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Pricing of Dependent Risks

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PRICING OF DEPENDENT RISKS

by

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ABSTRACT

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In some types of insurance businesses, such as cyber or homeowners insurance, the assumption that risks are independent is violated. Because of this, the commonly used expected value premium principle does not work. Therefore, we propose different premium principles for pricing dependent risks. We derive formulas for these principles when the risks are normally distributed, pareto distributed and each risk is an aggregate loss. Furthermore, we investigate the behavior of the different premium principles related to a change in the dependence of the risks. Additionally, we examine the impact that a parameter of one risk has on the premium for each proposed principle.

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1 Introduction

In a traditonal insurance company, the premium for overtaking a risk is calculated based on the expected claim of the risk and a safety loading which is usually a percentage of the expected loss. This means $P = (1+\theta)\,\mathrm{E}[X]$, where $\theta > 0$ is the safety loading coefficent and X is the risk that should be insured. The method of calculating the premium is the expected value principle. The premium principle and comparable principles work well for traditional insurance because the strong law of large numbers guarantees that the total claim converges to the expectation of total loss. This is equal to the total premium plus the safety loading. This concept works very well for life insurance or car insurance.

However, there are currently other kinds of insurance which are becoming more common and more important, such as cyber insurance or home insurance in an earthquake-active area or flood region. The problem is we cannot use the expected value principle to find the premium for those types of risks and the law of large numbers to justify it because the risks in these types of insurance are not independent anymore. Hence, there is the need to find new premium principles which consider the dependent structure of the risks.

One approach is "global pricing". This approach is based on calculating the total premium which is necassary to cover all of the insured risks. After that, the premium should be properly allocated to every individual policyholder. Assuming that an insurance has n risks $X_1, X_2, \ldots, X_n$ then the total loss is $TL = \sum_{i=1}^{n} X_i$. The Value at Risk (VaR) of the total loss would be a good choice for the total premium because the VaR describes how much money the insurer needs to be $100\alpha\%$ confident to cover the total loss. The defintion of Value at Risk for a continous random variable and level α is given in Klugman et al.(2012) [1] (Definition 3.12):

$$VaR_\alpha(TL) = max\{x : P(TL > x) = 1 - \alpha\} \tag{1.1}$$

Another plausible candidate for the total premium is the Tail Value at Risk which is more conservative and is defined as following:

$$TVaR_\alpha(TL) = \mathrm{E}\big[TL|TL > VaR_\alpha(TL)\big] \tag{1.2}$$

The defintion of TVaR is taken from Klugman et al.(2012) [1] (Definition 3.13).

After calculating the amount of the total premium, it has to be allocated to each individual policyholder. We propose the following three capital allocation principles.

- The Value at Risk premium principle:

$$P_i = \frac{VaR_\alpha(X_i)}{\sum_{i=1}^{n} VaR_\alpha(X_i)} \times VaR_\alpha(TL) \tag{1.3}$$

- The Tail Value at Risk premium principle:

$$P_i = \frac{TVaR_\alpha(X_i)}{\sum_{i=1}^{n} TVaR_\alpha(X_i)} \times TVaR_\alpha(TL) \tag{1.4}$$

- The conditional premium principle:

$$P_i = \mathrm{E}\big[X_i|TL > VaR_\alpha(TL)\big] \tag{1.5}$$

Note under (1.4) and (1.5), the total premium would be equal to $TVaR_\alpha(TL)$. Additionally, if all risks are exchangeable, then the total premium would be assigned equally to each policyholder.

In the next chapters we will calculate formulas for each premium principle when the risks X_i, $i = 1, \ldots, n$ are normally distributed, pareto distributed or each risk is an aggregate loss. Furthermore, we will analyze the impact the dependence of the risks has on the individual premium.

2 Multivariate Normal model

Let's consider the multivariate normal model, which means we have risks X_i, $i = 1, \dots, n$ and each of these risks are normally distributed, $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$. We also assume that all risks have the same covariance $Cov[X_i, X_j] = \rho\sigma^2$ for all $i \neq j$. But before we derive formulas for the different premium principles we need the following results.

Proposition 2.1 *If X is normally distributed with $\mathrm{E}[X] = \mu$ and $Var(X) = \sigma^2$ then the Value at Risk (VaR) is:*

$$VaR_\alpha(X) = max\{x : P(X > x) = 1 - \alpha\} = \mu + \sigma z_\alpha \tag{2.1}$$

where $z_\alpha = \Phi^{-1}(\alpha)$ and $\Phi(x)$ is the cdf of a standard normal random variable.

Proof. Since X is normally distributed with $\mathrm{E}[X] = \mu$ and $Var(X) = \sigma^2$ it holds that $X = \sigma Z + \mu$ with $Z \sim N(0, 1)$. With the defintion of the Value at Risk we get:

$$VaR_\alpha(X) \Rightarrow P(X \leq x) = \alpha \Leftrightarrow P\Big(Z \leq \frac{x - \mu}{\sigma}\Big) = \alpha \Leftrightarrow \Phi\Big(\frac{x - \mu}{\sigma}\Big) = \alpha \Leftrightarrow x = \mu + \sigma z_\alpha$$

□

Proposition 2.2 *If $X \sim \mathcal{N}(\mu, \sigma^2)$ then the Tail Value at Risk is:*

$$TVaR_\alpha(X) = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1 - \alpha} \tag{2.2}$$

where $\phi(x)$ represents the density of a standard normal random variable.

Proof. Let's consider a risk $X \sim \mathcal{N}(\mu, \sigma^2)$ and with the defintion of the Tail Value at Risk and (2.1) we get:

$$TVaR_\alpha(X) = \int_{VaR_\alpha(X)}^{\infty} \frac{xf_x(x)}{P(X > VaR_\alpha(X))}\, dx = \frac{1}{1 - \alpha}\int_{\mu+\sigma z_\alpha}^{\infty} \frac{x}{\sqrt{2\pi}\sigma} exp\bigg(-\frac{1}{2}\Big(\frac{x - \mu}{\sigma}\Big)^2\bigg)\, dx$$

After substituting $y = \frac{x-\mu}{\sigma}$ and with $\frac{dx}{\sigma} = dy$ we get:

$$= \frac{1}{1-\alpha} \int_{z_\alpha}^{\infty} \frac{\sigma y + \mu}{\sqrt{2\pi}} exp\Big(-\frac{1}{2} y^2\Big)\, dy$$

After dividing the integral in two parts, integrating the second part and defining $Z \sim N(0,1)$ we get:

$$= \frac{\mu}{1-\alpha} P(Z > z_\alpha) + \frac{\sigma}{1-\alpha} \left[\frac{1}{\sqrt{2\pi}} exp\Big(-\frac{1}{2} y^2\Big) \right]_{y=z_\alpha}^{y=\infty} = \mu + \sigma \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$$

□

Proposition 2.3 *If $X_i \sim \mathcal{N}(\mu, \sigma^2)$ for all $i = 1, \ldots, n$ and all risks have the same covariance $Cov[X_i, X_j] = \rho\sigma^2$ for all $i \neq j$, then the total loss $TL = \sum_{i=1}^{n} X_i$ is also normally distributed with $\mathrm{E}[TL] = n\mu$ and $Var(TL) = n\sigma^2 + n(n-1)\rho\sigma^2$.*

Proof. It holds that the sum of normally distributed random variables is normally distributed. Also, since all risks are identical, normally distributed and because the expectation is additive, it follows directly that the $\mathrm{E}[TL] = n\mu$. For the variance it holds that:

$$\begin{aligned} Var(TL) &= Cov\left[\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov[X_i, X_j] \\ &= nCov[X_1, X_1] + n(n-1)Cov[X_1, X_2] = n\sigma^2 + n(n-1)\rho\sigma^2 \end{aligned} \tag{2.3}$$

This holds because we have in the double sum exactly n terms with $i = j$ and $n(n-1)$ terms with $i \neq j$. □

Another interesting measure for comparing premiums is to compare the three proposed premium principles (1.3), (1.4) and (1.5) with the expected premium principle. We can compare these premiums by setting them equal and calculating the safety loading for each proposed premium principle with the following equation:

$$\theta_i = \frac{P_i}{\mathrm{E}[X_i]} - 1 \tag{2.4}$$

In the next section we derive explicit formulas for the case that all risks are identical and normally distributed.

2.1 Premium calculation

Let's assume that all risks are identical, normally distributed and have the same covariance. This means:

- $X_i \sim \mathcal{N}(\mu, \sigma^2)$ for all $i \in \{1, \ldots, n\}$ (2.5)
- $Cov[X_i, X_j] = \rho\sigma^2$ for all $i \neq j$ (2.6)

Then we can get the following expression for the premium principles:

Proposition 2.4 *If the risks satisfy the conditions* (2.5) *and* (2.6) *then the VaR premium given in (1.3) is:*

$$P_i = \mu + \sigma z_\alpha \sqrt{\frac{1 + (n-1)\rho}{n}}, \text{ for all } i = 1, \ldots, n \tag{2.7}$$

Proof.

$$P_i = \frac{VaR_\alpha(X_i)}{\sum_{i=1}^{n} VaR_\alpha(X_i)} \times VaR_\alpha\left(\sum_{i=1}^{n} X_i\right) \overset{(2.1)}{=} \frac{\mu + \sigma z_\alpha}{n(\mu + \sigma z_\alpha)} \times (n\mu + \sigma_{TL} z_\alpha)$$
$$\overset{(2.3)}{=} \mu + \frac{\sqrt{n\sigma^2 + n(n-1)\rho\sigma^2}}{n} z_\alpha = \mu + \sigma z_\alpha \sqrt{\frac{1 + (n-1)\rho}{n}}$$

□

Proposition 2.5 *If the risks satisfy the conditions* (2.5) *and* (2.6) *then the TVaR premium* *given in (1.4) is:*

$$P_i = \mu + \sigma \frac{\phi(z_\alpha)}{1-\alpha} \sqrt{\frac{1+(n-1)\rho}{n}}, \text{ for all } i = 1, \dots, n \tag{2.8}$$

