$$nG_{n,m}(a) - aG'_{n,m}(a) = \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{c_{n,m}}a^{-1-n}r_0^n \tag{82}$$

$$\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda}) = \frac{\lambda}{a^2}i_n(\frac{r_0}{\lambda})\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{(2n+1)c_{n,m}}. \tag{83}$$

Thus the full general solutions become

$$\begin{aligned}\Phi_p(r) &= \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{C_{p,n,m}}{\epsilon_p}r^nP_n^m(\cos\phi)e^{im\theta} \\ &+ \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n}P_n^m(\cos\phi)e^{im\theta}\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{(2n+1)c_{n,m}\epsilon_p}\begin{cases} -r^nr_0^{-1-n} + r^{-1-n}r_0^n & r \geq r_0 \\ 0 & r \leq r_0 \end{cases}\end{aligned} \tag{84}$$

$$\begin{aligned}u_p(\mathbf{r}) &= \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} -A_{p,n,m}i_n(\frac{r}{\lambda})P_n^m(\cos\phi)e^{im\theta} \\ &+ \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{C_{p,n,m}}{\epsilon_p}r^nP_n^m(\cos\phi)e^{im\theta} \\ &+ \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n}P_n^m(\cos\phi)e^{im\theta}\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{c_{n,m}\lambda\epsilon_p}\begin{cases} -\frac{\lambda}{(2n+1)}r^nr_0^{-1-n} + i_n(\frac{r}{\lambda})k_n(\frac{r_0}{\lambda}) & \\ +\frac{\lambda}{(2n+1)}r^{-1-n}r_0^n - k_n(\frac{r}{\lambda})i_n(\frac{r_0}{\lambda}) & r \geq r_0 \\ 0 & r \leq r_0 \end{cases}\end{aligned} \tag{85}$$

As the equations defined in D_s had no non-homogeneous part, we repeat their general forms

$$\begin{aligned}\Phi_s(\mathbf{r}) = &\textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s}B_{s,n,m}k_n(\omega r)P_n^m(\cos\phi)e^{im\theta}\\ &+\textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{D_{s,n,m}}{\epsilon_s}r^{-n-1}P_n^m(\cos\phi)e^{im\theta}\end{aligned} \tag{86}$$

$$\begin{aligned}u_s(\mathbf{r}) = &\textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{-\epsilon_\infty}{\epsilon_s}B_{s,n,m}k_n(\omega r)P_n^m(\cos\phi)e^{im\theta}\\ &+\textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{D_{s,n,m}}{\epsilon_s}r^{-n-1}P_n^m(\cos\phi)e^{im\theta}\end{aligned} \tag{87}$$

With $A_{p,n,m}, B_{s,n,m}, C_{p,n,m}, D_{s,n,m}$ defined as in $(77), (75), (76), (74)$.

However, when we examine Φ_p and u_p we may observe that the series as written does not obviously converge. In part this is because we have used (54) and (55) rather than (20) and (22) for our solutions. Even with the change, Theorem 2.1 has convergence proven by Lemma 5.1 that gives us convergent solution for $r \neq r_0$. However, under the point charge, we can do better.

First, we consider the version that produces better convergence, namely (20), (22).

$$\frac{G_{n,m}(r)}{\epsilon_p} - \frac{G_{n,m}(a)r^n}{a^n\epsilon_p} = \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p(2n+1)c_{n,m}}\left(\begin{cases} r^{-1-n}r_0^n - a^{-1-2n}r^n r_0^n & r \geq r_0 \\ r^n r_0^{-1-n} - a^{-1-2n}r^n r_0^n & r \leq r_0 \end{cases}\right).$$

By construction,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} G_{n,m}(r)P_n^m(\cos\phi)e^{im\theta} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\epsilon_p\Phi)_{n,m}(r)P_n^m(\cos\phi)e^{im\theta}$$

solves $\Delta f = -\frac{1}{\epsilon_p}\delta_{\mathbf{r_0},\mathbf{r}}$. The addition of $G_{n,m}(a)\frac{r^n}{a^n}$ terms only changes the particular solution. While this sum does not give us $\frac{1}{4\pi|\mathbf{r_0}-\mathbf{r}|\epsilon_p}$, a known particular solution, it must differ by a homogeneous solution. Comparing with (207), we observe that

$$\frac{1}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} P_n^m(\cos\phi)e^{im\theta}\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p(2n+1)c_{n,m}}\left(\begin{cases} r^{-1-n}r_0^n & r \geq r_0 \\ r^n r_0^{-1-n} & r \leq r_0 \end{cases}\right).$$

Thus,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{G_{n,m}(r)}{\epsilon_p} - \frac{r^n G_{n,m}(a)}{a^n\epsilon_p}\right)P_n^m(\cos\phi)e^{im\theta} \\ = \frac{1}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|} - \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a^{-1-2n}r^n r_0^n P_n^m(\cos\phi)e^{im\theta}\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p(2n+1)c_{n,m}}.$$

In a similar manner,

$$\frac{1}{\epsilon_p\lambda}\left(-H_{n,m}(r) + \frac{H_{n,m}(a)i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}\right) = \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p\lambda c_{n,m}}\begin{cases} -k_n(\frac{r}{\lambda})i_n(\frac{r_0}{\lambda}) + \frac{k_n(\frac{a}{\lambda})i_n(\frac{r_0}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} & r \geq r_0 \\ -i_n(\frac{r}{\lambda})k_n(\frac{r_0}{\lambda}) + \frac{k_n(\frac{a}{\lambda})i_n(\frac{r_0}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} & r \leq r_0 \end{cases}$$

By construction,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{H_{n,m}(r)}{\epsilon_p\lambda}P_n^m(\cos\phi)e^{im\theta} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\Phi-u)_{n,m}(r)P_n^m(\cos\phi)e^{im\theta}$$

solves $\Delta f - \frac{1}{\lambda^2}f = -\frac{1}{\epsilon_p}\delta_{\mathbf{r_0},\mathbf{r}}$. The addition of the the $H_{n,m}(a)\frac{i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}$ term only changes the particular solution. While this sum does not give us known particular solution $\frac{e^{-|\mathbf{r}-\mathbf{r_0}|/\lambda}}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|}$, it must differ by a homogeneous solution. Comparing with (208), we observe that

$$\frac{e^{-|\mathbf{r}-\mathbf{r_0}|/\lambda}}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}P_n^m(\cos\phi)e^{im\theta}\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p\lambda(2n+1)c_{n,m}}\left(\begin{cases} (2n+1)i_n(\frac{r_0}{\lambda})k_n(\frac{r}{\lambda}) & r \geq r_0 \\ (2n+1)k_n(\frac{r_0}{\lambda})i_n(\frac{r}{\lambda}) & r \leq r_0 \end{cases}\right)$$

Thus,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{\epsilon_p\lambda}\left(-H_{n,m}(r) + \frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})}i_n(\tfrac{r}{\lambda})\right)P_n^m(\cos\phi)e^{im\theta} \\ = -\frac{e^{-|\mathbf{r}-\mathbf{r_0}|/\lambda}}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|} + \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{k_n(\frac{a}{\lambda})i_n(\frac{r_0}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}P_n^m(\cos\phi)e^{im\theta}\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p\lambda c_{n,m}}.$$

Together, these produce forms that have more stability for r near r_0, and we rewrite Φ_p and u_p as follows using the (20) and (22) forms and the definition for $C_{p,n,m}$ and $A_{p,n,m}$ found in (76) and (77).

$$\begin{aligned} \Phi_p(r) = \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} r^n(\frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s a^n}B_{s,n,m}k_n(\omega a) + \frac{D_{s,n,m}}{\epsilon_s}a^{-2n-1})P_n^m(\cos\phi)e^{im\theta} \\ +\frac{1}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|} - \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a^{-1-2n}r^n r_0^n \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p(2n+1)c_{n,m}}P_n^m(\cos\phi)e^{im\theta} \end{aligned} . \tag{88}$$

$$\begin{aligned} u_p(\mathbf{r}) = -\textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{1}{i_n(\frac{a}{\lambda})}B_{s,n,m}k_n(\omega a)i_n(\frac{r}{\lambda})P_n^m(\cos\phi)e^{im\theta} \\ +\textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} r^n(\frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s a^n}B_{s,n,m}k_n(\omega a) + \frac{D_{s,n,m}}{\epsilon_s}a^{-2n-1})P_n^m(\cos\phi)e^{im\theta} \\ +\frac{1}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|} - \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a^{-1-2n}r^n r_0^n \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p(2n+1)c_{n,m}}P_n^m(\cos\phi)e^{im\theta} \\ -\frac{e^{-|\mathbf{r}-\mathbf{r_0}|/\lambda}}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|} + \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{k_n(\frac{a}{\lambda})i_n(\frac{r_0}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p\lambda c_{n,m}}P_n^m(\cos\phi)e^{im\theta} \end{aligned} . \tag{89}$$

This leads us to Theorem 3.1.

Theorem 3.1. *Let the conditions of Theorem 2.1 hold. Let* $\mathbf{r_0} \in D_p$. *Let* $\rho_p(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r_0})$.

If we instead define Φ_p, Φ_s, u_p, u_s *by*

(88), (86), (89), (87), *then* Φ_s *and* Φ_p *weakly solve* (15), (16), (17) *subject to* (18), (19). *The convergence of the series is geometric with the ratio dependent on* ρ.

However, the verification in Lemma 4.1 and the convergence in Lemma 5.3 must still be shown to prove the theorem.

(a) $r = 0.1$ (b) $r = 0.25$ (c) $r = 0.5$

Figure 3: zxy^+ Hemisphere values for Φ on Γ for the point charge at $(\phi,\theta) = (\frac{\pi}{4}, \frac{3\pi}{4})$ for various radii

The coloring in [Figure 3] scaled so that 0 is colored red and the maximum value is colored blue.

For these samples, the maximums are approximately $0.03368, 0.04563, 0.07945$ respectively. $\epsilon_\infty = 1.8, \epsilon_s = 80, \epsilon_p = 1, a = 1, \lambda = 15$ As one might expect, when the charge is close to the boundary, it produces a greater potential near that part on the boundary and a shrinking potential on the opposite pole.

We also observe that there is a rotational symmetry among the solution. Since there is the rotational symmetry of the ρ_p function, such symmetry in the potential function is to be expected. This observation will be used in Section 5 to improve the convergence rates.

3.6 Specific example with charge distributed in ball

Now suppose instead that charge density function is uniformly distributed over a ball with radius b and the center at $\mathbf{r_0}$. For the moment, we will restrict ourselves to the case of the center being $\mathbf{r_0} = (0, 0, z)$. To have the total charge be of unit density, equivalent to that of the point charge model,we will have, $\rho_p(\mathbf{r}) = \frac{3}{4\pi b^3}$ for $|\mathbf{r} - \mathbf{r_0}| < b$ and $\rho_p(\mathbf{r}) = 0$ otherwise.

For the moment, we shall assume that this ball of support does not include the origin.

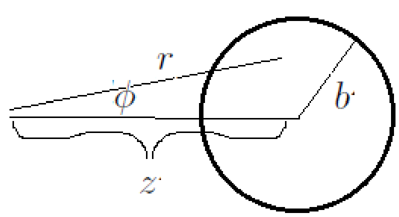


Figure 4: zx-plane cross view of D_p showing the support of ρ_p

Then by (43), we get

$$\rho_{p,n,m}(r) = \frac{1}{c_{n,m}} \int_0^\pi \int_0^{2\pi} \rho_p(\mathbf{r}) \overline{P_n^m(\cos\phi) e^{im\theta}} \sin\phi d\theta d\phi.$$

Since $|\mathbf{r}-\mathbf{r_0}|$ does not depend on θ, we have that $\rho_{p,n,m}(r)=0$ for $m\neq 0$.

For $m=0$, we get

$$\rho_{p,n,0}(r)=\frac{2\pi}{c_{n,0}}\int_0^{\pi}\rho_p(\mathbf{r})P_n(\cos\phi)\sin\phi d\phi.$$

For fixed r with $z-b<r<z+b$, we have that $|\mathbf{r}-\mathbf{r_0}|<b$ when $\cos(\phi)>\frac{b^2-r^2+z^2}{-2rz}$

So, for $z-b<r<z+b$

$$\rho_{p,n,0}(r)=\frac{2\pi}{c_{n,0}}\int_0^{\arccos(\frac{b^2-r^2+z^2}{-2rz})}P_n(\cos\phi)\sin\phi d\phi$$

$$=\frac{2\pi}{c_{n,0}}\int_1^{\frac{b^2-r^2+z^2}{-2rz}}-P_n(x)dx.$$

So, since $P_n(1)=1$, and (190), we have

$$\rho_{p,n,0}(r)=\frac{2\pi}{(2n+1)c_{n,0}}(P_{n-1}(\frac{b^2-r^2+z^2}{-2rz})-P_{n+1}(\frac{b^2-r^2+z^2}{-2rz})).$$

Thus, for $r\leq z-b$, $G_{n,0}(r)=0$.

For $z-b\leq r\leq z+b$,

$$G_{n,0}(r)=\frac{2\pi}{(2n+1)^2c_{n,0}}\left(\begin{array}{c}-r^n\int_{z-b}^{r}t^{1-n}(P_{n-1}(\frac{b^2-t^2+z^2}{-2tz})-P_{n+1}(\frac{b^2-t^2+z^2}{-2tz}))dt\\+r^{-1-n}\int_{z-b}^{r}t^{n+2}(P_{n-1}(\frac{b^2-t^2+z^2}{-2tz})-P_{n+1}(\frac{b^2-t^2+z^2}{-2tz}))dt\end{array}\right).\tag{90}$$

For $z+b\leq r$,

$$G_{n,0}(r)=\frac{2\pi}{(2n+1)^2c_{n,0}}\left(\begin{array}{c}-r^n\int_{z-b}^{z+b}t^{1-n}(P_{n-1}(\frac{b^2-t^2+z^2}{-2tz})-P_{n+1}(\frac{b^2-t^2+z^2}{-2tz}))dt\\+r^{-1-n}\int_{z-b}^{z+b}t^{n+2}(P_{n-1}(\frac{b^2-t^2+z^2}{-2tz})-P_{n+1}(\frac{b^2-t^2+z^2}{-2tz}))dt\end{array}\right).\tag{91}$$

Likewise, for $r < z - b$, $H_{n,0}(r) = 0$

For $z - b < r < z + b$,

$$H_{n,0}(r) = \frac{2\pi}{(2n+1)c_{n,0}}\left(\begin{array}{c} -i_n(\frac{r}{\lambda}) \int_{z-b}^{r} k_n(\frac{t}{\lambda}) t^2 (P_{n-1}(\frac{b^2-t^2+z^2}{-2tz}) - P_{n+1}(\frac{b^2-t^2+z^2}{-2tz}))dt \\ +k_n(\frac{r}{\lambda}) \int_{z-b}^{r} i_n(\frac{t}{\lambda}) t^2 (P_{n-1}(\frac{b^2-t^2+z^2}{-2tz}) - P_{n+1}(\frac{b^2-t^2+z^2}{-2tz}))dt \end{array} \right). \quad (92)$$

For $z + b < r$,

$$H_{n,0}(r) = \frac{2\pi}{(2n+1)c_{n,0}}\left(\begin{array}{c} -i_n(\frac{r}{\lambda}) \int_{z-b}^{z+b} k_n(\frac{t}{\lambda}) t^2 (P_{n-1}(\frac{b^2-t^2+z^2}{-2tz}) - P_{n+1}(\frac{b^2-t^2+z^2}{-2tz}))dt \\ +k_n(\frac{r}{\lambda}) \int_{z-b}^{z+b} i_n(\frac{t}{\lambda}) t^2 (P_{n-1}(\frac{b^2-t^2+z^2}{-2tz}) - P_{n+1}(\frac{b^2-t^2+z^2}{-2tz}))dt \end{array} \right). \quad (93)$$

Unfortunately, these integrals are over the intersection of a sphere with increasing radius and a ball. Thus, we will not expect to have nicer forms for when $z - b < r < z + b$.

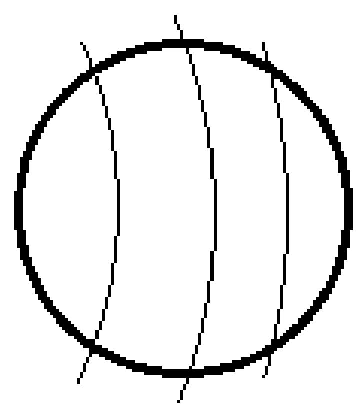


Figure 5: Curves of integration of ρ_p

To calculate $G_{n,m}(a)$ we shall make use of the knowledge that $\Delta f = -\delta$, has fundamental solution $\frac{1}{4\pi|\mathbf{r}|}$. Then to solve $\Delta f = -\rho_p$, we integrate and we obtain a particular solution for $\mathbf{r} \notin B(\mathbf{r_0}, b)$,

$$\frac{4\pi b^3}{3} f(\mathbf{r}) = \int_{B(\mathbf{r_0},b)} \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'.$$

A simple translation produces a radial integral,

$$\frac{4\pi b^3}{3} f(\mathbf{r}+\mathbf{r_0}) = \int_{B(0,b)} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'.$$

Being radial, we no longer need $\mathbf{r}$ to be completely arbitrary outside of $B(0,b)$. It is sufficient to consider only $\mathbf{r} = (0,0,z)$ Rewriting the integral, we get

$$\int_{B(0,b)} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = \int_0^b \frac{4\pi r'^2}{4\pi z} dr' = \frac{4\pi b^3}{3} \frac{1}{4\pi z}.$$

Therefore, a particular solution is $f(\mathbf{r}+\mathbf{r_0}) = \frac{1}{4\pi|\mathbf{r}|}$

$$f_p(\mathbf{r}) = \frac{1}{4\pi|\mathbf{r}-\mathbf{r_0}|}$$

From (207), we know that

$$f_p(\mathbf{r}) = \frac{1}{4\pi}\left(\begin{cases} \sum_{n=0}^{\infty} r_0^n r^{-n-1} P_n(\cos\phi) & \text{if } r > r_0 \\ \sum_{n=0}^{\infty} r_0^{-n-1} r^n P_n(\cos\phi) & \text{if } r < r_0 \end{cases}\right)$$

While this could be used for our particular solution, recall that our particular solution comes from the integral of $\rho_{p,n,m}$, and it is 0 for r near 0. Thus, to make this particular solution match ours, we add a homogeneous solution $\sum_{n=0}^{\infty} -r_0^{-n-1} r^n P_n(\cos\phi)$. This produces a particular solution

$$f_p(\mathbf{r}) = \frac{1}{4\pi}\left(\begin{cases} \sum_{n=0}^{\infty}(-r_0^{-n-1} r^n + r_0^n r^{-n-1}) P_n(\cos\phi) & \text{if } r > r_0 \\ \sum_{n=0}^{\infty} 0 & \text{if } r < r_0 \end{cases}\right).$$

Regrettably, this series no longer converges. For now, though, we are only interested in the coefficients to $P_n(\cos\phi)$. It is also worth pointing out that this particular solution is only defined for $\mathbf{r} \notin B(\mathbf{r_0}, b)$, thus we cannot assert that these coefficients are valid for $\mathbf{r}$ with radius $r_0 - b < |\mathbf{r}| < r_0 + b$.

Nonetheless, we obtain that for $r \geq r_0 + b$, using $c_{n,0} = \frac{4\pi}{(2n+1)}$,

$$G_{n,m}(r) = \frac{-r_0^{-n-1} r^n + r_0^n r^{-n-1}}{(2n+1)c_{n,m}} \tag{94}$$

$$G_{n,m}(a) = \frac{-r_0^{-n-1} a^n + r_0^n a^{-n-1}}{(2n+1)c_{n,m}} \tag{95}$$

and

$$nG_{n,m}(a) - aG'_{n,m}(a) = \frac{(2n+1)a^{-n-1} r_0^n}{c_{n,m}} \tag{96}$$

Since for our chosen $\mathbf{r_0}$, we have $\theta_0 = \phi_0 = 0$, comparing with the point charge model, (80) and (82), observe that we have obtained identical results for $G_{n,m}(a)$ and $nG_{n,m}(a) - aG'_{n,m}(a)$. They also agree for $G_{n,m}(r)$ with $r < z - b$ or $r > z + b$.

We repeat a similar process to calculate $H_{n,m}(a)$. First, we make use of the fundamental solution of $\Delta f - (1/\lambda^2)f = -\delta$ is $\frac{e^{-|\mathbf{r}|/\lambda}}{4\pi|\mathbf{r}|}$. Then to solve $\Delta f - (1/\lambda^2)f = -\rho_p$, we integrate and we obtain a particular solution for $\mathbf{r} \notin B(\mathbf{r_0}, b)$,

$$\frac{4\pi b^3}{3} f(\mathbf{r}) = \int_{B(\mathbf{r_0},b)} \frac{e^{-|\mathbf{r}-\mathbf{r}'|/\lambda}}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'.$$

A simple translation produces a radial integral.

$$\frac{4\pi b^3}{3} f(\mathbf{r} + \mathbf{r_0}) = \int_{B(0,b)} \frac{e^{-|\mathbf{r}-\mathbf{r}'|/\lambda}}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'.$$

Being radial, we no longer need $\mathbf{r}$ to be completely arbitrary outside of $B(0, b)$. It is sufficient to consider only $\mathbf{r} = (0, 0, z)$ with $z > b$. Rewriting the integral, we get

$$\int_{B(0,b)} \frac{e^{-|\mathbf{r}-\mathbf{r}'|/\lambda}}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = 4\pi b^2 \lambda \frac{e^{-z/\lambda}}{4\pi z} i_1(\frac{b}{\lambda}).$$

Thus a particular solution is

$$f(\mathbf{r}+\mathbf{r_0}) = \frac{3\lambda i_1(\frac{b}{\lambda})e^{-|\mathbf{r}|/\lambda}}{4\pi b|\mathbf{r}|}$$

$$f_p(\mathbf{r}) = \frac{3\lambda i_1(\frac{b}{\lambda})e^{-|\mathbf{r}-\mathbf{r_0}|/\lambda}}{4\pi b|\mathbf{r}-\mathbf{r_0}|}.$$

By (208), we have

$$f_p(\mathbf{r}) = \frac{3}{4\pi b} i_1(\frac{b}{\lambda}) \begin{cases} \sum_{n=0}^{\infty}(2n+1)i_n(\frac{z}{\lambda})k_n(\frac{r}{\lambda})P_n(\cos\phi) & \text{if } r > z \\ \sum_{n=0}^{\infty}(2n+1)i_n(\frac{r}{\lambda})k_n(\frac{z}{\lambda})P_n(\cos\phi) & \text{if } r < z \end{cases}$$

Again, while this could be used for our particular solution, we have ours defined using an integral of $\rho_{p,n,m}$, thus it must be 0 for r near 0. Thus, to make this particular solution match ours, we add a homogeneous solution $\sum_{n=0}^{\infty} -3i_1(\frac{b}{\lambda})(2n+1)i_n(\frac{r}{\lambda})k_n(\frac{z}{\lambda})$.

This produces

$$f_p(\mathbf{r}) = \frac{3}{4\pi b} i_1(\frac{b}{\lambda}) \begin{cases} \sum_{n=0}^{\infty}(2n+1)(-i_n(\frac{r}{\lambda})k_n(\frac{z}{\lambda}) + i_n(\frac{z}{\lambda})k_n(\frac{r}{\lambda}))P_n(\cos\phi) & \text{if } r > z \\ 0 & \text{if } r < z \end{cases}.$$

As before, we are only interested in the coefficients to $P_n(\cos\phi)$, and again, these coefficients are not valid for $\mathbf{r}$ with radius $r_0 - b < |\mathbf{r}| < r_0 + b$

As $H_{n,m}$ is the result from (45) and (50), using $c_{n,0} = \frac{4\pi}{2n+1}$ we get that for $r > r_0 + b$

$$H_{n,m}(r) = \frac{3\lambda i_1(\frac{b}{\lambda})}{c_{n,m} b}(-i_n(\frac{r}{\lambda})k_n(\frac{z}{\lambda}) + i_n(\frac{z}{\lambda})k_n(\frac{r}{\lambda})). \tag{97}$$

Thus

$$H_{n,m}(a) = \frac{3\lambda i_1(\frac{b}{\lambda})}{c_{n,m}b}(-i_n(\frac{a}{\lambda})k_n(\frac{z}{\lambda}) + i_n(\frac{z}{\lambda})k_n(\frac{a}{\lambda})) \tag{98}$$

and

$$\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda}) = \frac{3i_1(\frac{b}{\lambda})\lambda^2}{c_{n,m}a^2}i_n(\frac{z}{\lambda}). \tag{99}$$

Unlike the results for $G_{n,m}$, we notice that while $H_{n,m}(a)$ and $\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})$ for the ball charge model are similar to the point charge model (81) and (83), they are not the same. With our chosen $\mathbf{r_0}$ creating $\theta_0 = \phi_0 = 0$, we observe that the ball charge model has an additional factor of $\frac{3i_1(\frac{b}{\lambda})\lambda}{b}$.

We observe that

$$\lim_{b\to 0} \frac{3i_1(\frac{b}{\lambda})\lambda}{b} = \lim_{b\to 0} \frac{3\lambda^3}{b^3}\left(\frac{b}{\lambda}\cosh(\frac{b}{\lambda}) - \sinh(\frac{b}{\lambda})\right)$$

Applying L'hopital a few times gives us

$$\lim_{b\to 0} \frac{3i_1(\frac{b}{\lambda})\lambda}{b} = \lim_{b\to 0} \frac{3\lambda^3}{3b^2}(\frac{1}{\lambda}\cosh(\frac{b}{\lambda}) + \frac{b}{\lambda^2}\sinh(\frac{b}{\lambda}) - \frac{1}{\lambda}\cosh(\frac{b}{\lambda})) = \lim_{b\to 0}\frac{\lambda}{b}\sinh(\frac{b}{\lambda}) = \lim_{b\to 0}\frac{\lambda\cosh(\frac{b}{\lambda})}{\lambda} = 1$$

Thus, as $b \to 0$, the constants $H_{n,m}(a)$, $\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})$ of the ball charge model tend to the same value for the point charge model. We ought to expect this as the limit of function with unit total mass distributed uniformly on a ball with decreasing radius is one possible definition for the Dirac delta function.

As for the actual solutions Φ and u, we refer back to (20), (22), (21), (23).

Where $G_{n,m} = H_{n,m} = 0$ for $m \neq 0$, $G_{n,0}(r)$ defined by 0 for $r \leq z - b$, (90) for $z - b \leq r \leq z + b$ and (94) for $r \geq z + b$

$H_{n,0}(r)$ is defined by 0 for $r \leq z - b$, (92) for $z - b \leq r \leq z + b$ and (97) for $r \geq z + b$.

$G_{n,0}(a)$ is defined by (95)

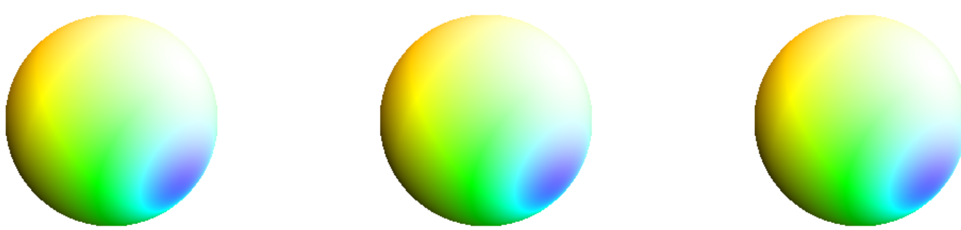
$nG_{n,0}(a) - aG'_{n,0}(a)$ is defined by (96)

$H(a)$ is defined by (98)

$\frac{H_{n,0}(a) i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,0}(a) i_n(\frac{a}{\lambda})$ is defined by (99)

Again, while the series for G and H may not directly converge, the forms for Φ_p and u_p given by (20) and (22) will produce, proven in Section 5, convergent series. Since the conditions for ρ satisfy the criteria of Theorem 5.4 we will have convergence, and thus weak solutions everywhere. Unfortunately, we did not produce convenient formula for G and H for general r beyond using an integral, so there is little to be gained from writing the solutions for general r with the integral rather than in the general form of (54), (55), (21), (23).

This example also required that the distributed charge not include the origin. However, that was not essential when we were calculating $G_{n,m}(r), H_{n,m}(r)$ for $r \geq z + b$, so that part will be the same. The case of $r \leq z - b$ can no longer occur. The case of $z - b \leq r \leq z + b$ will require us to change the bounds on the integrals, since the curves of integration, as seen from the curves figure, [Figure 4] and [Figure 5], will be entire spheres for r small.



(a) Point charge model (b) Ball model with radius 0.1 (c) Ball model with radius 0.5

Figure 6: Values for Φ on Γ for the point charge and ball charge centered at $(0, 0, 0.5)$ with varying radii, rotated $\pi/4$ about the $y - axis$, $3\pi/4$ about the $z - axis$.

The coloring in [Figure 6] scaled so that 0 is colored red and the maximum value is colored blue.

In these examples, the maximums are $0.079445, 0.079466, 0.079786$ respectively. $\epsilon_\infty = 1.8, \epsilon_s = 80, \epsilon_p = 1, a = 1, \lambda = 15$. While one might expect there to be some variance on the boundary by increasing the radius of the ball for the charge, visually, that is not the case. However, it is true that using a smaller radius for the charge distribution does produce values closer to that of the point charge model.

4 Verification

Lemma 4.1. *Let the conditions of Theorem 2.1 hold.*

For emphasis, define u_p on D_p and u_s on D_s as in (55) and (23). Define Φ_p and Φ_s as in (54) and (21).

Then

$$u_p(\mathbf{r}) = \int_{D_p} Q_\lambda(\mathbf{r} - \mathbf{r}')\Phi_p(\mathbf{r}')d\mathbf{r}' + \int_{D_s} Q_\lambda(\mathbf{r} - \mathbf{r}')\Phi_s(\mathbf{r}')d\mathbf{r}' \quad \mathbf{r} \in D_p$$

$$u_s(\mathbf{r}) = \int_{D_p} Q_\lambda(\mathbf{r} - \mathbf{r}')\Phi_p(\mathbf{r}')d\mathbf{r}' + \int_{D_s} Q_\lambda(\mathbf{r} - \mathbf{r}')\Phi_s(\mathbf{r}')d\mathbf{r}' \quad \mathbf{r} \in D_s.$$

4.1 Preliminary work

By construction, u and Φ solve the created system of differential equations, however, we must verify that the solution we have obtained is consistent with the definition of u, (25), used to create the system. Hence, we must verify Lemma 4.1.

Once those equations have been verified, we indeed have solutions (assuming convergence). This verification will also split up the formula by the orthogonal terms, $P_n^m(\cos\phi)e^{im\theta}$, and then verifying that their coefficients match. To do so, we appeal to (206), rewritten here

$$\begin{aligned}&\int_{r'=a} Q_\lambda(\mathbf{r}-\mathbf{r'})P_n^m(\cos(\phi'))e^{im\theta'}dS(\mathbf{r'})\\ &= \tfrac{a^2}{\lambda^3}P_n^m(\cos(\phi))e^{im\theta}\begin{cases} i_n(\tfrac{r}{\lambda})k_n(\tfrac{a}{\lambda}) & \text{for } r \le a \\ i_n(\tfrac{a}{\lambda})k_n(\tfrac{r}{\lambda}) & \text{for } r > a\end{cases}\end{aligned} \tag{100}$$

where r, ϕ, θ are the spherical coordinates to $\mathbf{r}$. Since Φ_p and Φ_s have been decomposed into these orthogonal terms, in (54) and (21), we must examine

$$\begin{aligned}&\int f(s)\int_{r'=s} Q_\lambda(\mathbf{r}-\mathbf{r'})P_n^m(\cos(\phi'))e^{im\theta'}dS(\mathbf{r'})ds\\ &= \tfrac{P_n^m(\cos(\phi))e^{im\theta}}{\lambda^3}\int s^2 f(s)\begin{cases} i_n(\tfrac{r}{\lambda})k_n(\tfrac{s}{\lambda}) & \text{for } r \le s \\ i_n(\tfrac{s}{\lambda})k_n(\tfrac{r}{\lambda}) & \text{for } r > s\end{cases} ds\end{aligned} \tag{101}$$

for the various radial functions f that occur in Φ_p and Φ_s. These functions are $f(s) = s^n, G_{n,m}(s)$ for Φ_p and $k_n(\omega s), s^{-n-1}$ for Φ_s.

