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Asymptotic Expansion of the L^2-norm of a Solution of the Strongly Damped Wave Equation

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ASYMPTOTIC EXPANSION OF THE L^2-NORM OF A SOLUTION OF THE STRONGLY DAMPED WAVE EQUATION

by

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ABSTRACT

ASYMPTOTIC EXPANSION OF THE
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The Fourier transform, $\mathcal{F}$, on $\mathbb{R}^N$ $(N \geq 1)$ transforms the Cauchy problem for the strongly damped wave equation $u_{tt} - \Delta u_t - \Delta u = 0$ to an ordinary differential equation in time t. We let $u(t,x)$ be the solution of the problem given by the Fourier transform, and $\nu(t,\xi)$ be the asymptotic profile of $\mathcal{F}(u)(t,\xi) = \hat{u}(t,\xi)$ found by Ikehata in [4].

In this thesis we study the asymptotic expansions of the squared L^2-norms of $u(t,x)$, $\hat{u}(t,\xi) - \nu(t,\xi)$, and $\nu(t,\xi)$ as $t \to \infty$. With suitable initial data $u(0,x)$ and $u_t(0,x)$, we establish the rate of growth or decay of the squared L^2-norms of $u(t,x)$ and $\nu(t,\xi)$ as $t \to \infty$. By noting the cancellation of leading terms of their respective expansions, we conclude that the rate of convergence between $\hat{u}(t,\xi)$ and $\nu(t,\xi)$ in the L^2-norm occurs quickly relative to their individual behaviors. Finally we consider three examples in order to illustrate the results.

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Chapter 1

Introduction

1.1 Motivation

In the analysis of linear partial differential equations (PDE), a common method is to use the Fourier transform, which will be defined in a later section. Broadly speaking, the benefit of this method is that the Fourier transform will reduce the original problem from a linear partial differential equation in one of the variables to a linear ordinary differential equation (ODE) of the other. This reduced problem is typically easier to solve, since linear ODE and their solutions are well understood. With appropriate assumptions the solution in the so-called Fourier space may be returned to the original setting by way of the Fourier inverse.

A straightforward and elementary example of this method at work is in deriving the fundamental solution of the heat equation

$$v_t(t,x) - \Delta v(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^N,$$
$$v(0,\cdot) = v_0 \in L^2(\mathbb{R}^N),$$

where Δ denotes the spatial Laplacian on $\mathbb{R}^N$. Applying the Fourier transform to the PDE and its initial condition, we obtain the equation

$$\hat{v}_t(t,\xi) + |\xi|^2\hat{v}(t,\xi) = 0, \qquad (t,\xi) \in [0,\infty) \times \mathbb{R}^N,$$
$$\hat{v}(0,\xi) = \hat{v}_0(\xi),$$

which has solution

$$\hat{v}(t,\xi) = \hat{v}_0(\xi)e^{-t|\xi|^2}.$$

With the assumption $v_0 \in L^2(\mathbb{R}^N)$, $\hat{v} \in L^2(\mathbb{R}^N)$ and we obtain the fundamental solution $v(t,x)$ via the Fourier inverse

$$v(t,x) = \mathcal{F}^{-1}(\hat{v}_0 \cdot e^{-t|\cdot|^2})(x).$$

We are then interested in the behavior of the squared L^2-norm

$$\|v(t,\cdot)\|_2^2 = \int_{\mathbb{R}^N} |v(t,x)|^2 dx$$

as $t \to \infty$. By the well-known Plancherel theorem (see Theorem 1 on p. 187 in Chapter 4 of [2]), the Fourier transform is an $L^2(\mathbb{R}^N)$-isometry, hence

$$\|v(t,\cdot)\|_2^2 = \|\hat{v}(t,\cdot)\|_2^2.$$

By imposing the additional assumption that $v_0 \in L^1(\mathbb{R}^N)$, it is a property of the Fourier transform that $\hat{v}_0 \in L^\infty(\mathbb{R}^N)$. It is then easily verified that for $t > 0$

$$\begin{aligned}\|v(t,\cdot)\|_2^2 = \|\hat{v}(t,\cdot)\|_2^2 &= \int_{\mathbb{R}^N} |\hat{v}_0(\xi)|^2 e^{-2t|\xi|^2} d\xi \\ &\leq \frac{M^2\omega_{N-1}}{2^{\frac{N}{2}+1}}\Gamma(\tfrac{N}{2})t^{-\frac{N}{2}} \\ &= O(t^{-\frac{N}{2}}).\end{aligned}$$

where $M = \|\hat{v}_0\|_\infty$ and ω_{N-1} is the surface area of the $(N-1)$-sphere in $\mathbb{R}^N$.

In his paper [11], Volkmer puts even stricter assumptions on the initial condition v_0 so as to guarantee the existence of derivatives of $\hat{v}_0$ up to a chosen order (the exact assumptions will be given in a later section). He then determines the asymptotic expansion of the squared $L^2(\mathbb{R}^N)$-norm $\|v(t,\cdot)\|_2^2$. He does this by using the additional assumptions on the initial condition and Plancherel's theorem to instead study $\|\hat{v}(t,\cdot)\|_2^2$. By studying the problem in the Fourier space, the solution $\hat{v}(t,\xi)$ is given explicitly and in terms of a differentiable function $\hat{v}_0(\xi)$.

We will employ this general strategy in the analysis of the problem we consider: the strongly damped wave equation. The motivation for finding the asymptotic expansions of course comes from Volkmer's work in [11], and also from Ikehata's work in [4].

1.2 The strongly damped wave equation in $\mathbb{R}^N$

For $N \in \mathbb{N}$ we begin by considering the strongly damped wave equation in $\mathbb{R}^N$:

$$\begin{gathered} u_{tt}(t,x) - \Delta u_t(t,x) - \Delta u(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^N, \\ u(0,\cdot) = u_0 \in H^1(\mathbb{R}^N), \quad u_t(0,\cdot) = u_1 \in L^2(\mathbb{R}^N). \end{gathered} \tag{1.1}$$

It was determined in [9] by Ikehata, Todorova, and Yordanov that (1.1) admits a unique weak solution $u \in C([0,\infty); H^1(\mathbb{R}^N)) \cap C^1([0,\infty); L^2(\mathbb{R}^N))$.

It is the goal of this thesis to investigate two main problems.

1. Given suitable additional assumptions on the initial data u_0 and u_1, determine the asymptotic expansion of the squared L^2-norm of the weak solution $u(t,x)$ as $t \to \infty$.

To state the second problem, we need further context about the weak solution $u(t,x)$. As discussed in the previous section, we will determine $u(t,x)$ with the help of the Fourier transform $\mathcal{F}$. In the Fourier space, Ikehata [4] found an asymptotic profile $\nu(t,\xi)$ of $\mathcal{F}(u)(t,\xi) := \hat{u}(t,\xi)$

such that for space dimension $N \in \mathbb{N}$,

$$\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = O(t^{-\frac{N}{2}}) \tag{1.2}$$

as $t \to \infty$ given additional assumptions on the initial data. In addition, for space dimension $N \geq 3$, he found the following decay rate for the weak solution $u(t,x)$:

$$\|u(t,\cdot)\|_2^2 = O(t^{-\frac{N}{2}+1}) \qquad (t \to \infty). \tag{1.3}$$

(1.2) and (1.3) reveal that $\hat{u}(t,\xi)$ and $\nu(t,\xi)$ tend to each other in norm faster than the decay of $u(t,x)$ in norm. This is a phenomenon similar to the diffusion phenomenon studied by Volkmer in [11]. In his paper Volkmer studied the dissipative wave equation in addition to the heat equation mentioned earlier. Given additional assumptions on the initial conditions of each problem, he was able to exhibit the diffusion phenomenon and provide asymptotic expansions of the norms of the solutions and the norm of the difference of the solutions. We may now state the second main problem.

2. Given suitable additional assumptions on the initial data u_0 and u_1, determine the asymptotic expansion of the squared L^2-norm of the difference of the solution $\hat{u}(t,\xi)$ and its profile $\nu(t,\xi)$ in the Fourier space.

To determine the necessary additional assumptions on the initial data, we follow the methods of Ikehata [4] and Volkmer [11] in that we use weighted L^1-data. By doing so we will be able to obtain asymptotic expansions of the desired norms up to a number of terms dependent upon the L^1-conditions plus an error term of lower order than all other terms. We first introduce some notation.

Notation. First we let $\mathbb{N} = \{1, 2, \ldots\}$ denote the positive integers and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ denote the non-negative integers. Let $N \in \mathbb{N}$. Throughout this thesis the $L^q(\mathbb{R}^N)$-norm is denoted by $\|\cdot\|_q$. We also define for all $\epsilon > 0$ $\bar{B}_\epsilon := \{x \in \mathbb{R}^N : |x| \leq \epsilon\}$ to be the closed ball of radius ϵ in $\mathbb{R}^N$. Then we denote the $L^q(\bar{B}_\epsilon)$-norm by $\|\cdot\|_{q,\epsilon}$.

For all $\theta \geq 0$ we define the weighted L^1-space

$$L^{1,\theta}(\mathbb{R}^N) := \left\{ \phi \in L^1(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} (1 + |x|)^\theta \, |\phi(x)| dx < \infty \right\}.$$

The norm $\|\cdot\|_{L^{1,\theta}(\mathbb{R}^N)}$ on the space $L^{1,\theta}(\mathbb{R}^N)$ is defined

$$\|\phi\|_{L^{1,\theta}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (1 + |x|)^\theta \, |\phi(x)| dx.$$

Lastly for all $\phi \in L^1(\mathbb{R}^N)$ we define the Fourier transform

$$\mathcal{F}(\phi)(\xi) = \hat{\phi}(\xi) := \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \phi(x) e^{-ix\cdot\xi} dx,$$

and the Fourier inverse

$$\mathcal{F}^{-1}(\phi)(x) := \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \phi(\xi) e^{i\xi\cdot x} d\xi.$$

We remark that for arbitrary $\phi \in L^2(\mathbb{R}^N)$, the Fourier transform is not given by the above integral definition. Rather, for functions $\phi_j \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ such that $\|\phi - \phi_j\|_2 \to 0$ as $j \to \infty$, $\hat{\phi}$ is defined to be the unique $L^2(\mathbb{R}^N)$-limit of $\hat{\phi}_j$ as $j \to \infty$. More details can be found on p. 189 in Chapter 4 of Evans' Partial Differential Equations [2].

1.3 Assumptions

We assume that the space dimension is $N \in \mathbb{N}$ and that there exists $K \in \mathbb{N}$ such that

$$u_0, u_1 \in L^{1,2K}(\mathbb{R}^N)$$

where the functions $u_0 \in H^1(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N)$ are the initial data of (1.1). Under these assumptions both $\hat{u}_0$ and $\hat{u}_1$ are $2K$-times continuously differentiable on $\mathbb{R}^N$, and in particular at $\xi = 0$. Thus we may choose $0 < \delta < 1$ such that, in the closed δ-neighborhood of $\xi = 0$, the following Taylor approximations hold:

$$\begin{aligned} \hat{u}_0(\xi) &= \sum_{|\sigma| \leq 2K-1} b_\sigma \xi^\sigma + O(|\xi|^{2K}), \\ \hat{u}_1(\xi) &= \sum_{|\sigma| \leq 2K-1} a_\sigma \xi^\sigma + O(|\xi|^{2K}), \end{aligned} \tag{1.4}$$

where $\sigma \in \mathbb{N}_0^N$ is a multi-index of order $|\sigma| = \sigma_1 + \ldots + \sigma_N$ (see Appendix A, p. 701 of [2] for further details), and for all $|\sigma| \leq 2K - 1$

$$\begin{aligned} b_\sigma &= \frac{(-i)^{|\sigma|}}{\sigma! \, (2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} x^\sigma u_0(x) dx, \\ a_\sigma &= \frac{(-i)^{|\sigma|}}{\sigma! \, (2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} x^\sigma u_1(x) dx. \end{aligned}$$

We fix this chosen value of $0 < \delta < 1$, and whenever δ is hereafter referred to, we mean this fixed value.

In this fixed closed δ-neighborhood of $\xi = 0$, we also have the Taylor approximations

$$|\xi|^2 \hat{u}_0(\xi) = \sum_{|\sigma| \leq 2K-1} b'_\sigma \xi^\sigma + O(|\xi|^{2K}),$$

$$\hat{u}_1(\xi) + |\xi|^2 \hat{u}_0(\xi) = \sum_{|\sigma| \leq 2K-1} c_\sigma \xi^\sigma + O(|\xi|^{2K}), \tag{1.5}$$

$$|\hat{u}_1(\xi) + |\xi|^2 \hat{u}_0(\xi)|^2 = \sum_{|\sigma| \leq 2K-1} d_\sigma \xi^\sigma + O(|\xi|^{2K}), \tag{1.6}$$

$$|\hat{u}_0(\xi)|^2 = \sum_{|\sigma| \leq 2K-1} f_\sigma \xi^\sigma + O(|\xi|^{2K}), \tag{1.7}$$

$$(\hat{u}_1(\xi) + |\xi|^2 \hat{u}_0(\xi)) \bar{\hat{u}}_0(\xi) = \sum_{|\sigma| \leq 2K-1} l_\sigma \xi^\sigma + O(|\xi|^{2K}), \tag{1.8}$$

where

$$b'_\sigma = \begin{cases} 0 & \text{all entries of } \sigma \text{ are } \leq 1 \\ \sum\limits_{\{j|\sigma_j \geq 2\}} b_{\sigma - 2e_j} & \text{otherwise,} \end{cases}$$

$$c_\sigma = a_\sigma + b'_\sigma,$$

$$d_\sigma = \sum_{\psi+\omega=\sigma} c_\psi \bar{c_\omega}\,,$$

$$f_\sigma = \sum_{\psi+\omega=\sigma} b_\psi \bar{b_\omega}\,,$$

$$l_\sigma = \sum_{\psi+\omega=\sigma} c_\psi \bar{b}_\omega\,.$$

With the definitions $P_0 := (2\pi)^{-N/2} \int_{\mathbb{R}^N} u_0(x)dx$ and $P_1 := (2\pi)^{-N/2} \int_{\mathbb{R}^N} u_1(x)dx$, it is easy to see that $a_{\mathbf{0}} = P_1$ and $b_{\mathbf{0}} = P_0$. Hence $c_{\mathbf{0}} = P_1$, $d_{\mathbf{0}} = |P_1|^2$, $f_{\mathbf{0}} = |P_0|^2$, and $l_{\mathbf{0}} = P_1 \bar{P}_0$.

1.4 Main results

Let us consider the strongly damped wave equation in $\mathbb{R}^N$ as given in (1.1). Applying the Fourier transform to (1.1), we obtain an ordinary differential equation in t

$$\begin{gathered} \hat{u}_{tt}(t,\xi) + |\xi|^2 \hat{u}_t(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = 0, \qquad (t,\xi) \in (0,\infty) \times \mathbb{R}^N, \\ \hat{u}(0,\cdot) = \hat{u}_0, \quad \hat{u}_t(0,\cdot) = \hat{u}_1. \end{gathered} \tag{1.9}$$

The solution to (1.9) is given by

$$\hat{u}(t,\xi) = (\hat{u}_1(\xi) + |\xi|^2 \hat{u}_0(\xi)) h(t,\xi) + \hat{u}_0(\xi) \partial_t h(t,\xi), \tag{1.10}$$

where

$$h(t,\xi) = e^{-\frac{t|\xi|^2}{2}} \frac{\sin(t|\xi|\sqrt{1 - |\xi|^2/4})}{|\xi|\sqrt{1 - |\xi|^2/4}}. \tag{1.11}$$

Therefore the weak solution of (1.1) under the Fourier transform is the inverse Fourier transform of $\hat{u}(t,\xi)$, i.e., $u(t,x) = \mathcal{F}^{-1}(\hat{u})(t,x)$. By Plancherel's theorem the Fourier transform as defined is an $L^2(\mathbb{R}^N)$-isometry (see Theorem 1 on p. 187 in Chapter 4 of [2]). Thus we will rarely need to appeal to the weak solution $u(t,x)$, instead using $\hat{u}(t,\xi)$ whose form is given explicitly.

Define

$$\begin{aligned}
\mu_1(t,\xi) &:= (\hat{u}_1(\xi) + |\xi|^2\hat{u}_0(\xi))h(t,\xi),\\
\mu_2(t,\xi) &:= \hat{u}_0(\xi)\partial_t h(t,\xi),\\
\nu_1(t,\xi) &:= P_1 e^{-\frac{t|\xi|^2}{2}}\frac{\sin(t|\xi|)}{|\xi|},\\
\nu_2(t,\xi) &:= P_0 e^{-\frac{t|\xi|^2}{2}}\cos(t|\xi|).
\end{aligned}$$

Then $\hat{u}(t,\xi) = \mu_1(t,\xi) + \mu_2(t,\xi)$. With the definition $\nu(t,\xi) := \nu_1(t,\xi) + \nu_2(t,\xi)$, we wish to find the asymptotic expansions of

$$\|u(t,\cdot)\|_2^2 \quad \text{and} \quad \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 \qquad (t \to \infty).$$

Indeed $\nu(t,\xi)$ is the profile determined by Ikehata in his paper [4]. His main result from that paper concerns the asymptotic behavior of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$.

Theorem (Ikehata, 2014 [4])**.** *Let $N \in \mathbb{N}$. Assume the initial data of (1.1) satisfy $u_0 \in H^1(\mathbb{R}^N) \cap L^{1,1}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,1}(\mathbb{R}^N)$. Then there exist constants $\alpha > 0$ and $C > 0$ such that as $t \to \infty$*

$$\begin{aligned}
\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 \le C\Big((\|u_1\|_1^2 + \|u_0\|_1^2 + \|u_1\|_{L^{1,1}(\mathbb{R}^N)}^2)t^{-\frac{N}{2}} + \|u_0\|_{L^{1,1}(\mathbb{R}^N)}^2 t^{-\frac{N}{2}-1}\\
+ e^{-\alpha t}(\|u_1\|_2^2 + \|u_0\|_2^2)\Big).
\end{aligned}$$

Additionally, in [4] Ikehata cites the paper [5], which he wrote with Natsume, to provide the asymptotic behavior of $\|u(t,\cdot)\|_2^2$. We give this result as a theorem as well.

Theorem (Ikehata and Natsume, 2012 [5])**.** *Let $N \geq 3$ be an integer. Assume the initial data of (1.1) satisfy $u_0 \in H^1(\mathbb{R}^N) \cap L^{1,1}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,1}(\mathbb{R}^N)$. Then there exist constants $\eta > 0$ and $C > 0$ such that as $t \to \infty$*

$$\|u(t,\cdot)\|_2^2 \leq C\Big(|P_1|^2(1+t)^{-\frac{N}{2}+1} + (|P_0|^2 + \|u_1\|^2_{L^{1,1}(\mathbb{R}^N)})(1+t)^{-\frac{N}{2}} + \|u_0\|^2_{L^{1,1}(\mathbb{R}^N)}(1+t)^{-\frac{N}{2}-1} + e^{-\eta t}(\|u_0\|_2^2 + \|u_1\|_2^2)\Big).$$

In fact, Ikehata and Onodera later proved in their paper [6] that, in space dimension $N \in \{1, 2\}$, there are sharp *growth* rates as $t \to \infty$

$$\|u(t,\cdot)\|_2^2 = O(\log(t)) \text{ in dimension } N = 2$$
$$\|u(t,\cdot)\|_2^2 = O(t) \text{ in dimension } N = 1,$$

where $\log(t)$ denotes the natural logarithm throughout this thesis.

As stated in section 1.2, we wish to extend the results of Ikehata [4], Ikehata−Natsume [5], and Ikehata−Onodera [6] given above to full asymptotic expansions. To do so, we use the fact that

$$\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = \int_{\mathbb{R}^N} |\hat{u}(t,\xi) - \nu(t,\xi)|^2 d\xi,$$
$$= \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_{2,\epsilon}^2 + \int_{|\xi|\geq\epsilon} |\hat{u}(t,\xi) - \nu(t,\xi)|^2 d\xi,$$

where $\epsilon > 0$ is an arbitrary constant. In the proof of the main theorem from [4], Ikehata proved the following fact, which is helpful in determining the desired expansions. We state the fact as a lemma.

Lemma (Ikehata, 2014 [4])**.** *Let $\epsilon > 0$ be given. Then there exists some $\eta_1 > 0$ depending on $\epsilon > 0$ such that as $t \to \infty$*

$$\int_{|\xi|\geq\epsilon} |\hat{u}(t,\xi)|^2 d\xi = O(e^{-\eta_1 t}).$$

Furthermore, it is a routine exercise to verify that for any $\epsilon > 0$ and $t \geq 1$,

$$\int_{|\xi|\geq\epsilon} |\nu(t,\xi)|^2 d\xi = O(e^{-\frac{t\epsilon^2}{2}}).$$

Therefore for all $\epsilon > 0$, let $\eta_1 > 0$ be as given in the preceding lemma. Then as $t \to \infty$

$$\int_{|\xi|\geq\epsilon} |\hat{u}(t,\xi) - \nu(t,\xi)|^2 d\xi \leq 2\left(\int_{|\xi|\geq\epsilon} |\hat{u}(t,\xi)|^2 d\xi + \int_{|\xi|\geq\epsilon} |\nu(t,\xi)|^2 d\xi\right) = O(e^{-\eta t}),$$

where $\eta = \min\{\eta_1, \frac{\epsilon^2}{2}\}$. We have shown that for all $\epsilon > 0$ there is some $\eta > 0$ such that

$$\begin{aligned} \|u(t,\cdot)\|_2^2 &= \|\hat{u}(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_{2,\epsilon}^2 + O(e^{-\eta t}) \qquad (t \to \infty), \\ \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 &= \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_{2,\epsilon}^2 + O(e^{-\eta t}) \qquad (t \to \infty). \end{aligned}$$

Thus all the interesting asymptotic behaviors of $\|u(t,\cdot)\|_2^2$ and $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ are captured in any ϵ-neighborhood of the origin. We let ϵ equal the fixed value of $0 < \delta < 1$, and hence we are interested in the expansions of

$$\|\hat{u}(t,\cdot)\|_{2,\delta}^2 \quad \text{and} \quad \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_{2,\delta}^2 \qquad (t \to \infty). \tag{1.12}$$

We will compute (1.12) by noting $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_{2,\delta}^2 = X(t) + Y(t) + Z(t)$, where

$$X(t) = \|\mu_1(t,\cdot) - \nu_1(t,\cdot)\|_{2,\delta}^2, \tag{1.13}$$

$$Y(t) = \|\mu_2(t,\cdot) - \nu_2(t,\cdot)\|_{2,\delta}^2, \tag{1.14}$$

$$Z(t) = 2\Re\langle \mu_1(t,\cdot) - \nu_1(t,\cdot), \mu_2(t,\cdot) - \nu_2(t,\cdot)\rangle_{2,\delta}. \tag{1.15}$$

From here we write $X(t) = X_1(t) + X_2(t) - X_3(t)$, $Y(t) = Y_1(t) + Y_2(t) - Y_3(t)$, and $Z(t) = Z_1(t) - Z_2(t) - Z_3(t) + Z_4(t)$, where

$$X_1(t) = \|\mu_1(t,\cdot)\|_{2,\delta}^2, \quad X_2(t) = \|\nu_1(t,\cdot)\|_{2,\delta}^2, \quad X_3(t) = 2\Re\langle \mu_1(t,\cdot), \nu_1(t,\cdot)\rangle_{2,\delta};$$

$$Y_1(t) = \|\mu_2(t,\cdot)\|_{2,\delta}^2, \quad Y_2(t) = \|\nu_2(t,\cdot)\|_{2,\delta}^2, \quad Y_3(t) = 2\Re\langle \mu_2(t,\cdot), \nu_2(t,\cdot)\rangle_{2,\delta};$$

$$Z_1(t) = 2\Re\langle \mu_1(t,\cdot), \mu_2(t,\cdot)\rangle_{2,\delta}, \quad Z_2(t) = 2\Re\langle \mu_1(t,\cdot), \nu_2(t,\cdot)\rangle_{2,\delta},$$

$$Z_3(t) = 2\Re\langle \nu_1(t,\cdot), \mu_2(t,\cdot)\rangle_{2,\delta}, \quad Z_4(t) = 2\Re\langle \nu_1(t,\cdot), \nu_2(t,\cdot)\rangle_{2,\delta}.$$

We also see that $\|\hat{u}(t,\cdot)\|_{2,\delta}^2 = X_1(t) + Y_1(t) + Z_1(t)$.

Theorem 1. *Let $N, K \in \mathbb{N}$. Let $u(t,x)$ be the weak solution of (1.1) under the Fourier transform with the initial data $u_0(x)$ and $u_1(x)$ satisfying $u_0 \in H^1(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$.*

1. *If $N \geq 3$, then*

$$\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_2^2 = \sum_{j=0}^{K-1} W_j t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}) \qquad (t \to \infty),$$

2. *If $N = 2$, then*

$$\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_2^2 = \frac{\pi|P_1|^2}{2}\ln(t) + \sum_{j=0}^{K-1} W_j t^{-j} + O(t^{-K}) \qquad (t \to \infty),$$

3. *If $N = 1$, then*

$$\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_2^2 = \pi|P_1|^2 t + \pi\Re(P_1\bar{P}_0) + \sum_{j=0}^{K-1} W_j t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty).$$

In all three cases the W_j ($j \in \{0,\ldots,K-1\}$) are coefficients dependent on the space dimension N and the initial data $u_0(x)$ and $u_1(x)$.

Theorem 2. *Let $N, K \in \mathbb{N}$. Let $u(t,x)$ be the weak solution of (1.1) under the Fourier transform with the initial data $u_0(x)$ and $u_1(x)$ satisfying $u_0 \in H^1(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$ and*

$u_1 \in L^2(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$. *Consider the asymptotic profile of* $\hat{u}(t,\xi)$ *found by Ikehata in [4]:*

$$\nu(t,\xi) = P_1 e^{-\frac{t|\xi|^2}{2}} \frac{\sin(t|\xi|)}{|\xi|} + P_0 e^{-\frac{t|\xi|^2}{2}} \cos(t|\xi|),$$

where $P_0 = (2\pi)^{-N/2} \int_{\mathbb{R}^N} u_0(x)dx$ *and* $P_1 = (2\pi)^{-N/2} \int_{\mathbb{R}^N} u_1(x)dx$. *Then as* $t \to \infty$

$$\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = \sum_{j=1}^{K-1} V_j t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}),$$

where the V_j $(j \in \{1,\dots,K-1\})$ *are coefficients dependent on the space dimension* N *and the initial data* $u_0(x)$ *and* $u_1(x)$.

In the process of proving Theorem 2 we will find the asymptotic expansions of $X_2(t)$, $Y_2(t)$, and $Z_4(t)$. Since $\|\nu(t,\cdot)\|_2^2 = \|\nu(t,\cdot)\|_{2,\delta}^2 + O(e^{-t\delta^2/2}) = X_2(t) + Y_2(t) + Z_4(t) + O(e^{-t\delta^2/2})$, we obtain the expansion of the squared L^2-norm of the asymptotic profile $\nu(t,\xi)$.

Corollary 1. *Let* $N, K \in \mathbb{N}$. *Let* $\nu(t,\xi)$ *be the asymptotic profile of* $\hat{u}(t,\xi)$ *found by Ikehata in [4].*

1. *If* $N \geq 3$, *then*

$$\|\nu(t,\cdot)\|_2^2 = \sum_{j=0}^{K-1} U_j t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}) \qquad (t \to \infty),$$

2. *If* $N = 2$, *then*

$$\|\nu(t,\cdot)\|_2^2 = \frac{\pi |P_1|^2}{2} \ln(t) + \sum_{j=0}^{K-1} U_j t^{-j} + O(t^{-K}) \qquad (t \to \infty),$$

3. *If* $N = 1$, *then*

$$\|\nu(t,\cdot)\|_2^2 = \pi |P_1|^2 t + \pi \Re(P_1 \bar{P}_0) + \sum_{j=0}^{K-1} U_j t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty).$$

In all three cases the U_j $(j \in \{0, \ldots, K-1\})$ *are coefficients dependent on the space dimension* N *and the initial data* $u_0(x) \in H^1(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$ *and* $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$ *of (1.1).*

The rest of this thesis is divided into four more chapters and an appendix. With Chapter 2 we prove Theorem 1, and with Chapter 3 we prove Theorem 2. Each of these chapters is broken down according to dimension, as the proof techniques vary by dimension. Additionally any necessary lemmas are collected before proceeding with the proof, and recalled as needed. The proofs of the cases of Corollary 1 are part of the proof of Theorem 2 and are briefly mentioned when complete. Chapter 4 gives three examples of the theorems at work. Chapter 5 concludes this thesis and considers the techniques used and how they might be applied to similar problems. Finally, the Appendix contains *Mathematica* code used in computing some coefficients.

Chapter 2

Proof of Theorem 1

The proof will be broken into three cases. We begin with the case that the space dimension is $N \geq 3$. Then we analyze the $N = 2$ case, and finally the $N = 1$ case. Each of these cases requires lemmas that provide the asymptotic expansions of integrals that will be encountered in the analysis that follows.

2.1 The space dimension $N \geq 3$ case

2.1.1 Auxiliary lemmas

For $0 < \epsilon < 2$, $m \in \mathbb{N}_0$, and $t > 0$ define

$$G^{\epsilon}_{1,m}(t) := \int_0^{\epsilon} r^m \exp(-tr^2 - irt\sqrt{4 - r^2})dr. \tag{2.1}$$

$G^{\epsilon}_{1,m}(t)$ will be treated using the small perturbation method found on p. 96 in Chapter 5 of de Bruijn's Asymptotic Methods in Analysis [1]. This method can be described as follows. Suppose the asymptotic expansion of an integral

$$\int_a^b r^m f(t, r)dr \qquad (m \in \mathbb{Z},\ t \to c)$$

is known, and some function $g(t,r) = \sum_{k=0}^{J-1} g_k(t) r^k + O(r^J)$ can be expanded in a Taylor polynomial with remainder for $r \in (a,b)$. Then the identity

$$\int_a^b r^m f(t,r) g(t,r) dr = \sum_{k=0}^{J-1} g_k(t) \int_a^b r^{m+k} f(t,r) dr + \int_a^b O(r^{m+J}) f(t,r) dr$$

can be used to determine the asymptotic expansion of $\int_a^b r^m f(t,r) g(t,r) dr$ as $t \to c$ by estimating the integral with the O-term and using what is already known about the asymptotics of the integral $\int_a^b r^{m+k} f(t,r) dr$.

With

$$g(t,r) := \exp(irt(2 - \sqrt{4-r^2})), \tag{2.2}$$

we rewrite (2.1) as

$$G_{1,m}^{\epsilon}(t) = \int_0^{\epsilon} r^m \exp(-tr^2 - 2irt) g(t,r) dr. \tag{2.3}$$

We consider the Taylor expansion of $g(t,r)$ at $r=0$, $g(t,r) = \sum_{k=0}^{\infty} g_k(t) r^k$, where $g_k(t)$ is a polynomial in t of degree at most $\frac{k}{3}$ (to be shown). The first few terms are

$$g(t,r) = 1 + \frac{it}{4} r^3 + \frac{it}{64} r^5 - \frac{t^2}{32} r^6 + \frac{it}{512} r^7 - \frac{t^2}{256} r^8 + \left(\frac{5it}{16384} - \frac{it^3}{384} \right) r^9 + \dots .$$

We need the following lemma.

Lemma 1. *Let $J \in \mathbb{N}_0$ and $0 < \epsilon < 2$. Then there exists $C_J > 0$ such that*

$$\tilde{g}_J(t,r) := g(t,r) - \sum_{k=0}^{3J-1} g_k(t) r^k \tag{2.4}$$

satisfies $|\tilde{g}_J(t,r)| \leq C_J t^J r^{3J}$ for $0 \leq r \leq \epsilon$ and $t \geq 1$.

Proof. Let $0 < \epsilon < 2$. If $J = 0$, then $\tilde{g}_J(t,r) = g(t,r)$ and the result holds since $|g(t,r)| \leq 1$ for $0 \leq r \leq \epsilon$. If $J \in \mathbb{N}$, we let $f(r) := r(2 - \sqrt{4-r^2})$. Then $g(t,r) = e^{i(tf(r))}$. For every

$x \geq 0$

$$\begin{aligned}\left|e^{ix} - \sum_{k=0}^{J-1} \frac{1}{k!}(ix)^k\right| &= \left|\int_0^x \frac{1}{(J-1)!} i^J e^{is}(x-s)^{J-1} ds\right| \\ &\leq \frac{1}{(J-1)!}\int_0^x (x-s)^{J-1} ds \\ &= \frac{1}{J!}x^J.\end{aligned}$$

Hence for $t \geq 0$ and $0 \leq r \leq \epsilon$,

$$\left|g(t,r) - \sum_{k=0}^{J-1} \frac{1}{k!}(itf(r))^k\right| \leq \frac{t^J (f(r))^J}{J!}. \tag{2.5}$$

Now for every $k \in \{0, \dots, J-1\}$, use the Taylor expansion $(f(r))^k = f_k(r) + \tilde{f}_k(r)$, where $f_k(r)$ is a polynomial in r of degree at most $3J-1$ and $|\tilde{f}_k(r)| \leq C_k r^{3J}$ for $0 \leq r \leq \epsilon$. We substitute the Taylor expansion for $(f(r))^k$ into (2.5) and note that $0 \leq f(r) \leq r^3$ for $0 \leq r \leq \epsilon$ to obtain

$$\left|g(t,r) - \sum_{k=0}^{J-1} \frac{1}{k!} i^k t^k (f_k(r) + \tilde{f}_k(r))\right| \leq \frac{1}{J!} t^J r^{3J}.$$

By the reverse triangle inequality

$$\left|g(t,r) - \sum_{k=0}^{J-1} \frac{1}{k!} i^k t^k f_k(r)\right| - \left|\sum_{k=0}^{J-1} \frac{1}{k!} i^k t^k \tilde{f}_k(r)\right| \leq \frac{1}{J!} t^J r^{3J},$$

which implies

$$\left|g(t,r) - \sum_{k=0}^{J-1} \frac{1}{k!} i^k t^k f_k(r)\right| \leq \sum_{k=0}^{J-1} \frac{1}{k!} t^k C_k r^{3J} + \frac{1}{J!} t^J r^{3J}.$$

Letting $t \geq 1$ gives the desired result. Note that we've also shown that $\deg(g_k(t)) \leq \frac{k}{3}$. $\square$

We now prove the following proposition.

Proposition 1. *Let $0 < \epsilon < 2$ and $m, Q \in \mathbb{N}_0$. Then as $t \to \infty$*

$$G^{\epsilon}_{1,m}(t) = \sum_{n=0}^{Q-1} B_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}),$$

where, for $n \in \{0, \ldots, Q-1\}$,

$$B_{m,n} \in \begin{cases} \mathbb{R} & m \in \mathbb{N} \text{ odd} \\ i\mathbb{R} & m \in \mathbb{N}_0 \text{ even.} \end{cases}$$

Proof. Let $0 < \epsilon < 2$ and $m, Q \in \mathbb{N}_0$. We define for $k, J \in \mathbb{N}_0$

$$F^{\epsilon}_k(t) := \int_0^{\epsilon} r^k \exp(-tr^2 - 2irt) dr \tag{2.6}$$

and

$$\tilde{G}^{\epsilon}_{1,k,J}(t) := \int_0^{\epsilon} r^k \exp(-tr^2 - 2irt) \tilde{g}_J(t, r) dr.$$

Thus we obtain from (2.3) and (2.4)

$$G^{\epsilon}_{1,m}(t) = \sum_{k=0}^{3J-1} g_k(t) F^{\epsilon}_{m+k}(t) + \tilde{G}^{\epsilon}_{1,m,J}(t). \tag{2.7}$$

By Lemma 1, for $t \geq 1$

$$|\tilde{G}^{\epsilon}_{1,m,J}(t)| \leq C_J t^J \int_0^{\infty} r^{m+3J} e^{-tr^2} dr = \frac{1}{2} C_J \Gamma(\tfrac{m+3J+1}{2}) t^{-\frac{m+J+1}{2}}. \tag{2.8}$$

We now seek an asymptotic expansion of $F^{\epsilon}_k(t)$ for $k \in \mathbb{N}_0$. Note that

$$F^{\epsilon}_k(t) = \left(\int_0^{\infty} - \int_{\epsilon}^{\infty} \right) r^k \exp(-tr^2 - 2irt) dr;$$

the former term we denote by $\tilde{F}_k(t)$, and the latter term is $O(e^{-t\epsilon^2/2})$ as $t \to \infty$. So the asymptotic expansions of $F^{\epsilon}_k(t)$ and $\tilde{F}_k(t)$ will be the same as long as $\tilde{F}_k(t)$ exhibits at most sub-exponential decay.

For $\Re(\lambda) < 0$ and $z \in \mathbb{C}$, let $D_\lambda(z)$ denote the parabolic cylinder function

$$D_\lambda(z) := \frac{e^{-\frac{z^2}{4}}}{\Gamma(-\lambda)} \int_0^\infty s^{-\lambda-1} \exp\left(-\frac{s^2}{2} - zs\right) ds$$

as found on p. 328 in Chapter 8 of [10] by Magnus, Oberhettinger, and Soni. With the substitution $r = \frac{s}{\sqrt{2t}}$ in $\tilde{F}_k(t)$, we obtain

$$\begin{aligned} \tilde{F}_k(t) &= (2t)^{-\frac{k+1}{2}} \int_0^\infty s^k \exp\left(-\frac{s^2}{2} - is\sqrt{2t}\right) ds \\ &= (2t)^{-\frac{k+1}{2}} e^{-\frac{t}{2}} k!\, D_{-k-1}(i\sqrt{2t}). \end{aligned}$$

From p. 331 in Chapter 8 of [10], we know the asymptotic expansion of $D_\lambda(z)$ as $|z| \to \infty$ in the set $\{z \in \mathbb{C} \mid |\arg(z)| < \frac{3\pi}{4}\} \subseteq \mathbb{C}$, and obtain

$$F_k^\epsilon(t) = \tilde{F}_k(t) + O(e^{-t\epsilon^2/2}) = \sum_{p=0}^{P-1} A_{k,p} t^{-k-p-1} + O(t^{-k-P-1}) \tag{2.9}$$

as $t \to \infty$, where $P \in \mathbb{N}_0$ and

$$A_{k,p} = \frac{k!}{(2i)^{k+1}} \frac{\left(\frac{k+1}{2}\right)_p \left(\frac{k+2}{2}\right)_p}{p!} \tag{2.10}$$

with the Pochhammer symbol $(a)_p$ denoting the rising factorial

$$(a)_p = \begin{cases} 1 & p = 0 \\ a(a+1) \cdot \ldots \cdot (a+p-1) & p \in \mathbb{N}. \end{cases}$$

Our goal is to find an asymptotic expansion of $G_{1,m}^\epsilon(t)$ with $Q \in \mathbb{N}_0$ explicit terms and then a O-term. Combining (2.7) and (2.9), we see that the leading term is of order t^{-m-1}, followed by order $t^{-m-2}, t^{-m-3}, \ldots$. Thus, we want the error term to be $O(t^{-m-Q-1})$. By the estimate (2.8), we know the error term $\tilde{G}^{\epsilon}_{1,m,J}(t)$ of $G^{\epsilon}_{1,m}(t)$ is of order $t^{-(m+J+1)/2}$. So we want $-m-Q-1 = -\frac{m+J+1}{2}$, and thus choose $J = J_{m,Q} := m+2Q+1$.

By substituting (2.9) into (2.7), as $t \to \infty$ we have

$$G^{\epsilon}_{1,m}(t) = \sum_{k=0}^{3J-1} g_k(t) \left(\sum_{p=0}^{P-1} A_{m+k,p} t^{-m-k-p-1} + O(t^{-m-k-P-1}) \right) + O(t^{-m-Q-1}) \tag{2.11}$$

$$= \sum_{n=0}^{Q-1} B_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}). \tag{2.12}$$

To compute the $B_{m,n}$ we first recall $g(t,r) := \exp(irt(2-\sqrt{4-r^2})) = \sum_{k=0}^{\infty} g_k(t) r^k$, if we expand in a Taylor series at $r=0$. We determined in Lemma 1 that for each $k \in \mathbb{N}_0$, $\deg(g_k(t)) \le \lfloor \frac{k}{3} \rfloor$. In fact, we can say more about the coefficients of each $g_k(t)$. We instead write

$$g(t,r) = \sum_{n=0}^{\infty} \frac{1}{n!}(irt(2-\sqrt{4-r^2}))^n = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}(r(2-\sqrt{4-r^2}))^n.$$

We then consider the Taylor expansion at $r=0$

$$r(2-\sqrt{4-r^2}) = 2r(1-\sqrt{1-r^2/4}) = \sum_{m=1}^{\infty} \frac{(2m-3)!!}{m!\, 2^{3m-1}} r^{2m+1} = \frac{1}{4} r^3 + \frac{1}{64} r^5 + \dots,$$

where we adopt the convention $(-1)!! = 1$ and for any $m \in \mathbb{N}_0$, $m!!$ denotes the double factorial

$$m!! = \begin{cases} 1 & m = 0 \\ m(m-2)\cdot \dots \cdot (3)(1) & m \in \mathbb{N} \text{ odd} \\ m(m-2)\cdot \dots \cdot (4)(2) & m \in \mathbb{N} \text{ even.} \end{cases} \tag{2.13}$$

So the preceding Taylor series contains only odd powers of r of degree at least 3 with positive coefficients. (By plugging the Taylor expansion into the above expansion of $g(t,r)$ we see another justification for the claim $\deg(g_k(t)) \le \frac{k}{3}$.)

Obtaining (2.12) from (2.11) is a matter of having the proper ranges on the indices k and p and then rearranging terms according to the powers of t. To determine the range of the index k, we observe that any monomial in $g_k(t)$ is of the form $a(it)^l$ for some $0 \le l \le \frac{k}{3}$ and $a \in \mathbb{R}$. For each $0 \le p \le P-1$ the contribution of this monomial to (2.11) is a monomial of degree $l-m-k-p-1$. So that this monomial is not absorbed into the $O(t^{-m-Q-1})$-term of (2.11) we require $l-m-k-p-1 \ge -m-Q$, which implies $-\frac{2}{3}k-p-1 \ge -Q$ since $0 \le l \le \frac{k}{3}$. This implies $k \le \frac{3}{2}(Q-1) - \frac{3}{2}p \le \frac{3}{2}(Q-1)$. Thus the range on the index k is $0 \le k \le \lfloor \frac{3}{2}(Q-1) \rfloor$.