Proof.

$$P_i = \frac{TVaR_\alpha(X_i)}{\sum_{i=1}^n TVaR_\alpha(X_i)} \times TVaR_\alpha\left(\sum_{i=1}^n X_i\right) \overset{(2.2)}{=} \frac{\mu + \sigma\frac{\phi(z_\alpha)}{1-\alpha}}{n(\mu + \sigma\frac{\phi(z_\alpha)}{1-\alpha}}) \times \left(n\mu + \sigma_{TL}\frac{\phi(z_\alpha)}{1-\alpha}\right)$$
$$\overset{(2.3)}{=} \mu + \frac{\sqrt{n\sigma^2 + n(n-1)\rho\sigma^2}}{n} \frac{\phi(z_\alpha)}{1-\alpha} = \mu + \sigma\sqrt{\frac{1+(n-1)\rho}{n}} \frac{\phi(z_\alpha)}{1-\alpha}$$

□

Proposition 2.6 *If the risks satisfy the conditions* (2.5) *and* (2.6) *then the conditional premium given in (1.5) is:*

$$P_i = \mathrm{E}\big[X_i | TL > VaR_\alpha(TL)\big] = \mu + \sigma\frac{\rho_{X_i,TL}\phi(z_\alpha))}{1-\alpha}, \text{ for all } i = 1, \dots, n \tag{2.9}$$

Proof. First we find the correlation between risk X_i and the total loss TL:

$$\begin{aligned}\rho_{X_i,TL} &= \frac{Cov(X_i, \sum_{j=1}^n X_j)}{\sigma\sigma_{TL}} \overset{(2.3)}{=} \frac{\sum_{j=1}^n Cov(X_i, X_j)}{\sigma\sqrt{n\sigma^2 + n(n-1)\rho\sigma^2}} \\ &= \frac{\sigma^2(1+(n-1)\rho)}{\sigma^2\sqrt{n(1+(n-1)\rho)}} = \sqrt{\frac{1+(n-1)\rho}{n}}\end{aligned} \tag{2.10}$$

Now we can get the formula for the premium:

$$P_i = \mathrm{E}\big[X_i | TL > VaR_\alpha(TL)\big] = \int_{-\infty}^{\infty} x \int_{VaR_\alpha(TL)}^{\infty} \frac{f_{x,y}(x,y)}{P(TL > VaR_\alpha(TL))}\, dy\, dx \tag{2.11}$$

where $f_{x,y}(x,y)$ represents the joint density of X_i and $Y = TL = \sum_{i=1}^n X_i \sim \mathcal{N}(n\mu, \sigma_{TL}^2)$. Furthermore, $P(TL > VaR_\alpha(TL)) = 1 - \alpha$ by the definition of Value at Risk. After exchanging the integrals and using the formula for the joint density with $\rho = \rho_{X_i,TL} \overset{(2.10)}{=}$

$\sqrt{\frac{1+(n-1)\rho}{n}}$ we get:

$$\frac{1}{1-\alpha}\int_{VaR_\alpha(TL)}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_{TL}}\exp\Big(-\frac{1}{2(1-\rho^2)}\big(\frac{y-\mu_{TL}}{\sigma_{TL}}\big)^2\Big)\int_{-\infty}^{\infty}\frac{x}{\sigma_x\sqrt{2\pi}\sqrt{1-\rho^2}} \\ \times\exp\Bigg(-\frac{1}{2(1-\rho^2)}\Big(\big(\frac{x-\mu_x}{\sigma_x}\big)^2-\frac{2\rho(x-\mu_x)(y-\mu_{TL})}{\sigma_x\sigma_{TL}}\Big)\Bigg)\,dx\,dy$$

Now we substitute $\tilde{x}=\frac{x-\mu_x}{\sigma_x}$ and $\tilde{y}=\frac{y-\mu_{TL}}{\sigma_{TL}}$ with $d\tilde{x}=\frac{1}{\sigma_x}\,dx$ and $d\tilde{y}=\frac{1}{\sigma_{TL}}\,dy$.

$$\frac{1}{1-\alpha}\int_{z_\alpha}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2(1-\rho^2)}\tilde{y}^2\Big)\int_{-\infty}^{\infty}\frac{\tilde{x}\sigma_x+\mu_x}{\sqrt{2\pi}\sqrt{1-\rho^2}} \\ \exp\Bigg(-\frac{1}{2(1-\rho^2)}\Big(\tilde{x}^2-2\rho\tilde{x}\tilde{y}\Big)\Bigg)\,d\tilde{x}\,d\tilde{y}$$

After completing the square and splitting the second integral into two integrals we get:

$$\frac{1}{1-\alpha}\int_{z_\alpha}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2(1-\rho^2)}\big(\tilde{y}^2-(\rho\tilde{y})^2\big)\Big)\cdot\Bigg(\sigma_x\,\mathrm{E}[K]+\mu_x\Bigg)\,d\tilde{y}$$

in which $K\sim\mathcal{N}(\rho\tilde{y},\,1-\rho^2)$. If we consider $Z\sim\mathcal{N}(0,\,1)$ and that the integral is additive we get:

$$\frac{1}{1-\alpha}\Big[\sigma_x\rho\int_{z_\alpha}^{\infty}\frac{\tilde{y}}{\sqrt{2\pi}}\exp\Big(-\frac{\tilde{y}^2(1-\rho^2)}{2(1-\rho^2)}\Big)\,d\tilde{y}+\mu_x\int_{z_\alpha}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{\tilde{y}^2(1-\rho^2)}{2(1-\rho^2)}\Big)\,d\tilde{y}\Big] \\ =\frac{1}{1-\alpha}\Big[\sigma_x\rho\phi(z_\alpha)+\mu_x\,P(Z>z_\alpha)\Big]=\frac{\sigma_x\rho\phi(z_\alpha)}{1-\alpha}+\mu_x\overset{(2.10)}{=}\mu_x+\sqrt{\frac{1+(n-1)\rho}{n}}\,\frac{\sigma_x\phi(z_\alpha)}{1-\alpha}$$

$\square$

We can see that (2.8) and (2.9) are the same. This is the case, because all risks are identical. With these explicit formulas for each premium principle, we can analyze how the individual premium changes when we have a higher dependence between the risks.

2.1.1 The number of contracts

One very important concept of an insurance company is that the company can accumulate a lot of contracts from the same type, such as life insurance contracts. This means all the contracts have the same distribution, expectation μ and variance σ^2. Each contract corresponds to a risk X_i in the portfolio of the insurance company. The insurance company has an advantage if their portfolio is large enough so that the average risk $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges in distribution to a normal distribution with expectation μ and variance $\frac{\sigma^2}{n}$. This holds after the central limit theorem but only if all the risks are independent. The consequence is that the portfolio is less volatile if the number of contracts n is increasing. This means that the insurance company can charge a smaller safety loading for each contract.

In general, it is not possible to assume that all the risks are independent. For example, if an insurance company insures homes in an earthquake region, clearly the independence assumption is violated. Because of this, two things need to be answered: If the insurance company cannot charge a smaller safety loading if the number of contracts is increasing and the importantance of the strength of the dependence between the risks for the individual premium.

First, we look at the derived formulas for the three proposed premium principles (2.7), (2.8) and (2.9). We assume a positive correlation between the risks $\rho \in (0, 1)$. For an insurance company, it is interesting if the premium per contract is decreasing when the total amount of contracts n is increasing. When this occurs, the insurer can offer a lower premium and can be more competitive on the market.

If we look at each premium principle and consider the formulas we derived as a function of n we can see that all three premium principles have the same slope:

$$\sqrt{\frac{1+(n-1)\rho}{n}} = \sqrt{\rho + \frac{1-\rho}{n}}$$

This means that each premium is decreasing when n is increasing and when μ, σ, α, ρ are fixed because $\frac{1-\rho}{n}$ is decreasing in n and $1-\rho>0$.

Since each premium principle is decreasing in n, we can take the limit of these premiums with $n \to \infty$ to find the smallest possible premium:

The smallest possible VaR premium is:

$$\lim_{n\to\infty} P_i \overset{(2.7)}{\Rightarrow} \lim_{n\to\infty} \mu + \sigma z_\alpha \sqrt{\frac{1+(n-1)\rho}{n}} = \lim_{n\to\infty} \mu + \sigma z_\alpha \sqrt{\rho + \frac{1-\rho}{n}} = \mu + \sigma z_\alpha \sqrt{\rho} \quad (2.12)$$

The smallest TVaR premium is:

$$\lim_{n\to\infty} P_i \overset{(2.8)}{\Rightarrow} \lim_{n\to\infty} \mu + \sigma \frac{\phi(z_\alpha)}{1-\alpha} \sqrt{\frac{1+(n-1)\rho}{n}} = \mu + \sigma \frac{\phi(z_\alpha)}{1-\alpha}\sqrt{\rho} \quad (2.13)$$

The smallest value for the conditional premium is the same as the TVaR premium because these premiums are the same if all the risks are identical. Since we can calculate the lowest possible premium for each premium principle we can also calculate the lowest possible safety loading.

The smallest possible safety loading for the VaR premium is:

$$\theta \overset{(2.4)}{=} \frac{P_i}{\mu} - 1 \overset{(2.12)}{=} \frac{\mu + \sigma z_\alpha \sqrt{\rho}}{\mu} - 1 = \frac{\sigma}{\mu} z_\alpha \sqrt{\rho}$$

The smallest safety loading for the TVaR and conditional premium is:

$$\theta \overset{(2.4)}{=} \frac{P_i}{\mu} - 1 \overset{(2.13)}{=} \frac{\mu + \sigma \frac{\phi(z_\alpha)}{1-\alpha}\sqrt{\rho}}{\mu} - 1 = \frac{\sigma}{\mu}\frac{\phi(z_\alpha)}{1-\alpha}\sqrt{\rho}$$

That means that an insurance company can charge a smaller premium if their total amount of contracts is increasing, but there is a smallest possible value for each premium. The same holds for the safety loading, since the safety loading is a linear transformation of the premium.