So, we will evaluate

$$\int s^2 f(s) i_n(\frac{r}{\lambda})k_n(\frac{s}{\lambda})ds \quad \text{and} \quad \int s^2 f(s) i_n(\frac{s}{\lambda})k_n(\frac{r}{\lambda})ds.$$

When $f(s) = s^n$, we have by (193) and (191),

$$\int s^2 s^n i_n(\frac{r}{\lambda})k_n(\frac{s}{\lambda})ds = -i_n(\frac{r}{\lambda})k_{n+1}(\frac{s}{\lambda})s^{n+2}\lambda \tag{102}$$

$$\int s^2 s^n i_n(\frac{s}{\lambda})k_n(\frac{r}{\lambda})ds = i_{n+1}(\frac{s}{\lambda})k_n(\frac{r}{\lambda})s^{n+2}\lambda. \tag{103}$$

When $f(s) = s^{-n-1}$, we have by (194) and (192),

$$\int s^2 s^{-n-1} i_n(\frac{r}{\lambda})k_n(\frac{s}{\lambda})ds = -i_n(\frac{r}{\lambda})k_{n-1}(\frac{s}{\lambda})s^{-n+1}\lambda \tag{104}$$

$$\int s^2 s^{-n-1} i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) ds = i_{n-1}(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) s^{-n+1}\lambda. \quad (105)$$

When $f(s) = k_n(\omega s)$, we use (193) and (194) to produce

$$\int s^{-n} k_n(\omega s) i_n(\frac{r}{\lambda}) s^2 s^n k_n(\frac{s}{\lambda}) ds = \begin{pmatrix} -s^{-n} k_n(\omega s) i_n(\frac{r}{\lambda}) s^2 s^n k_{n+1}(\frac{s}{\lambda})\lambda \\ -\int s^{-n} k_{n+1}(\omega s) i_n(\frac{r}{\lambda}) s^2 s^n k_{n+1}(\frac{s}{\lambda}) \lambda\omega ds \end{pmatrix}$$

$$\int s^2 s^n k_n(\omega s) i_n(\frac{r}{\lambda}) s^{-n} k_n(\frac{s}{\lambda}) ds = \begin{pmatrix} -\frac{s^2 s^n}{\omega} k_{n+1}(\omega s) i_n(\frac{r}{\lambda}) s^{-n} k_n(\frac{s}{\lambda}) \\ -\frac{1}{\omega\lambda}\int s^2 s^n k_{n+1}(\omega s) i_n(\frac{r}{\lambda}) s^{-n} k_{n+1}(\frac{s}{\lambda}) ds \end{pmatrix}.$$

Thus

$$(\omega^2\lambda^2 - 1)\int s^2 k_n(\omega s) i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) ds = -\omega\lambda^2 k_{n+1}(\omega s) i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 + \lambda k_n(\omega s) i_n(\frac{r}{\lambda}) k_{n+1}(\frac{s}{\lambda}) s^2. \quad (106)$$

Applying (191) and (192) additionally gives

$$\int s^{-n} k_n(\omega s) k_n(\frac{r}{\lambda}) s^2 s^n i_n(\frac{s}{\lambda}) ds = \begin{pmatrix} s^{-n} k_n(\omega s) k_n(\frac{r}{\lambda}) s^2 s^n i_{n+1}(\frac{s}{\lambda})\lambda \\ -\int s^{-n} k_{n+1}(\omega s) k_n(\frac{r}{\lambda}) s^2 s^n i_{n+1}(\frac{s}{\lambda}) \lambda\omega ds \end{pmatrix}$$

$$\int s^2 s^n k_n(\omega s) k_n(\frac{r}{\lambda}) s^{-n} i_n(\frac{s}{\lambda}) ds = \begin{pmatrix} -\frac{s^2 s^n}{\omega} k_{n+1}(\omega s) k_n(\frac{r}{\lambda}) s^{-n} i_n(\frac{s}{\lambda}) \\ -\frac{1}{\omega\lambda}\int s^2 s^n k_{n+1}(\omega s) k_n(\frac{r}{\lambda}) s^{-n} i_{n+1}(\frac{s}{\lambda}) ds \end{pmatrix}.$$

Thus,

$$(\omega^2\lambda^2 - 1)\int s^2 k_n(\omega s) k_n(\frac{r}{\lambda}) i_n(\frac{s}{\lambda}) ds = -\omega\lambda^2 k_{n+1}(\omega s) k_n(\frac{r}{\lambda}) i_n(\frac{s}{\lambda}) s^2 - \lambda k_n(\omega s) k_n(\frac{r}{\lambda}) i_{n+1}(\frac{s}{\lambda}) s^2. \quad (107)$$

It is worth observing that $\omega^2\lambda^2 - 1 = \frac{\epsilon_s - \epsilon_\infty}{\epsilon_\infty}$.

When $f(s) = G_{n,m}(s)$ defined in (48), we have

$$\int s^2 G_{n,m}(s) i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) ds$$

$$= \frac{k_n(\frac{r}{\lambda})}{2n+1} \int s^2 i_n(\frac{s}{\lambda}) \left[-s^n \int_0^s t^{1-n} \rho_{p,n,m}(t) dt + s^{1-n} \int_0^s t^{n+2} \rho_{p,n,m}(t) dt \right] ds$$

and

$$\int s^2 G_{n,m}(s) k_n(\frac{s}{\lambda}) i_n(\frac{r}{\lambda}) ds$$

$$= \frac{i_n(\frac{r}{\lambda})}{2n+1} \int s^2 k_n(\frac{s}{\lambda}) \left[-s^n \int_0^s t^{1-n} \rho_{p,n,m}(t) dt + s^{1-n} \int_0^s t^{n+2} \rho_{p,n,m}(t) dt \right] ds.$$

We calculate these internal integrals using (191), (192), (193), (194),

$$\begin{gathered} \int -s^2 i_n(\tfrac{s}{\lambda}) s^n \int_0^s t^{1-n} \rho_{p,n,m}(t) dt ds \\ = -s^2 i_{n+1}(\tfrac{s}{\lambda}) s^n \lambda \int_0^s t^{1-n} \rho_{p,n,m}(t) dt + \int s^3 i_{n+1}(\tfrac{s}{\lambda}) \lambda \rho_{p,n,m}(s) ds \end{gathered}$$

$$\begin{gathered} \int s^2 i_n(\tfrac{s}{\lambda}) s^{-1-n} \int_0^s t^{n+2} \rho_{p,n,m}(t) dt ds \\ = s^2 i_{n-1}(\tfrac{s}{\lambda}) s^{-1-n} \lambda \int_0^s t^{n+2} \rho_{p,n,m}(t) dt - \int s^3 i_{n-1}(\tfrac{s}{\lambda}) \lambda \rho_{p,n,m}(s) ds \end{gathered}$$

$$\begin{gathered} \int -s^2 k_n(\tfrac{s}{\lambda}) s^n \int_0^s t^{1-n} \rho_{p,n,m}(t) dt ds \\ = s^2 k_{n+1}(\tfrac{s}{\lambda}) s^n \lambda \int_0^s t^{1-n} \rho_{p,n,m}(t) dt - \int s^3 k_{n+1}(\tfrac{s}{\lambda}) \lambda \rho_{p,n,m}(s) ds \end{gathered}$$

$$\begin{gathered} \int s^2 k_n(\tfrac{s}{\lambda}) s^{-1-n} \int_0^s t^{n+2} \rho_{p,n,m}(t) dt ds \\ = -s^2 k_{n-1}(\tfrac{s}{\lambda}) s^{-1-n} \lambda \int_0^s t^{n+2} \rho_{p,n,m}(t) dt + \int s^3 k_{n-1}(\tfrac{s}{\lambda}) \lambda \rho_{p,n,m}(s) ds \end{gathered}.$$

Using (198), we get

$$\int s^2 G_{n,m}(s) i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) ds = \frac{k_n(\frac{r}{\lambda})}{2n+1} \begin{bmatrix} -s^2 i_{n+1}(\frac{s}{\lambda}) s^n \lambda \int_0^s t^{1-n} \rho_{p,n,m}(t) dt \\ +s^2 i_{n-1}(\frac{s}{\lambda}) s^{-1-n} \lambda \int_0^s t^{n+2} \rho_{p,n,m}(t) dt \\ -(2n+1)\lambda^2 \int s^2 i_n(\frac{s}{\lambda}) \rho_{p,n,m}(s) ds \end{bmatrix}. \tag{108}$$

Using (199), we get

$$\int s^2 G_{n,m}(s) k_n(\frac{s}{\lambda}) i_n(\frac{r}{\lambda}) ds = \frac{i_n(\frac{r}{\lambda})}{2n+1} \begin{bmatrix} +s^2 k_{n+1}(\frac{s}{\lambda}) s^n \lambda \int_0^s t^{1-n} \rho_{p,n,m}(t) dt \\ -s^2 k_{n-1}(\frac{s}{\lambda}) s^{-1-n} \lambda \int_0^s t^{n+2} \rho_{p,n,m}(t) dt \\ -(2n+1)\lambda^2 \int s^2 k_n(\frac{s}{\lambda}) \rho_{p,n,m}(s) ds \end{bmatrix}. \tag{109}$$

4.2 Verification proof

Now we are able to proceed with the verification, the proof of Lemma 4.1.

First let's consider the case of $r > a$.

$$\int_{D_p} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi_p(\mathbf{r}')d\mathbf{r}'$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \int_0^a (\frac{C_{p,n,m}}{\epsilon_p} s^n + \frac{G_{n,m}(s)}{\epsilon_p}) \int_{r'=s} Q_\lambda(\mathbf{r}-\mathbf{r}') P_n^m(\cos(\phi')) e^{im\theta'} dS(\mathbf{r}') ds$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{P_n^m(\cos(\phi)) e^{im\theta}}{\lambda^3} \int_0^a s^2 (\frac{C_{p,n,m}}{\epsilon_p} s^n + \frac{G_{n,m}(s)}{\epsilon_p}) i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) ds.$$

$$\int_{D_s} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi_s(\mathbf{r}')d\mathbf{r}'$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \int_a^\infty (\frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s} B_{s,n,m} k_n(\omega s) + \frac{D_{s,n,m}}{\epsilon_s} s^{-n-1}) \int_{r'=s} Q_\lambda(\mathbf{r}-\mathbf{r}') P_n^m(\cos(\phi')) e^{im\theta'} dS(\mathbf{r}') ds$$

$$= \frac{P_n^m(\cos(\phi))e^{im\theta}}{\lambda^3} \begin{pmatrix} \int_a^r \frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s} B_{s,n,m} k_n(\omega s) i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) s^2 ds \\ + \int_a^r \frac{D_{s,n,m}}{\epsilon_s} s^{-n-1} i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) s^2 ds \\ + \int_r^\infty \frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s} B_{s,n,m} k_n(\omega s) i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds \\ + \int_r^\infty \frac{D_{s,n,m}}{\epsilon_s} s^{-n-1} i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds \end{pmatrix}.$$

In the interest of appearances, since $A, B, C, D, G, H, \alpha, \beta, \delta, \gamma$ and ρ will have the same subscripts throughout, we omit them. The subscripts for i_n and k_n are essential, as some of their properties that we need change their subscripts. These properties are listed in the appendix and will be referenced as needed. As there are 6 separate integrals to reduce, we introduce notation.

$$I_1 = \int_0^a \frac{C}{\epsilon_p} s^{n+2} i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) ds$$

$$I_2 = \int_0^a \frac{G(s)}{\epsilon_p} s^2 i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) ds$$

$$I_3 = \int_a^r \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} B k_n(\omega s) i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) s^2 ds$$

$$I_4 = \int_a^r \frac{D}{\epsilon_s} s^{-n-1} i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) s^2 ds$$

$$I_5 = \int_r^\infty \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} B k_n(\omega s) i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds$$

$$I_6 = \int_r^\infty \frac{D}{\epsilon_s} s^{-n-1} i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds.$$

Then (103) gives us

$$I_1 = k_n(\frac{r}{\lambda}) i_{n+1}(\frac{a}{\lambda}) \lambda a^{n+2} \frac{C}{\epsilon_p},$$

(107) gives us

$$I_3 = \frac{B\epsilon_\infty k_n(\frac{r}{\lambda})}{\epsilon_s} \begin{pmatrix} -\omega\lambda^2 k_{n+1}(r\omega) i_n(\frac{r}{\lambda}) r^2 \\ -\lambda k_n(r\omega) i_{n+1}(\frac{r}{\lambda}) r^2 \\ +\omega\lambda^2 k_{n+1}(a\omega) i_n(\frac{a}{\lambda}) a^2 \\ +\lambda k_n(a\omega) i_{n+1}(\frac{a}{\lambda}) a^2 \end{pmatrix},$$

(105) gives us

$$I_4 = \frac{D k_n(\frac{r}{\lambda})}{\epsilon_s} \left[i_{n-1}(\frac{r}{\lambda}) r^{-n-1} r^2 \lambda - i_{n-1}(\frac{a}{\lambda}) a^{-n-1} a^2 \lambda \right],$$

(106) gives us

$$I_5 = \frac{B\epsilon_\infty i_n(\frac{r}{\lambda})}{\epsilon_s} \left[\omega\lambda^2 k_{n+1}(r\omega) k_n(\frac{r}{\lambda}) r^2 - \lambda k_n(r\omega) k_{n+1}(\frac{r}{\lambda}) r^2 \right],$$

and (104) gives us

$$I_6 = \frac{D i_n(\frac{r}{\lambda})}{\epsilon_s} \left[k_{n-1}(\frac{r}{\lambda}) r^{-n-1} r^2 \lambda \right].$$

To evaluate I_2, we apply (108).

$$I_2 = \frac{k_n(\frac{r}{\lambda})}{\epsilon_p(2n+1)} \begin{bmatrix} -a^2 i_{n+1}(\frac{a}{\lambda}) a^n \lambda \int_0^a t^{1-n} \rho(t) dt \\ +a^2 i_{n-1}(\frac{a}{\lambda}) a^{-1-n} \lambda \int_0^s t^{n+2} \rho(t) dt \\ -(2n+1)\lambda^2 \int_0^a s^2 i_n(\frac{s}{\lambda}) \rho(s) ds \end{bmatrix}.$$

From the property of H in (79), we have

$$-(2n+1)\lambda^2\int_0^a s^2 i_n(\frac{s}{\lambda})\rho(s)ds = -(2n+1)\lambda a^2(\frac{H(a)i_n'(\frac{a}{\lambda})}{\lambda} - H'(a)i_n(\frac{a}{\lambda})),$$

from the definition of G in (46), we have

$$-a^2a^n\lambda i_{n+1}(\frac{a}{\lambda})\int_0^a t^{1-n}\rho(t)dt + a^2a^{-1-n}\lambda i_{n+1}(\frac{a}{\lambda})\int_0^a t^{n+2}\rho(t)dt = a^2\lambda i_{n+1}(\frac{a}{\lambda})(2n+1)G(a),$$

and the property of G in (78) along with (198) gives

$$\begin{aligned}
&-a^2a^{-1-n}\lambda i_{n+1}(\tfrac{a}{\lambda})\textstyle\int_0^a t^{n+2}\rho(t)dt + a^2a^{-1-n}\lambda i_{n-1}(\tfrac{a}{\lambda})\int_0^a t^{n+2}\rho(t)dt\\
&= a^2\lambda(i_{n+1}(\tfrac{a}{\lambda}) + i_{n-1}(\tfrac{a}{\lambda}))(nG(a) - aG'(a))\\
&= a\lambda^2(2n+1)i_n(\tfrac{a}{\lambda})(nG(a) - aG'(a))
\end{aligned}.$$

Combined, that gives us

$$I_2 = \frac{k_n(\frac{r}{\lambda})}{\epsilon_p}\left[-\lambda a^2(\frac{H(a)i_n'(\frac{a}{\lambda})}{\lambda} - H'(a)i_n(\frac{a}{\lambda})) + a^2\lambda i_{n+1}(\frac{a}{\lambda})G(a) + a\lambda^2 i_n(\frac{a}{\lambda})(nG(a) - aG'(a))\right].$$

Applying some of the equations from the system we solved, namely (67), (76), (66) we have

$$I_2 = k_n(\frac{r}{\lambda})[-\lambda a^2(\gamma D+\delta B)+a^2\lambda i_{n+1}(\frac{a}{\lambda})(\frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s}k_n(\omega a)B+\frac{a^{-n-1}}{\epsilon_s}D-\frac{Ca^n}{\epsilon_p})+a\lambda^2 i_n(\frac{a}{\lambda})(\alpha D+\beta B)].$$

We now observe that all of the terms of $I_1, I_2, I_3, I_4, I_5, I_6$ have contain a B, C, or D. So, let's reorganize the terms such that

$$I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = J_1 B + J_2 C + J_3 D$$

Before examining $J_1,\ J_2,\ J_3$, first we observe that

$$I_3 + I_5 = \frac{B\epsilon_\infty}{\epsilon_s} \begin{bmatrix} -\lambda k_n(\omega r) i_{n+1}(\frac{r}{\lambda}) k_n(\frac{r}{\lambda}) r^2 \\ +\omega\lambda^2 k_{n+1}(\omega a) i_n(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) a^2 \\ +\lambda k_n(\omega a) i_{n+1}(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) a^2 \\ -\lambda k_n(\omega r) i_n(\frac{r}{\lambda}) k_{n+1}(\frac{r}{\lambda}) r^2 \end{bmatrix}$$

simplifies via (200) to

$$I_3 + I_5 = \frac{B\epsilon_\infty}{\epsilon_s} \begin{bmatrix} -k_n(\omega r)\lambda^3 \\ +\omega\lambda^2 k_{n+1}(\omega a) i_n(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) a^2 \\ +\lambda k_n(\omega a) i_{n+1}(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) a^2 \end{bmatrix}.$$

We also observe that

$$I_4 + I_6 = \frac{D}{\epsilon_s} \left[i_{n-1}(\frac{r}{\lambda}) k_n(\frac{r}{\lambda}) r^{-n-1} r^2 \lambda - i_{n-1}(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) a^{-n-1} a^2 \lambda + i_n(\frac{r}{\lambda}) k_{n-1}(\frac{r}{\lambda}) r^{-n-1} r^2 \lambda \right]$$

simplifies via (200) to

$$I_4 + I_6 = \frac{D}{\epsilon_s} \left[-i_{n-1}(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) a^{-n-1} a^2 \lambda + r^{-n-1} \lambda^3 \right].$$

Now we consider J_1 with the substitutions for δ and β found in (72) and (69).

$$J_1 = \frac{k_n(\frac{r}{\lambda})}{\epsilon_s} a^2 \begin{bmatrix} \epsilon_\infty \omega \lambda^2 k_{n+1}(\omega a) i_n(\frac{a}{\lambda}) \\ +\epsilon_\infty \lambda k_n(\omega a) i_{n+1}(\frac{a}{\lambda}) \\ -(\epsilon_s - \epsilon_\infty)(k_n(\omega a)\frac{n\lambda^2}{a} i_n(\frac{a}{\lambda})) \\ -\epsilon_s \lambda k_n(\omega a) i_{n+1}(\frac{a}{\lambda}) \\ -\epsilon_\infty k_{n+1}(\omega a)\omega\lambda^2 i_n(\frac{a}{\lambda}) \\ +\lambda i_{n+1}(\frac{a}{\lambda})(\epsilon_s - \epsilon_\infty) k_n(\omega a) \\ +\frac{\lambda^2}{a} i_n(\frac{a}{\lambda})(\epsilon_s - \epsilon_\infty) n k_n(\omega a) \end{bmatrix} - \frac{\epsilon_\infty}{\epsilon_s} k_n(r\omega)\lambda^3.$$

After the obvious simplification, we get

$$J_1 = -\frac{\epsilon_\infty}{\epsilon_s} k_n(r\omega)\lambda^3.$$

Now we consider J_3 with the substitutions for α and γ found in (68) and (70).

$$J_3 = \frac{k_n(\frac{r}{\lambda})}{a^n} \begin{bmatrix} -i_{n-1}(\frac{a}{\lambda})\frac{a\lambda}{\epsilon_s} \\ -\frac{\lambda^2}{\epsilon_p} i_n(\frac{a}{\lambda})(n+1) \\ +\frac{\lambda^2}{\epsilon_s} i_n(\frac{a}{\lambda})(n+1) \\ +i_{n+1}(\frac{a}{\lambda})\frac{a\lambda}{\epsilon_s} \\ +\frac{n}{\epsilon_s}\lambda^2 i_n(\frac{a}{\lambda}) \\ +\frac{n+1}{\epsilon_p}\lambda^2 i_n(\frac{a}{\lambda}) \end{bmatrix} + r^{-n-1}\frac{\lambda^3}{\epsilon_s}.$$

The obvious simplification gives us

$$J_3 = \frac{k_n(\frac{r}{\lambda})}{a^n} \begin{bmatrix} -i_{n-1}(\frac{a}{\lambda})\frac{a\lambda}{\epsilon_s} \\ +\frac{\lambda^2}{\epsilon_s} i_n(\frac{a}{\lambda})(2n+1) \\ +i_{n+1}(\frac{a}{\lambda})\frac{a\lambda}{\epsilon_s} \end{bmatrix} + r^{-n-1}\frac{\lambda^3}{\epsilon_s}.$$

Applying (198) then gives us

$$J_3 = r^{-n-1}\frac{\lambda^3}{\epsilon_s}.$$

Finally, we have

$$J_2 = k_n(\frac{r}{\lambda})i_{n+1}(\frac{a}{\lambda})\lambda\frac{a^{n+2}}{\epsilon_p} - k_n(\frac{r}{\lambda})a^2\lambda i_{n+1}(\frac{a}{\lambda})\frac{a^n}{\epsilon_p} = 0.$$

So,

$$I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = -\frac{\epsilon_\infty}{\epsilon_s}k_n(\omega r)\lambda^3 B + r^{-n-1}\frac{\lambda^3}{\epsilon_s}D.$$

Thus we get

$$u_s(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{m=n}\frac{P_n^m(\cos(\phi))e^{im\theta}}{\epsilon_s}(-\epsilon_\infty k_n(\omega r)B + r^{-n-1}D)$$

which matches the form found in (23). Thus, we have verified the u_s case.

Now we repeat the analogous argument with $r < a$.

$$\int_{D_p} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi_p(\mathbf{r}')d\mathbf{r}'$$

$$= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\int_0^a(\frac{C_{p,n,m}}{\epsilon_p}s^n + \frac{G_{n,m}(s)}{\epsilon_p})\int_{r'=s} Q_\lambda(\mathbf{r}-\mathbf{r}')P_n^m(\cos(\phi'))e^{im\theta'}dS(\mathbf{r}')ds$$

$$= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{P_n^m(\cos(\phi))e^{im\theta}}{\lambda^3}\begin{pmatrix} \int_0^r s^2\frac{C_{p,n,m}}{\epsilon_p}s^n i_n(\frac{s}{\lambda})k_n(\frac{r}{\lambda})ds \\ +\int_0^r s^2\frac{G(s)}{\epsilon_p}i_n(\frac{s}{\lambda})k_n(\frac{r}{\lambda})ds \\ +\int_r^a s^2\frac{C_{p,n,m}}{\epsilon_p}s^n i_n(\frac{r}{\lambda})k_n(\frac{s}{\lambda})ds \\ +\int_r^a s^2\frac{G(s)}{\epsilon_p}i_n(\frac{r}{\lambda})k_n(\frac{s}{\lambda})ds \end{pmatrix}.$$

$$\int_{D_s} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi_s(\mathbf{r}')d\mathbf{r}'$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \int_a^{\infty} (\frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} B_{s,n,m} k_n(\omega s) + \frac{D_{s,n,m}}{\epsilon_s} s^{-n-1}) \int_{r'=s} Q_\lambda(\mathbf{r} - \mathbf{r}') P_n^m(\cos(\phi')) e^{im\theta'} dS(\mathbf{r}') ds$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{P_n^m(\cos(\phi)) e^{im\theta}}{\lambda^3} \left(\begin{array}{c} \int_a^{\infty} \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} B_{s,n,m} k_n(\omega s) i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds \\ + \int_a^{\infty} \frac{D_{s,n,m}}{\epsilon_s} s^{-n-1} i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds \end{array} \right).$$

Again, we shall drop the nonessential subscripts for aesthetics. Now, as there are 6 separate integrals to reduce, we introduce notation.

$$K_1 = \int_0^r s^2 \frac{C}{\epsilon_p} s^n i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) ds$$

$$K_2 = \int_0^r s^2 \frac{G(s)}{\epsilon_p} i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) ds$$

$$K_3 = \int_r^a s^2 \frac{C}{\epsilon_p} s^n i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) ds$$

$$K_4 = \int_r^a s^2 \frac{G(s)}{\epsilon_p} i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) ds$$

$$K_5 = \int_a^{\infty} \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} B k_n(\omega s) i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds$$

$$K_6 = \int_a^{\infty} \frac{D}{\epsilon_s} s^{-n-1} i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds.$$

Now let's work on simplifying these integrals. (103) gives us

$$K_1 = \frac{r^{n+2}}{\epsilon_p} C i_{n+1}(\frac{r}{\lambda}) \lambda k_n(\frac{r}{\lambda}),$$

(102), gives us

$$K_3 = -\frac{a^{n+2}}{\epsilon_p} C i_n(\frac{r}{\lambda}) k_{n+1}(\frac{a}{\lambda})\lambda + \frac{r^{n+2}}{\epsilon_p} C i_n(\frac{r}{\lambda}) k_{n+1}(\frac{r}{\lambda})\lambda,$$

(106) gives us

$$K_5 = \frac{B\epsilon_\infty i_n(\frac{r}{\lambda})}{\epsilon_s}(\omega\lambda^2 k_{n+1}(a\omega) k_n(\frac{a}{\lambda})a^2 - \lambda k_n(a\omega) k_{n+1}(\frac{a}{\lambda})a^2),$$

and (104) gives us

$$K_6 = \frac{D i_n(\frac{r}{\lambda})}{\epsilon_s} k_{n-1}(\frac{a}{\lambda}) a^{-n-1} a^2 \lambda.$$

From (108), we get

$$\epsilon_p(2n+1)K_2 = \begin{bmatrix} -k_n(\frac{r}{\lambda}) r^2 i_{n+1}(\frac{r}{\lambda}) r^n \lambda \int_0^r t^{1-n}\rho(t)dt \\ +k_n(\frac{r}{\lambda}) r^2 i_{n-1}(\frac{r}{\lambda}) r^{-1-n} \lambda \int_0^r t^{n+2}\rho(t)dt \\ -k_n(\frac{r}{\lambda})(2n+1)\lambda^2 \int_0^r s^2 i_n(\frac{s}{\lambda})\rho(s)ds \end{bmatrix}.$$

From (109), we get

$$\epsilon_p(2n+1)K_4 = \begin{bmatrix} i_n(\frac{r}{\lambda}) a^2 k_{n+1}(\frac{a}{\lambda}) a^n \lambda \int_0^a t^{1-n}\rho(t)dt \\ -i_n(\frac{r}{\lambda}) r^2 k_{n+1}(\frac{r}{\lambda}) r^n \lambda \int_0^r t^{1-n}\rho(t)dt \\ -i_n(\frac{r}{\lambda}) a^2 k_{n-1}(\frac{a}{\lambda}) a^{-1-n} \lambda \int_0^a t^{n+2}\rho(t)dt \\ +i_n(\frac{r}{\lambda}) r^2 k_{n-1}(\frac{r}{\lambda}) r^{-1-n} \lambda \int_0^r t^{n+2}\rho(t)dt \\ -(2n+1) i_n(\frac{r}{\lambda})\lambda^2 \int_r^a s^2 k_n(\frac{s}{\lambda})\rho(s)ds \end{bmatrix}.$$

It is more convenient to think about simplifying $\epsilon_p(2n+1)(K_2 + K_4)$. The first thing we do is simplify using (200) to get

$$\epsilon_p(2n+1)(K_2+K_4) = \left[\begin{array}{c} -k_n(\frac{r}{\lambda})(2n+1)\lambda^2\int_0^r s^2 i_n(\frac{s}{\lambda})\rho(s)ds \\ +i_n(\frac{r}{\lambda})a^2 k_{n+1}(\frac{a}{\lambda})a^n\lambda\int_0^a t^{1-n}\rho(t)dt \\ -i_n(\frac{r}{\lambda})a^2 k_{n-1}(\frac{a}{\lambda})a^{-1-n}\lambda\int_0^a t^{n+2}\rho(t)dt \\ -(2n+1)i_n(\frac{r}{\lambda})\lambda^2\int_r^a s^2 k_n(\frac{s}{\lambda})\rho(s)ds \\ -r^n\lambda^3\int_0^r t^{1-n}\rho(t)dt \\ +r^{-1-n}\lambda^3\int_0^r t^{n+2}\rho(t)dt \end{array}\right].$$

From the definition of H in (47), we have

$$(2n+1)\lambda^2 i_n(\frac{r}{\lambda})\int_0^r s^2 k_n(\frac{s}{\lambda})\rho(s)ds-(2n+1)\lambda^2 k_n(\frac{r}{\lambda})\int_0^r s^2 i_n(\frac{s}{\lambda})\rho(s)ds = -(2n+1)\lambda^2 H(r). \quad (110)$$

A property of H found in (79) gives

$$\left(\frac{H(a)i_n'(\frac{a}{\lambda})}{\lambda} - H'(a)i_n(\frac{a}{\lambda})\right)\frac{a^2}{\lambda}k_n(\frac{a}{\lambda}) = \int_0^a i_n(\frac{s}{\lambda})k_n(\frac{a}{\lambda})\rho(s)s^2 ds,$$

which combined with (47) gives us

$$-H(a)+\left(\frac{H(a)i_n'(\frac{a}{\lambda})}{\lambda} - H'(a)i_n(\frac{a}{\lambda})\right)\frac{a^2}{\lambda}k_n(\frac{a}{\lambda}) = \int_0^a k_n(\frac{s}{\lambda})i_n(\frac{a}{\lambda})\rho(s)s^2 ds.$$

Thus,

$$\begin{array}{c} -\frac{(2n+1)\lambda^2 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}\left[-H(a)+\left(\frac{H(a)i_n'(\frac{a}{\lambda})}{\lambda} - H'(a)i_n(\frac{a}{\lambda})\right)\frac{a^2}{\lambda}k_n(\frac{a}{\lambda})\right] \\ = -\int_0^a s^2 k_n(\frac{s}{\lambda})i_n(\frac{r}{\lambda})\lambda^2(2n+1)\rho(s)ds \end{array}. \quad (111)$$

The definition for G from (46) gives us

$$-r^n\lambda^3\int_0^r t^{1-n}\rho(t)dt + r^{-1-n}\lambda^3\int_0^r t^{n+2}\rho(t)dt = \lambda^3 G(r)(2n+1) \tag{112}$$

$$\begin{array}{c} a^2\lambda k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})a^n\int_0^a t^{1-n}\rho(t)dt \\ -a^2\lambda k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})a^{-1-n}\int_0^a t^{n+2}\rho(t)dt \\ = -a^2\lambda k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})(2n+1)G(a) \end{array} . \tag{113}$$

Applying (199) with a property of G found in (78) gives us

$$\begin{array}{c} -a^2\lambda k_{n-1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})a^{-1-n}\int_0^a t^{n+2}\rho(t)dt \\ +a^2\lambda k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})a^{-1-n}\int_0^a t^{n+2}\rho(t)dt \\ = (nG(a)-aG'(a))(a^2\lambda i_n(\frac{r}{\lambda}))(-k_{n-1}(\frac{a}{\lambda})+k_{n+1}(\frac{a}{\lambda})) \\ = (nG(a)-aG'(a))(a\lambda^2 i_n(\frac{r}{\lambda}))(2n+1)k_n(\frac{a}{\lambda}) \end{array} . \tag{114}$$

Summing $(110), (111), (112), (113), (114)$ then gives us

$$\epsilon_p(K_2+K_4) = \left[\begin{array}{c} -\lambda^2 H(r) \\ +\frac{\lambda^2 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}H(a) \\ -\frac{\lambda^2 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}(\frac{H(a)i_n'(\frac{a}{\lambda})}{\lambda} - H'(a)i_n(\frac{a}{\lambda}))\frac{a^2}{\lambda}k_n(\frac{a}{\lambda}) \\ \lambda^3 G(r) \\ -a^2\lambda k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})G(a) \\ +(nG(a)-aG'(a))(a\lambda^2 i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda})) \end{array} \right],$$

to which we apply equations from the system we solved earlier, namely $(66), (67), (76), (77)$.

$$(K_2+K_4)=\begin{bmatrix} -\frac{\lambda^2 H(r)}{\epsilon_p} \\ -\frac{\lambda^3 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} i_n(\frac{a}{\lambda})A \\ +\frac{\lambda^3 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} k_n(a\omega)B \\ -\frac{\lambda a^2 i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})}(\gamma D+\delta B) \\ +\frac{\lambda^3 G(r)}{\epsilon_p} \\ \frac{a^{n+2}}{\epsilon_p}\lambda k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})C \\ -a^2 k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})\frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s}k_n(a\omega)\lambda B \\ -a^{-n+1}\frac{\lambda}{\epsilon_s}k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})D \\ +(\alpha D+\beta B)(a\lambda^2 i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda})) \end{bmatrix}.$$

We now observe that all of the terms of $K_1, K_2, K_3, K_4, K_5, K_6$ have $A, B, C, D, G(r)$ or $H(r)$ in them, so we may reorganize the terms so that

$$K_1+K_2+K_3+K_4+K_5+K_6 = L_1A+L_2B+L_3C+L_4D+\frac{\lambda^3}{\epsilon_p}G(r)-\frac{\lambda^2}{\epsilon_p}H(r).$$

$$L_1=-\frac{\lambda^3 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}i_n(\frac{a}{\lambda})=-\lambda^3 i_n(\frac{r}{\lambda}).$$

$$L_3=\frac{r^{n+2}}{\epsilon_p}i_{n+1}(\frac{r}{\lambda})k_n(\frac{r}{\lambda})\lambda-\frac{a^{n+2}}{\epsilon_p}i_n(\frac{r}{\lambda})k_{n+1}(\frac{a}{\lambda})\lambda+\frac{r^{n+2}}{\epsilon_p}i_n(\frac{r}{\lambda})k_{n+1}(\frac{r}{\lambda})\lambda+\frac{a^{n+2}}{\epsilon_p}i_n(\frac{r}{\lambda})k_{n+1}(\frac{a}{\lambda})\lambda.$$

Simplifying with (200) gives us

$$L_3=\frac{r^n\lambda^3}{\epsilon_p}.$$

$$L_2 = \left(\begin{array}{c} \frac{\lambda^3 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} k_n(a\omega) \\ -\frac{\lambda a^2 i_n(\frac{r}{\lambda}) k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})} \delta \\ -a^2 k_{n+1}(\frac{a}{\lambda}) i_n(\frac{r}{\lambda}) \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} k_n(a\omega) \\ +\beta(a\lambda^2 i_n(\frac{r}{\lambda}) k_n(\frac{a}{\lambda})) \\ +\frac{\epsilon_\infty i_n(\frac{r}{\lambda})}{\epsilon_s} \omega \lambda^2 k_{n+1}(a\omega) k_n(\frac{a}{\lambda}) a^2 \\ -\frac{\epsilon_\infty i_n(\frac{r}{\lambda})}{\epsilon_s} \lambda k_n(a\omega) k_{n+1}(\frac{a}{\lambda}) a^2 \end{array} \right).$$

Applying the definitions for δ and β found in (72) and (69) gives us

$$L_2 = \left(\begin{array}{c} \frac{\lambda^3 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} k_n(a\omega) \\ -\frac{\lambda^2 i_n(\frac{r}{\lambda}) k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})} \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} n a k_n(a\omega) i_n(\frac{a}{\lambda}) \\ -\frac{\lambda i_n(\frac{r}{\lambda}) k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})} a^2 k_n(a\omega) i_{n+1}(\frac{a}{\lambda}) \\ -\frac{\lambda^2 i_n(\frac{r}{\lambda}) k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})} \frac{\epsilon_\infty}{\epsilon_s} \omega k_{n+1}(a\omega) i_n(\frac{a}{\lambda}) \\ -a^2 \lambda k_{n+1}(\frac{a}{\lambda}) i_n(\frac{r}{\lambda}) \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} k_n(a\omega) \\ +a\lambda^2 i_n(\frac{r}{\lambda}) \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s} n k_n(a\omega) k_n(\frac{a}{\lambda}) \\ +\frac{\epsilon_\infty i_n(\frac{r}{\lambda})}{\epsilon_s} \omega \lambda^2 k_{n+1}(a\omega) k_n(\frac{a}{\lambda}) a^2 \\ -\frac{\epsilon_\infty i_n(\frac{r}{\lambda})}{\epsilon_s} \lambda k_n(a\omega) k_{n+1}(\frac{a}{\lambda}) a^2 \end{array} \right).$$

After the obvious simplification, we are left with

$$L_2 = \left(\begin{array}{c} \frac{\lambda^3 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} k_n(a\omega) \\ -\frac{\lambda i_n(\frac{r}{\lambda}) k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})} a^2 k_n(a\omega) i_{n+1}(\frac{a}{\lambda}) \\ -a^2 \lambda k_{n+1}(\frac{a}{\lambda}) i_n(\frac{r}{\lambda}) k_n(a\omega) \end{array} \right)$$

$$= \frac{\lambda i_n(\frac{r}{\lambda}) k_n(a\omega) a^2}{i_n(\frac{a}{\lambda})} (-k_{n+1}(\frac{a}{\lambda}) i_n(\frac{a}{\lambda}) + \frac{\lambda^2}{a^2} - k_n(\frac{a}{\lambda}) i_{n+1}(\frac{a}{\lambda})) = 0,$$

which simplified from (200).

$$L_4 = \begin{pmatrix} -\frac{\lambda a^2 i_n(\frac{r}{\lambda}) k_n(\frac{a}{\lambda})\gamma}{i_n(\frac{a}{\lambda})} \\ -a^{-n+1}\frac{\lambda}{\epsilon_s}k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda}) \\ +\alpha(a\lambda^2 i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda})) \\ +\frac{i_n(\frac{r}{\lambda})}{\epsilon_s}k_{n-1}(\frac{a}{\lambda})a^{-n-1}a^2\lambda \end{pmatrix}.$$

Applying the definition of γ and α found in (70) and (68) gives us

$$L_4 = \begin{pmatrix} -\frac{\lambda^2 i_n(\frac{r}{\lambda}) k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})}\frac{i_n(\frac{a}{\lambda})(n+1)}{a^n}(\frac{1}{\epsilon_p} - \frac{1}{\epsilon_s}) \\ -a^{-n+1}\frac{\lambda}{\epsilon_s}k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda}) \\ +\lambda^2 i_n(\frac{r}{\lambda})a^{-n}(\frac{n}{\epsilon_s} + \frac{n+1}{\epsilon_p})k_n(\frac{a}{\lambda}) \\ +\frac{i_n(\frac{r}{\lambda})}{\epsilon_s}k_{n-1}(\frac{a}{\lambda})a^{-n-1}a^2\lambda \end{pmatrix}.$$

The obvious simplification and (199) gives us

$$L_4 = \frac{\lambda a^{-n+1} i_n(\frac{r}{\lambda})}{\epsilon_s}(k_{n+1}(\frac{a}{\lambda}) - k_{n-1}(\frac{a}{\lambda})) + \frac{\lambda^2 i_n(\frac{r}{\lambda})a^{-n}}{\epsilon_s}(2n+1)k_n(\frac{a}{\lambda}) = 0.$$

Thus

$$K_1 + K_2 + K_3 + K_4 + K_5 + K_6 = -\lambda^3 i_n(\frac{r}{\lambda})A + \frac{r^n\lambda^3}{\epsilon_p}C + \frac{\lambda^3}{\epsilon_p}G(r) - \frac{\lambda^2}{\epsilon_p}H(r).$$

Thus we get

$$u_p(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} P_n^m(cos(\phi))e^{im\theta}(-i_n(\frac{r}{\lambda})A + \frac{r^n}{\epsilon_p}C + \frac{G(r)}{\epsilon_p} - \frac{H(r)}{\lambda\epsilon_p})$$

Which matches the form found in (55). Thus we have now also verified the u_p case. Thus we have proven Lemma 4.1.