Now let $0 \le k \le \lfloor \frac{3}{2}(Q-1) \rfloor$. The fact that any monomial in $g_k(t)$ is of the form $a(it)^l$ for some $0 \le l \le \frac{k}{3}$ and $a \in \mathbb{R}$ can help us find the range on the index p as well. Start with the contribution of that monomial to (2.11) as before. With the same reasoning we require $-\frac{2}{3}k-p-1 \ge -Q$, which implies $p \le Q - \frac{2}{3}k - 1$. Therefore the range on the index p is $0 \le p \le P-1$ where $P = P_k := \lfloor Q - \frac{2}{3}k \rfloor$.

Using a computer application such as *Mathematica*, one can write a program to compute as many $B_{m,n}$ as desired. For any $m \in \mathbb{N}_0$ we have

$$\begin{aligned} B_{m,0} &= A_{m,0} \\ B_{m,1} &= A_{m,1} \\ B_{m,2} &= A_{m,2} + \frac{i}{4} A_{m+3,0}. \end{aligned} \tag{2.14}$$

Lastly we verify that for $m, n \in \mathbb{N}_0$

$$B_{m,n} \in \begin{cases} \mathbb{R} & m \in \mathbb{N} \text{ odd} \\ i\mathbb{R} & m \in \mathbb{N}_0 \text{ even.} \end{cases}$$

For each k, $g_k(t)$ is an even or odd polynomial in t with k. (To see this, we use the fact that $g(-t,r) = g(t,-r)$ to compare the coefficients of the Taylor expansions of both sides in a small ϵ-neighborhood of $r = 0$.) Using the definition of $A_{k,p}$ in (2.10) we see that $i^{m+k+1} A_{m+k,p} > 0$, and so any term of the inner sum of (2.11) is of the form $b i^{-m-k-1} t^{-m-k-p-1}$ where $b > 0$. We recall that any monomial of $g_k(t)$ is of the form $a(it)^l$, where $a \in \mathbb{R}$ and $0 \leq l \leq \lfloor \frac{k}{3} \rfloor$ is even or odd with k (since $g_k(t)$ is even or odd with k). Thus the contribution of (2.11) to (2.12) is of the form

$$a(it)^l b i^{-m-k-1} t^{-m-k-p-1} = abi^{l-k} i^{-m-1} t^{-m-(k+p-l)-1}.$$

Since l and k are either both even or both odd, $l-k$ is even. The contribution is just

$$c i^{-m-1} t^{-m-(k+p-l)-1} \tag{2.15}$$

where $c \in \mathbb{R}$. The coefficient ci^{-m-1} of (2.15) is thus a contributing term of $B_{m,n}$ with $n = k+p-l$, and any other such term of $B_{m,n}$ will also be of the form ci^{-m-1} with varying $c \in \mathbb{R}$. Therefore for each $m \in \mathbb{N}_0$, $i^{m+1} B_{m,n} \in \mathbb{R}$ and we deduce what was claimed. □

Lemma 2. *For $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$ define*

$$I^{\epsilon}_{1,m}(t) := \int_0^{\epsilon} r^m e^{-tr^2} \sin^2(tr\sqrt{1-r^2/4}) dr. \tag{2.16}$$

1. *If $m \in \mathbb{N}_0$ is even, then as $t \to \infty$*

$$I^{\epsilon}_{1,m}(t) = \frac{1}{4}\Gamma(\tfrac{m+1}{2}) t^{-\frac{m}{2}-\frac{1}{2}} + O(t^{-m-Q-1}),$$

   where $Q \in \mathbb{N}_0$.

2. *If $m \in \mathbb{N}$ is odd, then as $t \to \infty$*

$$I^{\epsilon}_{1,m}(t) = \frac{1}{4}\Gamma(\tfrac{m+1}{2}) t^{-\frac{m}{2}-\frac{1}{2}} - \sum_{n=0}^{Q-1} \frac{1}{2} B_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}),$$

   where $Q \in \mathbb{N}_0$.

Proof. Using the identity $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$, we obtain

$$I^{\epsilon}_{1,m}(t) = \frac{1}{2}\int_0^{\epsilon} r^m e^{-tr^2} dr - \frac{1}{2}\int_0^{\epsilon} r^m e^{-tr^2}\cos(tr\sqrt{4-r^2})dr. \tag{2.17}$$

We note $\int_0^{\epsilon} r^m e^{-tr^2} dr = \int_0^{\infty} r^m e^{-tr^2} dr - \int_{\epsilon}^{\infty} r^m e^{-tr^2} dr$, the latter term of which is $O(e^{-t\epsilon^2/2})$ for all $t \geq 1$. Further, $\int_0^{\infty} r^m e^{-tr^2} dr = \frac{1}{2}\Gamma(\frac{m+1}{2})t^{-m/2-1/2}$. We also observe that $\int_0^{\epsilon} r^m e^{-tr^2}\cos(tr\sqrt{4-r^2})dr = \Re(G^{\epsilon}_{1,m}(t))$. By Proposition 1, for $t \geq 1$ we may rewrite (2.17):

$$\begin{aligned}
I^{\epsilon}_{1,m}(t) &= \frac{1}{2}\left(\int_0^{\infty} r^m e^{-tr^2} dr - \int_{\epsilon}^{\infty} r^m e^{-tr^2} dr\right) - \frac{1}{2}\Re(G^{\epsilon}_{1,m}(t)) \\
&= \frac{1}{2}\left(\frac{1}{2}\Gamma(\tfrac{m+1}{2})t^{-\frac{m}{2}-\frac{1}{2}} + O(e^{-\frac{t\epsilon^2}{2}})\right) - \frac{1}{2}\Re\left(\sum_{n=0}^{Q-1} B_{m,n}t^{-m-n-1} + O(t^{-m-Q-1})\right) \\
&= \frac{1}{4}\Gamma(\tfrac{m+1}{2})t^{-\frac{m}{2}-\frac{1}{2}} - \sum_{n=0}^{Q-1}\frac{1}{2}\Re(B_{m,n})t^{-m-n-1} + O(t^{-m-Q-1}),
\end{aligned} \tag{2.18}$$

for any $Q \in \mathbb{N}_0$. We complete the proof by using that fact that $B_{m,n} \in \mathbb{R}$ if $m \in \mathbb{N}$ is odd and $B_{m,n} \in i\mathbb{R}$ if $m \in \mathbb{N}_0$ is even. □

Lemma 3. *For $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$ define*

$$I^{\epsilon}_{2,m}(t) := \int_0^{\epsilon} r^m e^{-tr^2}\cos^2(tr\sqrt{1-r^2/4})dr.$$

1. *If $m \in \mathbb{N}_0$ is even, then as $t \to \infty$*

$$I^{\epsilon}_{1,m}(t) = \frac{1}{4}\Gamma(\tfrac{m+1}{2})t^{-\frac{m}{2}-\frac{1}{2}} + O(t^{-m-Q-1}),$$

where $Q \in \mathbb{N}_0$.

2. *If $m \in \mathbb{N}$ is odd, then as $t \to \infty$*

$$I^{\epsilon}_{1,m}(t) = \frac{1}{4}\Gamma(\tfrac{m+1}{2})t^{-\frac{m}{2}-\frac{1}{2}} + \sum_{n=0}^{Q-1}\frac{1}{2}B_{m,n}t^{-m-n-1} + O(t^{-m-Q-1}),$$

where $Q \in \mathbb{N}_0$.

Proof. The proof is the same as the proof of Lemma 2, but instead we use the identity $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$. □

Lemma 4. *For* $0 < \epsilon < 2$ *and* $m \in \mathbb{N}_0$ *define*

$$I^{\epsilon}_{3,m}(t) := \int_0^{\epsilon} r^m e^{-tr^2} \sin(tr\sqrt{1 - r^2/4}) \cos(tr\sqrt{1 - r^2/4}) dr.$$

1. *If* $m \in \mathbb{N}_0$ *is even, then as* $t \to \infty$

$$I^{\epsilon}_{3,m}(t) = \sum_{n=0}^{Q-1} -\frac{1}{2}\Im(B_{m,n})t^{-m-n-1} + O(t^{-m-Q-1}),$$

where $Q \in \mathbb{N}_0$.

2. *If* $m \in \mathbb{N}$ *is odd, then as* $t \to \infty$

$$I^{\epsilon}_{3,m}(t) = O(t^{-m-Q-1}),$$

where $Q \in \mathbb{N}_0$.

Proof. Using Proposition 1 and the identity $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$ we obtain

$$\begin{aligned} I^{\epsilon}_{3,m}(t) &= \frac{1}{2}\int_0^{\epsilon} r^m e^{-tr^2} \sin(tr\sqrt{4 - r^2}) dr \\ &= -\frac{1}{2}\Im(G^{\epsilon}_{1,m}(t)) \\ &= -\frac{1}{2}\Im\left(\sum_{n=0}^{Q-1} B_{m,n}t^{-m-n-1} + O(t^{-m-Q-1})\right) \qquad (t \geq 1) \\ &= \sum_{n=0}^{Q-1} -\frac{1}{2}\Im(B_{m,n})t^{-m-n-1} + O(t^{-m-Q-1}) \qquad (t \geq 1). \end{aligned}$$

for any $Q \in \mathbb{N}_0$. We complete the proof by using that fact that $B_{m,n} \in \mathbb{R}$ if $m \in \mathbb{N}$ is odd and $B_{m,n} \in i\mathbb{R}$ if $m \in \mathbb{N}_0$ is even. □

2.1.2 Intermediate computations

We are now in a position to find the asymptotic expansion of $\|u(t,\cdot)\|_2^2$ as $t\to\infty$. Let us first recall that since the Fourier transform as defined is an $L^2(\mathbb{R}^N)$-isometry, $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_2^2$. Also recall that with the fixed $0<\delta<1$, $\|\hat{u}(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_{2,\delta}^2 + O(e^{-\eta t})$ as $t\to\infty$ for some $\eta>0$. Since $\|\hat{u}(t,\cdot)\|_{2,\delta}^2 = X_1(t)+Y_1(t)+Z_1(t)$, we may determine the asymptotic expansion in three steps. Over the next few sections, we will do so and arrive at the proof of Theorem 1 part 1 (i.e., the space dimension $N\geq 3$ case). Let us assume that, unless otherwise stated, $K\in\mathbb{N}$ and the initial data of (1.1) satisfy $u_0,u_1\in L^{1,2K}(\mathbb{R}^N)$ in addition to the assumptions $u_0\in H^1(\mathbb{R}^N)$, $u_1\in L^2(\mathbb{R}^N)$ given in (1.1).

Expansion of $X_1(t)$

We observe

$$X_1(t) = \int_{|\xi|\leq\delta} |\hat{u}_1(\xi)+|\xi|^2\hat{u}_0(\xi)|^2 (h(t,\xi))^2 d\xi, \tag{2.19}$$

and substitute (1.6) and (1.11) into (2.19). After simplifying, we have a sum of integrals indexed by the multi-indices σ plus an integral of a $O(|\xi|^{2K})$-term. We estimate the integral with the $O(|\xi|^{2K})$-term once, as all others may be estimated similarly.

$$\begin{aligned}\left|\int_{|\xi|\leq\delta} O(|\xi|^{2K}) e^{-t|\xi|^2}\frac{\sin^2(t|\xi|\sqrt{1-|\xi|^2/4})}{|\xi|^2(1-|\xi|^2/4)}d\xi\right| &\leq \frac{M}{1-\delta^2/4}\int_{|\xi|\leq\delta}|\xi|^{2K-2}e^{-t|\xi|^2}d\xi\\ &= \frac{M\omega_{N-1}}{1-\delta^2/4}\int_0^\delta r^{2K+N-3}e^{-tr^2}dr\\ &= O(t^{-K-\frac{N}{2}+1}) \qquad (t\to\infty)\end{aligned}$$

for some $M>0$, where ω_{N-1} is the surface area of the $(N-1)$-sphere in $\mathbb{R}^N$.

Therefore as $t\to\infty$

$$X_1(t) = \sum_{|\sigma|\leq 2K-1} d_\sigma \int_{|\xi|\leq\delta}\xi^\sigma e^{-t|\xi|^2}\frac{\sin^2(t|\xi|\sqrt{1-|\xi|^2/4})}{|\xi|^2(1-|\xi|^2/4)}d\xi + O(t^{-K-\frac{N}{2}+1}).$$

Observe that, aside from ξ^σ, the integrand of the preceding equation is radially defined in ξ. Hence if even one entry of σ is odd, the integral equals 0. So the sum can in fact only range over $|\sigma|\leq K-1$ if we replace σ by 2σ. After that we switch to polar, and hence as $t\to\infty$

$$X_1(t) = \sum_{|\sigma|\leq K-1} d_{2\sigma} D_\sigma \int_0^\delta r^{2|\sigma|+N-3} e^{-tr^2} \frac{\sin^2(tr\sqrt{1-r^2/4})}{1-r^2/4} dr + O(t^{-K-\frac{N}{2}+1}), \tag{2.20}$$

where

$$D_\sigma = 2\frac{\Gamma(\sigma_1+\frac{1}{2})\cdot\ldots\cdot\Gamma(\sigma_N+\frac{1}{2})}{\Gamma(|\sigma|+\frac{N}{2})}. \tag{2.21}$$

Since $0<\delta<1<2$ and $0\leq r\leq\delta$ we may write for $L\in\mathbb{N}_0$

$$\frac{1}{1-r^2/4} = \sum_{k=0}^{L-1} \frac{r^{2k}}{4^k} + O(r^{2L}). \tag{2.22}$$

Then for each $|\sigma|\leq K-1$ we define $L = L_\sigma := K-|\sigma|\in\mathbb{N}$. We substitute (2.22) into (2.20) and estimate the integrals with the $O(r^{2L_\sigma})$-terms (by choice of L_σ, each integral is $O(t^{-K-N/2+1})$ as $t\to\infty$). By Lemma 2 with $Q = Q_{\sigma,k} := \max\{0, \lceil K-2|\sigma|-2k-\frac{N}{2}+1\rceil\}$, we obtain as $t\to\infty$

$$\begin{aligned} X_1(t) &= \sum_{|\sigma|\leq K-1}\sum_{k=0}^{L_\sigma-1} \frac{d_{2\sigma}D_\sigma}{4^k}\int_0^\delta r^{2|\sigma|+2k+N-3}e^{-tr^2}\sin^2(tr\sqrt{1-r^2/4})dr + O(t^{-K-\frac{N}{2}+1}) \\ &= \sum_{|\sigma|\leq K-1}\sum_{k=0}^{L_\sigma-1} \frac{d_{2\sigma}D_\sigma}{4^k} I^\delta_{1,2|\sigma|+2k+N-3}(t) + O(t^{-K-\frac{N}{2}+1}) \\ &= \sum_{j=0}^{K-1}(X_{1,1}^{[j]} - X_{1,2}^{[j]})t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}), \end{aligned} \tag{2.23}$$

where

$$X_{1,1}^{[j]} = \sum_{|\sigma|+k=j} \frac{d_{2\sigma}D_\sigma}{4^{k+1}}\Gamma(j+\tfrac{N}{2}-1),$$

$$X_{1,2}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-\frac{N}{2}+1} \frac{d_{2\sigma}D_\sigma}{2^{2k+1}} B_{2|\sigma|+2k+N-3,n} & N \geq 4 \text{ even and } \frac{N}{2}-1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

We note the extra conditions for $X_{1,2}^{[j]}$. If $N \geq 3$ is odd, then $B_{2|\sigma|+N-3+2k,n} \in i\mathbb{R}$ and Lemma 2 only gives us $X_{1,1}^{[j]}$. If $N \geq 4$ is even but $K < \frac{N}{2}$, then each power of t from the expansion is already on the order $O(t^{-K-N/2+1})$.

Expansion of $Y_1(t)$

We first observe

$$Y_1(t) = \int_{|\xi|\leq\delta} |\hat{u}_0(\xi)|^2 (\partial_t h(t,\xi))^2 d\xi. \tag{2.24}$$

We then use (1.11) to obtain

$$\partial_t h(t,\xi) = e^{-\frac{t|\xi|^2}{2}} \left(\cos(t|\xi|\sqrt{1-|\xi|^2/4}) - \frac{|\xi|}{2} \frac{\sin(t|\xi|\sqrt{1-|\xi|^2/4})}{\sqrt{1-|\xi|^2/4}} \right), \tag{2.25}$$

which implies

$$(\partial_t h(t,\xi))^2 = e^{-t|\xi|^2} \Bigg(\cos^2(t|\xi|\sqrt{1-|\xi|^2/4}) + \frac{|\xi|^2}{4} \frac{\sin^2(t|\xi|\sqrt{1-|\xi|^2/4})}{1-|\xi|^2/4} - \frac{|\xi| \sin(t|\xi|\sqrt{1-|\xi|^2/4}) \cos(t|\xi|\sqrt{1-|\xi|^2/4})}{\sqrt{1-|\xi|^2/4}} \Bigg). \tag{2.26}$$

In order to obtain an asymptotic expansion of $Y_1(t)$ that is not simply $O(t^{-K-N/2+1})$ as $t \to \infty$, we must further assume that $K \geq 2$. With this additional assumption, in the neighborhood $|\xi| \leq \delta$ we now truncate (1.7) to

$$|\hat{u}_0(\xi)|^2 = \sum_{|\sigma|\leq 2K-3} f_\sigma \xi^\sigma + O(|\xi|^{2K-2}). \tag{2.27}$$

Substituting (2.26) and (2.27) into (2.24) and estimating the integral with the $O(|\xi|^{2K-2})$-

term we obtain as $t \to \infty$

$$Y_1(t) = \sum_{|\sigma| \le 2K-3} f_\sigma \int_{|\xi| \le \delta} \xi^\sigma (\partial_t h(t,\xi))^2 d\xi + O(t^{-K-\frac{N}{2}+1}). \tag{2.28}$$

Since $\partial_t h(t,\xi)$ is radially defined in ξ, if some σ satisfying $|\sigma| \le 2K-3$ has an odd entry, the integral in (2.28) vanishes. So we consider multi-indices with only even entries and switch to polar:

$$Y_1(t) = \sum_{|\sigma| \le K-2} f_{2\sigma} D_\sigma \int_0^\delta r^{2|\sigma|+N-1} (\partial_t h(t,r))^2 dr + O(t^{-K-\frac{N}{2}+1}) \qquad (t \to \infty).$$

Now writing $(\partial_t h(t,r))^2$ as in (2.26), $Y_1(t) = Y_1^A(t) + Y_1^B(t) - Y_1^C(t) + O(t^{-K-N/2+1})$ as $t \to \infty$, where

$$Y_1^A(t) = \sum_{|\sigma| \le K-2} f_{2\sigma} D_\sigma \int_0^\delta r^{2|\sigma|+N-1} e^{-tr^2} \cos^2(tr\sqrt{1-r^2/4}) dr, \tag{2.29}$$

$$Y_1^B(t) = \sum_{|\sigma| \le K-2} \frac{f_{2\sigma} D_\sigma}{4} \int_0^\delta r^{2|\sigma|+N+1} e^{-tr^2} \frac{\sin^2(tr\sqrt{1-r^2/4})}{1-r^2/4} dr, \tag{2.30}$$

$$Y_1^C(t) = \sum_{|\sigma| \le K-2} f_{2\sigma} D_\sigma \int_0^\delta r^{2|\sigma|+N} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4}) \cos(tr\sqrt{1-r^2/4})}{\sqrt{1-r^2/4}} dr. \tag{2.31}$$

We begin with $Y_1^A(t)$. By Lemma 3 with $Q = Q_\sigma := \max\{0, \lceil K - 2|\sigma| - \frac{N}{2} - 1 \rceil\}$, we obtain for $t \to \infty$

$$\begin{aligned} Y_1^A(t) &= \sum_{|\sigma| \le K-2} f_{2\sigma} D_\sigma I^\delta_{2,2|\sigma|+N-1}(t) \\ &= \sum_{j=0}^{K-1} (Y_{1,1}^{[j]} + Y_{1,2}^{[j]}) t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}), \end{aligned} \tag{2.32}$$

where

$$Y_{1,1}^{[j]} = \begin{cases} \sum_{|\sigma|=j-1} \frac{f_{2\sigma}D_\sigma}{4}\Gamma(j+\frac{N}{2}-1) & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{1,2}^{[j]} = \begin{cases} \sum_{2|\sigma|+n=j-\frac{N}{2}-1} \frac{f_{2\sigma}D_\sigma}{2} B_{2|\sigma|+N-1,n} & N \ge 4 \text{ even and } \frac{N}{2}+1 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. From the defintion of $Y_{1,1}^{[j]}$ we see the necessity of the added assumption $K \ge 2$, else no terms are contributed to the expansion.

For $Y_1^B(t)$ we first substitute (2.22) into (2.30) with $L = L_\sigma := K-|\sigma|-2$. Each resulting integral with a $O(r^{2L_\sigma})$-term is $O(t^{-K-N/2+1})$ as $t \to \infty$ by choice of L_σ. Thus by Lemma 2 with $Q = Q_{\sigma,k} := \max\{0, \lceil K-2|\sigma|-2k-\frac{N}{2}-3\rceil\}$, we have as $t \to \infty$

$$\begin{aligned} Y_1^B(t) &= \sum_{|\sigma|\le K-2} \sum_{k=0}^{L_\sigma-1} \frac{f_{2\sigma}D_\sigma}{4^{k+1}} \int_0^\delta r^{2|\sigma|+2k+N+1} e^{-tr^2} \sin^2(tr\sqrt{1-r^2/4})dr + O(t^{-K-\frac{N}{2}+1}) \\ &= \sum_{|\sigma|\le K-2} \sum_{k=0}^{L_\sigma-1} \frac{f_{2\sigma}D_\sigma}{4^{k+1}} I^\delta_{1,2|\sigma|+2k+N+1}(t) + O(t^{-K-\frac{N}{2}+1}) \\ &= \sum_{j=0}^{K-1} (Y_{1,3}^{[j]} - Y_{1,4}^{[j]}) t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}), \end{aligned} \tag{2.33}$$

where

$$Y_{1,3}^{[j]} = \begin{cases} \sum_{|\sigma|+k=j-2} \frac{f_{2\sigma}D_\sigma}{4^{k+2}}\Gamma(j+\frac{N}{2}-1) & 2 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{1,4}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-\frac{N}{2}-3} \frac{f_{2\sigma}D_\sigma}{2^{2k+3}} B_{2|\sigma|+2k+N+1,n} & N \ge 4 \text{ even and } \frac{N}{2}+3 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. It is clear from the definition of $Y_{1,3}^{[j]}$ that in order to obtain any terms in the expansion of $Y_1^B(t)$, $K \geq 3$ is in fact necessary.

Now before finding the expansion of $Y_1^C(t)$ we first expand $(1 - r^2/4)^{-1/2}$ in a Taylor series in the δ-neighborhood of $r = 0$:

$$\frac{1}{\sqrt{1 - r^2/4}} = \sum_{k=0}^{L-1} \alpha_k r^{2k} + O(r^{2L}), \tag{2.34}$$

where $L \in \mathbb{N}_0$ and, with the convention $(-1)!! = 1$,

$$\alpha_k = \frac{(2k-1)!!}{8^k \cdot k!} \qquad (k \in \mathbb{N}_0).$$

We then substitute (2.34) into (2.31) with $L = L_\sigma := K - |\sigma| - 1$. Each resulting integral with a $O(r^{2L_\sigma})$-term is $O(t^{-K-N/2+1})$ as $t \to \infty$ by choice of L_σ. Hence as $t \to \infty$

$$Y_1^C(t) = \sum_{|\sigma| \leq K-2} \sum_{k=0}^{L_\sigma - 1} f_{2\sigma} D_\sigma \alpha_k I^\delta_{3,2|\sigma|+2k+N}(t) + O(t^{-K-\frac{N}{2}+1}).$$

Then by Lemma 4 with $Q = Q_{\sigma,k} := \max\{0, \lceil K - 2|\sigma| - 2k - \frac{N}{2} - 2 \rceil\}$

$$Y_1^C(t) = \sum_{j=0}^{K-1} Y_{1,5}^{[j]} t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}) \qquad (t \to \infty), \tag{2.35}$$

where

$$Y_{1,5}^{[j]} = \begin{cases} \sum\limits_{2|\sigma|+2k+n=j-\frac{N}{2}-2} \frac{-f_{2\sigma} D_\sigma \alpha_k}{2} \Im(B_{2|\sigma|+2k+N,n}) & N \geq 4 \text{ even and } \frac{N}{2}+2 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that $Y_1(t) = Y_1^A(t) + Y_1^B(t) - Y_1^C(t) + O(t^{-K-N/2+1})$ as $t \to \infty$, we combine results (2.32), (2.33), (2.35) and obtain the expansion

$$Y_1(t) = \sum_{j=0}^{K-1} (Y_{1,1}^{[j]} + Y_{1,2}^{[j]} + Y_{1,3}^{[j]} - Y_{1,4}^{[j]} - Y_{1,5}^{[j]}) t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}) \qquad (t \to \infty). \quad (2.36)$$

Remark. Based on the above definitions of the $Y_{1,i}^{[j]}$, some $Y_{1,i}^{[j]}$ will not contribute to the above expansion for certain values of N and K. For example, if $N \geq 3$ is odd, then only $Y_{1,1}^{[j]}$ and $Y_{1,3}^{[j]}$ will contribute to the expansion. Even then, $Y_{1,3}^{[j]}$ does not contribute unless $K \geq 3$. In fact we have a similar expansion when $N \geq 4$ is even but $K \leq \frac{N}{2} + 1$. There are more cases depending on N and K, but ultimately all $Y_{1,i}^{[j]}$ will contribute to the expansion as given in (2.36) once $N \geq 4$ is even and $K \geq \frac{N}{2} + 4$.

Expansion of $Z_1(t)$

Again we need to require $K \geq 2$ in order to obtain more than just a $O(t^{-K-N/2+1})$-term for the asymptotic expansion of $Z_1(t)$. Nonetheless we begin with

$$Z_1(t) = 2\Re \left(\int_{|\xi| \leq \delta} (\hat{u}_1(\xi) + |\xi|^2 \hat{u}_0(\xi)) \bar{\hat{u}}_0(\xi) h(t,\xi) \partial_t h(t,\xi) d\xi \right). \quad (2.37)$$

We substitute (1.8), (1.11), and (2.25) into (2.37) and estimate the resulting integral with the $O(|\xi|^{2K})$-term; it is $O(t^{-K-N/2+1})$ as $t \to \infty$. Then switching to polar and using the definitions of $h(t,\xi)$ and $\partial_t h(t,\xi)$, we obtain $Z_1(t) = Z_1^A(t) - Z_1^B(t) + O(t^{-K-N/2+1})$ as $t \to \infty$, where

$$Z_1^A(t) = \sum_{|\sigma| \leq K-1} 2\Re(l_{2\sigma}) D_\sigma \int_0^\delta r^{2|\sigma|+N-2} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4}) \cos(tr\sqrt{1-r^2/4})}{\sqrt{1-r^2/4}} dr, \quad (2.38)$$

$$Z_1^B(t) = \sum_{|\sigma| \leq K-1} \Re(l_{2\sigma}) D_\sigma \int_0^\delta r^{2|\sigma|+N-1} e^{-tr^2} \frac{\sin^2(tr\sqrt{1-r^2/4})}{1-r^2/4} dr. \quad (2.39)$$

Let us begin with $Z_1^A(t)$ by substituting (2.34) in (2.38) with $L = L_\sigma := K - |\sigma|$. Each resulting integral with a $O(r^{2L_\sigma})$-term is $O(t^{-K-N/2+1})$ as $t\to\infty$. Hence as $t\to\infty$

$$Z_1^A(t)=\sum_{|\sigma|\le K-1}\sum_{k=0}^{L_\sigma-1}2\Re(l_{2\sigma})D_\sigma\alpha_k I^\delta_{3,2|\sigma|+2k+N-2}(t)+O(t^{-K-\frac{N}{2}+1}).$$

By Lemma 4 with $Q=Q_{\sigma,k}:=\max\{0,\lceil K-2|\sigma|-2k-\frac{N}{2}\rceil\}$

$$Z_1^A(t)=\sum_{j=0}^{K-1}Z_{1,1}^{[j]}t^{-j-\frac{N}{2}+1}+O(t^{-K-\frac{N}{2}+1})\qquad(t\to\infty),\tag{2.40}$$

where

$$Z_{1,1}^{[j]}=\begin{cases}\displaystyle\sum_{2|\sigma|+2k+n=j-\frac{N}{2}}-\Re(l_{2\sigma})D_\sigma\alpha_k\Im(B_{2|\sigma|+2k+N-2,n}) & N\ge 4\text{ even and }\frac{N}{2}\le j\le K-1\\ 0 & \text{otherwise.}\end{cases}$$

For $Z_1^B(t)$ we substitute (2.22) into (2.39) with $L=L_\sigma:=K-|\sigma|-1$. By choice of L_σ each resulting integral with a $O(r^{2L_\sigma})$-term is $O(t^{-K-N/2+1})$ as $t\to\infty$. Hence

$$Z_1^B(t)=\sum_{|\sigma|\le K-1}\sum_{k=0}^{L_\sigma-1}\frac{\Re(l_{2\sigma})D_\sigma}{4^k}I^\delta_{1,2|\sigma|+2k+N-1}(t)+O(t^{-K-\frac{N}{2}+1})\qquad(t\to\infty).$$

By Lemma 2 with $Q=Q_{\sigma,k}:=\max\{0,\lceil K-2|\sigma|-2k-\frac{N}{2}-1\rceil\}$

$$Z_1^B(t)=\sum_{j=0}^{K-1}(Z_{1,2}^{[j]}-Z_{1,3}^{[j]})t^{-j-\frac{N}{2}+1}+O(t^{-K-\frac{N}{2}+1})\qquad(t\to\infty),\tag{2.41}$$

where

$$Z_{1,2}^{[j]}=\begin{cases}\displaystyle\sum_{|\sigma|+k=j-1}\frac{\Re(l_{2\sigma})D_\sigma}{4^{k+1}}\Gamma(j+\frac{N}{2}-1) & 1\le j\le K-1\\ 0 & \text{otherwise,}\end{cases}$$

$$Z_{1,3}^{[j]} = \begin{cases} \displaystyle\sum_{2|\sigma|+2k+n=j-\frac{N}{2}-1} \frac{\Re(l_{2\sigma})D_\sigma}{2^{2k+1}} B_{2|\sigma|+2k+N-1,n} & N \geq 4 \text{ even and } \frac{N}{2}+1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $Z_1(t) = Z_1^A(t) - Z_1^B(t) + O(t^{-K-N/2+1})$ we combine the results (2.40) and (2.41) to obtain the expansion

$$Z_1(t) = \sum_{j=0}^{K-1} (Z_{1,1}^{[j]} - Z_{1,2}^{[j]} + Z_{1,3}^{[j]}) t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}) \qquad (t \to \infty). \tag{2.42}$$

Remark. It is worth noting that much like $Y_1(t)$ only some $Z_{1,i}^{[j]}$ actually contribute terms to the expansion depending on N and K. For example if $N \geq 3$ is odd, then only $Z_{1,2}^{[j]}$ contributes terms as long as $K \geq 2$ (which was assumed a priori). A similar thing happens if $N \geq 4$ is even but $K \leq \frac{N}{2}$. It is not until we have $N \geq 4$ even and $K \geq \frac{N}{2} + 2$ that all $Z_{1,i}^{[j]}$ will contribute to the expansion. Nonetheless, there is no harm in writing the expansion for $Z_1(t)$ as in (2.42) since the definitions of the $Z_{1,i}^{[j]}$ capture the cases mentioned in this discussion.

2.1.3 Asymptotic expansion of $\|u(t,\cdot)\|_2^2$ as $t \to \infty$

From the discussion at the beginning of section 2.1.2, $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_{2,\delta}^2 + O(e^{-\eta t})$ as $t \to \infty$ for some $\eta > 0$ dependent on the fixed $0 < \delta < 1$. Since $\|\hat{u}(t,\cdot)\|_{2,\delta}^2 = X_1(t) + Y_1(t) + Z_1(t)$, we combine results (2.23), (2.36), and (2.42) to obtain the asymptotic expansion

$$\|u(t,\cdot)\|_2^2 = \sum_{j=0}^{K-1} W_j t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}) \qquad (t \to \infty),$$

where, for $j \in \{0, \ldots, K-1\}$,

$$W_j = X_{1,1}^{[j]} - X_{1,2}^{[j]} + Y_{1,1}^{[j]} + Y_{1,2}^{[j]} + Y_{1,3}^{[j]} - Y_{1,4}^{[j]} - Y_{1,5}^{[j]} + Z_{1,1}^{[j]} - Z_{1,2}^{[j]} + Z_{1,3}^{[j]}.$$

This completes the proof of Theorem 1 part 1.

The first three coefficients for each $N \geq 3$

Let us assume that $N \geq 3$ is odd and $K = 3$. Then $\|u(t,\cdot)\|_2^2 = W_0 t^{-N/2+1} + W_1 t^{-N/2} + W_2 t^{-N/2-1} + O(t^{-N/2-2})$ as $t \to \infty$, where

$$W_0 = \frac{d_{\mathbf{0}} D_{\mathbf{0}}}{4}\Gamma(\tfrac{N}{2}-1), \qquad W_1 = \sum_{|\sigma|+k=1} \frac{d_{2\sigma} D_\sigma}{4^{k+1}}\Gamma(\tfrac{N}{2}) + \frac{f_{\mathbf{0}} D_{\mathbf{0}}}{4}\Gamma(\tfrac{N}{2}) - \frac{\Re(l_{\mathbf{0}}) D_{\mathbf{0}}}{4}\Gamma(\tfrac{N}{2}),$$

$$W_2 = \sum_{|\sigma|+k=2} \frac{d_{2\sigma} D_\sigma}{4^{k+1}}\Gamma(\tfrac{N}{2}+1) + \sum_{|\sigma|=1} \frac{f_{2\sigma} D_\sigma}{4}\Gamma(\tfrac{N}{2}+1) + \frac{f_{\mathbf{0}} D_{\mathbf{0}}}{4^2}\Gamma(\tfrac{N}{2}+1) - \sum_{|\sigma|+k=1} \frac{\Re(l_{2\sigma}) D_\sigma}{4^{k+1}}\Gamma(\tfrac{N}{2}+1).$$

It is a straightforward computation to verify that for $m \geq 1$ odd

$$\begin{aligned} D_{e_j} &= \frac{(\sqrt{2})^{m+1}\pi^{\frac{m-1}{2}}}{m!!} \qquad (j \in \{1,\dots,m\}),\\ D_{2e_j} &= \frac{3(\sqrt{2})^{m+1}\pi^{\frac{m-1}{2}}}{(m+2)!!} \qquad (j \in \{1,\dots,m\}),\\ D_{e_i+e_j} &= \frac{(\sqrt{2})^{m+1}\pi^{\frac{m-1}{2}}}{(m+2)!!} \qquad (i \neq j \in \{1,\dots,m\}), \end{aligned} \tag{2.43}$$

where $e_j = (\delta_{ij})_{i=1}^m$ is a multi-index of magnitude one, and for any $m \in \mathbb{N}_0$, $m!!$ denotes the double factorial as given in (2.13). We remark that for $m = 1$, $D_{e_i+e_j} = 0$ vacuously. We combine the results (2.43) with the identities $d_{\mathbf{0}} = |P_1|^2$, $f_{\mathbf{0}} = |P_0|^2$, $l_{\mathbf{0}} = P_1\bar{P}_0$, and $D_{\mathbf{0}} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ to simplify each W_j $(j \in \{0,1,2\})$ for dimension $N \geq 3$ odd.

$$W_0 = \frac{|P_1|^2\pi^{\frac{N}{2}}}{N-2}, \qquad W_1 = \frac{|P_1|^2\pi^{\frac{N}{2}}}{8} + \frac{|P_0|^2\pi^{\frac{N}{2}}}{2} - \frac{\Re(P_1\bar{P}_0)\pi^{\frac{N}{2}}}{2} + \frac{\pi^{\frac{N}{2}}}{2N}\sum_{j=1}^N d_{2e_j},$$

$$W_2 = \frac{N|P_1|^2\pi^{\frac{N}{2}}}{64} + \frac{N|P_0|^2\pi^{\frac{N}{2}}}{16} - \frac{N\Re(P_1\bar{P}_0)\pi^{\frac{N}{2}}}{16} + \frac{\pi^{\frac{N}{2}}}{16}\sum_{j=1}^N d_{2e_j} + \frac{\pi^{\frac{N}{2}}}{4}\sum_{j=1}^N f_{2e_j} - \frac{\pi^{\frac{N}{2}}}{4}\sum_{j=1}^N \Re(l_{2e_j}) + \frac{3\pi^{\frac{N}{2}}}{4(N+2)}\sum_{j=1}^N d_{4e_j} + \frac{\pi^{\frac{N}{2}}}{4(N+2)}\sum_{1\leq i<j\leq N} d_{2(e_i+e_j)}.$$

In fact, it is again straightforward to verify that for $m \geq 2$ even

$$
\begin{aligned}
D_{e_j} &= \frac{(2\pi)^{\frac{m}{2}}}{m!!} \qquad (j \in \{1, \ldots, m\}), \\
D_{2e_j} &= \frac{3(2\pi)^{\frac{m}{2}}}{(m+2)!!} \qquad (j \in \{1, \ldots, m\}), \\
D_{e_i+e_j} &= \frac{(2\pi)^{\frac{m}{2}}}{(m+2)!!} \qquad (i \neq j \in \{1, \ldots, m\}).
\end{aligned} \tag{2.44}
$$

Remark. If we assume that $N \geq 8$ is even, then we obtain the exact same expressions for the first $K = 3$ coefficients (W_0, W_1, and W_2) as in the $N \geq 3$ odd and $K = 3$ case above.

Let us now assume that $N = 4$ and $K = 3$. Then $\|u(t, \cdot)\|_2^2 = W_0 t^{-1} + W_1 t^{-2} + W_2 t^{-3} + O(t^{-4})$ as $t \to \infty$, where

$$
\begin{aligned}
W_0 &= \frac{d_{\mathbf{0}} D_{\mathbf{0}}}{4}\Gamma(1), \qquad W_1 = \sum_{|\sigma|+k=1} \frac{d_{2\sigma} D_\sigma}{4^{k+1}}\Gamma(2) - \frac{d_{\mathbf{0}} D_{\mathbf{0}}}{2} B_{1,0} + \frac{f_{\mathbf{0}} D_{\mathbf{0}}}{4}\Gamma(2) - \frac{\Re(l_{\mathbf{0}}) D_{\mathbf{0}}}{4}\Gamma(2), \\
W_2 &= \sum_{|\sigma|+k=2} \frac{d_{2\sigma} D_\sigma}{4^{k+1}}\Gamma(3) - \frac{d_{\mathbf{0}} D_{\mathbf{0}}}{2} B_{1,1} + \sum_{|\sigma|=1} \frac{f_{2\sigma} D_\sigma}{4}\Gamma(3) + \frac{f_{\mathbf{0}} D_{\mathbf{0}}}{4^2}\Gamma(3) \\
&\quad + (-\Re(l_{\mathbf{0}}) D_{\mathbf{0}} \alpha_0 \Im(B_{2,0})) - \sum_{|\sigma|+k=1} \frac{\Re(l_{2\sigma}) D_\sigma}{4^{k+1}}\Gamma(3).
\end{aligned}
$$

Each W_j can be simplified by using the results for D_{e_j}, D_{2e_j}, and $D_{e_i+e_j}$ from (2.44). Additionally we need the expressions for the $B_{m,n}$ given in (2.14) and the fact that $\alpha_0 = 1$. Hence

$$
\begin{aligned}
W_0 &= \frac{|P_1|^2\pi^2}{2}, \qquad W_1 = \frac{3|P_1|^2\pi^2}{8} + \frac{|P_0|^2\pi^2}{2} - \frac{\Re(P_1\bar{P}_0)\pi^2}{2} + \frac{\pi^2}{8}\sum_{j=1}^{4} d_{2e_j}, \\
W_2 &= \frac{7|P_1|^2\pi^2}{16} + \frac{|P_0|^2\pi^2}{4} - \frac{3\Re(P_1\bar{P}_0)\pi^2}{4} + \frac{\pi^2}{16}\sum_{j=1}^{4} d_{2e_j} + \frac{\pi^2}{4}\sum_{j=1}^{4} f_{2e_j} \\
&\quad - \frac{\pi^2}{4}\sum_{j=1}^{4}\Re(l_{2e_j}) + \frac{\pi^2}{8}\sum_{j=1}^{4} d_{4e_j} + \frac{\pi^2}{24}\sum_{1\leq i<j\leq 4} d_{2(e_i+e_j)}.
\end{aligned}
$$

We now assume that $N = 6$ and $K = 3$. Then $\|u(t, \cdot)\|_2^2 = W_0 t^{-2} + W_1 t^{-3} + W_2 t^{-4} + O(t^{-5})$

as $t \to \infty$, where

$$W_0 = \frac{d_{\mathbf{0}} D_{\mathbf{0}}}{4}\Gamma(2), \qquad W_1 = \sum_{|\sigma|+k=1} \frac{d_{2\sigma} D_\sigma}{4^{k+1}}\Gamma(3) + \frac{f_{\mathbf{0}} D_{\mathbf{0}}}{4}\Gamma(3) - \frac{\Re(l_{\mathbf{0}}) D_{\mathbf{0}}}{4}\Gamma(3),$$

$$W_2 = \sum_{|\sigma|+k=2} \frac{d_{2\sigma} D_\sigma}{4^{k+1}}\Gamma(4) - \frac{d_{\mathbf{0}} D_{\mathbf{0}}}{2} B_{3,0} + \sum_{|\sigma|=1} \frac{f_{2\sigma} D_\sigma}{4}\Gamma(4) + \frac{f_{\mathbf{0}} D_{\mathbf{0}}}{4^2}\Gamma(4) - \sum_{|\sigma|+k=1} \frac{\Re(l_{2\sigma}) D_\sigma}{4^{k+1}}\Gamma(4).$$

Again using (2.44) and (2.14), we may simplify each W_j.

$$W_0 = \frac{|P_1|^2\pi^3}{4}, \qquad W_1 = \frac{|P_1|^2\pi^3}{8} + \frac{|P_0|^2\pi^3}{2} - \frac{\Re(P_1\bar{P}_0)\pi^3}{2} + \frac{\pi^3}{12}\sum_{j=1}^{6} d_{2e_j},$$

$$W_2 = -\frac{3|P_1|^2\pi^3}{32} + \frac{3|P_0|^2\pi^3}{8} - \frac{3\Re(P_1\bar{P}_0)\pi^3}{8} + \frac{\pi^3}{16}\sum_{j=1}^{6} d_{2e_j} + \frac{\pi^3}{4}\sum_{j=1}^{6} f_{2e_j}$$
$$- \frac{\pi^3}{4}\sum_{j=1}^{6}\Re(l_{2e_j}) + \frac{3\pi^3}{32}\sum_{j=1}^{6} d_{4e_j} + \frac{\pi^3}{32}\sum_{1\le i<j\le 6} d_{2(e_i+e_j)}.$$

2.2 The space dimension $N = 2$ case

The low dimensional cases $N = 2$ and $N = 1$ must be treated separately since the integrals we encounter will have singular integrands at 0. Such does not happen in the higher dimensional cases $N \ge 3$.