Let's consider risks $X_i \sim \mathcal{N}(5,\ 10)$, $i = 1, \ldots, n$ with $\rho = 0.5$ and $\alpha = 0.99$. We are interested in the behavior of each premium and the corresponding safety loading when the amount of contracts n is increasing.

For identical risks the conditional premium is equal to the TVaR premium so we do not

n	1	2	5	25	100	∞
VaR premium	12.36	11.37	10.70	10.30	10.23	10.20
θ for VaR premium	1.47	1.27	1.14	1.06	1.05	1.04
TVaR premium	13.43	12.30	11.53	11.08	10.99	10.96
θ for TVaR premium	1.69	1.46	1.31	1.22	1.20	1.19

Table 2.1: The premium and safety loading for different principles depending on n and $\rho = 0.5$

have to calculate it. In Table 2.1 we can see that the VaR premium is less than the other two premium principles. This makes sense because both premiums are based on the expectation that the loss is already greater than the Value at Risk for a given α. Futhermore, we can see that in both cases the premium is decreasing quickly until $n \approx 5$ and after that the decrease in the premium is rather small. After 100 risks we are already close to the smallest premium possible. Moreover, it is possible to lower the safety loading by over 40% for the VaR premium, 50% for the TVaR and 50% for the conditional premium if the insurance company assembles a lot of risks.

So far, we only considered $\rho \in (0, 1)$. Now we consider two special cases: $\rho = 1$ and $\rho = -\frac{1}{n}$. The first case means that all risks are perfectly positively correlated making the VaR premium:

$$P_i \overset{(2.7)}{=} \mu + \sigma z_\alpha \sqrt{\frac{1 + (n-1)1}{n}} = \mu + \sigma z_\alpha$$

For the TVaR premium and conditional premium we get:

$$P_i \overset{(2.8)}{=} \mu + \sigma \frac{\phi(z_\alpha)}{1-\alpha} \sqrt{\frac{1 + (n-1)1}{n}} = \mu + \sigma \frac{\phi(z_\alpha)}{1-\alpha}$$

We can see that if all the risks are perfectly correlated ($\rho = 1$) then the n cancels, which means that an insurance company cannot charge a smaller premium if they accumulate more risks. The consequence is that there is no diversification effect.

For the second case, we consider that all risks are negatively correlated with $\rho = -\frac{1}{n}$ making the VaR premium:

$$P_i \overset{(2.7)}{=} \mu + \sigma z_\alpha \sqrt{\frac{1 + (n-1)(-\frac{1}{n})}{n}} = \mu + \frac{1}{n} \cdot \sigma z_\alpha$$

For the TVaR premium and conditional premium we get:

$$P_i \overset{(2.8)}{=} \mu + \sigma \frac{\phi(z_\alpha)}{1-\alpha} \sqrt{\frac{1 + (n-1)(-\frac{1}{n})}{n}} = \mu + \frac{1}{n} \cdot \frac{\sigma \phi(z_\alpha)}{1-\alpha}$$

For this case, we can see that all three premiums converge for $n \to \infty$ to μ. This means that for any given safety loading θ one can find the amount of risks n such that the proposed premium principle are smaller than the expected premium principle $(1+\theta)\mu$.

For the VaR premium it has to hold that:

$$\begin{aligned}(1+\theta)\mu &\geq \mu + \sigma z_\alpha \frac{1}{n} \\ \Rightarrow n &\geq \frac{\sigma z_\alpha}{\theta \mu}\end{aligned}$$

For the TVaR premium and conditional premium it has to hold that:

$$\begin{aligned}(1+\theta)\mu &\geq \mu + \sigma \frac{\phi(z_\alpha)}{(1-\alpha)n} \\ \Rightarrow n &\geq \frac{\sigma \phi(z_\alpha)}{(1-\alpha)\theta\mu}\end{aligned}$$

Let's consider risks $X_i \sim \mathcal{N}(5, 10)$, $i = 1, \ldots, n$ and $\alpha = 0.99$. Also, all risks have a correlation of $\rho = -\frac{1}{n}$. We are interested in how the premium and the safety loading for each risk is changing when the total amount of risks n is increasing.

Table 2.2 shows only the VaR premium and TVaR premium because the conditional pre-

n	1	2	5	25	100	∞
VaR premium	12.36	8.68	6.47	5.29	5.07	5
θ for VaR premium	1.47	0.74	0.29	0.06	0.01	0
TVaR premium	13.43	9.21	6.69	5.34	5.08	5
θ for TVaR premium	1.69	0.84	0.34	0.07	0.02	0

Table 2.2: The premium and safety loading for different principles depending on n and $\rho = -\frac{1}{n}$

mium is equal to the TVaR premium when all risks are identical. Also, we can see a similiar pattern to the one in Table 2.1, where the premiums are decreasing quickly and after a certain size of n the premiums are decreasing slowly. The difference between Table 2.1 and Table 2.2 is that all three premium principles are converging to the same limit instead of different ones. The VaR premium is always smaller than the premium for the other two principles, but after $n \geq 100$ all three premiums are roughly the same. This means if we have the case where all risks are negatively correlated with $\rho = -\frac{1}{n}$ and we pool a large number of risks, then the premium is roughly the same for every premium principle. Furthermore, θ is decreasing quickly from 147% to 6% for the VaR premium principle if the portfolio size is increased from 1 to 25 and from 169% to 7% for the TVaR premium principle and the conditional premium principle.

Let's consider the same setup as in the previous example, $X_i \sim \mathcal{N}(5, 10)$ for all $i = 1, \ldots, n$, $\alpha = 0.99$ and $\rho = -\frac{1}{n}$. We want to know how many risks we have to pool until we can charge a safety loading of less than 20%.

For the VaR premium principle we need $n \geq \frac{\sqrt{10}\, z_\alpha}{0.2*5} \approx 7.35 \Rightarrow n = 8$. This means we need at least 8 risks to charge a smaller safety loading than 20% per risk.

For the TVaR and conditional premium principle we need $n \geq \frac{\sqrt{10}\, \phi(z_\alpha)}{(1-0.99)*0.2*5} \approx 8.43 \Rightarrow n = 9$. This means we need at least 9 risks to charge a smaller safety loading than 20% per risk.

In conclusion, we found that the concept of pooling risks is still functioning if the risks are correlated with $\rho \in (0, 1)$ or $\rho = -\frac{1}{n}$ and identical and normally distributed. Additionally, it holds that if the risks are correlated with $\rho = -\frac{1}{n}$ all three proposed premium principles converge for $n \to \infty$ against μ, the expectation of one risk. This means that by pooling a lot of risks the premium can be reduced until it is only the expectation. The only problem is if all risks are perfectly positively correlated, when $\rho = 1$, the premium is not decreasing in n and the concept of pooling risks is not working.

2.1.2 The correlation coefficient

So far we assumed that the risks are positively correlated and derived formulas for the case that all risks are identically distributed. We investigated the impact that the amount of contracts has on the premium. But, it is also interesting how much the premium changes if the correlation between the risks is changing. In this section we assume that the correlation between all risks are the same, which means that $Cov[X_i, X_j] = \rho\sigma^2$ for all $i \neq j$. Furthermore, we assume that again all risks are identical $X_i \sim \mathcal{N}(\mu, \sigma^2)$ for all $i = 1, \ldots, n$. This means we can use (2.7), (2.8) and (2.9) for calculating the premium for each principle. The premium for the conditional principle and the TVaR principle are the same because all risks are identical.

Let's consider risks $X_i \sim \mathcal{N}(3, 5)$ for $i = \{1, \ldots, 25\}$ and $\alpha = 0.99$. In Figure 2.1 we can see the premium for different values of ρ and different premium principles. The solid lines describes the premium calculated according to the TVaR and conditional premium principle. The dashed line corresponds to the VaR premium. We can see that the VaR premium is always lower than the other two premium principles. Furthermore, we can see that the premiums are increasing if the dependence between the risks are increasing. This is reasonable because the variance of the sum of all risks is increasing when the correlation is increasing (see. (2.3)), causing the premium to also increase. Another more economical reason is, since the variance is higher, the uncertainty for the insurance company is increasing. The consequence is that they want to have a higher premium for taking over the risks. Additionally, we can see that both curves are concave. This makes sense because if we interpret each premium principle as a function dependent on ρ then they all have the same slope, namely

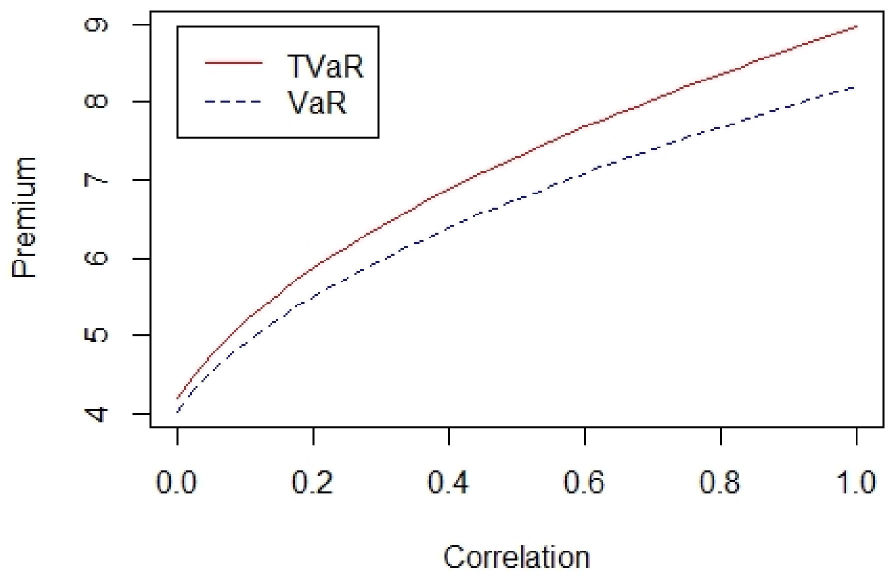

Figure 2.1: The premium for different principle depending on ρ

$\sqrt{\rho}$. Since the squareroot is a concave function the premium is also concave.