5 Convergence

Lemma 5.1. *Let the conditions of Theorem 2.1 hold.*

Let ρ_p be supported in a ball $B(\mathbf{0}, b)$ with $b < a$.

Then ϕ_s and u_s defined by (21) *and* (23) *are geometrically convergent series.*

If the support of ρ_p is further confined to be within $B(\mathbf{0}, c) \cup (B(\mathbf{0}, \mathbf{a}) - B(\mathbf{0}, \mathbf{d}))$, then ϕ_p and u_p defined by (20) *and* (22) *are geometrically convergent series when $\mathbf{r}$ satisfies $c < |\mathbf{r}| < d$.*

Since we have the solution defined as an infinite series, it is necessary to also verify that it converges. Naturally, this convergence will depend on ρ_p.

We will use the notation $a_n \sim b_n$ to mean $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

It is known from [5] that for fixed $r > 0$

$$i_n(r) \sim \frac{1}{\sqrt{2r(2n+1)}}(\frac{er}{2n+1})^{n+1/2}$$

$$k_n(r) \sim \frac{2}{\sqrt{2r(2n+1)}}(\frac{er}{2n+1})^{-n-1/2}$$

These relations with the limit definition for e produce

$$\frac{i_{n+1}(r)}{i_n(r)} \sim \frac{r}{2n} \tag{115}$$

$$\frac{k_{n+1}(r)}{k_n(r)} \sim \frac{2n}{r} \tag{116}$$

$$\frac{i_n(R)}{i_n(r)} \sim (\frac{R}{r})^n \tag{117}$$

$$\frac{k_n(R)}{k_n(r)} \sim (\frac{R}{r})^{-n-1} \tag{118}$$

$$i_n(r)k_n(R) \sim \frac{1}{R(2n+1)}(\frac{r}{R})^n. \tag{119}$$

We may apply asymptotic relations to prove convergence, but they only prove results for when n is large, and may not immediately give us some bound on how large n must be. Since the solution is given as an infinite series, we desire having a bound on the number of terms that must be included to remain within a given tolerance of the solution. To that end, we must consider other bounds.

(198), when combined with $i_n(r)$ being decreasing over n [5, Section 10.37], gives us

$$\frac{ri_n(r)}{2ni_{n+1}(r)} = \frac{ri_{n+2}(r)}{2ni_{n+1}(r)} + \frac{2n+3}{2n} \leq 1 + \frac{3+r}{2n}.$$

(198) also gives us

$$i_n(r) \geq \frac{2n+3}{r} i_{n+1}(r) \geq \frac{2n}{r} i_{n+1}(r).$$

Combined, we get

$$1 \leq \frac{ri_n(r)}{2ni_{n+1}(r)} \leq 1 + \epsilon \tag{120}$$

with the the first inequality holding always and the second holding for $r + 3 \leq 2n\epsilon$.

(199), when combined with $k_n(r)$ being increasing over n [5, Section 10.37], gives us

$$\frac{rk_{n+1}(r)}{2nk_n(r)} = \frac{rk_{n-1}(r)}{2nk_n(r)} + \frac{2n+1}{2n} \leq 1 + \frac{r+1}{2n}.$$

(199) also gives us

$$k_{n+1}(r) \geq \frac{2n+1}{r} k_n(r) \geq \frac{2n}{r} k_n(r).$$

Combined, we get

$$1 \leq \frac{r k_{n+1}(r)}{2n k_n(r)} \leq 1 + \epsilon \qquad (121)$$

with the the first inequality holding always and the second holding for $r + 1 \leq 2n\epsilon$.

Define $\tilde{i}_n(r) = i_n(r) r^{-n}$. First, (198) gives us

$$\tilde{i}_{n-1}(r) r^{n-1} - \tilde{i}_{n+1}(r) r^{n+1} = \frac{2n+1}{r} \tilde{i}_n(r) r^n.$$

Multiplication by r^{1-n} and setting $r = 0$ gives us for $n \geq 1$

$$\tilde{i}_{n-1}(0) = (2n+1) \tilde{i}_n(0)$$

Direct calculation gives us $\tilde{i}_0(0) = i_0(0) = 1$, thus for $n \geq 0$,

$$\tilde{i}_n(0) = \frac{1}{1 * 3 * 5 * \ldots * (2n+1)}. \qquad (122)$$

Then (192) says, $\tilde{i}_n'(r) = r \tilde{i}_{n+1}(r)$. Nonnegativity of i_n then implies that $\tilde{i}_n$ is increasing and nonnegative. So, we have

$$\frac{\tilde{i}_n(r)}{\tilde{i}_n(0)} \geq 1.$$

We also have

$$\tilde{i}_n(r) - \tilde{i}_n(0) = \int_0^r t\tilde{i}_{n+1}(t)dt \leq \frac{r^2}{2}\tilde{i}_{n+1}(r).$$

(195) gives us $i_{n-1}(r) \geq \frac{n+1}{r} i_n(r)$, so $\tilde{i}_n(r) \geq (n+2)\tilde{i}_{n+1}(r)$.

So,

$$\tilde{i}_n(r) - \tilde{i}_n(0) \leq \frac{r^2}{2(n+2)}\tilde{i}_n(r)$$

$$\tilde{i}_n(r)(1 - \frac{r^2}{2(n+2)}) \leq \tilde{i}_n(0).$$

Then, if $r^2 \leq n+2$, we get

$$1 \leq \frac{\tilde{i}_n(r)}{\tilde{i}_n(0)} \leq 2.$$

Thus, if $r^2, R^2 \leq n+2$, we get

$$\frac{1}{2} \leq \frac{\tilde{i}_n(R)}{\tilde{i}_n(r)} = \frac{i_n(R)}{i_n(r)}\frac{r^n}{R^n} \leq 2. \tag{123}$$

Next, define $\tilde{k}_n(r) = k_n(r)r^{n+1}$. Then, (199) gives us

$$\tilde{k}_{n-1}(r)r^{-n} - \tilde{k}_{n+1}(r)r^{-(n+2)} = -\frac{2n+1}{r}\tilde{k}_n(r)r^{-(n+1)}.$$

Multiplication by r^{n+2} and setting $r = 0$ gives us for $n \geq 1$

$$\tilde{k}_{n+1}(0) = (2n+1)\tilde{k}_n(0).$$

Direct calculation gives us $\tilde{k}_0(r) = \frac{e^{-r}}{r} r = e^{-r}$, so $\tilde{k}_0(0) = 1$.

Similarly, $\tilde{k}_1(r) = \frac{e^{-r}}{r^2}(1+r)r^2 = e^{-r}(1+r)$, so $\tilde{k}_1(0) = 1$.

Thus, with the empty product understood to be 1, we have that for $n \geq 0$,

$$\tilde{k}_n(0) = 1 * 3 * 5 * ... * (2n-1). \tag{124}$$

Then (193) says, $\tilde{k}_n'(r) = -r\tilde{k}_{n-1}(r)$. Nonnegativity of k_n then implies that $\tilde{k}_n$ is decreasing and nonnegative. So, we have

$$\frac{\tilde{k}_n(r)}{\tilde{k}_n(0)} \leq 1$$

$$\tilde{k}_n(0) - \tilde{k}_n(r) = \int_0^r t\tilde{k}_{n-1}(t)dt \leq \frac{r^2}{2}\tilde{k}_{n-1}(r)$$

$$1 - \frac{r^2}{2}\frac{\tilde{k}_{n-1}(0)}{\tilde{k}_n(0)} \leq \frac{\tilde{k}_n(r)}{\tilde{k}_n(0)}.$$

From our definition of $\tilde{k}_n(0)$ from (124), we have

$$1 - \frac{r^2}{2(2n-1)} \leq \frac{\tilde{k}_n(r)}{\tilde{k}_n(0)}.$$

Then, if $r^2 \leq 2n-1$, we get

$$\frac{1}{2} \leq \frac{\tilde{k}_n(r)}{\tilde{k}_n(0)} \leq 1.$$

Thus, if $r^2, R^2 \leq 2n-1$, we get

$$\frac{1}{2} \leq \frac{\tilde{k}_n(R)}{\tilde{k}_n(r)} = \frac{k_n(R)}{k_n(r)}\frac{R^{n+1}}{r^{n+1}} \leq 2. \tag{125}$$

Combining the previous, we have that if $r^2 \leq n+2$ and $R^2 \leq 2n-1$, then

$$\frac{1}{2} \leq \frac{\tilde{i}_n(r)\tilde{k}_n(R)}{\tilde{i}_n 0\tilde{k}_n(0)} \leq 2.$$

The definitions of $\tilde{i}_n(0)$ and $\tilde{k}_n(0)$ from (122), (124) give us

$$\frac{1}{2} \leq \tilde{i}_n(r)\tilde{k}_n(R)(2n+1) \leq 2.$$

Thus,

$$\frac{1}{2} \leq i_n(r)k_n(R)(2n+1)R\frac{R^n}{r^n} \leq 2. \tag{126}$$

For the purposes of finding error estimates, we shall use the notation $a_n \sim_C b_n$ to mean that $a_n \sim b_n$ and that for $n \geq N$ for a given N, we have

$$\frac{1}{C} \leq \left|\frac{a_n}{b_n}\right| \leq C$$

Once useful property is that for $a_n, b_n, c_n > 0$, if $a_n \sim_C b_n$ for $n \geq N$, $a_n \sim c_n$, and $b_n \leq \frac{D}{C}c_n$ and $b_n \geq \frac{C}{D}c_n$ for $n \geq M$, then $a_n \sim_D c_n$ for $n \geq \max(N, M)$. Now we work to find error estimates on the various coefficients.

Since the terms of $\delta_{n,m}$ are positive, we apply (115) and (116) with the error bounds from (120) and (121) for $a\omega + 1, \frac{a}{\lambda} + 3 \leq 2n$,

$$\delta_{n,m} \sim_2 \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s}(k_n(a\omega)\frac{n\lambda}{a}i_n(\frac{a}{\lambda})) + k_n(a\omega)i_n(\frac{a}{\lambda})\frac{a}{2\lambda n} + \frac{\epsilon_\infty}{\epsilon_s}k_n(a\omega)\omega\lambda i_n(\frac{a}{\lambda})\frac{2n}{a\omega}$$

$$= k_n(a\omega)i_n(\frac{a}{\lambda})\left[\frac{(\epsilon_s + \epsilon_\infty)}{\epsilon_s}\frac{n\lambda}{a} + \frac{a}{2\lambda n}\right],$$

which then gives us

$$\delta_{n,m} \sim k_n(a\omega)i_n(\frac{a}{\lambda})(\frac{\epsilon_s + \epsilon_\infty}{\epsilon_s})\frac{n\lambda}{a}.$$

To determine bounds for the error, we observe that if $a \leq \lambda n$, then

$$\frac{(\epsilon_s + \epsilon_\infty)}{\epsilon_s}\frac{n\lambda}{a} + \frac{a}{2\lambda n} \leq \frac{2(\epsilon_s + \epsilon_\infty)n\lambda + n\lambda\epsilon_s}{2\epsilon_s a} \leq \frac{3}{2}(\frac{\epsilon_s + \epsilon_\infty}{\epsilon_s})\frac{n\lambda}{a}.$$

In the other direction, we have

$$\frac{(\epsilon_s + \epsilon_\infty)}{\epsilon_s}\frac{n\lambda}{a} + \frac{a}{2\lambda n} \geq \frac{2}{3}(\frac{\epsilon_s + \epsilon_\infty}{\epsilon_s})\frac{n\lambda}{a}.$$

Thus we have with the additional constraint $a \leq \lambda n$,

$$\delta_{n,m} \sim_3 k_n(a\omega)i_n(\frac{a}{\lambda})(\frac{\epsilon_s + \epsilon_\infty}{\epsilon_s})\frac{n\lambda}{a}.$$

The terms of $\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m}$ as seen in (73) are positive, so we may apply (115) and (116), along with (120) and (121) to get that for $\frac{a}{\lambda} + 3, a\omega + 1 \leq 2n$,

$$\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m} \sim_2 \begin{pmatrix} \frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s}(k_n(a\omega)\frac{n\lambda}{a}i_n(\frac{a}{\lambda}))\frac{2n+1}{a^{n+1}\epsilon_s} \\ +(k_n(a\omega)i_n(\frac{a}{\lambda})\frac{a}{2n\lambda} \\ +\frac{\epsilon_\infty}{\epsilon_s}k_n(a\omega)\frac{2n}{a\omega}\omega\lambda i_n(\frac{a}{\lambda}))(\frac{n}{\epsilon_s a^{n+1}} + \frac{n+1}{\epsilon_p a^{n+1}}) \end{pmatrix}$$

$$= \frac{k_n(a\omega)i_n(\frac{a}{\lambda})}{a^{n+1}\epsilon_s}\left(\frac{(\epsilon_s - \epsilon_\infty)}{\epsilon_s}\frac{n\lambda}{a}(2n+1) + \frac{a}{2\lambda} + \frac{a(n+1)\epsilon_s}{2\lambda n\epsilon_p} + \frac{2\epsilon_\infty n^2\lambda}{\epsilon_s a} + \frac{2n(n+1)\epsilon_\infty\lambda}{a\epsilon_p}\right)$$

$$= \frac{k_n(a\omega)i_n(\frac{a}{\lambda})}{a^{n+1}\epsilon_s}\left(\frac{\lambda(2n^2+n)(\epsilon_p + \epsilon_\infty)}{a\epsilon_p} - \frac{\lambda n\epsilon_\infty}{a\epsilon_s} + \frac{a}{2\lambda}(\frac{\epsilon_p + \epsilon_s}{\epsilon_p}) + \frac{a\epsilon_s}{2\lambda n\epsilon_p}\right)$$

$$\sim \frac{k_n(a\omega)i_n(\frac{a}{\lambda})}{a^{n+2}\epsilon_s}\frac{\lambda(2n^2)(\epsilon_p + \epsilon_\infty)}{\epsilon_p}.$$

So we have

$$\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m} \sim \frac{k_n(a\omega)i_n(\frac{a}{\lambda})}{a^{n+2}\epsilon_s}\frac{\lambda(2n^2)(\epsilon_p+\epsilon_\infty)}{\epsilon_p}.$$

We wish to have conditions that give us

$$\frac{2}{3}\frac{\lambda(2n^2)(\epsilon_p+\epsilon_\infty)}{a\epsilon_p} \leq \left(\frac{\lambda(2n^2+n)(\epsilon_p+\epsilon_\infty)}{a\epsilon_p} - \frac{\lambda n\epsilon_\infty}{a\epsilon_s} + \frac{a(\epsilon_p+\epsilon_s)}{2\lambda\epsilon_p} + \frac{a\epsilon_s}{2\lambda n\epsilon_p}\right) \leq \frac{3}{2}\frac{\lambda(2n^2)(\epsilon_p+\epsilon_\infty)}{a\epsilon_p}$$

As that will imply that, for the conditions given,

$$\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m} \sim_3 \frac{k_n(a\omega)i_n(\frac{a}{\lambda})}{a^{n+2}\epsilon_s}\frac{\lambda(2n^2)(\epsilon_p+\epsilon_\infty)}{\epsilon_p}. \tag{127}$$

If $n \geq 1$, then

$$\frac{a}{2\lambda}\left(\frac{\epsilon_p+\epsilon_s}{\epsilon_p}\right) + \frac{a\epsilon_s}{2\lambda n\epsilon_p} \leq \frac{a(\epsilon_p+2\epsilon_s)}{2\lambda\epsilon_p}.$$

If $n \geq \frac{a^2(\epsilon_p+2\epsilon_s)}{2\lambda^2(\epsilon_p+\epsilon_\infty)}$, then

$$\frac{a(\epsilon_p+2\epsilon_s)}{2\lambda\epsilon_p} \leq \frac{n\lambda(\epsilon_p+\epsilon_\infty)}{a\epsilon_p}.$$

If $n \geq 2$, then

$$\frac{\lambda(2n^2+2n)(\epsilon_p+\epsilon_\infty)}{a\epsilon_p} \leq \frac{3}{2}\frac{\lambda(2n^2)(\epsilon_p+\epsilon_\infty)}{a\epsilon_p}.$$

Concerning the other inequality, if $n \geq \frac{3\epsilon_\infty\epsilon_p}{2\epsilon_s(\epsilon_p+\epsilon_\infty)}$, then

$$\frac{\lambda n\epsilon_\infty}{a\epsilon_s} \leq \frac{2}{3}\frac{\lambda(n^2)(\epsilon_p+\epsilon_\infty)}{a\epsilon_p},$$

$$\frac{2}{3}\frac{2\lambda(n^2)(\epsilon_p+\epsilon_\infty)}{a\epsilon_p} \leq \frac{2\lambda(n^2)(\epsilon_p+\epsilon_\infty)}{a\epsilon_p} - \frac{\lambda n\epsilon_\infty}{a\epsilon_s}.$$

All of these inequalities combined give the desired result, (127).

So (78) then gives us

$$\frac{nG_{n,m}(a) - aG'_{n,m}(a)}{\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m}} \sim_3 \frac{a\epsilon_s}{k_n(a\omega)i_n(\frac{a}{\lambda})}\frac{\epsilon_p}{\lambda(2n^2)(\epsilon_p+\epsilon_\infty)}\int_0^a r^{n+2}\rho_{p,n,m}(r)dr, \tag{128}$$

and (79) gives us

$$\frac{\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})}{\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m}} \sim_3 \frac{a^n\epsilon_s}{k_n(a\omega)i_n(\frac{a}{\lambda})}\frac{\epsilon_p\lambda}{\lambda(2n^2)(\epsilon_p+\epsilon_\infty)}\int_0^a i_n(\frac{r}{\lambda})r^2\rho_{p,n,m}(r)dr \tag{129}$$

for

$$n \geq N_0 := \max(\frac{a}{\lambda}, \frac{a\omega+1}{2}, \frac{a+3\lambda}{2\lambda}, 2, \frac{a^2(\epsilon_p+2\epsilon_s)}{2\lambda^2(\epsilon_p+\epsilon_\infty)}, \frac{3\epsilon_\infty\epsilon_p}{2\epsilon_s(\epsilon_p+\epsilon_\infty)})$$

Unfortunately, we have no guarantees on the sign of $\rho_{p,n,m}$, and thus cannot guarantee cannot predict the signs of (128) and (129). Therefore, we are unable to guarantee asymptotic relations on combinations of them, as seen in $A_{p,n,m}, B_{s,n,m}, C_{p,n,m}, D_{s,n,m}$. Fortunately, all that we require are bounds on these coefficients.

Furthermore, to deal with ρ, it is known from [6] that

$$|P_n^m(\cos\phi)|^2 \leq \frac{(n+m)!}{(n-m)!} = \frac{c_{n,m}(2n+1)}{4\pi} \tag{130}$$

As $\rho_{p,n,m}$ is an integral respect to $P_n^m(\cos\phi)e^{im\theta}$, as defined in (43), we also get

$$|P_n^m(\cos\phi)\int_0^a r^2 f(r)\rho_{p,n,m}(r)dr| \le \max|f|\frac{(2n+1)}{4\pi}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r} \tag{131}$$

Since having the support of ρ near the boundary requires changing the differential equations on the interface, it is reasonable to assume that $\rho(\mathbf{r})$ is nonzero only for $|\mathbf{r}| < b < a$

Applying the bounds we found earlier, we have that for $n \ge N_0$,

$$\left|\delta_{n,m}\frac{nG_{n,m}(a) - aG'_{n,m}(a)}{\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m}}\right||P_n^m(\cos\phi)| \le 9\frac{\epsilon_p(\epsilon_s+\epsilon_\infty)}{2n(\epsilon_p+\epsilon_\infty)}\int_0^a |r^{n+2}\rho_{p,n,m}(r)|dr|P_n^m(\cos\phi)|$$

$$\le 9\frac{\epsilon_p(\epsilon_s+\epsilon_\infty)b^n(2n+1)}{8\pi n(\epsilon_p+\epsilon_\infty)}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Concerning $\beta_{n,m}$, we have that, for $n \ge N_0$,

$$\left|\beta_{n,m}\frac{\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})}{\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m}}\right||P_n^m(\cos\phi)| \le 3\frac{\epsilon_p a^n(\epsilon_s-\epsilon_\infty)}{2ni_n(\frac{a}{\lambda})(\epsilon_p+\epsilon_\infty)}\int_0^a |i_n(\frac{r}{\lambda})r^2\rho_{p,n,m}(r)|dr$$

$$\le 3\frac{\epsilon_p a^n(\epsilon_s-\epsilon_\infty)(2n+1)i_n(\frac{b}{\lambda})}{8\pi n i_n(\frac{a}{\lambda})(\epsilon_p+\epsilon_\infty)}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

By applying (123), we have that for $n \ge \max(N_0, \left(\frac{a}{\lambda}\right)^2 - 2)$

$$\left|\beta_{n,m}\frac{\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})}{\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m}}\right||P_n^m(\cos\phi)| \le 6\frac{\epsilon_p(\epsilon_s-\epsilon_\infty)(2n+1)b^n}{8\pi n(\epsilon_p+\epsilon_\infty)}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Then by (74), for $n \ge \max(N_0, \left(\frac{a}{\lambda}\right)^2 - 2)$

$$|D_{s,n,m}||P_n^m(\cos\phi)| \le 15(\epsilon_s+\epsilon_\infty)\frac{b^n(2n+1)}{8\pi n(\epsilon_p+\epsilon_\infty)}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Concerning $\gamma_{n,m}$, we have that for $n \geq N_0$,

$$\left|\gamma_{n,m}\frac{nG_{n,m}(a) - aG'_{n,m}(a)}{\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m}}\right| |P_n^m(\cos\phi)| \leq 3\frac{(\epsilon_s + \epsilon_p)(n+1)}{2n^2a^{n+1}k_n(\omega a)(\epsilon_p + \epsilon_\infty)}\int_0^a |r^{n+2}\rho_{p,n,m}(r)|dr$$

$$\leq 3\frac{(\epsilon_s + \epsilon_p)(n+1)(2n+1)b^n}{8\pi n^2 a^{n+1}k_n(\omega a)(\epsilon_p + \epsilon_\infty)}\int_{D_p} |\rho(\mathbf{r})|d\mathbf{r}.$$

Concerning $\alpha_{n,m}$, we have that for $n \geq N_0$,

$$\left|\alpha_{n,m}\frac{\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})}{\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m}}\right| |P_n^m(\cos\phi)|$$

$$\leq \frac{3(n+1)(\epsilon_p + \epsilon_s)}{2n^2ak_n(a\omega)i_n(\frac{a}{\lambda})(\epsilon_p + \epsilon_\infty)}\int_0^a |i_n(\frac{r}{\lambda})r^2\rho_{p,n,m}(r)|dr$$

$$\leq 3\frac{(n+1)(\epsilon_p + \epsilon_s)(2n+1)i_n(\frac{b}{\lambda})}{8\pi n^2ak_n(a\omega)i_n(\frac{a}{\lambda})(\epsilon_p + \epsilon_\infty)}\int_{D_p} |\rho(\mathbf{r})|d\mathbf{r}.$$

Again we apply (123) to get that for $n \geq \max(N_0, \left(\frac{a}{\lambda}\right)^2 - 2)$,

$$\left|\alpha_{n,m}\frac{\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})}{\delta_{n,m}\alpha_{n,m} - \beta_{n,m}\gamma_{n,m}}\right| |P_n^m(\cos\phi)| \leq 6\frac{(n\epsilon_p + (n+1)\epsilon_s)(2n+1)b^n}{8\pi n^2a^{n+1}k_n(a\omega)(\epsilon_p + \epsilon_\infty)}\int_{D_p} |\rho(\mathbf{r})|d\mathbf{r}.$$

Then by (75), for $n \geq \max(N_0, \left(\frac{a}{\lambda}\right)^2 - 2)$,

$$\epsilon_p|B_{s,n,m}||P_n^m(\cos\phi)| \leq \frac{9(\epsilon_p + \epsilon_s)(n+1)(2n+1)b^n}{8\pi n^2a^{n+1}k_n(a\omega)(\epsilon_p + \epsilon_\infty)}\int_{D_p} |\rho(\mathbf{r})|d\mathbf{r}.$$

Let's assume that $n \geq \max(N_0, \left(\frac{a}{\lambda}\right)^2 - 2)$. Then we have

$$\left|\frac{\epsilon_p}{\epsilon_s} a^{-2n-1} D_{s,n,m} P_n^m(\cos\phi)\right| \leq 15(\epsilon_s + \epsilon_\infty) \frac{b^n(2n+1)\epsilon_p}{8\pi n a^{2n+1}\epsilon_s(\epsilon_p + \epsilon_\infty)} \int_{D_p} |\rho(\mathbf{r}) d\mathbf{r}$$

$$\left|\frac{\epsilon_p(\epsilon_s - \epsilon_\infty)}{\epsilon_s a^n} k_n(\omega a) B_{s,n,m} P_n^m(\cos\phi)\right| \leq \frac{9(\epsilon_s - \epsilon_\infty)(\epsilon_p + \epsilon_s)(n+1)(2n+1)b^n}{8\pi n^2 a^{2n+1}\epsilon_s(\epsilon_p + \epsilon_\infty)} \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

Combined, using the equation for $C_{p,n,m}$ found in (76), we get

$$\left|(C_{p,n,m} + \frac{G_{n,m}(a)}{a^n}) P_n^m(\cos\phi)\right| \leq \frac{24(\epsilon_s + \epsilon_\infty)(\epsilon_p + \epsilon_s)(2n+1)(n+1)b^n}{8\pi n^2 a^{2n+1}\epsilon_s(\epsilon_p + \epsilon_\infty)} \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

Similarly, the equation for $A_{p,n,m}$ found in (77) gives us

$$\left|(A_{p,n,m} + \frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})\epsilon_p \lambda}) P_n^m(\cos\phi)\right| \leq \frac{9(\epsilon_p + \epsilon_s)(n+1)(2n+1)b^n}{8\pi n^2 a^{n+1} i_n(\frac{a}{\lambda})\epsilon_p(\epsilon_p + \epsilon_\infty)} \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

So, now let us consider the various series that occur in Φ and u.

For ϕ_s and u_s defined by (21) and (23), we have $r \geq a$ and the series

$$\sum_{m=-n}^{n} |B_{s,n,m} k_n(\omega r) P_n^m(\cos\phi) e^{im\theta}| \leq \frac{9(\epsilon_p + \epsilon_s)(n+1)(2n+1)^2 b^n k_n(r\omega)}{8\pi n^2 a^{n+1} k_n(a\omega)(\epsilon_p + \epsilon_\infty)\epsilon_p} \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

Since $\tilde{k}_n$ is decreasing and nonnegative, we have $r^{n+1} k_n(r\omega) \leq k_n(a\omega) a^{n+1}$.

We also observe that since $n \geq 2$, we have $\frac{(n+1)(2n+1)^2}{2n^2} \leq 3(2n+1)$

By defining $M_B := \frac{27(\epsilon_p + \epsilon_s)}{4\pi\epsilon_p(\epsilon_p + \epsilon_\infty)}$, we have

$$\sum_{m=-n}^{n} |B_{s,n,m} k_n(\omega r) P_n^m(\cos\phi) e^{im\theta}| \leq M_B \frac{(2n+1)b^n a^{n+1}}{a^{n+1} r^{n+1}} \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r},$$

and since $b < a \leq r$,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} B_{s,n,m}k_n(\omega r)P_n^m(\cos\phi)e^{im\theta} \quad \text{converges like a geometric series .}$$

Also present within ϕ_s and u_s is the series

$$\sum_{m=-n}^{n} |D_{s,n,m}r^{-n-1}P_n^m(\cos\phi)e^{im\theta}| \leq 15(\epsilon_s+\epsilon_\infty)\frac{b^n(2n+1)^2}{8\pi n(\epsilon_p+\epsilon_\infty)r^{n+1}}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Since $n \geq 1$, we have $\frac{(2n+1)^2}{2n} \leq 2(2n+1)$.

By defining $M_D := \frac{30(\epsilon_s+\epsilon_\infty)}{4\pi(\epsilon_p+\epsilon_\infty)}$, we get

$$\sum_{m=-n}^{n} |D_{s,n,m}r^{-n-1}P_n^m(\cos\phi)e^{im\theta}| \leq M_D\frac{(2n+1)b^n}{r^{n+1}}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r},$$

and since $b < r$,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} D_{s,n,m}r^{-n-1}P_n^m(\cos\phi)e^{im\theta} \quad \text{converges like a geometric series .}$$

Thus we known without any additional assumptions that ϕ_s and u_s converge. For ϕ_p and u_p defined by (20) and (22), we have $r \leq a$ and the series

$$\sum_{m=-n}^{n}\left|(C_{p,n,m}+\frac{G_{n,m}(a)}{a^n})r^nP_n^m(\cos\phi)e^{im\theta}\right|$$

$$\leq \frac{24(\epsilon_s+\epsilon_\infty)(\epsilon_p+\epsilon_s)(2n+1)^2(n+1)b^nr^n}{8\pi n^2a^{2n+1}\epsilon_s(\epsilon_p+\epsilon_\infty)}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

We also observe that since $n \geq 2$, we have $\frac{(n+1)(2n+1)^2}{2n^2} \leq 3(2n+1)$

By defining $M_C := \frac{72(\epsilon_s+\epsilon_\infty)(\epsilon_p+\epsilon_s)}{4\pi\epsilon_s(\epsilon_p+\epsilon_\infty)}$, we have

$$\sum_{m=-n}^{n}\left|(C_{p,n,m}+\frac{G_{n,m}(a)}{a^n})r^nP_n^m(\cos\phi)e^{im\theta}\right| \le M_C\frac{(2n+1)b^nr^n}{a^{2n+1}}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r},$$

and since $b < a$ and $r \le a$,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(C_{p,n,m}+\frac{G_{n,m}(a)}{a^n})r^nP_n^m(\cos\phi)e^{im\theta} \quad \text{converges like a geometric series .}$$

u_p also has the series

$$\sum_{m=-n}^{n}\left|(A_{p,n,m}+\frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})\epsilon_p\lambda})i_n(\frac{r}{\lambda})P_n^m(\cos\phi)e^{im\theta}\right|$$

$$\le \frac{9(\epsilon_p+\epsilon_s)(n+1)(2n+1)^2b^ni_n(\frac{r}{\lambda})}{8\pi n^2a^{n+1}i_n(\frac{a}{\lambda})\epsilon_p(\epsilon_p+\epsilon_\infty)}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Since $\tilde{i}_n$ is increasing and nonnegative, we have $r^{-n}i_n(\frac{r}{\lambda}) \le i_n(\frac{a}{\lambda})a^{-n}$.