2.2.1 Auxiliary lemmas

Lemma 5. *For $0 < \epsilon < 2$ and $t > 0$ we use (2.16) to define*

$$I^{\epsilon}_{1,-1}(t) = \int_0^{\epsilon} r^{-1} e^{-tr^2} \sin^2(tr\sqrt{1 - r^2/4})dr.$$

If $M \in \mathbb{N}$, then as $t \to \infty$,

$$I^{\epsilon}_{1,-1}(t) = \frac{1}{4}\ln(t) + \frac{\gamma}{4} + \frac{\ln(2)}{2} + \sum_{n=0}^{M-2} \frac{-1}{2(n+1)} B_{1,n} t^{-n-1}$$

$$+\sum_{1\le 2k+n\le M-1}\frac{\beta_k}{2k+n}\Im(B_{2k,n})t^{-2k-n}+O(t^{-M}),$$

where γ is the Euler-Mascheroni constant and, using the convention $(-1)!!=1$,

$$\beta_k=\begin{cases}1 & k=0\\ \frac{-(2k-3)!!}{k!8^k} & k\in\mathbb{N}.\end{cases}\tag{2.45}$$

Proof. For every $t>0$, it is a routine argument to verify (using Lebesgue's dominated convergence theorem and the mean value theorem) that $\frac{d}{dt}I^{\epsilon}_{1,-1}(t)=-I^{\epsilon}_{1,1}(t)+I_1(t)$, where $-I^{\epsilon}_{1,1}(t)$ is as in (2.16) and

$$I_1(t)=\int_0^{\epsilon}\sqrt{1-r^2/4}\,e^{-tr^2}\sin(tr\sqrt{4-r^2})dr.$$

By Lemma 2 with $Q:=M-1$, $-I^{\epsilon}_{1,1}(t)$ thus has the following expansion as $t\to\infty$:

$$-I^{\epsilon}_{1,1}(t)=-\frac{1}{4}t^{-1}+\sum_{n=0}^{M-2}\frac{1}{2}B_{1,n}t^{-n-2}+O(t^{-M-1}).$$

Turning to $I_1(t)$, we begin by expanding $\sqrt{1-r^2/4}$ in a Taylor series about $r=0$. Since $0<\epsilon<2$, for $0\le r\le\epsilon$:

$$\sqrt{1-r^2/4}=\sum_{k=0}^{L-1}\beta_k r^{2k}+O(r^{2L})\qquad(L\in\mathbb{N}_0),\tag{2.46}$$

where β_k is as given in (2.45) for $k\in\mathbb{N}_0$. Substituting (2.46) into the definition of $I_1(t)$ and estimating the integral with the $O(r^{2L})$-term, we obtain

$$\begin{aligned}I_1(t)&=\sum_{k=0}^{L-1}\beta_k\int_0^{\epsilon}r^{2k}e^{-tr^2}\sin(tr\sqrt{4-r^2})dr+O(t^{-L-\frac{1}{2}})\\&=\sum_{k=0}^{L-1}-\beta_k\Im(G^{\epsilon}_{1,2k}(t))+O(t^{-L-\frac{1}{2}})\qquad(t\to\infty).\end{aligned}\tag{2.47}$$

We let $L := M+1$ and substitute (2.12) into (2.47) with $Q = Q_k := \max\{0, L-2k\}$ to obtain as $t \to \infty$

$$
\begin{aligned}
I_1(t) &= \sum_{2k+n\leq M} -\beta_k \Im(B_{2k,n}) t^{-2k-n-1} + O(t^{-M-1}) \\
&= -\beta_0 \Im(B_{0,0}) t^{-1} + \sum_{1\leq 2k+n\leq M-1} -\beta_k \Im(B_{2k,n}) t^{-2k-n-1} + O(t^{-M-1}) \\
&= \frac{1}{2} t^{-1} + \sum_{1\leq 2k+n\leq M-1} -\beta_k \Im(B_{2k,n}) t^{-2k-n-1} + O(t^{-M-1})
\end{aligned}
$$

We combine the results for $-I^{\epsilon}_{1,1}(t)$ and $I_1(t)$ to obtain as $t \to \infty$

$$
\frac{d}{dt} I^{\epsilon}_{1,-1}(t) = \frac{1}{4} t^{-1} + \sum_{n=0}^{M-2} \frac{1}{2} B_{1,n} t^{-n-2} + \sum_{1\leq 2k+n\leq M-1} -\beta_k \Im(B_{2k,n}) t^{-2k-n-1} + O(t^{-M-1}).
$$

This implies that for some constant C, as $t \to \infty$

$$
\begin{aligned}
I^{\epsilon}_{1,-1}(t) = \frac{1}{4}\ln(t) + C + \sum_{n=0}^{M-2} \frac{-1}{2(n+1)} B_{1,n} t^{-n-1} \\
+ \sum_{1\leq 2k+n\leq M-1} \frac{\beta_k}{2k+n} \Im(B_{2k,n}) t^{-2k-n} + O(t^{-M}).
\end{aligned}
$$

This fact is proven by modifying the proof given on p. 17 in Chapter 1 of [1].

To determine the constant C we consider the asymptotics of the following integral for $0 < \epsilon < 2$ (to be proved separately):

$$
J^{\epsilon}_{1,-1}(t) := \int_0^{\epsilon} r^{-1} e^{-tr^2} \sin^2(tr) dr = \frac{1}{4}\ln(t) + \frac{\gamma}{4} + \frac{\ln(2)}{2} + O(t^{-1}) \qquad (t \to \infty). \tag{2.48}
$$

We now use the fact that $|\sin^2(tr) - \sin^2(tr\sqrt{1-r^2/4})| \leq tr|1 - \sqrt{1-r^2/4}| \leq \frac{tr^3}{4}$ for $0 \leq r \leq 2$. Thus

$$
|J^{\epsilon}_{1,-1}(t) - I^{\epsilon}_{1,-1}(t)| \leq \frac{t}{4}\int_0^{\epsilon} r^2 e^{-tr^2} dr \leq \frac{t}{8}\Gamma(\tfrac{3}{2}) t^{-\frac{3}{2}} = \frac{1}{8}\Gamma(\tfrac{3}{2}) t^{-\frac{1}{2}} \to 0 \qquad (t \to \infty). \tag{2.49}
$$

Furthermore

$$
\begin{aligned}
J^{\epsilon}_{1,-1}(t) - I^{\epsilon}_{1,-1}(t) &= \left(\frac{1}{4}\ln(t) + \frac{\gamma}{4} + \frac{\ln(2)}{2} + O(t^{-1})\right) - \left(\frac{1}{4}\ln(t) + C + O(t^{-1})\right) \\
&= \frac{\gamma}{4} + \frac{\ln(2)}{2} - C + O(t^{-1}) \to \frac{\gamma}{4} + \frac{\ln(2)}{2} - C \qquad (t \to \infty). \qquad (2.50)
\end{aligned}
$$

Considering (2.49) and (2.50) together, $C = \frac{\gamma}{4} + \frac{\ln(2)}{2}$, which proves the lemma. □

We now must prove the claim about $J^{\epsilon}_{1,-1}(t)$ from Lemma 5.

Lemma 6. *Let $0 < \epsilon < 2$ and $P \in \mathbb{N}_0$. Then with $J^{\epsilon}_{1,-1}(t)$ as defined in (2.48),*

$$
J^{\epsilon}_{1,-1}(t) = \frac{1}{4}\ln(t) + \frac{\gamma}{4} + \frac{1}{2}\ln(2) + \sum_{p=1}^{P} \frac{-1}{4p}\left(\frac{1}{2}\right)_p t^{-p} + O(t^{-P-1}) \qquad (t \to \infty).
$$

Proof. Let $0 < \epsilon < 2$, $P \in \mathbb{N}_0$, and $t > 0$. Then by the mean value theorem and Lebesgue's dominated convergence theorem, $\frac{d}{dt}J^{\epsilon}_{1,-1}(t) = J_1(t) - \Im(F^{\epsilon}_0(t))$, where $F^{\epsilon}_0(t)$ is as given in (2.6) and

$$
J_1(t) = -\int_0^{\epsilon} r e^{-tr^2} \sin^2(tr) dr.
$$

We use the fact that $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$ to write

$$
J_1(t) = -\frac{1}{2}\int_0^{\epsilon} r e^{-tr^2} dr + \frac{1}{2}\Re(F^{\epsilon}_1(t)). \qquad (2.51)
$$

We observe the integral in the first term of (2.51) $\int_0^{\epsilon} r e^{-tr^2} dr = \int_0^{\infty} r e^{-tr^2} dr + O(e^{-t\epsilon^2/2})$ as $t \to \infty$. From this fact and (2.9) we conclude that as $t \to \infty$

$$
\frac{d}{dt}J^{\epsilon}_{1,-1}(t) = \frac{1}{4}t^{-1} + \sum_{p=0}^{P-1} \frac{1}{2}A_{1,p}t^{-p-2} + O(t^{-P-2}) - \sum_{s=1}^{S-1} \Im(A_{0,s})t^{-s-1} + O(t^{-S-1}),
$$

where $P \in \mathbb{N}_0$ is as given and $S := P + 1$. Simplifying we obtain

$$
\frac{d}{dt}J^{\epsilon}_{1,-1}(t) = \frac{1}{4}t^{-1} + \sum_{p=1}^{P} \frac{1}{4}\left(\frac{1}{2}\right)_p t^{-p-1} + O(t^{-P-2}) \qquad (t \to \infty).
$$

Therefore there is some constant C such that

$$J^{\epsilon}_{1,-1}(t) = \frac{1}{4}\ln(t) + C + \sum_{p=1}^{P} \frac{-1}{4p}\left(\frac{1}{2}\right)_p t^{-p} + O(t^{-P-1}) \qquad (t \to \infty). \tag{2.52}$$

Now we must determine the constant C. To do so, we first note that $J^{\epsilon}_{1,-1}(t) = \tilde{J}_{1,-1}(t) - \int_{\epsilon}^{\infty} r^{-1}e^{-tr^2}\sin^2(tr)dr$, where for $t > 0$

$$\tilde{J}_{1,-1}(t) = \int_0^{\infty} r^{-1}e^{-tr^2}\sin^2(tr)dr.$$

It is easily verified that as $t \to \infty$, $\int_{\epsilon}^{\infty} r^{-1}e^{-tr^2}\sin^2(tr)dr = O(e^{-t\epsilon^2/2}) = o(1)$, so we instead study $\tilde{J}_{1,-1}(t)$.

With the substitution $s = \sqrt{t}r$ we have $\tilde{J}_{1,-1(t)} = \int_0^{\infty} s^{-1}e^{-s^2}\sin^2(\sqrt{t}s)ds$. Integrating by parts we obtain

$$\tilde{J}_{1,-1}(t) = J_2(t) + J_3(t) + J_4(t),$$

where

$$J_2(t) = \int_0^{\infty} s\ln(s)e^{-s^2}ds,$$
$$J_3(t) = -\int_0^{\infty} s\ln(s)e^{-s^2}\cos(2\sqrt{t}s)ds,$$
$$J_4(t) = -\sqrt{t}\int_0^{\infty} \ln(s)e^{-s^2}\sin(2\sqrt{t}s)ds.$$

With the substitution $u = s^2$,

$$J_2(t) = \frac{1}{4}\int_0^{\infty} \ln(u)e^{-u}du.$$

The integral

$$\int_0^{\infty} \ln(u)e^{-au}du = -\frac{\gamma + \ln(a)}{a} \qquad (\Re(a) > 0), \tag{2.53}$$

where γ is the Euler-Mascheroni constant, is a Laplace transform identity found on p. 443

in Chapter 11 of [10] by Magnus et al. Therefore

$$J_2(t) = -\frac{\gamma}{4}.$$

We now let $f_1(s) = -s\ln(s)e^{-s^2} \cdot \chi_{(0,\infty)}(s)$, where $\chi_{(0,\infty)}(s)$ is the characteristic function of the set $(0,\infty)$. Then $f_1 \in L^1(\mathbb{R})$. Furthermore, for all $t > 0$

$$\begin{aligned} J_3(t) &= \Re\left(\int_{-\infty}^{\infty} f_1(s)e^{-2i\sqrt{t}s}ds\right) \\ &= \sqrt{2\pi}\Re(\hat{f}_1(2\sqrt{t})). \end{aligned}$$

By Proposition 2.2.17 (Riemann-Lebesgue lemma) found on p. 105 in Chapter 2 of Grafakos' Classical Fourier Analysis [3]

$$\lim_{t\to\infty} J_3(t) = 0.$$

Lastly for $t > 0$ we define

$$J_5(t) := -\sqrt{t}\int_0^{\infty} \ln(s)e^{-s}\sin(2\sqrt{t}s)ds.$$

Then

$$J_4(t) - J_5(t) = \sqrt{t}\int_0^{\infty} f_2(s)\sin(2\sqrt{t}s)ds, \tag{2.54}$$

where $f_2(s) = \ln(s)(e^{-s} - e^{-s^2})$. We integrate (2.54) by parts and use the fact that $\lim_{t\to 0+} f_2(t) = 0$ to obtain

$$\begin{aligned} J_4(t) - J_5(t) &= \frac{1}{2}\int_0^{\infty} f_2'(s)\cos(2\sqrt{t}s)ds \\ &= \frac{1}{2}\Re\left(\int_{-\infty}^{\infty} f_3(s)e^{-2i\sqrt{t}s}ds\right), \end{aligned}$$

where $f_3 := f_2' \cdot \chi_{(0,\infty)} \in L^1(\mathbb{R})$. Thus for $t > 0$

$$J_4(t) - J_5(t) = \sqrt{\frac{\pi}{2}}\,\Re(\hat{f}_3(2\sqrt{t})),$$

and by the Riemann-Lebesgue lemma

$$\lim_{t\to\infty} (J_4(t) - J_5(t)) = 0.$$

We use the fact that $\sin(2\sqrt{t}s) = \Im(\exp(2i\sqrt{t}s))$ and (2.53) to obtain

$$J_5(t) = \sqrt{t}\Im\left(\frac{\gamma + \ln(1-2i\sqrt{t})}{1-2i\sqrt{t}}\right) = \frac{1}{4}\ln(t) + \frac{\gamma}{2} + \frac{1}{2}\ln(2) + o(1) \qquad (t\to\infty).$$

Combining the results of the analysis of $\tilde{J}_{1,-1}(t)$, we have

$$J^{\epsilon}_{1,-1}(t) = \tilde{J}_{1,-1}(t) + o(1) = \frac{1}{4}\ln(t) + \frac{\gamma}{4} + \frac{1}{2}\ln(2) + o(1)$$

as $t \to \infty$. From (2.52) we have

$$J^{\epsilon}_{1,-1}(t) = \frac{1}{4}\ln(t) + C + o(1)$$

as $t \to \infty$. Thus $C = \frac{\gamma}{4} + \frac{1}{2}\ln(2)$ and the proof is complete. □

2.2.2 Intermediate computations

As stated in the unnumbered lemma in section 1.4, Ikehata [4] showed that for the fixed $0 < \delta < 1$, $\int_{|\xi|>\delta} |\hat{u}(t,\xi)|^2 d\xi$ is exponentially small as $t \to \infty$. We will instead find the expansion of $\|\hat{u}(t,\cdot)\|^2_{2,\delta} = X_1(t) + Y_1(t) + Z_1(t)$ as before. Additionally, the overlying assumption is still that $K \in \mathbb{N}$ with the initial data of (1.1) $u_0 \in H^1(\mathbb{R}^2) \cap L^{1,2K}(\mathbb{R}^2)$ and $u_1 \in L^2(\mathbb{R}^2) \cap L^{1,2K}(\mathbb{R}^2)$. We will again comment on whether K should be larger, as necessary.

Over the next several sections, we will arrive at the proof of Theorem 1 part 2.

Expansion of $X_1(t)$

With the assumptions on K and the initial data u_0, u_1 of (1.1), we may use the argument for the expansion of $X_1(t)$ from section 2.1.2 with $N = 2$ to obtain

$$\begin{aligned} X_1(t) &= \sum_{|\sigma|\le K-1} d_{2\sigma} D_\sigma \int_0^\delta r^{2|\sigma|-1} e^{-tr^2} \frac{\sin^2(tr\sqrt{1-r^2/4})}{1-r^2/4} dr + O(t^{-K}) \\ &= X_1^A(t) + X_1^B(t) + O(t^{-K}) \qquad (t\to\infty), \end{aligned}$$

where

$$X_1^A(t) = d_{\mathbf{0}} D_{\mathbf{0}} \int_0^\delta r^{-1} e^{-tr^2} \frac{\sin^2(tr\sqrt{1-r^2/4})}{1-r^2/4} dr, \tag{2.55}$$

$$X_1^B(t) = \sum_{1\le|\sigma|\le K-1} d_{2\sigma} D_\sigma \int_0^\delta r^{2|\sigma|-1} e^{-tr^2} \frac{\sin^2(tr\sqrt{1-r^2/4})}{1-r^2/4} dr. \tag{2.56}$$

To determine the expansion of (2.55), we expand $(1 - r^2/4)^{-1}$ in a Taylor polynomial with remainder as in (2.22), and define $L := K \in \mathbb{N}$. Thus with the identity $d_{\mathbf{0}} D_{\mathbf{0}} = 2\pi|P_1|^2$,

$$\begin{aligned} X_1^A(t) &= \sum_{k=0}^{K-1} \frac{\pi|P_1|^2}{2^{2k-1}} \int_0^\delta r^{2k-1} e^{-tr^2} \sin^2(tr\sqrt{1-r^2/4}) dr + O(t^{-K}) \\ &= 2\pi|P_1|^2 I_{1,-1}^\delta(t) + \sum_{k=1}^{K-1} \frac{\pi|P_1|^2}{2^{2k-1}} I_{1,2k-1}^\delta(t) + O(t^{-K}) \qquad (t\to\infty). \end{aligned}$$

Then we use Lemma 5 with $M := K$ and Lemma 2 with $Q = Q_k := \max\{0, K - 2k\}$, and simplify to obtain as $t \to \infty$

$$\begin{aligned} X_1^A(t) =& \frac{\pi|P_1|^2}{2} \ln(t) + \frac{\pi|P_1|^2}{2} \gamma + \pi|P_1|^2 \ln(2) + \sum_{k=1}^{K-1} \left(\frac{\pi|P_1|^2}{2^{2k+1}} \Gamma(k) - \frac{\pi|P_1|^2}{k} B_{1,k-1} \right) t^{-k} \\ &+ \sum_{1\le 2k+n\le K-1} \frac{2\pi|P_1|^2 \beta_k}{2k+n} \Im(B_{2k,n}) t^{-2k-n} - \sum_{\substack{2\le 2k+n\le K-1 \\ k\ge 1}} \frac{\pi|P_1|^2}{4^k} B_{2k-1,n} t^{-2k-n} + O(t^{-K}) \\ =& \frac{\pi|P_1|^2}{2} \ln(t) + \sum_{j=0}^{K-1} (X_{1,1}^{[j]} + X_{1,2}^{[j]} - X_{1,3}^{[j]}) t^{-j} + O(t^{-K}), \end{aligned}$$

where

$$X_{1,1}^{[j]} = \begin{cases} \frac{\pi\gamma|P_1|^2}{2} + \pi|P_1|^2 \ln(2) & j = 0 \\ \frac{\pi|P_1|^2}{2^{2j+1}}\Gamma(j) - \frac{\pi|P_1|^2}{j} B_{1,j-1} & 1 \leq j \leq K-1, \end{cases}$$

$$X_{1,2}^{[j]} = \begin{cases} \sum\limits_{2k+n=j} \frac{2\pi|P_1|^2\beta_k}{j} \Im(B_{2k,n}) & 1 \leq j \leq K-1 \\ 0 & \text{otherwise}, \end{cases}$$

$$X_{1,3}^{[j]} = \begin{cases} \sum\limits_{\substack{2k+n=j \\ k\geq 1}} \frac{\pi|P_1|^2}{4^k} B_{2k-1,n} & 2 \leq j \leq K-1 \\ 0 & \text{otherwise}. \end{cases}$$

To find the expansion of (2.56) we again expand $(1 - r^2/4)^{-1}$ in a Taylor polynomial as in (2.22) with $L_\sigma := K - |\sigma|$ to obtain

$$X_1^B(t) = \sum_{1\leq|\sigma|\leq K-1} \sum_{k=0}^{L_\sigma - 1} \frac{d_{2\sigma}D_\sigma}{4^k} I^\delta_{1,2|\sigma|+2k-1}(t) + O(t^{-K}) \qquad (t \to \infty). \tag{2.57}$$

We then substitute (2.18) into (2.57) with $Q = Q_{\sigma,k} := \max\{0, K - 2|\sigma| - 2k\}$ and simplify to obtain

$$\begin{aligned} X_1^B(t) &= \sum_{\substack{1\leq|\sigma|+k\leq K-1 \\ |\sigma|\geq 1}} \frac{d_{2\sigma}D_\sigma}{4^{k+1}}\Gamma(|\sigma|+k)t^{-|\sigma|-k} - \sum_{\substack{2\leq 2|\sigma|+2k+n\leq K-1 \\ |\sigma|\geq 1}} \frac{d_{2\sigma}D_\sigma}{2^{2k+1}} B_{2|\sigma|+2k-1,n} t^{-2|\sigma|-2k-n} + O(t^{-K}) \\ &= \sum_{j=0}^{K-1} (X_{1,4}^{[j]} - X_{1,5}^{[j]})t^{-j} + O(t^{-K}) \qquad (t \to \infty), \end{aligned}$$

where

$$X_{1,4}^{[j]} = \begin{cases} \sum\limits_{\substack{|\sigma|+k=j \\ |\sigma|\geq 1}} \frac{d_{2\sigma}D_\sigma}{4^{k+1}} \Gamma(j) & 1 \leq j \leq K-1 \\ 0 & \text{otherwise}, \end{cases}$$

$$X_{1,5}^{[j]} = \begin{cases} \displaystyle\sum_{\substack{2|\sigma|+2k+n=j \\ |\sigma|\geq 1}} \tfrac{d_{2\sigma} D_\sigma}{2^{2k+1}} B_{2|\sigma|+2k-1,n} & 2 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Combining the results for $X_1^A(t)$ and $X_1^B(t)$, we have the expansion as $t \to \infty$

$$X_1(t) = \frac{\pi |P_1|^2}{2} \ln(t) + \sum_{j=0}^{K-1} (X_{1,1}^{[j]} + X_{1,2}^{[j]} - X_{1,3}^{[j]} + X_{1,4}^{[j]} - X_{1,5}^{[j]}) t^{-j} + O(t^{-K}). \tag{2.58}$$

Remark. Both $X_{1,2}^{[j]}$ and $X_{1,4}^{[j]}$ require $K \geq 2$ in order to contribute non-zero terms to the expansion, and $X_{1,3}^{[j]}$ and $X_{1,5}^{[j]}$ require $K \geq 3$.

Expansion of $Y_1(t)$

We refer back to the expansion of $Y_1(t)$ obtained in section 2.1.2. From there we see that with $K \geq 2$, all steps from that section are valid here with $N = 2$. We arrive at $Y_1(t) = Y_1^A(t) + Y_1^B(t) - Y_1^C(t) + O(t^{-K})$ as $t \to \infty$, where $Y_1^A(t)$, $Y_1^B(t)$, and $Y_1^C(t)$ are as given in (2.29)–(2.31) with $N = 2$. Since all powers of r in the resulting integrals are positive, we obtain the same expansion for $Y_1(t)$ as in the $N \geq 3$ cases. Namely

$$Y_1(t) = \sum_{j=0}^{K-1} (Y_{1,1}^{[j]} + Y_{1,2}^{[j]} + Y_{1,3}^{[j]} - Y_{1,4}^{[j]} - Y_{1,5}^{[j]}) t^{-j} + O(t^{-K}) \qquad (t \to \infty), \tag{2.59}$$

where

$$Y_{1,1}^{[j]} = \begin{cases} \displaystyle\sum_{|\sigma|=j-1} \tfrac{f_{2\sigma} D_\sigma}{4} \Gamma(j) & 1 \leq j \leq K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{1,2}^{[j]} = \begin{cases} \displaystyle\sum_{2|\sigma|+n=j-2} \tfrac{f_{2\sigma} D_\sigma}{2} B_{2|\sigma|+1,n} & 2 \leq j \leq K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{1,3}^{[j]} = \begin{cases} \sum_{|\sigma|+k=j-2} \frac{f_{2\sigma} D_\sigma}{4^{k+2}} \Gamma(j) & 2 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{1,4}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-4} \frac{f_{2\sigma} D_\sigma}{2^{2k+3}} B_{2|\sigma|+2k+3,n} & 4 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{1,5}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-3} \frac{-f_{2\sigma} D_\sigma \alpha_k}{2} \Im(B_{2|\sigma|+2k+2,n}) & 3 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Expansion of $Z_1(t)$

We refer back to earlier work from section 2.1.2 regarding the expansion of $Z_1(t)$. With $K \ge 2$ all steps from that section are valid now with $N = 2$. We deduce once again that $Z_1(t) = Z_1^A(t) - Z_1^B(t) + O(t^{-K})$ as $t \to \infty$, where $Z_1^A(t)$ and $Z_1^B(t)$ are as given in (2.38) and (2.39) with $N = 2$. Since all powers of r present in the resulting integrals are non-negative, we obtain the same expansion for $Z_1(t)$ as in the $N \ge 3$ cases. In particular

$$Z_1(t) = \sum_{j=0}^{K-1} (Z_{1,1}^{[j]} - Z_{1,2}^{[j]} + Z_{1,3}^{[j]}) t^{-j} + O(t^{-K}) \qquad (t \to \infty), \tag{2.60}$$

where

$$Z_{1,1}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-1} -\Re(l_{2\sigma}) D_\sigma \alpha_k \Im(B_{2|\sigma|+2k,n}) & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{1,2}^{[j]} = \begin{cases} \sum_{|\sigma|+k=j-1} \frac{\Re(l_{2\sigma}) D_\sigma}{4^{k+1}} \Gamma(j) & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{1,3}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-2} \frac{\Re(l_{2\sigma})D_\sigma}{2^{2k+1}} B_{2|\sigma|+2k+1,n} & 2 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

2.2.3 Asymptotic expansion of $\|u(t,\cdot)\|_2^2$ as $t \to \infty$

We recall from section 2.1.2 that $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_{2,\delta}^\delta + O(e^{-\eta t})$ for some $\eta > 0$ dependent on the fixed $0 < \delta < 1$. Since $\|\hat{u}(t,\cdot)\|_{2,\delta}^2 = X_1(t) + Y_1(t) + Z_1(t)$, we combine results (2.58), (2.59), and (2.60) to obtain the asymptotic expansion

$$\|u(t,\cdot)\|_2^2 = \frac{\pi|P_1|^2}{2}\ln(t) + \sum_{j=0}^{K-1} W_j t^{-j} + O(t^{-K}) \qquad (t \to \infty),$$

where, for $\{0,\ldots,K-1\}$,

$$W_j = X_{1,1}^{[j]} + X_{1,2}^{[j]} - X_{1,3}^{[j]} + X_{1,4}^{[j]} - X_{1,5}^{[j]} + Y_{1,1}^{[j]} + Y_{1,2}^{[j]} + Y_{1,3}^{[j]} - Y_{1,4}^{[j]} - Y_{1,5}^{[j]} + Z_{1,1}^{[j]} - Z_{1,2}^{[j]} + Z_{1,3}^{[j]}.$$

The first three coefficients

Let us assume $K = 3$. Then $\|u(t,\cdot)\|_2^2 = \frac{\pi|P_1|^2}{2}\ln(t) + W_0 + W_1 t^{-1} + W_2 t^{-2} + O(t^{-3})$ as $t \to \infty$, where

$$\begin{aligned} W_0 &= \frac{\pi\gamma|P_1|^2}{2} + \pi|P_1|^2\ln(2), \\ W_1 &= \frac{\pi|P_1|^2}{2^3}\Gamma(1) - \pi|P_1|^2 B_{1,0} + 2\pi|P_1|^2\beta_0\Im(B_{0,1}) + \sum_{|\sigma|=1}\frac{d_{2\sigma}D_\sigma}{4}\Gamma(1) + \frac{f_{\mathbf{0}}D_{\mathbf{0}}}{4}\Gamma(1) \\ &\qquad - \Re(l_{\mathbf{0}})D_{\mathbf{0}}\alpha_0\Im(B_{0,0}) - \frac{\Re(l_{\mathbf{0}})D_{\mathbf{0}}}{4}\Gamma(1), \\ W_2 &= \frac{\pi|P_1|^2}{2^5}\Gamma(2) - \frac{\pi|P_1|^2}{2}B_{1,1} + \sum_{2k+n=2}\pi|P_1|^2\beta_k\Im(B_{2k,n}) - \frac{\pi|P_1|^2}{4}B_{1,0} \\ &\quad + \sum_{\substack{|\sigma|+k=2 \\ |\sigma|\geq 1}}\frac{d_{2\sigma}D_\sigma}{4^{k+1}}\Gamma(2) - \sum_{|\sigma|=1}\frac{d_{2\sigma}D_\sigma}{2}B_{1,0} + \sum_{|\sigma|=1}\frac{f_{2\sigma}D_\sigma}{4}\Gamma(2) + \frac{f_{\mathbf{0}}D_{\mathbf{0}}}{2}B_{1,0} \\ &\quad + \frac{f_{\mathbf{0}}D_{\mathbf{0}}}{4^2}\Gamma(2) - \Re(l_{\mathbf{0}})D_{\mathbf{0}}\alpha_0\Im(B_{0,1}) - \sum_{|\sigma|+k=1}\frac{\Re(l_{2\sigma})D_\sigma}{4^{k+1}}\Gamma(2) + \frac{\Re(l_{\mathbf{0}})D_{\mathbf{0}}}{2}B_{1,0}. \end{aligned}$$

We may simplify the W_j ($j \in \{0,1,2\}$) by using the formula for D_σ as given in (2.21) and the expressions for the $B_{m,n}$ given in (2.14).

$$W_0 = \frac{\pi\gamma|P_1|^2}{2} + \pi|P_1|^2\ln(2), \qquad W_1 = -\frac{\pi|P_1|^2}{8} + \frac{\pi|P_0|^2}{2} + \frac{\pi\Re(P_1\bar{P}_0)}{2} + \frac{\pi}{4}\sum_{j=1}^{2} d_{2e_j},$$

$$W_2 = -\frac{\pi|P_1|^2}{32} - \frac{\pi|P_0|^2}{8} + \frac{\pi\Re(P_1\bar{P}_0)}{8} + \frac{3\pi}{16}\sum_{j=1}^{2} d_{2e_j} + \frac{\pi}{4}\sum_{j=1}^{2} f_{2e_j} - \frac{\pi}{4}\sum_{j=1}^{2}\Re(l_{2e_j}) + \frac{3\pi}{16}\sum_{j=1}^{2} d_{4e_j} + \frac{\pi}{16} d_{2(e_1+e_2)}.$$

2.3 The space dimension $N = 1$ case

2.3.1 Auxiliary lemmas

Lemma 7. *For $0 < \epsilon < 2$ and $t > 0$, we define*

$$I^{\epsilon}_{1,-2}(t) := \int_0^{\epsilon} r^{-2} e^{-tr^2} \sin^2(tr\sqrt{1 - r^2/4})dr.$$

If $M \in \mathbb{N}$, then as $t \to \infty$

$$I^{\epsilon}_{1,-2}(t) = \frac{\pi}{2}t - \frac{\sqrt{\pi}}{2}t^{\frac{1}{2}} + O(t^{-M}).$$

Proof. Let $0 < \epsilon < 2$, $t > 0$, and $M \in \mathbb{N}$. Using integration by parts, $I^{\epsilon}_{1,-2}(t) = I_1(t) + I_2(t) + I_3(t)$, where

$$I_1(t) = -\epsilon^{-1} e^{-t\epsilon^2} \sin^2(t\epsilon\sqrt{1 - \epsilon^2/4}),$$
$$I_2(t) = -2t\int_0^{\epsilon} e^{-tr^2} \sin^2(tr\sqrt{1 - r^2/4})dr,$$
$$I_3(t) = t\int_0^{\epsilon} r^{-1}\frac{d}{dr}[r\sqrt{1 - r^2/4}]e^{-tr^2} \sin(2tr\sqrt{1 - r^2/4}).$$

As $t \to \infty$,

$$I_1(t) = O(e^{-t\epsilon^2}) = O(t^{-M}).$$

Next, we note that $I_2(t) = -2tI_{1,0}^{\epsilon}(t)$. Thus by Lemma 2 with $Q := M$,

$$I_2(t) = -\frac{\sqrt{\pi}}{2}t^{\frac{1}{2}} + O(t^{-M}) \qquad (t \to \infty). \tag{2.61}$$

To analyze $I_3(t)$, we first consider

$$t^{-1}I_3(t) = \int_0^{\epsilon} r^{-1}\frac{d}{dr}[r\sqrt{1-r^2/4}]e^{-tr^2}\sin(2tr\sqrt{1-r^2/4}).$$

Using the mean value theorem and Lebesgue's dominated convergence theorem, we have that $\frac{d}{dt}[t^{-1}I_3(t)] = I_4(t) + I_5(t)$, where

$$\begin{aligned} I_4(t) &= -\int_0^{\epsilon} r\frac{d}{dr}[r\sqrt{1-r^2/4}]\, e^{-tr^2}\sin(2tr\sqrt{1-r^2/4})dr, \\ I_5(t) &= 2\int_0^{\epsilon} \sqrt{1-r^2/4}\,\frac{d}{dr}[r\sqrt{1-r^2/4}]\, e^{-tr^2}\cos(2tr\sqrt{1-r^2/4})dr. \end{aligned}$$

We now use the fact that $\frac{d}{dr}[r\sqrt{1-r^2/4}] = \sqrt{1-r^2/4} - \frac{r^2}{4}(1-r^2/4)^{-1/2}$ to analyze both $I_4(t)$ and $I_5(t)$.

First, $I_4(t) = I_4^A(t) + I_4^B(t)$, where

$$\begin{aligned} I_4^A(t) &= -\int_0^{\epsilon} r\sqrt{1-r^2/4}\, e^{-tr^2}\sin(tr\sqrt{4-r^2})dr, \\ I_4^B(t) &= \frac{1}{4}\int_0^{\epsilon} r^3 e^{-tr^2}\frac{\sin(tr\sqrt{4-r^2})}{\sqrt{1-r^2/4}}dr. \end{aligned}$$

In $I_4^A(t)$, we expand $\sqrt{1-r^2/4}$ in a Taylor polynomial with remainder about $r = 0$ as in (2.46), with $L := M+1$. This choice of L makes the resulting integral with the $O(r^{2L})$-term of the order $O(t^{-M-2})$ as $t \to \infty$. Thus as $t \to \infty$

$$I_4^A(t) = \sum_{k=0}^{M} \beta_k \Im(G^\epsilon_{1,2k+1}(t)) + O(t^{-M-2}).$$

We use the asymptotic expansion for $G^\epsilon_{1,m}(t)$ given in Proposition 1 with $Q = Q_k := \max\{0, M-2k\}$ and recall that each $B_{2k+1,n} \in \mathbb{R}$, since $2k+1$ is odd, to obtain $I_4^A(t) = O(t^{-M-2})$ as $t \to \infty$.

We proceed in a similar fashion for $I_4^B(t)$ by expanding $(1 - r^2/4)^{-1/2}$ in a Taylor polynomial with remainder about $r = 0$ as in (2.34), with $L := M$. Then

$$I_4^B(t) = \sum_{k=0}^{M-1} -\frac{\alpha_k}{4} \Im(G^\epsilon_{1,2k+3}(t)) + O(t^{-M-2}),$$

which can again be shown to reduce to $I_4^B(t) = O(t^{-M-2})$ as $t \to \infty$, since $2k+3$ is odd.

We combine the results for $I_4^A(t)$ and $I_4^B(t)$ with the fact $I_4(t) = I_4^A(t) + I_4^B(t)$ to obtain

$$I_4(t) = O(t^{-M-2}) \qquad (t \to \infty). \tag{2.62}$$

Now, $I_5(t) = 2\Re(G^\epsilon_{1,0}(t)) - \Re(G^\epsilon_{1,2}(t))$. For both terms we apply Proposition 1 with $Q := M+1$ for the first term and $Q := M$ for the second. Then since each $B_{0,n}, B_{2,n} \in i\mathbb{R}$, we obtain

$$I_5(t) = O(t^{-M-2}) + O(t^{-M-3}) = O(t^{-M-2}) \qquad (t \to \infty). \tag{2.63}$$

We combine results (2.62) and (2.63) with the fact that $\frac{d}{dt}[t^{-1} I_3(t)] = I_4(t) + I_5(t)$ to obtain the expansion for any $M \in \mathbb{N}$

$$\frac{d}{dt}[t^{-1} I_3(t)] = O(t^{-M-2}) \qquad (t \to \infty),$$

which implies

$$t^{-1} I_3(t) = C + O(t^{-M-1}) \qquad (t \to \infty).$$

Thus

$$I_3(t) = Ct + O(t^{-M}) \qquad (t \to \infty). \tag{2.64}$$

The constant $C = \frac{\pi}{2}$, which may be determined by comparing the asymptotic behavior of $I_3(t)$ to that of the following integral for $0 < \epsilon < 2$:

$$\tilde{I}_3(t) := t \int_0^\epsilon r^{-1} e^{-tr^2} \sin(2tr) dr = \frac{\pi}{2} t + O(t^{-M}) \qquad (t \to \infty). \tag{2.65}$$

The asymptotics of $\tilde{I}_3(t)$ are true for all $M \in \mathbb{N}$ and will be proved separately.

We use the fact that $|\sin(2tr) - \sin(2tr\sqrt{1 - r^2/4})| \leq 2tr|1 - \sqrt{1 - r^2/4}| \leq \frac{tr^3}{2}$ for $0 \leq r \leq 2$ to obtain

$$t^{-1}|\tilde{I}_3(t) - I_3(t)| \leq \frac{t}{2} \int_0^\epsilon r^2 e^{-tr^2} dr \leq \frac{t}{4} \Gamma(\tfrac{3}{2}) t^{-\frac{3}{2}} = \frac{1}{4} \Gamma(\tfrac{3}{2}) t^{-\frac{1}{2}} \to 0 \qquad (t \to \infty). \tag{2.66}$$

Further,

$$t^{-1}(\tilde{I}_3(t) - I_3(t)) = \frac{\pi}{2} - C + O(t^{-M-1}) \to \frac{\pi}{2} - C \qquad (t \to \infty). \tag{2.67}$$

Considering (2.66) and (2.67) together verifies the value $C = \frac{\pi}{2}$. To complete the proof of the lemma, we combine the asymptotics $I_1(t) = O(t^{-M})$, (2.61), and (2.64) along with the fact that $I^\epsilon_{1,-2}(t) = I_1(t) + I_2(t) + I_3(t)$. □

We now must verify the following claim.

Lemma 8. *For $0 < \epsilon < 2$ fixed and for all $M \in \mathbb{N}$, (2.65) holds.*

Proof. Let $0 < \epsilon < 2$ and $M \in \mathbb{N}$. We note that

$$t^{-1}\tilde{I}_3(t) = \left(\int_0^\infty - \int_\epsilon^\infty \right) r^{-1} e^{-tr^2} \sin(2tr) dr,$$

the latter integral of which is on the order $O(e^{-t\epsilon^2/2})$ as $t \to \infty$.

The integral $\int_0^\infty r^{-1} e^{-tr^2} \sin(2tr) dr$ is related to the Fourier sine transform (defined on p. 397 in Chapter 11 of [10]) of the function $r^{-1}e^{-tr^2}$. By p. 418 in Chapter 11 of [10], we have for all $t > 0$

$$\int_0^\infty r^{-1}e^{-tr^2}\sin(2tr)dr = \frac{\pi}{2}\mathrm{erf}(\sqrt{t}),$$

where $\mathrm{erf}(t)$ denotes the error function. We appeal to p. 352 in Chapter 9 of [10] for the asymptotics of the error function. Therefore

$$t^{-1}\tilde{I}_3(t) = \frac{\pi}{2} + O(t^{-\frac{1}{2}}e^{-t}) + O(e^{-\frac{t\epsilon^2}{2}}) = \frac{\pi}{2} + O(t^{-M-1}) \qquad (t \to \infty),$$

which proves the lemma. □

Lemma 9. *For $0 < \epsilon < 2$ and $t > 0$, we define*

$$\begin{aligned} I^{\epsilon}_{3,-1}(t) &:= \int_0^\epsilon r^{-1}e^{-tr^2}\sin(tr\sqrt{1-r^2/4})\cos(tr\sqrt{1-r^2/4})dr \\ &= \frac{1}{2}\int_0^\epsilon r^{-1}e^{-tr^2}\sin(tr\sqrt{4-r^2})dr. \end{aligned}$$

If $M \in \mathbb{N}$, then as $t \to \infty$

$$I^{\epsilon}_{3,-1}(t) = \frac{\pi}{4} + O(t^{-M}).$$

Proof. Let $0 < \epsilon < 2$, $t > 0$, and $M \in \mathbb{N}$. We use the mean value theorem and Lebesgue's dominated convergence theorem to justify differentiation within the integral. Hence

$$\frac{d}{dt}I^{\epsilon}_{3,-1}(t) = -I^{\epsilon}_{3,1}(t) + I_1(t),$$

where

$$I_1(t) = \frac{1}{2}\int_0^\epsilon \sqrt{4-r^2}\, e^{-tr^2}\cos(tr\sqrt{4-r^2})dr.$$

By Lemma 4 with $Q := M - 1$,

$$-I^{\epsilon}_{3,1}(t) = O(t^{-M-1}) \qquad (t \to \infty).$$

To determine the asymptotics of $I_1(t)$, we note

$$I_1(t) = \int_0^\epsilon \sqrt{1 - r^2/4}\, e^{-tr^2} \cos(tr\sqrt{4 - r^2})dr. \tag{2.68}$$

Into (2.68) we substitute (2.46) with $L := M + 1$. Hence

$$I_1(t) = \sum_{k=0}^{M} \beta_k \Re(G^\epsilon_{1,2k}(t)) + O(t^{-M-\frac{3}{2}}) \qquad (t \to \infty). \tag{2.69}$$

We apply Proposition 1 to (2.69) with $Q = Q_k := \max\{0, M - 2k + 1\}$. We also observe that since each $2k$ ($k \in \{0, \ldots, M\}$) is even, every $B_{2k,n} \in i\mathbb{R}$. Thus $I_1(t) = O(t^{-M-3/2}) = O(t^{-M-1})$ as $t \to \infty$.