Another interesting aspect is how the correlation between the risks changes the safety loading of each premium principle. The formula for the safety loading is given in (2.4).

Let's consider risks $X_i \sim \mathcal{N}(3, 5)$ for $i = 1, \ldots, 25$ and $\alpha = 0.99$ to calculate the safety loading for an increasing correlation parameter ρ.

In Figure 2.2 we can see that the curves have the same shape as in Figure 2.1. But in this figure, the y-axis represents the safety loading. This means that for a correlation close to 0 the safety loading for the VaR premium is 40% and for the TVaR and conditional premium it is 45%. On the other hand, the safety loading of the VaR premium increases up to 160% and 200% for the TVaR and conditional premium if the correlation is reaching 1. This means that the dependence between the risks has a huge impact on the premium. It can increase the safety loading by up to 120% for the VaR premium and 155% for the TVaR and conditional premium in the case that all $X_i \sim \mathcal{N}(3, 5)$.

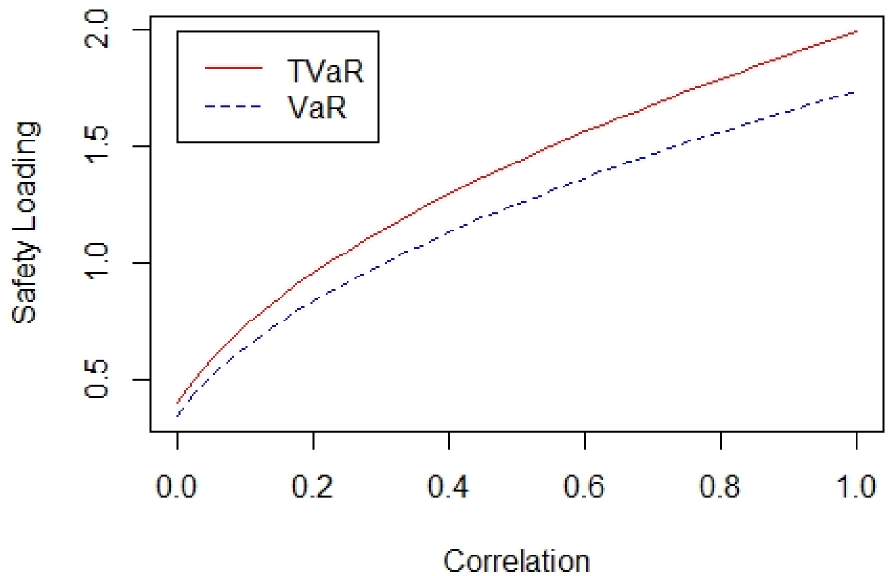

Figure 2.2: The safety loading for different principle depending on ρ

2.2 Impact of expectation and variance on the premium

So far we studied the impact of the number of risks on the individual premium and the influence of the correlation on the individual premium. We also assumed that all the risks are identical. In practice, that does not always make sense because the risks are usually not identical. In this section we will analyze the influence of the expectation and variance on the premium.

Since, in previous sections, we derived formulas for premium principles when all the risks were identical, we have to derive new formulas for each premium principle and for the total loss $TL = \sum_{i=1}^{n} X_i$ when the risks are not identical.

Proposition 2.7 *Let $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for all $i = 1, \ldots, n$ and all risks have the same correlation $Cov[X_i, X_j] = \rho\sigma_i\sigma_j$ for all $i \neq j$. Then the total loss $TL = \sum_{i=1}^{n} X_i \sim \mathcal{N}(\mu_{TL}, \sigma_{TL}^2)$ with $\mu_{TL} = \sum_{i=1}^{n} \mu_i$ and $\sigma_{TL}^2 = \sum_{i=1}^{n} \sigma_i^2 + 2\sum_{i<j}^{n} \sigma_i\sigma_j\rho$*

Proof. Since the sum of normal random variables is normally distributed and the expectation is additive we get $\mathrm{E}\big[TL\big] = \mathrm{E}\big[\sum_{i=1}^{n} X_i\big] = \sum_{i=1}^{n} \mu_i$. For the variance it holds that:

$$
\begin{aligned}
Var(TL) &= Cov\Big[\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\Big] = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov[X_i, X_j] = \sum_{i=1}^{n} \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=0, i\neq j}^{n} \sigma_i \sigma_j \rho \\
&= \sum_{i=1}^{n} \sigma_i^2 + 2\sum_{i<j}^{n} \sigma_i \sigma_j \rho
\end{aligned}
$$

□

Proposition 2.8 *Let $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for all $i = 1, \ldots, n$ and all risks have the same correlation $Cov[X_i, X_j] = \rho\sigma_i\sigma_j$ for all $i \neq j$. Then the VaR premium is:*

$$
P_i = \frac{\mu_i + \sigma_i z_\alpha}{\sum_{i=1}^{n} \big(\mu_i + \sigma_i z_\alpha\big)} \times \left(\sum_{i=1}^{n} \mu_i + \sigma_{TL} z_\alpha\right) \tag{2.14}
$$

Proof. First we plug (2.1) into the definition of the VaR premium (1.3). Then, we can use (2.1) and that the total loss is normally distributed as described in Proposition 2.7 to get:

$$
P_i = \frac{VaR_\alpha(X_i)}{\sum_{i=1}^{n} VaR_\alpha(X_i)} \times VaR_\alpha\left(\sum_{i=1}^{n} X_i\right) = \frac{\mu_i + \sigma_i z_\alpha}{\sum_{i=1}^{n} \big(\mu_i + \sigma_i z_\alpha\big)} \times \left(\sum_{i=1}^{n} \mu_i + \sigma_{TL} z_\alpha\right)
$$

□

Proposition 2.9 *Let $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for all $i = 1, \ldots, n$ and all risks have the same correlation $Cov[X_i, X_j] = \rho\sigma_i\sigma_j$ for all $i \neq j$. Then the TVaR premium is:*

$$
P_i = \frac{\mu_i + \sigma_i \frac{\phi(z_\alpha)}{1-\alpha}}{\sum_{i=1}^{n} \big(\mu_i + \sigma_i \frac{\phi(z_\alpha)}{1-\alpha}\big)} \times \left(\sum_{i=1}^{n} \mu_i + \sigma_{TL} \frac{\phi(z_\alpha)}{1-\alpha}\right) \tag{2.15}
$$

Proof. First we plug the representation for the Tail Value at Risk (2.2) into the definition of the TVaR premium (1.4). Then we can use (1.4) and that the total loss is normally distributed as described in Proposition 2.7 to get:

$$P_i = \frac{TVaR_\alpha(X_i)}{\sum_{i=1}^n TVaR_\alpha(X_i)} \times TVaR_\alpha\left(\sum_{i=1}^n X_i\right) = \frac{\mu_i + \sigma_i \frac{\phi(z_\alpha)}{1-\alpha}}{\sum_{i=1}^n \left(\mu_i + \sigma_i \frac{\phi(z_\alpha)}{1-\alpha}\right)} \times \left(\sum_{i=1}^n \mu_i + \sigma_{TL} \frac{\phi(z_\alpha)}{1-\alpha}\right)$$

□

Proposition 2.10 *Let $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for all $i = 1, \ldots, n$ and all risks have the same correlation $Cov[X_i, X_j] = \rho\sigma_i\sigma_j$ for all $i \neq j$. Then the conditional premium is:*

$$P_i = \mu_i + \sigma_i \frac{\rho_{X_i,TL}\phi(z_\alpha)}{1-\alpha} \tag{2.16}$$

where $\rho_{X_i,TL}$ is the correlation between the risk X_i and the total loss TL and it is equal to:

$$\rho_{X_i,TL} = \frac{\sigma_i + \sum_{j=0, i\neq j}^n \sigma_j \rho}{\sigma_{TL}}$$

Proof. After following the same steps as in Proposition 2.6 and since the total loss TL is normally distributed only with different parameters we get:

$$P_i = \mathrm{E}\big[X_i | TL > VaR_\alpha(TL)\big] = \frac{\sigma_i \rho_{X_i,TL}\phi(z_\alpha)}{1-\alpha} + \mu_i$$

where

$$\rho_{X_i,TL} = \frac{Cov\big[X_i, \sum_{i=n}^n X_i\big]}{\sigma_i\, \sigma_{TL}} = \frac{\sigma_i^2 + \sum_{j=0, i\neq j}^n \sigma_i \sigma_j \rho}{\sigma_i\, \sigma_{TL}} = \frac{\sigma_i + \sum_{j=0, i\neq j}^n \sigma_j \rho}{\sigma_{TL}}$$

This proofs the second equation. □

2.2.1 Impact of μ on the premium

In this section we analyze the impact the expectation μ has on the premium. For this we consider two different cases. The first case is that we investigate the impact μ has on its own premium. The other case is the impact μ has on the premium for the other risks.

Let's consider two risks $X_1 \sim \mathcal{N}(\mu_1, 1)$, $X_2 \sim \mathcal{N}(1, 1)$, $\alpha = 0.99$ and $\rho = 0.9$. Also, we assume that μ_1 is increasing from 1 to 10. We are interested in how the premium will change for risk X_1 and for risk X_2.

First, we consider the premium for the risk X_1. The premium for risk X_1 and for each principle is in Table 2.3. We can see that the VaR premium is less than the other two premiums, which makes sense because the other premium principles are more conservative than the VaR premium principle. Furthermore, the VaR premium is increasing with a slope of less than 1. This means the VaR premium does not pass the complete increase in the risk to the premium of the responsible risk. We can see the same thing for the TVaR premium. The only premium which assigns the complete increased risk to the responsible risk is the conditional premium. This is also shown in the formula used to calculate the premium (2.16). The conditional premium takes into account only the expectation of the risk, which should be priced.