We also observe that since $n \ge 2$, we have $\frac{(n+1)(2n+1)^2}{2n^2} \le 3(2n+1)$.

By defining $M_A := \frac{27(\epsilon_p+\epsilon_s)}{4\pi\epsilon_p(\epsilon_p+\epsilon_\infty)}$, we have

$$\sum_{m=-n}^{n}\left|(A_{p,n,m}+\frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})\epsilon_p\lambda})i_n(\frac{r}{\lambda})P_n^m(\cos\phi)e^{im\theta}\right| \le M_A\frac{(2n+1)r^nb^n}{a^{2n+1}}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r},$$

and since $b < a$ and $r \le a$, we have

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(A_{p,n,m}+\frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})\epsilon_p\lambda})i_n(\frac{r}{\lambda})P_n^m(\cos\phi)e^{im\theta} \quad \text{converges like a geometric series.}$$

So, what we still need to show is that

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(G_{n,m}(r)-\frac{r^nG_{n,m}(a)}{a^n})P_n^m(\cos\phi)e^{im\theta}$$

and

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\frac{H_{n,m}(a)i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}-H_{n,m}(r))P_n^m(\cos\phi)e^{im\theta}$$

both converge for $r \leq a$. However, depending on how ρ_p is defined, there may be some r for which the series do not converge. For instance, as the point charge model has the reciprocal of the distance to the point involved in its solution, an unbounded result, we will be forced to add more assumptions to create convergence. Let's assume that $\rho_p(\mathbf{r})$ is not supported near $|\mathbf{r}| = r$. So, let's say $\rho_p(\mathbf{r})$ is nonzero only for $0 \leq |\mathbf{r}| < c < r$ and $r < d < |\mathbf{r}| < b < a$

The definition of $G_{n,m}$ in (48) gives us

$$G_{n,m}(r) - \frac{r^n G_{n,m}(a)}{a^n} = \frac{1}{2n+1}\begin{pmatrix} r^n \int_r^a t^{1-n}\rho_{p,n,m}(t)dt \\ +r^{-1-n}\int_0^r t^{n+2}\rho_{p,n,m}(t)dt \\ -\frac{r^n}{a^{2n+1}}\int_0^a t^{n+2}\rho_{p,n,m}(t)dt \end{pmatrix}.$$

Then applying (131) gives us

$$|G_{n,m}(r) - \frac{r^n G_{n,m}(a)}{a^n}||P_n^m(\cos\phi)| \leq \frac{1}{4\pi}(r^n d^{-1-n} + r^{-1-n}c^n + \frac{r^n b^n}{a^{2n+1}})\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Since $r < d$, $c < r$, and $b, r < a$, we have geometric convergence, and thus

$$\sum_{n=0}^{\infty}\sum_{m=n}^{-n}(G_{n,m}(r) - \frac{r^n G_{n,m}(a)}{a^n})P_n^m(\cos\phi)e^{im\theta} \quad \text{converges like a geometric series}\ .$$

The definition of $H_{n,m}$ in (49) gives us

$$-H_{n,m}(r) + \frac{H_{n,m}(a)i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} = \begin{pmatrix} -i_n(\frac{r}{\lambda})\int_r^a k_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \\ -k_n(\frac{r}{\lambda})\int_0^r i_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \\ +\frac{k_n(\frac{a}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}\int_0^a i_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \end{pmatrix}.$$

Then applying (131) gives us

$$|-H_{n,m}(r)+\frac{H_{n,m}(a)i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}||P_n^m(\cos\phi)|\leq\frac{(2n+1)}{4\pi}\left(\begin{array}{c}i_n(\frac{r}{\lambda})k_n(\frac{d}{\lambda})\\+k_n(\frac{r}{\lambda})i_n(\frac{c}{\lambda})\\+\frac{k_n(\frac{a}{\lambda})i_n(\frac{r}{\lambda})i_n(\frac{b}{\lambda})}{i_n(\frac{a}{\lambda})}\end{array}\right)\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

We then apply (126) and (123). We note that $a\geq b$ implies $\frac{i_n(\frac{b}{\lambda})}{i_n(\frac{a}{\lambda})}\leq(\frac{b}{a})^n$ without any extra restriction on n. Then there is $N_H=\max((\frac{a}{\lambda})^2-2,\frac{(\frac{a}{\lambda})^2+1}{2})$ such that for $n\geq N_H$, we have

$$|-H_{n,m}(r)+\frac{H_{n,m}(a)i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}||P_n^m(\cos\phi)|\leq\frac{2}{4\pi}\left(\frac{\lambda}{d}(\frac{r}{d})^n+\frac{\lambda}{r}(\frac{c}{r})^n+\frac{\lambda}{a}(\frac{r}{a})^n(\frac{b}{a})^n\right)\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Then, as $r<d$, $c<r$, and $b,r<a$, we have geometric convergence, and thus

$$\sum_{n=0}^{\infty}\sum_{m=n}^{-n}(-H_{n,m}(r)+\frac{H_{n,m}(a)i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})})P_n^m(\cos\phi)e^{im\theta}\quad\text{converges like a geometric series .}$$

Thus we have shown that for $\mathbf{r}$ outside of the support of $\rho_{p,n,m}$, we have convergences in the solutions Φ_s,u_s,Φ_p,u_p defined by (21), (23), (20), (22). Thus we have proven Lemma 5.1.

For the purposes of determining error bounds, if $n\geq\tilde{N}$ where

$$\tilde{N}:=\max(\frac{a}{\lambda},\frac{a\omega+1}{2},\frac{a+3\lambda}{2\lambda},2,\frac{a^2(\epsilon_p+2\epsilon_s)}{2\lambda^2(\epsilon_p+\epsilon_\infty)},\frac{3\epsilon_\infty\epsilon_p}{2\epsilon_s(\epsilon_p+\epsilon_\infty)},\frac{a^2}{\lambda^2}-2,\frac{\frac{a^2}{\lambda^2}+1}{2})$$

then we note that the maximum contribution to the nth term of Φ_s is bounded by, with $b<a\leq r$

$$\frac{3(2n+1)}{4\pi\epsilon_s(\epsilon_p+\epsilon_\infty)\epsilon_p}\frac{1}{r}\left(\frac{b}{r}\right)^n(9(\epsilon_s-\epsilon_\infty)(\epsilon_p+\epsilon_s)+15\epsilon_p(\epsilon_s+\epsilon_\infty))\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

For u_s, the nth term is bounded by, with $b<a\leq r$

$$\frac{3(2n+1)}{4\pi\epsilon_s(\epsilon_p+\epsilon_\infty)\epsilon_p}\frac{1}{r}\left(\frac{b}{r}\right)^n(9\epsilon_\infty(\epsilon_p+\epsilon_s)+15\epsilon_p(\epsilon_s+\epsilon_\infty))\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

For Φ_p, the nth term is bounded by, with $b < a$, $c < r < d$, $r \leq a$

$$\frac{(2n+1)}{4\pi\epsilon_p}\left((\frac{72(\epsilon_p+\epsilon_s)}{\epsilon_s(\epsilon_p+\epsilon_\infty)}+1)\frac{1}{a}\left(\frac{br}{a^2}\right)^n+\frac{1}{d}\left(\frac{r}{d}\right)^n+\frac{1}{r}\left(\frac{c}{r}\right)^n\right)\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

For u_p, the nth term is bounded by, with $b < a$, $c < r < d$, $r \leq a$

$$\frac{(2n+1)}{4\pi\epsilon_p}\left((\frac{99(\epsilon_p+\epsilon_s)}{\epsilon_s(\epsilon_p+\epsilon_\infty)}+3)\frac{1}{a}\left(\frac{br}{a^2}\right)^n+\frac{3}{d}\left(\frac{r}{d}\right)^n+\frac{3}{r}\left(\frac{c}{r}\right)^n\right)\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Where the terms involving c and d are omitted if ρ does not have support for $|\mathbf{r}| < r$ and $|\mathbf{r}| > r$ respectively.

While the inclusion of the $\frac{r}{d}$ and $\frac{c}{r}$ terms are unfortunate, they are necessary for expressions of the form $\frac{1}{|\mathbf{r}-\mathbf{r_0}|}$. However, when we were calculating Φ_p and u_p for the point charge model, the inclusion of the $\frac{1}{|\mathbf{r}-\mathbf{r_0}|}$ term allowed us to rewrite, using $b = r_0$,

$$\begin{aligned}&\textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\frac{G_{n,m}(r)}{\epsilon_p}-\frac{r^nG_{n,m}(a)}{a^n\epsilon_p})P_n^m(\cos\phi)e^{im\theta}\\ =&\textstyle\frac{1}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|}-\sum_{n=0}^{\infty}\sum_{m=-n}^{n}a^{-1-2n}r^nb^nP_n^m(\cos\phi)e^{im\theta}\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p(2n+1)c_{n,m}}\end{aligned}$$

Then, if we consider this new series part, we have, via the P_n^m bounds from (130),

$$\sum_{m=-n}^{n}\left|a^{-1-2n}r^nb^nP_n^m(\cos\phi)\frac{P_n^m(\cos\phi_0)}{\epsilon_p(2n+1)c_{n,m}}\right|\leq\frac{(2n+1)}{4\pi}\frac{1}{a\epsilon_p}\left(\frac{rb}{a^2}\right)^n.$$

Thus, by removing $\frac{1}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|}$ from the series, we have made it so that the bound to the nth term of the series for Φ_p reduces to

$$\frac{(2n+1)}{4\pi\epsilon_p}\left(\frac{72(\epsilon_p+\epsilon_s)}{\epsilon_s(\epsilon_p+\epsilon_\infty)}+1\right)\frac{1}{a}\left(\frac{rb}{a^2}\right)^n.$$

We also rewrite

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{\epsilon_p\lambda}(-H_{n,m}(r)+\frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})}i_n(\frac{r}{\lambda}))P_n^m(\cos\phi)e^{im\theta}$$
$$=-\frac{e^{-|\mathbf{r}-\mathbf{r_0}|/\lambda}}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|}+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{k_n(\frac{a}{\lambda})i_n(\frac{r_0}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}P_n^m(\cos\phi)e^{im\theta}\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p\lambda c_{n,m}}.$$

Then, if we consider this new series part, we have, via the P_n^m bounds from (130),

$$\sum_{m=-n}^{n}\left|\frac{k_n(\frac{a}{\lambda})i_n(\frac{r_0}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}P_n^m(\cos\phi)\frac{P_n^m(\cos\phi_0)}{\epsilon_p\lambda c_{n,m}}\right|\le\frac{(2n+1)}{4\pi}\frac{1}{\epsilon_p\lambda}\frac{k_n(\frac{a}{\lambda})i_n(\frac{b}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}.$$

We apply (126) and (123) with $N_H=\max((\frac{a}{\lambda})^2-2,\frac{(\frac{a}{\lambda})^2+1}{2})$ to get that for $n\ge N_H$, we have

$$\sum_{m=-n}^{n}\left|\frac{k_n(\frac{a}{\lambda})i_n(\frac{r_0}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}P_n^m(\cos\phi)\frac{P_n^m(\cos\phi_0)}{\epsilon_p\lambda c_{n,m}}\right|\le\frac{2}{4\pi a\epsilon_p}\left(\frac{rb}{a^2}\right)^n.$$

Then the bound for the nth term of the u_p series reduces to

$$\frac{(2n+1)}{4\pi\epsilon_p}\left(\frac{99(\epsilon_p+\epsilon_s)}{\epsilon_s(\epsilon_p+\epsilon_\infty)}+3\right)\frac{1}{a}\left(\frac{rb}{a^2}\right)^n.$$

Such improved convergence rates can be obtained for more general ρ, but only if we can find explicit particular solutions to the differential equations. The trade off is that these particular solutions may not be bounded or well behaved.

Another observation we may make is that we were calculating a bound for $\left|\sum_{m=-n}^{n}\cdots\right|$ by $\sum_{m=-n}^{n}|\cdots|\le(2n+1)|\cdots|$. We can do better.

In the point charge model example with a point charge at $\mathbf{r_0}$, all of the terms of series, once the constants are expanded, have the factor

$$P_n^m(\cos\phi)e^{im\theta}P_n^m(\cos\phi_0)\frac{e^{-im\theta_0}}{(2n+1)c_{n,m}}.$$

It is also noteworthy that the multiple is independent of m and that no $\phi, \theta, \phi_0, \theta_0$ appear beyond in this factor. This can be seen from the calculations for $A_{p,n,m}, C_{p,n,m}, B_{s,n,m}, D_{s,n,m}$ as the only time m was essential was in the $H_{n,m}$ and $G_{n,m}$ functions.

So, let us examine these terms.

$$P_n^m(\cos\phi)e^{im\theta}P_n^m(\cos\phi_0)\frac{e^{-im\theta_0}}{(2n+1)c_{n,m}} + P_n^{-m}(\cos\phi)e^{-im\theta}P_n^{-m}(\cos\phi_0)\frac{e^{im\theta_0}}{(2n+1)c_{n,-m}}$$

$$= P_n^m(\cos\phi)e^{im\theta}P_n^m(\cos\phi_0)\frac{e^{-im\theta_0}}{(2n+1)c_{n,m}} + P_n^m(\cos\phi)e^{-im\theta}P_n^m(\cos\phi_0)\frac{e^{im\theta_0}}{(2n+1)c_{n,-m}}\left(\frac{(n-m)!}{(n+m)!}\right)^2$$

$$= \frac{P_n^m(\cos\phi)P_n^m(\cos\phi_0)}{(2n+1)}\left(\frac{e^{im(\theta-\theta_0)}}{c_{n,m}} + e^{-im(\theta-\theta_0)}\frac{(2n+1)(n+m)!}{4\pi(n-m)!}\left(\frac{(n-m)!}{(n+m)!}\right)^2\right)$$

$$= \frac{P_n^m(\cos\phi)P_n^m(\cos\phi_0)}{(2n+1)c_{n,m}}\left(e^{im(\theta-\theta_0)} + e^{-im(\theta-\theta_0)}\right) = \frac{2P_n^m(\cos\phi)P_n^m(\cos\phi_0)\cos(m(\theta-\theta_0))}{(2n+1)c_{n,m}}.$$

Thus

$$\sum_{m=-n}^{n} P_n^m(\cos\phi)e^{im\theta}P_n^m(\cos\phi_0)\frac{e^{-im\theta_0}}{(2n+1)c_{n,m}} = \sum_{m=-n}^{n}\frac{P_n^m(\cos\phi)P_n^m(\cos\phi_0)\cos(m(\theta-\theta_0))}{(2n+1)c_{n,m}}.$$

So we have removed the imaginary component and transformed the complex exponential into a cosine. We may also choose to rewrite it as

$$\sum_{m=-n}^{n} \frac{P_n^m(\cos\phi)e^{im\theta}P_n^m(\cos\phi_0)e^{-im\theta_0}}{(2n+1)c_{n,m}} = \sum_{m=-n}^{n} \frac{P_n^m(\cos\phi)P_n^m(\cos\phi_0)\cos(m(\theta-\theta_0))}{4\pi}\frac{(n-m)!}{(n+m)!}.$$

This allows us to use the Legendre Addition Formula [5, Section 14.18.2]

$$\sum_{m=-n}^{n} \frac{P_n^m(\cos\phi)P_n^m(\cos\phi_0)\cos(m(\theta-\theta_0))}{4\pi}\frac{(n-m)!}{(n+m)!} = \frac{P_n(\cos\phi\cos\phi_0+\sin\phi\sin\phi_0\cos(\theta-\theta_0))}{4\pi}.$$

If we consider $\mathbf{r}$ and $\mathbf{r_0}$ to have coordinates (r,ϕ,θ) and (r_0,ϕ_0,θ_0) respectively, then, after rotating θ_0 about the z-axis, the points will be $(r,\phi,\theta-\theta_0)$ and $(r_0,\phi_0,0)$. Since rotation preserves angles, both pairs of points will have the same angle between them. The cosine of that angle is $\cos(\phi)\cos(\phi_0)+\sin(\phi)\sin(\phi_0)\cos(\theta-\theta_0)$. Let us refer to that angle as $\tilde{\phi}$.

$$\sum_{m=-n}^{n} P_n^m(\cos\phi)e^{im\theta}P_n^m(\cos\phi_0)\frac{e^{-im\theta_0}}{(2n+1)c_{n,m}} = \frac{1}{4\pi}P_n(\cos(\phi)\cos(\phi_0)+\sin(\phi)\sin(\phi_0)\cos(\theta-\theta_0))$$

This means that $\sum_{m=-n}^{n} X_{n,m}P_n^m(\cos\phi)e^{im\theta} = \tilde{X}_n P_n(\cos\tilde{\phi})$ Where X is $A,B,C,$ or D for some $\tilde{A},\tilde{B},\tilde{C}$, or $\tilde{D}$. So, Φ_p, u_p, Φ_s, u_s may be written in the form $\sum_{n=0}^{\infty} f_n(r)P_n(\cos\tilde{\phi})$. These solve the differential equations for $\rho=\delta(\mathbf{r}-\mathbf{r_0})$:

$$\begin{aligned}
\Phi_p(\mathbf{r}) &= \textstyle\sum_{n=0}^{\infty} \Phi_{p,n,\mathbf{r_0}}(r)P_n(\cos\tilde{\phi})\\
\Phi_s(\mathbf{r}) &= \textstyle\sum_{n=0}^{\infty} \Phi_{s,n,\mathbf{r_0}}(r)P_n(\cos\tilde{\phi})\\
u_p(\mathbf{r}) &= \textstyle\sum_{n=0}^{\infty} u_{p,n,\mathbf{r_0}}(r)P_n(\cos\tilde{\phi})\\
u_s(\mathbf{r}) &= \textstyle\sum_{n=0}^{\infty} u_{s,n,\mathbf{r_0}}(r)P_n(\cos\tilde{\phi})
\end{aligned} \tag{132}$$

Let us consider a rotation O that takes $\mathbf{r_0}$ to the z axis. Whatever rotation we choose, $\mathbf{r}$ will be mapped to spherical coordinates $\tilde{\mathbf{r}}=(r,\tilde{\phi},\tilde{\theta})$. Since Δ is rotation invariant and Γ is a sphere, we observe that

$$\begin{aligned}
\Phi_p(\tilde{\mathbf{r}}) &= \textstyle\sum_{n=0}^{\infty} \Phi_{p,n,\mathbf{r_0}}(r) P_n(\cos\tilde{\phi}) \\
\Phi_s(\tilde{\mathbf{r}}) &= \textstyle\sum_{n=0}^{\infty} \Phi_{s,n,\mathbf{r_0}}(r) P_n(\cos\tilde{\phi}) \\
u_p(\tilde{\mathbf{r}}) &= \textstyle\sum_{n=0}^{\infty} u_{p,n,\mathbf{r_0}}(r) P_n(\cos\tilde{\phi}) \\
u_s(\tilde{\mathbf{r}}) &= \textstyle\sum_{n=0}^{\infty} u_{s,n,\mathbf{r_0}}(r) P_n(\cos\tilde{\phi})
\end{aligned} \tag{133}$$

will solve the system for $\rho = \delta(\tilde{\mathbf{r}} - (0,0,r_0))$. Since $\tilde{\phi}$ is both the ϕ component of $\tilde{\mathbf{r}}$ and the angle between $\tilde{\mathbf{r}}$ and $(0,0,r_0)$, the $\Phi_{p,n,\mathbf{r_0}}, \Phi_{s,n,\mathbf{r_0}}, u_{p,n,\mathbf{r_0}}, u_{s,n,\mathbf{r_0}}$ coefficients defined above must agree with what we obtain from the method of solving the differential equations established earlier. Thus, to solve the differential equations system with $\rho = \delta(\mathbf{r} - \mathbf{r_0})$, we may first solve the case with $\rho = \delta(\tilde{\mathbf{r}} - O(\mathbf{r_0}))$ and obtain (133). Then $\Phi_p(O(\mathbf{r})), \Phi_s(O(\mathbf{r})), u_p(O(\mathbf{r})), u_s(O(\mathbf{r}))$ will be the solutions for $\rho = \delta(\mathbf{r} - \mathbf{r_0})$.

This means that for the point charge model, we can shrink the number of terms summed from a quadratic to linear in terms of n. Furthermore, the solution for arbitrary ρ can be taken from an appropriate integral of the point charge solutions, a weighted average. We use

$$\rho(\mathbf{r}) = \int_{D_p} \delta(\mathbf{r} - \mathbf{r_0}) \rho(\mathbf{r_0}) d\mathbf{r_0},$$

and we repeat the argument with $P_n^m(\cos\phi) e^{im\theta} P_n^m(\cos\phi_0) \frac{e^{-im\theta_0}}{(2n+1)c_{n,m}}$ replaced by

$$\int_{D_p} P_n^m(\cos\phi) e^{im\theta} P_n^m(\cos\phi_0) \frac{e^{-im\theta_0}}{(2n+1)c_{n,m}} \rho(\mathbf{r_0}) d\mathbf{r_0}$$

to obtain

$$\begin{aligned}
&\textstyle\sum_{m=-n}^{n} \int_{D_p} P_n^m(\cos\phi) e^{im\theta} P_n^m(\cos\phi_0) \frac{e^{-im\theta_0}}{(2n+1)c_{n,m}} \rho(\mathbf{r_0}) d\mathbf{r_0} \\
&= \textstyle\int_{D_p} \frac{1}{4\pi} P_n(\cos(\phi)\cos(\phi_0) + \sin(\phi)\sin(\phi_0)\cos(\theta - \theta_0)) \rho(\mathbf{r_0}) d\mathbf{r_0}
\end{aligned}.$$

Thus we have

$$\left|\sum_{m=-n}^{n}\int_{D_p} P_n^m(\cos\phi)e^{im\theta}P_n^m(\cos\phi_0)\frac{e^{-im\theta_0}}{(2n+1)c_{n,m}}\rho(\mathbf{r_0})d\mathbf{r_0}\right| \leq \frac{1}{4\pi}\int_{D_p}|\rho(\mathbf{r_0})|d\mathbf{r_0}.$$

This is an improvement over using (131) by a factor of $2n+1$. Thus by repeating the argument we used to show convergence, but with the components expanded out to reveal the $\int_{D_p} P_n^m(\cos\phi)e^{im\theta}P_n^m(\cos\phi_0)\frac{e^{-im\theta_0}}{(2n+1)c_{n,m}}\rho(\mathbf{r_0})d\mathbf{r}$ factor, we can remove the $2n+1$ factor from our previously found bounds.

Thus we have proven the following lemma.

Thus, the maximum contribution to the nth term of Φ_s is bounded by, with $b < a \leq r$

Lemma 5.2. *Let the conditions of Theorem 2.1 hold.*

Let ρ_p be supported in a ball $B(\mathbf{0}, b)$ with $b < a$. Let the support of ρ_p be further confined to be within $B(\mathbf{0}, c) \cup (B(\mathbf{0}, \mathbf{a}) - B(\mathbf{0}, \mathbf{d}))$

Let ϕ_s and u_s be defined by (21) and (23).

Let ϕ_p and u_p be defined by (20) and (22).

Then for $n \geq \tilde{N}$

$$\tilde{N} := \max(\frac{a}{\lambda}, \frac{a\omega+1}{2}, \frac{a+3\lambda}{2\lambda}, 2, \frac{a^2(\epsilon_p+2\epsilon_s)}{2\lambda^2(\epsilon_p+\epsilon_\infty)}, \frac{3\epsilon_\infty\epsilon_p}{2\epsilon_s(\epsilon_p+\epsilon_\infty)}, \frac{a^2}{\lambda^2}-2, \frac{\frac{a^2}{\lambda^2}+1}{2}),$$

where $|\mathbf{r}| = r$, we have the following:

For Φ_s, the nth term is bounded by, with $b < a \leq r$

$$\frac{3}{4\pi\epsilon_s(\epsilon_p+\epsilon_\infty)\epsilon_p}\frac{1}{r}\left(\frac{b}{r}\right)^n (9(\epsilon_s-\epsilon_\infty)(\epsilon_p+\epsilon_s)+15\epsilon_p(\epsilon_s+\epsilon_\infty))\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

For u_s, the nth term is bounded by, with $b < a \leq r$

$$\frac{3}{4\pi\epsilon_s(\epsilon_p+\epsilon_\infty)\epsilon_p}\frac{1}{r}\left(\frac{b}{r}\right)^n(9\epsilon_\infty(\epsilon_p+\epsilon_s)+15\epsilon_p(\epsilon_s+\epsilon_\infty))\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

For Φ_p, the nth term is bounded by, with $b<a$, $c<r<d$, $r\leq a$

$$\frac{1}{4\pi\epsilon_p}\left((\frac{72(\epsilon_p+\epsilon_s)}{\epsilon_s(\epsilon_p+\epsilon_\infty)}+1)\frac{1}{a}\left(\frac{br}{a^2}\right)^n+\frac{1}{d}\left(\frac{r}{d}\right)^n+\frac{1}{r}\left(\frac{c}{r}\right)^n\right)\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

For u_p, the nth term is bounded by, with $b<a$, $c<r<d$, $r\leq a$

$$\frac{1}{4\pi\epsilon_p}\left((\frac{99(\epsilon_p+\epsilon_s)}{\epsilon_s(\epsilon_p+\epsilon_\infty)}+3)\frac{1}{a}\left(\frac{br}{a^2}\right)^n+\frac{3}{d}\left(\frac{r}{d}\right)^n+\frac{3}{r}\left(\frac{c}{r}\right)^n\right)\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Again, with the terms involving c and d removed if $\rho(\mathbf{r})=0$ for all $|\mathbf{r}|<r$ and $|\mathbf{r}|>r$ respectively. The c and d terms may also be removed if rewrite the solution using explicit forms as in the case of the point charge model.

Indeed, our work with developing the improved convergence and focusing on the point charge model gives us the following lemma.

Lemma 5.3. *Let the conditions of Theorem 2.1 hold.*

Let $\mathbf{r_0}\in D_p$. Let $\rho_p(\mathbf{r})=\delta(\mathbf{r}-\mathbf{r_0})$.

Let ϕ_s and u_s be defined by (86) *and* (87).

Let ϕ_p and u_p be defined by (88) *and* (89).

Then for $n\geq\tilde{N}$

$$\tilde{N}:=\max(\frac{a}{\lambda},\frac{a\omega+1}{2},\frac{a+3\lambda}{2\lambda},2,\frac{a^2(\epsilon_p+2\epsilon_s)}{2\lambda^2(\epsilon_p+\epsilon_\infty)},\frac{3\epsilon_\infty\epsilon_p}{2\epsilon_s(\epsilon_p+\epsilon_\infty)},\frac{a^2}{\lambda^2}-2,\frac{\frac{a^2}{\lambda^2}+1}{2}),$$

where $|\mathbf{r}|=r$, we have the following:

For Φ_s, the nth term is bounded by, with $b<a\leq r$

$$\frac{3}{4\pi\epsilon_s(\epsilon_p+\epsilon_\infty)\epsilon_p}\frac{1}{r}\left(\frac{b}{r}\right)^n(9(\epsilon_s-\epsilon_\infty)(\epsilon_p+\epsilon_s)+15\epsilon_p(\epsilon_s+\epsilon_\infty))\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

For u_s, the nth term is bounded by, with $b < a \leq r$

$$\frac{3}{4\pi\epsilon_s(\epsilon_p+\epsilon_\infty)\epsilon_p}\frac{1}{r}\left(\frac{b}{r}\right)^n(9\epsilon_\infty(\epsilon_p+\epsilon_s)+15\epsilon_p(\epsilon_s+\epsilon_\infty))\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

For Φ_p, the nth term is bounded by, with $b < a$,

$$\frac{1}{4\pi\epsilon_p}\left((\frac{72(\epsilon_p+\epsilon_s)}{\epsilon_s(\epsilon_p+\epsilon_\infty)}+1)\frac{1}{a}\left(\frac{br}{a^2}\right)^n\right)\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

For u_p, the nth term is bounded by, with $b < a$,

$$\frac{1}{4\pi\epsilon_p}\left((\frac{99(\epsilon_p+\epsilon_s)}{\epsilon_s(\epsilon_p+\epsilon_\infty)}+3)\frac{1}{a}\left(\frac{br}{a^2}\right)^n\right)\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Thus the series converge everywhere.

It is interesting to note that $\tilde{N}$ does not depend on ρ. So if we have the constants $\epsilon_\infty, \epsilon_s, \epsilon_p, a, \lambda$ fixed, then every ρ will require the same number of terms in the sum to guarantee the error bounds. The particular ρ may have slower convergence.

We may also observe that if ρ has support over anything beyond finitely many points, Lemma 5.1 does not provide much convergence over Φ_p or u_p. We have a better result found in Theorem 5.4.

Theorem 5.4. *Let the conditions of Theorem 2.1 hold.*

Let ρ_p be supported in a ball $B(\mathbf{0}, b)$ with $b < a$.

Let $\max|\rho_p(\mathbf{r})| < \infty$.

Let Φ_p, Φ_s, u_p, u_s be defined as in $(20), (21), (22), (23)$*. Then their series converge everywhere, and Φ_p, Φ_s weakly solve* $(15), (16), (17)$ *subject to* $(18), (19)$ *everywhere.*

Suppose ρ is bounded. Then for function f and $y \geq x \geq 0$, we have

$$\left| \sum_{m=-n}^{n} P_n^m(\cos\phi) \int_x^y f(t)\rho_{p,n,m}(t)dt \right|$$

$$\leq \left| \sum_{m=-n}^{n} P_n^m(\cos\phi) \int_x^y \frac{f(t)}{c_{n,m}} \int_0^{\pi} \int_0^{2\pi} \rho(t,\phi_0,\theta_0) P_n^m(\cos\phi_0) \sin\phi_0 d\theta_0 d\phi_0 dt \right|$$

$$\leq (2n+1) \max|\rho| \int_x^y |f(t)| \int_0^{\pi} \int_0^{2\pi} P_n(\cos\tilde{\phi}) \sin\phi_0 d\theta_0 d\phi_0 dt$$

where $\tilde{\phi}$ is the angle between (t,ϕ_0,θ_0) and $\mathbf{r}$.