We combine the results for $-I^\epsilon_{3,1}(t)$ and $I_1(t)$ to obtain $\frac{d}{dt} I^\epsilon_{3,-1}(t) = O(t^{-M-1})$ as $t \to \infty$. Hence

$$I^\epsilon_{3,-1}(t) = C + O(t^{-M}) \qquad (t \to \infty).$$

We must determine the constant C.

Since we are given $M \in \mathbb{N}$, Lemma 8 implies

$$t^{-1}\tilde{I}_3(t) = \frac{\pi}{2} + O(t^{-M}) \qquad (t \to \infty).$$

Since $|\sin(2tr) - \sin(tr\sqrt{4 - r^2})| \leq 2tr|1 - \sqrt{1 - r^2/4}| \leq \frac{tr^3}{2}$ for $0 \leq r \leq 2$, we obtain

$$|(2t)^{-1}\tilde{I}_3(t) - I^\epsilon_{3,-1}(t)| \leq \frac{t}{4}\int_0^\epsilon r^2 e^{-tr^2} dr \leq \frac{t}{8}\Gamma(\tfrac{3}{2})t^{-\frac{3}{2}} = \frac{1}{8}\Gamma(\tfrac{3}{2})t^{-\frac{1}{2}} \to 0 \qquad (t \to \infty). \tag{2.70}$$

Further, for the given $M \in \mathbb{N}$,

$$(2t)^{-1}\tilde{I}_3(t) - I^\epsilon_{3,-1}(t) = \frac{\pi}{4} - C + O(t^{-M}) \to \frac{\pi}{4} - C \qquad (t \to \infty). \tag{2.71}$$

Considering (2.70) and (2.71) together implies $C = \frac{\pi}{4}$ and the proof is complete. □

2.3.2 Intermediate computations

Our goal is to find the asymptotic expansion of the norm $\|\hat{u}(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_2^2$ as $t \to \infty$. By Ikehata's lemma from section 1.4, for the fixed $0 < \delta < 1$, $\int_{|\xi|>\delta} |\hat{u}(t,\xi)|^2 d\xi$ decays exponentially as $t \to \infty$. As with the higher dimensional cases, we instead compute the expansion of $\|\hat{u}(t,\cdot)\|_{2,\delta}^2 = X_1(t) + Y_1(t) + Z_1(t)$. We also assume $K \in \mathbb{N}$ and the initial data of (1.1) satisfy $u_0 \in H^1(\mathbb{R}) \cap L^{1,2K}(\mathbb{R})$ and $u_1 \in L^2(\mathbb{R}) \cap L^{1,2K}(\mathbb{R})$. These assumptions will enable us to prove Theorem 1 part 3 over the next few sections.

Expansion of $X_1(t)$

We follow the work in finding the expansion of $X_1(t)$ outlined in section 2.1.2. Let us keep in mind that the dimension $N = 1$, and so the "multi-indices" are simply non-negative integers. We thus arrive at one-dimensional version of (2.20). In particular, $X_1(t) = X_1^A(t) + X_1^B(t) + O(t^{-K+1/2})$ as $t \to \infty$, where

$$X_1^A(t) = 2|P_1|^2 \int_0^\delta r^{-2} e^{-tr^2} \frac{\sin^2(tr\sqrt{1-r^2/4})}{1-r^2/4} dr, \tag{2.72}$$

$$X_1^B(t) = \sum_{\sigma=1}^{K-1} 2d_{2\sigma} \int_0^\delta r^{2\sigma-2} e^{-tr^2} \frac{\sin^2(tr\sqrt{1-r^2/4})}{1-r^2/4} dr. \tag{2.73}$$

To analyze $X_1^A(t)$ we substitute (2.22) into (2.72) with $L := K$ to obtain as $t \to \infty$

$$X_1^A(t) = 2|P_1|^2 I_{1,-2}^\delta(t) + \sum_{k=1}^{K-1} \frac{|P_1|^2}{2^{2k-1}} I_{1,2k-2}^\delta(t) + O(t^{-K+\frac{1}{2}}) \tag{2.74}$$

We treat the first term of (2.74) by using Lemma 7 with $M := K$. The sum in (2.74) is treated by Lemma 2 with $Q = Q_k := \max\{0, K-2k+1\}$. Therefore we have the asymptotic expansion

$$X_1^A(t) = \pi|P_1|^2 t - \sqrt{\pi}|P_1|^2 t^{\frac{1}{2}} + \sum_{k=1}^{K-1} \frac{|P_1|^2}{2^{2k+1}} \Gamma(k-\tfrac{1}{2}) t^{-k+\frac{1}{2}} + O(t^{-K+\frac{1}{2}})$$

$$= \pi |P_1|^2 t + \sum_{j=0}^{K-1} X_{1,1}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \tag{2.75}$$

where

$$X_{1,1}^{[j]} = \begin{cases} -\sqrt{\pi}|P_1|^2 & j = 0 \\ \frac{|P_1|^2}{2^{2j+1}} \Gamma(j - \frac{1}{2}) & 1 \leq j \leq K-1. \end{cases}$$

Turning to $X_1^B(t)$, we substitute (2.22) into (2.73) with $L = L_\sigma := K - \sigma$. Then as $t \to \infty$,

$$X_1^B(t) = \sum_{\sigma=1}^{K-1} \sum_{k=0}^{L_\sigma - 1} \frac{d_{2\sigma}}{2^{2k-1}} I_{1,2\sigma+2k-2}^{\delta}(t) + O(t^{-K+\frac{1}{2}}).$$

We again apply Lemma 2 with $Q = Q_{\sigma,k} := \max\{0, K - 2\sigma - 2k + 1\}$ to conclude

$$\begin{aligned} X_1^B(t) &= \sum_{\substack{1 \leq \sigma + k \leq K-1 \\ \sigma \geq 1}} \frac{d_{2\sigma}}{2^{2k+1}} \Gamma(\sigma + k - \tfrac{1}{2}) t^{-\sigma-k+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \\ &= \sum_{j=0}^{K-1} X_{1,2}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \end{aligned} \tag{2.76}$$

where

$$X_{1,2}^{[j]} = \begin{cases} \sum\limits_{\substack{\sigma+k=j \\ \sigma \geq 1}} \frac{d_{2\sigma}}{2^{2k+1}} \Gamma(j - \frac{1}{2}) & 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

We combine the results (2.75) and (2.76) with the fact that $X_1(t) = X_1^A(t) + X_1^B(t) + O(t^{-K+1/2})$ as $t \to \infty$ to obtain the asymptotic expansion:

$$X_1(t) = \pi |P_1|^2 t + \sum_{j=0}^{K-1} (X_{1,1}^{[j]} + X_{1,2}^{[j]}) t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty). \tag{2.77}$$

Expansion of $Y_1(t)$

We proceed as in the expansion of $Y_1(t)$ from section 2.1.2 with $N = 1$. Therefore as $t \to \infty$, $Y_1(t) = Y_1^A(t) + Y_1^B(t) - Y_1^C(t) + O(t^{-K+1/2})$, where

$$Y_1^A(t) = \sum_{\sigma=0}^{K-2} 2f_{2\sigma} \int_0^\delta r^{2\sigma} e^{-tr^2} \cos^2(tr\sqrt{1-r^2/4})dr,$$
$$Y_1^B(t) = \sum_{\sigma=0}^{K-2} \frac{f_{2\sigma}}{2} \int_0^\delta r^{2\sigma+2} e^{-tr^2} \frac{\sin^2(tr\sqrt{1-r^2/4})}{1-r^2/4}dr,$$
$$Y_1^C(t) = \sum_{\sigma=0}^{K-2} 2f_{2\sigma} \int_0^\delta r^{2\sigma+1} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4})\cos(tr\sqrt{1-r^2/4})}{\sqrt{1-r^2/4}}dr.$$

We note that each of the powers of r in the preceding integrals is a non-negative integer, so the analysis from section 2.1.2 carries over. Thus

$$Y_1^A(t) = \sum_{j=0}^{K-1} Y_{1,1}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \tag{2.78}$$

where

$$Y_{1,1}^{[j]} = \begin{cases} \frac{f_{2j-2}}{2}\Gamma(j-\frac{1}{2}) & 1 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore,

$$Y_1^B(t) = \sum_{j=0}^{K-1} Y_{1,2}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \tag{2.79}$$

where

$$Y_{1,2}^{[j]} = \begin{cases} \sum_{\sigma+k=j-2} \frac{f_{2\sigma}}{2^{2k+3}}\Gamma(j-\frac{1}{2}) & 2 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Y_1^C(t) = O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty). \tag{2.80}$$

Combining the fact $Y_1(t) = Y_1^A(t) + Y_1^B(t) - Y_1^C(t) + O(t^{-K+1/2})$ as $t \to \infty$ with (2.78), (2.79), and (2.80), we obtain

$$Y_1(t) = \sum_{j=0}^{K-1} (Y_{1,1}^{[j]} + Y_{1,2}^{[j]}) t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty). \tag{2.81}$$

Expansion of $Z_1(t)$

Let us begin by assuming that $K \geq 2$. With this assumption, we may repeat the analysis of the expansion of $Z_1(t)$ from section 2.1.2 with $N = 1$ to obtain $Z_1(t) = Z_1^A(t) + Z_1^B(t) - Z_1^C(t) + O(t^{-K+1/2})$ as $t \to \infty$, where

$$Z_1^A(t) = 4\Re(l_0) \int_0^\delta r^{-1} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4})\cos(tr\sqrt{1-r^2/4})}{\sqrt{1-r^2/4}} dr, \tag{2.82}$$

$$Z_1^B(t) = \sum_{\sigma=1}^{K-1} 4\Re(l_{2\sigma}) \int_0^\delta r^{2\sigma-1} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4})\cos(tr\sqrt{1-r^2/4})}{\sqrt{1-r^2/4}} dr, \tag{2.83}$$

$$Z_1^C(t) = \sum_{\sigma=0}^{K-1} 2\Re(l_{2\sigma}) \int_0^\delta r^{2\sigma} e^{-tr^2} \frac{\sin^2(tr\sqrt{1-r^2/4})}{1-r^2/4} dr. \tag{2.84}$$

To find the expansion of $Z_1^A(t)$, we substitute (2.34) into (2.82) with $L := K$. Then

$$Z_1^A(t) = 4\Re(P_1\bar{P}_0) I_{3,-1}^\delta(t) + \sum_{k=1}^{K-1} 4\Re(P_1\bar{P}_0)\alpha_k I_{3,2k-1}^\delta(t) + O(t^{-K+\frac{1}{2}}). \tag{2.85}$$

We use Lemma 9 with $M := K$ to treat the first term of (2.85). We treat the sum in (2.85) by using Lemma 4 with $Q = Q_k := \max\{0, K-2k\}$. Thus

$$Z_1^A(t) = \pi\Re(P_1\bar{P}_0) + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty). \tag{2.86}$$

To analyze $Z_1^B(t)$, we substitute (2.34) into (2.83) with $L = L_\sigma := K - \sigma$. Then

$$Z_1^B(t) = \sum_{\sigma=1}^{K-1} \sum_{k=0}^{L_\sigma - 1} 4\Re(P_1\bar{P}_0)\alpha_k I_{3,2\sigma+2k-1}^\delta(t) + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty).$$

Again we use Lemma 4, now with $Q = Q_{\sigma,k} := \max\{0, K - 2\sigma - 2k\}$. We have

$$Z_1^B(t) = O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty). \tag{2.87}$$

We note that $Z_1^C(t)$ in (2.84) is the one-dimensional analog of $Z_1^B(t)$ from section 2.1.2. Since the powers of r in (2.84) are non-negative, the analysis carries over and we obtain

$$Z_1^C(t) = \sum_{j=0}^{K-1} Z_{1,1}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \tag{2.88}$$

where

$$Z_{1,1}^{[j]} = \begin{cases} \sum_{\sigma+k=j-1} \frac{\Re(l_{2\sigma})}{2^{2k+1}} \Gamma(j - \frac{1}{2}) & 1 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Combining results (2.86), (2.87), and (2.88) with the fact that $Z_1(t) = Z_1^A(t) + Z_1^B(t) - Z_1^C(t) + O(t^{-K+1/2})$ as $t \to \infty$, we obtain

$$Z_1(t) = \pi \Re(P_1 \bar{P}_0) - \sum_{j=0}^{K-1} Z_{1,1}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty). \tag{2.89}$$

2.3.3 Asymptotic expansion of $\|u(t,\cdot)\|_2^2$ as $t \to \infty$

We use the facts that $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_{2,\delta}^2 + O(e^{-\eta t})$ as $t \to \infty$ and $\|\hat{u}(t,\cdot)\|_{2,\delta}^2 = X_1(t) + Y_1(t) + Z_1(t)$ to combine results (2.77), (2.81), and (2.89) and obtain the asymptotic expansion

$$\|u(t,\cdot)\|_2^2 = \pi |P_1|^2 t + \pi \Re(P_1 \bar{P}_0) + \sum_{j=0}^{K-1} W_j t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty),$$

where, for $j \in \{0, \ldots, K-1\}$,

$$W_j = X_{1,1}^{[j]} + X_{1,2}^{[j]} + Y_{1,1}^{[j]} + Y_{1,2}^{[j]} - Z_{1,1}^{[j]}.$$

This expansion proves part 3 of Theorem 1.

The first three coefficients

Let us assume $K = 3$. Then $\|u(t,\cdot)\|_2^2 = \pi|P_1|^2 t + \pi\Re(P_1\bar{P}_0) + W_0 t^{1/2} + W_1 t^{-1/2} + W_2 t^{-3/2} + O(t^{-5/2})$ as $t \to \infty$, where

$$W_0 = -\sqrt{\pi}|P_1|^2, \qquad W_1 = \frac{|P_1|^2}{2^3}\Gamma(\tfrac{1}{2}) + \frac{d_2}{2}\Gamma(\tfrac{1}{2}) + \frac{f_0}{2}\Gamma(\tfrac{1}{2}) - \frac{\Re(l_0)}{2}\Gamma(\tfrac{1}{2}),$$
$$W_2 = \frac{|P_1|^2}{2^5}\Gamma(\tfrac{3}{2}) + \sum_{\substack{\sigma+k=2\\ \sigma\geq 1}} \frac{d_{2\sigma}}{2^{2k+1}}\Gamma(\tfrac{3}{2}) + \frac{f_2}{2}\Gamma(\tfrac{3}{2}) + \frac{f_0}{2^3}\Gamma(\tfrac{3}{2}) - \sum_{\sigma+k=1} \frac{\Re(l_{2\sigma})}{2^{2k+1}}\Gamma(\tfrac{3}{2}).$$

Each of the W_j $(j \in \{0,1,2\})$ may be simplified to obtain

$$W_0 = -\sqrt{\pi}|P_1|^2, \qquad W_1 = \frac{\sqrt{\pi}|P_1|^2}{8} + \frac{\sqrt{\pi}|P_0|^2}{2} - \frac{\sqrt{\pi}\Re(P_1\bar{P}_0)}{2} + \frac{\sqrt{\pi}d_2}{2},$$
$$W_2 = \frac{\sqrt{\pi}|P_1|^2}{64} + \frac{\sqrt{\pi}|P_0|^2}{16} - \frac{\sqrt{\pi}\Re(P_1\bar{P}_0)}{16} + \frac{\sqrt{\pi}d_2}{16} + \frac{\sqrt{\pi}f_2}{4} - \frac{\sqrt{\pi}\Re(l_2)}{4} + \frac{\sqrt{\pi}d_4}{4}\ .$$

Remark. It is worth noting that the values for the W_j $(j \in \{0,1,2\})$ are precisely what we would obtain, if we let $N = 1$ in section 2.1.3.

Chapter 3

Proof of Theorem 2

3.1 The space dimension $N \geq 3$ case

Now that we have found the asymptotic expansion of $\|u(t,\cdot)\|_2^2$ as $t \to \infty$ in dimension $N \geq 3$, we seek the full asymptotic expansion of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ as $t \to \infty$ in dimension $N \geq 3$. To do this we need to finish computing the expansions of $X(t)$, $Y(t)$, and $Z(t)$, each given in (1.13), (1.14), and (1.15), respectively. But before we are able to find these expansions, we must first study the expansions of some more integrals that will appear in the analysis.

3.1.1 Auxiliary lemmas

Lemma 10. *For $0 < \epsilon < 2$, $m \in \mathbb{N}_0$, and $t > 0$ define*

$$G_{2,m}^{\epsilon}(t) := \int_0^{\epsilon} r^m \exp(-tr^2 - irt\sqrt{1 - r^2/4} + irt)dr. \tag{3.1}$$

If $Q \in \mathbb{N}_0$, then as $t \to \infty$

$$G_{2,m}^{\epsilon}(t) = \sum_{n=0}^{2Q-1} C_{m,n} t^{-\frac{m}{2} - \frac{n}{2} - \frac{1}{2}} + O(t^{-\frac{m}{2} - Q - \frac{1}{2}}),$$

where, for $n \in \{0, \ldots, 2Q-1\}$,

$$C_{m,n} \in \begin{cases} \mathbb{R} & n \in \mathbb{N}_0 \text{ even} \\ i\mathbb{R} & n \in \mathbb{N} \text{ odd.} \end{cases}$$

Proof. Let $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$. With $g(t,r) = \exp(irt(2-\sqrt{4-r^2}))$ as in (2.2), we rewrite (3.1) as

$$G^{\epsilon}_{2,m}(t) = \int_0^{\epsilon} r^m e^{-tr^2} g(\tfrac{t}{2}, r) dr. \tag{3.2}$$

Expanding $g(\frac{t}{2}, r)$ in a Taylor series about $r = 0$, we obtain $g(\frac{t}{2}, r) = \sum_{k=0}^{\infty} g_k(\frac{t}{2}) r^k$. By Lemma 1 with $t \geq 2$, we have that for any $J \in \mathbb{N}_0$ there exists $C_J > 0$ such that

$$|\tilde{g}_J(\tfrac{t}{2}, r)| \leq \frac{C_J}{2^J} t^J r^{3J}. \tag{3.3}$$

We then rewrite (3.2) as

$$G^{\epsilon}_{2,m}(t) = \sum_{k=0}^{3J-1} g_k(\tfrac{t}{2}) \int_0^{\epsilon} r^{m+k} e^{-tr^2} dr + \int_0^{\epsilon} r^m e^{-tr^2} \tilde{g}_J(\tfrac{t}{2}, r) dr.$$

By the estimate (3.3), for $t \geq 2$ the final term is $O(t^{-(m+J+1)/2})$. Thus as $t \to \infty$

$$\begin{aligned} G^{\epsilon}_{2,m}(t) &= \sum_{k=0}^{3J-1} g_k(\tfrac{t}{2}) \int_0^{\epsilon} r^{m+k} e^{-tr^2} dr + O(t^{-\frac{m+J+1}{2}}) \\ &= \sum_{k=0}^{3J-1} g_k(\tfrac{t}{2}) \left(\frac{1}{2} \Gamma(\tfrac{m+k+1}{2}) t^{-\frac{m}{2} - \frac{k}{2} - \frac{1}{2}} + O(e^{-t\epsilon^2/2}) \right) + O(t^{-\frac{m+J+1}{2}}) \\ &= \sum_{k=0}^{3J-1} \frac{1}{2} g_k(\tfrac{t}{2}) \Gamma(\tfrac{m+k+1}{2}) t^{-\frac{m}{2} - \frac{k}{2} - \frac{1}{2}} + O(t^{-\frac{m+J+1}{2}}). \end{aligned} \tag{3.4}$$

Let $Q \in \mathbb{N}_0$ and suppose that, after collecting terms according to the power of t, we want the first $2Q \in 2\mathbb{N}_0$ terms of the above expansion and then a O-term. Then as $t \to \infty$

$$G^{\epsilon}_{2,m}(t) = \sum_{n=0}^{2Q-1} C_{m,n} t^{-\frac{m}{2}-\frac{n}{2}-\frac{1}{2}} + O(t^{-\frac{m}{2}-Q-\frac{1}{2}}). \tag{3.5}$$

So that the error terms of (3.4) and (3.5) agree, we require that $-\frac{m}{2} - Q - \frac{1}{2} = -\frac{m}{2} - \frac{J}{2} - \frac{1}{2}$, which is equivalent to $J = 2Q$.

Observe that $C_{m,n}$ is the coefficient of $t^{-m/2-n/2-1/2}$ when the sum (3.4) is rearranged according to the powers of t. Each term $\frac{1}{2}g_k(\frac{t}{2})\Gamma(\frac{m+k+1}{2})t^{-m/2-k/2-1/2}$ has leading degree $j-\frac{m}{2}-\frac{k}{2}-\frac{1}{2}$, where j is a non-negative integer at most $\frac{k}{3}$. Note that $-\frac{m}{2}-\frac{n}{2}-\frac{1}{2} = j-\frac{m}{2}-\frac{k}{2}-\frac{1}{2}$ if and only if $n = k-2j$. Since $j \leq \frac{k}{3}$, this implies that $\frac{k}{3} \leq n$, which in turn implies $0 \leq k \leq 3n$. So for fixed $m \in \mathbb{N}_0$ and $n \in \{0, \ldots, 2Q-1\}$, to compute $C_{m,n}$ we must find the first $3n+1$ terms of (3.4) and look for the coefficient of $t^{-m/2-n/2-1/2}$ after collecting terms according to the powers of t.

For $m \in \mathbb{N}_0$, we may compute the first few $C_{m,n}$ using a computer algebra system like *Mathematica*,

$$\begin{aligned}
C_{m,0} &= \frac{1}{2}\Gamma(\tfrac{m+1}{2}) \\
C_{m,1} &= \frac{i}{16}\Gamma(\tfrac{m+4}{2}) \\
C_{m,2} &= -\frac{1}{256}\Gamma(\tfrac{m+7}{2}) \\
C_{m,3} &= \frac{i}{256}\Gamma(\tfrac{m+6}{2}) - \frac{i}{6144}\Gamma(\tfrac{m+10}{2}) \\
C_{m,4} &= \frac{1}{196608}\Gamma(\tfrac{m+13}{2}) - \frac{1}{2048}\Gamma(\tfrac{m+9}{2}).
\end{aligned} \tag{3.6}$$

We now determine if $C_{m,n}$ is real or purely imaginary depending only on n. Suppose $n \geq 0$ is even and fix $0 \leq k \leq 3n$. Then a generic term of $\frac{1}{2}g_k(\frac{t}{2})\Gamma(\frac{m+k+1}{2})t^{-m/2-k/2-1/2}$ is of the form $a(it)^l t^{-m/2-k/2-1/2}$, where $a \in \mathbb{R}$ and $0 \leq l \leq \frac{k}{3}$. In order for this term to contribute to $C_{m,n}$, it must hold that the powers of t are equal, i.e., $l - \frac{m}{2} - \frac{k}{2} - \frac{1}{2} = -\frac{m}{2} - \frac{n}{2} - \frac{1}{2}$, which is equivalent to $n = k - 2l$. Since n is assumed to be even, it must hold that k is even. So in fact, only even $0 \leq k \leq 3n$ are able to contribute terms to $C_{m,n}$ if n is even. Since k is even, $g_k(\frac{t}{2})$ is an even polynomial in t, and thus l is even. Hence for even n, any contributing term to $C_{m,n}$ is real, and thus $C_{m,n} \in \mathbb{R}$. A similar argument shows that if n is odd, then $C_{m,n} \in i\mathbb{R}$. □

Lemma 11. *For* $0 < \epsilon < 2$, $m \in \mathbb{N}_0$, *and* $t > 0$ *define*

$$G^{\epsilon}_{3,m}(t) := \int_0^{\epsilon} r^m \exp(-tr^2 - irt\sqrt{1 - r^2/4} - irt)dr.$$

If $Q \in \mathbb{N}_0$, *then as* $t \to \infty$

$$G^{\epsilon}_{3,m}(t) = \sum_{n=0}^{Q-1} \tilde{B}_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}),$$

where, for $n \in \{0, \dots, Q-1\}$,

$$\tilde{B}_{m,n} \in \begin{cases} \mathbb{R} & m \in \mathbb{N} \text{ odd} \\ i\mathbb{R} & m \in \mathbb{N}_0 \text{ even.} \end{cases}$$

Proof. Let $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$. We observe that $G^{\epsilon}_{3,m}(t)$ is closely related to $G^{\epsilon}_{1,m}(t)$:

$$G^{\epsilon}_{3,m}(t) = \int_0^{\epsilon} r^m \exp(-tr^2 - 2irt) g(\tfrac{t}{2}, r)dr,$$

where $g(t, r) = \exp(irt(2 - \sqrt{4 - r^2}))$ as in (2.2). The analysis of $G^{\epsilon}_{1,m}(t)$ carries over. Since we now have the argument $\frac{t}{2}$ instead of t in g, for $Q \in \mathbb{N}_0$ the expansion is

$$G^{\epsilon}_{3,m}(t) = \sum_{n=0}^{Q-1} \tilde{B}_{m,n} t^{-m-n-1} + O(t^{-m-Q-1}) \qquad (t \to \infty).$$

The $\tilde{B}_{m,n}$ satisfy $\tilde{B}_{m,n} \in i\mathbb{R}$ if m is even, and $\tilde{B}_{m,n} \in \mathbb{R}$ if m is odd, just like $B_{m,n}$. Further, the $\tilde{B}_{m,n}$ can be computed on a computer algebra system like *Mathematica*; the first three are

$$\tilde{B}_{m,0} = A_{m,0}$$
$$\tilde{B}_{m,1} = A_{m,1} \tag{3.7}$$
$$\tilde{B}_{m,2} = A_{m,2} + \frac{i}{8}A_{m+3,0}.$$

□

Lemma 12. *Let $m \in \mathbb{N}_0$, $0 < \epsilon < 2$, and $t > 0$. Define*

$$H^{\epsilon}_{1,m}(t) := \int_0^{\epsilon} r^m e^{-tr^2} \sin(tr\sqrt{1-r^2/4}) \sin(tr) dr,$$
$$H^{\epsilon}_{2,m}(t) := \int_0^{\epsilon} r^m e^{-tr^2} \cos(tr\sqrt{1-r^2/4}) \cos(tr) dr.$$

1. *If $m \in \mathbb{N}_0$ is even, then as $t \to \infty$*

$$H^{\epsilon}_{1,m}(t) = H^{\epsilon}_{2,m}(t) = \sum_{n=0}^{Q-1} \frac{1}{2} C_{m,2n} t^{-\frac{m}{2}-n-\frac{1}{2}} + O(t^{-\frac{m}{2}-Q-\frac{1}{2}}), \tag{3.8}$$

where $Q \in \mathbb{N}_0$.

2. *If $m \in \mathbb{N}_0$ is odd, then as $t \to \infty$*

$$H^{\epsilon}_{1,m}(t) = \sum_{n=0}^{Q-1} \frac{1}{2} C_{m,2n} t^{-\frac{m}{2}-n-\frac{1}{2}} + O(t^{-\frac{m}{2}-Q-\frac{1}{2}}) - \sum_{n=0}^{R-1} \frac{1}{2} \tilde{B}_{m,n} t^{-m-n-1} + O(t^{-m-R-1}),$$
$$H^{\epsilon}_{2,m}(t) = \sum_{n=0}^{Q-1} \frac{1}{2} C_{m,2n} t^{-\frac{m}{2}-n-\frac{1}{2}} + O(t^{-\frac{m}{2}-Q-\frac{1}{2}}) + \sum_{n=0}^{R-1} \frac{1}{2} \tilde{B}_{m,n} t^{-m-n-1} + O(t^{-m-R-1}), \tag{3.9}$$

where $Q, R \in \mathbb{N}_0$.

Proof. Let $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$. We use the fact that $\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y))$ to obtain

$$H^{\epsilon}_{1,m}(t) = \frac{1}{2}\Re(G^{\epsilon}_{2,m}(t)) - \frac{1}{2}\Re(G^{\epsilon}_{3,m}(t)). \tag{3.10}$$

We apply lemmas 10 and 11 to (3.10) to find that as $t \to \infty$

$$H^{\epsilon}_{1,m}(t) = \frac{1}{2}\Re\left(\sum_{n=0}^{2Q-1} C_{m,n} t^{-\frac{m}{2}-\frac{n}{2}-\frac{1}{2}} + O(t^{-\frac{m}{2}-Q-\frac{1}{2}})\right) - \frac{1}{2}\Re\left(\sum_{n=0}^{R-1} \tilde{B}_{m,n} t^{-m-n-1} + O(t^{-m-R-1})\right),$$

where $Q, R \in \mathbb{N}_0$. We now recall the following: for any $m \in \mathbb{N}_0$, $C_{m,n} \in i\mathbb{R}$ if $n \in \mathbb{N}$ is odd and $C_{m,n} \in \mathbb{R}$ if $n \in \mathbb{N}_0$ is even; and $\tilde{B}_{m,n} \in i\mathbb{R}$ if $m \in \mathbb{N}_0$ is even and $\tilde{B}_{m,n} \in \mathbb{R}$ if $m \in \mathbb{N}_0$ is odd. If $m \in \mathbb{N}_0$ is odd, (3.9) is immediate by linearity of $\Re(\cdot)$. If $m \in \mathbb{N}_0$ is even, then by setting $R \geq Q$ we obtain (3.8).

To prove the claims about $H^{\epsilon}_{2,m}(t)$, we use the fact that $\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$ and repeat the previous argument. □

Lemma 13. *Let $m \in \mathbb{N}_0$, $0 < \epsilon < 2$, and $t > 0$. Define*

$$\begin{aligned} H^{\epsilon}_{3,m}(t) &:= \int_0^{\epsilon} r^m e^{-tr^2} \sin(tr\sqrt{1-r^2/4})\cos(tr)dr, \\ H^{\epsilon}_{4,m}(t) &:= \int_0^{\epsilon} r^m e^{-tr^2} \cos(tr\sqrt{1-r^2/4})\sin(tr)dr. \end{aligned}$$

1. *If $m \in \mathbb{N}_0$ is even, then as $t \to \infty$*

$$\begin{aligned} H^{\epsilon}_{3,m}(t) = &\sum_{n=0}^{R-1} -\frac{1}{2}\Im(\tilde{B}_{m,n})t^{-m-n-1} + O(t^{-m-R-1}) \\ &- \sum_{n=0}^{Q-1} \frac{1}{2}\Im(C_{m,2n+1})t^{-\frac{m}{2}-n-1} + O(t^{-\frac{m}{2}-Q-\frac{1}{2}}), \end{aligned} \tag{3.11}$$

$$\begin{aligned} H^{\epsilon}_{4,m}(t) = &\sum_{n=0}^{R-1} -\frac{1}{2}\Im(\tilde{B}_{m,n})t^{-m-n-1} + O(t^{-m-R-1}) \\ &+ \sum_{n=0}^{Q-1} \frac{1}{2}\Im(C_{m,2n+1})t^{-\frac{m}{2}-n-1} + O(t^{-\frac{m}{2}-Q-\frac{1}{2}}), \end{aligned}$$

where $Q, R \in \mathbb{N}_0$.

2. If $m \in \mathbb{N}_0$ is odd, then as $t \to \infty$

$$H^{\epsilon}_{3,m}(t) = -\sum_{n=0}^{Q-1} \frac{1}{2}\Im(C_{m,2n+1}) t^{-\frac{m}{2}-n-1} + O(t^{-\frac{m}{2}-Q-\frac{1}{2}}), \tag{3.12}$$

$$H^{\epsilon}_{4,m}(t) = \sum_{n=0}^{Q-1} \frac{1}{2}\Im(C_{m,2n+1}) t^{-\frac{m}{2}-n-1} + O(t^{-\frac{m}{2}-Q-\frac{1}{2}}),$$

where $Q \in \mathbb{N}_0$.

Proof. Let $0 < \epsilon < 2$ and $m \in \mathbb{N}_0$. We use the fact that $\sin(x)\cos(y) = \frac{1}{2}(\sin(x+y)+\sin(x-y))$ to obtain

$$H^{\epsilon}_{3,m}(t) = -\frac{1}{2}\Im(G^{\epsilon}_{3,m}(t)) - \frac{1}{2}\Im(G^{\epsilon}_{2,m}(t)). \tag{3.13}$$

We apply lemmas 11 and 10 to (3.13) to find that as $t \to \infty$

$$H^{\epsilon}_{3,m}(t) = -\frac{1}{2}\Im\left(\sum_{n=0}^{R-1} \tilde{B}_{m,n} t^{-m-n-1} + O(t^{-m-R-1})\right) - \frac{1}{2}\Im\left(\sum_{n=0}^{2Q-1} C_{m,n} t^{-\frac{m}{2}-\frac{n}{2}-\frac{1}{2}} + O(t^{-\frac{m}{2}-Q-\frac{1}{2}})\right),$$

where $Q, R \in \mathbb{N}_0$. By the facts about $\tilde{B}_{m,n}$ and $C_{m,n}$ we recalled in the proof of Lemma 12 we conclude that if $m \in \mathbb{N}_0$ is even, then (3.11) follows from linearity of $\Im(\cdot)$. If $m \in \mathbb{N}_0$ is odd, then by setting $R \geq Q$ we obtain (3.12).

To prove the claims about $H^{\epsilon}_{4,m}(t)$, we use the fact that $\cos(x)\sin(y) = \frac{1}{2}(\sin(x+y) - \sin(x-y))$ and repeat the previous argument. □

3.1.2 Intermediate computations

We are now in a position to find the asymptotic expansions of the remaining parts of $X(t)$, $Y(t)$, and $Z(t)$. In doing so, we will be able to determine the asymptotic expansion of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ as $t \to \infty$. Using results about $X_1(t)$, $Y_1(t)$, and $Z_1(t)$ from section 2.1.2 and the sections that follow, we will be able to prove Theorem 2 for the dimension $N \geq 3$ case. Let us assume that $K \in \mathbb{N}$ and the initial data of (1.1) satisfy $u_0 \in H^1(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$.

Expansion of $X_2(t)$

$$X_2(t) = |P_1|^2 \int_{|\xi|\le\delta} e^{-t|\xi|^2} \frac{\sin^2(t|\xi|)}{|\xi|^2} d\xi.$$

We switch to polar and use the fact that $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$ to obtain

$$\begin{aligned} X_2(t) &= D_{\mathbf{0}}|P_1|^2 \int_0^\delta r^{N-3} e^{-tr^2} \sin^2(tr) dr \\ &= \frac{D_{\mathbf{0}}|P_1|^2}{2} \int_0^\delta r^{N-3} e^{-tr^2} dr - \frac{D_{\mathbf{0}}|P_0|^2}{2} \int_0^\delta r^{N-3} e^{-tr^2} \cos(2tr) dr. \end{aligned} \tag{3.14}$$

As $t \to \infty$ the integral in the first term of (3.14) is $\frac{1}{2}\Gamma(\frac{N}{2}-1)t^{-N/2+1} + O(e^{-t\delta^2/2})$. The integral in the second term of (3.14) equals $\Re(F^\delta_{N-3}(t))$, where $F^\delta_k(t)$ is as in (2.6). Thus by (3.14) and (2.9), as $t \to \infty$

$$X_2(t) = \frac{D_{\mathbf{0}}|P_1|^2}{4}\Gamma(\tfrac{N}{2}-1)t^{-\frac{N}{2}+1} - \sum_{p=0}^{P-1} \frac{D_{\mathbf{0}}|P_1|^2}{2}\Re(A_{N-3,p})t^{-N+2-p} + O(t^{-N+2-P}). \tag{3.15}$$

By definition of $A_{k,p}$ in (2.10), the sum in (3.15) vanishes unless $N \ge 4$ is even. And if $N \ge 4$ is even, we require $K \ge \frac{N}{2}$ else each term of the sum in (3.15) is $O(t^{-K-N/2+1})$. In any case we choose $P := \max\{0, \lceil K - \frac{N}{2} + 1\rceil\}$ so that $O(t^{-N+2-P}) = O(t^{-K-N/2+1})$. Thus

$$X_2(t) = \sum_{j=0}^{K-1} X_{2,1}^{[j]} t^{-\frac{N}{2}+1-j} + O(t^{-K-\frac{N}{2}+1}) \qquad (t \to \infty), \tag{3.16}$$

where

$$X_{2,1}^{[j]} = \begin{cases} \frac{D_{\mathbf{0}}|P_1|^2}{4}\Gamma(\frac{N}{2}-1) & j = 0 \\ -\frac{D_{\mathbf{0}}|P_1|^2}{2} A_{N-3,j-\frac{N}{2}+1} & N \ge 4 \text{ even and } \frac{N}{2}-1 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Expansion of $X_3(t)$

We observe

$$X_3(t) = 2\Re\left(\bar{P}_1 \int_{|\xi|\le\delta} e^{-\frac{t|\xi|^2}{2}}(\hat{u}_1(\xi) + |\xi|^2\hat{u}_0(\xi))\frac{\sin(t|\xi|)}{|\xi|}h(t,\xi)d\xi\right).$$

We substitute (1.5) and (1.11) in to the above expression for $X_3(t)$ and estimate the resulting integral with the $O(|\xi|^{2K})$-term to find it is $O(t^{-K-N/2+1})$ as $t \to \infty$. Thus as $t \to \infty$

$$\begin{aligned} X_3(t) &= 2\Re\left(\sum_{|\sigma|\le 2K-1} \bar{P}_1 c_\sigma \int_{|\xi|\le\delta} \xi^\sigma e^{-t|\xi|^2}\frac{\sin(t|\xi|\sqrt{1-|\xi|^2/4})\sin(t|\xi|)}{|\xi|^2\sqrt{1-|\xi|^2/4}}d\xi\right) + O(t^{-K-\frac{N}{2}+1}) \\ &= \sum_{|\sigma|\le K-1} 2\Re(\bar{P}_1 c_{2\sigma})\int_{|\xi|\le\delta} \xi^{2\sigma} e^{-t|\xi|^2}\frac{\sin(t|\xi|\sqrt{1-|\xi|^2/4})\sin(t|\xi|)}{|\xi|^2\sqrt{1-|\xi|^2/4}}d\xi + O(t^{-K-\frac{N}{2}+1}) \\ &= \sum_{|\sigma|\le K-1} 2\Re(\bar{P}_1 c_{2\sigma})D_\sigma \int_0^\delta r^{2|\sigma|+N-3}e^{-tr^2}\frac{\sin(tr\sqrt{1-r^2/4})\sin(tr)}{\sqrt{1-r^2/4}}dr + O(t^{-K-\frac{N}{2}+1}). \end{aligned} \tag{3.17}$$

The second equality follows from the fact that if $|\sigma|\le K-1$ has an odd entry, then the integral evaluates to zero.

We use (2.34) in (3.17) with $L = L_\sigma := K - |\sigma|$ to get rid of the denominator in the integrals. By choice of L_σ, the integrals with the $O(r^{2L})$-terms are all $O(t^{-K-N/2+1})$ as $t \to \infty$. Thus

$$X_3(t) = \sum_{|\sigma|\le K-1}\sum_{k=0}^{L_\sigma-1} \Re(\bar{P}_1 c_{2\sigma})D_\sigma \alpha_k H^\delta_{1,2|\sigma|+2k+N-3} + O(t^{-K-\frac{N}{2}+1}) \qquad (t\to\infty). \tag{3.18}$$

Since $N \ge 3$ we may use Lemma 12 in (3.18) with $Q = Q_{\sigma,k} := K - |\sigma| - k$ and $R = R_{\sigma,k} := \max\{0, \lceil K - 2|\sigma| - 2k - \frac{N}{2} + 1\rceil\}$ to obtain that as $t \to \infty$

$$X_3(t) = \sum_{j=0}^{K-1}(X^{[j]}_{3,1} - X^{[j]}_{3,2})t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}), \tag{3.19}$$

where

$$X_{3,1}^{[j]} = \sum_{|\sigma|+k+n=j} \Re(\bar{P}_1 c_{2\sigma}) D_\sigma \alpha_k C_{2|\sigma|+2k+N-3,2n},$$

$$X_{3,2}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-\frac{N}{2}+1} \Re(\bar{P}_1 c_{2\sigma}) D_\sigma \alpha_k \tilde{B}_{2|\sigma|+2k+N-3,n} & N \geq 4 \text{ even and } \frac{N}{2}-1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Asymptotic expansion of $X(t)$ as $t \to \infty$

Combining results (2.23), (3.16), and (3.19) for $X_1(t)$, $X_2(t)$, and $X_3(t)$, respectively, with the fact that $X(t) = X_1(t) + X_2(t) - X_3(t)$, we obtain the expansion as $t \to \infty$

$$X(t) = \sum_{j=0}^{K-1} V_j^X t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}), \tag{3.20}$$

where, for $j \in \{0, \ldots, K-1\}$,

$$V_j^X = X_{1,1}^{[j]} - X_{1,2}^{[j]} + X_{2,1}^{[j]} - X_{3,1}^{[j]} + X_{3,2}^{[j]}.$$

The first three coefficients for each $N \geq 3$

Let us first assume that $N \geq 3$ is odd and $K = 3$, or that $N \geq 8$ is even and $K = 3$. Then $X(t) = V_0^X t^{-N/2+1} + V_1^X t^{-N/2} + V_2^X t^{-N/2-1} + O(t^{-N/2-2})$ as $t \to \infty$, where

$$V_0^X = \frac{d_{\mathbf{0}} D_{\mathbf{0}}}{4} \Gamma(\tfrac{N}{2}-1) + \frac{D_{\mathbf{0}} |P_1|^2}{4} \Gamma(\tfrac{N}{2}-1) - \Re(\bar{P}_1 c_{\mathbf{0}}) D_{\mathbf{0}} \alpha_0 C_{N-3,0},$$

$$V_1^X = \sum_{|\sigma|+k=1} \frac{d_{2\sigma} D_\sigma}{4^{k+1}} \Gamma(\tfrac{N}{2}) - \sum_{|\sigma|+k+n=1} \Re(\bar{P}_1 c_{2\sigma}) D_\sigma \alpha_k C_{2|\sigma|+2k+N-3,2n},$$

$$V_2^X = \sum_{|\sigma|+k=2} \frac{d_{2\sigma} D_\sigma}{4^{k+1}} \Gamma(\tfrac{N}{2}+1) - \sum_{|\sigma|+k+n=2} \Re(\bar{P}_1 c_{2\sigma}) D_\sigma \alpha_k C_{2|\sigma|+2k+N-3,2n}.$$

We recall facts (2.43) and (2.44) from section 2.1.3 about the D_σ, the identities for the

$C_{m,n}$ given in (3.6), and that $d_{\mathbf{0}} = |P_1|^2$ in order to simplify each V_i^X $(i \in \{0,1,2\})$.

$$V_0^X = 0, \qquad V_1^X = \frac{N(N+2)|P_1|^2\pi^{\frac{N}{2}}}{512} + \frac{\pi^{\frac{N}{2}}}{2N}\sum_{j=1}^{N} d_{2e_j} - \frac{\pi^{\frac{N}{2}}}{N}\sum_{j=1}^{N}\Re(\bar{P}_1 c_{2e_j})$$

$$\begin{aligned} V_2^X = &\frac{-N(N^4+20N^3-628N^2-4208N-18048)|P_1|^2\pi^{\frac{N}{2}}}{3145728} + \frac{\pi^{\frac{N}{2}}}{16}\sum_{j=1}^{N} d_{2e_j} \\ &+ \frac{(N^2+6N-56)\pi^{\frac{N}{2}}}{1024}\sum_{j=1}^{N}\Re(\bar{P}_1 c_{2e_j}) + \frac{3\pi^{\frac{N}{2}}}{4(N+2)}\sum_{j=1}^{N} d_{4e_j} + \frac{\pi^{\frac{N}{2}}}{4(N+2)}\sum_{1\le i<j\le N} d_{2(e_i+e_j)} \\ &- \frac{3\pi^{\frac{N}{2}}}{2(N+2)}\sum_{j=1}^{N}\Re(\bar{P}_1 c_{4e_j}) - \frac{\pi^{\frac{N}{2}}}{2(N+2)}\sum_{1\le i<j\le N}\Re(\bar{P}_1 c_{2(e_i+e_j)}). \end{aligned}$$

Remark. In the case $N \ge 3$ odd and $K = 3$, the only terms from (3.20) that contribute non-zero terms for some $j \in \{0,1,2\}$ are $X_{1,1}^{[j]}$, $X_{2,1}^{[j]}$, and $X_{3,1}^{[j]}$. These terms are *exactly* the terms that contribute similarly for the case $N \ge 8$ even and $K = 3$. It is straightforward to verify that the quantities obtained above for the V_j^X are valid for the case $N \ge 8$ even and $K = 3$.