μ_1	1	2	3	4	5	6	7	8	9	10
VaR premium	3.27	4.26	5.25	6.25	7.25	8.24	9.24	10.24	11.24	12.23
TVaR premium	3.60	4.59	5.58	6.58	7.57	8.57	9.57	10.56	11.56	12.56
Cond. premium	3.60	4.60	5.60	6.60	7.60	8.60	9.60	10.60	11.60	12.60

Table 2.3: The premium for each principle for a risk with an increasing expectation μ

In Figure 2.3 we can see the VaR premium for X_2. Even though the expectation and variance of the risk is not changing the premium is. It increases from ≈ 3.27 to 3.30 while the expectation of X_1 increases from 1 to 10. This means that the VaR premium principle passes some of the increased risk of X_1 on to the unchanged risk X_2.

In Figure 2.4 we can see the TVaR and the conditional premium for X_2. The TVaR premium is the solid line and the dashed line represents the conditional premium. We can see that the conditional premium is a constant, which makes sense because the formula for the conditional premium (2.16) does not take into account the expectation of the other risks in the portfolio. On the other hand, the TVaR premium for X_2 is increasing from approxi-

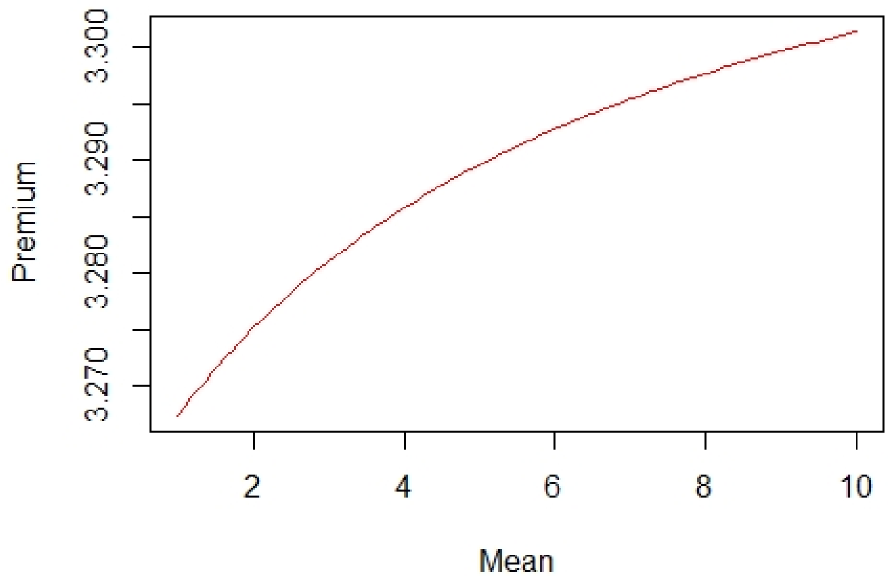

Figure 2.3: The VaR premium for X_2 while X_1 has an increasing expectation μ

mately 3.60 to 3.63. This means that the TVaR principle passes some of the additional risk on to the unchanged risk.

In conclusion, when you have a case where one risk is fixed and the expectation of the other risk is increasing, the conditional premium principle assigns the complete additional risk to the corresponding premium. The additional risk is caused by the increased expectation of one risk. The VaR and TVaR premium principles assign a small amount of the additional risk to the premium of the fixed risk. Therefore, they are not completely fair. The conditional premium, on the other hand, is fair.

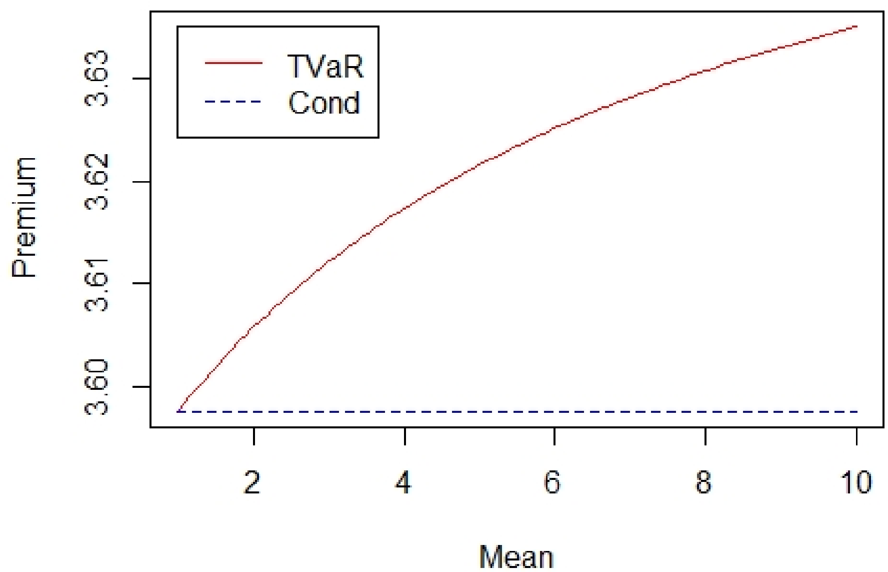

Figure 2.4: The TVaR and the conditional premium for X_2 while X_1 has an increasing expectation μ

2.2.2 Impact of σ on the premium

In the last section we analyzed the impact of μ on the premium. In this section we will analyze the impact the standard deviation σ has on the premium. We consider the same two cases as in previous section. The first case analyzes the impact σ has on its own premium and the second case analyzes the impact σ has on the premium for the other risks.

Let's consider two risks $X_1 \sim \mathcal{N}(1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(1, 1)$, $\alpha = 0.99$ and $\rho = 0.9$. Also, we assume that the standard deviation of X_1, σ_1 is increasing from 1 to 10. We are interested in how the premium will change for X_1 and X_2.

Figure 2.4 shows how the premium of risk X_1 is changing when the standard deviation of risk X_1 is increasing from 1 to 10. For the VaR principle the premium is increasing from 3.27 to 24.08. That is reasonable because the uncertainty of the risk is increasing which means you have to pay a higher premium to transfer the risk to an insurance company. The Value

σ_1	1	2	3	4	5	6	7	8	9	10
VaR premium	3.27	5.55	7.85	10.16	12.48	14.79	17.11	19.43	21.75	24.08
TVaR premium	3.60	6.22	8.85	11.50	14.15	16.80	19.46	22.12	24.78	27.44
Cond. premium	3.60	6.27	8.95	11.62	14.29	16.96	19.63	22.30	24.96	27.63

Table 2.4: The premium for each principle for a risk with an increasing standard deviation σ_1

at Risk for $\alpha = 0.99$, $\sigma_1 = 1$ and $\sigma_1 = 10$ is $1 + z_\alpha = 3.33$ and $1 + 10z_\alpha = 24.26$, respectively. When $\sigma_1 = 1$ the difference between the $VaR_{0.99}(X_1)$ and the VaR premium is 0.06. When $\sigma_1 = 10$ the difference between the $VaR_{0.99}(X_1)$ and the VaR premium is 0.18. This means that the insurance company demands a comparatively smaller premium if the standard deviation is increasing. For the TVaR premium principle we can see the same behavior as for the VaR premium principle. When $\alpha = 0.99$ and $\sigma_1 = 1$, the $TVaR_{0.99}(X_1) = 3.67$, resulting in a difference of 0.07 between the $TVaR_{0.99}(X_1)$ and the TVaR premium. When $\alpha = 0.99$ and $\sigma_1 = 10$, the $TVaR_{0.99}(X_1) = 27.65$, resulting in a difference of 0.21 between the $TVaR_{0.99}(X_1)$ and the TVaR premium. This leads to the same conclusion as for the VaR premium principle. On the other hand, the conditional premium is increasing faster than the corresponding $TVaR_{0.99}(X_1)$. This means that one has to pay an additional amount to compensate the insurance company for overtaking the risk.

We can see in Figure 2.5 the VaR premium for X_2. Even though the expectation and variance of the risk are fixed, the premium is changing. At first, the premium is decreasing until $\sigma_1 \approx 1.5$ and is increasing after that. This means that at first X_2 benefits from the increasing standard deviation of the other risk. But, if the change in σ_1 is too big, the premium of X_2 is increasing.

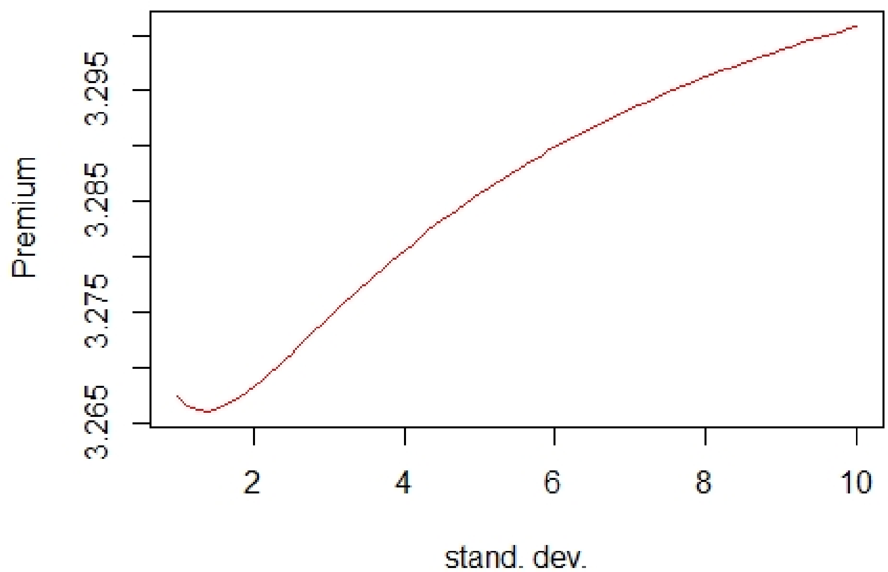

Figure 2.5: The VaR premium for X_2 while the standard deviation of X_1 is increasing

In Figure 2.6 we can see the TVaR and the conditional premium for X_2. The dashed line represents the TVaR premium and the solid line represents the conditional premium. We can see that the conditional premium is decreasing from 3.60 to 3.45. This means that the principle would charge a risk less when there is a risk with the same expectation but bigger standard deviation in the portfolio of the insurance company. On the other hand, the TVaR premium for X_2 shows the same behavior as the VaR premium in Figure 2.5. The TVaR premium for the fixed risk is increasing from 3.60 to 3.64 but at first the premium decreases to 3.59 at $\sigma_1 \approx 1.5$.