[6] gives us a bound for P_n, which we apply to give us

$$\leq (2n+1) \max|\rho| \int_x^y |f(t)| \int_0^{\pi} \int_0^{2\pi} \sqrt{\frac{2}{\pi n}} \frac{\sin\phi_0}{\sqrt{\sin\tilde{\phi}}} d\theta_0 d\phi_0 dt.$$

We have that for some M,

$$\int_0^{\pi} \int_0^{2\pi} \frac{\sin\phi_0}{\sqrt{\sin\tilde{\phi}}} d\theta_0 d\phi_0 dt \leq M.$$

Thus,

$$\left| \sum_{m=-n}^{n} P_n^m(\cos\phi) \int_x^y f(t)\rho_{p,n,m}(t)dt \right| \leq (2n+1) \max|\rho| M \sqrt{\frac{2}{\pi n}} \max|f|(y-x). \tag{134}$$

So, for sequence $\{\epsilon_n\}_{n=0}^{\infty}$, with $0 < \epsilon_n < r$,

$$\left| \sum_{m=-n}^{n} (G_{n,m}(r) - \frac{r^n G_{n,m}(a)}{a^n}) P_n^m(\cos\phi) \right|$$

$$\leq \left| \sum_{m=-n}^{n} \frac{1}{2n+1} P_n^m(\cos\phi) \begin{pmatrix} r^n \int_r^{r+\epsilon_n} t^{1-n} \rho_{p,n,m}(t)dt \\ +r^n \int_{r+\epsilon_n}^{a} t^{1-n} \rho_{p,n,m}(t)dt \\ +r^{-1-n} \int_{r-\epsilon_n}^{r} t^{n+2} \rho_{p,n,m}(t)dt \\ +r^{-1-n} \int_0^{r-\epsilon_n} t^{n+2} \rho_{p,n,m}(t)dt \\ +\frac{r^n}{a^{2n+1}} \int_0^a t^{n+2} \rho_{p,n,m}(t)dt \end{pmatrix} \right|$$

Applying (131) gives us

$$\sum_{m=-n}^{n} \frac{1}{2n+1} P_n^m(\cos\phi) \begin{pmatrix} r^n \int_{r+\epsilon_n}^{a} t^{1-n} \rho_{p,n,m}(t)dt \\ +r^{-1-n} \int_0^{r-\epsilon_n} t^{n+2} \rho_{p,n,m}(t)dt \\ +\frac{r^n}{a^{2n+1}} \int_0^a t^{n+2} \rho_{p,n,m}(t)dt \end{pmatrix}$$

$$\leq \frac{1}{4\pi} \left(\frac{1}{(r+\epsilon_n)} \left(\frac{r}{r+\epsilon_n} \right)^n + \frac{1}{r} \left(\frac{r-\epsilon_n}{r} \right)^n + \frac{1}{a} \left(\frac{br}{a^2} \right)^n \right) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}$$

Applying (134) gives us

$$\sum_{m=-n}^{n} \frac{1}{2n+1} P_n^m(\cos\phi) \begin{pmatrix} r^n \int_r^{r+\epsilon_n} t^{1-n} \rho_{p,n,m}(t)dt \\ +r^{-1-n} \int_{r-\epsilon_n}^{r} t^{n+2} \rho_{p,n,m}(t)dt \end{pmatrix} \leq \max|\rho| M \sqrt{\frac{2}{\pi n}} \epsilon_n \left((r+\epsilon_n) + r \right)$$

We desire that $\sum_{n=0}^{\infty} \left| \sum_{m=-n}^{n} (G_{n,m}(r) - \frac{r^n G_{n,m}(a)}{a^n}) P_n^m(\cos\phi) \right|$ converges. So, it will be enough for the following three series

$$\sum_{n=1}^{\infty} M \sqrt{\frac{2}{\pi n}} \epsilon_n < \infty$$

$$\sum_{n=1}^{\infty} \left(\frac{r}{r+\epsilon_n} \right)^n < \infty$$

$$\sum_{n=1}^{\infty}\left(\frac{r-\epsilon_n}{r}\right)^n < \infty$$

to hold. We shall define $\epsilon_n = rn^{-3/4}$ to achieve this. This implies that

$$\sum_{n=0}^{\infty}\sum_{m=n}^{-n}(G_{n,m}(r) - \frac{r^n G_{n,m}(a)}{a^n})P_n^m(\cos\phi)e^{im\theta} \qquad \text{converges}$$

Likewise, concerning the $H_{n,m}$, we have

$$|\sum_{m=-n}^{n} -H_{n,m}(r) + \frac{H_{n,m}(a)i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}P_n^m(\cos\phi)|$$

$$\leq \left|\sum_{m=-n}^{n} P_n^m(\cos\phi)\begin{pmatrix} i_n(\frac{r}{\lambda})\int_r^{r+\epsilon_n} k_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \\ +i_n(\frac{r}{\lambda})\int_{r+\epsilon_n}^{a} k_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \\ +k_n(\frac{r}{\lambda})\int_{r-\epsilon_n}^{r} i_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \\ +k_n(\frac{r}{\lambda})\int_0^{r-\epsilon_n} i_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \\ +\frac{k_n(\frac{a}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}\int_0^a i_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \end{pmatrix}\right|.$$

Applying (131) gives us

$$\sum_{m=-n}^{n} P_n^m(\cos\phi)\begin{pmatrix} i_n(\frac{r}{\lambda})\int_{r+\epsilon_n}^{a} k_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \\ +k_n(\frac{r}{\lambda})\int_0^{r-\epsilon_n} i_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \\ +\frac{k_n(\frac{a}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}\int_0^a i_n(\frac{t}{\lambda})t^2\rho_{p,n,m}(t)dt \end{pmatrix}$$

$$\leq \frac{(2n+1)}{4\pi}\begin{pmatrix} i_n(\frac{r}{\lambda})k_n(\frac{r+\epsilon_n}{\lambda}) \\ +k_n(\frac{r}{\lambda})i_n(\frac{r-\epsilon_n}{\lambda}) \\ +\frac{k_n(\frac{a}{\lambda})i_n(\frac{r}{\lambda})i_n(\frac{b}{\lambda})}{i_n(\frac{a}{\lambda})} \end{pmatrix}\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}.$$

Applying (134) gives us

$$\sum_{m=-n}^{n} P_n^m(\cos\phi) \begin{pmatrix} i_n(\frac{r}{\lambda}) \int_r^{r+\epsilon_n} k_n(\frac{t}{\lambda}) t^2 \rho_{p,n,m}(t) dt \\ +k_n(\frac{r}{\lambda}) \int_{r-\epsilon_n}^{r} i_n(\frac{t}{\lambda}) t^2 \rho_{p,n,m}(t) dt \end{pmatrix}$$

$$\leq (2n+1) \max|\rho| M \sqrt{\frac{2}{\pi n}} \epsilon_n (i_n(\frac{r}{\lambda}) k_n(\frac{r+\epsilon_n}{\lambda})(r+\epsilon_n)^2 + k_n(\frac{r}{\lambda}) i_n(\frac{r-\epsilon_n}{\lambda}) r^2).$$

We then apply (126) and (123). We note that $a \geq b$ implies $\frac{i_n(\frac{b}{\lambda})}{i_n(\frac{a}{\lambda})} \leq (\frac{b}{a})^n$ without any extra restriction on n. Then there is $N_H = \max((\frac{a}{\lambda})^2 - 2, \frac{(\frac{a}{\lambda})^2+1}{2})$ such that for $n \geq N_H$, we have

$$\sum_{m=-n}^{n} P_n^m(\cos\phi) \begin{pmatrix} i_n(\frac{r}{\lambda}) \int_{r+\epsilon_n}^{a} k_n(\frac{t}{\lambda}) t^2 \rho_{p,n,m}(t) dt \\ +k_n(\frac{r}{\lambda}) \int_0^{r-\epsilon_n} i_n(\frac{t}{\lambda}) t^2 \rho_{p,n,m}(t) dt \\ +\frac{k_n(\frac{a}{\lambda}) i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} \int_0^a i_n(\frac{t}{\lambda}) t^2 \rho_{p,n,m}(t) dt \end{pmatrix}$$

$$\leq \frac{2}{4\pi} \left(\frac{\lambda}{(r+\epsilon_n)} \left(\frac{r}{r+\epsilon_n} \right)^n + \frac{\lambda}{r} \left(\frac{r-\epsilon_n}{r} \right)^n + \frac{\lambda}{a} \left(\frac{br}{a^2} \right)^n \right) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

and

$$\sum_{m=-n}^{n} P_n^m(\cos\phi) \begin{pmatrix} i_n(\frac{r}{\lambda}) \int_r^{r+\epsilon_n} k_n(\frac{t}{\lambda}) t^2 \rho_{p,n,m}(t) dt \\ +k_n(\frac{r}{\lambda}) \int_{r-\epsilon_n}^{r} i_n(\frac{t}{\lambda}) t^2 \rho_{p,n,m}(t) dt \end{pmatrix}$$

$$\leq 2 \max|\rho| M \sqrt{\frac{2}{\pi n}} \epsilon_n (\lambda(r+\epsilon_n) \left(\frac{r}{r+\epsilon_n} \right)^n + \lambda r \left(\frac{r-\epsilon_n}{r} \right)^n).$$

We desire that $\sum_{n=0}^{\infty} \left| \sum_{m=-n}^{n} (-H_{n,m}(r) + \frac{H_{n,m}(a) i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}) P_n^m(\cos\phi) \right|$ converges. So, again it will be enough for the following series

$$\sum_{n=1}^{\infty} M \sqrt{\frac{2}{\pi n}} \epsilon_n < \infty$$

$$\sum_{n=1}^{\infty} \left(\frac{r}{r+\epsilon_n} \right)^n < \infty$$

$$\sum_{n=1}^{\infty}\left(\frac{r-\epsilon_n}{r}\right)^n < \infty$$

to hold. Again, we shall define $\epsilon_n = rn^{-3/4}$ to achieve this. This implies that

$$\sum_{n=0}^{\infty}\sum_{m=n}^{-n}(-H_{n,m}(r) + \frac{H_{n,m}(a)i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})})P_n^m(\cos\phi)e^{im\theta} \qquad \text{converges}$$

.

So, we may write new bounds for the nth term of the series when ρ is bounded, when $n \geq \tilde{N}$.

For Φ_p the nth term is bounded by, with $b < a$, $r \leq a$,

$$\frac{1}{4\pi\epsilon_p}\left((\frac{72(\epsilon_p+\epsilon_s)}{\epsilon_s(\epsilon_p+\epsilon_\infty)}+1)\frac{1}{a}\left(\frac{br}{a^2}\right)^n + \frac{1}{r(1+n^{-3/4})}\left(\frac{1}{1+n^{-3/4}}\right)^n + \frac{1}{r}\left(\frac{1-n^{-3/4}}{1}\right)^n\right)\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}$$
$$+\frac{1}{\epsilon_p}\max|\rho|M\sqrt{\frac{2}{\pi n}}n^{-3/4}r(2+n^{-3/4})$$

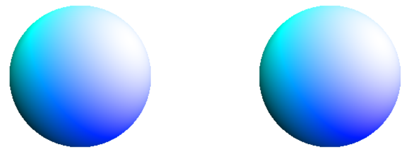
For u_p the nth term is bounded by, with $b < a$, $r \leq a$,

$$\frac{1}{4\pi\epsilon_p}\left((\frac{99(\epsilon_p+\epsilon_s)}{\epsilon_s(\epsilon_p+\epsilon_\infty)}+3)\frac{1}{a}\left(\frac{br}{a^2}\right)^n + \frac{3}{r(1+n^{-3/4})}\left(\frac{1}{1+n^{-3/4}}\right)^n + \frac{3}{r}\left(\frac{1-n^{-3/4}}{1}\right)^n\right)\int_{D_p}|\rho(\mathbf{r})|d\mathbf{r}$$
$$+\frac{3}{\epsilon_p}\max|\rho|M\sqrt{\frac{2}{\pi n}}n^{-3/4}r(2+n^{-3/4})$$

Thus, if ρ is bounded, then the series are convergent everywhere and thus gives weak solution on all of $\mathbb{R}^3$. While as written, this doesn't demonstrate convergence at $r = 0$, we remove the integrals concerning $r - \epsilon_n$ and their results, since they'd fall outside of the domain. 0^n and $i_n(0)$ are 0 for $n = 1, 2, ...,$ which removes most of the other integrals. This will give us that for $n \geq 1$, the nth term for u_p and Φ_p is 0.

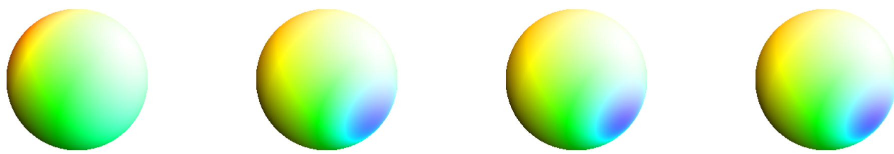
Thus we have proven Theorem 5.4.

When the point charge is near the origin, as in [Figure 7] we observe that we have very quick convergence. Again, we use $\epsilon_\infty = 1.8, \epsilon_s = 80, \epsilon_p = 1, a = 1, \lambda = 15$. For these constants, $\tilde{N} = 2$ and thus our bounds for the nth term are valid.



(a) $n = 1$ (b) Complete series

Figure 7: zxy^+ Hemisphere values of the incomplete series for Φ on Γ up to a given n for the point charge at $(r, \phi, \theta) = (0.1, \frac{\pi}{4}, \frac{3\pi}{4})$



(a) $n = 1$ (b) $n = 4$ (c) $n = 8$ (d) Complete series

Figure 8: zxy^+ Hemisphere values of the incomplete series for Φ on Γ up to a given n for the point charge at $(r, \phi, \theta) = (0.5, \frac{\pi}{4}, \frac{3\pi}{4})$

When the point charge has been moved from the origin, as in [Figure 8], the convergence takes longer, as we expected from the calculations, the convergence being that of a geometric series with ratio $\frac{r_0}{a} < 1$. $\tilde{N} = 2$ and thus our bounds for the nth term are valid.

n	Maximum contribution of nth part	Rate of convergence	Theoretical maximum using Φ_s
1	5.229×10^{-4}	9.74	6.2065
2	5.369×10^{-5}	9.87	0.62065
3	5.442×10^{-6}	9.92	0.062065
4	5.487×10^{-7}	9.94	0.0062065

Table 1: Convergence on Γ for point charge at $(r, \phi, \theta) = (0.1, \pi/4, 3\pi/4)$

We observe that actual maximum contribution of the nth part of the series is far less than the guaranteed bounds. This is in part because we chose to find the error estimates for any set of constants rather than the particular $\epsilon_p, \epsilon_s, \epsilon_\infty, a, \lambda$. We also made some simple estimates with using the asymptotic relations to keep n small, and did not appeal to more sophisticated bounds for P_n^m. While the estimated bounds were only valid for $n \geq 2$, in these instances, they also hold with $n = 1$, although we should not expect such bounds to apply to smaller n in general.

n	Maximum contribution of nth part	Rate of convergence	Theoretical maximum using Φ_s
1	0.01312	1.94	31.033
2	0.006729	1.97	15.517
3	0.003408	1.98	7.7582
4	0.001718	1.99	3.8791

Table 2: Convergence on Γ for point charge at $(r, \phi, \theta) = (0.5, \pi/4, 3\pi/4)$

Nonetheless, the actual contribution has been decreasing near the factor of $\frac{a}{r_0}$, which would be expected from our bounds.

6 Nonlocal Modified Linear Poisson-Boltzmann Model

We demonstrate the appeal of the techniques used to solve a nonlocal modified linear Poisson-Boltzmann equation, (NMLPBE) model as seen in [10].

$$-\epsilon_p \Delta \Phi_p(\mathbf{r}) = \frac{1}{\epsilon_0} \rho_p(\mathbf{r}) \quad \mathbf{r} \in D_p \tag{135}$$

$$-\epsilon_\infty \Delta \Phi_s(\mathbf{r}) + \frac{(\epsilon_s - \epsilon_\infty)}{\lambda^2} [\Phi_s(\mathbf{r}) - (\Phi * Q_\lambda)(\mathbf{r})] + \kappa^2 \Phi_s(\mathbf{r}) = 0 \quad \mathbf{r} \in D_s \tag{136}$$

$$\Phi_s(\mathbf{r}) \to 0 \quad \text{as } |\mathbf{r}| \to \infty \tag{137}$$

subject to the interface equations

$$\Phi_s(\mathbf{r}) = \Phi_p(\mathbf{r}) \quad \mathbf{r} \in \Gamma \tag{138}$$

$$\epsilon_\infty \frac{\partial \Phi_s(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} + (\epsilon_s - \epsilon_\infty) \frac{\partial (\Phi * Q_\lambda)(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} - \epsilon_p \frac{\partial \Phi_p(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} = 0 \quad \mathbf{r} \in \Gamma \tag{139}$$

where $\mathbf{n}(\mathbf{r})$ is the outward normal and

$$Q_\lambda(\mathbf{r}) = \frac{1}{4\pi\lambda^2|\mathbf{r}|}e^{-|\mathbf{r}|/\lambda} \quad \mathbf{r} \neq 0.$$

This model assumes there are n_p point charges at locations $\mathbf{r_j}$ with magnitude z_j. It leads to ρ being defined by

$$\frac{1}{\epsilon_0}\rho_p(\mathbf{r}) = \alpha \sum_{j=1}^{n_p} z_j \delta(\mathbf{r} - \mathbf{r}_j) \quad \mathbf{r} \in D_p \tag{140}$$

The special constants α and κ^2 are given by

$$\alpha = \frac{10^{10}e_c^2}{e_0 k_B T} \quad \kappa^2 = 2I_s \frac{10^{-17}N_A e_c^2}{e_0 k_B T}$$

Where ϵ_0 is the permittivity constant of the vacuum as before. e_c is the elementary charge. k_B is the Boltzmann constant, T is the absolute temperature, and N_A is the Avogadro number. I_s is the ionic solvent strength in moles per liter. We list their values from [10]

e_0	$\approx 8.8542 \times 10^{-12}$	Farad/Meter
ϵ_c	$\approx 1.6022 \times 10^{-19}$	Coulomb
T	$= 298.15$	Kelvin
k_B	$\approx 1.3806 \times 10^{-23}$	Joule/Kevin
N_A	$\approx 6.0221 \times 10^{23}$	ions/mole

For applying this model to actual simulations of protein in water, the values for those constants are important. This is especially in the case of the nonlinear version, which features $\kappa^2 \sinh(\Phi)$ instead of $\kappa^2\Phi$. Since our techniques are only valid for the linear equations we will be focusing on the linear model. Again, (136) may include charges among the water, but if the charges are balanced, as in NaCl salt water, the 0 remains reasonable.

For our purposes, we shall assume $\epsilon_0 = e_c = 1$ with arbitrary ρ when solving the model, and then scaling the solution as appropriate for the actual values of ϵ_0, e_c.

We shall solve the equations of this model using the same techniques and many of the details will be comparable. The substantially different hurdle comes from when we need to solve a system of 2 equations and we must find a bound on the determinant of that system. As they are substantially more involved, the details have been put in the appendix. The verification is comparable in the objective and the use integral identities involving i_n and k_n, but the details are distinct as far as D_s is concerned.

We concern ourselves with Theorem 6.1.

Theorem 6.1. *Let $\epsilon_p, \epsilon_\infty, \epsilon_s, a, \kappa > 0$, be constants with $\epsilon_s > \epsilon_\infty$. Also let $\epsilon_0 = 1$.*

$D_p = \{\mathbf{r}|r < a\}$, $\Gamma = \{\mathbf{r}|r = a\}$, $D_s = \{\mathbf{r}|r > a\}$

Let ρ_p be a distribution defined on D_p, with support inside some closed set X within D_p. Let $\int_{D_p} |\rho_p(\mathbf{r})| d\mathbf{r} < \infty$.

Let $\omega_1, \omega_2 > 0$ be defined from

$$\omega_1^2 = \frac{1}{2\epsilon_\infty \lambda^2}(\epsilon_s + \kappa^2\lambda^2 + \sqrt{(\epsilon_s + \kappa^2\lambda^2)^2 - 4\epsilon_\infty \lambda^2 \kappa^2})$$

$$\omega_2^2 = \frac{1}{2\epsilon_\infty \lambda^2}(\epsilon_s + \kappa^2\lambda^2 - \sqrt{(\epsilon_s + \kappa^2\lambda^2)^2 - 4\epsilon_\infty \lambda^2 \kappa^2}).$$

Let P_n^m denote the Associated Legendre polynomial, $i_n(r), k_n(r)$ denote the modified spherical Bessel functions as defined in (185), (182), (183)

Define Φ_p and Φ_s by

$$\begin{aligned}\Phi_p(r) = \textstyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} (\frac{C_{p,n,m}}{\epsilon_p} r^n + \frac{r^n G_{n,m}(a)}{\epsilon_p a^n}) P_n^m(\cos\phi) e^{im\theta} \\ + \textstyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} (\frac{G_{n,m}(r)}{\epsilon_p} - \frac{r^n G_{n,m}(a)}{\epsilon_p a^n}) P_n^m(\cos\phi) e^{im\theta}\end{aligned} \tag{141}$$

$$\Phi_s(\mathbf{r}) = \begin{pmatrix} (1-\lambda^2\omega_1^2)\sum_{n=0}^{\infty}\sum_{m=-n}^{n} B_{s,n,m}k_n(\omega_1 r)P_n^m(\cos\phi)e^{im\theta} \\ +(1-\lambda^2\omega_2^2)\sum_{n=0}^{\infty}\sum_{m=-n}^{n} D_{s,n,m}k_n(\omega_2 r)P_n^m(\cos\phi)e^{im\theta} \end{pmatrix}. \tag{142}$$

where the spherical coordinates of $\mathbf{r}$ *are given by* (r, ϕ, θ).

u_p *and* u_s *defined on* D_p *and* D_s *defined by*

$$\begin{aligned} u_p(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-A_{p,n,m} - \tfrac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})\epsilon_p\lambda})i_n(\tfrac{r}{\lambda})P_n^m(\cos\phi)e^{im\theta} \\ +\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\tfrac{C_{p,n,m}}{\epsilon_p}r^n + \tfrac{r^n G_{n,m}(a)}{\epsilon_p a^n})P_n^m(\cos\phi)e^{im\theta} \\ +\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\tfrac{1}{\epsilon_p\lambda}(\lambda G_{n,m}(r) - \lambda\tfrac{r^n G_{n,m}(a)}{a^n} - H_{n,m}(r) + \tfrac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})}i_n(\tfrac{r}{\lambda}))P_n^m(\cos\phi)e^{im\theta} \end{aligned} . \tag{143}$$

$$u_s(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(B_{s,n,m}k_n(\omega_1 r) + D_{s,n,m}k_n(\omega_2 r))P_n^m(\cos\phi)e^{im\theta}. \tag{144}$$

satisfy

$$u(\mathbf{r}) = (Q_\lambda * \Phi)(\mathbf{r}) = \int_{\mathbb{R}^3} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi(\mathbf{r}')d\mathbf{r}'. \tag{145}$$

.

The coefficients within Φ_s, Φ_p, u_s, *and* u_p *are given by* (174), (175), (176), (177)*. These in turn are defined by* (68), (172), (70), (173), (156), (157).

Then outside of X, Φ_s *and* Φ_p *weakly solve* (135), (136), (137) *subject to* (138), (139)*. The convergence of the series is geometric with the ratio dependent on* ρ.

However, the verification in Lemma 6.2 and the convergence in Lemma 6.5 must still be shown to prove the theorem.

Define u by

$$u(\mathbf{r}) = (Q_\lambda * \Phi)(\mathbf{r}) = \int_{\mathbb{R}^3} Q_\lambda(\mathbf{r} - \mathbf{r}')\Phi(\mathbf{r}')d\mathbf{r}'. \tag{146}$$

As u will solve

$$-\lambda^2 \Delta u(\mathbf{r}) + u(\mathbf{r}) - \Phi(\mathbf{r}) = 0, \tag{147}$$

we have the familiar equations.

$$-\lambda^2 \Delta u_p(\mathbf{r}) + u_p(\mathbf{r}) - \Phi_p(\mathbf{r}) = 0 \quad \mathbf{r} \in D_p \tag{148}$$

$$-\lambda^2 \Delta u_s(\mathbf{r}) + u_s(\mathbf{r}) - \Phi_s(\mathbf{r}) = 0 \quad \mathbf{r} \in D_s. \tag{149}$$

This allows us to transform the equations with convolutions into

$$\epsilon_\infty \frac{\partial \Phi_s(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} + (\epsilon_s - \epsilon_\infty)\frac{\partial u_s(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} - \epsilon_p \frac{\partial \Phi_p(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} = 0 \quad \mathbf{r} \in \Gamma \tag{150}$$

and

$$-\epsilon_\infty \Delta \Phi_s(\mathbf{r}) - (\epsilon_s - \epsilon_\infty)\Delta u_s + \kappa^2 \Phi_s(\mathbf{r}) = 0 \quad \mathbf{r} \in D_s. \tag{151}$$

And as u is continuous, we repeat the same boundary conditions as before

$$u_s(\mathbf{r}) = u_p(\mathbf{r}) \quad \mathbf{r} \in \Gamma \tag{152}$$

$$\frac{\partial u_s(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} = \frac{\partial u_p(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} \quad \mathbf{r} \in \Gamma. \tag{153}$$

So, we shall need to solve $(135), (148), (149), (151)$ subject to the boundary conditions given by $(152), (153), (138), (150)$.

To solve u_p and ϕ_p, for both the general homogeneous solutions and the particular one, we observe that (27) and (29) agree with (135) and (148), and hence will have the same form for solutions.

$$\begin{aligned}\Phi_p(\mathbf{r}) = \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{C_{p,n,m}}{\epsilon_p} r^n P_n^m(\cos\phi)e^{im\theta} \\ + \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{G_{n,m}(r)}{\epsilon_p} P_n^m(\cos\phi)e^{im\theta}\end{aligned} \tag{154}$$

$$\begin{aligned}u_p(\mathbf{r}) = \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} -A_{p,n,m} i_n(\frac{r}{\lambda}) P_n^m(\cos\phi)e^{im\theta} \\ + \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{C_{p,n,m}}{\epsilon_p} r^n P_n^m(\cos\phi)e^{im\theta} \\ + \textstyle\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{1}{\epsilon_p\lambda}(\lambda G_{n,m} - H_{n,m})(r) P_n^m(\cos\phi)e^{im\theta}\end{aligned} \tag{155}$$

These have the identical definitions for G and H from $(48), (49), (78), (79)$

$$G_{n,m}(r) = \epsilon_p \Phi_{n,m}(r) = \frac{1}{2n+1}\left(-r^n \int_0^r t^{1-n}\rho_{p,n,m}(t)dt + r^{-1-n}\int_0^r t^{n+2}\rho_{p,n,m}(t)dt\right) \tag{156}$$

$$\frac{H_{n,m}(r)}{\epsilon_p\lambda} = (\Phi - u)_{n,m}(r) = \frac{1}{\epsilon_p\lambda}\left(-i_n(\frac{r}{\lambda})\int_0^r k_n(\frac{t}{\lambda})\rho_{p,n,m}(t)t^2dt + k_n(\frac{r}{\lambda})\int_0^r i_n(\frac{t}{\lambda})\rho_{p,n,m}(t)t^2dt\right). \tag{157}$$

$$nG_{n,m}(a) - aG'_{n,m}(a) = a^{-1-n}\int_0^a r^{n+2}\rho_{p,n,m}(r)dr \tag{158}$$

$$\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda}) = \frac{\lambda}{a^2}\int_0^a i_n(\frac{r}{\lambda})\rho_{p,n,m}(r)r^2dr \tag{159}$$

Again, for the purposes of convergence, it may be more convenient to think of the solutions as

$$\Phi_p(r) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\frac{C_{p,n,m}}{\epsilon_p}r^n + \frac{r^n G_{n,m}(a)}{\epsilon_p a^n})P_n^m(\cos\phi)e^{im\theta}$$
$$+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\frac{G_{n,m}(r)}{\epsilon_p} - \frac{r^n G_{n,m}(a)}{\epsilon_p a^n})P_n^m(\cos\phi)e^{im\theta} \quad (160)$$

$$u_p(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-A_{p,n,m} - \frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})\epsilon_p\lambda})i_n(\frac{r}{\lambda})P_n^m(\cos\phi)e^{im\theta}$$
$$+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(\frac{C_{p,n,m}}{\epsilon_p}r^n + \frac{r^n G_{n,m}(a)}{\epsilon_p a^n})P_n^m(\cos\phi)e^{im\theta} \quad . \quad (161)$$
$$+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{\epsilon_p\lambda}(\lambda G_{n,m}(r) - \lambda\frac{r^n G_{n,m}(a)}{a^n} - H_{n,m}(r) + \frac{H_{n,m}(a)}{i_n(\frac{a}{\lambda})}i_n(\frac{r}{\lambda}))P_n^m(\cos\phi)e^{im\theta}$$

To solve u_s and ϕ_s, we apply (147) to (151)

$$\epsilon_\infty\lambda^2\Delta^2 u_s(\mathbf{r}) - \epsilon_\infty\Delta u_s(\mathbf{r}) - (\epsilon_s - \epsilon_\infty)\Delta u_s + \kappa^2 u_s(\mathbf{r}) - \kappa^2\lambda^2\Delta u_s = 0$$

So, the fourth order equation we need to solve is

$$\epsilon_\infty\lambda^2\Delta^2 u_s(\mathbf{r}) + (-\epsilon_s - \kappa^2\lambda^2)\Delta u_s(\mathbf{r}) + \kappa^2 u_s(\mathbf{r}) = 0.$$

This can be rewritten as

$$\epsilon_\infty\lambda^2(\Delta - \omega_1^2)(\Delta - \omega_2^2)u_s(\mathbf{r}) = 0,$$

where

$$\omega_1^2 = \frac{1}{2\epsilon_\infty\lambda^2}(\epsilon_s + \kappa^2\lambda^2 + \sqrt{(\epsilon_s + \kappa^2\lambda^2)^2 - 4\epsilon_\infty\lambda^2\kappa^2})$$

$$\omega_2^2 = \frac{1}{2\epsilon_\infty\lambda^2}(\epsilon_s + \kappa^2\lambda^2 - \sqrt{(\epsilon_s + \kappa^2\lambda^2)^2 - 4\epsilon_\infty\lambda^2\kappa^2}).$$

We observe that since $\epsilon_s > \epsilon_\infty$,

$$(\epsilon_s + \kappa^2\lambda^2)^2 - 4\epsilon_\infty\lambda^2\kappa^2 = (\kappa^2\lambda^2 - 2\epsilon_\infty + \epsilon_s)^2 + 4\epsilon_\infty(\epsilon_s - \epsilon_\infty) > 0.$$

Thus, ω_1^2, ω_2^2 are real. $\epsilon_\infty > 0$ implies that ω_1^2, ω_2^2 are positive, as suggested by their notation. We use this because, we have already determined the homogeneous solutions to equations of the form $(\Delta - \kappa^2)u = 0$ while working through the previous model. These solutions are $i_n(\kappa r)$ and $k_n(\kappa r)$ However, as we desire $u_s(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$, only the $k_n(\kappa r)$ solutions are valid.

Thus, when we solve

$$\epsilon_\infty\lambda^2(\Delta - \omega_1^2)(\Delta - \omega_2^2)u_s(\mathbf{r}) = \epsilon_\infty\lambda^2(\Delta - \omega_2^2)(\Delta - \omega_1^2)u_s(\mathbf{r}) = 0.$$

We have the homogeneous solution

$$u_s(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(B_{s,n,m}k_n(\omega_1 r) + D_{s,n,m}k_n(\omega_2 r))P_n^m(\cos\phi)e^{im\theta}. \tag{162}$$

Substituting back gives us

$$\Phi_s(\mathbf{r}) = \begin{pmatrix} (1 - \lambda^2\omega_1^2)\sum_{n=0}^{\infty}\sum_{m=-n}^{n} B_{s,n,m}k_n(\omega_1 r)P_n^m(\cos\phi)e^{im\theta} \\ +(1 - \lambda^2\omega_2^2)\sum_{n=0}^{\infty}\sum_{m=-n}^{n} D_{s,n,m}k_n(\omega_2 r)P_n^m(\cos\phi)e^{im\theta} \end{pmatrix}. \tag{163}$$

Observe that $(163), (162), (160), (161)$ match $(142), (144), (141), (143)$ of Theorem 2.1.

6.1 Solving the New Model

As with the Lorentzian model, we integrate the interface conditions over Γ to equate the coefficients to $P_n^m(\cos\phi)e^{im\theta}$.

$$-A_{p,n,m}i_n(\frac{a}{\lambda}) + \frac{C_{p,n,m}}{\epsilon_p}a^n + \frac{1}{\epsilon_p\lambda}(\lambda G_{n,m}(a) - H_{n,m}(a)) = B_{s,n,m}k_n(\omega_1 a) + D_{s,n,m}k_n(\omega_2 a) \quad (164)$$

$$\frac{-A_{p,n,m}i_n'(\frac{a}{\lambda})}{\lambda} + \frac{C_{p,n,m}}{\epsilon_p}na^{n-1} + \frac{1}{\epsilon_p\lambda}(\lambda G_{n,m}'(a) - H_{n,m}'(a)) = B_{s,n,m}k_n'(\omega_1 a)\omega_1 + D_{s,n,m}k_n'(\omega_2 a)\omega_2 \quad (165)$$

$$B_{s,n,m}(1-\lambda^2\omega_1^2)k_n(\omega_1 a) + D_{s,n,m}(1-\lambda^2\omega_2^2)k_n(\omega_2 a) = \frac{C_{p,n,m}}{\epsilon_p}a^n + \frac{G_{n,m}(a)}{\epsilon_p} \quad (166)$$

$$\begin{array}{c} \epsilon_\infty\left(B_{s,n,m}(1-\lambda^2\omega_1^2)k_n'(\omega_1 a)\omega_1 + D_{s,n,m}(1-\lambda^2\omega_2^2)k_n'(\omega_2 a)\omega_2\right) \\ +(\epsilon_s - \epsilon_\infty)\left(B_{s,n,m}k_n'(\omega_1 a)\omega_1 + D_{s,n,m}k_n'(\omega_2 a)\right) - C_{p,n,m}na^{n-1} - G_{n,m}'(a) = 0 \end{array}. \quad (167)$$

Then we solve the system of equations. (164) + (166) gives

$$-A_{p,n,m}i_n(\frac{a}{\lambda}) + \frac{-H_{n,m}(a)}{\epsilon_p\lambda} = B_{s,n,m}k_n(\omega_1 a)\lambda^2\omega_1^2 + D_{s,n,m}k_n(\omega_2 a)\lambda^2\omega_2^2. \quad (168)$$

ϵ_p(165) + (167) gives

$$\begin{array}{c} -\frac{A_{p,n,m}i_n'(\frac{a}{\lambda})\epsilon_p}{\lambda} + \frac{-H_{n,m}'(a)}{\lambda} \\ +B_{s,n,m}k_n'(\omega_1 a)\omega_1(-\epsilon_\infty\lambda^2\omega_1^2 + \epsilon_s - \epsilon_p) \\ +D_{s,n,m}k_n'(\omega_2 a)\omega_2(-\epsilon_\infty\lambda^2\omega_2^2 + \epsilon_s - \epsilon_p) = 0 \end{array}. \quad (169)$$

(166)n + (167)a/ϵ_p gives

$$
\begin{gathered}
B_{s,n,m}(1-\lambda^2\omega_1^2)k_n(\omega_1 a)n - B_{s,n,m}\tfrac{(\epsilon_s-\epsilon_\infty\lambda^2\omega_1^2)}{\epsilon_p}k_n'(\omega_1 a)\omega_1 a \\
+D_{s,n,m}(1-\lambda^2\omega_2^2)k_n(\omega_2 a)n - D_{s,n,m}\tfrac{(\epsilon_s-\epsilon_\infty\lambda^2\omega_2^2)}{\epsilon_p}k_n'(\omega_2 a)\omega_2 a \ \cdot \\
= n\tfrac{G_{n,m}(a)}{\epsilon_p} - \tfrac{G'_{n,m}(a)}{\epsilon_p}a
\end{gathered}
$$

$(168)i_n'(\frac{a}{\lambda}) - (169)\lambda i_n(\frac{a}{\lambda})/\epsilon_p$ gives

$$
\begin{gathered}
\tfrac{-H_{n,m}(a)}{\epsilon_p\lambda}i_n'(\tfrac{a}{\lambda}) + \tfrac{H'_{n,m}(a)}{\epsilon_p}i_n(\tfrac{a}{\lambda}) \\
= B_{s,n,m}k_n(\omega_1 a)\lambda^2\omega_1^2 i_n'(\tfrac{a}{\lambda}) + B_{s,n,m}k_n'(\omega_1 a)\omega_1\tfrac{(-\epsilon_\infty\lambda^2\omega_1^2+\epsilon_s-\epsilon_p)}{\epsilon_p}\lambda i_n(\tfrac{a}{\lambda}) \ \cdot \\
+D_{s,n,m}k_n(\omega_2 a)\lambda^2\omega_2^2 i_n'(\tfrac{a}{\lambda}) + D_{s,n,m}k_n'(\omega_2 a)\omega_2\tfrac{(-\epsilon_\infty\lambda^2\omega_2^2+\epsilon_s-\epsilon_p)}{\epsilon_p}\lambda i_n(\tfrac{a}{\lambda})
\end{gathered}
$$

As before we use these last two equations produce

$$
\beta^2_{n,m}D_{s,n,m} + \beta^1_{n,m}B_{s,n,m} = \frac{1}{\epsilon_p}(nG_{n,m}(a) - aG'_{n,m}(a)) \tag{170}
$$

$$
\delta^2_{n,m}D_{s,n,m} + \delta^1_{n,m}B_{s,n,m} = \frac{1}{\epsilon_p}(\frac{H_{n,m}(a)i_n'(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})), \tag{171}
$$

where for $i = 1, 2$,

$$
\beta^i_{n,m} = (1-\lambda^2\omega_i^2)k_n(\omega_i a)n - \frac{(\epsilon_s-\epsilon_\infty\lambda^2\omega_i^2)}{\epsilon_p}k_n'(\omega_i a)\omega_i a \tag{172}
$$

$$
\delta^i_{n,m} = -k_n(\omega_i a)\lambda^2\omega_i^2 i_n'(\frac{a}{\lambda}) - k_n'(\omega_i a)\omega_i\frac{(-\epsilon_\infty\lambda^2\omega_i^2+\epsilon_s-\epsilon_p)}{\epsilon_p}\lambda i_n(\frac{a}{\lambda}). \tag{173}
$$

We use (196) to produce a more convenient form for them.

$$
\beta^i_{n,m} = \frac{(\epsilon_p - \epsilon_p\lambda^2\omega_i^2 - \epsilon_s + \epsilon_\infty\lambda^2\omega_i^2)}{\epsilon_p}k_n(\omega_i a)n + \frac{(\epsilon_s-\epsilon_\infty\lambda^2\omega_i^2)}{\epsilon_p}k_{n+1}(\omega_i a)\omega_i a
$$

$$\delta^i_{n,m} = \begin{pmatrix} \frac{(-\epsilon_p\lambda^2\omega_i^2+\lambda^2\omega_i^2\epsilon_\infty-\epsilon_s+\epsilon_p)}{\epsilon_p a} n\lambda k_n(\omega_i a) i_n(\frac{a}{\lambda}) \\ -\lambda^2\omega_i^2 k_n(\omega_i a) i_{n+1}(\frac{a}{\lambda}) \\ +\frac{(-\lambda^2\omega_i^2\epsilon_\infty+\epsilon_s-\epsilon_p)}{\epsilon_p}\lambda k_{n+1}(\omega_i a)\omega_i i_n(\frac{a}{\lambda}) \end{pmatrix}.$$

While we ought to prove that $\beta^2_{n,m}\delta^1_{n,m} - \beta^1_{n,m}\delta^2_{n,m} < 0$, as it is more complicated than in the previous model, we shall give the proof in Appendix C. The relationship between ω_1 and ω_2 and the identities between k_n, k_{n-1}, k_{n-2} play large roles.