In the case $N \in \{4,6\}$ and $K = 3$, all five terms of (3.20) contribute non-zero terms to the expansion for some $j \in \{0,1,2\}$. Nonetheless, through a straightforward (yet slightly longer) computation, it can be shown that the quantities obtained above for the V_j^X are valid for the case $N \in \{4,6\}$ and $K = 3$ too.

It is also worth noting that in all cases the $t^{-N/2+1}$-term vanishes.

Expansion of $Y_2(t)$

We use the fact that $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ and then switch to polar to obtain

$$\begin{aligned} Y_2(t) &= |P_0|^2 \int_{|\xi|\le\delta} e^{-t|\xi|^2}\cos^2(t|\xi|)d\xi \\ &= \frac{D_{\mathbf{0}}|P_0|^2}{2}\int_0^{\delta} r^{N-1}e^{-tr^2}dr + \frac{D_{\mathbf{0}}|P_0|^2}{2}\int_0^{\delta} r^{N-1}e^{-tr^2}\cos(2tr)dr. \end{aligned} \tag{3.21}$$

The asymptotics of the integral in the first term of (3.21) are $\int_0^\delta r^{N-1}e^{-tr^2}dr = \frac{1}{2}\Gamma(\frac{N}{2})t^{-N/2} + O(e^{-t\delta^2/2})$ as $t \to \infty$. The integral in the second term of (3.21) equals $\Re(F^\delta_{N-1}(t))$. Thus, combining (2.9) and (3.21) we obtain as $t \to \infty$

$$Y_2(t) = \frac{D_{\mathbf{0}}|P_0|^2}{4}\Gamma(\tfrac{N}{2})t^{-\frac{N}{2}} + \sum_{p=0}^{P-1}\frac{D_{\mathbf{0}}|P_0|^2}{2}\Re(A_{N-1,p})t^{-N-p} + O(t^{-N-P}). \tag{3.22}$$

Much the same as with $X_2(t)$, the sum in (3.22) vanishes unless $N \geq 4$ is even, and even so, every term is $O(t^{-K-N/2+1})$ unless $K \geq \frac{N}{2}+2$. Regardless, we choose $P := \max\{0, \lceil K - \frac{N}{2} - 1\rceil\}$ so that $O(t^{-N-P}) = O(t^{-K-N/2+1})$. Therefore

$$Y_2(t) = \sum_{j=0}^{K-1} Y_{2,1}^{[j]} t^{-\frac{N}{2}+1-j} + O(t^{-K-\frac{N}{2}+1}) \qquad (t \to \infty), \tag{3.23}$$

where

$$Y_{2,1}^{[j]} = \begin{cases} \frac{D_{\mathbf{0}}|P_0|^2}{4}\Gamma(\frac{N}{2}) & j = 1 \\ \frac{D_{\mathbf{0}}|P_0|^2}{2}A_{N-1,j-\frac{N}{2}-1} & N \geq 4 \text{ even and } \frac{N}{2}+1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. We observe from the definition of $Y_{2,1}^{[j]}$ that the asymptotic expansion of $Y_2(t)$ has no terms unless $K \geq 2$.

Expansion of $Y_3(t)$

$$Y_3(t) = 2\Re\left(\bar{P}_0 \int_{|\xi|\leq\delta} \hat{u}_0(\xi)e^{-\frac{t|\xi|^2}{2}}\cos(t|\xi|)\partial_t h(t,\xi)d\xi\right).$$

Using (2.25) we obtain $Y_3(t) = Y_3^A(t) - Y_3^B(t)$, where

$$Y_3^A(t) = 2\Re\left(\bar{P}_0 \int_{|\xi|\leq\delta} \hat{u}_0(\xi)e^{-t|\xi|^2}\cos(t|\xi|\sqrt{1-|\xi|^2/4})\cos(t|\xi|)d\xi\right), \tag{3.24}$$

$$Y_3^B(t) = \Re\left(\bar{P}_0 \int_{|\xi|\leq\delta} \hat{u}_0(\xi)|\xi|e^{-t|\xi|^2}\frac{\sin(t|\xi|\sqrt{1-|\xi|^2/4})\cos(t|\xi|)}{\sqrt{1-|\xi|^2/4}}d\xi\right). \tag{3.25}$$

Let us assume that $K \geq 2$. Then into both (3.24) and (3.25) we substitute (1.4) truncated to $\sum_{|\sigma|\leq 2K-3} b_\sigma \xi^\sigma + O(|\xi|^{2K-2})$. Estimates of the integrals with the $O(|\xi|^{2K-2})$-terms are, respectively, $O(t^{-K-N/2+1})$ and $O(t^{-K-N/2+1/2})$ as $t \to \infty$, both of which are $O(t^{-K-N/2+1})$ as $t \to \infty$. Therefore as $t \to \infty$

$$
\begin{aligned}
Y_3^A(t) &= 2\Re\left(\sum_{|\sigma|\leq 2K-3} \bar{P}_0 b_\sigma \int_{|\xi|\leq\delta} \xi^\sigma e^{-t|\xi|^2}\cos(t|\xi|\sqrt{1-|\xi|^2/4})\cos(t|\xi|)d\xi\right) + O(t^{-K-\frac{N}{2}+1}) \\
&= \sum_{|\sigma|\leq K-2} 2\Re(\bar{P}_0 b_{2\sigma}) D_\sigma \int_0^\delta r^{2|\sigma|+N-1} e^{-tr^2}\cos(tr\sqrt{1-r^2/4})\cos(tr)dr + O(t^{-K-\frac{N}{2}+1}) \\
&= \sum_{|\sigma|\leq K-2} 2\Re(\bar{P}_0 b_{2\sigma}) D_\sigma H^\delta_{2,2|\sigma|+N-1}(t) + O(t^{-K-\frac{N}{2}+1}). \qquad (3.26)
\end{aligned}
$$

The second equality is obtained by noting that multi-indices $|\sigma|\leq 2K-3$ with any odd entry yield an integral that evaluates to zero, then switching to polar and using linearity of $\Re(\cdot)$.

We now apply Lemma 12 to equality (3.26) with $Q = Q_{\sigma,k} := K-|\sigma|-1$ and $R = R_{\sigma,k} := \max\{0, \lceil K-2|\sigma|-\frac{N}{2}-1\rceil\}$. Thus as $t \to \infty$

$$
Y_3^A(t) = \sum_{j=0}^{K-1}(Y_{3,1}^{[j]} + Y_{3,2}^{[j]})t^{-\frac{N}{2}+1-j} + O(t^{-K-\frac{N}{2}+1}), \qquad (3.27)
$$

where

$$
Y_{3,1}^{[j]} = \begin{cases} \sum_{|\sigma|+n=j-1} \Re(\bar{P}_0 b_{2\sigma}) D_\sigma C_{2|\sigma|+N-1,2n} & 1 \leq j \leq K-1 \\ 0 & \text{otherwise,} \end{cases}
$$

$$
Y_{3,2}^{[j]} = \begin{cases} \sum_{2|\sigma|+n=j-\frac{N}{2}-1} \Re(\bar{P}_0 b_{2\sigma}) D_\sigma \tilde{B}_{2|\sigma|+N-1,n} & N\geq 4 \text{ even and } \frac{N}{2}+1\leq j\leq K-1 \\ 0 & \text{otherwise.} \end{cases}
$$

We now reduce $Y_3^B(t)$ as we did with $Y_3^A(t)$. Any multi-index σ ($|\sigma|\leq 2K-3$) with an odd entry yields and integral that evaluates to zero. So we only consider those multi-indicies with even entries, and then switch to polar. So as $t \to \infty$

$$\begin{aligned} Y_3^B(t) &= \Re\left(\sum_{2K-3} \bar{P}_0 b_\sigma \int_{|\xi|\le\delta} \xi^\sigma |\xi| e^{-t|\xi|^2} \frac{\sin(t|\xi|\sqrt{1-|\xi|^2/4})\cos(t|\xi|)}{\sqrt{1-|\xi|^2/4}} d\xi\right) + O(t^{-K-\frac{N}{2}+1}) \\ &= \sum_{|\sigma|\le K-2} \Re(\bar{P}_0 b_{2\sigma}) D_\sigma \int_0^\delta r^{2|\sigma|+N} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4})\cos(tr)}{\sqrt{1-r^2/4}} dr + O(t^{-K-\frac{N}{2}+1}). \end{aligned} \tag{3.28}$$

We substitute (2.34) into (3.28) with $L = L_\sigma := K - |\sigma| - 1$. By choice of L_σ each integral with a $O(r^{2L_\sigma})$-term is $O(t^{-K-N/2+1})$ as $t \to \infty$. Therefore as $t \to \infty$

$$\begin{aligned} Y_3^B(t) &= \sum_{|\sigma|\le K-2} \sum_{k=0}^{L_\sigma-1} \Re(\bar{P}_0 b_{2\sigma}) D_\sigma \alpha_k \int_0^\delta r^{2|\sigma|+2k+N} e^{-tr^2} \sin(tr\sqrt{1-r^2/4})\cos(tr) dr + O(t^{-K-\frac{N}{2}+1}) \\ &= \sum_{|\sigma|\le K-2} \sum_{k=0}^{L_\sigma-1} \Re(\bar{P}_0 b_{2\sigma}) D_\sigma \alpha_k H^\delta_{3,2|\sigma|+2k+N}(t) + O(t^{-K-\frac{N}{2}+1}). \end{aligned} \tag{3.29}$$

We now apply Lemma 13 to (3.29) with $R = R_{\sigma,k} := \max\{0, \lceil K - 2|\sigma| - 2k - \frac{N}{2} - 2\rceil\}$ and $Q = Q_{\sigma,k} := K - |\sigma| - k - 1$ to obtain

$$Y_3^B(t) = \sum_{j=0}^{K-1} (Y_{3,3}^{[j]} + Y_{3,4}^{[j]}) t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}) \qquad (t \to \infty), \tag{3.30}$$

where

$$Y_{3,3}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-\frac{N}{2}-2} \frac{-\Re(\bar{P}_0 b_{2\sigma}) D_\sigma \alpha_k}{2} \Im(\tilde{B}_{2|\sigma|+2k+N,n}) & N \ge 4 \text{ even and } \frac{N}{2}+2 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{3,4}^{[j]} = \begin{cases} \sum_{|\sigma|+k+n=j-2} \frac{-\Re(\bar{P}_0 b_{2\sigma}) D_\sigma \alpha_k}{2} \Im(C_{2|\sigma|+2k+N,2n+1}) & 2 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. We observe that unless $K \geq 3$, the asymptotic expansion of $Y_3^B(t)$ will not have any non-zero terms.

Combining the fact that $Y_3(t) = Y_3^A(t) - Y_3^B(t)$ with results (3.27) and (3.30) for $Y_3^A(t)$ and $Y_3^B(t)$, respectively, we conclude that as $t \to \infty$,

$$Y_3(t) = \sum_{j=0}^{K-1} (Y_{3,1}^{[j]} + Y_{3,2}^{[j]} - Y_{3,3}^{[j]} - Y_{3,4}^{[j]}) t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}). \tag{3.31}$$

Asymptotic expansion of $Y(t)$ as $t \to \infty$

We combine results (2.36), (3.22), and (3.31) for $Y_1(t)$, $Y_2(t)$, and $Y_3(t)$, respectively, with the fact that $Y(t) = Y_1(t) + Y_2(t) - Y_3(t)$ to obtain the expansion as $t \to \infty$

$$Y(t) = \sum_{j=0}^{K-1} V_j^Y t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}), \tag{3.32}$$

where, for $j \in \{0, \ldots, K-1\}$,

$$V_j^Y = Y_{1,1}^{[j]} + Y_{1,2}^{[j]} + Y_{1,3}^{[j]} - Y_{1,4}^{[j]} - Y_{1,5}^{[j]} + Y_{2,1}^{[j]} - Y_{3,1}^{[j]} - Y_{3,2}^{[j]} + Y_{3,3}^{[j]} + Y_{3,4}^{[j]}.$$

The first three coefficients for each $N \geq 3$

We first assume that $N \geq 3$ is odd and $K = 3$, or that $N \geq 8$ is even and $K = 3$. Then $Y(t) = V_0^Y t^{-N/2+1} + V_1^Y t^{-N/2} + V_2^Y t^{-N/2-1} + O(t^{-N/2-2})$ as $t \to \infty$, where

$$V_0^Y = 0, \qquad V_1^Y = \frac{f_{\mathbf{0}} D_{\mathbf{0}}}{4} \Gamma(\tfrac{N}{2}) + \frac{D_{\mathbf{0}} |P_0|^2}{4} \Gamma(\tfrac{N}{2}) - \Re(\bar{P}_0 b_{\mathbf{0}}) D_{\mathbf{0}} C_{N-1,0},$$
$$V_2^Y = \sum_{|\sigma|=1} \frac{f_{2\sigma} D_\sigma}{4} \Gamma(\tfrac{N}{2}+1) + \frac{f_{\mathbf{0}} D_{\mathbf{0}}}{16} \Gamma(\tfrac{N}{2}+1) - \sum_{|\sigma|+n=1} \Re(\bar{P}_0 b_{2\sigma}) D_\sigma C_{N+2|\sigma|-1,2n}$$
$$+ \frac{-\Re(\bar{P}_0 b_{\mathbf{0}}) D_{\mathbf{0}} \alpha_0}{2} \Im(C_{N,1}).$$

We may simplify each V_i^Y $(i \in \{0,1,2\})$ with the identities for the $C_{m,n}$ given in (3.6).

$$V_0^Y = 0, \qquad V_1^Y = 0,$$
$$V_2^Y = \frac{N(N^2-10N+40)|P_0|^2\pi^{\frac{N}{2}}}{1024} + \frac{\pi^{\frac{N}{2}}}{4}\sum_{j=1}^{N} f_{2e_j} - \frac{\pi^{\frac{N}{2}}}{2}\sum_{j=1}\Re(\bar{P}_0 b_{2e_j}).$$

Remark. It is worth noting that in the computation of the V_j^Y, the only terms from (3.32) that contribute a non-zero quantity for some $j \in \{0,1,2\}$ are $Y_{1,1}^{[j]}$, $Y_{1,3}^{[j]}$, $Y_{2,1}^{[j]}$, $Y_{3,1}^{[j]}$, and $Y_{3,4}^{[j]}$. These are *exactly* the terms that would contribute similarly in the cases $N \geq 4$ even and $K = 3$. In fact, it is straightforward to show that the quantities obtained above for the V_j^Y are valid for $N \geq 4$ even and $K = 3$. Additionally it is worth noting that both the $t^{-N/2+1}$-term and $t^{-N/2}$-term vanish.

Expansion of $Z_2(t)$

We begin with

$$Z_2(t) = 2\Re\left(\bar{P}_0\int_{|\xi|\leq\delta}(\hat{u}_1(\xi)+|\xi|^2\hat{u}_0(\xi))e^{-\frac{t|\xi|^2}{2}}h(t,\xi)\cos(t|\xi|)d\xi\right). \tag{3.33}$$

We then substitute (1.11) and (1.5) into (3.33), and estimate the resulting integral with the $O(|\xi|^{2K})$-term and find it is $O(t^{-K-N/2+1})$ as $t\to\infty$. After eliminating the integrals that evaluate to zero, and then switching to polar we obtain as $t\to\infty$

$$Z_2(t) = \sum_{|\sigma|\leq K-1} 2\Re(\bar{P}_0 c_{2\sigma})D_\sigma\int_0^\delta r^{2|\sigma|+N-2}e^{-tr^2}\frac{\sin(tr\sqrt{1-r^2/4})\cos(tr)}{\sqrt{1-r^2/4}}dr + O(t^{-K-\frac{N}{2}+1}). \tag{3.34}$$

Into (3.34) we substitute (2.34) with $L = L_\sigma := K - |\sigma|$. The resulting integrals with the $O(r^{2L_\sigma})$-terms are all $O(t^{-K-N/2+1})$ as $t\to\infty$. Thus as $t\to\infty$

$$Z_2(t) = \sum_{|\sigma|\leq K-1}\sum_{k=0}^{L_\sigma-1} 2\Re(\bar{P}_0 c_{2\sigma})D_\sigma\alpha_k\int_0^\delta r^{2|\sigma|+2k+N-2}e^{-tr^2}\sin(tr\sqrt{1-r^2/4})\cos(tr)dr + O(t^{-K-\frac{N}{2}+1})$$

$$= \sum_{|\sigma|\leq K-1} \sum_{k=0}^{L_\sigma - 1} 2\Re(\bar{P}_0 c_{2\sigma}) D_\sigma \alpha_k H^\delta_{3,2|\sigma|+2k+N-2}(t) + O(t^{-K-\frac{N}{2}+1}). \tag{3.35}$$

Let us now apply Lemma 13 to (3.35) with $R = R_{\sigma,k} := \max\{0, \lceil K - 2|\sigma| - 2k - \frac{N}{2} \rceil\}$ and $Q = Q_{\sigma,k} := K - |\sigma| - k$. Then as $t \to \infty$

$$Z_2(t) = \sum_{j=0}^{K-1} (Z_{2,1}^{[j]} + Z_{2,2}^{[j]}) t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}), \tag{3.36}$$

where

$$Z_{2,1}^{[j]} = \begin{cases} \sum\limits_{2|\sigma|+2k+n=j-\frac{N}{2}} -\Re(\bar{P}_0 c_{2\sigma}) D_\sigma \alpha_k \Im(\tilde{B}_{2|\sigma|+2k+N-2,n}) & N \geq 4 \text{ even and } \frac{N}{2} \leq j \leq K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{2,2}^{[j]} = \begin{cases} \sum\limits_{|\sigma|+k+n=j-1} -\Re(\bar{P}_0 c_{2\sigma}) D_\sigma \alpha_k \Im(C_{2|\sigma|+2k+N-2,2n+1}) & 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. To obtain any non-zero terms for $Z_2(t)$, $K \geq 2$ is necessary.

Expansion of $Z_3(t)$

We begin with

$$Z_3(t) = 2\Re\left(P_1 \int_{|\xi|\leq\delta} \bar{\hat{u}}_0(\xi) e^{-\frac{t|\xi|^2}{2}} \partial_t h(t,\xi) \frac{\sin(t|\xi|)}{|\xi|} d\xi \right).$$

Using the definition of $\partial_t h(t,\xi)$ as given in (2.25), we obtain two new integrals. Into each of the resulting integrals, we substitute (1.4) and estimate the $O(|\xi|^{2K})$-terms; they are both $O(t^{-K-N/2+1})$ as $t \to \infty$. We then get rid of the terms which evaluate to zero ($|\sigma| \leq 2K-1$ with an odd entry) and switch to polar to find $Z_3(t) = Z_3^A(t) - Z_3^B(t) + O(t^{-K-N/2+1})$ as

$t \to \infty$, where

$$Z_3^A(t) = \sum_{|\sigma| \le K-1} 2\Re(P_1 \bar{b}_{2\sigma}) D_\sigma \int_0^\delta r^{2|\sigma|+N-2} e^{-tr^2} \cos(tr(\sqrt{1-r^2/4})) \sin(tr) dr,$$

$$Z_3^B(t) = \sum_{|\sigma| \le K-1} \Re(P_1 \bar{b}_{2\sigma}) D_\sigma \int_0^\delta r^{2|\sigma|+N-1} e^{-tr^2} \frac{\sin(tr(\sqrt{1-r^2/4})) \sin(tr)}{\sqrt{1-r^2/4}} dr. \qquad (3.37)$$

To find an expansion for $Z_3^A(t)$, we observe that

$$Z_3^A(t) = \sum_{|\sigma| \le K-1} 2\Re(P_1 \bar{b}_{2\sigma}) D_\sigma H^\delta_{4,2|\sigma|+N-2}(t). \qquad (3.38)$$

Therefore by applying Lemma 13 to (3.38) with $R = R_\sigma := \max\{0, \lceil K - 2|\sigma| - \frac{N}{2} \rceil\}$ and $Q = Q_\sigma := K - |\sigma|$, we obtain

$$Z_3^A(t) = \sum_{j=0}^{K-1} (Z_{3,1}^{[j]} + Z_{3,2}^{[j]}) t^{-\frac{N}{2}+1-j} + O(t^{-K-\frac{N}{2}+1}) \qquad (3.39)$$

as $t \to \infty$, where

$$Z_{3,1}^{[j]} = \begin{cases} \sum\limits_{2|\sigma|+n=j-\frac{N}{2}} -\Re(P_1 \bar{b}_{2\sigma}) D_\sigma \Im(\tilde{B}_{2|\sigma|+N-2,n}) & N \ge 4 \text{ even and } \frac{N}{2} \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{3,2}^{[j]} = \begin{cases} \sum\limits_{|\sigma|+n=j-1} \Re(P_1 \bar{b}_{2\sigma}) D_\sigma \Im(C_{2|\sigma|+N-2,2n+1}) & 1 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

To find the expansion for $Z_3^B(t)$, we first substitute (2.34) into (3.37) with $L = L_\sigma := K - |\sigma| - 1$. With this choice of L_σ, each resulting integral with a $O(r^{2L_\sigma})$-term is $O(t^{-K-N/2+1})$ as $t \to \infty$. Then

$$Z_3^B(t) = \sum_{|\sigma| \le K-1} \sum_{k=0}^{L_\sigma - 1} \Re(P_1 \bar{b}_{2\sigma}) D_\sigma \alpha_k H^\delta_{1,2|\sigma|+2k+N-1}(t) + O(t^{-K-\frac{N}{2}+1}) \qquad (3.40)$$

as $t \to \infty$. We now apply Lemma 12 to (3.40) with $Q = Q_{\sigma,k} := K - |\sigma| - k - 1$ and $R = R_{\sigma,k} := \max\{0, \lceil K - 2|\sigma| - 2k - \frac{N}{2} - 1\rceil\}$ to obtain

$$Z_3^B(t) = \sum_{j=0}^{K-1} (Z_{3,3}^{[j]} - Z_{3,4}^{[j]}) t^{-\frac{N}{2}+1-j} + O(t^{-K-\frac{N}{2}+1}), \tag{3.41}$$

as $t \to \infty$, where

$$Z_{3,3}^{[j]} = \begin{cases} \sum_{|\sigma|+k+n=j-1} \frac{\Re(P_1 \bar{b}_{2\sigma}) D_\sigma \alpha_k}{2} C_{2|\sigma|+2k+N-1,2n} & 1 \le j \le K-1 \\ 0 & \text{otherwise}, \end{cases}$$

$$Z_{3,4}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-\frac{N}{2}-1} \frac{\Re(P_1 \bar{b}_{2\sigma}) D_\sigma \alpha_k}{2} \tilde{B}_{2|\sigma|+2k+N-1,n} & N \ge 4 \text{ even and } \frac{N}{2}+1 \le j \le K-1 \\ 0 & \text{otherwise}. \end{cases}$$

Remark. We observe from the definitions of the $Z_{3,i}^{[j]}$ $(i \in \{1, \ldots, 4\})$ that the expansion of $Z_3(t)$ has no non-zero terms unless $K \ge 2$.

Since $Z_3(t) = Z_3^A(t) - Z_3^B(t) + O(t^{-K-N/2+1})$, we combine results (3.39) and (3.41) for $Z_3^A(t)$ and $Z_3^B(t)$, respectively to obtain the asymptotic expansion as $t \to \infty$

$$Z_3(t) = \sum_{j=0}^{K-1} (Z_{3,1}^{[j]} + Z_{3,2}^{[j]} - Z_{3,3}^{[j]} + Z_{3,4}^{[j]}) t^{-\frac{N}{2}+1-j} + O(t^{-K-\frac{N}{2}+1}). \tag{3.42}$$

Expansion of $Z_4(t)$

Finally we seek the expansion of

$$Z_4(t) = 2\Re\left(P_1 \bar{P}_0 \int_{|\xi|\le\delta} e^{-t|\xi|^2} \frac{\sin(t|\xi|)\cos(t|\xi|)}{|\xi|} d\xi\right).$$

Switching to polar and using the identity $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$ we obtain

$$
\begin{aligned}
Z_4(t) &= \Re(P_1\bar{P}_0)D_{\mathbf{0}} \int_0^\delta r^{N-2} e^{-tr^2} \sin(2tr) dr \\
&= -\Re(P_1\bar{P}_0)D_{\mathbf{0}} \Im \left(\int_0^\delta r^{N-2} \exp(-tr^2 - 2itr) dr \right) \\
&= -\Re(P_1\bar{P}_0)D_{\mathbf{0}} \Im(F^\delta_{N-2}(t)), \qquad (3.43)
\end{aligned}
$$

where $F^\delta_k(t)$ is as in (2.6). We now use (2.9) in (3.43) with $P := \max\{0, \lceil K - \frac{N}{2} \rceil\}$ to obtain as $t \to \infty$

$$
\begin{aligned}
Z_4(t) &= \sum_{p=0}^{P-1} -\Re(P_1\bar{P}_0)D_{\mathbf{0}} \Im(A_{N-2,p}) t^{-N-p+1} + O(t^{-K-\frac{N}{2}+1}) \\
&= \sum_{j=0}^{K-1} Z^{[j]}_{4,1} t^{-\frac{N}{2}+1-j} + O(t^{-K-\frac{N}{2}+1}), \qquad (3.44)
\end{aligned}
$$

where

$$
Z^{[j]}_{4,1} = \begin{cases} -\Re(P_1\bar{P}_0)D_{\mathbf{0}} \Im(A_{N-2,j-\frac{N}{2}}) & N \geq 4 \text{ even and } \frac{N}{2} \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}
$$

Asymptotic expansion of $Z(t)$ as $t \to \infty$

Let us combine results (2.42), (3.36), (3.42), and (3.44) for $Z_1(t)$, $Z_2(t)$, $Z_3(t)$, and $Z_4(t)$, respectively, with the fact that $Z(t) = Z_1(t) - Z_2(t) - Z_3(t) + Z_4(t)$ to obtain the expansion as $t \to \infty$

$$
Z(t) = \sum_{j=0}^{K-1} V^Z_j t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}), \qquad (3.45)
$$

where, for $j \in \{0, \ldots, K-1\}$,

$$
V^Z_j = Z^{[j]}_{1,1} - Z^{[j]}_{1,2} + Z^{[j]}_{1,3} - Z^{[j]}_{2,1} - Z^{[j]}_{2,2} - Z^{[j]}_{3,1} - Z^{[j]}_{3,2} + Z^{[j]}_{3,3} - Z^{[j]}_{3,4} + Z^{[j]}_{4,1}.
$$

The first three coefficients for each $N \geq 3$

We initially assume that $N \geq 3$ is odd and $K = 3$, or that $N \geq 8$ is even and $K = 3$. Then $Z(t) = V_0^Z t^{-N/2+1} + V_1^Z t^{-N/2} + V_2^Z t^{-N/2-1} + O(t^{-N/2-2})$ as $t \to \infty$, where

$$V_0^Z = 0, \qquad V_1^Z = -\frac{\Re(l_{\mathbf{0}})}{4}\Gamma(\tfrac{N}{2}) + \Re(\bar{P}_0 c_{\mathbf{0}}) D_{\mathbf{0}} \alpha_0 \Im(C_{N-2,1}) - \Re(P_1 \bar{b}_{\mathbf{0}}) D_{\mathbf{0}} \Im(C_{N-2,1}) + \frac{\Re(P_1 \bar{b}_{\mathbf{0}})\alpha_0}{2} C_{N-1,0},$$

$$V_2^Z = -\sum_{|\sigma|+k=1} \frac{\Re(l_{2\sigma}) D_\sigma}{4^{k+1}} \Gamma(\tfrac{N}{2}+1) - \sum_{|\sigma|+k+n=1} -\Re(\bar{P}_0 c_{2\sigma}) D_\sigma \alpha_k \Im(C_{2|\sigma|+2k+N-2,2n+1}) - \sum_{|\sigma|+n=1} \Re(P_1 \bar{b}_{2\sigma}) D_\sigma \Im(C_{2|\sigma|+N-2,2n+1}) + \sum_{|\sigma|+k+n=1} \frac{\Re(P_1 \bar{b}_{2\sigma}) D_\sigma \alpha_k}{2} C_{2|\sigma|+2k+N-1,2n}$$

We simplify the V_i^Z ($i \in \{0,1,2\}$), using the identities (2.43) and (2.44) we have for D_σ from section 2.1.3 and $C_{m,n}$ from (3.6):

$$V_0^Z = 0, \qquad V_1^Z = 0,$$

$$V_2^Z = \frac{-N(N^2 - 2N + 56)\Re(P_1 \bar{P}_0)\pi^{\frac{N}{2}}}{2048} - \frac{\pi^{\frac{N}{2}}}{4} \sum_{j=1}^{N} \Re(l_{2e_j}) + \frac{(N+2)\pi^{\frac{N}{2}}}{32} \sum_{j=1}^{N} \Re(\bar{P}_0 c_{2e_j}) - \frac{(N-6)\pi^{\frac{N}{2}}}{32} \sum_{j=1}^{N} \Re(P_1 \bar{b}_{2e_j}).$$

Remark. Much the same as the V_j^Y, only some terms from (3.45) will contribute non-zero terms for some $j \in \{0,1,2\}$. These terms are $Z_{1,2}^{[j]}$, $Z_{2,2}^{[j]}$, $Z_{3,2}^{[j]}$, and $Z_{3,3}^{[j]}$, which just so happen to be the terms that contribute similarly for the case $N \geq 6$ even and $K = 3$. It is then straightforward to verify that the above quantities for the V_j^Z ($j \in \{0,1,2\}$) are valid for $N \geq 6$ even and $K = 3$. For the case $N = 4$ and $K = 3$, in addition to those already listed, the terms $Z_{1,1}^{[j]}$, $Z_{2,1}^{[j]}$, $Z_{3,1}^{[j]}$, and $Z_{4,1}^{[j]}$ also contribute non-zero terms for some $j \in \{0,1,2\}$. Nonetheless, upon simplifying, we find that the quantities above for the V_j^Z ($j \in \{0,1,2\}$) are also valid for the case $N = 4$ and $K = 3$.

3.1.3 Asymptotic expansion of $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ as $t\to\infty$

Let us recall that for the fixed $0<\delta<1$ there exists $\eta>0$ such that $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_{2,\delta}^2 + O(e^{-\eta t}) = X(t)+Y(t)+Z(t)+O(e^{-\eta t})$ as $t\to\infty$. We will now combine the results from the prior analysis to obtain the full asymptotic expansion of $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ as $t\to\infty$ in a single closed form for $N\geq 3$.

Let $N\geq 3$ and $K\geq 1$ be integers as given in the statement of Theorem 2. Then as $t\to\infty$

$$\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_{2,\delta}^2 + O(e^{-\eta t}) = \sum_{j=0}^{K-1} V_j t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}), \quad (3.46)$$

where $V_j = V_j^X + V_j^Y + V_j^Z$ for every $j\in\{0,\dots,K-1\}$. If we can verify that $V_0=0$, then (3.46) matches the statement of Theorem 2 and the proof is complete. To do so, we determine the first three coefficients V_0, V_1, and V_2.

Assume that $N\geq 3$ and $K=3$. Since for $i\in\{0,1,2\}$ the V_i^X, V_i^Y, and V_i^Z are given in a common form for $N\geq 3$, we have $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2 = V_0 t^{-N/2+1} + V_1 t^{-N/2} + V_2 t^{-N/2-1} + O(t^{-N/2-2})$ as $t\to\infty$, where

$$V_0 = 0, \qquad V_1 = \frac{N(N+2)|P_1|^2\pi^{\frac{N}{2}}}{512} + \frac{\pi^{\frac{N}{2}}}{2N}\sum_{j=1}^N d_{2e_j} - \frac{\pi^{\frac{N}{2}}}{N}\sum_{j=1}^N \Re(\bar{P}_1 c_{2e_j}),$$

$$\begin{aligned} V_2 =& \frac{-N(N^4+20N^3-628N^2-4208N-18048)|P_1|^2\pi^{\frac{N}{2}}}{3145728} + \frac{N(N^2-10N+40)|P_0|^2\pi^{\frac{N}{2}}}{1024} \\ &- \frac{N(N^2-2N+56)\Re(P_1\bar{P}_0)\pi^{\frac{N}{2}}}{2048} + \frac{\pi^{\frac{N}{2}}}{16}\sum_{j=1}^N d_{2e_j} + \frac{(N^2+6N-56)\pi^{\frac{N}{2}}}{1024}\sum_{j=1}^N \Re(\bar{P}_1 c_{2e_j}) \\ &+ \frac{\pi^{\frac{N}{2}}}{4}\sum_{j=1}^N f_{2e_j} - \frac{\pi^{\frac{N}{2}}}{2}\sum_{j=1}^N \Re(\bar{P}_0 b_{2e_j}) - \frac{\pi^{\frac{N}{2}}}{4}\sum_{j=1}^N \Re(l_{2e_j}) + \frac{(N+2)\pi^{\frac{N}{2}}}{32}\sum_{j=1}^N \Re(\bar{P}_0 c_{2e_j}) \\ &- \frac{(N-6)\pi^{\frac{N}{2}}}{32}\sum_{j=1}^N \Re(P_1\bar{b}_{2e_j}) + \frac{3\pi^{\frac{N}{2}}}{4(N+2)}\sum_{j=1}^N d_{4e_j} + \frac{\pi^{\frac{N}{2}}}{4(N+2)}\sum_{1\leq i<j\leq N} d_{2(e_i+e_j)} \\ &- \frac{3\pi^{\frac{N}{2}}}{2(N+2)}\sum_{j=1}^N \Re(\bar{P}_1 c_{4e_j}) - \frac{\pi^{\frac{N}{2}}}{2(N+2)}\sum_{1\leq i<j\leq N} \Re(\bar{P}_1 c_{2(e_i+e_j)}). \end{aligned}$$

Indeed $V_0 = 0$ and so completes the proof of Theorem 2 for space dimension $N \geq 3$.

3.1.4 Asymptotic expansion of $\|\nu(t,\cdot)\|_2^2$ as $t \to \infty$

Let us again recall that for the fixed $0 < \delta < 1$, $\|\nu(t,\cdot)\|_2^2 = \|\nu(t,\cdot)\|_{2,\delta}^2 + O(e^{-t\delta^2/2}) = X_2(t) + Y_2(t) + Z_4(t) + O(e^{-t\delta^2/2})$. We then combine results (3.16), (3.23), and (3.44) to obtain the proof of Corollary 1 part 1 and the expansion as $t \to \infty$

$$\|\nu(t,\cdot)\|_2^2 = \|\nu(t,\cdot)\|_{2,\delta}^2 + O(e^{-\frac{t\delta^2}{2}}) = \sum_{j=0}^{K-1} U_j t^{-j-\frac{N}{2}+1} + O(t^{-K-\frac{N}{2}+1}),$$

where, for $j \in \{0,\ldots,K-1\}$,

$$U_j = X_{2,1}^{[j]} + Y_{2,1}^{[j]} + Z_{4,1}^{[j]}.$$

The first $K = 3$ coefficients for dimension $N \geq 3$ odd or $N \geq 8$ even are

$$U_0 = \frac{|P_1|^2 \pi^{\frac{N}{2}}}{N-2}, \qquad U_1 = \frac{|P_0|^2 \pi^{\frac{N}{2}}}{2}, \qquad U_2 = 0.$$

The coefficients for $N = 4$ are

$$U_0 = \frac{|P_1|^2\pi^2}{2}, \qquad U_1 = \frac{|P_1|^2\pi^2}{4} + \frac{|P_0|^2\pi^2}{2}, \qquad U_2 = \frac{3|P_1|^2\pi^2}{8} - \frac{\Re(P_1\bar{P}_0)\pi^2}{2}.$$

Finally, the coefficients for $N = 6$ are

$$U_0 = \frac{|P_1|^2\pi^3}{4}, \qquad U_1 = \frac{|P_0|^2\pi^3}{2}, \qquad U_2 = -\frac{3|P_1|^2\pi^3}{16}.$$

Remark. For each $N \geq 3$ the first coefficient U_0 in the expansion is equal to that of $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot)\|_2^2$. This leads to the cancellation of the leading terms when considering the expansion of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$.

3.2 The dimension $N = 2$ case

3.2.1 Auxiliary lemma

Lemma 14. *For $0 < \epsilon < 2$ and $t > 0$, define*

$$H^{\epsilon}_{1,-1}(t) := \int_0^{\epsilon} r^{-1} e^{-tr^2} \sin(tr\sqrt{1 - r^2/4}) \sin(tr) dr.$$

If $M \in \mathbb{N}$, then as $t \to \infty$

$$H^{\epsilon}_{1,-1}(t) = \frac{1}{4}\ln(t) + \frac{\gamma}{4} + \frac{1}{2}\ln(2) + \sum_{n=1}^{M-1} \frac{1}{2n} C_{1,2n} t^{-n} + \sum_{n=0}^{M-2} -\frac{1}{2(n+1)} \tilde{B}_{1,n} t^{-n-1}$$
$$+ \sum_{1 \le 2k+n \le M-1} \frac{\beta_k}{2(2k+n)} \Im(\tilde{B}_{2k,n}) t^{-2k-n} + \sum_{1 \le k+n \le M-1} -\frac{\beta_k}{2(k+n)} \Im(C_{2k,2n+1}) t^{-k-n}$$
$$+ \sum_{n=1}^{M-1} \frac{1}{2n} \Im(\tilde{B}_{0,n}) t^{-n} + \sum_{n=1}^{M-1} \frac{1}{2n} \Im(C_{0,2n+1}) t^{-n} + O(t^{-M}).$$

Proof. Let $0 < \epsilon < 2$, $t > 0$, and $M \in \mathbb{N}$. Using the mean value theorem and Lebesgue's dominated convergence theorem, it is a routine argument to show that $\frac{d}{dt} H^{\epsilon}_{1,-1}(t) = -H^{\epsilon}_{1,1}(t) + H_1(t) + H^{\epsilon}_{3,0}(t)$, where

$$H_1(t) = \int_0^{\epsilon} \sqrt{1 - r^2/4}\, e^{-tr^2} \cos(tr\sqrt{1 - r^2/4}) \sin(tr) dr,$$

and $H^{\epsilon}_{1,1}(t)$ and $H^{\epsilon}_{3,0}(t)$ are as given in lemmas 12 and 13, respectively.