In conclusion, we can say that a risk with a much higher standard deviation than the other risk benefits from the VaR or TVaR premium principle because the premium is relatively smaller when both risks are equal. On the other hand, the risk with the smaller standard deviation benefits from the conditional premium principle because the bigger the

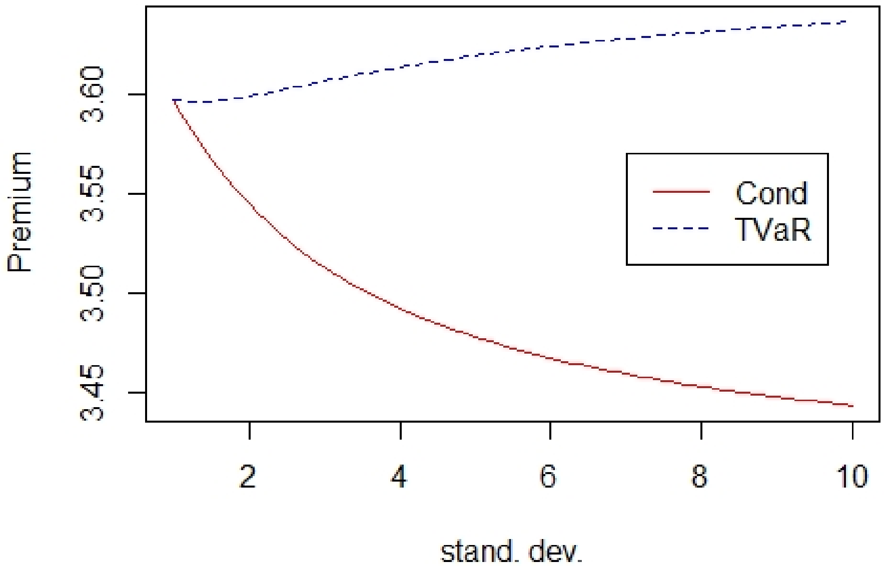

Figure 2.6: The TVaR premium and conditional premium for X_2 while X_1 has an increasing standard deviation

difference between the standard deviation of the risks the smaller the premium is for this risk. Therefore, we see that each premium principle is better for a different type of risk in an insurance portfolio.

3 Bivariate Pareto model

In this chapter we analyze the properties of the three proposed premium principles if the risks are pareto distributed. The pareto distribution is frequently used in insurance to describe the risk which corresponds with an insurance contract. For simplicity, we only consider the bivariate case. But before we can calculate the proposed premium principles, we need the following results.

Definition 1 *A random variable X is pareto distributed $Par(a, \theta)$ of type 1 with parameter $a > 0$ and $\theta > 0$ if the pdf is equal to:*

$$f(x) = \begin{cases} \dfrac{a\theta^a}{x^{a+1}} & x \geq \theta \\ 0 & else \end{cases}$$

Definition 2 *Let $X \sim Par(a, \theta_1)$ and $Y \sim Par(a, \theta_2)$ of type 1 with $a > 2$. Then the joint density function is equal to:*

$$f(x, y) = a(a+1)(\theta_1\theta_2)^{a+1}(\theta_2 x + \theta_1 y - \theta_1\theta_2)^{-(a+2)},\ x \geq \theta_1, y \geq \theta_2, a > 0$$

The joint density function of two pareto distributed random variable is taken from Mardia (1962) [2].

Proposition 3.1 *Let $X \sim Par(a, \theta_1)$ and $Y \sim Par(a, \theta_2)$ be bivariate pareto distributed with $a > 2$ and $\theta > 0$. Then the correlation of X and Y is:*

$$corr(X, Y) = \frac{1}{a}$$

Proof. The proof can be found in Mardia (1962) [2]. □

Proposition 3.2 *Let $X \sim Par(a, \theta)$ with $a > 1$ and $\theta > 0$. Then the Value at Risk of X is:*

$$VaR_{\alpha}(X) = \theta(1-\alpha)^{-\frac{1}{a}} \tag{3.1}$$

Proof. Since X is pareto distributed the cdf is: $F_X(x) = 1 - (\frac{x}{\theta})^{-a}$ for $x \geq \theta$. By applying the definition of the Value at Risk we get:

$$VaR_{\alpha}(X) \Rightarrow P(X \leq x) = \alpha \Rightarrow 1 - \left(\frac{x}{\theta}\right)^{-a} = \alpha \Rightarrow x = \theta\big(1-\alpha\big)^{-\frac{1}{a}}$$

□

Proposition 3.3 *Let $X \sim Par(a, \theta)$ with $a > 1$ and $\theta > 0$. Then the Tail Value at Risk of X is:*

$$TVaR_{\alpha}(X) = \frac{a\theta}{(1-\alpha)^{1/a}(a-1)} \tag{3.2}$$

Proof. With the defintion of TVaR, the Value at Risk of X given in (3.1) and the density of a pareto distributed random variable we get:

$$\begin{aligned}
TVaR_{\alpha}(X) &= \frac{1}{1-\alpha}\int_{TVaR_{\alpha}(X)}^{\infty} xa\theta^a x^{-(a+1)}\,dx = \frac{a\theta^a}{1-\alpha}\int_{TVaR_{\alpha}(X)}^{\infty} x^{-a}\,dx \\
&= \frac{a\theta^a}{1-\alpha}\left[\frac{x^{-(a-1)}}{-(a-1)}\right]_{x=\theta(1-\alpha)^{-\frac{1}{a}}}^{x=\infty} = \frac{a\theta^a}{(1-\alpha)(a-1)}\Big(\theta(1-\alpha)^{-\frac{1}{a}}\Big)^{-(a-1)} \\
&= \frac{a\theta}{(1-\alpha)^{1/a}(a-1)}
\end{aligned}$$

□

3.1 Premium calculation for identical Pareto risks

In this section we assume that both risks are identical pareto distributed which means that $X \sim Par(a, \theta)$ and $Y \sim Par(a, \theta)$.

Proposition 3.4 *Let $X \sim Par(a, \theta)$ and $Y \sim Par(a, \theta)$ with $a > 2$ and $\theta > 0$. Then the pdf of $X + Y$ is:*

$$f_{X+Y}(v) = a(a+1)\theta^a (v-\theta)^{-(a+2)}(v-2\theta)\mathbb{1}_{\{v \geq 2\theta\}} \tag{3.3}$$

Proof. Since $X \sim Par(a, \theta)$ and $Y \sim Par(a, \theta)$ with $a > 2$ and $\theta > 0$, the joint density function is given in Definition 2. We use the following transformation on the joint distribution:

$$W = X,\, V = X + Y \Rightarrow X = W,\, Y = V - W \text{ and } |J| = 1 \tag{3.4}$$

where $|J|$ is the determinant of the Jacobian matrix. Now, we can calculate the joint density of $W = X$ and $V = X + Y$:

$$\begin{aligned} f_{W,V}(w, v) = f_{X,Y}(w, v-w) \cdot |J| &= a(a+1)(\theta^2)^{a+1}(\theta w + \theta(v-w) - \theta^2)^{-(a+2)}\mathbb{1}_{\{v-w \geq \theta, w \geq \theta\}} \\ &= a(a+1)\theta^{2a+2}(\theta v - \theta^2)^{-(a+2)}\mathbb{1}_{\{v-w \geq \theta, w \geq \theta\}} \\ &= a(a+1)\theta^{2a+2}\theta^{-(a+2)}(v-\theta)^{-(a+2)}\mathbb{1}_{\{v-w \geq \theta, w \geq \theta\}} \\ &= a(a+1)\theta^a(v-\theta)^{-(a+2)}\mathbb{1}_{\{v-w \geq \theta, w \geq \theta\}} \end{aligned} \tag{3.5}$$

Then the marginal density of $X + Y$ is:

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty} f_{W,V}(w, v)\, dw = \int_{-\infty}^{\infty} a(a+1)\theta^a(v-\theta)^{-(a+2)}\mathbb{1}_{\{v-\theta \geq w, w \geq \theta\}}\, dw \\ &= \int_{\theta}^{v-\theta} a(a+1)\theta^a(v-\theta)^{-(a+2)}\, dw \\ &= a(a+1)\theta^a(v-\theta)^{-(a+2)}(v-2\theta)\mathbb{1}_{\{v \geq 2\theta\}} \end{aligned}$$

We need the indicator function in the last line because a pdf has to be greater or equal than zero. Since $V = X + Y$, we completed the proof. □

Proposition 3.5 *Let $X \sim Par(a, \theta)$ and $Y \sim Par(a, \theta)$ with $a > 2$ and $\theta > 0$. Then the* *cdf of $X+Y$ is:*

$$F_{X+Y}(z) = 1 + a\theta^{a+1}(z-\theta)^{-(a+1)} - (a+1)\theta^a(z-\theta)^{-a}\mathbb{1}_{\{z\geq 2\theta\}}$$