Once we have proven that are linear system is thus solvable, we have solutions

$$D_{s,n,m} = \frac{\epsilon_p^{-1}}{\beta^2_{n,m}\delta^1_{n,m} - \beta^1_{n,m}\delta^2_{n,m}} \begin{pmatrix} \delta^1_{n,m}(nG_{n,m}(a) - aG'_{n,m}(a)) \\ -\beta^1_{n,m}(\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})) \end{pmatrix} \tag{174}$$

$$B_{s,n,m} = \frac{\epsilon_p^{-1}}{\beta^2_{n,m}\delta^1_{n,m} - \beta^1_{n,m}\delta^2_{n,m}} \begin{pmatrix} -\delta^2_{n,m}(nG_{n,m}(a) - aG'_{n,m}(a)) \\ +\beta^2_{n,m}(\frac{H_{n,m}(a)i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a)i_n(\frac{a}{\lambda})) \end{pmatrix}. \tag{175}$$

Substituting back gives us

$$C_{p,n,m} = \frac{\epsilon_p}{a^n}\left(B_{s,n,m}(1-\lambda^2\omega_1^2)k_n(\omega_1 a) + D_{s,n,m}(1-\lambda^2\omega_2^2)k_n(\omega_2 a) - \frac{G_{n,m}(a)}{\epsilon_p}\right) \tag{176}$$

$$A_{p,n,m} = \frac{1}{i_n(\frac{a}{\lambda})}\left(-\lambda^2\omega_1^2 k_n(\omega_1 a)B_{s,n,m} - \lambda^2\omega_2^2 k_n(\omega_2 a)D_{s,n,m} - \frac{H_{n,m}(a)}{\epsilon_p\lambda}\right). \tag{177}$$

6.2 Verification

As in the previous Lorentzian Model, we must verify that the u we obtained in (155) and (144) matches the form it was given in (146) using (154) and (142) That is, we need to show

Lemma 6.2. *Let the conditions of Theorem 6.1 hold.*

For emphasis, define u_p on D_p and u_s on D_s as in (155) *and* (144). *Define Φ_p and Φ_s as in* (154) *and* (142).

Then

$$u_p(\mathbf{r}) = \int_{D_p} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi_p(\mathbf{r}')d\mathbf{r}' + \int_{D_s} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi_s(\mathbf{r}')d\mathbf{r}' \quad \mathbf{r} \in D_p$$

$$u_s(\mathbf{r}) = \int_{D_p} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi_p(\mathbf{r}')d\mathbf{r}' + \int_{D_s} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi_s(\mathbf{r}')d\mathbf{r}' \quad \mathbf{r} \in D_s.$$

We remind ourselves of the preliminary work done for the integrals as the process here is largely the same. In fact, since the formulas for the protein region have not changed, they will produce identical results. Additionally, we omit the subscripts to $A, B, C, D, G, H, \beta, \delta$ and ρ. However, the β and δ superscripts are necessary and the symmetry helps simplify the process.

First let's consider the case of $r > a$.

$$\int_{D_p} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi_p(\mathbf{r}')d\mathbf{r}'$$

$$= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\int_0^a (\frac{C_{p,n,m}}{\epsilon_p}s^n + \frac{G_{n,m}(s)}{\epsilon_p})\int_{r'=s} Q_\lambda(\mathbf{r}-\mathbf{r}')P_n^m(\cos(\phi'))e^{im\theta'}dS(\mathbf{r}')ds$$

$$= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{P_n^m(\cos(\phi))e^{im\theta}}{\lambda^3}\int_0^a s^2(\frac{C_{p,n,m}}{\epsilon_p}s^n + \frac{G_{n,m}(s)}{\epsilon_p})i_n(\frac{s}{\lambda})k_n(\frac{r}{\lambda})ds$$

By defining $I_1 + I_2 = \int_0^a s^2(\frac{C_{p,n,m}}{\epsilon_p}s^n + \frac{G_{n,m}(s)}{\epsilon_p})i_n(\frac{s}{\lambda})k_n(\frac{r}{\lambda})ds$, we have

$$I_1 = k_n(\frac{r}{\lambda})i_{n+1}(\frac{a}{\lambda})\lambda a^{n+2}\frac{C}{\epsilon_p}$$

$$I_2 = k_n(\frac{r}{\lambda})\left[-\lambda a^2(\delta^2 D + \delta^1 B) + a^2\lambda i_{n+1}(\frac{a}{\lambda})G(a) + a\lambda^2 i_n(\frac{a}{\lambda})(\beta^2 D + \beta^1 B)\right].$$

The change to I_2 appears in the definitions of β^i, δ^i, and $G(a)$ when calculated in terms of B, C, and D. Using (170), (171), and (176), we get

$$I_2 = \sum_{i=1}^{2} Q^i \begin{pmatrix} \frac{(\epsilon_p\lambda^2\omega_i^2-\lambda^2\omega_i^2\epsilon_\infty+\epsilon_s-\epsilon_p)n\lambda^2 a}{\epsilon_p} k_n(\omega_i a) i_n(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) \\ +\lambda^3\omega_i^2 k_n(\omega_i a) i_{n+1}(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) a^2 \\ -\frac{(-\lambda^2\omega_i^2\epsilon_\infty+\epsilon_p-\epsilon_s)\lambda^2}{\epsilon_p} k_{n+1}(\omega_i a)\omega_i i_n(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) a^2 \\ +\frac{(\epsilon_p-\epsilon_p\lambda^2\omega_i^2-\epsilon_s+\epsilon_\infty\lambda^2\omega_i^2)n\lambda^2 a}{\epsilon_p} i_n(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) k_n(\omega_i a) \\ +\frac{(\epsilon_s-\epsilon_\infty\lambda^2\omega_i^2)}{\epsilon_p} k_{n+1}(\omega_i a)\omega_i a^2\lambda^2 i_n(\frac{a}{\lambda}) k_n(\frac{r}{\lambda}) \\ +(1-\lambda^2\omega_i^2) a^2\lambda k_n(\omega_i a) k_n(\frac{r}{\lambda}) i_{n+1}(\frac{a}{\lambda}) \end{pmatrix} - k_n(\frac{r}{\lambda}) a^2 \lambda i_{n+1}(\frac{a}{\lambda}) \frac{Ca^n}{\epsilon_p},$$

where $Q^1 = B, Q^2 = D$.

$$\int_{D_s} Q_\lambda(\mathbf{r}-\mathbf{r}')\Phi_s(\mathbf{r}')d\mathbf{r}'$$

$$= \int_a^\infty ((1-\lambda^2\omega_1^2)B_{s,n,m}k_n(\omega_1 s)+(1-\lambda^2\omega_2^2)D_{s,n,m}k_n(\omega_2 s)) \int_{r'=s} Q_\lambda(\mathbf{r}-\mathbf{r}')P_n^m(\cos(\phi'))e^{im\theta'}dS(\mathbf{r}')ds$$

$$= \frac{P_n^m(\cos(\phi))e^{im\theta}}{\lambda^3} \begin{pmatrix} \int_a^r (1-\lambda^2\omega_1^2)B_{s,n,m}k_n(\omega_1 s) i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) s^2 ds \\ +\int_a^r (1-\lambda^2\omega_2^2)D_{s,n,m}k_n(\omega_2 s) i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) s^2 ds \\ +\int_r^\infty (1-\lambda^2\omega_1^2)B_{s,n,m}k_n(\omega_1 s) i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds \\ +\int_r^\infty (1-\lambda^2\omega_2^2)D_{s,n,m}k_n(\omega_2 s) i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds \end{pmatrix}.$$

We define

$$I_3^i = \int_a^r (1-\lambda^2\omega_i^2)Q_{s,n,m}^i k_n(\omega_i s) i_n(\frac{s}{\lambda}) k_n(\frac{r}{\lambda}) s^2 ds$$

$$I_5^i = \int_r^\infty (1-\lambda^2\omega_i^2)Q_{s,n,m}^i k_n(\omega_i s) i_n(\frac{r}{\lambda}) k_n(\frac{s}{\lambda}) s^2 ds.$$

We apply (106), (107) to give us that

$$I_3^i + I_5^i = Q^i \begin{pmatrix} \omega_i \lambda^2 k_{n+1}(\omega_i r) k_n(\frac{r}{\lambda}) i_n(\frac{r}{\lambda}) r^2 \\ +\lambda k_n(\omega_i r) k_n(\frac{r}{\lambda}) i_{n+1}(\frac{r}{\lambda}) r^2 \\ -\omega_i \lambda^2 k_{n+1}(\omega_i a) k_n(\frac{r}{\lambda}) i_n(\frac{a}{\lambda}) a^2 \\ -\lambda k_n(\omega_i a) k_n(\frac{r}{\lambda}) i_{n+1}(\frac{a}{\lambda}) a^2 \\ -\omega_i \lambda^2 k_{n+1}(\omega_i r) k_n(\frac{r}{\lambda}) i_n(\frac{r}{\lambda}) r^2 \\ +\lambda k_n(\omega_i r) k_{n+1}(\frac{r}{\lambda}) i_n(\frac{r}{\lambda}) r^2 \end{pmatrix}.$$

Applying (200) reduces this to

$$I_3^i + I_5^i = Q^i \begin{pmatrix} -\omega_i \lambda^2 k_{n+1}(\omega_i a) k_n(\frac{r}{\lambda}) i_n(\frac{a}{\lambda}) a^2 \\ -\lambda k_n(\omega_i a) k_n(\frac{r}{\lambda}) i_{n+1}(\frac{a}{\lambda}) a^2 \\ +\lambda^3 k_n(\omega_i r) \end{pmatrix}.$$

We are in the fortunate case that directly summing gives us desirable results without having to appeal to any additional properties of i_n or k_n.

$$I_1 + I_2 + I_3^1 + I_3^2 + I_5^1 + I_5^2 = B\lambda^3 k_n(\omega_1 r) + D\lambda^3 k_n(\omega_2 r).$$

Thus we get

$$u_s(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (B_{s,n,m} k_n(\omega_1 r) + D_{s,n,m} k_n(\omega_2 r)) P_n^m(\cos(\phi)) e^{im\theta}$$

which matches the form found in (144). Thus we have verified the u_s case.

Now we examine the case where $r < a$. Again, the observation that the formula have not changed for D_p, allows us to repeat the results from the previous model.

$$\int_{D_p} Q_\lambda(\mathbf{r} - \mathbf{r}') \Phi_p(\mathbf{r}') d\mathbf{r}'$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{P_n^m(\cos(\phi))e^{im\theta}}{\lambda^3}(K_1 + K_2 + K_3 + K_4),$$

where

$$K_1 = \frac{r^{n+2}}{\epsilon_p} C i_{n+1}(\frac{r}{\lambda}) \lambda k_n(\frac{r}{\lambda})$$

$$K_3 = -\frac{a^{n+2}}{\epsilon_p} C i_n(\frac{r}{\lambda}) k_{n+1}(\frac{a}{\lambda}) \lambda + \frac{r^{n+2}}{\epsilon_p} C i_n(\frac{r}{\lambda}) k_{n+1}(\frac{r}{\lambda}) \lambda.$$

While we have a slightly different formula for $H(a)$ and $G(a)$ in terms of A, B, C, D, the result is comparable.

$$(K_2 + K_4) = \begin{bmatrix} -\frac{\lambda^2 H(r)}{\epsilon_p} \\ -\lambda^2 i_n(\frac{r}{a}) A \\ -\frac{\lambda^2 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} \lambda^3 \omega_1^2 k_n(\omega_1 a) B \\ -\frac{\lambda^2 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} \lambda^3 \omega_2^2 k_n(\omega_2 a) D \\ -\frac{\lambda a^2 i_n(\frac{r}{\lambda}) k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})} (\delta^2 D + \delta^1 B) \\ +\frac{\lambda^3 G(r)}{\epsilon_p} \\ +\frac{a^{n+2}}{\epsilon_p} \lambda k_{n+1}(\frac{a}{\lambda}) i_n(\frac{r}{\lambda}) C \\ -a^2 \lambda k_{n+1}(\frac{a}{\lambda}) i_n(\frac{r}{\lambda})(1 - \lambda^2 \omega_1^2) k_n(\omega_1 a) B \\ -a^2 \lambda k_{n+1}(\frac{a}{\lambda}) i_n(\frac{r}{\lambda})(1 - \lambda^2 \omega_2^2) k_n(\omega_2 a) D \\ +(\beta^2 D + \beta^1 B)(a \lambda^2 i_n(\frac{r}{\lambda}) k_n(\frac{a}{\lambda})) \end{bmatrix}.$$

Then, for D_s, we have

$$\int_{D_s} Q_\lambda(\mathbf{r} - \mathbf{r}') \Phi_s(\mathbf{r}') d\mathbf{r}'$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \int_a^{\infty} \left(\begin{array}{c} (1-\lambda^2\omega_1^2)B_{s,n,m}k_n(\omega_1 s) \\ +(1-\lambda^2\omega_2^2)D_{s,n,m}k_n(\omega_2 s) \end{array} \right) \int_{r'=s} Q_\lambda(\mathbf{r}-\mathbf{r}')P_n^m(\cos(\phi'))e^{im\theta'}dS(\mathbf{r}')ds$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{P_n^m(\cos(\phi))e^{im\theta}}{\lambda^3} \left(\begin{array}{c} \int_a^{\infty}(1-\lambda^2\omega_1^2)B_{s,n,m}k_n(\omega_1 s)i_n(\frac{r}{\lambda})k_n(\frac{s}{\lambda})s^2ds \\ +\int_a^{\infty}(1-\lambda^2\omega_2^2)D_{s,n,m}k_n(\omega_2 s)i_n(\frac{r}{\lambda})k_n(\frac{s}{\lambda})s^2ds \end{array} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{P_n^m(\cos(\phi))e^{im\theta}}{\lambda^3}(K_5^1 + K_5^2).$$

Applying (106) gives us

$$K_5^i = Q^i(-\omega_i\lambda^2 k_{n+1}(\omega_i a)k_n(\frac{a}{\lambda})i_n(\frac{r}{\lambda})a^2 + \lambda k_n(\omega_i a)k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})a^2).$$

As the terms of $K_1, K_2, K_3, K_4, K_5^1, K_5^2$ have $A, B, C, D, G(r)$ or $H(r)$ in them, we may reorganize the terms so that

$$K_1 + K_2 + K_3 + K_4 + K_5^1 + K_5^2 = L_1 A + L_2^1 B + L_3 C + L_2^2 D + \frac{\lambda^3}{\epsilon_p}G(r) - \frac{\lambda^2}{\epsilon_p}H(r).$$

As L_1 and L_3 are defined identically to that of the previous model, we repeat the conclusions.

$$L_1 = -\lambda^3 i_n(\frac{r}{\lambda}).$$

$$L_3 = \frac{r^n\lambda^3}{\epsilon_p}.$$

Concerning L_2^i, we apply our definitions for β^i and δ^i from (170) and (171) to get

$$L_2^i = \begin{pmatrix} -\frac{\lambda^2 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}\lambda^3\omega_i^2 k_n(\omega_i a) \\ -\frac{\lambda a^2 i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})}\frac{(-\epsilon_p\lambda^2\omega_i^2+\lambda^2\omega_i^2\epsilon_\infty-\epsilon_s+\epsilon_p)}{\epsilon_p a} n\lambda k_n(\omega_i a) i_n(\frac{a}{\lambda}) \\ +\frac{\lambda a^2 i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})}\lambda^2\omega_i^2 k_n(\omega_i a) i_{n+1}(\frac{a}{\lambda}) \\ -\frac{\lambda a^2 i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})}\frac{(-\lambda^2\omega_i^2\epsilon_\infty+\epsilon_s-\epsilon_p)}{\epsilon_p} k_{n+1}(\omega_i a)\omega_i i_n(\frac{a}{\lambda}) \\ -a^2\lambda k_{n+1}(\frac{a}{\lambda}) i_n(\frac{r}{\lambda})(1-\lambda^2\omega_i^2)k_n(\omega_i a) \\ +(a\lambda^2 i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda}))\frac{(\epsilon_p-\epsilon_p\lambda^2\omega_i^2-\epsilon_s+\epsilon_\infty\lambda^2\omega_i^2)}{\epsilon_p}k_n(\omega_i a) n \\ +(a\lambda^2 i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda}))\frac{(\epsilon_s-\epsilon_\infty\lambda^2\omega_i^2)}{\epsilon_p}k_{n+1}(\omega_i a)\omega_i a \\ -\omega_i\lambda^2 k_{n+1}(\omega_i a)k_n(\frac{a}{\lambda})i_n(\frac{r}{\lambda})a^2 \\ +\lambda k_n(\omega_i a)k_{n+1}(\frac{a}{\lambda})i_n(\frac{r}{\lambda})a^2 \end{pmatrix}.$$

The natural simplification gives us

$$L_2^i = \begin{pmatrix} -\frac{\lambda^2 i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}\lambda^3\omega_i^2 k_n(\omega_i a) \\ +\frac{\lambda a^2 i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda})}{i_n(\frac{a}{\lambda})}\lambda^2\omega_i^2 k_n(\omega_i a) i_{n+1}(\frac{a}{\lambda}) \\ +a^2\lambda k_{n+1}(\frac{a}{\lambda}) i_n(\frac{r}{\lambda})\lambda^2\omega_i^2 k_n(\omega_i a) \end{pmatrix}$$

$$= \frac{a^2 i_n(\frac{r}{\lambda})\lambda^3\omega_i^2 k_n(\omega_i a)}{i_n(\frac{a}{\lambda})}[k_{n+1}(\frac{a}{\lambda})i_n(\frac{a}{\lambda}) + k_n(\frac{a}{\lambda})i_{n+1}(\frac{a}{\lambda}) - \frac{\lambda^2}{a^2}] = 0,$$

where equality with zero comes from applying (200).

Altogether, this gives us.

$$K_1 + K_2 + K_3 + K_4 + K_5^1 + K_5^2 = -\lambda^3 i_n(\frac{r}{\lambda})A + \frac{r^n\lambda^3}{\epsilon_p}C + \frac{\lambda^3}{\epsilon_p}G(r) - \frac{\lambda^2}{\epsilon_p}H(r)$$

Thus we get

$$u_p(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} P_n^m(\cos(\phi))e^{im\theta}(-i_n(\frac{r}{\lambda})A_{p,n,m} + \frac{r^n}{\epsilon_p}C_{p,n,m} + \frac{G_{n,m}(r)}{\epsilon_p} - \frac{H_{n,m}(r)}{\lambda\epsilon_p})$$

which matches our definition from (155). Thus we have now verified the u_p case.

Thus, assuming we have convergence, we have shown Lemma 6.2.

6.3 Convergence

Lemma 6.3. *Let the conditions of Theorem 6.1 hold.*

Let ρ_p be supported in a ball $B(\mathbf{0}, b)$ with $b < a$.

Then ϕ_s and u_s defined by (142) *and* (144) *are geometrically convergent series.*

If the support of ρ_p is further confined to be within $B(\mathbf{0}, c) \cup (B(\mathbf{0}, \mathbf{a}) - B(\mathbf{0}, \mathbf{d}))$, then ϕ_p and u_p defined by (141) *and* (143) *are geometrically convergent series when $\mathbf{r}$ satisfies $c < |\mathbf{r}| < d$.*

To prove convergence, we shall need to know the behavior of $\beta^2_{n,m}\delta^1_{n,m} - \beta^1_{n,m}\delta^2_{n,m}$

It is shown in Appendix D that

$$|\beta^2_{n,m}\delta^1_{n,m} - \beta^1_{n,m}\delta^2_{n,m}| \geq \frac{2}{3}n^2 k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})\frac{\lambda^3}{a\epsilon_p}(\omega_1^2 - \omega_2^2)(\epsilon_\infty + \epsilon_p)$$

for $n \geq N_0$ where

$$N_0 = \max(\frac{3(\omega_1 a + 1)}{2\epsilon}, \sqrt{\frac{3a^2(\epsilon_p + \epsilon_s)}{2\epsilon_p \lambda^2(\epsilon_\infty + \epsilon_p)}})$$

where

$$\epsilon = \min(\frac{1}{6}\frac{\omega_1^2 - \omega_2^2}{(3\omega_1^2 - \omega_2^2)}, \frac{1}{12}\frac{\lambda^2(\omega_1^2 - \omega_2^2)(\epsilon_p + \epsilon_\infty)}{(\epsilon_p + \epsilon_s)}, \frac{(\epsilon_p + \epsilon_s)(\omega_1^2 - \omega_2^2)}{12\lambda^2\epsilon_\infty\omega_1^2\omega_2^2}, \frac{(\epsilon_p + \epsilon_s)(\omega_1^2 - \omega_2^2)}{6\epsilon_s(\omega_1^2 + \omega_2^2)}, \frac{1}{3}).$$

Notice that this implies that $N_0 \geq \frac{9}{2}$.

Using the definition for $\beta^i_{n,m}$ from (172) and then applying (121), we have that for $\omega_i a + 1 \leq 2n$,

$$|\beta^i_{n,m}| \leq \frac{|\epsilon_p - \epsilon_p\lambda^2\omega_i^2|}{\epsilon_p} k_n(\omega_i a)n + \frac{|\epsilon_s - \epsilon_\infty\lambda^2\omega_i^2|}{\epsilon_p} |k_n(\omega_i a)n - k_{n+1}(\omega_i a)\omega_i a|$$

$$|\beta^i_{n,m}| \leq \frac{|\epsilon_p - \epsilon_p\lambda^2\omega_i^2|}{\epsilon_p} k_n(\omega_i a)n + 3\frac{|\epsilon_s - \epsilon_\infty\lambda^2\omega_i^2|}{\epsilon_p} k_n(\omega_i a)n.$$

Using the definition for $\delta^i_{n,m}$ from (173) and then applying (121) and (120), we have that for $\omega_i a + 1 \leq 2n$,

$$|\delta^i_{n,m}| \leq \left(\begin{array}{c} \frac{\epsilon_p\lambda^2\omega_i^2}{\epsilon_p a} n\lambda k_n(\omega_i a) i_n(\frac{a}{\lambda}) \\ + \frac{\epsilon_p\lambda^2\omega_i^2}{\epsilon_p a} a k_n(\omega_i a) i_{n+1}(\frac{a}{\lambda}) \\ + \frac{|\lambda^2\omega_i^2\epsilon_\infty - \epsilon_s + \epsilon_p|}{\epsilon_p a} |n\lambda k_n(\omega_i a) i_n(\frac{a}{\lambda}) - a\lambda k_{n+1}(\omega_i a) i_n(\frac{a}{\lambda})| \end{array} \right).$$

$$|\delta^i_{n,m}| \leq \left(\frac{\epsilon_p\lambda^2\omega_i^2 n\lambda}{\epsilon_p a} + \frac{\epsilon_p\lambda^2\omega_i^2 a}{2n\epsilon_p\lambda} + 3\frac{|\lambda^2\omega_i^2\epsilon_\infty - \epsilon_s + \epsilon_p|}{\epsilon_p a} n\lambda \right) k_n(\omega_i a) i_n(\frac{a}{\lambda}).$$

If $a \leq n\lambda$, then we have

$$|\delta^i_{n,m}| \leq \left(2\frac{\epsilon_p\lambda^2\omega_i^2 n\lambda}{\epsilon_p a} + 3\frac{|\lambda^2\omega_i^2\epsilon_\infty - \epsilon_s + \epsilon_p|}{\epsilon_p a} n\lambda \right) k_n(\omega_i a) i_n(\frac{a}{\lambda}).$$

To reduce notation, we will denote constants X^i and Y^i

$$\left| \frac{\beta^i_{n,m}}{\beta^2_{n,m}\delta^1_{n,m} - \beta^1_{n,m}\delta^2_{n,m}} \right| \leq X^i \frac{k_n(\omega_i a)a}{nk_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})}$$

$$\left| \frac{\delta^i_{n,m}}{\beta^2_{n,m}\delta^1_{n,m} - \beta^1_{n,m}\delta^2_{n,m}} \right| \leq Y^i \frac{k_n(\omega_i a)}{nk_n(\omega_1 a)k_n(\omega_2 a)},$$

where

$$X^i = \frac{(|\epsilon_p - \epsilon_p \lambda^2 \omega_i^2| + 3|\epsilon_s - \epsilon_\infty \lambda^2 \omega_i^2|)}{\frac{2}{3}(\omega_1^2 - \omega_2^2)(\epsilon_\infty + \epsilon_p)\lambda^3}$$

$$Y^i = \frac{(2\epsilon_p \lambda^2 \omega_i^2 + 3|\lambda^2 \omega_i^2 \epsilon_\infty - \epsilon_s + \epsilon_p|)}{\frac{2}{3}(\omega_1^2 - \omega_2^2)(\epsilon_\infty + \epsilon_p)\lambda^2}.$$

As in the previous model, we shall assume that the support of ρ is away from the boundary. That is, we will assume that $\rho(\mathbf{r})$ is nonzero only for $|\mathbf{r}| < b < a$. Additionally, we appeal to the known bound for ρ, (131). It is also good to remind ourselves that the bounds given are only for $n \geq N_1$ where

$$N_1 = \max(N_0, \frac{\omega_1 a + 1}{2}, \frac{\lambda}{a})$$

Next, our definition of $G_{n,m}$ from (158) tell us that for $n \geq N_1$,

$$|\delta_{n,m}^i \frac{nG_{n,m}(a) - aG'_{n,m}(a)}{\beta_{n,m}^2 \delta_{n,m}^1 - \beta_{n,m}^1 \delta_{n,m}^2}||P_n^m(\cos\phi)| \leq \frac{Y^i k_n(\omega_i a) a^{-1-n}}{nk_n(\omega_1 a)k_n(\omega_2 a)} \int_0^a |r^{n+2}\rho_{p,n,m}(r)|dr|P_n^m(\cos\phi)|$$

$$\leq \frac{Y^i k_n(\omega_i a)}{4\pi n k_n(\omega_1 a) k_n(\omega_2 a)} \frac{1}{a} \left(\frac{b}{a}\right)^n (2n+1) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

Likewise, our definition of $H_{n,m}$ found in (159) tell us that for $n \geq N_1$,

$$|\beta_{n,m}^i \frac{\frac{H_{n,m}(a) i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a) i_n(\frac{a}{\lambda})}{\beta_{n,m}^2 \delta_{n,m}^1 - \beta_{n,m}^1 \delta_{n,m}^2}||P_n^m(\cos\phi)|$$

$$\leq \frac{X^i a k_n(\omega_i a)}{nk_n(\omega_1 a)k_n(\omega_2 a) i_n(\frac{a}{\lambda})} \frac{\lambda}{a^2} \int |i_n(\frac{r}{\lambda})\rho_{p,n,m}(r) r^2|dr|P_n^m(\cos\phi)|$$

$$\leq \frac{X^i k_n(\omega_i a)}{4\pi n k_n(\omega_1 a) k_n(\omega_2 a) i_n(\frac{a}{\lambda})} \frac{\lambda}{a}(2n+1) i_n(\frac{b}{\lambda}) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

Applying (123) gives us that there is constant $N_2 = \max(N_1, (\frac{a}{\lambda})^2 - 2)$ such that for $n \geq N_2$,

$$|\beta^i_{n,m} \frac{\frac{H_{n,m}(a) i'_n(\frac{a}{\lambda})}{\lambda} - H'_{n,m}(a) i_n(\frac{a}{\lambda})}{\beta^2_{n,m}\delta^1_{n,m} - \beta^1_{n,m}\delta^2_{n,m}}||P^m_n(\cos\phi)| \leq 2 \frac{X^i k_n(\omega_i a)}{4\pi n k_n(\omega_1 a) k_n(\omega_2 a)} \frac{\lambda}{a} \left(\frac{b}{a}\right)^n (2n+1) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

Therefore, by (174), for $n \geq N_2$,

$$|D_{s,n,m}||P^m_n(\cos\phi)| \leq \frac{1}{4\pi\epsilon_p} \frac{Y^1 + 2\lambda X^1}{n a k_n(\omega_2 a)} \left(\frac{b}{a}\right)^n (2n+1) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

Likewise, by (175), for $n \geq N_2$,

$$|B_{s,n,m}||P^m_n(\cos\phi)| \leq \frac{1}{4\pi\epsilon_p} \frac{Y^2 + 2\lambda X^2}{n a k_n(\omega_1 a)} \left(\frac{b}{a}\right)^n (2n+1) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

Using our definitions for $C_{p,n,m}$ and $A_{p,n,m}$, found in (176) and (177) respectively, and absorbing constants into X_C and X_A we have that for $n \geq N_2$

$$|C_{p,n,m} + \frac{G_{n,m}(a)}{a^n}||P^m_n(\cos\phi)| \leq \frac{1}{4\pi a^n} \left(\frac{X_C}{na}\right) \left(\frac{b}{a}\right)^n (2n+1) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}$$

$$|A_{p,n,m} + \frac{H_{n,m}(a)}{\epsilon_p \lambda i_n(\frac{a}{\lambda})}||P^m_n(\cos\phi)| \leq \frac{1}{4\pi i_n(\frac{a}{\lambda})\epsilon_p} \left(\frac{X_A}{na}\right) \left(\frac{b}{a}\right)^n (2n+1) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r},$$

where

$$X_C = |1 - \lambda^2\omega_1^2|(Y^2 + 2\lambda X^2) + |1 - \lambda^2\omega_2^2|(Y^1 + 2\lambda X^1)$$

$$X_A = \lambda^2\omega_1^2(Y^2 + 2\lambda X^2) + \lambda^2\omega_2^2(Y^1 + 2\lambda X^1).$$

Now let's consider the various series that occur in Φ and u. Where $r \geq a$, we have $B_{s,n,m}$ and $D_{s,n,m}$.

$$\sum_{m=-n}^{n} |B_{s,n,m} k_n(\omega_1 r) P_n^m(\cos\phi) e^{im\theta}| \leq \frac{(Y^2 + 2\lambda X^2) k_n(\omega_1 r)}{4\pi\epsilon_p n a k_n(\omega_1 a)} \left(\frac{b}{a}\right)^n (2n+1)^2 \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

Applying (125) with $r \geq a$, we have $\frac{k_n(\omega_1 r)}{k_n(\omega_1 a)} \leq (\frac{a}{r})^{n+1}$. Additionally, $\frac{2n+1}{n} \leq 3$ for $n \geq 1$, so we have

$$\sum_{m=-n}^{n} |B_{s,n,m} k_n(\omega_1 r) P_n^m(\cos\phi) e^{im\theta}| \leq 3\frac{(Y^2 + 2\lambda X^2)}{4\pi a \epsilon_p} \left(\frac{b}{a}\right)^n \left(\frac{a}{r}\right)^{n+1} (2n+1) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r},$$

and since $b < a \leq r$,

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{s,n,m} k_n(\omega_1 r) P_n^m(\cos\phi) e^{im\theta} \quad \text{converges like a geometric series .}$$

Likewise for the $D_{s,n,m}$ series, we have

$$\sum_{m=-n}^{n} |D_{s,n,m} k_n(\omega_2 r) P_n^m(\cos\phi) e^{im\theta}| \leq \frac{(Y^1 + 2\lambda X^1) k_n(\omega_2 r)}{4\pi\epsilon_p n a k_n(\omega_2 a)} \left(\frac{b}{a}\right)^n (2n+1)^2 \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r},$$

which for the same reasoning becomes

$$\sum_{m=-n}^{n} |D_{s,n,m} k_n(\omega_2 r) P_n^m(\cos\phi) e^{im\theta}| \leq 3\frac{(Y^1 + 2\lambda X^1)}{4\pi a \epsilon_p} \left(\frac{b}{a}\right)^n \left(\frac{a}{r}\right)^{n+1} (2n+1) \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}$$

,

and since $b < a \leq r$,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} D_{s,n,m}k_n(\omega_2 r)P_n^m(\cos\phi)e^{im\theta} \quad \text{converges like a geometric series .}$$

Concerning the u_p and ϕ_p series, we have, $r \le a$, and $C_{p,n,m}$ and $A_{p,n,m}$.

$$\sum_{m=-n}^{n} |(C_{p,n,m} + \frac{G_{n,m}(a)}{a^n})\frac{r^n}{\epsilon_p}||P_n^m(\cos\phi)| \le \frac{r^n}{4\pi\epsilon_p a^n}\left(\frac{X_C}{na}\right)\left(\frac{b}{a}\right)^n (2n+1)^2 \int_{D_p} |\rho(\mathbf{r})|d\mathbf{r}.$$

Again, $\frac{2n+1}{n} \le 3$, so we have

$$\sum_{m=-n}^{n} |(C_{p,n,m} + \frac{G_{n,m}(a)}{a^n})\frac{r^n}{\epsilon_p}||P_n^m(\cos\phi)| \le 3\left(\frac{r}{a}\right)^n \left(\frac{X_C}{4\pi a\epsilon_p}\right)\left(\frac{b}{a}\right)^n (2n+1) \int_{D_p} |\rho(\mathbf{r})|d\mathbf{r},$$

and since $b < a$ and $r \le a$,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} (C_{p,n,m} + \frac{G_{n,m}(a)}{a^n})r^n P_n^m(\cos\phi)e^{im\theta} \quad \text{converges like a geometric series .}$$

Then for $A_{p,n,m}$, we have

$$\sum_{m=-n}^{n} |(A_{p,n,m} + \frac{H_{n,m}(a)}{\epsilon_p\lambda i_n(\frac{a}{\lambda})})i_n(\frac{r}{\lambda})||P_n^m(\cos\phi)| \le \frac{i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}\left(\frac{X_A}{4\pi na\epsilon_p}\right)\left(\frac{b}{a}\right)^n (2n+1)^2 \int_{D_p} |\rho(\mathbf{r})|d\mathbf{r}.$$

Since $a \ge r$, we may apply (123) to get $\frac{i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})} \le \left(\frac{r}{a}\right)^n$. Additionally, $\frac{2n+1}{n} \le 3$. Then we get

$$\sum_{m=-n}^{n} |(A_{p,n,m} + \frac{H_{n,m}(a)}{\epsilon_p\lambda i_n(\frac{a}{\lambda})})i_n(\frac{r}{\lambda})||P_n^m(\cos\phi)| \le \left(3\frac{X_A}{4\pi a\epsilon_p}\right)\left(\frac{r}{a}\right)^n \left(\frac{b}{a}\right)^n (2n+1) \int_{D_p} |\rho(\mathbf{r})|d\mathbf{r},$$

and since $b < a$ and $r \le a$,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(A_{p,n,m}+\frac{H_{n,m}(a)}{\epsilon_p\lambda i_n(\frac{a}{\lambda})})i_n(\frac{r}{\lambda})P_n^m(\cos\phi)e^{im\theta} \quad \text{converges like a geometric series .}$$

For the remaining series for $H_{n,m}(r)$ and $G_{n,m}(r)$, we have already proven their convergence in the previous model. For the error bounds for them, we had required that $(\frac{a}{\lambda})^2-2, \frac{(\frac{a}{\lambda})^2+1}{2} \leq n$. So we restate that under the assumption that $\mathbf{r}$ is away from the support of ρ,

$$\sum_{n=0}^{\infty}\sum_{m=n}^{-n}(G_{n,m}(r)-\frac{r^nG_{n,m}(a)}{a^n})P_n^m(\cos\phi)e^{im\theta} \quad \text{converges like a geometric series .}$$

$$\sum_{n=0}^{\infty}\sum_{m=n}^{-n}(-H_{n,m}(r)+\frac{H_{n,m}(a)i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})})P_n^m(\cos\phi)e^{im\theta} \quad \text{converges like a geometric series .}$$

Thus we have proven Lemma 6.3.