By Lemma 12 with $Q := M$ and $R := M - 1$, and the fact that $C_{1,0} = \frac{1}{2}$, as $t \to \infty$

$$-H^{\epsilon}_{1,1}(t) = -\frac{1}{4} t^{-1} + \sum_{n=1}^{M-1} -\frac{1}{2} C_{1,2n} t^{-n-1} + \sum_{n=0}^{M-2} \frac{1}{2} \tilde{B}_{1,n} t^{-n-2} + O(t^{-M-1}). \tag{3.47}$$

Similarly, by Lemma 13 with $R := M$ and $Q := M + 1$, and the facts that $\tilde{B}_{0,0} = -\frac{i}{2}$

and $C_{0,1} = \frac{i}{16}$, as $t \to \infty$

$$H^{\epsilon}_{3,0}(t) = \frac{7}{32}t^{-1} + \sum_{n=1}^{M-1} -\frac{1}{2}\Im(\tilde{B}_{0,n})t^{-n-1} + \sum_{n=1}^{M} -\frac{1}{2}\Im(C_{0,2n+1})t^{-n-1} + O(t^{-M-1}). \tag{3.48}$$

To determine the asymptotic expansion of $H_1(t)$, we expand $\sqrt{1 - r^2/4}$ in a Taylor polynomial with remainder about $r = 0$ as in (2.46). We then estimate the resulting integral with the $O(r^{2L})$-term and find it to be $O(t^{-L-1/2})$ as $t \to \infty$, where $L \in \mathbb{N}$. Hence as $t \to \infty$

$$H_1(t) = \sum_{k=0}^{L-1} \beta_k H^{\epsilon}_{4,2k}(t) + O(t^{-L-\frac{1}{2}}). \tag{3.49}$$

Let $L := M+1$ and substitute the result of Lemma 13 into (3.49) with $R = R_k := \max\{0, L-2k\}$ and $Q = Q_k := L - k$ so that both O-terms are instead $O(t^{-M-1})$. Thus as $t \to \infty$

$$H_1(t) = \sum_{2k+n\leq M} -\frac{\beta_k}{2}\Im(\tilde{B}_{2k,n})t^{-2k-n-1} + \sum_{k+n\leq M} \frac{\beta_k}{2}\Im(C_{2k,2n+1})t^{-k-n-1} + O(t^{-M-1}).$$

We use the facts that $\beta_0 = 1$, $\tilde{B}_{0,0} = -\frac{i}{2}$, and $C_{0,1} = \frac{i}{16}$ to obtain as $t \to \infty$

$$H_1(t) = \frac{9}{32}t^{-1} + \sum_{1\leq 2k+n\leq M-1} -\frac{\beta_k}{2}\Im(\tilde{B}_{2k,n})t^{-2k-n-1} + \sum_{1\leq k+n\leq M-1} \frac{\beta_k}{2}\Im(C_{2k,2n+1})t^{-k-n-1} + O(t^{-M-1}). \tag{3.50}$$

We now combine results (3.47), (3.48), and (3.50) with the fact that $\frac{d}{dt}H^{\epsilon}_{1,-1}(t) = -H^{\epsilon}_{1,1}(t) + H_1(t) + H^{\epsilon}_{3,0}(t)$ to obtain as $t \to \infty$

$$\begin{aligned}\frac{d}{dt}H^{\epsilon}_{1,-1}(t) = \frac{1}{4}t^{-1} &+ \left(\sum_{n=1}^{M-1} -\frac{1}{2}C_{1,2n}t^{-n-1} + \sum_{n=0}^{M-2} \frac{1}{2}\tilde{B}_{1,n}t^{-n-2} + O(t^{-M-1})\right) \\ &+ \left(\sum_{1\leq 2k+n\leq M-1} -\frac{\beta_k}{2}\Im(\tilde{B}_{2k,n})t^{-2k-n-1} + \sum_{1\leq k+n\leq M-1} \frac{\beta_k}{2}\Im(C_{2k,2n+1})t^{-k-n-1} + O(t^{-M-1})\right) \\ &+ \left(\sum_{n=1}^{M-1} -\frac{1}{2}\Im(\tilde{B}_{0,n})t^{-n-1} + \sum_{n=1}^{M-1} -\frac{1}{2}\Im(C_{0,2n+1})t^{-n-1} + O(t^{-M-1})\right).\end{aligned}$$

Therefore there is some constant C such that as $t \to \infty$

$$\begin{aligned} H^{\epsilon}_{1,-1}(t) = \frac{1}{4}\ln(t) + C + \sum_{n=1}^{M-1} \frac{1}{2n} C_{1,2n} t^{-n} + \sum_{n=0}^{M-2} -\frac{1}{2(n+1)} \tilde{B}_{1,n} t^{-n-1} \\ + \sum_{1\le 2k+n \le M-1} \frac{\beta_k}{2(2k+n)} \Im(\tilde{B}_{2k,n}) t^{-2k-n} + \sum_{1\le k+n\le M-1} -\frac{\beta_k}{2(k+n)} \Im(C_{2k,2n+1}) t^{-k-n} \\ + \sum_{n=1}^{M-1} \frac{1}{2n} \Im(\tilde{B}_{0,n}) t^{-n} + \sum_{n=1}^{M-1} \frac{1}{2n} \Im(C_{0,2n+1}) t^{-n} + O(t^{-M}). \end{aligned}$$

To determine the constant C we proceed as in the proof of Lemma 5 and compare $H^{\epsilon}_{1,-1}(t)$ to $J^{\epsilon}_{1,-1}(t)$. Since $|\sin^2(tr) - \sin(tr\sqrt{1-r^2/4})\sin(tr)| \le tr|1-\sqrt{1-r^2/4}| \le \frac{tr^3}{4}$ for $0 \le r \le 2$,

$$|J^{\epsilon}_{1,-1}(t) - H^{\epsilon}_{1,-1}(t)| \le \frac{t}{4}\int_0^{\epsilon} r^2 e^{-tr^2} dr \le \frac{t}{8}\Gamma(\tfrac{3}{2}) t^{-\frac{3}{2}} = \frac{1}{8}\Gamma(\tfrac{3}{2}) t^{-\frac{1}{2}} \to 0 \qquad (t \to \infty). \quad (3.51)$$

Furthermore

$$\begin{aligned} J^{\epsilon}_{1,-1}(t) - H^{\epsilon}_{1,-1}(t) &= \left(\frac{1}{4}\ln(t) + \frac{\gamma}{4} + \frac{\ln(2)}{2} + O(t^{-1})\right) - \left(\frac{1}{4}\ln(t) + C + O(t^{-1})\right) \\ &= \frac{\gamma}{4} + \frac{\ln(2)}{2} - C + O(t^{-1}) \to \frac{\gamma}{4} + \frac{\ln(2)}{2} - C \qquad (t \to \infty). \quad (3.52) \end{aligned}$$

Considering (3.51) and (3.52) together, $C = \frac{\gamma}{4} + \frac{\ln(2)}{2}$, which proves the lemma. □

3.2.2 Intermediate computations

We may now set about proving the dimension $N = 2$ case of Theorem 2. The proof will rely on the previous results for $X_1(t)$, $Y_1(t)$, and $Z_1(t)$, as well as the results of the following several sections. As usual, $K \in \mathbb{N}$ and the initial data of (1.1) satisfy $u_0 \in H^1(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$ and $u_1 \in L^2(\mathbb{R}^N) \cap L^{1,2K}(\mathbb{R}^N)$.

Expansion of $X_2(t)$

Let us recall that for $t > 0$

$$X_2(t) = |P_1|^2 \int_{|\xi|\le\delta} e^{-t|\xi|^2} \frac{\sin^2(t|\xi|)}{|\xi|^2} d\xi.$$

Upon switching to polar, we see that $X_2(t) = 2\pi|P_1|^2 J^\delta_{1,-1}(t)$, where $J^\delta_{1,-1}(t)$ is as given in (2.48). We apply Lemma 6 with $P = K-1$ to obtain the asymptotic expansion of $X_2(t)$ as $t \to \infty$:

$$X_2(t) = \frac{\pi|P_1|^2}{2}\ln(t) + \sum_{j=0}^{K-1} X^{[j]}_{2,1} t^{-j} + O(t^{-K}), \tag{3.53}$$

where

$$X^{[j]}_{2,1} = \begin{cases} \frac{\pi\gamma|P_1|^2}{2} + \pi|P_1|^2\ln(2) & j = 0 \\ -\frac{\pi|P_1|^2}{2j}\left(\frac{1}{2}\right)_j & 1 \le j \le K-1. \end{cases}$$

Expansion of $X_3(t)$

We analyze $X_3(t)$ as in section 3.1.2 and obtain (3.17) with $N = 2$. In particular, $X_3(t) = X^A_3(t) + X^B_3(t) + O(t^{-K})$ as $t \to \infty$, where

$$X^A_3(t) = 2\Re(\bar{P}_1 c_{\mathbf{0}}) D_{\mathbf{0}} \int_0^\delta r^{-1} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4})\sin(tr)}{\sqrt{1-r^2/4}} dr, \tag{3.54}$$

$$X^B_3(t) = \sum_{1\le|\sigma|\le K-1} 2\Re(\bar{P}_1 c_{2\sigma}) D_\sigma \int_0^\delta r^{2|\sigma|-1} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4})\sin(tr)}{\sqrt{1-r^2/4}} dr. \tag{3.55}$$

We first consider $X^A_3(t)$. We substitute (2.34) into (3.54) with $L := K$. Then

$$X^A_3(t) = 4\pi|P_1|^2 H^\delta_{1,-1}(t) + \sum_{k=1}^{K-1} 4\pi|P_1|^2 \alpha_k H^\delta_{1,2k-1}(t) + O(t^{-K}) \qquad (t\to\infty). \tag{3.56}$$

For the first term of (3.56) we use Lemma 14 with $M := K$. Therefore as $t \to \infty$

$$4\pi|P_1|^2 H^{\delta}_{1,-1}(t) = \pi|P_1|^2 \ln(t) + \sum_{j=0}^{K-1} (X^{[j]}_{3,1} + X^{[j]}_{3,2} + X^{[j]}_{3,3}) t^{-j} + O(t^{-K}),$$

where

$$X^{[j]}_{3,1} = \begin{cases} \pi\gamma|P_1|^2 + 2\pi|P_1|^2 \ln(2) & j = 0 \\ \frac{2\pi|P_1|^2}{j}(C_{1,2j} - \tilde{B}_{1,j-1} + \Im(\tilde{B}_{0,j}) + \Im(C_{0,2j+1})) & 1 \le j \le K-1, \end{cases}$$

$$X^{[j]}_{3,2} = \begin{cases} \sum\limits_{2k+n=j} \frac{2\pi|P_1|^2 \beta_k}{j} \Im(\tilde{B}_{2k,n}) & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$X^{[j]}_{3,3} = \begin{cases} \sum\limits_{k+n=j} -\frac{2\pi|P_1|^2 \beta_k}{j} \Im(C_{2k,2n+1}) & 1 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

For the sum in (3.56) we use Lemma 12 with $Q = Q_k := K - k$ and $R = R_k := \max\{0, K - 2k\}$. Thus as $t \to \infty$

$$\sum_{k=1}^{K-1} 4\pi|P_1|^2 \alpha_k H^{\delta}_{1,2k-1}(t) = \sum_{j=0}^{K-1} (X^{[j]}_{3,4} - X^{[j]}_{3,5}) t^{-j} + O(t^{-K}),$$

where

$$X^{[j]}_{3,4} = \begin{cases} \sum\limits_{\substack{k+n=j \\ k \ge 1}} 2\pi|P_1|^2 \alpha_k C_{2k-1,2n} & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$X^{[j]}_{3,5} = \begin{cases} \sum\limits_{\substack{2k+n=j \\ k \ge 1}} 2\pi|P_1|^2 \alpha_k \tilde{B}_{2k-1,n} & 2 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus as $t \to \infty$

$$X_3^A(t) = \pi|P_1|^2 \ln(t) + \sum_{j=0}^{K-1} (X_{3,1}^{[j]} + X_{3,2}^{[j]} + X_{3,3}^{[j]} + X_{3,4}^{[j]} - X_{3,5}^{[j]}) t^{-j} + O(t^{-K}). \tag{3.57}$$

We now consider $X_3^B(t)$. Into (3.55) we substitute (2.34) with $L = L_\sigma := K - |\sigma|$. Then by choice of each L_σ, as $t \to \infty$

$$X_3^B(t) = \sum_{1 \le |\sigma| \le K-1} \sum_{k=0}^{L_\sigma - 1} 2\Re(\bar{P}_1 c_{2\sigma}) D_\sigma \alpha_k H^\delta_{1,2|\sigma|+2k-1}(t) + O(t^{-K}). \tag{3.58}$$

We apply Lemma 12 to (3.58) with $Q = Q_{\sigma,k} := K - |\sigma| - k$ and $R = R_{\sigma,k} := \max\{0, K - 2|\sigma| - 2k\}$. Then as $t \to \infty$

$$X_3^B(t) = \sum_{j=0}^{K-1} (X_{3,6}^{[j]} - X_{3,7}^{[j]}) t^{-j} + O(t^{-K}), \tag{3.59}$$

where

$$X_{3,6}^{[j]} = \begin{cases} \displaystyle\sum_{\substack{|\sigma|+k+n=j \\ |\sigma| \ge 1}} \Re(\bar{P}_1 c_{2\sigma}) D_\sigma \alpha_k C_{2|\sigma|+2k-1,2n} & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$X_{3,7}^{[j]} = \begin{cases} \displaystyle\sum_{\substack{2|\sigma|+2k+n=j \\ |\sigma| \ge 1}} \Re(\bar{P}_1 c_{2\sigma}) D_\sigma \alpha_k \tilde{B}_{2|\sigma|+2k-1,n} & 2 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $X_3(t) = X_3^A(t) + X_3^B(t)$, we combine their expansions (3.57) and (3.59) to obtain the expansion as $t \to \infty$

$$X_3(t) = \pi|P_1|^2 \ln(t) + \sum_{j=0}^{K-1} (X_{3,1}^{[j]} + X_{3,2}^{[j]} + X_{3,3}^{[j]} + X_{3,4}^{[j]} - X_{3,5}^{[j]} + X_{3,6}^{[j]} - X_{3,7}^{[j]}) t^{-j} + O(t^{-K}). \tag{3.60}$$

Asymptotic expansion of $X(t)$ as $t \to \infty$

We combine results (2.58), (3.53), and (3.60) with the fact that $X(t) = X_1(t) + X_2(t) - X_3(t)$ to obtain the expansion as $t \to \infty$

$$X(t) = \sum_{j=0}^{K-1} V_j^X t^{-j} + O(t^{-K}),$$

where, for $j \in \{0, \ldots, K-1\}$,

$$V_j^X = X_{1,1}^{[j]} + X_{1,2}^{[j]} - X_{1,3}^{[j]} + X_{1,4}^{[j]} - X_{1,5}^{[j]} + X_{2,1}^{[j]} - X_{3,1}^{[j]} - X_{3,2}^{[j]} - X_{3,3}^{[j]} - X_{3,4}^{[j]} + X_{3,5}^{[j]} - X_{3,6}^{[j]} + X_{3,7}^{[j]}.$$

The first three coefficients

Let us assume that $K = 3$. Then $X(t) = V_0^X + V_1^X t^{-1} + V_2^X t^{-2} + O(t^{-3})$ as $t \to \infty$, where

$$\begin{aligned}
V_0^X &= \frac{\pi\gamma|P_1|^2}{2} + \pi|P_1|^2\ln(2) + \frac{\pi\gamma|P_1|^2}{2} + \pi|P_1|^2\ln(2) - \pi\gamma|P_1|^2 - 2\pi|P_1|^2\ln(2), \\
V_1^X &= \frac{\pi|P_1|^2}{2^3}\Gamma(1) - \pi|P_1|^2 B_{1,0} + 2\pi|P_1|^2\beta_0\Im(B_{0,1}) + \sum_{|\sigma|=1} \frac{d_{2\sigma}D_\sigma}{4}\Gamma(1) \\
&\quad - \frac{\pi|P_1|^2}{2}\left(\frac{1}{2}\right)_1 - 2\pi|P_1|^2(C_{1,2} - \tilde{B}_{1,0} + \Im(\tilde{B}_{0,1}) + \Im(C_{0,3})) - 2\pi|P_1|^2\beta_0\Im(\tilde{B}_{0,1}) \\
&\quad - \sum_{k+n=1} -2\pi|P_1|^2\beta_k\Im(C_{2k,2n+1}) - 2\pi|P_1|^2\alpha_1 C_{1,0} - \sum_{|\sigma|=1} \Re(\bar{P}_1 c_{2\sigma})D_\sigma\alpha_0 C_{1,0}, \\
V_2^X &= \frac{\pi|P_1|^2}{2^5}\Gamma(2) - \frac{\pi|P_1|^2}{2}B_{1,1} + \sum_{2k+n=2} \frac{2\pi|P_1|^2\beta_k}{2}\Im(B_{2k,n}) - \frac{\pi|P_1|^2}{4}B_{1,0} \\
&\quad + \sum_{\substack{|\sigma|+k=2 \\ |\sigma|\geq 1}} \frac{d_{2\sigma}D_\sigma}{4^{k+1}}\Gamma(2) - \sum_{|\sigma|=1} \frac{d_{2\sigma}D_\sigma}{2}B_{1,0} - \frac{\pi|P_1|^2}{4}\left(\frac{1}{2}\right)_2 \\
&\quad - \frac{2\pi|P_1|^2}{2}(C_{1,4} - \tilde{B}_{1,1} + \Im(\tilde{B}_{0,2}) + \Im(C_{0,5})) - \sum_{2k+n=2} \frac{2\pi|P_1|^2\beta_k}{2}\Im(\tilde{B}_{2k,n}) \\
&\quad - \sum_{k+n=2} -\frac{2\pi|P_1|^2\beta_k}{2}\Im(C_{2k,2n+1}) - \sum_{\substack{k+n=2 \\ k\geq 1}} 2\pi|P_1|^2\alpha_k C_{2k-1,2n} + 2\pi|P_1|^2\alpha_1\tilde{B}_{1,0} \\
&\quad - \sum_{\substack{|\sigma|+k+n=2 \\ |\sigma|\geq 1}} \Re(\bar{P}_1 c_{2\sigma})D_\sigma\alpha_k C_{2|\sigma|+2k-1,2n} + \sum_{|\sigma|=1} \Re(\bar{P}_1 c_{2\sigma})D_\sigma\alpha_0\tilde{B}_{1,0}.
\end{aligned}$$

Each of the V_j^X ($j \in \{0,1,2\}$) can be simplified using the formulas we have for the $B_{m,n}$, $\tilde{B}_{m,n}$, and $C_{m,n}$ from (2.14), (3.7), and (3.6), respectively. Hence

$$V_0^X = 0, \qquad V_1^X = \frac{\pi|P_1|^2}{64} + \frac{\pi}{4}\sum_{j=1}^{2} d_{2e_j} - \frac{\pi}{2}\sum_{j=1}^{2}\Re(\bar{P}_1 c_{2e_j}),$$

$$V_2^X = \frac{75\pi|P_1|^2}{4096} + \frac{3\pi}{16}\sum_{j=1}^{2} d_{2e_j} - \frac{37\pi}{128}\sum_{j=1}^{2}\Re(\bar{P}_1 c_{2e_j}) + \frac{3\pi}{16}\sum_{j=1}^{2} d_{4e_j} - \frac{3\pi}{8}\sum_{j=1}^{2}\Re(\bar{P}_1 c_{4e_j}) + \frac{\pi}{16} d_{2(1,1)} - \frac{\pi}{8}\Re(\bar{P}_1 c_{2(1,1)}).$$

Expansions of $Y_2(t)$ and $Y_3(t)$

To find the asymptotic expansions of $Y_2(t)$ and $Y_3(t)$, we refer back to section 3.1.2. From that analysis we see that all integrals obtained will have positive powers of r if $N = 2$. So the analysis carries over and we may use the results from those sections. In particular,

$$Y_2(t) = \sum_{j=0}^{K-1} Y_{2,1}^{[j]} t^{-j} + O(t^{-K}) \qquad (t \to \infty), \tag{3.61}$$

where

$$Y_{2,1}^{[j]} = \begin{cases} \frac{\pi|P_0|^2}{2} & j = 1 \\ \pi|P_0|^2 A_{1,j-2} & 2 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

And

$$Y_3(t) = \sum_{j=0}^{K-1} (Y_{3,1}^{[j]} + Y_{3,2}^{[j]} - Y_{3,3}^{[j]} - Y_{3,4}^{[j]}) t^{-j} + O(t^{-K}) \qquad (t \to \infty), \tag{3.62}$$

where

$$Y_{3,1}^{[j]} = \begin{cases} \displaystyle\sum_{|\sigma|+n=j-1} \Re(\bar{P}_0 b_{2\sigma}) D_\sigma C_{2|\sigma|+1,2n} & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{3,2}^{[j]} = \begin{cases} \sum_{2|\sigma|+n=j-2} \Re(\bar{P}_0 b_{2\sigma}) D_\sigma \tilde{B}_{2|\sigma|+1,n} & 2 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{3,3}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-3} -\frac{\Re(\bar{P}_0 b_{2\sigma}) D_\sigma \alpha_k}{2} \Im(\tilde{B}_{2|\sigma|+2k+2,n}) & 3 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{3,4}^{[j]} = \begin{cases} \sum_{|\sigma|+k+n=j-2} -\frac{\Re(\bar{P}_0 b_{2\sigma}) D_\sigma \alpha_k}{2} \Im(C_{2|\sigma|+2k+2,2n+1}) & 2 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Asymptotic expansion of $Y(t)$ as $t \to \infty$

We combine results (2.59), (3.61), and (3.62) with the fact that $Y(t) = Y_1(t) + Y_2(t) - Y_3(t)$ to obtain the expansion as $t \to \infty$

$$Y(t) = \sum_{j=0}^{K-1} V_j^Y t^{-j} + O(t^{-K}),$$

where, for $j \in \{0, \ldots, K-1\}$,

$$V_j^Y = Y_{1,1}^{[j]} + Y_{1,2}^{[j]} + Y_{1,3}^{[j]} - Y_{1,4}^{[j]} - Y_{1,5}^{[j]} + Y_{2,1}^{[j]} - Y_{3,1}^{[j]} - Y_{3,2}^{[j]} + Y_{3,3}^{[j]} + Y_{3,4}^{[j]}.$$

The first three coefficients

We assume that $K = 3$. Since the expansion of $Y(t)$ is as given in the $N \ge 3$ case, but with $N = 2$, it follows that the coefficients obtained in section 3.1.2 are valid here with $N = 2$. Thus $Y(t) = V_0^Y + Y_1^Y t^{-1} + V_2^Y t^{-2} + O(t^{-3})$ as $t \to \infty$, where

$$V_0^Y = 0, \qquad V_1^Y = 0,$$
$$V_2^Y = \frac{3\pi |P_0|^2}{64} + \frac{\pi}{4} \sum_{j=1}^{2} f_{2e_j} - \frac{\pi}{2} \sum_{j=1}^{2} \Re(\bar{P}_0 b_{2e_j}).$$

Expansions of $Z_2(t)$, $Z_3(t)$, and $Z_4(t)$

To determine the asymptotics of $Z_2(t)$, $Z_3(t)$, and $Z_4(t)$, we again refer to section 3.1.2. The analysis in that section yields integrals with non-negative powers of r if $N = 2$. So the results from those sections are valid here, and we obtain the following asymptotic expansions.

$$Z_2(t) = \sum_{j=0}^{K-1}(Z_{2,1}^{[j]} + Z_{2,2}^{[j]})t^{-j} + O(t^{-K}) \qquad (t \to \infty), \tag{3.63}$$

where

$$Z_{2,1}^{[j]} = \begin{cases} \sum_{2|\sigma|+2k+n=j-1} -\Re(\bar{P}_0 c_{2\sigma})D_\sigma \alpha_k \Im(\tilde{B}_{2|\sigma|+2k,n}) & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{2,2}^{[j]} = \begin{cases} \sum_{|\sigma|+k+n=j-1} -\Re(\bar{P}_0 c_{2\sigma})D_\sigma \alpha_k \Im(C_{2|\sigma|+2k,2n+1}) & 1 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

$$Z_3(t) = \sum_{j=0}^{K-1}(Z_{3,1}^{[j]} + Z_{3,2}^{[j]} - Z_{3,3}^{[j]} + Z_{3,4}^{[j]})t^{-j} + O(t^{-K}) \qquad (t \to \infty), \tag{3.64}$$

where

$$Z_{3,1}^{[j]} = \begin{cases} \sum_{2|\sigma|+n=j-1} -\Re(P_1 \bar{b}_{2\sigma})D_\sigma \Im(\tilde{B}_{2|\sigma|,n}) & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{3,2}^{[j]} = \begin{cases} \sum_{|\sigma|+n=j-1} \Re(P_1 \bar{b}_{2\sigma})D_\sigma \Im(C_{2|\sigma|,2n+1}) & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{3,3}^{[j]} = \begin{cases} \sum_{|\sigma|+k+n=j-1} \frac{\Re(P_1 \bar{b}_{2\sigma})D_\sigma \alpha_k}{2} C_{2|\sigma|+2k+1,2n} & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{3,4}^{[j]} = \begin{cases} \sum\limits_{2|\sigma|+2k+n=j-2} \frac{\Re(P_1\bar{b}_{2\sigma})D_\sigma\alpha_k}{2}\tilde{B}_{2|\sigma|+2k+1,n} & 2 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

$$Z_4(t) = \sum_{j=0}^{K-1} Z_{4,1}^{[j]} t^{-j} + O(t^{-K}) \qquad (t \to \infty), \tag{3.65}$$

where

$$Z_{4,1}^{[j]} = \begin{cases} -2\pi\Re(P_1\bar{P}_0)\Im(A_{0,j-1}) & 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. We note that for both the $Y_i(t)$ ($i \in \{2,3\}$) and $Z_i(t)$ ($i \in \{2,3,4\}$), no terms are contributed to the asymptotic expansions unless $K \geq 2$.

Asymptotic expansion of $Z(t)$ as $t \to \infty$

We combine the results (2.60), (3.63), (3.64), and (3.65) with the fact that $Z(t) = Z_1(t) - Z_2(t) - Z_3(t) + Z_4(t)$ to obtain the following expansion as $t \to \infty$:

$$Z(t) = \sum_{j=0}^{K-1} V_j^Z t^{-j} + O(t^{-K}),$$

where, for $j \in \{0, \ldots, K-1\}$,

$$V_j^Z = Z_{1,1}^{[j]} - Z_{1,2}^{[j]} + Z_{1,3}^{[j]} - Z_{2,1}^{[j]} - Z_{2,2}^{[j]} - Z_{3,1}^{[j]} - Z_{3,2}^{[j]} + Z_{3,3}^{[j]} - Z_{3,4}^{[j]} + Z_{4,1}^{[j]}.$$

The first three coefficients

We assume that $K = 3$. Since the expansion of $Z(t)$ is as given in the $N \geq 3$ cases, but with $N = 2$, it follows that the coefficients obtained in section 3.1.2 are valid with $N = 2$. Thus $Z(t) = V_0^Z + Z_1^Z t^{-1} + V_2^Z t^{-2} + O(t^{-3})$, where

$$V_0^Z = 0, \qquad V_1^Z = 0,$$

$$V_2^Z = -\frac{7\pi\Re(P_1\bar{P}_0)}{128} - \frac{\pi}{4}\sum_{j=1}^{2}\Re(l_{2e_j}) + \frac{\pi}{8}\sum_{j=1}^{2}\Re(P_1\bar{b}_{2e_j}) + \frac{\pi}{8}\sum_{j=1}^{2}\Re(\bar{P}_0 c_{2e_j}).$$

3.2.3 Asymptotic expansion of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ as $t \to \infty$

We proceed as in section 3.1.3. For the fixed $0 < \delta < 1$ there exists $\eta > 0$ such that $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_{2,\delta}^2 + O(e^{-\eta t}) = X(t) + Y(t) + Z(t) + O(e^{-\eta t})$ as $t \to \infty$. We combine the results of the analysis for the case $N = 2$ to obtain the full asymptotic expansion of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$. Further, we suppress the $O(e^{-\eta t})$-term since it is dominated by the larger $O(t^{-K})$-term.

Let $K \geq 1$ be an integer as given in the statement of Theorem 2. Then as $t \to \infty$

$$\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = \sum_{j=0}^{K-1} V_j t^{-j} + O(t^{-K}),$$

where $V_j = V_j^X + V_j^Y + V_j^Z$ for each $j \in \{0, \ldots, K-1\}$. To complete the proof of Theorem 2 for dimension $N = 2$, we must verify that $V_0 = 0$.

For dimension $N = 2$, let $K = 3$. Then $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = V_0 + V_1 t^{-1} + V_2 t^{-2} + O(t^{-3})$ as $t \to \infty$, where

$$V_0 = 0, \qquad V_1 = \frac{\pi|P_1|^2}{64} + \frac{\pi}{4}\sum_{j=1}^{2} d_{2e_j} - \frac{\pi}{2}\sum_{j=1}^{2}\Re(\bar{P}_1 c_{2e_j}),$$

$$\begin{aligned} V_2 = {} & \frac{75\pi|P_1|^2}{4096} + \frac{3\pi|P_0|^2}{64} - \frac{7\pi\Re(P_1\bar{P}_0)}{128} + \frac{3\pi}{16}\sum_{j=1}^{2} d_{2e_j} - \frac{37\pi}{128}\sum_{j=1}^{2}\Re(\bar{P}_1 c_{2e_j}) + \frac{\pi}{4}\sum_{j=1}^{2} f_{2e_j} \\ & - \frac{\pi}{2}\sum_{j=1}^{2}\Re(\bar{P}_0 b_{2e_j}) - \frac{\pi}{4}\sum_{j=1}^{2}\Re(l_{2e_j}) + \frac{\pi}{8}\sum_{j=1}^{2}\Re(P_1\bar{b}_{2e_j}) + \frac{\pi}{8}\sum_{j=1}^{2}\Re(\bar{P}_0 c_{2e_j}) \\ & + \frac{3\pi}{16}\sum_{j=1}^{2} d_{4e_j} - \frac{3\pi}{8}\sum_{j=1}^{2}\Re(\bar{P}_1 c_{4e_j}) + \frac{\pi}{16} d_{2(1,1)} - \frac{\pi}{8}\Re(\bar{P}_1 c_{2(1,1)}). \end{aligned}$$

This completes the proof of Theorem 2 for dimension $N = 2$.

3.2.4 Asymptotic expansion of $\|\nu(t,\cdot)\|_2^2$ as $t\to\infty$

Let us again recall that for the fixed $0<\delta<1$, $\|\nu(t,\cdot)\|_2^2=\|\nu(t,\cdot)\|_{2,\delta}^2+O(e^{-t\delta^2/2})=X_2(t)+Y_2(t)+Z_4(t)+O(e^{-t\delta^2/2})$. We then combine results (3.53), (3.61), and (3.65) to obtain the proof of Corollary 1 part 2 and the expansion as $t\to\infty$

$$\|\nu(t,\cdot)\|_2^2=\|\nu(t,\cdot)\|_{2,\delta}^2+O(e^{-\frac{t\delta^2}{2}})=\frac{|P_1|^2\pi}{2}\ln(t)+\sum_{j=0}^{K-1}U_jt^{-j}+O(t^{-K}),$$

where, for $j\in\{0,\dots,K-1\}$,

$$U_j=X_{2,1}^{[j]}+Y_{2,1}^{[j]}+Z_{4,1}^{[j]}.$$

The coefficients associated with $K=3$ for dimension $N=2$ are

$$\begin{aligned}U_0&=\frac{\pi\gamma|P_1|^2}{2}+\pi|P_1|^2\ln(2),\\U_1&=-\frac{|P_1|^2\pi}{4}+\frac{|P_0|^2\pi}{2}+\Re(P_1\bar{P}_0)\pi,\\U_2&=-\frac{3|P_1|^2\pi}{16}-\frac{|P_0|^2\pi}{4}+\frac{\Re(P_1\bar{P}_0)\pi}{2}.\end{aligned}$$

3.3 The space dimension $N=1$ case

3.3.1 Auxiliary lemmas

Lemma 15. *For $0<\epsilon<2$ and $t>0$, define*

$$J_{-2}^{\epsilon}(t):=\int_0^{\epsilon}r^{-2}e^{-tr^2}\sin^2(tr)dr. \tag{3.66}$$

If $M\in\mathbb{N}$, then as $t\to\infty$,

$$J_{-2}^{\epsilon}(t)=\frac{\pi}{2}t-\frac{\sqrt{\pi}}{2}t^{\frac{1}{2}}+O(t^{-M}).$$

Proof. Let $0 < \epsilon < 2$ and $M \in \mathbb{N}$. We first note that

$$J^{\epsilon}_{-2}(t) = \left(\int_0^{\infty} - \int_{\epsilon}^{\infty} \right) r^{-2} e^{-tr^2} \sin^2(tr) dr.$$

The latter integral is $O(e^{-t\epsilon^2/2})$ as $t \to \infty$. The former integral we denote by $\tilde{J}_{-2}(t)$. It is then routine to verify using the mean value theorem and Lebesgue's dominated convergence theorem that for every fixed $t > 0$

$$\frac{d}{dt}\tilde{J}_{-2}(t) = \frac{1}{2}\int_0^{\infty} e^{-tr^2}\cos(2tr)dr - \frac{1}{2}\int_0^{\infty} e^{-tr^2}dr + \int_0^{\infty} r^{-1}e^{-tr^2}\sin(2tr)dr.$$

The first and third integrals are Fourier cosine and sine identities and are treated using p. 402 and p. 418 in Chapter 11 of [10], respectively. The second term equals $-\frac{\sqrt{\pi}}{4}t^{-1/2}$. Therefore, for every $t > 0$

$$\frac{d}{dt}\tilde{J}_{-2}(t) = \frac{\sqrt{\pi}}{4}t^{-\frac{1}{2}}e^{-t} - \frac{\sqrt{\pi}}{4}t^{-\frac{1}{2}} + \frac{\pi}{4}\mathrm{erf}(\sqrt{t}).$$

Thus there is some constant C such that for all $t > 0$

$$\tilde{J}_{-2}(t) = \frac{\sqrt{\pi}}{2}t^{\frac{1}{2}}e^{-t} - \frac{\sqrt{\pi}}{2}t^{\frac{1}{2}} + \frac{\pi}{2}t\,\mathrm{erf}(\sqrt{t}) + C.$$

As $t \to 0+$, the left hand side tends to zero and the right hand side tends to C. Thus $C = 0$, and as $t \to \infty$

$$J^{\epsilon}_{-2}(t) = \frac{\sqrt{\pi}}{2}t^{\frac{1}{2}}e^{-t} - \frac{\sqrt{\pi}}{2}t^{\frac{1}{2}} + \frac{\pi}{2}t\,\mathrm{erf}(\sqrt{t}) + O(e^{-\frac{t\epsilon^2}{2}}).$$

We appeal to p. 352 in Chapter 9 of [10] for the asymptotics of the error function to obtain as $t \to \infty$

$$\begin{aligned} J^{\epsilon}_{-2}(t) &= O(t^{\frac{1}{2}}e^{-t}) - \frac{\sqrt{\pi}}{2}t^{\frac{1}{2}} + \frac{\pi}{2}t\left(1 + O(t^{-\frac{1}{2}}e^{-t})\right) + O(e^{-\frac{t\epsilon^2}{2}}) \\ &= \frac{\pi}{2}t - \frac{\sqrt{\pi}}{2}t^{\frac{1}{2}} + O(t^{\frac{1}{2}}e^{-t}) + O(e^{-\frac{t\epsilon^2}{2}}) \end{aligned}$$

$$= \frac{\pi}{2} t - \frac{\sqrt{\pi}}{2} t^{\frac{1}{2}} + O(t^{-M}),$$

proving the lemma. □

Lemma 16. *For $0 < \epsilon < 2$ and $t > 0$, define*

$$H^{\epsilon}_{1,-2}(t) := \int_0^{\epsilon} r^{-2} e^{-tr^2} \sin(tr\sqrt{1 - r^2/4}) \sin(tr) dr.$$

Then as $t \to \infty$, $H^{\epsilon}_{1,-2}(t) = \frac{\pi}{2} t + O(t^{1/2})$. If $M \in \mathbb{N}$, the full asymptotic expansion as $t \to \infty$ is

$$\begin{aligned} H^{\epsilon}_{1,-2}(t) = \; & \frac{\pi}{2} t + \sum_{n=0}^{M-1} -C_{0,2n} t^{-n+\frac{1}{2}} + \sum_{0 \le k+n \le M-1} \frac{\alpha_k}{2k+2n+1} \Im(C_{2k+1,2n+1}) t^{-k-n+\frac{1}{2}} \\ & + \sum_{0 \le k+n \le M-2} -\frac{\alpha_k}{4k+4n+6} \Im(C_{2k+3,2n+1}) t^{-k-n-\frac{1}{2}} + \sum_{n=0}^{M-1} -\frac{1}{4n+2} C_{2,2n} t^{-n+\frac{1}{2}} \\ & + \sum_{n=1}^{M} \frac{2}{2n-1} C_{0,2n} t^{-n+\frac{3}{2}} + \sum_{1 \le k+n \le M} -\frac{\alpha_k}{2k+2n-1} C_{2k,2n} t^{-k-n+\frac{3}{2}} \\ & + \sum_{0 \le k+n \le M-1} \frac{\alpha_k}{4k+4n+2} C_{2k+2,2n} t^{-k-n+\frac{1}{2}} + \sum_{n=0}^{M-1} -\frac{1}{2n+1} \Im(C_{1,2n+1}) t^{-n+\frac{1}{2}} \\ & + \sum_{1 \le k+n \le M} -\frac{\beta_k}{2k+2n-1} C_{2k,2n} t^{-k-n+\frac{3}{2}} + O(t^{-M+\frac{1}{2}}). \end{aligned}$$

Proof. Let $0 < \epsilon < 2$. Using integration by parts, $H^{\epsilon}_{1,-2}(t) = H_1(t) - 2tH^{\epsilon}_{1,0}(t) + H_2(t) + H_3(t)$, where $H^{\epsilon}_{1,0}(t)$ is as given in Lemma 12 and

$$\begin{aligned} H_1(t) &= -\epsilon^{-1} e^{-t\epsilon^2} \sin(t\epsilon\sqrt{1 - \epsilon^2/4}) \sin(t\epsilon), \\ H_2(t) &= t \int_0^{\epsilon} r^{-1} \frac{d}{dr}[r\sqrt{1 - r^2/4}] e^{-tr^2} \cos(tr\sqrt{1 - r^2/4}) \sin(tr) dr, \\ H_3(t) &= t \int_0^{\epsilon} r^{-1} e^{-tr^2} \sin(tr\sqrt{1 - r^2/4}) \sin(tr) dr. \end{aligned}$$

We first fix $M \in \mathbb{N}$. Then we note that $H_1(t) = O(e^{-t\epsilon^2}) = O(t^{-M+1/2})$ as $t \to \infty$.

Secondly, from Lemma 12, with $Q := M$

$$
\begin{aligned}
-2tH^{\epsilon}_{1,0}(t) &= -2t\left(\sum_{n=0}^{M-1}\frac{1}{2}C_{0,2n}t^{-n-\frac{1}{2}} + O(t^{-M-\frac{1}{2}})\right) \\
&= \sum_{n=0}^{M-1} -C_{0,2n}t^{-n+\frac{1}{2}} + O(t^{-M+\frac{1}{2}}) \qquad (t\to\infty).
\end{aligned}
$$

Next, we analyze $H_2(t)$ as follows. Using the mean value theorem and Lebesgue's dominated convergence theorem, for every $t > 0$, $\frac{d}{dt}[t^{-1}H_2(t)] = H_2^A(t) + H_2^B(t) + H_2^C(t)$, where

$$
\begin{aligned}
H_2^A(t) &= -\int_0^{\epsilon} r\frac{d}{dr}[r\sqrt{1-r^2/4}]e^{-tr^2}\cos(tr\sqrt{1-r^2/4})\cos(tr)dr, \\
H_2^B(t) &= -\int_0^{\epsilon} \sqrt{1-r^2/4}\frac{d}{dr}[r\sqrt{1-r^2/4}]e^{-tr^2}\sin(tr\sqrt{1-r^2/4})\sin(tr)dr, \\
H_2^C(t) &= \int_0^{\epsilon} \frac{d}{dr}[r\sqrt{1-r^2/4}]e^{-tr^2}\cos(tr\sqrt{1-r^2/4})\cos(tr)dr.
\end{aligned}
$$

We use the fact that

$$
\frac{d}{dr}[r\sqrt{1-r^2/4}] = \frac{1-r^2/2}{\sqrt{1-r^2/4}} \tag{3.67}
$$

to obtain

$$
\begin{aligned}
H_2^A(t) = &-\int_0^{\epsilon}\frac{r}{\sqrt{1-r^2/4}}e^{-tr^2}\cos(tr\sqrt{1-r^2/4})\sin(tr)dr \\
&+\frac{1}{2}\int_0^{\epsilon}\frac{r^3}{\sqrt{1-r^2/4}}e^{-tr^3}\cos(tr\sqrt{1-r^2/4})\sin(tr)dr.
\end{aligned} \tag{3.68}
$$

Into both terms of (3.68) we substitute (2.22) with $L := M+1$ and $L := M$, respectively. Then

$$
H_2^A(t) = \sum_{k=0}^{M} -\alpha_k H^{\epsilon}_{4,2k+1}(t) + \sum_{k=0}^{M-1}\frac{\alpha_k}{2}H^{\epsilon}_{4,2k+3}(t) + O(t^{-M-\frac{3}{2}}) \qquad (t\to\infty). \tag{3.69}
$$

Now into both terms of (3.69) we enter the results from Lemma 13 with $Q = Q_k := M-k+1$

and $Q = Q_k := M - k$, respectively. We thus obtain as $t \to \infty$

$$H_2^A(t) = \sum_{0\le k+n\le M-1} -\frac{\alpha_k}{2}\Im(C_{2k+1,2n+1})t^{-k-n-\frac{3}{2}} + \sum_{0\le k+n\le M-2} \frac{\alpha_k}{4}\Im(C_{2k+3,2n+1})t^{-k-n-\frac{5}{2}} + O(t^{-M-\frac{3}{2}}).$$

We again use (3.67) to simplify $H_2^B(t)$:

$$H_2^B(t) = \frac{1}{2}H_{1,2}^{\epsilon}(t) - H_{1,0}^{\epsilon}(t). \tag{3.70}$$

Into both terms of (3.70) we insert the results from Lemma 12 with $Q := M$ and $Q := M+1$, respectively. Thus as $t \to \infty$

$$H_2^B(t) = \sum_{n=0}^{M-1} \frac{1}{4}C_{2,2n}t^{-n-\frac{3}{2}} + \sum_{n=0}^{M} -\frac{1}{2}C_{0,2n}t^{-n-\frac{1}{2}} + O(t^{-M-\frac{3}{2}}).$$

Using (3.67), we obtain

$$\begin{aligned} H_2^C(t) = &\int_0^{\epsilon} \frac{1}{\sqrt{1-r^2/4}} e^{-tr^2} \cos(tr\sqrt{1-r^2/4}) \cos(tr)dr \\ &- \frac{1}{2}\int_0^{\epsilon} \frac{r^2}{\sqrt{1-r^2/4}} e^{-tr^2} \cos(tr\sqrt{1-r^2/4}) \cos(tr)dr. \end{aligned} \tag{3.71}$$

Into both terms of (3.71) we substitute (2.34) with $L := M + 1$ and $L := M$, respectively. So as $t \to \infty$

$$H_2^C(t) = \sum_{k=0}^{M} \alpha_k H_{2,2k}^{\epsilon}(t) - \sum_{k=0}^{M-1} \frac{\alpha_k}{2} H_{2,2k+2}^{\epsilon}(t). \tag{3.72}$$

We use the result from Lemma 12 to simplify both terms of (3.72), with $Q = Q_k := M-k+1$ and $Q = Q_k := M - k$, respectively. Thus as $t \to \infty$

$$H_2^C(t) = \sum_{0\le k+n\le M} \frac{\alpha_k}{2}C_{2k,2n}t^{-k-n-\frac{1}{2}} + \sum_{0\le k+n\le M-1} -\frac{\alpha_k}{4}C_{2k+2,2n}t^{-k-n-\frac{3}{2}} + O(t^{-M-\frac{3}{2}}).$$

We now combine the results for $H_2^A(t)$, $H_2^B(t)$, and $H_2^C(t)$ to obtain the expansion as

$t \to \infty$

$$\frac{d}{dt}[t^{-1}H_2(t)] = \sum_{0\le k+n\le M-1} -\frac{\alpha_k}{2}\Im(C_{2k+1,2n+1})t^{-k-n-\frac{3}{2}} + \sum_{0\le k+n\le M-2} \frac{\alpha_k}{4}\Im(C_{2k+3,2n+1})t^{-k-n-\frac{5}{2}}$$
$$+ \sum_{n=0}^{M-1} \frac{1}{4}C_{2,2n}t^{-n-\frac{3}{2}} + \sum_{n=0}^{M} -\frac{1}{2}C_{0,2n}t^{-n-\frac{1}{2}} + \sum_{0\le k+n\le M} \frac{\alpha_k}{2}C_{2k,2n}t^{-k-n-\frac{1}{2}}$$
$$+ \sum_{0\le k+n\le M-1} -\frac{\alpha_k}{4}C_{2k+2,2n}t^{-k-n-\frac{3}{2}} + O(t^{-M-\frac{3}{2}}).$$

We integrate both sides with respect to t and then multiply both sides by t to obtain the expansion as $t \to \infty$

$$H_2(t) = C_1 t + \sum_{0\le k+n\le M-1} \frac{\alpha_k}{2k+2n+1}\Im(C_{2k+1,2n+1})t^{-k-n+\frac{1}{2}}$$
$$+ \sum_{0\le k+n\le M-2} -\frac{\alpha_k}{4k+4n+6}\Im(C_{2k+3,2n+1})t^{-k-n-\frac{1}{2}} + \sum_{n=0}^{M-1} -\frac{1}{4n+2}C_{2,2n}t^{-n+\frac{1}{2}}$$
$$+ \sum_{n=0}^{M} \frac{1}{2n-1}C_{0,2n}t^{-n+\frac{3}{2}} + \sum_{0\le k+n\le M} -\frac{\alpha_k}{2k+2n-1}C_{2k,2n}t^{-k-n+\frac{3}{2}}$$
$$+ \sum_{0\le k+n\le M-1} \frac{\alpha_k}{4k+4n+2}C_{2k+2,2n}t^{-k-n+\frac{1}{2}} + O(t^{-M+\frac{1}{2}}),$$

where C_1 is a constant yet to be determined.