Proof. In Proposition 3.4 we calculated the pdf of $X+Y$. It holds that $F_X(z) = \int_{-\infty}^{z} f_X(x)\,dx$. Furthermore, we assume that $z \geq 2\theta$, resulting in:

$$\begin{aligned}
F_{X+Y}(z) = \int_{-\infty}^{z} f_{X+Y}(v)\,dv &= \int_{-\infty}^{z} a(a+1)\theta^a(v-\theta)^{-(a+2)}(v-2\theta)\mathbb{1}_{\{v\geq 2\theta\}}\,dv \\
&= a(a+1)\theta^a\Big[\int_{2\theta}^{z}(v-\theta)^{-(a+1)} - \theta(v-\theta)^{-(a+2)}\,dv\Big] \\
&= a(a+1)\theta^a\Big[\frac{1}{-a}(v-\theta)^{-a} - \frac{\theta}{-(a+1)}(v-\theta)^{-(a+1)}\Big]_{v=2\theta}^{v=z} \\
&= -(a+1)\theta^a(z-\theta)^{-a} + a\theta^{a+1}(z-\theta)^{-(a+1)} + (a+1)\theta^a\theta^{-a} - a\theta^{a+1}\theta^{-(a+1)} \\
&= 1 + a\theta^{a+1}(z-\theta)^{-(a+1)} - (a+1)\theta^a(z-\theta)^{-a}
\end{aligned}$$

Since, $F_{X+Y}(z)$ is continous for all $z > 2\theta$, we have to show that $F_{X+Y}(2\theta) = 0$ and that $\lim_{z\to\infty} F_{X+Y}(z) = 1$. If our function satisfies these conditions then it is indeed a distribution function. First we check that $F_{X+Y}(2\theta) = 0$.

$$\begin{aligned}
F_{X+Y}(2\theta) &= 1 + a\theta^{a+1}(2\theta-\theta)^{-(a+1)} - (a+1)\theta^a(2\theta-\theta)^{-a} \\
&= 1 + a\theta^{a+1}\theta^{-(a+1)} - (a+1)\theta^a\theta^{-a} = 1 + a - (a+1) = 0
\end{aligned}$$

For the second condition it holds that:

$$\lim_{z\to\infty} F_{X+Y}(z) = 1 + a\theta^{a+1}(z-\theta)^{-(a+1)} - (a+1)\theta^a(z-\theta)^{-a} \underset{z\to\infty}{\to} 1 + 0 - 0$$

This holds because $(z-\theta)^{-(a+1)}$ and $(z-\theta)^{-a}$ converge to 0 for $z\to\infty$. All in all we have shown that the cdf of $X+Y$ is $F_{X+Y}(z)$. □

The joint density of $X+Y$ and X is given in (3.5) when both risks are identical, pareto distributed with parameters $a > 2$ and $\theta > 0$. Also, we calculated the pdf and cdf of $X+Y$. With these results we can calculate the $VaR_\alpha(X+Y)$, $TVaR_\alpha(X+Y)$ and $E\big[X|X+Y > VaR_\alpha(X+Y)\big]$. In the following results we state the formulas for each one.

Proposition 3.6 *Let $X \sim Par(a,\theta)$ and $Y \sim Par(a,\theta)$ with $a > 2$ and $\theta > 0$. Then the Value at Risk of $X+Y$ is the value $q \in \mathbb{R}$, which solves the following equation:*

$$\alpha = 1 + a\theta^{a+1}(q-\theta)^{-(a+1)} - (a+1)\theta^a(q-\theta)^{-a} \tag{3.6}$$

Proof. In Proposition 3.5 we calculated the cdf of $X+Y$. After the definition of the Value at Risk it holds that:

$$VaR_\alpha(X+Y) = q \Leftrightarrow F_{X+Y}(q) = \alpha$$
$$\Rightarrow 1 + a\theta^{a+1}(q-\theta)^{-(a+1)} - (a+1)\theta^a(q-\theta)^{-a} = \alpha$$

Since $X+Y$ is a continuous random variable and $F_{X+Y}(z)$ is strictly increasing, the solution of the above equation is unique. $\square$

Proposition 3.7 *Let $X \sim Par(a,\theta)$ and $Y \sim Par(a,\theta)$ with $a > 2$ and $\theta > 0$. Then the Tail Value at Risk is:*

$$TVaR_\alpha(X+Y) = \frac{a(a+1)\theta^a}{1-\alpha}\Big[\frac{1}{a-1}(q-\theta)^{-(a-1)} - \frac{\theta^2}{a+1}(q-\theta)^{-(a+1)}\Big]$$

in which q is given in Proposition 3.6.

Proof. With the Definition of $TVaR_\alpha(X)$ and q as described in Proposition 3.6 we get:

$$\begin{aligned}
TVaR_\alpha(X+Y) &= \frac{1}{1-\alpha}\int_q^\infty v f_{X+Y}(v)\,dv \\
&= \frac{1}{1-\alpha}\int_q^\infty va(a+1)\theta^a(v-\theta)^{-(a+2)}(v-2\theta)\mathbb{1}_{\{v\geq 2\theta\}}\,dv \\
&= \frac{a(a+1)\theta^a}{1-\alpha}\int_q^\infty (v-\theta)^{-(a+2)}\big((v-\theta)^2-\theta^2\big)\,dv \qquad (3.7)\\
&= \frac{a(a+1)\theta^a}{1-\alpha}\left(\int_q^\infty (v-\theta)^{-a}-(v-\theta)^{-(a+2)}\theta^2\,dv\right) \\
&= \frac{a(a+1)\theta^a}{1-\alpha}\left[\frac{1}{-(a-1)}(v-\theta)^{-(a-1)}-\frac{\theta^2}{-(a+1)}(v-\theta)^{-(a+1)}\right]_{v=q}^{v=\infty} \\
&= \frac{a(a+1)\theta^a}{1-\alpha}\left[\frac{1}{a-1}(q-\theta)^{-(a-1)}-\frac{\theta^2}{a+1}(q-\theta)^{-(a+1)}\right]
\end{aligned}$$

We can drop the indicator function because $q > 2\theta$. This follows from the definition of q. □

Proposition 3.8 *Let $X \sim Par(a,\theta)$ and $Y \sim Par(a,\theta)$ with $a > 2$ and $\theta > 0$. Then the conditional premium given in (1.5) is equal to:*

$$\begin{aligned}
&\mathrm{E}\big[X|X+Y > VaR_\alpha(X+Y)\big] = \mathrm{E}\big[Y|X+Y > VaR_\alpha(X+Y)\big] \\
&= \frac{a(a+1)\theta^a}{2(1-\alpha)}\left(\frac{1}{a-1}(q-\theta)^{-(a-1)}-\frac{\theta^2}{a+1}(q-\theta)^{-(a+1)}\right)
\end{aligned}$$

Proof. Since X and Y are identical it follows directly that $\mathrm{E}\big[X|X+Y > VaR_\alpha(X+Y)\big] = \mathrm{E}\big[Y|X+Y > VaR_\alpha(X+Y)\big]$.

Now we only need to show that:

$$E\big[X|X+Y > VaR_\alpha(X+Y)\big] = \frac{a(a+1)\theta^a}{2(1-\alpha)}\left(\frac{1}{a-1}(q-\theta)^{-(a-1)}-\frac{\theta^2}{a+1}(q-\theta)^{-(a+1)}\right)$$

With the joint density of $X+Y$ and X given in (3.5) we get:

$$\begin{aligned} E\big[X|X+Y>VaR_\alpha(X+Y)\big] &= \frac{1}{1-\alpha}\int_{-\infty}^{\infty} x \int_q^{\infty} f_{X,X+Y}(x,v)\,dv\,dx \\ &= \frac{a(a+1)\theta^a}{1-\alpha}\int_{-\infty}^{\infty} x \int_q^{\infty} (v-\theta)^{-(a+2)}\mathbb{1}_{\{v-x\geq\theta, x\geq\theta\}}\,dv\,dx \end{aligned}$$

After exchanging the integrals and applying the indicator function to the boundaries of the integral we get:

$$\begin{aligned} &\frac{a(a+1)\theta^a}{1-\alpha}\int_q^{\infty}(v-\theta)^{-(a+2)}\int_\theta^{v-\theta} x\,dx\,dv \\ =&\frac{a(a+1)\theta^a}{1-\alpha}\int_q^{\infty}(v-\theta)^{-(a+2)}\left(\frac{1}{2}(v-\theta)^2-\theta^2\right)dv \end{aligned}$$

We notice that this expression is equal to (3.7) except of a factor of $\frac{1}{2}$. Therefore, we can apply the same steps found in the proof of Proposition 3.7 and get the final result. $\square$

3.1.1 Premium calculation for identical Pareto risks

In this section, we investigate the influence of θ and the correlation parameter a of pareto distributed risks on each proposed premium principle.

Since we assumed in this chapter that both risks, X_1 and X_2, are identical, $VaR_\alpha(X_1) = VaR_\alpha(X_2)$ and $TVaR_\alpha(X_1) = TVaR_\alpha(X_2)$. Hence, we get the following formulas for the premium principles:

- VaR premium: $P_i \overset{(1.3)}{=} \frac{VaR_\alpha(X_i)}{VaR_\alpha(X_1)+VaR_\alpha(X_2)} VaR_\alpha(X_1+X_2) = \frac{1}{2}VaR_\alpha(X_1+X_2)$ for $i=1,2$.

- TVaR premium: $P_i \overset{(1.4)}{=} \frac{TVaR_\alpha(X_i)}{TVaR_\alpha(X_1)+TVaR_\alpha(X_2)} TVaR_\alpha(X_1+X_2) = \frac{1}{2}TVaR_\alpha(X_1+X_2)$ for $i=1,2$.

- The conditional premium is given in Proposition 3.8

We notice that the TVaR premium is equal to the conditional premium. Because of the defintion of these premium principles that has to be the case.

Let's consider $X_1 \sim Par(a, \theta)$ and $X_2 \sim Par(a, \theta)$ with $\theta = 5$, $\alpha = 0.99$ and a correlation parameter a ranging from 2.1 to 10. From this, we get the following premium for each principle:

a	2.1	2.5	3	4	5	10
Correlation	0.48	0.40	0.33	0.25	0.20	0.10
VaR premium	40.06	27.78	20.25	13.76	10.99	7.22
TVaR premium	74.98	45.15	29.47	17.71	13.26	7.79
Cond. premium	74.98	45.15	29.47	17.71	13.26	7.79

Table 3.1: The premium for a pareto distributed risk with a decreasing correlation

In Table 3.1 we can see the premiums for different values of a and for the three proposed premium principles. From Proposition 3.1 we get the formula for the correlation between risks X_1 and X_2. This formula holds only if $a > 2$. Because of this we only considered values of a between 2.1 and 10. It holds that for every principle the premium is decreasing if the correlation between the risks is getting smaller. This corresponds with an increasing a. Furthermore, we can see that the VaR premium is always lower than the TVaR and conditional premium, which makes sense because the TVaR and conditional premium principles are more conservative. But, the difference between the VaR premium and the TVaR and conditional premium is decreasing from over 30 to less than 1. This makes sense since the correlation is decreasing, which means that the distribution of $X_1 + X_2$ gets lighter right tails.