As we may desire error bounds, we will write the maximum contribution to the nth term in the series. Since the $G_{n,m}, H_{n,m}$ are identical for the modified model and the original, the argument in the convergence for the original model for removing the $(2n+1)$ factor applies here as well. This also allows us to repeat comparable lemmas for a point charge model and one with bounded ρ.

Lemma 6.4. *Let the conditions of Theorem 6.1 hold.*

Let ρ_p be supported in a ball $B(\mathbf{0}, b)$ with $b < a$. Let the support of ρ_p be further confined to be within $B(\mathbf{0}, c) \cup (B(\mathbf{0}, \mathbf{a}) - B(\mathbf{0}, \mathbf{d}))$

Let ϕ_s and u_s be defined by (142) *and* (144).

Let ϕ_p and u_p be defined by (141) *and* (143).

Then for $n \geq \tilde{N}$ with

$$\tilde{N} = \max(N_0, \frac{\omega_1 a+1}{2}, \frac{\lambda}{a}, \frac{a^2}{\lambda^2}-2, \frac{(\frac{a}{\lambda})^2+1}{2})$$

$$N_0 = \max(\frac{3(\omega_1 a + 1)}{2\epsilon}, \sqrt{\frac{3a^2(\epsilon_p + \epsilon_s)}{2\epsilon_p \lambda^2(\epsilon_\infty + \epsilon_p)}})$$

$$\epsilon = \min(\frac{1}{6}\frac{\omega_1^2 - \omega_2^2}{(3\omega_1^2 - \omega_2^2)}, \frac{1}{12}\frac{\lambda^2(\omega_1^2 - \omega_2^2)(\epsilon_p + \epsilon_\infty)}{(\epsilon_p + \epsilon_s)}, \frac{(\epsilon_p + \epsilon_s)(\omega_1^2 - \omega_2^2)}{12\lambda^2 \epsilon_\infty \omega_1^2 \omega_2^2}, \frac{(\epsilon_p + \epsilon_s)(\omega_1^2 - \omega_2^2)}{6\epsilon_s(\omega_1^2 + \omega_2^2)}, \frac{1}{3}).$$

where $|\mathbf{r}| = r$*, we have the following:*

For Φ_s*, the nth is bounded by, with* $b < a \leq r$

$$\frac{3}{4\pi r \epsilon_p}\left((Y^1 + 2\lambda X^1)|1 - \lambda^2\omega_2^2| + (Y^2 + 2\lambda X^2)|1 - \lambda^2\omega_1^2\right)\left(\frac{b}{a}\right)^n \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}$$

For u_s*, the nth term is bounded by, with* $b < a \leq r$

$$\frac{3}{4\pi r \epsilon_p}\left(Y^1 + 2\lambda X^1 + Y^2 + 2\lambda X^2\right)\left(\frac{b}{r}\right)^n \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}$$

For Φ_p*, the nth term is bounded by, with* $b < a$*,* $c < r < d$*, and* $r \leq a$

$$\frac{1}{4\pi\epsilon_p}\left(\left(\frac{3X_C + 1}{a}\right)\left(\frac{br}{a^2}\right)^n + \frac{1}{d}\left(\frac{r}{d}\right)^n + \frac{1}{r}\left(\frac{c}{r}\right)^n\right)\int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}$$

For u_p*, the nth term is bounded by, with* $b < a$*,* $c < r < d$*, and* $r \leq a$

$$\frac{3}{4\pi\epsilon_p}\left(\left(\frac{X_C + X_A + 1}{a}\right)\left(\frac{br}{a^2}\right)^n + \frac{1}{d}\left(\frac{r}{d}\right)^n + \frac{1}{r}\left(\frac{c}{r}\right)^n\right)\int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}$$

.

Among these bounds, $Y^1, Y^2, X^1, X^2, X_C, X_A$ *are some constants.*

Lemma 6.5. *Let the conditions of Theorem 6.1 hold.*

Let $\mathbf{r_0} \in D_p$*. Let* $\rho_p(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r_0})$*.*

Let ϕ_s *and* u_s *be defined by* (178) *and* (179)*.*

$$\Phi_s(\mathbf{r}) = \begin{pmatrix} (1-\lambda^2\omega_1^2)\sum_{n=0}^{\infty}\sum_{m=-n}^{n} B_{s,n,m}k_n(\omega_1 r)P_n^m(\cos\phi)e^{im\theta} \\ +(1-\lambda^2\omega_2^2)\sum_{n=0}^{\infty}\sum_{m=-n}^{n} D_{s,n,m}k_n(\omega_2 r)P_n^m(\cos\phi)e^{im\theta} \end{pmatrix}. \tag{178}$$

$$u_s(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(B_{s,n,m}k_n(\omega_1 r) + D_{s,n,m}k_n(\omega_2 r))P_n^m(\cos\phi)e^{im\theta}. \tag{179}$$

Let ϕ_p *and* u_p *be defined by* (180) *and* (181)*.*

$$\begin{array}{c} \Phi_p(r) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} r^n(\frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s a^n}B_{s,n,m}k_n(\omega a) + \frac{D_{s,n,m}}{\epsilon_s}a^{-2n-1})P_n^m(\cos\phi)e^{im\theta} \\ +\frac{1}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|} - \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a^{-1-2n}r^n r_0^n \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p(2n+1)c_{n,m}}P_n^m(\cos\phi)e^{im\theta} \end{array}. \tag{180}$$

$$\begin{array}{c} u_p(\mathbf{r}) = -\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{1}{i_n(\frac{a}{\lambda})}B_{s,n,m}k_n(\omega a)i_n(\frac{r}{\lambda})P_n^m(\cos\phi)e^{im\theta} \\ +\sum_{n=0}^{\infty}\sum_{m=-n}^{n} r^n(\frac{(\epsilon_s-\epsilon_\infty)}{\epsilon_s a^n}B_{s,n,m}k_n(\omega a) + \frac{D_{s,n,m}}{\epsilon_s}a^{-2n-1})P_n^m(\cos\phi)e^{im\theta} \\ +\frac{1}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|} - \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a^{-1-2n}r^n r_0^n \frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p(2n+1)c_{n,m}}P_n^m(\cos\phi)e^{im\theta} \\ -\frac{e^{-|\mathbf{r}-\mathbf{r_0}|/\lambda}}{4\pi\epsilon_p|\mathbf{r}-\mathbf{r_0}|} + \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{k_n(\frac{a}{\lambda})i_n(\frac{r_0}{\lambda})i_n(\frac{r}{\lambda})}{i_n(\frac{a}{\lambda})}\frac{P_n^m(\cos\phi_0)e^{-im\theta_0}}{\epsilon_p\lambda c_{n,m}}P_n^m(\cos\phi)e^{im\theta} \end{array}. \tag{181}$$

Then for $n \geq \tilde{N}$

$$\tilde{N} = \max(N_0, \frac{\omega_1 a + 1}{2}, \frac{\lambda}{a}, \frac{a^2}{\lambda^2} - 2, \frac{(\frac{a}{\lambda})^2 + 1}{2})$$

$$N_0 = \max(\frac{3(\omega_1 a + 1)}{2\epsilon}, \sqrt{\frac{3a^2(\epsilon_p + \epsilon_s)}{2\epsilon_p\lambda^2(\epsilon_\infty + \epsilon_p)}})$$

$$\epsilon = \min(\frac{1}{6}\frac{\omega_1^2 - \omega_2^2}{(3\omega_1^2 - \omega_2^2)}, \frac{1}{12}\frac{\lambda^2(\omega_1^2 - \omega_2^2)(\epsilon_p + \epsilon_\infty)}{(\epsilon_p + \epsilon_s)}, \frac{(\epsilon_p + \epsilon_s)(\omega_1^2 - \omega_2^2)}{12\lambda^2\epsilon_\infty\omega_1^2\omega_2^2}, \frac{(\epsilon_p + \epsilon_s)(\omega_1^2 - \omega_2^2)}{6\epsilon_s(\omega_1^2 + \omega_2^2)}, \frac{1}{3}).$$

where $|\mathbf{r}| = r$*, we have the following:*

For Φ_s*, the nth term is bounded by, with* $b < a \leq r$

$$\frac{3}{4\pi r\epsilon_p}\left((Y^1 + 2\lambda X^1)|1 - \lambda^2\omega_2^2| + (Y^2 + 2\lambda X^2)|1 - \lambda^2\omega_1^2\right)\left(\frac{b}{a}\right)^n \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}$$

For u_s*, the nth term is bounded by, with* $b < a \leq r$

$$\frac{3}{4\pi r\epsilon_p}\left(Y^1 + 2\lambda X^1 + Y^2 + 2\lambda X^2\right)\left(\frac{b}{r}\right)^n \int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}$$

For Φ_p*, the nth term is bounded by, with* $b < a$*,*

$$\frac{1}{4\pi\epsilon_p}\left(\left(\frac{3X_C + 1}{a}\right)\left(\frac{br}{a^2}\right)^n\right)\int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}$$

For u_p*, the nth term is bounded by, with* $b < a$*,*

$$\frac{3}{4\pi\epsilon_p}\left(\left(\frac{X_C + X_A + 1}{a}\right)\left(\frac{br}{a^2}\right)^n\right)\int_{D_p} |\rho(\mathbf{r})| d\mathbf{r}.$$

Thus the series converge everywhere.

Theorem 6.6. *Let the conditions of Theorem 6.1 hold.*

Let ρ_p *be supported in a ball* $B(\mathbf{0}, b)$ *with* $b < a$*.*

Let $\max|\rho_p(\mathbf{r})| < \infty$ *.*

Let Φ_p, Φ_s, u_p, u_s *be defined as in* $(141), (142), (143), (144)$*. Then their series converge everywhere, and* Φ_p, Φ_s *weakly solve* $(135), (136), (137)$ *subject to* $(138), (139)$ *everywhere.*

7 Final Remarks

As promised, we should discuss the necessity of D_p being a ball. Let's consider $\mathbb{R}^3$ written in coordinates (r, ϕ, θ).

Separating u, Φ, and ρ into orthogonal components with the Associated Legendre polynomials $p(\phi, \theta) = P_n^m(\cos\phi)e^{im\theta}$ did not require us to use Γ directly. However, to determine the coefficients in terms of r to these orthogonal components, we needed appropriate identities on the polynomials so that $\Delta(f(r)p(\phi, \theta))$ can be written in terms of f and its derivatives. That gave us ordinary differential equations and led to the homogeneous solutions.

When it came to finding a particular solution, again we did not require Γ to be a sphere. For the purposes of convergence, it was convenient to be have an identity between a particular solution and the series representation of it.

When it came to satisfying the interface conditions, that's where we required the normal derivative to Γ to be the derivative respect to r rather than ϕ or θ in order to respect the orthogonal components. If (r, ϕ, θ) represent the spherical coordinates, that will force Γ to be a sphere. Likewise, to guarantee that we have unique solutions, we must be able to integrate the functions over Γ with respect to the orthogonal components to find their coefficients. Since the Associate Legendre polynomials are orthogonal respect to integration over a sphere, that also forces Γ to be a sphere.

If Γ is not a sphere, one may attempt numerical methods to find the appropriate coefficients of the homogeneous solution. Alternatively, there may be coordinate systems and a different set of orthogonal functions that will work.

If we are able to find solutions, then we must still verify the claim about u. That $u(\mathbf{r}) = \int_{\mathbb{R}^3} Q_\lambda(\mathbf{r} - \mathbf{r}')\Phi(\mathbf{r}')d\mathbf{r}$ holds. We used an identity, (206), which involves separating Φ into its orthogonal components based on the Associated Legendre polynomials. So, to use this identity, we needed to have used the Associated Legendre polynomial decomposition, which we just mentioned required having Γ be a sphere. Additionally, the identity separated the space into D_p and D_s and then further decomposed those domains into spheres. In order for this to make sense, D_p must be a ball. Thus, whether we insist on using the Associated Legendre polynomials as the orthogonal decomposition or use some other set of orthogonal functions, then we'd need a different identity to use for the verification.

Finally, when it came to convergence, we used the radius of D_p as a fixed bound. If D_p is not a ball, then $|\mathbf{r}|$ for $\mathbf{r} \in \Gamma$ varies. We can repeat the argument for convergence that we used before, but at the cost of not having convergence for $\mathbf{r}$ if there are $\mathbf{r_p}$ and $\mathbf{r_s}$ such that $\mathbf{r_p} \in D_p$, $\mathbf{r_s} \in D_s$ and $|\mathbf{r}| = |\mathbf{r_p}| = |\mathbf{r_s}|$. Depending on ρ, we may be able to recover the convergence through better control on the bounds of integration.

Another issue concerning convergence when Γ is not a sphere is that we were able to use a rotation argument to shrink the number of terms summed for each term of the series. Such rotations may have domain issues if D_p is not a ball. This applies to both the calculation of the error bounds and the efficiency improvement for the point charge ρ.

If instead of using spherical coordinates and the Associate Legendre polynomials, we used some other coordinate system, then we will need asymptotic relations on the f functions to prove convergence. It is also unlikely that there will be rotational symmetry to take advantage of.

The idea of solving a partial differential equation via decomposition into orthogonal components is not particularly special, but to be able to do explicitly as in the Lorentzian Model and a Linear Poisson-Boltzmann Model on a monatomic ion is rare. Having such solutions does allow one to verify numerical methods on such nice domains and can provide insight into developing other numerical methods, such as referenced in [9].

It will be interesting in the future to work on non-spherical domains and examine the properties discussed in this section further to determine whether these techniques of finding solutions in terms of a series is feasible.

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Appendix A Known Properties

Let (r, θ, ϕ) be the spherical coordinates of $\mathbf{r}$.

P_n denote the Legendre polynomial of degree n. P_n^m denote the Associated Legendre polynomial. It is known [5] that $\{P_n^m(\cos\phi)e^{im\theta}\}_{n=0,m=-n}^{n=\infty,m=n}$ are orthogonal and complete over the sphere.

$i_n(r), k_n(r)$ denote the modified spherical Bessel functions.

One possible set of definitions for these functions:

$$i_n(x) = x^n \left(\frac{d}{xdx}\right)^n \frac{\sinh x}{x} \tag{182}$$

$$k_n(x) = (-1)^n x^n \left(\frac{d}{xdx}\right)^n \frac{e^{-x}}{x} \tag{183}$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2-1)^n] \tag{184}$$

$$P_n^m(x) = \frac{(-1)^m}{2^n n!}(1-x^2)^{m/2} \frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n \tag{185}$$

Among other properties, these satisfy

$$r^2 i_n''(r) + 2r i_n'(r) - (r^2 + n(n+1)) i_n(r) = 0 \tag{186}$$

$$r^2 k_n''(r) + 2r k_n'(r) - (r^2 + n(n+1)) k_n(r) = 0 \tag{187}$$

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \tag{188}$$

$$(1-x^2)P_n^{m''}(x) - 2xP_n^{m'}(x) + (n(n+1) - \frac{m^2}{1-x^2})P_n^m(x) = 0 \tag{189}$$

$$P_n(x) = \frac{1}{2n+1}(P'_{n+1}(x) - P'_{n-1}(x)) \tag{190}$$

Some useful properties for the modified spherical Bessel functions, found in [5] among others include the following

$$\frac{d}{dr}r^{n+1}i_n(r) = r^{n+1}i_{n-1}(r) \tag{191}$$

$$\frac{d}{dr}r^{-n}i_n(r) = r^{-n}i_{n+1}(r) \tag{192}$$

$$\frac{d}{dr}r^{n+1}k_n(r) = -r^{n+1}k_{n-1}(r) \tag{193}$$

$$\frac{d}{dr}r^{-n}k_n(r) = -r^{-n}k_{n+1}(r) \tag{194}$$

$$i'_n(r) = -\frac{n+1}{r}i_n(r) + i_{n-1}(r) = \frac{n}{r}i_n(r) + i_{n+1}(r) \tag{195}$$

$$k'_n(r) = -\frac{n+1}{r}k_n(r) - k_{n-1}(r) = \frac{n}{r}k_n(r) - k_{n+1}(r) \tag{196}$$

$$i_n(r)k'_n(r) - i'_n(r)k_n(r) = -\frac{1}{r^2} \tag{197}$$

$$i_{n-1}(r) - i_{n+1}(r) = \frac{2n+1}{r} i_n(r) \tag{198}$$

$$k_{n-1}(r) - k_{n+1}(r) = -\frac{2n+1}{r} k_n(r) \tag{199}$$

$$k_n(r) i_{n+1}(r) + k_{n+1}(r) i_n(r) = \frac{1}{r^2} \tag{200}$$

The modified spherical Bessel functions also have noteworthy limits as $r \to 0$ or $r \to \infty$.

$$i_n(r) \to \infty \quad \text{as } r \to \infty \tag{201}$$

$$i_n(r) \to 0 \quad \text{as } r \to 0 \quad \text{for } n \geq 1 \tag{202}$$

$$i_0(r) \to 1 \quad \text{as } r \to 0 \tag{203}$$

$$k_n(r) \to \infty \quad \text{as } r \to 0 \tag{204}$$

$$k_n(r) \to 0 \quad \text{as } r \to \infty \tag{205}$$

Appendix B Integral Formulas

It is known [5] that

$$\int_{r'=a} Q_\lambda(\mathbf{r}-\mathbf{r}')P_n^m(\cos\phi')e^{im\theta}dS(\mathbf{r}')$$

$$= \frac{a^2}{\lambda^3}P_n^m(\cos\phi)e^{im\theta}\begin{cases} i_n(\frac{r}{\lambda})k_n(\frac{a}{\lambda}) & \text{if } r \le a \\ i_n(\frac{a}{\lambda})k_n(\frac{r}{\lambda}) & \text{if } r \ge a \end{cases} \tag{206}$$

By letting P_n be the Legendre polynomial of degree n, if (r, ϕ, θ) are the spherical coordinates to $\mathbf{r}$ and $\mathbf{r_0} = (0, 0, z)$ then

$$\frac{1}{|\mathbf{r}-\mathbf{r_0}|} = \begin{cases} \sum_{n=0}^{\infty} z^n r^{-n-1}P_n(\cos\phi) & \text{if } r > z \\ \sum_{n=0}^{\infty} z^{-n-1}r^n P_n(\cos\phi) & \text{if } r < z \end{cases} \tag{207}$$

and [5] gives

$$\frac{e^{-|\mathbf{r}-\mathbf{r_0}|/\lambda}}{|\mathbf{r}-\mathbf{r_0}|} = \frac{1}{\lambda}\begin{cases} \sum_{n=0}^{\infty}(2n+1)i_n(\frac{z}{\lambda})k_n(\frac{r}{\lambda})P_n(\cos\phi) & \text{if } r > z \\ \sum_{n=0}^{\infty}(2n+1)i_n(\frac{r}{\lambda})k_n(\frac{z}{\lambda})P_n(\cos\phi) & \text{if } r < z \end{cases} \tag{208}$$

One method to find solutions to second order ODEs is given by

If y_1, y_2 are independent fundamental solutions to

$$y''(t) + q(t)y'(t) + r(t)y(t) = 0,$$

then

$$y''(t) + q(t)y'(t) + r(t)y(t) = f(t)$$

has a particular solution

$$y = -y_1\int\frac{y_2 f}{y_1y_2' - y_2y_1'}dt + y_2\int\frac{y_1 f}{y_1y_2' - y_2y_1'}dt. \tag{209}$$

The constant terms resulting from indefinite integration merely affect the particular solution we get.

Appendix C Proof of $\beta^2_{n,m}\delta^1_{n,m} - \delta^2_{n,m}\beta^1_{n,m} < 0$

Lemma C.1. *Let the conditions of Theorem 6.1 hold.*

$$\beta^i_{n,m} = (1-\lambda^2\omega_i^2)k_n(\omega_i a)n - \frac{(\epsilon_s - \epsilon_\infty\lambda^2\omega_i^2)}{\epsilon_p}k'_n(\omega_i a)\omega_i a$$

$$\delta^i_{n,m} = -k_n(\omega_i a)\lambda^2\omega_i^2 i'_n(\frac{a}{\lambda}) - k'_n(\omega_i a)\omega_i\frac{(-\epsilon_\infty\lambda^2\omega_i^2 + \epsilon_s - \epsilon_p)}{\epsilon_p}\lambda i_n(\frac{a}{\lambda})$$

Then for $n = 0, 1, 2..$, $m = -n, -n+1, ...n$,

$$\beta^2_{n,m}\delta^1_{n,m} - \delta^2_{n,m}\beta^1_{n,m} < 0.$$

In the modified model, we need to prove that $\beta^2_{n,m}\delta^1_{n,m} - \delta^2_{n,m}\beta^1_{n,m} < 0$ As the terms are independent of m, we only need to verify the statement over n.

First, let's reorganize the terms

$$\beta^i_{n,m} = Ak_n(\omega_i a)n + B\omega_i^2 k_n(\omega_i a)n + Dk_{n+1}(\omega_i a)\omega_i a + E\omega_i^3 a k_{n+1}(\omega_i a)$$

$$\delta^i_{n,m} = \begin{pmatrix} \frac{F\omega_i^2 n}{a}k_n(\omega_i a)i_n(\frac{a}{\lambda}) \\ +\frac{Gn}{a}k_n(\omega_i^2 a)i_n(\frac{a}{\lambda}) \\ +H\omega_i^2 k_n(\omega_i a)i_{n+1}(\frac{a}{\lambda}) \\ +J\omega_i^3 k_{n+1}(\omega_i a)i_n(\frac{a}{\lambda}) \\ +K\omega_i k_{n+1}(\omega_i a)i_n(\frac{a}{\lambda}) \end{pmatrix}$$

with

$$A = \frac{\epsilon_p - \epsilon_s}{\epsilon_p}, \quad B = \frac{(-\epsilon_p + \epsilon_\infty)}{\epsilon_p}\lambda^2, \quad D = \frac{\epsilon_s}{\epsilon_p}, \quad E = \frac{-\epsilon_\infty \lambda^2}{\epsilon_p}$$

$$F = \frac{(-\epsilon_p + \epsilon_\infty)}{\epsilon_p}\lambda^3, \quad G = \frac{(\epsilon_p - \epsilon_s)}{\epsilon_p}\lambda, \quad H = -\lambda^2, \quad J = \frac{-\epsilon_\infty \lambda^3}{\epsilon_p}, \quad K = \frac{\epsilon_s - \epsilon_p}{\epsilon_p}\lambda.$$

We shall next examine the 20 pairs terms organized by the $A, B, ..., K$ coefficients obtained by $\beta^2\delta^1 - \delta^2\beta^1$.

From symmetry, we observe that the BF, AG, BH, EJ, DK terms are 0 as the $\beta^2\delta^1$ and $\delta^2\beta^1$ contributions are equivalent. As pairs, the AF and BG terms contribute 0, the EF and BJ terms contribute 0, and the EG and AJ terms contribute 0.

The DJ and EK pair, the DG and AK pair, the DF and BK pair, as well as AH, DH, and EH all contribute a nonzero component. The sum of all of these terms gives us $\beta^2\delta^1 - \beta^1\delta^2$. It will useful to organize them in the following fashion:

$$i_n(\frac{a}{\lambda})\lambda n \frac{(\epsilon_p)}{\epsilon_p}(k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_2 - k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_1) \tag{210}$$

$$i_n(\frac{a}{\lambda})\lambda n \frac{(-\epsilon_s)}{\epsilon_p}(k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_2 - k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_1) \tag{211}$$

$$i_n(\frac{a}{\lambda})\lambda^3 n \frac{(-\epsilon_p)}{\epsilon_p}\omega_1\omega_2(k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_1 - k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_2) \tag{212}$$

$$i_n(\frac{a}{\lambda})\lambda^3 n \frac{(\epsilon_\infty)}{\epsilon_p}\omega_1\omega_2(k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_1 - k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_2) \tag{213}$$

$$i_n(\frac{a}{\lambda})\frac{\epsilon_\infty \lambda^3}{\epsilon_p}(\omega_2^2 - \omega_1^2)\omega_1\omega_2 a k_{n+1}(\omega_1 a)k_{n+1}(\omega_2 a) \tag{214}$$

$$i_{n+1}(\frac{a}{\lambda})a\lambda^4\frac{\epsilon_\infty}{\epsilon_p}\omega_1^2\omega_2^2(k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_2 - k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_1) \tag{215}$$

$$i_{n+1}(\frac{a}{\lambda})a\lambda^2\frac{(-\epsilon_s)}{\epsilon_p}\omega_1\omega_2(k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_1 - k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_2) \tag{216}$$

$$i_{n+1}(\frac{a}{\lambda})\frac{(\epsilon_p)}{\epsilon_p}\lambda^2(\omega_2^2 - \omega_1^2)nk_n(\omega_1 a)k_n(\omega_2 a) \tag{217}$$

$$i_{n+1}(\frac{a}{\lambda})\frac{(-\epsilon_s)}{\epsilon_p}\lambda^2(\omega_2^2 - \omega_1^2)nk_n(\omega_1 a)k_n(\omega_2 a) \tag{218}$$

Then $\beta_{n,m}^2\delta_{n,m}^1 - \beta_{n,m}^1\delta_{n,m}^2 = (210) + (211) + (212) + (213) + (214) + (215) + (216) + (217) + (218)$.

Before we proceed on with proof by induction, we shall apply (183) to get

$$k_0(x) = \frac{e^{-x}}{x}$$

$$k_1(x) = \frac{e^{-x}(x+1)}{x^2}$$

$$k_2(x) = \frac{e^{-x}(x^2+3x+3)}{x^3}$$

Additionally, there are a few inequalities that we shall need to prove by induction as well.

$$k_0(\omega_1 a)k_1(\omega_2 a)\omega_2 - k_0(\omega_2 a)k_1(\omega_1 a)\omega_1 = \frac{e^{-(\omega_1+\omega_2)a}}{\omega_1\omega_2 a^3}((\omega_2 a+1) - (\omega_1 a+1)) < 0$$

$$k_1(\omega_1 a)k_2(\omega_2 a)\omega_2 - k_1(\omega_2 a)k_2(\omega_1 a)\omega_1 = \frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2 a^5}(a^3(\omega_1\omega_2^2 - \omega_2\omega_1^2) + a^2(\omega_2^2 - \omega_1^2)) < 0$$

Then via (199), we have by induction that for $n \geq 0$,

$$\begin{gathered} k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_2 - k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_1 \\ = (2n-1)k_{n-1}(\omega_1 a)k_{n-1}(\omega_2 a)(\tfrac{\omega_2^2-\omega_1^2}{\omega_1\omega_2 a}) \\ +k_{n-2}(\omega_1 a)k_{n-1}(\omega_2 a)\omega_2 - k_{n-2}(\omega_2 a)k_{n-1}(\omega_1 a)\omega_1 < 0 \end{gathered} \tag{219}$$

We also shall need

$$k_0(\omega_1 a)k_1(\omega_2 a)\omega_1^2\omega_2 - k_0(\omega_2 a)k_1(\omega_1 a)\omega_2^2\omega_1 = \frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2 a^3}(a(\omega_1^2\omega_2 - \omega_1\omega_2^2) + (\omega_1^2 - \omega_2^2)) > 0$$

$$k_1(\omega_1 a)k_2(\omega_2 a)\omega_1^2\omega_2 - k_1(\omega_2 a)k_2(\omega_1 a)\omega_2^2\omega_1 = \frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^3\omega_2^3 a^5}\begin{pmatrix} a^3(\omega_1^3\omega_2^2 - \omega_1^2\omega_2^3) \\ +a^2(3\omega_1^3\omega_2 - 3\omega_1\omega_2^3) \\ +a(3\omega_1^3 + 3\omega_1^2\omega_2 - 3\omega_1\omega_2^2 - 3\omega_2^3) \\ +(3\omega_1^2 - 3\omega_2^2) \end{pmatrix} > 0$$

Then via (199), we have by induction that for $n \geq 0$,

$$\begin{gathered} k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_1^2\omega_2 - k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_2^2\omega_1 \\ = \tfrac{(2n+1)k_n(\omega_1 a)k_n(\omega_2 a)}{a}(\omega_1^2 - \omega_2^2) \qquad . \\ +k_{n-2}(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1^2\omega_2 - k_{n-2}(\omega_2 a)k_{n-1}(\omega_1 a)\omega_2^2\omega_1 > 0 \end{gathered} \tag{220}$$

In addition, there are a few reductions that follow from (199) that will be useful for us.

$$k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_2 - k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_1 = k_n(\omega_1 a)k_{n-1}(\omega_2 a)\omega_2 - k_n(\omega_2 a)k_{n-1}(\omega_1 a)\omega_1 \quad (221)$$

$$\begin{array}{c} k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_1^2\omega_2 - k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_2^2\omega_1 \\ = \frac{(2n+1)k_n(\omega_1 a)k_n(\omega_2 a)}{a}(\omega_1^2 - \omega_2^2) \\ +k_n(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1^2\omega_2 - k_n(\omega_2 a)k_{n-1}(\omega_1 a)\omega_2^2\omega_1 \end{array} \quad (222)$$

$$\begin{array}{c} k_{n+1}(\omega_1 a)k_{n+1}(\omega_2 a)\omega_1\omega_2 a \\ = (2n+1)^2\frac{k_n(\omega_1 a)k_n(\omega_2 a)}{a} \\ +(2n+1)(k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1 + k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2) \\ +k_{n-1}(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1\omega_2 a \end{array}. \quad (223)$$

Now we are ready to work on proving $\beta^2\delta^1 - \beta^1\delta^2 < 0$

First, we consider the $n = 0$ case. (210), (211), (212), (213), (217), (218) are trivially 0. (214) is negative since $k_n(r), i_n(r) > 0$ for $r > 0$, all of the ϵ are positive, and $\omega_1 > \omega_2$.

Concerning (215) we calculate explicitly to verify that it is negative.

$$i_1(\frac{a}{\lambda})a\lambda^4\frac{\epsilon_\infty}{\epsilon_p}\omega_1^2\omega_2^2(\frac{e^{-\omega_1 a}}{a\omega_1}\frac{e^{-\omega_2 a}(\omega_2+1)}{a^2\omega_2^2}\omega_2 - \frac{e^{-\omega_2 a}}{\omega_2}\frac{e^{-\omega_1 a}(\omega_1+1)}{a^2\omega_1^2}\omega_1)$$

$$i_1(\frac{a}{\lambda})\lambda^4\frac{\epsilon_\infty}{\epsilon_p}\omega_1\omega_2\frac{e^{-(\omega_1+\omega_2)a}}{a^2}((\omega_2 a+1) - (\omega_1 a+1)) < 0$$

Likewise, for (216) we calculate explicitly to verify that it is negative.

$$i_1(\frac{a}{\lambda})a\lambda^2\frac{(-\epsilon_s)}{\epsilon_p}\omega_1\omega_2(\frac{e^{-\omega_1 a}}{a\omega_1}\frac{e^{-\omega_2 a}(\omega_2+1)}{a^2\omega_2^2}\omega_1^2 - \frac{e^{-\omega_2 a}}{\omega_2}\frac{e^{-\omega_1 a}(\omega_1+1)}{a^2\omega_1^2}\omega_2^2)$$

$$= i_1(\frac{a}{\lambda})\lambda^2 \frac{(-\epsilon_s)}{\epsilon_p} \frac{e^{-(\omega_1+\omega_2)a}}{\omega_1\omega_2 a^2}((\omega_2 a+1)\omega_1^2 - (\omega_1 a+1)\omega_2^2)$$

Since $(\omega_2 a+1)\omega_1^2 - (\omega_1 a+1)\omega_2^2 = (\omega_1 - \omega_2)(\omega_1\omega_2 a + \omega_1 + \omega_2) > 0$, we thus have that (216) < 0 for $n = 0$.

Thus we have proven that $\beta^2\delta^1 - \beta^1\delta^2 < 0$ for $n = 0$.

Now we work on the $n = 1$ case.