Now we determine the asymptotic expansion of $H_3(t)$ similarly to $H_2(t)$. We use the mean value theorem and Lebesgue's dominated convergence theorem to determine that for every $t > 0$, $\frac{d}{dt}[t^{-1}H_3(t)] = -H^{\epsilon}_{3,1}(t) + H^{A}_{3}(t) - H^{\epsilon}_{1,0}(t)$, where

$$H^{A}_{3}(t) = \int_0^{\epsilon} \sqrt{1-r^2/4}\,e^{-tr^2}\cos(tr\sqrt{1-r^2/4})\cos(tr)dr. \tag{3.73}$$

We use Lemma 13 with $Q := M+1$ to obtain the expansion

$$-H^{\epsilon}_{3,1}(t) = \sum_{n=0}^{M-1} \frac{1}{2}\Im(C_{1,2n+1})t^{-n-\frac{3}{2}} + O(t^{-M-\frac{3}{2}}) \qquad (t \to \infty).$$

Similarly, we use Lemma 12 with $Q := M+1$ to obtain the expansion

$$-H^{\epsilon}_{1,0}(t) = \sum_{n=0}^{M} -\frac{1}{2}C_{0,2n}t^{-n-\frac{1}{2}} + O(t^{-M-\frac{3}{2}}) \qquad (t \to \infty).$$

We analyze $H_3^A(t)$ by substituting (2.46) into (3.73) with $L := M+1$ to obtain

$$H_3^A(t) = \sum_{k=0}^{M} \beta_k H^{\epsilon}_{2,2k}(t) + O(t^{-M-\frac{3}{2}}) \qquad (t \to \infty).$$

We now use the result of Lemma 12 with $Q = Q_k := M-k+1$ to obtain

$$H_3^A(t) = \sum_{0\le k+n \le M} \frac{\beta_k}{2}C_{2k,2n}t^{-k-n-\frac{1}{2}} + O(t^{-M-\frac{3}{2}}).$$

Now we combine the results for $-H^{\epsilon}_{3,1}(t)$, $H_3^A(t)$, and $-H^{\epsilon}_{1,0}(t)$ to obtain the expansion as $t \to \infty$

$$\frac{d}{dt}[t^{-1}H_3(t)] = \sum_{n=0}^{M-1} \frac{1}{2}\Im(C_{1,2n+1})t^{-n-\frac{3}{2}} + \sum_{0\le k+n\le M} \frac{\beta_k}{2}C_{2k,2n}t^{-k-n-\frac{1}{2}} + \sum_{n=0}^{M} -\frac{1}{2}C_{0,2n}t^{-n-\frac{1}{2}} + O(t^{-M-\frac{3}{2}}).$$

We integrate both sides with respect to t and then multiply both sides by t to obtain the expansion as $t \to \infty$

$$\begin{aligned} H_3(t) = C_2 t + \sum_{n=0}^{M-1} -\frac{1}{2n+1}\Im(C_{1,2n+1})t^{-n+\frac{1}{2}} + \sum_{0\le k+n\le M} -\frac{\beta_k}{2k+2n-1}C_{2k,2n}t^{-k-n+\frac{3}{2}} \\ + \sum_{n=0}^{M} \frac{1}{2n-1}C_{0,2n}t^{-n+\frac{3}{2}} + O(t^{-M+\frac{1}{2}}), \end{aligned}$$

where C_2 is a constant yet to be determined.

We combine the asymptotic expansions we found for $H_1(t)$, $-2tH^{\epsilon}_{1,0}(t)$, $H_2(t)$, and $H_3(t)$ with the fact that $H^{\epsilon}_{1,-2}(t) = H_1(t) - 2tH^{\epsilon}_{1,0}(t) + H_2(t) + H_3(t)$ to determine that

$$t^{-1}H^{\epsilon}_{1,-2}(t) = C + O(t^{-\frac{1}{2}}) \qquad (t \to \infty),$$

where $C = C_1 + C_2$.

To determine the constant C, consider $J^{\epsilon}_{-2}(t)$ as given in (3.66). Since $|\sin^2(tr) - \sin(tr\sqrt{1-r^2/4})\sin(tr)| \le tr|1-\sqrt{1-r^2/4}| \le \frac{tr^3}{4}$ for $0 \le r \le 2$, we obtain

$$t^{-1}|J^{\epsilon}_{-2}(t) - H^{\epsilon}_{1,-2}(t)| \le \frac{1}{4}\int_0^{\epsilon} re^{-tr^2}dr \le \frac{1}{8}\Gamma(1)t^{-1} = O(t^{-1}) \to 0 \qquad (t \to \infty). \qquad (3.74)$$

Furthermore,

$$t^{-1}(J^{\epsilon}_{-2}(t) - H^{\epsilon}_{1,-2}(t)) = \frac{\pi}{2} - C + O(t^{-\frac{1}{2}}) \to \frac{\pi}{2} - C \qquad (t \to \infty). \qquad (3.75)$$

Considering (3.74) and (3.75) together gives us the value $C = \frac{\pi}{2}$.

Finally we arrive at the asymptotic expansion of $H^{\epsilon}_{1,-2}(t)$ as $t \to \infty$:

$$\begin{aligned} H^{\epsilon}_{1,-2}(t) = \frac{\pi}{2}t + \sum_{n=0}^{M-1} -C_{0,2n}t^{-n+\frac{1}{2}} + \sum_{0\le k+n\le M-1} \frac{\alpha_k}{2k+2n+1}\Im(C_{2k+1,2n+1})t^{-k-n+\frac{1}{2}} \\ + \sum_{0\le k+n\le M-2} -\frac{\alpha_k}{4k+4n+6}\Im(C_{2k+3,2n+1})t^{-k-n-\frac{1}{2}} + \sum_{n=0}^{M-1} -\frac{1}{4n+2}C_{2,2n}t^{-n+\frac{1}{2}} \\ + \sum_{n=1}^{M} \frac{2}{2n-1}C_{0,2n}t^{-n+\frac{3}{2}} + \sum_{1\le k+n\le M} -\frac{\alpha_k}{2k+2n-1}C_{2k,2n}t^{-k-n+\frac{3}{2}} \\ + \sum_{0\le k+n\le M-1} \frac{\alpha_k}{4k+4n+2}C_{2k+2,2n}t^{-k-n+\frac{1}{2}} + \sum_{n=0}^{M-1} -\frac{1}{2n+1}\Im(C_{1,2n+1})t^{-n+\frac{1}{2}} \\ + \sum_{1\le k+n\le M} -\frac{\beta_k}{2k+2n-1}C_{2k,2n}t^{-k-n+\frac{3}{2}} + O(t^{-M+\frac{1}{2}}). \end{aligned}$$

□

Lemma 17. *For $0 < \epsilon < 2$ and $t > 0$, define*

$$H^{\epsilon}_{3,-1}(t) := \int_0^{\epsilon} r^{-1}e^{-tr^2}\sin(tr\sqrt{1-r^2/4})\cos(tr)dr.$$

Then $H^{\epsilon}_{3,-1}(t) = \frac{\pi}{4} + O(t^{-1/2})$ as $t \to \infty$. If $M \in \mathbb{N}$, the full asymptotic expansion of $H^{\epsilon}_{3,-1}(t)$

as $t \to \infty$ is

$$H^{\epsilon}_{3,-1}(t) = \frac{\pi}{4} + \sum_{n=0}^{M-2} -\frac{1}{2n+1}\Im(C_{1,2n+1})t^{-n-\frac{1}{2}} + \sum_{1\le k+n\le M-1} -\frac{\beta_k}{2k+2n-1}C_{2k,2n}t^{-k-n+\frac{1}{2}} + \sum_{n=1}^{M-1} \frac{1}{2n-1}C_{0,2n}t^{-n+\frac{1}{2}} + O(t^{-M+\frac{1}{2}}).$$

Proof. Let $0 < \epsilon < 2$ and $t > 0$. Using the mean value theorem and Lebesgue's dominated convergence theorem, $\frac{d}{dt}H^{\epsilon}_{3,-1}(t) = -H^{\epsilon}_{3,1}(t) + H_1(t) - H^{\epsilon}_{1,0}(t)$, where

$$H_1(t) = \int_0^{\epsilon} \sqrt{1-r^2/4}e^{-tr^2}\cos(tr\sqrt{1-r^2/4})\cos(tr)dr, \tag{3.76}$$

and $H^{\epsilon}_{3,1}(t)$ and $H^{\epsilon}_{1,0}(t)$ are as given in lemmas 13 and 12, respectively.

Fix $M \in \mathbb{N}$. Then by the result of Lemma 13 with $Q := M$

$$-H^{\epsilon}_{3,1}(t) = \sum_{n=0}^{M-2} \frac{1}{2}\Im(C_{1,2n+1})t^{-n-\frac{3}{2}} + O(t^{-M-\frac{1}{2}}) \qquad (t \to \infty).$$

Similarly, by the result of Lemma 12 with $Q := M$

$$-H^{\epsilon}_{1,0}(t) = \sum_{n=0}^{M-1} -\frac{1}{2}C_{0,2n}t^{-n-\frac{1}{2}} + O(t^{-M-\frac{1}{2}}) \qquad (t \to \infty).$$

We begin the analysis of $H_1(t)$ by substituting (2.46) into (3.76) with $L := M$. Then

$$H_1(t) = \sum_{k=0}^{M-1} \beta_k H^{\epsilon}_{2,2k}(t) + O(t^{-M-\frac{1}{2}}) \qquad (t \to \infty). \tag{3.77}$$

We substitute the result from Lemma 12 into (3.77) with $Q := M-k$ to obtain the expansion

$$H_1(t) = \sum_{0\le k+n\le M-1} \frac{\beta_k}{2}C_{2k,2n}t^{-k-n-\frac{1}{2}} + O(t^{-M-\frac{1}{2}}) \qquad (t \to \infty).$$

We now combine the expansions of $-H^{\epsilon}_{3,1}(t)$, $H_1(t)$, and $-H^{\epsilon}_{1,0}(t)$ with the fact that

$\frac{d}{dt}H^{\epsilon}_{3,-1}(t) = -H^{\epsilon}_{3,1}(t) + H_1(t) - H^{\epsilon}_{1,0}(t)$ to obtain the expansion as $t \to \infty$

$$\frac{d}{dt}H^{\epsilon}_{3,-1}(t) = \sum_{n=0}^{M-2} \frac{1}{2}\Im(C_{1,2n+1})t^{-n-\frac{3}{2}} + \sum_{1\le k+n\le M-1} \frac{\beta_k}{2}C_{2k,2n}t^{-k-n-\frac{1}{2}} + \sum_{n=1}^{M-1} -\frac{1}{2}C_{0,2n}t^{-n-\frac{1}{2}} + O(t^{-M-\frac{1}{2}}).$$

Integrating both sides with respect to t, we obtain as $t \to \infty$

$$H^{\epsilon}_{3,-1}(t) = C + \sum_{n=0}^{M-2} -\frac{1}{2n+1}\Im(C_{1,2n+1})t^{-n-\frac{1}{2}} + \sum_{1\le k+n\le M-1} -\frac{\beta_k}{2k+2n-1}C_{2k,2n}t^{-k-n+\frac{1}{2}} + \sum_{n=1}^{M-1} \frac{1}{2n-1}C_{0,2n}t^{-n+\frac{1}{2}} + O(t^{-M+\frac{1}{2}}),$$

where C is a constant yet to be determined.

To determine the constant C, we recall the definition of $\tilde{I}_3(t)$ from (2.65). In particular, we found that for the fixed $M \in \mathbb{N}$

$$(2t)^{-1}\tilde{I}_3(t) = \frac{\pi}{4} + O(t^{-M}) \qquad (t \to \infty). \tag{3.78}$$

We first use the fact that $|\sin(tr)\cos(tr) - \sin(tr\sqrt{1-r^2/4})\cos(tr)| \le tr|1-\sqrt{1-r^2/4}| \le \frac{tr^3}{4}$ for $0 \le r \le 2$ to obtain

$$|(2t)^{-1}\tilde{I}_3(t) - H^{\epsilon}_{3,-1}(t)| \le \frac{t}{4}\int_0^{\epsilon} r^2 e^{-tr^2}dr \le \frac{t}{8}\Gamma(\tfrac{3}{2})t^{-\frac{3}{2}} + O(t^{-\frac{1}{2}}) \to 0 \qquad (t \to \infty). \tag{3.79}$$

We then recall (3.78) and that $H^{\epsilon}_{3,-1}(t) = C + O(t^{-1/2})$ as $t \to \infty$, and obtain

$$(2t)^{-1}\tilde{I}_3(t) - H^{\epsilon}_{3,-1}(t) = \frac{\pi}{4} - C + O(t^{-\frac{1}{2}}) \to \frac{\pi}{4} - C \qquad (t \to \infty). \tag{3.80}$$

We finally consider (3.79) and (3.80) together to see $C = \frac{\pi}{4}$, which completes the proof. $\square$

Lemma 18. *For $0 < \epsilon < 2$ and $t > 0$, define*

$$H^{\epsilon}_{4,-1}(t) := \int_0^{\epsilon} r^{-1} e^{-tr^2} \cos(tr\sqrt{1-r^2/4}) \sin(tr) dr.$$

Then $H^{\epsilon}_{4,-1}(t) = \frac{\pi}{4} + O(t^{-1/2})$ as $t \to \infty$. If $M \in \mathbb{N}$, the full asymptotic expansion of $H^{\epsilon}_{4,-1}(t)$ as $t \to \infty$ is given by

$$\begin{aligned} H^{\epsilon}_{4,-1}(t) = \; & \frac{\pi}{4} + \sum_{n=0}^{M-2} \frac{1}{2n+1} \Im(C_{1,2n+1}) t^{-n-\frac{1}{2}} + \sum_{1 \le k+n \le M-1} \frac{\beta_k}{2k+2n-1} C_{2k,2n} t^{-k-n+\frac{1}{2}} \\ & + \sum_{n=1}^{M-1} -\frac{1}{2n-1} C_{0,2n} t^{-n+\frac{1}{2}} + O(t^{-M+\frac{1}{2}}). \end{aligned}$$

Proof. Let $0 < \epsilon < 2$ and $t > 0$. Using the mean value theorem and Lebesgue's dominated convergence theorem, $\frac{d}{dt} H^{\epsilon}_{4,-1}(t) = -H^{\epsilon}_{4,1}(t) + H_1(t) + H^{\epsilon}_{2,0}(t)$, where

$$H_1(t) = -\int_0^{\epsilon} \sqrt{1-r^2/4} e^{-tr^2} \sin(tr\sqrt{1-r^2/4}) \sin(tr) dr \qquad (3.81)$$

and $H^{\epsilon}_{4,1}(t)$ and $H^{\epsilon}_{2,0}(t)$ are as given in lemmas 13 and 12, respectively.

Fix $M \in \mathbb{N}$. Then by the result of Lemma 13 with $Q := M$

$$-H^{\epsilon}_{4,1}(t) = \sum_{n=0}^{M-2} -\frac{1}{2} \Im(C_{1,2n+1}) t^{-n-\frac{3}{2}} + O(t^{-M-\frac{1}{2}}) \qquad (t \to \infty).$$

Similarly, by the result of Lemma 12 with $Q := M$

$$H^{\epsilon}_{2,0}(t) = \sum_{n=0}^{M-1} \frac{1}{2} C_{0,2n} t^{-n-\frac{1}{2}} + O(t^{-M-\frac{1}{2}}) \qquad (t \to \infty).$$

We begin the analysis of $H_1(t)$ by substituting (2.46) into (3.81) with $L := M$. Then

$$H_1(t) = \sum_{k=0}^{M-1} -\beta_k H^{\epsilon}_{1,2k}(t) + O(t^{-M-\frac{1}{2}}) \qquad (t \to \infty). \qquad (3.82)$$

We substitute the result from Lemma 12 into (3.82) with $Q := M-k$ to obtain the expansion

$$H_1(t) = \sum_{0\le k+n\le M-1} -\frac{\beta_k}{2}C_{2k,2n}t^{-k-n-\frac{1}{2}} + O(t^{-M-\frac{1}{2}}) \qquad (t\to\infty).$$

We now combine the expansions of $-H^{\epsilon}_{4,1}(t)$, $H_1(t)$, and $H^{\epsilon}_{2,0}(t)$ with the fact that $\frac{d}{dt}H^{\epsilon}_{4,-1}(t) = -H^{\epsilon}_{4,1}(t) + H_1(t) + H^{\epsilon}_{2,0}(t)$ to obtain the expansion as $t\to\infty$

$$\begin{aligned}\frac{d}{dt}H^{\epsilon}_{4,-1}(t) = \sum_{n=0}^{M-2} -\frac{1}{2}\Im(C_{1,2n+1})t^{-n-\frac{3}{2}} + \sum_{1\le k+n\le M-1} -\frac{\beta_k}{2}C_{2k,2n}t^{-k-n-\frac{1}{2}}&\\ +\sum_{n=1}^{M-1}\frac{1}{2}C_{0,2n}t^{-n-\frac{1}{2}} + O(t^{-M-\frac{1}{2}}).&\end{aligned}$$

Integrating both sides with respect to t, we obtain

$$\begin{aligned}H^{\epsilon}_{4,-1}(t) = C + \sum_{n=0}^{M-2}\frac{1}{2n+1}\Im(C_{1,2n+1})t^{-n-\frac{1}{2}} + \sum_{1\le k+n\le M-1}\frac{\beta_k}{2k+2n-1}C_{2k,2n}t^{-k-n+\frac{1}{2}}&\\ +\sum_{n=1}^{M-1} -\frac{1}{2n-1}C_{0,2n}t^{-n+\frac{1}{2}} + O(t^{-M+\frac{1}{2}}),&\end{aligned}$$

where $C = \frac{\pi}{4}$ is determined in the same way as in the proof of Lemma 17. In particular, we compare $H^{\epsilon}_{4,-1}(t)$ to $(2t)^{-1}\tilde{I}_3(t)$, with $\tilde{I}_3(t)$ as given in (2.65). □

3.3.2 Intermediate computations

We are now in a position to prove the dimension $N = 1$ case of Theorem 2. The proof will be carried out over the next few sections. We assume that $K \in \mathbb{N}$ and the initial data of (1.1) satisfy $u_0 \in H^1(\mathbb{R}) \cap L^{1,2K}(\mathbb{R})$ and $u_1 \in L^2(\mathbb{R}) \cap L^{1,2K}(\mathbb{R})$.

Expansion of $X_2(t)$

For $t > 0$

$$X_2(t) = |P_1|^2\int_{-\delta}^{\delta} e^{-t\xi^2}\frac{\sin^2(t\xi)}{\xi^2}d\xi = 2|P_1|^2\int_0^{\delta} r^{-2}e^{-tr^2}\sin^2(tr)dr.$$

By Lemma 15 with $M := K$

$$X_2(t) = \pi|P_1|^2 t - \sqrt{\pi}|P_1|^2 t^{\frac{1}{2}} + O(t^{-K+\frac{1}{2}})$$
$$= \pi|P_1|^2 t + \sum_{j=0}^{K-1} X_{2,1}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \tag{3.83}$$

where

$$X_{2,1}^{[j]} = \begin{cases} -\sqrt{\pi}|P_1|^2 & j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Expansion of $X_3(t)$

We may follow the work done in expanding $X_3(t)$ in section 3.1.2 with dimension $N = 1$. We obtain $X_3(t) = X_3^A(t) + X_3^B(t) + O(t^{-K+1/2})$, where

$$X_3^A(t) = 4\Re(\bar{P}_1 c_0) \int_0^\delta r^{-2} e^{-tr^2} \frac{\sin(tr\sqrt{1 - r^2/4})\sin(tr)}{\sqrt{1 - r^2/4}} dr, \tag{3.84}$$

$$X_3^B(t) = \sum_{\sigma=1}^{K-1} 4\Re(\bar{P}_1 c_{2\sigma}) \int_0^\delta r^{2\sigma-2} e^{-tr^2} \frac{\sin(tr\sqrt{1 - r^2/4})\sin(tr)}{\sqrt{1 - r^2/4}} dr. \tag{3.85}$$

Into (3.84) we substitute (2.34) with $L := K$. Then

$$X_3^A(t) = 4|P_1|^2 H_{1,-2}^\delta(t) + \sum_{k=1}^{K-1} 4|P_1|^2 \alpha_k H_{1,2k-2}^\delta(t) + O(t^{-K+\frac{1}{2}}). \tag{3.86}$$

To the first term of (3.86) we apply Lemma 16 with $M := K$. We then treat the sum in (3.86) by using Lemma 12 with $Q = Q_k := K - k$. Thus as $t \to \infty$

$$X_3^A(t) = 2\pi|P_1|^2 t + \sum_{j=0}^{K-1} (X_{3,1}^{[j]} + X_{3,2}^{[j]} + X_{3,3}^{[j]} + X_{3,4}^{[j]} + X_{3,5}^{[j]} + X_{3,6}^{[j]}$$
$$+ X_{3,7}^{[j]} + X_{3,8}^{[j]} + X_{3,9}^{[j]} + X_{3,10}^{[j]}) t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}), \tag{3.87}$$

where

$$
\begin{aligned}
X_{3,1}^{[j]} &= -4|P_1|^2 C_{0,2j},\\
X_{3,2}^{[j]} &= \sum_{k+n=j} \frac{4|P_1|^2\alpha_k}{2j+1}\Im(C_{2k+1,2n+1}),\\
X_{3,3}^{[j]} &= \begin{cases} \sum\limits_{k+n=j-1} -\frac{2|P_1|^2\alpha_k}{2j+1}\Im(C_{2k+3,2n+1}) & 1\le j\le K-1\\ 0 & \text{otherwise,}\end{cases}\\
X_{3,4}^{[j]} &= -\frac{2|P_1|^2}{2j+1}C_{2,2j},\\
X_{3,5}^{[j]} &= \frac{8|P_1|^2}{2j+1}C_{0,2j+2},\\
X_{3,6}^{[j]} &= \sum_{k+n=j+1} -\frac{4|P_1|^2\alpha_k}{2j+1}C_{2k,2n},\\
X_{3,7}^{[j]} &= \sum_{k+n=j} \frac{2|P_1|^2\alpha_k}{2j+1}C_{2k+2,2n},\\
X_{3,8}^{[j]} &= -\frac{4|P_1|^2}{2j+1}\Im(C_{1,2j+1}),\\
X_{3,9}^{[j]} &= \sum_{k+n=j+1} -\frac{4|P_1|^2\beta_k}{2j+1}C_{2k,2n},\\
X_{3,10}^{[j]} &= \begin{cases} \sum\limits_{\substack{k+n=j\\ k\ge 1}} 2|P_1|^2\alpha_k C_{2k-2,2n} & 1\le j\le K-1\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

To analyze $X_3^B(t)$, we substitute (2.34) into (3.85) with $L = L_\sigma := K-\sigma$. Hence

$$X_3^B(t) = \sum_{\sigma=1}^{K-1}\sum_{k=0}^{L_\sigma-1} 4\Re(\bar{P}_1 c_{2\sigma})\alpha_k H^{\delta}_{1,2\sigma+2k-2}(t) + O(t^{-K+\frac{1}{2}}) \qquad (t\to\infty).$$

We apply Lemma 12 with $Q = Q_{\sigma,k} := K-\sigma-k$ and simplify to obtain

$$X_3^B(t) = \sum_{j=0}^{K-1} X_{3,11}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t\to\infty), \tag{3.88}$$

where

$$X_{3,11}^{[j]} = \begin{cases} \sum\limits_{\substack{\sigma+k+n=j \\ \sigma\geq 1}} 2\Re(\bar{P}_1 c_{2\sigma})\alpha_k C_{2\sigma+2k-2,2n} & 1 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $X_3(t) = X_3^A(t) + X_3^B(t) + O(t^{-K+1/2})$, we combine (3.87) and (3.88) to obtain the asymptotic expansion as $t \to \infty$

$$\begin{aligned} X_3(t) = 2\pi|P_1|^2 t + \sum_{j=0}^{K-1} (X_{3,1}^{[j]} + X_{3,2}^{[j]} + X_{3,3}^{[j]} + X_{3,4}^{[j]} + X_{3,5}^{[j]} + X_{3,6}^{[j]} \\ + X_{3,7}^{[j]} + X_{3,8}^{[j]} + X_{3,9}^{[j]} + X_{3,10}^{[j]} + X_{3,11}^{[j]}) t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}). \end{aligned} \tag{3.89}$$

Asymptotic expansion of $X(t)$ as $t \to \infty$

We combine results (2.77), (3.83), and (3.89) for $X_1(t)$, $X_2(t)$, and $X_3(t)$, respectively, with the fact that $X(t) = X_1(t) + X_2(t) - X_3(t)$ to obtain the asymptotic expansion

$$X(t) = \sum_{j=0}^{K-1} V_j^X t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty),$$

where, for $j \in \{0, \ldots, K-1\}$,

$$V_j^X = X_{1,1}^{[j]} + X_{1,2}^{[j]} + X_{2,1}^{[j]} - X_{3,1}^{[j]} - X_{3,2}^{[j]} - X_{3,3}^{[j]} - X_{3,4}^{[j]} - X_{3,5}^{[j]} - X_{3,6}^{[j]} - X_{3,7}^{[j]} - X_{3,8}^{[j]} - X_{3,9}^{[j]} - X_{3,10}^{[j]} - X_{3,11}^{[j]}.$$

The first three coefficients

Let us assume that $K = 3$. It can then be shown that $X(t) = V_0^X t^{1/2} + V_1^X t^{-1/2} + V_2^X t^{-3/2} + O(t^{-5/2})$ as $t \to \infty$, where

$$V_0^X = 0, \qquad V_1^X = \frac{3\sqrt{\pi}|P_1|^2}{512} + \frac{\sqrt{\pi} d_2}{2} - \sqrt{\pi}\Re(\bar{P}_1 c_2),$$
$$V_2^X = \frac{7621\sqrt{\pi}|P_1|^2}{1048576} + \frac{\sqrt{\pi} d_2}{16} - \frac{49\sqrt{\pi}\Re(\bar{P}_1 c_2)}{1024} + \frac{\sqrt{\pi} d_4}{4} - \frac{\sqrt{\pi}\Re(\bar{P}_1 c_4)}{2}.$$

We note that these are the values we obtain for the V_j^X ($j \in \{0, 1, 2\}$) if $N = 1$ in section 3.1.2.

Expansions of $Y_2(t)$ and $Y_3(t)$

To determine the asymptotic expansions of $Y_2(t)$ and $Y_3(t)$, we may refer to the analysis in section 3.1.2 with $N = 1$. We observe that none of the integrals obtained will have negative powers of r. The analysis carries over to this case and hence

$$Y_2(t) = \sum_{j=0}^{K-1} Y_{2,1}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \tag{3.90}$$

where

$$Y_{2,1}^{[j]} = \begin{cases} \frac{\sqrt{\pi}|P_0|^2}{2} & j = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Y_3(t) = \sum_{j=0}^{K-1} (Y_{3,1}^{[j]} - Y_{3,2}^{[j]}) t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \tag{3.91}$$

where

$$Y_{3,1}^{[j]} = \begin{cases} \sum_{\sigma+n=j-1} 2\Re(\bar{P}_0 b_{2\sigma}) C_{2\sigma,2n} & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_{3,2}^{[j]} = \begin{cases} \sum_{\sigma+k+n=j-2} -\Re(\bar{P}_0 b_{2\sigma}) \alpha_k \Im(C_{2\sigma+2k+1,2n+1}) & 2 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

Asymptotic expansion of $Y(t)$ as $t \to \infty$

Since $Y(t) = Y_1(t) + Y_2(t) - Y_3(t)$ we combine results (2.81), (3.90), and (3.91) to obtain the asymptotic expansion

$$Y(t) = \sum_{j=0}^{K-1} V_j^Y t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty),$$

where, for $j \in \{0, \ldots, K-1\}$,

$$V_j^Y = Y_{1,1}^{[j]} + Y_{1,2}^{[j]} + Y_{2,1}^{[j]} - Y_{3,1}^{[j]} + Y_{3,2}^{[j]}.$$

The first three coefficients

Let us assume $K = 3$. It can then be shown that $Y(t) = V_0^Y t^{1/2} + V_1^Y t^{-1/2} + V_2^Y t^{-3/2} + O(t^{-5/2})$ as $t \to \infty$, where

$$V_0^Y = 0, \qquad V_1^Y = 0,$$
$$V_2^Y = \frac{31\sqrt{\pi}|P_0|^2}{1024} + \frac{\sqrt{\pi} f_2}{4} - \frac{\sqrt{\pi}\Re(\bar{P}_0 b_2)}{2}.$$

The coefficients V_j^Y ($j \in \{0, 1, 2\}$) match those from section 3.1.2 with $N = 1$.

Expansion of $Z_2(t)$

Since $K \in \mathbb{N}$, we may refer to the work done in expanding $Z_2(t)$ in section 3.1.2 with $N = 1$. From (3.34), we have $Z_2(t) = Z_2^A(t) + Z_2^B(t) + O(t^{-K+1/2})$ as $t \to \infty$, where

$$Z_2^A(t) = 4\Re(\bar{P}_0 c_0) \int_0^\delta r^{-1} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4})\cos(tr)}{\sqrt{1-r^2/4}} dr, \tag{3.92}$$

$$Z_2^B(t) = \sum_{\sigma=1}^{K-1} 4\Re(\bar{P}_0 c_{2\sigma}) \int_0^\delta r^{2\sigma-1} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4})\cos(tr)}{\sqrt{1-r^2/4}} dr. \tag{3.93}$$

Into (3.92) we substitute (2.34) with $L := K$. Hence

$$Z_2^A(t) = 4\Re(P_1\bar{P}_0)H^{\delta}_{3,-1}(t) + \sum_{k=1}^{K-1} 4\Re(P_1\bar{P}_0)\alpha_k H^{\delta}_{3,2k-1}(t) + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty). \quad (3.94)$$

Into the first term of (3.94) we substitute the expansion of $H^{\delta}_{3,-1}(t)$ obtained in Lemma 17 with $M := K$. We also apply Lemma 13 with $Q = Q_k := K - k$ to the sum in (3.94). Hence as $t \to \infty$

$$Z_2^A(t) = \pi\Re(P_1\bar{P}_0) + \sum_{j=0}^{K-1}(Z_{2,1}^{[j]} + Z_{2,2}^{[j]} + Z_{2,3}^{[j]} + Z_{2,4}^{[j]})t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}), \qquad (3.95)$$

where

$$Z_{2,1}^{[j]} = \begin{cases} -\frac{4\Re(P_1\bar{P}_0)}{2j-1}\Im(C_{1,2j-1}) & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{2,2}^{[j]} = \begin{cases} \sum_{k+n=j} -\frac{4\Re(P_1\bar{P}_0)\beta_k}{2j-1}C_{2k,2n} & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{2,3}^{[j]} = \begin{cases} \frac{4\Re(P_1\bar{P}_0)}{2j-1}C_{0,2j} & 1 \le j \le K-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$Z_{2,4}^{[j]} = \begin{cases} \sum_{\substack{k+n=j-1 \\ k \ge 1}} -2\Re(P_1\bar{P}_0)\alpha_k\Im(C_{2k-1,2n+1}) & 2 \le j \le K-1 \\ 0 & \text{otherwise.} \end{cases}$$

To analyze $Z_2^B(t)$ we substitute (2.34) into (3.93) with $L = L_\sigma := K - \sigma$. Hence

$$Z_2^B(t) = \sum_{\sigma=1}^{K-1}\sum_{k=0}^{L_\sigma-1} 4\Re(\bar{P}_0 c_{2\sigma})\alpha_k H^{\delta}_{3,2\sigma+2k-1}(t) + O(t^{-j+\frac{1}{2}}) \qquad (t \to \infty).$$

We then apply Lemma 13 with $Q = Q_{\sigma,k} := K - \sigma - k$ and simplify to obtain

$$Z_2^B(t) = \sum_{j=0}^{K-1} Z_{2,5}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \tag{3.96}$$

where

$$Z_{2,5}^{[j]} = \begin{cases} \sum_{\substack{\sigma+k+n=j-1 \\ \sigma \geq 1}} -2\Re(\bar{P}_0 c_{2\sigma})\alpha_k \Im(C_{2\sigma+2k-1,2n+1}) & 2 \leq j \leq K-1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $Z_2(t) = Z_2^A(t) + Z_2^B(t) + O(t^{-K+1/2})$, we combine (3.95) and (3.96) to obtain the asymptotic expansion as $t \to \infty$

$$Z_2(t) = \pi\Re(P_1\bar{P}_0) + \sum_{j=0}^{K-1} (Z_{2,1}^{[j]} + Z_{2,2}^{[j]} + Z_{2,3}^{[j]} + Z_{2,4}^{[j]} + Z_{2,5}^{[j]}) t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}). \tag{3.97}$$

Expansion of $Z_3(t)$

To analyze $Z_3(t)$, we refer to the expansion of $Z_3(t)$ in section 3.1.2 with $N = 1$. Then $Z_3(t) = Z_3^A(t) + Z_3^B(t) - Z_3^C(t) + O(t^{-K+1/2})$ as $t \to \infty$, where

$$Z_3^A(t) = 4\Re(P_1\bar{b}_0)H_{4,-1}^{\delta}(t), \tag{3.98}$$

$$Z_3^B(t) = \sum_{\sigma=1}^{K-1} 4\Re(P_1\bar{b}_{2\sigma})H_{4,2\sigma-1}^{\delta}(t), \tag{3.99}$$

$$Z_3^C(t) = \sum_{\sigma=0}^{K-1} 2\Re(P_1\bar{b}_{2\sigma}) \int_0^{\delta} r^{2\sigma} e^{-tr^2} \frac{\sin(tr\sqrt{1-r^2/4})\sin(tr)}{\sqrt{1-r^2/4}} dr. \tag{3.100}$$

We obtain the asymptotic expansion of $Z_3^A(t)$ by applying Lemma 18 to (3.98) with $M := K$. Thus

$$Z_3^A(t) = \pi\Re(P_1\bar{P}_0) + \sum_{j=0}^{K-1} (Z_{3,1}^{[j]} + Z_{3,2}^{[j]} + Z_{3,3}^{[j]}) t^{-j+\frac{1}{2}} \qquad (t \to \infty), \tag{3.101}$$

where

$$Z_{3,1}^{[j]} = \begin{cases} \frac{4\Re(P_1\bar{P}_0)}{2j-1}\Im(C_{1,2j-1}) & 1 \le j \le K-1 \\ 0 & \text{otherwise}, \end{cases}$$

$$Z_{3,2}^{[j]} = \begin{cases} \sum_{k+n=j} \frac{4\Re(P_1\bar{P}_0)\beta_k}{2j-1} C_{2k,2n} & 1 \le j \le K-1 \\ 0 & \text{otherwise}, \end{cases}$$

$$Z_{3,3}^{[j]} = \begin{cases} -\frac{4\Re(P_1\bar{P}_0)}{2j-1} C_{0,2j} & 1 \le j \le K-1 \\ 0 & \text{otherwise}. \end{cases}$$

To obtain the expansion of $Z_3^B(t)$, we apply Lemma 13 to (3.99) with $Q = Q_\sigma := K - \sigma$. Therefore

$$Z_3^B(t) = \sum_{j=0}^{K-1} Z_{3,4}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \tag{3.102}$$

where

$$Z_{3,4}^{[j]} = \begin{cases} \sum_{\substack{\sigma+n=j-1 \\ \sigma \ge 1}} 2\Re(P_1\bar{b}_{2\sigma})\Im(C_{2\sigma-1,2n+1}) & 2 \le j \le K-1 \\ 0 & \text{otherwise}. \end{cases}$$

We finally note that (3.100) is the one-dimensional analog of (3.37) from section 3.1.2. Since the powers of r are non-negative, we obtain a similar asymptotic expansion, but with $N = 1$. Thus

$$Z_3^C(t) = \sum_{j=0}^{K-1} Z_{3,5}^{[j]} t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty), \tag{3.103}$$

where

$$Z_{3,5}^{[j]} = \begin{cases} \sum_{\sigma+k+n=j-1} \Re(P_1\bar{b}_{2\sigma})\alpha_k C_{2\sigma+2k,2n} & 1 \le j \le K-1 \\ 0 & \text{otherwise}. \end{cases}$$

Since $Z_3(t) = Z_3^A(t) + Z_3^B(t) - Z_3^C(t) + O(t^{-K+1/2})$, we combine (3.101), (3.102), and

(3.103) to obtain the expansion as $t \to \infty$

$$Z_3(t) = \pi \Re(P_1 \bar{P}_0) + \sum_{j=0}^{K-1} (Z_{3,1}^{[j]} + Z_{3,2}^{[j]} + Z_{3,3}^{[j]} + Z_{3,4}^{[j]} - Z_{3,5}^{[j]}) t^{-j+\frac{1}{2}}. \tag{3.104}$$

Expansion of $Z_4(t)$

From the definition of $Z_4(t)$, we see that

$$Z_4(t) = 2\Re(P_1 \bar{P}_0) \cdot t^{-1} \tilde{I}_3(t),$$

where $\tilde{I}_3(t)$ is as given in (2.65) with $\epsilon = \delta$. Therefore by Lemma 8,

$$Z_4(t) = \pi \Re(P_1 \bar{P}_0) + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty). \tag{3.105}$$

Asymptotic expansion of $Z(t)$ as $t \to \infty$

We combine results (2.89), (3.97), (3.104), and (3.105) for $Z_1(t)$, $Z_2(t)$, $Z_3(t)$, and $Z_4(t)$, respectively, with the fact that $Z(t) = Z_1(t) - Z_2(t) - Z_3(t) + Z_4(t)$ to obtain the asymptotic expansion

$$Z(t) = \sum_{j=0}^{K-1} V_j^Z t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}) \qquad (t \to \infty),$$

where, for $j \in \{0, \ldots, K-1\}$,

$$V_j^Z = -Z_{1,1}^{[j]} - Z_{2,1}^{[j]} - Z_{2,2}^{[j]} - Z_{2,3}^{[j]} - Z_{2,4}^{[j]} - Z_{2,5}^{[j]} - Z_{3,1}^{[j]} - Z_{3,2}^{[j]} - Z_{3,3}^{[j]} - Z_{3,4}^{[j]} + Z_{3,5}^{[j]}.$$

The first three coefficients

Let us assume $K = 3$. It can then be shown that $Z(t) = V_0^Z t^{1/2} + V_1^Z t^{-1/2} + V_2^Z t^{-3/2} + O(t^{-5/2})$ as $t \to \infty$, where

$$V_0^Z = 0, \qquad V_1^Z = 0,$$

$$V_2^Z = -\frac{55\sqrt{\pi}\Re(P_1\bar{P}_0)}{2048} - \frac{\sqrt{\pi}\Re(l_2)}{4} + \frac{3\sqrt{\pi}\Re(\bar{P}_0 c_2)}{32} + \frac{5\sqrt{\pi}\Re(P_1\bar{b}_2)}{32},$$

The V_j^Z ($j \in \{0, 1, 2\}$) agree with the coefficients obtained in section 3.1.2 with $N = 1$.

3.3.3 Asymptotic expansion of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ as $t \to \infty$

For the fixed $0 < \delta < 1$, recall that for some $\eta > 0$ $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_{2,\delta}^2 + O(e^{-\eta t}) = X(t) + Y(t) + Z(t) + O(e^{-\eta t})$ as $t \to \infty$. We now combine the results we have for dimension $N = 1$ to obtain the full asymptotic expansion of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$. The $O(e^{-\eta t})$-term will hereafter be suppressed since it is dominated by the larger $O(t^{-K+1/2})$-term.

Let $K \geq 1$ be an integer as given in the statement of Theorem 2. Then as $t \to \infty$

$$\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = \sum_{j=0}^{K-1} V_j t^{-j} + O(t^{-K}),$$

where $V_j = V_j^X + V_j^Y + V_j^Z$ for each $j \in \{0, \ldots, K-1\}$. To complete the proof of Theorem 2 for dimension $N = 1$, we must verify that $V_0 = 0$.