(210) gives us

$$i_1(\frac{a}{\lambda})\lambda \frac{(\epsilon_p)}{\epsilon_p} \frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2 a^5}(a^3(\omega_1\omega_2^2 - \omega_1^2\omega_2) + a^2(\omega_2^2 - \omega_1^2)) < 0$$

For (210), we will introduce a $(\omega_1^2 + \omega_2^2)$ factor as it will be useful later. Nonetheless, it gives us

$$i_1(\frac{a}{\lambda})\lambda \frac{(-\epsilon_s)}{\epsilon_p} \frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2 a^5(\omega_1^2+\omega_2^2)}(a^3(\omega_1\omega_2^4 + \omega_1^3\omega_2^2 - \omega_1^2\omega_2^3 - \omega_1^4\omega_2) + a^2(\omega_2^4 - \omega_1^4)). \tag{224}$$

(212) gives us

$$i_1(\frac{a}{\lambda})\lambda^3 \frac{(-\epsilon_p)}{\epsilon_p} \frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2 a^5} \left(\begin{array}{c} a^3(\omega_1^3\omega_2^2 - \omega_1^2\omega_2^3) \\ +a^2(3\omega_1^3\omega_2 - 3\omega_1\omega_2^3) \\ +a(3\omega_1^3 + 3\omega_1^2\omega_2 - 3\omega_1\omega_2^2 - 3\omega_2^3) \\ +(3\omega_1^2 - 3\omega_2^2) \end{array} \right) < 0.$$

(213) gives us

$$i_1(\frac{a}{\lambda})\lambda^3 \frac{(\epsilon_\infty)}{\epsilon_p} \frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2 a^5} \left(\begin{array}{c} a^3(\omega_1^3\omega_2^2 - \omega_1^2\omega_2^3) \\ +a^2(3\omega_1^3\omega_2 - 3\omega_1\omega_2^3) \\ +a(3\omega_1^3 + 3\omega_1^2\omega_2 - 3\omega_1\omega_2^2 - 3\omega_2^3) \\ +(3\omega_1^2 - 3\omega_2^2) \end{array} \right).$$

(214) gives us

$$i_1(\frac{a}{\lambda})\frac{\epsilon_\infty\lambda^3}{\epsilon_p}\frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2a^5}\left(\begin{array}{c} a^4(\omega_2^4\omega_1^2-\omega_1^4\omega_2^2) \\ +a^3(3\omega_2^4\omega_1+3\omega_2^3\omega_1^2-3\omega_1^3\omega_2^2-3\omega_1^4\omega_2) \\ +a^2(3\omega_2^4+9\omega_2^3\omega_1-9\omega_1^3\omega_2-3\omega_1^4) \\ +a(9\omega_2^3+9\omega_2^2\omega_1-9\omega_2\omega_1^2-9\omega_1^3) \\ +(9\omega_2^2-9\omega_1^2) \end{array}\right).$$

So (213) + (214) give us

$$i_1(\frac{a}{\lambda})\frac{\epsilon_\infty\lambda^3}{\epsilon_p}\frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2a^5}\left(\begin{array}{c} a^4(\omega_2^4\omega_1^2-\omega_1^4\omega_2^2) \\ +a^3(3\omega_2^4\omega_1+2\omega_2^3\omega_1^2-2\omega_1^3\omega_2^2-3\omega_1^4\omega_2) \\ +a^2(3\omega_2^4+6\omega_2^3\omega_1-6\omega_1^3\omega_2-3\omega_1^4) \\ +a(6\omega_2^3+6\omega_2^2\omega_1-6\omega_2\omega_1^2-6\omega_1^3) \\ +(6\omega_2^2-6\omega_1^2) \end{array}\right)<0.$$

Applying $\omega_1^2+\omega_2^2=\frac{\epsilon_s+\kappa^2\lambda^2}{\epsilon_\infty\lambda^2}$ will transform the result into

$$i_1(\frac{a}{\lambda})\frac{\lambda(\kappa^2\lambda^2)}{\epsilon_p}\frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2a^5(\omega_1^2+\omega_2^2}\left(\begin{array}{c} a^4(\omega_2^4\omega_1^2-\omega_1^4\omega_2^2) \\ +a^3(3\omega_2^4\omega_1+2\omega_2^3\omega_1^2-2\omega_1^3\omega_2^2-3\omega_1^4\omega_2) \\ +a^2(3\omega_2^4+6\omega_2^3\omega_1-6\omega_1^3\omega_2-3\omega_1^4) \\ +a(6\omega_2^3+6\omega_2^2\omega_1-6\omega_2\omega_1^2-6\omega_1^3) \\ +(6\omega_2^2-6\omega_1^2) \end{array}\right)<0$$

and

$$
i_1(\frac{a}{\lambda})\frac{\lambda(\epsilon_s)}{\epsilon_p}\frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2a^5(\omega_1^2+\omega_2^2}\left(\begin{array}{c} a^4(\omega_2^4\omega_1^2-\omega_1^4\omega_2^2) \\ +a^3(3\omega_2^4\omega_1+2\omega_2^3\omega_1^2-2\omega_1^3\omega_2^2-3\omega_1^4\omega_2) \\ +a^2(3\omega_2^4+6\omega_2^3\omega_1-6\omega_1^3\omega_2-3\omega_1^4) \\ +a(6\omega_2^3+6\omega_2^2\omega_1-6\omega_2\omega_1^2-6\omega_1^3) \\ +(6\omega_2^2-6\omega_1^2) \end{array}\right)
$$

the latter to which we add (224) and get

$$
i_1(\frac{a}{\lambda})\frac{\lambda\epsilon_s}{\epsilon_p}\frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2a^5(\omega_1^2+\omega_2^2)}\left(\begin{array}{c} a^4(\omega_2^4\omega_1^2-\omega_1^4\omega_2^2) \\ +a^3(2\omega_2^4\omega_1+3\omega_2^3\omega_1^2-3\omega_1^3\omega_2^2-2\omega_1^4\omega_2) \\ +a^2(2\omega_2^4+6\omega_2^3\omega_1-6\omega_1^3\omega_2-2\omega_1^4) \\ +a(6\omega_2^3+6\omega_2^2\omega_1-6\omega_2\omega_1^2-6\omega_1^3) \\ +(6\omega_2^2-6\omega_1^2) \end{array}\right) < 0.
$$

Thus we have determined that for $n = 1$, (210) + 211) + (212) + (213) + (214) < 0.

Next, we have (215) which gives us

$$
i_2(\frac{a}{\lambda})a\lambda^4\frac{\epsilon_\infty}{\epsilon_p}\omega_1^2\omega_2^2\frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2a^5}\left(a^3(\omega_1\omega_2^2)-\omega_2\omega_1^2+a^2(\omega_2^2-\omega_1^2)\right) < 0
$$

(216) gives us

$$
i_2(\frac{a}{\lambda})\lambda^2\frac{(-\epsilon_s)}{\epsilon_p}\frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2a^4}\left(\begin{array}{c} a^3(\omega_1^3\omega_2^2-\omega_1^2\omega_2^3) \\ +a^2(3\omega_1^3\omega_2-3\omega_1\omega_2^3) \\ +a(3\omega_1^3+3\omega_1^2\omega_2-3\omega_1\omega_2^2-3\omega_2^3) \\ +(3\omega_1^2-3\omega_2^2) \end{array}\right).
$$

(217) gives us

$$i_2(\frac{a}{\lambda})\frac{(\epsilon_p)}{\epsilon_p}\lambda^2\frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2a^4}\left(\begin{array}{c} a^2(\omega_2^3\omega_1-\omega_1^3\omega_2) \\ +a(\omega_2^3+\omega_2^2\omega_1-\omega_2\omega_1^2-\omega_1^3) \\ +(\omega_2^2-\omega_1^2) \end{array}\right)<0.$$

(218) gives us

$$i_2(\frac{a}{\lambda})\frac{(-\epsilon_s)}{\epsilon_p}\lambda^2\frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2a^4}\left(\begin{array}{c} a^2(\omega_2^3\omega_1-\omega_1^3\omega_2) \\ +a(\omega_2^3+\omega_2^2\omega_1-\omega_2\omega_1^2-\omega_1^3) \\ +(\omega_2^2-\omega_1^2) \end{array}\right).$$

So, (216) + (218) is

$$i_2(\frac{a}{\lambda})\lambda^2\frac{(-\epsilon_s)}{\epsilon_p}\frac{e^{-(\omega_1+\omega_2)a}}{\omega_1^2\omega_2^2a^4}\left(\begin{array}{c} a^3(\omega_1^3\omega_2^2-\omega_1^2\omega_2^3) \\ +a^2(2\omega_1^3\omega_2-2\omega_1\omega_2^3) \\ +a(2\omega_1^3+2\omega_1^2\omega_2-2\omega_1\omega_2^2-2\omega_2^3) \\ +(2\omega_1^2-2\omega_2^2) \end{array}\right)<0.$$

Thus we have shown that (215) + (216) + (217) + (218) <0. Thus, we have our desired result: $\beta^2_{n,m}\delta^1_{n,m}-\beta^1_{n,m}\delta^2_{n,m}<0$ for $n=1$.

So now we work on the induction case.

We will assume that $\beta^2\delta^1-\beta^1\delta^2<0$ at $n-1$ and $n-2$, and show that it holds at $n\geq 2$.

(217) ≤ 0 is immediate.

From (219), we have (210) <0 and (215) <0.

From (220), we have (212) <0.

Thus we need to show (211) + (213) + (214) <0 and (216) + (218) <0.

Now we may apply our reductions.

(211) becomes via (221)

$$i_n(\frac{a}{\lambda})\lambda n\frac{(-\epsilon_s)}{\epsilon_p}(k_n(\omega_1 a)k_{n-1}(\omega_2 a)\omega_2 - k_n(\omega_2 a)k_{n-1}(\omega_1 a)\omega_1),$$

and with the addition of $(\omega_1^2+\omega_2^2)$, it becomes

$$i_n(\frac{a}{\lambda})\frac{\lambda n}{(\omega_1^2+\omega_2^2)}\frac{(-\epsilon_s)}{\epsilon_p}\begin{pmatrix} k_n(\omega_1 a)k_{n-1}(\omega_2 a)\omega_2^3 \\ +k_n(\omega_1 a)k_{n-1}(\omega_2 a)\omega_2\omega_1^2 \\ -k_n(\omega_2 a)k_{n-1}(\omega_1 a)\omega_1^3 \\ -k_n(\omega_2 a)k_{n-1}(\omega_1 a)\omega_1\omega_2^2 \end{pmatrix}.$$

(213) becomes via (222)

$$i_n(\frac{a}{\lambda})\lambda^3 n\frac{(\epsilon_\infty)}{\epsilon_p}\begin{pmatrix} \frac{(2n+1)k_n(\omega_1 a)k_n(\omega_2 a)}{a}(\omega_1^2-\omega_2^2) \\ +k_n(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1^2\omega_2 \\ -k_n(\omega_2 a)k_{n-1}(\omega_1 a)\omega_2^2\omega_1 \end{pmatrix}.$$

(214) becomes via (223)

$$i_n(\frac{a}{\lambda})\frac{\epsilon_\infty\lambda^3}{\epsilon_p}(\omega_2^2-\omega_1^2)\begin{pmatrix} (2n+1)^2\frac{k_n(\omega_1 a)k_n(\omega_2 a)}{a} \\ +(2n+1)(k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1 + k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2) \\ +k_{n-1}(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1\omega_2 a \end{pmatrix},$$

which we rewrite as

$$i_n(\frac{a}{\lambda})\frac{\epsilon_\infty \lambda^3}{\epsilon_p}\begin{pmatrix} (\omega_2^2-\omega_1^2)(2n+1)^2\frac{k_n(\omega_1 a)k_n(\omega_2 a)}{a} \\ +(2n+1)k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1\omega_2^2 \\ +(2n+1)k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2^3 \\ -(2n+1)k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1^3 \\ -(2n+1)k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2\omega_1^2 \\ +(\omega_2^2-\omega_1^2)k_{n-1}(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1\omega_2 a \end{pmatrix}.$$

(213) + (214) then becomes

$$i_n(\frac{a}{\lambda})\frac{\epsilon_\infty \lambda^3}{\epsilon_p}\begin{pmatrix} (\omega_2^2-\omega_1^2)(2n+1)(n+1)\frac{k_n(\omega_1 a)k_n(\omega_2 a)}{a} \\ +(n+1)k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1\omega_2^2 \\ +(2n+1)k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2^3 \\ -(2n+1)k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1^3 \\ -(n+1)k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2\omega_1^2 \\ +(\omega_2^2-\omega_1^2)k_{n-1}(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1\omega_2 a \end{pmatrix} < 0.$$

To show it is negative, we used (220) on the middle terms.

Thus we may apply $\omega_1^2+\omega_2^2=\frac{\epsilon_s+\kappa^2\lambda^2}{\epsilon_\infty\lambda^2}$ to split up (213) + (214) and obtain

$$i_n(\frac{a}{\lambda})\frac{\lambda(\kappa^2\lambda^2)}{\epsilon_p(\omega_1^2+\omega_2^2)}\begin{pmatrix} (\omega_2^2-\omega_1^2)(2n+1)(n+1)\frac{k_n(\omega_1 a)k_n(\omega_2 a)}{a} \\ +(n+1)k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1\omega_2^2 \\ +(2n+1)k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2^3 \\ -(2n+1)k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1^3 \\ -(n+1)k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2\omega_1^2 \\ +(\omega_2^2-\omega_1^2)k_{n-1}(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1\omega_2 a \end{pmatrix} < 0$$

and

$$i_n(\frac{a}{\lambda})\frac{\lambda\epsilon_s}{\epsilon_p(\omega_1^2+\omega_2^2)}\begin{pmatrix} (\omega_2^2-\omega_1^2)(2n+1)(n+1)\frac{k_n(\omega_1 a)k_n(\omega_2 a)}{a} \\ +(n+1)k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1\omega_2^2 \\ +(2n+1)k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2^3 \\ -(2n+1)k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1^3 \\ -(n+1)k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2\omega_1^2 \\ +(\omega_2^2-\omega_1^2)k_{n-1}(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1\omega_2 a \end{pmatrix}.$$

We add (211) to the latter term and arrive at

$$i_n(\frac{a}{\lambda})\frac{\lambda\epsilon_s}{\epsilon_p(\omega_1^2+\omega_2^2)}\begin{pmatrix} (\omega_2^2-\omega_1^2)(2n+1)(n+1)\frac{k_n(\omega_1 a)k_n(\omega_2 a)}{a} \\ +(2n+1)k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1\omega_2^2 \\ +(n+1)k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2^3 \\ -(n+1)k_{n-1}(\omega_1 a)k_n(\omega_2 a)\omega_1^3 \\ -(2n+1)k_{n-1}(\omega_2 a)k_n(\omega_1 a)\omega_2\omega_1^2 \\ +(\omega_2^2-\omega_1^2)k_{n-1}(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1\omega_2 a \end{pmatrix} < 0.$$

To show it is negative, we used (219) on the second and fifth terms. Altogether, this gives us (211) + (213) + (214) < 0.

(216) gives us

$$i_{n+1}(\frac{a}{\lambda})a\lambda^2\frac{(-\epsilon_s)}{\epsilon_p}\begin{pmatrix} \frac{(2n+1)k_n(\omega_1 a)k_n(\omega_2 a)}{a}(\omega_1^2-\omega_2^2) \\ +k_{n-2}(\omega_1 a)k_{n-1}(\omega_2 a)\omega_1^2\omega_2 \\ -k_{n-2}(\omega_2 a)k_{n-1}(\omega_1 a)\omega_2^2\omega_1 \end{pmatrix}.$$

Induction with (220) bounds (216) from above by

$$i_{n+1}(\frac{a}{\lambda})\lambda^2\frac{(-\epsilon_s)}{\epsilon_p}(2n+1)k_n(\omega_1 a)k_n(\omega_2 a)(\omega_1^2-\omega_2^2).$$

(218) is

$$i_{n+1}(\frac{a}{\lambda})\frac{(-\epsilon_s)}{\epsilon_p}\lambda^2(\omega_2^2-\omega_1^2)nk_n(\omega_1 a)k_n(\omega_2 a).$$

When added to the (216) bound, we get

$$i_{n+1}(\frac{a}{\lambda})\lambda^2\frac{(-\epsilon_s)}{\epsilon_p}(n+1)k_n(\omega_1 a)k_n(\omega_2 a)(\omega_1^2-\omega^2)<0.$$

Thus (216) + (218) <0 .

So we have proven that the various parts of $\beta^2_{n,m}\delta^1_{n,m}-\beta^1_{n,m}\delta^2_{n,m}$ are negative. Thus we have our full conclusion that $\beta^2_{n,m}\delta^1_{n,m}-\beta^1_{n,m}\delta^2_{n,m}<0$ for all $n\geq 0$, that is, Lemma C.1.

Appendix D Proof of $|\beta^2_{n,m}\delta^1_{n,m}-\delta^2_{n,m}\beta^1_{n,m}|$ lower bound

Lemma D.1. *Let the conditions of Theorem 6.1 hold.*

$$\beta^i_{n,m}=(1-\lambda^2\omega_i^2)k_n(\omega_i a)n-\frac{(\epsilon_s-\epsilon_\infty\lambda^2\omega_i^2)}{\epsilon_p}k_n'(\omega_i a)\omega_i a$$

$$\delta^i_{n,m}=-k_n(\omega_i a)\lambda^2\omega_i^2 i_n'(\frac{a}{\lambda})-k_n'(\omega_i a)\omega_i\frac{(-\epsilon_\infty\lambda^2\omega_i^2+\epsilon_s-\epsilon_p)}{\epsilon_p}\lambda i_n(\frac{a}{\lambda})$$

Then

$$|\beta^2_{n,m}\delta^1_{n,m}-\beta^1_{n,m}\delta^2_{n,m}|\geq\frac{2}{3}n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})\frac{\lambda^3}{a\epsilon_p}(\omega_1^2-\omega_2^2)(\epsilon_\infty+\epsilon_p)$$

holds for $n\geq N_0$ where

$$N_0=\max(\frac{3(\omega_1 a+1)}{2\epsilon},\sqrt{\frac{3a^2(\epsilon_p+\epsilon_s)}{2\epsilon_p\lambda^2(\epsilon_\infty+\epsilon_p)}})$$

where

$$\epsilon = \min(\frac{1}{6}\frac{\omega_1^2-\omega_2^2}{(3\omega_1^2-\omega_2^2)}, \frac{1}{12}\frac{\lambda^2(\omega_1^2-\omega_2^2)(\epsilon_p+\epsilon_\infty)}{(\epsilon_p+\epsilon_s)}, \frac{(\epsilon_p+\epsilon_s)(\omega_1^2-\omega_2^2)}{12\lambda^2\epsilon_\infty\omega_1^2\omega_2^2}, \frac{(\epsilon_p+\epsilon_s)(\omega_1^2-\omega_2^2)}{6\epsilon_s(\omega_1^2+\omega_2^2)}, \frac{1}{3})$$

Since $\beta^2_{n,m}\delta^1_{n,m} - \beta^1_{n,m}\delta^2_{n,m} = (210)+(211)+(212)+(213)+(214)+(215)+(216)+(217)+(218)$, we shall determine asymptotic relations on each of the parts of the sum. However, caution must be made for the sum since $a_n \sim b_n$ and $c_n \sim d_n$ do not imply $a_n + c_n \sim b_n + d_n$, especially since as we demonstrate in the proof of $\beta^2_{n,m}\delta^1_{n,m} - \beta^1_{n,m}\delta^2_{n,m} < 0$, the terms of the sum do not all have the same sign.

Instead we shall apply the fact that $\lim_{n\to\infty}\frac{a_n}{c_n} = A$, $\lim_{n\to\infty}\frac{b_n}{c_n} = B$ and $A+B \neq 0$, imply that $a_n + b_n \sim c_n(A+B)$.

We split up $((210)+(211))/(n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda}))$ into 2 terms, and then via (116)

$$i_n(\frac{a}{\lambda})\lambda n\frac{(\epsilon_p-\epsilon_s)}{\epsilon_p}\frac{(k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_2)}{n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} \sim 2\lambda\frac{(\epsilon_p-\epsilon_s)}{a\epsilon_p}$$

$$i_n(\frac{a}{\lambda})\lambda n\frac{(\epsilon_p-\epsilon_s)}{\epsilon_p}\frac{(-k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_1)}{n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} \sim -2\lambda\frac{(\epsilon_p-\epsilon_s)}{a\epsilon_p}.$$

We split up $((212)+(213))/(n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda}))$ into 2 terms, and then via (116)

$$i_n(\frac{a}{\lambda})\lambda^3 n\frac{(-\epsilon_p+\epsilon_\infty)}{\epsilon_p}\omega_1\omega_2\frac{k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_1}{n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} \sim 2\lambda^3\frac{(-\epsilon_p+\epsilon_\infty)}{a\epsilon_p}\omega_1^2$$

$$i_n(\frac{a}{\lambda})\lambda^3 n\frac{(-\epsilon_p+\epsilon_\infty)}{\epsilon_p}\omega_1\omega_2\frac{-k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_2}{n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} \sim -2\lambda^3\frac{(-\epsilon_p+\epsilon_\infty)}{a\epsilon_p}\omega_2^2.$$

As far as $(214)/(n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda}))$ is concerned, we apply (116) and obtain

$$i_n(\frac{a}{\lambda})\frac{\epsilon_\infty\lambda^3}{\epsilon_p}(\omega_2^2-\omega_1^2)\frac{\omega_1\omega_2 a k_{n+1}(\omega_1 a)k_{n+1}(\omega_2 a)}{n^2 k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} \sim 4\frac{\epsilon_\infty\lambda^3}{a\epsilon_p}(\omega_2^2-\omega_1^2).$$

Combining these 5 convergences, noting that their entire sum is not 0, we have

$$\frac{(210)+(211)+(212)+(213)+(214)}{n^2 k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} \sim 2\frac{\lambda^3}{a\epsilon_p}(\omega_2^2-\omega_1^2)(\epsilon_\infty+\epsilon_p) < 0.$$

To determine the error bounds for convergence, we appeal to (121). This tells us that for a given $\epsilon > 0$, if we have n large enough, then

$$\begin{aligned}&\left|\frac{(210)+(211)+(212)+(213)+(214)}{n^2 k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} - 2\frac{\lambda^3}{a\epsilon_p}(\omega_2^2-\omega_1^2)(\epsilon_\infty+\epsilon_p)\right| \\ &\leq \epsilon\left(2(\frac{2\lambda(\epsilon_p+\epsilon_s)}{a\epsilon_p}) + \frac{2\lambda^3}{a\epsilon_p}(\epsilon_p+\epsilon_\infty)(\omega_1^2+\omega_2^2) + \frac{4\epsilon_\infty\lambda^3}{a\epsilon_p}(\omega_1^2-\omega_2^2)\right)\end{aligned}.$$

We wish to guarantee that $(210)+(211)+(212)+(213)+(214)$ stays sufficiently far from 0. Thus, we shall need

$$\epsilon\left(2(\frac{2\lambda(\epsilon_p+\epsilon_s)}{a\epsilon_p}) + \frac{2\lambda^3}{a\epsilon_p}(\epsilon_p+\epsilon_\infty)(\omega_1^2+\omega_2^2) + \frac{4\epsilon_\infty\lambda^3}{a\epsilon_p}(\omega_1^2-\omega_2^2)\right) \leq \frac{2}{3}\frac{\lambda^3}{a\epsilon_p}(\omega_1^2-\omega_2^2)(\epsilon_\infty+\epsilon_p) \quad (225)$$

We have

$$\epsilon\left(2(\frac{2\lambda(\epsilon_p+\epsilon_s)}{a\epsilon_p}) + \frac{2\lambda^3}{a\epsilon_p}(\epsilon_p+\epsilon_\infty)(\omega_1^2+\omega_2^2) + \frac{4\epsilon_\infty\lambda^3}{a\epsilon_p}(\omega_1^2-\omega_2^2)\right)$$

$$= \frac{2\epsilon\lambda}{a\epsilon_p}\left(2(\epsilon_p+\epsilon_s) + \lambda^2\epsilon_p(\omega_1^2+\omega_2^2) + \lambda^2\epsilon_\infty(3\omega_1^2-\omega_2^2)\right)$$

$$\leq \frac{2\epsilon\lambda}{a\epsilon_p}\left(2(\epsilon_p+\epsilon_s) + \lambda^2(\epsilon_p+\epsilon_\infty)(3\omega_1^2+\omega_2^2)\right)$$

since $\omega_1 > \omega_2$. To get the desired inequality (225), we shall need $\epsilon(3\omega_1^2 - \omega_2^2) \le \frac{1}{6}(\omega_1^2 - \omega_2^2)$ and $2\epsilon(\epsilon_p + \epsilon_s) \le \frac{1}{6}\lambda^2(\omega_1^2 - \omega_2^2)(\epsilon_p + \epsilon_\infty)$.

To determine the bounds for n, we again return to (121). Since among the convergences we did use it twice in one term, we shall need to choose $\tilde{\epsilon}$ such that $(1+\tilde{\epsilon})^2 \le 1+\epsilon$.

Assuming that $\epsilon < 3$, we have $(1+\frac{\epsilon}{3})^2 < 1+\epsilon$, so we shall use $\tilde{\epsilon} = \frac{\epsilon}{3}$. This gives us the desired bound, (225).

Then, so long as $\omega_1 a + 1 \le \frac{2n\epsilon}{3}$, we have

$$\left| \frac{(210)+(211)+(212)+(213)+(214)}{n^2 k_n(\omega_1 a) k_n(\omega_2 a) i_n(\frac{a}{\lambda})} - 2\frac{\lambda^3}{a\epsilon_p}(\omega_2^2 - \omega_1^2)(\epsilon_\infty + \epsilon_p) \right| \ge \frac{4}{3}\frac{\lambda^3}{a\epsilon_p}(\omega_1^2 - \omega_2^2)(\epsilon_\infty + \epsilon_p).$$

Next, we examine the other components of $\beta_{n,m}^2 \delta_{n,m}^1 - \beta_{n,m}^1 \delta_{n,m}^2$.

We split up $(215)/(k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda}))$ into 2 terms, and then via (116) and (115),

$$i_{n+1}(\frac{a}{\lambda}) a\lambda^4 \frac{\epsilon_\infty}{\epsilon_p}\omega_1^2\omega_2^2 \frac{k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_2}{k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} \sim a\lambda^3 \frac{\epsilon_\infty}{\epsilon_p}\omega_1^2\omega_2^2 (\frac{2n\lambda i_{n+1}(\frac{a}{\lambda})}{a i_n(\frac{a}{\lambda})})$$

$$i_{n+1}(\frac{a}{\lambda}) a\lambda^4 \frac{\epsilon_\infty}{\epsilon_p}\omega_1^2\omega_2^2 \frac{-k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_1)}{k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} \sim -a\lambda^3 \frac{\epsilon_\infty}{\epsilon_p}\omega_1^2\omega_2^2 (\frac{2n\lambda i_{n+1}(\frac{a}{\lambda})}{a i_n(\frac{a}{\lambda})}).$$

We split up $(216)/(k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda}))$ into 2 terms, and then via (116) and (115),

$$i_{n+1}(\frac{a}{\lambda}) a\lambda^2 \frac{(-\epsilon_s)}{\epsilon_p}\omega_1\omega_2 \frac{k_n(\omega_1 a)k_{n+1}(\omega_2 a)\omega_1}{k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} \sim a\lambda \frac{(-\epsilon_s)}{\epsilon_p}\omega_1^2 (\frac{2n\lambda i_{n+1}(\frac{a}{\lambda})}{a i_n(\frac{a}{\lambda})})$$

$$i_{n+1}(\frac{a}{\lambda}) a\lambda^2 \frac{(-\epsilon_s)}{\epsilon_p}\omega_1\omega_2 \frac{-k_n(\omega_2 a)k_{n+1}(\omega_1 a)\omega_2}{k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} \sim -a\lambda \frac{(-\epsilon_s)}{\epsilon_p}\omega_2^2 (\frac{2n\lambda i_{n+1}(\frac{a}{\lambda})}{a i_n(\frac{a}{\lambda})}).$$

As far as $((217)+(218))/(n^2 k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda}))$ is concerned, we apply (115) and obtain

$$i_{n+1}(\frac{a}{\lambda})\frac{(\epsilon_p-\epsilon_s)}{\epsilon_p}\lambda^2(\omega_2^2-\omega_1^2)n\frac{k_n(\omega_1 a)k_n(\omega_2 a)}{k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})}\sim\frac{(\epsilon_p-\epsilon_s)}{2\epsilon_p}a\lambda(\omega_2^2-\omega_1^2)(\frac{2n\lambda i_{n+1}(\frac{a}{\lambda})}{a i_n(\frac{a}{\lambda})}).$$

Combining these 5 convergences, noting that their entire sum is not 0, we have

$$\frac{(215)+(216)+(217)+(218)}{k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})}\sim a\lambda\frac{(\epsilon_p+\epsilon_s)}{2\epsilon_p}(\omega_2^2-\omega_1^2)(\frac{2n\lambda i_{n+1}(\frac{a}{\lambda})}{a i_n(\frac{a}{\lambda})})<0.$$

To determine the error bounds for convergence, we appeal to (121). This tells us that for a given $\epsilon>0$, if we have n large enough, then

$$\left|\frac{(215)+(216)+(217)+(218)}{k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})}-a\lambda\frac{(\epsilon_p+\epsilon_s)}{2\epsilon_p}(\omega_2^2-\omega_1^2)(\frac{2n\lambda i_{n+1}(\frac{a}{\lambda})}{a i_n(\frac{a}{\lambda})})\right|$$
$$\leq\epsilon\left(2a\lambda^3\frac{\epsilon_\infty}{\epsilon_p}\omega_1^2\omega_2^2+a\lambda\frac{\epsilon_s}{\epsilon_p}(\omega_1^2+\omega_2^2)+\frac{(\epsilon_p+\epsilon_s)}{2\epsilon_p}a\lambda(\omega_1^2-\omega_2^2)\right)(\frac{2n\lambda i_{n+1}(\frac{a}{\lambda})}{a i_n(\frac{a}{\lambda})})$$

Unlike the previous set of terms, we do not require (215) + (216) + (217) + (218) to be away from 0. We just need it to not be too large. It will be enough that

$$\epsilon\left(2a\lambda^3\frac{\epsilon_\infty}{\epsilon_p}\omega_1^2\omega_2^2+a\lambda\frac{\epsilon_s}{\epsilon_p}(\omega_1^2+\omega_2^2)+\frac{(\epsilon_p+\epsilon_s)}{2\epsilon_p}a\lambda(\omega_1^2-\omega_2^2)\right)\leq a\lambda\frac{(\epsilon_p+\epsilon_s)}{2\epsilon_p}(\omega_1^2-\omega_2^2)$$

To this end, we shall require that following bounds all hold:

$$\epsilon\leq\frac{1}{3}\frac{(\epsilon_p+\epsilon_s)(\omega_1^2-\omega_2^2)}{4\lambda^2\epsilon_\infty\omega_1^2\omega_2^2}$$

$$\epsilon\leq\frac{1}{3}\frac{(\epsilon_p+\epsilon_s)(\omega_1^2-\omega_2^2)}{2\epsilon_s(\omega_1^2+\omega_2^2)}$$

$$\epsilon\leq\frac{1}{3}.$$

To next determine bounds for n, we appeal to (121). Since among these convergences, we used this result twice in several terms, we shall need to choose $\tilde{\epsilon}$ such that $(1+\tilde{\epsilon})^2 \leq 1+\epsilon$. As before, it is sufficient to choose $\tilde{\epsilon} = \frac{\epsilon}{3}$.

So, as before, we need $\omega_1 a + 1 \leq \frac{2n\epsilon}{3}$. In such a case, we have, using (120), we have

$$\left|\frac{(215)+(216)+(217)+(218)}{k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})}\right| \leq 2a\lambda\frac{(\epsilon_p+\epsilon_s)}{2\epsilon_p}(\omega_1^2-\omega_2^2)(\frac{2n\lambda i_{n+1}(\frac{a}{\lambda})}{ai_n(\frac{a}{\lambda})}) \leq 2a\lambda\frac{(\epsilon_p+\epsilon_s)}{2\epsilon_p}(\omega_1^2-\omega_2^2).$$

To connect the two sets of convergences, we shall need

$$\frac{2}{n^2}a\lambda\frac{(\epsilon_p+\epsilon_s)}{2\epsilon_p}(\omega_1^2-\omega_2^2) \leq \frac{2}{3}\frac{\lambda^3}{a\epsilon_p}(\omega_1^2-\omega_2^2)(\epsilon_\infty+\epsilon_p).$$

This is implied by

$$\frac{3a^2(\epsilon_p+\epsilon_s)}{2\epsilon_p\lambda^2(\epsilon_\infty+\epsilon_p)} \leq n^2.$$

In such a case, we have

$$\left|\frac{(210)+(211)+(212)+(213)+(214)}{n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})} + \frac{(215)+(216)+(217)+(218)}{n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})}\right| \geq \frac{2}{3}\frac{\lambda^3}{a\epsilon_p}(\omega_1^2-\omega_2^2)(\epsilon_\infty+\epsilon_p).$$

Thus, we have

$$|\beta_{n,m}^2\delta_{n,m}^1 - \beta_{n,m}^1\delta_{n,m}^2| \geq \frac{2}{3}n^2k_n(\omega_1 a)k_n(\omega_2 a)i_n(\frac{a}{\lambda})\frac{\lambda^3}{a\epsilon_p}(\omega_1^2-\omega_2^2)(\epsilon_\infty+\epsilon_p)$$

for $n \geq N_0$ where

$$N_0 = \max(\frac{3(\omega_1 a + 1)}{2\epsilon}, \sqrt{\frac{3a^2(\epsilon_p + \epsilon_s)}{2\epsilon_p \lambda^2(\epsilon_\infty + \epsilon_p)}})$$

where

$$\epsilon = \min(\frac{1}{6}\frac{\omega_1^2 - \omega_2^2}{(3\omega_1^2 - \omega_2^2)}, \frac{1}{12}\frac{\lambda^2(\omega_1^2 - \omega_2^2)(\epsilon_p + \epsilon_\infty)}{(\epsilon_p + \epsilon_s)}, \frac{(\epsilon_p + \epsilon_s)(\omega_1^2 - \omega_2^2)}{12\lambda^2\epsilon_\infty\omega_1^2\omega_2^2}, \frac{(\epsilon_p + \epsilon_s)(\omega_1^2 - \omega_2^2)}{6\epsilon_s(\omega_1^2 + \omega_2^2)}, \frac{1}{3}).$$

Thus Lemma D.1 is proven.

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