Let $K = 3$. Then $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = V_0 t^{1/2} + V_1 t^{-1/2} + V_2 t^{-3/2} + O(t^{-5/2})$ as $t \to \infty$, where $V_j = V_j^X + V_j^Y + V_j^Z$ for each $j \in \{0, \ldots, K-1\}$.

$$V_0 = 0, \qquad V_1 = \frac{3\sqrt{\pi}|P_1|^2}{512} + \frac{\sqrt{\pi}d_2}{2} - \sqrt{\pi}\Re(\bar{P}_1 c_2),$$

$$V_2 = \frac{7621\sqrt{\pi}|P_1|^2}{1048576} + \frac{31\sqrt{\pi}|P_0|^2}{1024} - \frac{55\sqrt{\pi}\Re(P_1\bar{P}_0)}{2048} + \frac{\sqrt{\pi}d_2}{16} - \frac{49\sqrt{\pi}\Re(\bar{P}_1 c_2)}{1024} + \frac{\sqrt{\pi}f_2}{4}$$
$$- \frac{\sqrt{\pi}\Re(\bar{P}_0 b_2)}{2} - \frac{\sqrt{\pi}\Re(l_2)}{4} + \frac{3\sqrt{\pi}\Re(\bar{P}_0 c_2)}{32} + \frac{5\sqrt{\pi}\Re(P_1\bar{b}_2)}{32} + \frac{\sqrt{\pi}d_4}{4} - \frac{\sqrt{\pi}\Re(\bar{P}_1 c_4)}{2}.$$

Since $V_0 = 0$, the proof of Theorem 2 for dimension $N = 1$ is complete. Since all dimension $N \geq 1$ cases have been verified, here concludes the proof of Theorem 2.

3.3.4 Asymptotic expansion of $\|\nu(t,\cdot)\|_2^2$ as $t \to \infty$

Let us again recall that for the fixed $0 < \delta < 1$, $\|\nu(t,\cdot)\|_2^2 = \|\nu(t,\cdot)\|_{2,\delta}^2 + O(e^{-t\delta^2/2}) = X_2(t) + Y_2(t) + Z_4(t) + O(e^{-t\delta^2/2})$. We then combine results (3.83), (3.90), and (3.105) to obtain the proof of Corollary 1 part 3 and the expansion as $t \to \infty$

$$\|\nu(t,\cdot)\|_2^2 = \|\nu(t,\cdot)\|_{2,\delta}^2 + O(e^{-\frac{t\delta^2}{2}}) = \pi|P_1|^2 t + \pi\Re(P_1\bar{P}_0) + \sum_{j=0}^{K-1} U_j t^{-j+\frac{1}{2}} + O(t^{-K+\frac{1}{2}}),$$

where, for $j \in \{0,\ldots,K-1\}$,

$$U_j = X_{2,1}^{[j]} + Y_{2,1}^{[j]}.$$

The coefficients associated with $K = 3$ for dimension $N = 1$ are

$$U_0 = -\sqrt{\pi}|P_1|^2, \qquad U_1 = \frac{\sqrt{\pi}|P_0|^2}{2}, \qquad U_2 = 0.$$

This completes the proof of all three parts of Corollary 1. Furthermore, we have completed the proofs of all three main results.

Chapter 4

Examples

In this chapter we consider examples of (1.1) with different initial conditions and use *Mathematica* to plot the first few terms of the expansion of the squared L^2-norms of the solution against the actual values of the norms. The examples of initial conditions we consider are

1. $u_0(x) = 0$ and $u_1(x) = e^{-|x|^2/2}$,

2. $u_0(x) = e^{-|x|^2/2}$ and $u_1(x) = 0$,

3. $u_0(x) = u_1(x) = x_1 \cdot \ldots \cdot x_N e^{-|x|^2/2}$.

4.1 $u_0(x) = 0$ and $u_1(x) = e^{-|x|^2/2}$

This first example simplifies the problem significantly. We first note that the Fourier transforms of the initial conditions are $\hat{u}_0(\xi) = 0$ and $\hat{u}_1(\xi) = e^{-|\xi|^2/2}$. We appeal to (1.10) for the solution to (1.9), which is the solution to (1.1) in the Fourier space.

$$\hat{u}(t,\xi) = \hat{u}_1(\xi)h(t,\xi) = e^{-\frac{(t+1)|\xi|^2}{2}} \frac{\sin(t|\xi|\sqrt{1-|\xi|^2/4})}{|\xi|\sqrt{1-|\xi|^2/4}}$$

and, since $P_0 = (2\pi)^{-N/2} \int_{\mathbb{R}^N} u_0(x)dx = 0$ and $P_1 = (2\pi)^{-N/2} \int_{\mathbb{R}^N} u_1(x)dx = 1$, the asymptotic profile $\nu(t, \xi)$ found by Ikehata in [4] is

$$\nu(t, \xi) = e^{-\frac{t|\xi|^2}{2}} \frac{\sin(t|\xi|)}{|\xi|}.$$

To illustrate the three different solutions depending on space dimension, we examine asymptotic behavior in dimensions $N = 1, 2, 3$. We note that $u_1 \in L^{1,2K}(\mathbb{R}^N)$ for any $N, K \in \mathbb{N}$. Therefore we could find arbitrarily many terms in the expansions of $\|u(t, \cdot)\|_2^2$. However, we will content ourselves with finding the terms associated with $K = 3$, as we studied earlier in this thesis.

4.1.1 Dimension $N = 1$

The computation of terms of the expansion of $\|u(t, \cdot)\|_2^2$ associated with $K = 3$ can be found in section 2.3.3. Therefore the asymptotic expansion of $\|u(t, \cdot)\|_2^2$ as $t \to \infty$ is given by

$$\|u(t, \cdot)\|_2^2 = \pi t - \sqrt{\pi} t^{\frac{1}{2}} - \frac{3\sqrt{\pi}}{8} t^{-\frac{1}{2}} + \frac{5\sqrt{\pi}}{64} t^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}).$$

Given in Figure 4.1 are two images. The image on the right compares the asymptotic expansion of $\|u(t, \cdot)\|_2^2$ (without the error term of order $O(t^{-5/2})$) with the actual values. Note the scale as even for relatively small values of t, the approximation is very good. The image on the left is there to provide the general shape of the graph of the actual values of $\|u(t, \cdot)\|_2^2$.

Next we consider the asymptotic expansion of $\|\hat{u}(t, \cdot) - \nu(t, \cdot)\|_2^2$ as $t \to \infty$. The computation of the terms of the expansion can be found in section 3.3.3. Thus as $t \to \infty$,

$$\|\hat{u}(t, \cdot) - \nu(t, \cdot)\|_2^2 = \frac{3\sqrt{\pi}}{512} t^{-\frac{1}{2}} + \frac{32709\sqrt{\pi}}{1048576} t^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}).$$

Indeed the solution $\hat{u}(t, \xi)$ of (1.1) in the Fourier space and its asymptotic profile $\nu(t, \xi)$ tend

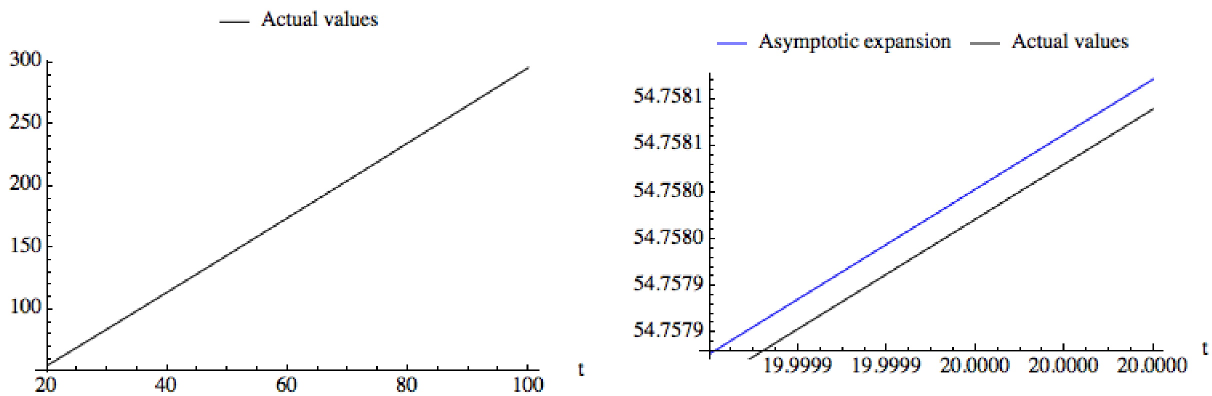


Figure 4.1: Graph of $\|u(t,\cdot)\|_2^2$ (left), and plot comparing $\|u(t,\cdot)\|_2^2$ to its asymptotic expansion (right)

toward one another in norm. As stated in Theorem 2, we will see this rate of convergence increase when the space dimension is larger. Figure 4.2 compares the asymptotic expansion we derived with the actual values of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$.

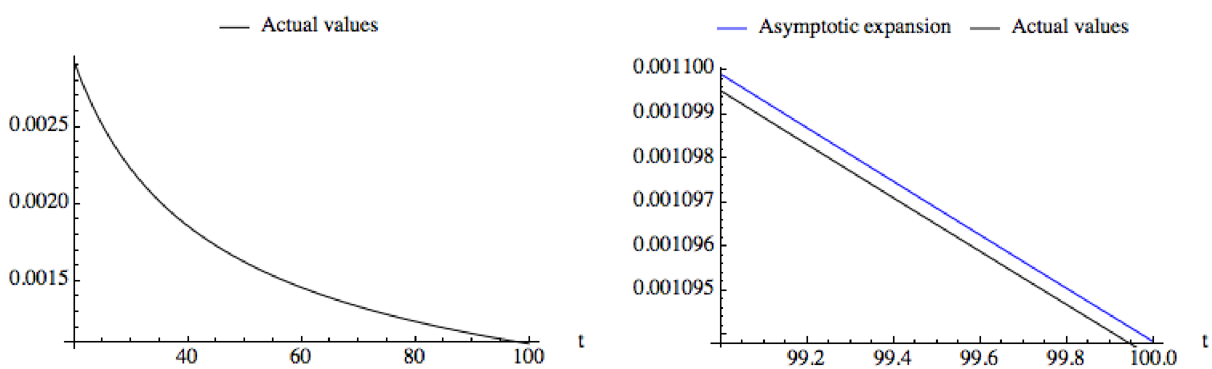


Figure 4.2: Graph of $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ (left), and plot comparing $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ to its asymptotic expansion (right)

4.1.2 Dimension $N = 2$

In space dimension $N = 2$, we determined that growth of the norms $\|u(t,\cdot)\|_2^2$ would be on the order of $\ln(t)$ so long as $P_1 = \int_{\mathbb{R}^2} u_1(x)dx \neq 0$. The asymptotic expansion of $\|u(t,\cdot)\|_2^2$ as $t \to \infty$ with $K = 3$ is given in section 2.2.3:

$$\|u(t,\cdot)\|_2^2 = \frac{\pi}{2}\ln(t) + \frac{\pi\gamma}{2} + \pi\ln(2) - \frac{5\pi}{8}t^{-1} - \frac{5\pi}{32}t^{-2} + O(t^{-3}).$$

Figure 4.3 compares the asymptotic expansion obtained for $\|u(t,\cdot)\|_2^2$ with the actual values. We note the logarithmic growth of the norms of the solutions $u(t,x)$ as discovered from the analysis.

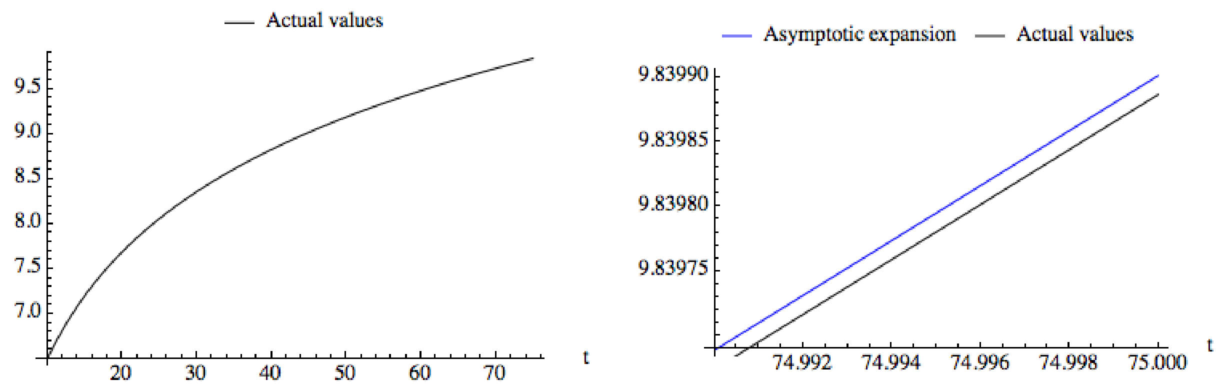


Figure 4.3: Graph of $\|u(t,\cdot)\|_2^2$ (left), and plot comparing $\|u(t,\cdot)\|_2^2$ to its asymptotic expansion (right)

The terms of the asymptotic expansion of $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ are given in section 3.2.3. Thus as $t \to \infty$

$$\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2 = \frac{\pi}{64}t^{-1} + \frac{235\pi}{4096}t^{-2} + O(t^{-3}).$$

We observe the rate of convergence in norm is faster than in the one-dimensional case. Figure 4.4 exhibits the similarity between the expansion and actual values.

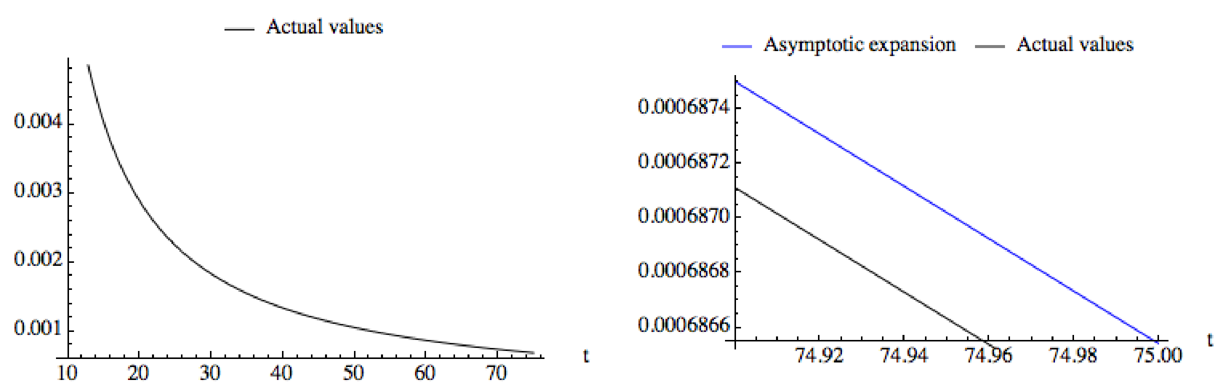


Figure 4.4: Graph of $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ (left), and plot comparing $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ to its asymptotic expansion (right)

4.1.3 Dimension $N \geq 3$

The cases $N \geq 3$ all have solutions whose L^2-norms decay over time. In the analysis we determined that the decay rate $\|u(t,\cdot)\|_2^2$ is on the order of $t^{-N/2+1}$. We just consider the problem (1.1) in $\mathbb{R}^3$, since the analysis of the other cases have similar results. We found the terms of the asymptotic expansion of $\|u(t,\cdot)\|_2^2$ corresponding to $K = 3$ in section 2.1.3. We have the expansion as $t \to \infty$

$$\|u(t,\cdot)\|_2^2 = \pi\sqrt{\pi}t^{-\frac{1}{2}} - \frac{3\pi\sqrt{\pi}}{8}t^{-\frac{3}{2}} + \frac{15\pi\sqrt{\pi}}{64}t^{-\frac{5}{2}} + O(t^{-\frac{7}{2}}).$$

Figure 4.5 illustrates the similarity between the first few terms of the asymptotic expansion obtained for $\|u(t,\cdot)\|_2^2$ and the actual values.

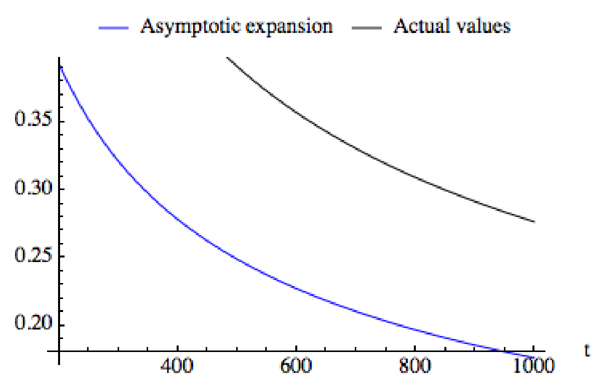


Figure 4.5: Graph of $\|u(t,\cdot)\|_2^2$ compared to its asymptotic expansion

The terms of the asymptotic expansion of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ are given in section 3.1.3. The diffusion-type phenomenon mentioned in the Introduction is evident in the cases $N \geq 3$, since the decay rate of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ is faster than that of $\|u(t,\cdot)\|_2^2$ alone. We have the asymptotic expansion as $t \to \infty$

$$\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = \frac{15\pi\sqrt{\pi}}{512}t^{-\frac{3}{2}} + \frac{80247\pi\sqrt{\pi}}{1048576}t^{-\frac{5}{2}} + O(t^{-\frac{7}{2}}).$$

Once again in Figure 4.6 we compare the asymptotic expansion we obtained to the actual values and note the similarities.

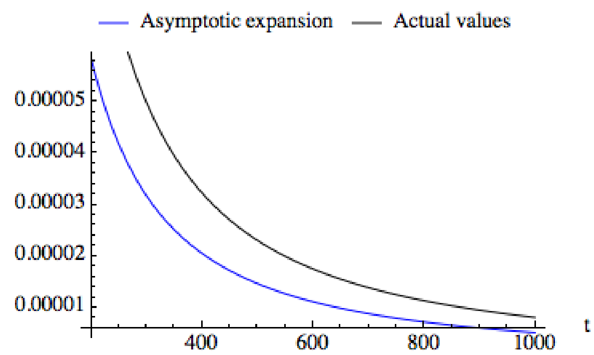


Figure 4.6: Graph of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ compared to its asymptotic expansion

4.2 $u_0(x) = e^{-|x|^2/2}$ and $u_1(x) = 0$

For the second example we interchange the initial data. This simple change yields very different results when compared to the last example. The Fourier transforms of the initial conditions are $u_0(\xi) = e^{-|\xi|^2/2}$ and $u_1(\xi) = 0$. The solution of (1.9) (and thus the solution of (1.1) in the Fourier space) is given by (1.10):

$$\begin{aligned}\hat{u}(t,\xi) &= |\xi|^2 \hat{u}_0(\xi) + \hat{u}_0(\xi)\partial_t h(t,\xi) \\ &= e^{-\frac{(1+t)|\xi|^2}{2}} \left(\cos(t|\xi|\sqrt{1-|\xi|^2/4}) + \frac{|\xi|\sin(t|\xi|\sqrt{1-|\xi|^2/4})}{2\sqrt{1-|\xi|^2/4}} \right).\end{aligned}$$

Since $P_0 = 1$ and $P_1 = 0$, the asymptotic profile found by Ikehata in [4] is simply

$$\nu(t,\xi) = e^{-\frac{t|\xi|^2}{2}} \cos(t|\xi|).$$

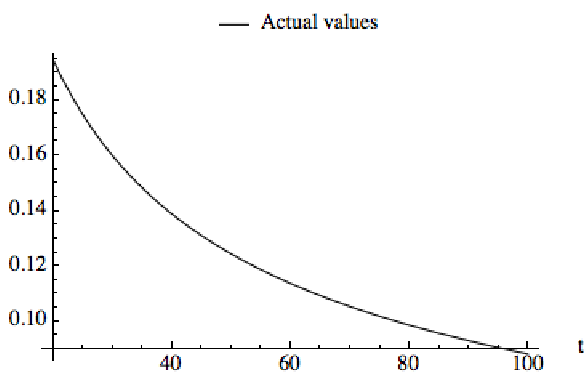
We illustrate the asymptotic behavior in space dimensions $N = 1, 2, 3$ as in the previous example. Since $u_0 \in L^{1,2K}(\mathbb{R}^N)$ for any $N, K \in \mathbb{N}$, we may find as many terms in the expansions as desired. We will look at the expansions for $K = 3$.

4.2.1 Dimension $N = 1$

The terms of the expansion of $\|u(t,\cdot)\|_2^2$ may be computed from section 2.3.3. Hence as $t \to \infty$

$$\|u(t,\cdot)\|_2^2 = \frac{\sqrt{\pi}}{2} t^{-\frac{1}{2}} - \frac{3\sqrt{\pi}}{16} t^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}).$$

Figure 4.7 illustrates the decay of the solution in the $L^2(\mathbb{R})$-norm, as well as the similarity between the actual values of $\|u(t,\cdot)\|_2^2$ compared to the first few terms of its asymptotic expansion.



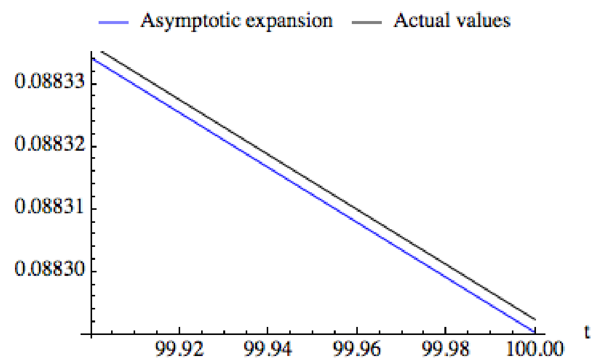


Figure 4.7: Graph of $\|u(t,\cdot)\|_2^2$ (left), and plot comparing $\|u(t,\cdot)\|_2^2$ to its asymptotic expansion (right)

Next in Figure 4.8, we compare the actual values of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ to that of the first terms of its asymptotic expansion

$$\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = \frac{127\sqrt{\pi}}{1024} t^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}) \qquad (t \to \infty).$$

The term in this expansion may be computed from section 3.3.3.

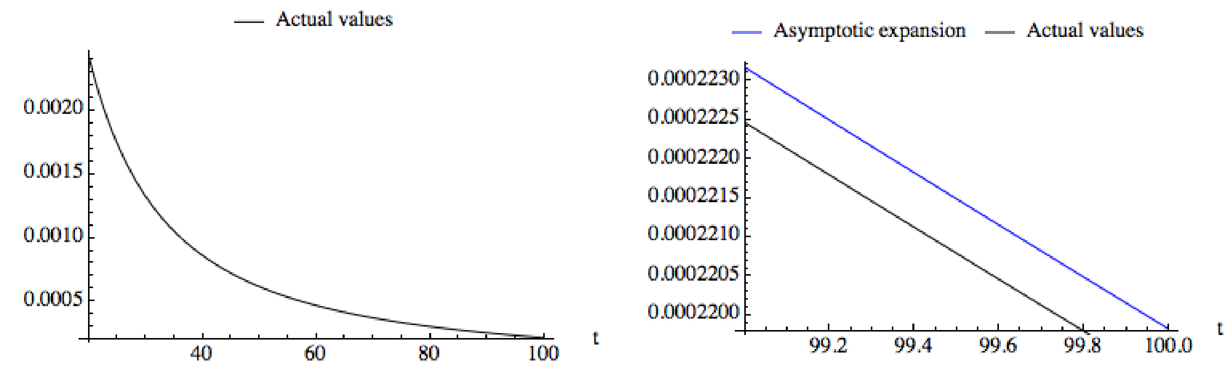


Figure 4.8: Graph of $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ (left), and plot comparing $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ to its asymptotic expansion (right)

4.2.2 Dimension $N = 2$

We appeal to sections 2.2.3 and 3.2.3, respectively, for the asymptotic expansions of $\|u(t,\cdot)\|_2^2$ and $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$. Hence as $t \to \infty$

$$\begin{aligned}
\|u(t,\cdot)\|_2^2 &= \frac{\pi}{2}t^{-1} - \frac{\pi}{4}t^{-2} + O(t^{-3}) \\
\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2 &= \frac{19\pi}{64}t^{-2} + O(t^{-3}).
\end{aligned}$$

Figures 4.9 and 4.10 show the relationship between the actual values of $\|u(t,\cdot)\|_2^2$ and $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ compared with the first terms of their asymptotic expansions.

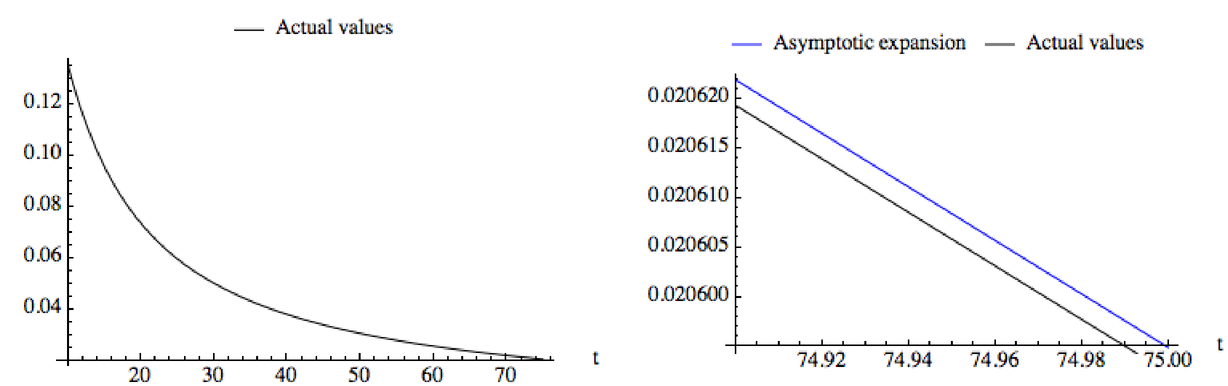


Figure 4.9: Graph of $\|u(t,\cdot)\|_2^2$ (left), and plot comparing $\|u(t,\cdot)\|_2^2$ to its asymptotic expansion (right)

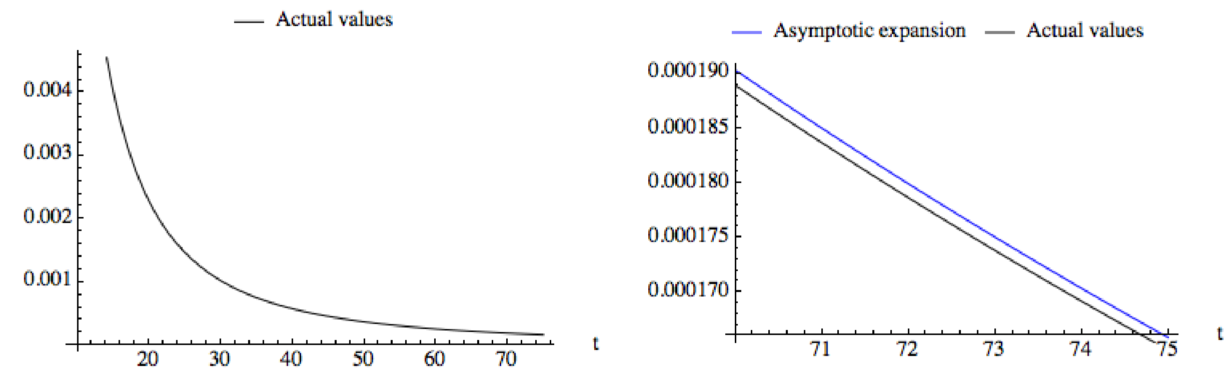


Figure 4.10: Graph of $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ (left), and plot comparing $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ to its asymptotic expansion (right)

4.2.3 Dimension $N \geq 3$

The asymptotic expansions of $\|u(t,\cdot)\|_2^2$ and $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ are given in sections 2.1.3 and 3.1.3, respectively. As $t \to \infty$

$$\begin{aligned} \|u(t,\cdot)\|_2^2 &= \frac{\pi\sqrt{\pi}}{2}t^{-\frac{3}{2}} - \frac{9\pi\sqrt{\pi}}{16}t^{-\frac{5}{2}} + O(t^{-\frac{7}{2}}) \\ \|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2 &= \frac{537\pi\sqrt{\pi}}{1024}t^{-\frac{5}{2}} + O(t^{-\frac{7}{2}}). \end{aligned}$$

The figures 4.11 and 4.12 illustrate the closeness of the graphs of $\|u(t,\cdot)\|_2^2$ and $\|\hat{u}(t,\cdot)-\nu(t,\cdot)\|_2^2$ to the first terms of their asymptotic expansions.

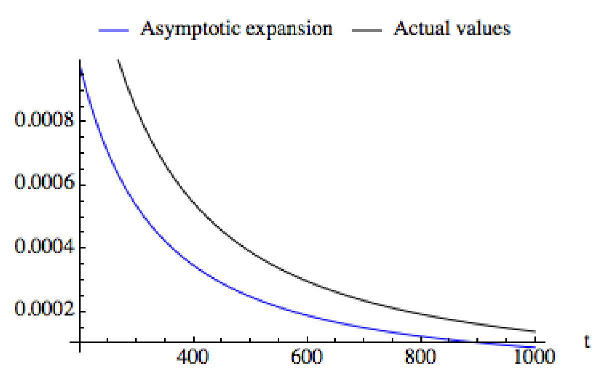


Figure 4.11: Graph of $\|u(t,\cdot)\|_2^2$ compared to its asymptotic expansion

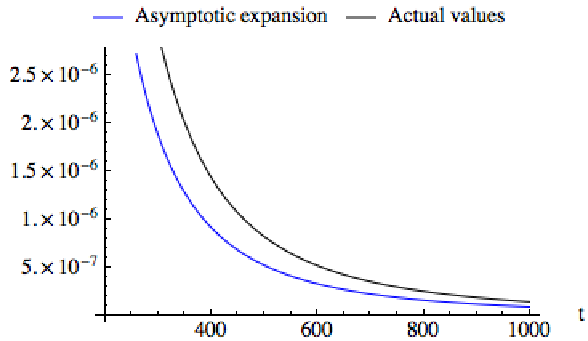


Figure 4.12: Graph of $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ compared to its asymptotic expansion

4.3 $u_0(x) = u_1(x) = x_1 \cdot \ldots \cdot x_N \, e^{-|x|^2/2}$

In this example we use the same function for both initial conditions. These initial data yield interesting results since they are non-zero, yet have zero integral on $\mathbb{R}^N$. The Fourier transforms are $\hat{u}_0(\xi) = \hat{u}_1(\xi) = (-i)^N \xi_1 \cdot \ldots \cdot \xi_N \, e^{-|\xi|^2/2}$. The solution of (1.1) in the Fourier space is given by (1.10):

$$\begin{aligned}\hat{u}(t,\xi) &= (\hat{u}_1(\xi) + |\xi|^2 \hat{u}_0(\xi))h(t,\xi) + \hat{u}_0(\xi)\partial_t h(t,\xi) \\ &= (-i)^N \xi_1 \cdot \ldots \cdot \xi_N \, e^{-\frac{(1+t)|\xi|^2}{2}} \left(\frac{\sin(t|\xi|\sqrt{1-|\xi|^2/4})}{|\xi|\sqrt{1-|\xi|^2/4}} + \frac{|\xi|\sin(t|\xi|\sqrt{1-|\xi|^2/4})}{2\sqrt{1-|\xi|^2/4}} \right. \\ &\qquad \left. + \cos(t|\xi|\sqrt{1-|\xi|^2/4}) \right).\end{aligned}$$

Furthermore, since both initial data integrate to zero, $P_0 = P_1 = 0$ and thus the profile found by Ikehata in [4] is

$$\nu(t,\xi) = 0,$$

and gives no information.

Since $\nu(t,\xi) = 0$, we note that $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$. We only consider the former, as both have similar asymptotic expansions and graphs. As with the previous examples, we will find the expansions for space dimensions $N = 1, 2, 3$. Since $u_0, u_1 \in L^{1,2K}(\mathbb{R}^N)$ for all $K \in \mathbb{N}$, we determine only the coefficients corresponding to $K = 3$.

4.3.1 Dimension $N = 1$

Since $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$, the expansions of both can be computed from section 2.3.3. As $t \to \infty$

$$\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = \frac{\sqrt{\pi}}{2} t^{-\frac{1}{2}} + \frac{5\sqrt{\pi}}{16} t^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}).$$

Figure 4.13 illustrates the common decay of both $\|u(t,\cdot)\|_2^2$ and $\|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ and compares them to the first terms of their common asymptotic expansion.

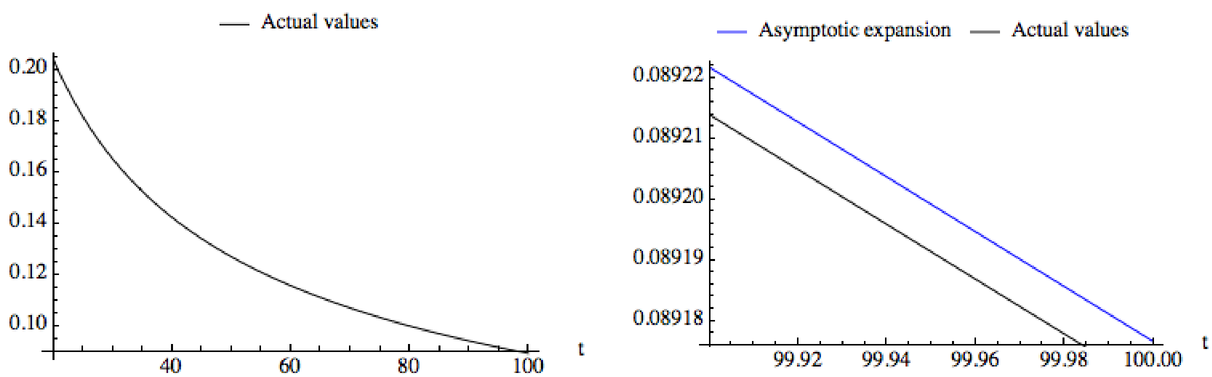


Figure 4.13: Graph of $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ (left), and plot comparing them to their asymptotic expansion (right)

4.3.2 Dimension $N = 2$

Once again $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ and the asymptotic expansion can be computed from section 2.2.3. As $t \to \infty$

$$\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = \frac{\pi}{16} t^{-2} + O(t^{-3}).$$

In Figure 4.14 we can see the shape of the graph of $\|u(t,\cdot)\|_2^2$, as well as the first term of its asymptotic expansion.

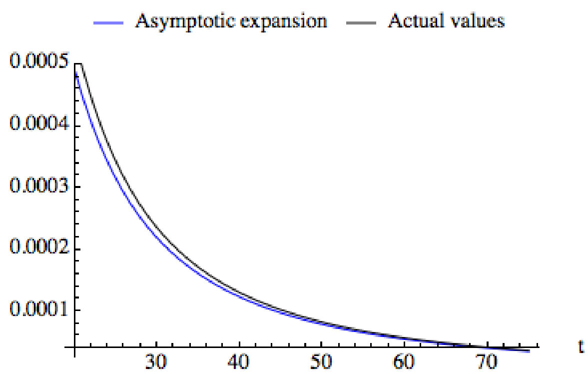


Figure 4.14: Graph of $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ compared to its asymptotic expansion

4.3.3 Dimension $N \geq 3$

The dimension $N \geq 3$ cases are where interesting things start to happen. It turns out that because of the choice of initial data, the coefficients of the expansion of $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$ obtained for $K = 3$ are all zero. Thus the asymptotic expansion as $t \to \infty$ in this scenario (recall we use dimension $N = 3$) is

$$\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2 = O(t^{-\frac{7}{2}}).$$

We may still illustrate the decay of the function $\|u(t,\cdot)\|_2^2$ (as in Figure 4.15), but since we do not have any terms in the expansion, we cannot compare the two graphs aside from noting how small the values of the function are.

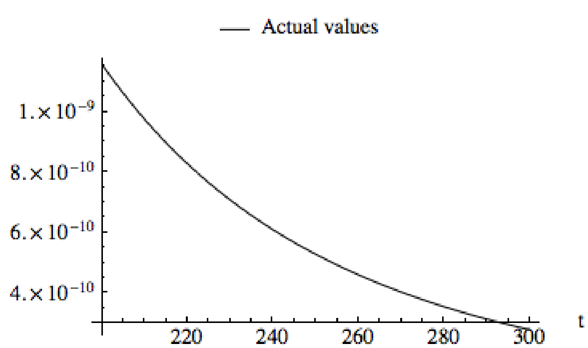


Figure 4.15: Graph of $\|u(t,\cdot)\|_2^2 = \|\hat{u}(t,\cdot) - \nu(t,\cdot)\|_2^2$

Chapter 5

Conclusion

We conclude this thesis by looking forward to extensions of this problem and to related problems. Let us first recall Volkmer's paper [11], in which he determined the asymptotic expansions of the L^2-norms of the weak solutions of the heat and dissipative wave equations, as well as the expansion of the L^2-norm of their difference. Additionally he was able to determine the same expansions of any partial time and space derivatives. Doing the same for the problem studied in this thesis is the next natural step when using Fourier transform methods to analyze the strong damped wave equation (1.1), since the Fourier transform changes spatial derivatives to multiplication. It is therefore the opinion of the author that finding the expansion of the L^2-norm of any partial space derivatives of the weak solution of (1.1) should be a straightforward task. However, the time derivatives might be difficult due to the increasing complexity of the time derivatives of the weak solution in the Fourier space.

The other extension of this problem is to consider the Cauchy problem for the PDE

$$\begin{gathered} u_{tt}(t,x) + (-\Delta)^{\theta} u_t(t,x) + (-\Delta)^{\tau} u(t,x) = 0, \qquad (t,x) \in (t,x) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N, \end{gathered} \tag{5.1}$$

where $0 < \frac{\tau}{2} \leq \theta \leq \tau$, and the operator $(-\Delta)^{\sigma}$ for $\sigma > 0$ is defined by

$$((-\Delta)^{\sigma} f)(x) := \mathcal{F}^{-1}\left(|\cdot|^{2\sigma} \hat{f}\right)(x).$$

Proper assumptions on the initial data would need to be found. However it is the opinion of the author that similar asymptotic expansions to those in this thesis involving the solution of (5.1) may also be obtained, since some results of Ikehata and Natsume in [5] seem to extend nicely to this scenario.

We now look ahead to how the Fourier transform method can be used to study related problems. We first look at the Cauchy problem for the generalized plate equation with a structural damping in $\mathbb{R}^N$ $(N \geq 1)$

$$\begin{gathered} u_{tt}(t,x) + (-\Delta)^{\theta} u_t(t,x) + \alpha \Delta^2 u(t,x) - \Delta u(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N, \end{gathered} \tag{5.2}$$

where $\alpha \geq 0$ and $0 \leq \theta \leq 1$. In their paper [7], Ikehata and Soga established asymptotic estimates for the squared L^2-norm of the difference of the Fourier transform of the weak solution of (5.2) and the profile $\nu(t,\xi)$ found in this thesis and Ikehata's paper [4]. It is the opinion of the author that the asymptotic expansion may be obtained using similar methods to those found in this thesis.

We discuss one final problem for which the methods used in this thesis might be of use, the wave equation with frictional and viscoelastic damping terms

$$\begin{gathered} u_{tt}(t,x) + \partial_t u(t,x) - \Delta u_t(t,x) - \Delta u(t,x) = f(u), \qquad (t,x) \in (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N. \end{gathered} \tag{5.3}$$

In general, f is a nonlinear function, but the Fourier transform method used in this thesis seems particularly well-suited to the homogeneous case of (5.3) when $f \equiv 0$. In their paper [8], Ikehata and Takeda obtained asymptotic estimates of the squared L^2-norm of the weak solution of the Cauchy problem for (5.3), which possibly can be extended to an asymptotic expansion.

For all the problems in this discussion, considering the solutions of the PDEs in the Fourier space could allow for expansions to be obtained, with strict enough conditions on the initial data.

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Appendix

Computing $B_{m,n}$, $\tilde{B}_{m,n}$, and $C_{m,n}$ with *Mathematica*

In the proofs of Proposition 1 and lemmas 10 and 11, we claimed the coefficients in the expansions of $G^{\epsilon}_{1,m}(t)$, $G^{\epsilon}_{2,m}(t)$, and $G^{\epsilon}_{3,m}(t)$ may be computed with a computer algebra system like *Mathematica*. The following lines of code will return the first few terms in each of the expansions.

These first three commands allow for us to set the desired accuracy. Since the accuracy is currently set to 5, we are able to correctly determine, for any $m \in \mathbb{N}_0$, $B_{m,n}$ and $\tilde{B}_{m,n}$ for $0 \leq n \leq 5 - 1 = 4$, and $C_{m,n}$ for $0 \leq n \leq 2 \cdot 5 - 1 = 9$.

```
acc = 5;
Bacc = acc;
Cacc = 2*acc;
```

The following sequence of commands returns the coefficients in the expansion of $G^{\epsilon}_{1,m}(t)$, where $0 < \epsilon < 2$.

```
g1 = Exp[I*r*t*(2 - Sqrt[4 - r^2])]; (* g[t,r] from the thesis *)
g1coeff[k_Integer] := SeriesCoefficient[g1, {r, 0, k}];
G1 = Sum[g1coeff[k]*Aa[m + k, p]*t^(-m - k - p - 1), {k, 0, 3/2*(Bacc - 1)}, {p, 0, Floor[
    Bacc - 2/3*k] - 1}];
```

```
Bb[m, n_Integer] := Coefficient[G1, t, -m - n - 1];
For[i = 0, i <= Bacc - 1, i++, Print["Bb[m,", i, "]", "=", Bb[m, i]]]
```

The following sequence of commands returns the coefficients in the expansions of $G^{\epsilon}_{3,m}(t)$ and $G^{\epsilon}_{2,m}(t)$, respectively, where $0 < \epsilon < 2$.

```
g2 = Exp[I*r*(t/2)*(2 - Sqrt[4 - r^2])]; (* g[t/2,r] from the thesis *)
g2coeff[k_Integer] := SeriesCoefficient[g2, {r, 0, k}];
G3 = Sum[g2coeff[k]*Aa[m + k, p]*t^(-m - k - p - 1), {k, 0, 3/2*(Bacc - 1)}, {p, 0, Floor[
    Bacc - 2/3*k] - 1}]
Bbtilde[m, n_Integer] := Coefficient[G3, t, -m - n - 1];
For[i = 0, i <= Bacc - 1, i++, Print["Bbtilde[m,", i, "]", "=", Bbtilde[m, i]]]
G2 = Apart[Sum[1/2*g2coeff[k]*Gamma[(m + k + 1)/2]*t^(-(m + k + 1)/2), {k, 0, 3*Cacc}], t];
Cc[m, n_Integer] := Coefficient[Apart[G2*t^(m/2 + n/2 + 1/2), t], t, 0];
For[i = 0, i <= Cacc - 1, i++, Print["Cc[m,", i, "]", "=", Cc[m, i]]]
```

To obtain numerical values for the coefficients, we need only insert the following line of code defining $A_{k,p}$ for $k, p \in \mathbb{N}_0$ as in (2.10) after the first three lines given above.

```
Aa[k_, p_] = k!*Pochhammer[(k + 1)/2, p]*Pochhammer[(k + 2)/2, p]/((2*I)^(k + 1)*p!)
```

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