Now we consider the same risks $X_1 \sim Par(a, \theta)$ and $X_2 \sim Par(a, \theta)$ but with a fixed correlation parameter $a = 3$, $\alpha = 0.99$ and θ increasing from 1 to 100.

In Table 3.2 we can see the premiums for different θ's and each proposed premium principle. All the numbers are rounded to the second decimal place. For all principles we see that the premium is increasing linearly. If θ is doubled the premium is doubled as well. This holds for every premium principle. That makes sense because θ is the scale paramter of the pareto

θ	1	2	3	5	10	15	25	100
VaR premium	4.05	8.10	12.15	20.25	40.49	60.74	101.24	404.94
TVaR premium	5.89	11.79	17.68	29.47	58.94	88.41	147.35	589.40
Cond. premium	5.89	11.79	17.68	29.47	58.94	88.41	147.35	589.40

Table 3.2: The premium for each principle for a pareto distributed risk with an increasing θ

distribution. The scale parameter describes how spread out the distribution is. This means that if θ is doubled the distribution is twice as spread out than before. Then, it is reasonable to charge the doubled premium.

3.2 Premium calculation for non-identical Pareto risks

So far, we have only considered identical pareto distributed risks. But most risks which are insured by an insurance company are not identical. That is why we consider risks that are not identical in this section, which means that for risks $X_1 \sim Par(a, \theta_1)$ and $X_2 \sim Par(a, \theta_2)$ it holds that $\theta_1 \neq \theta_2$. We will derive formulas for each proposed premium principle but before we can do that we have to derive the pdf and cdf of $X_1 + X_2$ and the joint density of X_i and $X_1 + X_2$, $i = 1, 2$.

Proposition 3.9 *Let $X_1 \sim Par(a, \theta_1)$ and $X_2 \sim Par(a, \theta_2)$ with $a > 2$, $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_1 \neq \theta_2$. Then the pdf of $X_1 + X_2$ is:*

$$f_{X_1+X_2}(v) = \frac{a(\theta_1\theta_2)^{a+1}}{\theta_2 - \theta_1}\Big[\big(\theta_1 v - \theta_1^2\big)^{-(a+1)} - \big(\theta_2 v - \theta_2^2\big)^{-(a+1)}\Big]\mathbb{1}_{\{v \geq \theta_1+\theta_2\}} \tag{3.8}$$

Proof. The joint density of X_1 and X_2 is given in Definition 2. We apply the same transformation $W = X_1$, $V = X_1 + X_2$ as in the proof of Proposition 3.4 and we get:

$$\begin{aligned} f_{W,V}(w, v) &= f_{X,Y}(w, v - w) \cdot |J| \\ &= a(a+1)(\theta_1\theta_2)^{a+1}\big(\theta_2 w + \theta_1(v - w) - \theta_1\theta_2\big)^{-(a+2)}\mathbb{1}_{\{v-w\geq\theta_2, w\geq\theta_1\}} \\ &= a(a+1)(\theta_1\theta_2)^{a+1}\big(\theta_1 v + (\theta_2 - \theta_1)w - \theta_1\theta_2\big)^{-(a+2)}\mathbb{1}_{\{v-\theta_2\geq w, w\geq\theta_1\}} \end{aligned} \tag{3.9}$$

By integrating over w we get the marginal density of $V = X_1 + X_2$.

$$
\begin{aligned}
f_V(v) &= \int_{-\infty}^{\infty} a(a+1)(\theta_1\theta_2)^{a+1}\big(\theta_1 v + (\theta_2-\theta_1)w - \theta_1\theta_2\big)^{-(a+2)})\mathbb{1}_{\{v-\theta_2\geq w, w\geq\theta_1\}}\,dw \\
&= \int_{\theta_1}^{v-\theta_2} a(a+1)(\theta_1\theta_2)^{a+1}(\theta_1 v + (\theta_2-\theta_1)w - \theta_1\theta_2)^{-(a+2)}\,dw \\
&= a(a+1)(\theta_1\theta_2)^{a+1}\left[\frac{1}{\theta_2-\theta_1}\frac{1}{-(a+1)}\big(\theta_1 v + (\theta_2-\theta_1)w - \theta_1\theta_2\big)^{-(a+1)}\right]_{w=\theta_1}^{w=v-\theta_2} \\
&= \frac{a(\theta_1\theta_2)^{a+1}}{(\theta_2-\theta_1)}\Big[\big(\theta_1 v + (\theta_2-\theta_1)\theta_1 - \theta_1\theta_2\big)^{-(a+1)} - \big(\theta_1 v + (\theta_2-\theta_1)(v-\theta_2) - \theta_1\theta_2\big)^{-(a+1)}\Big] \\
&= \frac{a(\theta_1\theta_2)^{a+1}}{(\theta_2-\theta_1)}\Big[\big(\theta_1 v - \theta_1^2\big)^{-(a+1)} - \big(\theta_2 v - \theta_2^2\big)^{-(a+1)}\Big]\mathbb{1}_{\{v\geq\theta_1+\theta_2\}}
\end{aligned}
$$

We need the indicator function because the pdf always has to be greater or equal than 0. This holds if $v > \theta_1 + \theta_2$. $\square$

Proposition 3.10 *Let $X_1 \sim Par(a,\theta_1)$ and $X_2 \sim Par(a,\theta_2)$ with $a > 2$, $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_1 \neq \theta_2$. Then the pdf of $X_1 + X_2$ is:*

$$
F_{X_1+X_2}(z) = 1 + \frac{1}{\theta_2-\theta_1}\Big[\theta_1^{a+1}(z-\theta_2)^{-a} - \theta_2^{a+1}(z-\theta_1)^{-a}\Big]\mathbb{1}_{\{z\geq\theta_1+\theta_2\}}
$$

Proof. It holds that $F_X(z) = \int_{-\infty}^{z} f_X(x)\,dx$. Also, we assume that $z \geq \theta_1 + \theta_2$. The pdf of $X_1 + X_2$ is given in (3.8). Then we get:

$$
\begin{aligned}
F_{X_1+X_2}(z) &= \int_{-\infty}^{z} \frac{a(\theta_1\theta_2)^{a+1}}{\theta_2-\theta_1}\Big[\big(\theta_1 v - \theta_1^2\big)^{-(a+1)} - \big(\theta_2 v - \theta_2^2\big)^{-(a+1)}\Big]\mathbb{1}_{\{v\geq\theta_1+\theta_2\}}\,dv \\
&= \frac{a(\theta_1\theta_2)^{a+1}}{\theta_2-\theta_1}\left[\frac{1}{-a}\theta_1^{-(a+1)}(v-\theta_1)^{-a} - \frac{1}{-a}\theta_2^{-(a+1)}(v-\theta_2)^{-a}\right]_{v=\theta_1+\theta_2}^{v=z} \\
&= \frac{(\theta_1\theta_2)^{a+1}}{\theta_2-\theta_1}\Big[\theta_2^{-(a+1)}(z-\theta_2)^{-a} - \theta_1^{-(a+1)}(z-\theta_1)^{-a} - \theta_2^{-(a+1)}\theta_1^{-a} + \theta_1^{-(a+1)}\theta_2^{-a}\Big] \\
&= \frac{1}{\theta_2-\theta_1}\Big[\theta_1^{a+1}(z-\theta_2)^{-a} - \theta_2^{a+1}(z-\theta_1)^{-a}\Big] + 1
\end{aligned}
$$

Since we assumed that $z \geq \theta_1 + \theta_2$, we have to check that $F_{X_1+X_2}(\theta_1+\theta_2) = 0$ and $\lim_{z\to\infty} F_{X_1+X_2}(z) = 1$. When these are true, then $F_{X_1+X_2}(z)$ is indeed a distribution function.

$$\begin{aligned}
F_{X_1+X_2}(\theta_1+\theta_2) &= 1 + \frac{1}{\theta_2-\theta_1}\Big[\theta_1^{a+1}(\theta_1+\theta_2-\theta_2)^{-a} - \theta_2^{a+1}(\theta_1+\theta_2-\theta_1)^{-a}\Big] \\
&= 1 + \frac{1}{\theta_2-\theta_1}\Big[\theta_1-\theta_2\Big] = 0 \\
\lim_{z\to\infty} F_{X_1+X_2}(z) &= \lim_{z\to\infty} 1 + \frac{1}{\theta_2-\theta_1}\Big[\theta_1^{a+1}(z-\theta_2)^{-a} - \theta_2^{a+1}(z-\theta_1)^{-a}\Big] \\
&= 1 + \frac{1}{\theta_2-\theta_1}\Big(0-0\Big) = 1
\end{aligned}$$

This holds because $\lim_{z\to\infty}(z-\theta_2)^{-a} = 0$ and $\lim_{z\to\infty}(z-\theta_1)^{-a} = 0$. □

Now that we calculated the pdf and cdf of the sum of two dependent pareto random variables we are able to calculate the Value at Risk and the Tail Value at Risk for $X_1 + X_2$.

Proposition 3.11 *Let $X_1 \sim Par(a, \theta_1)$ and $X_2 \sim Par(a, \theta_2)$ with $a > 2$, $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_1 \neq \theta_2$. Then the Value at Risk of $X_1 + X_2$ is the number $q \in \mathbb{R}$, which solves the following equation:*

$$\alpha = 1 + \frac{1}{\theta_2-\theta_1}\Big[\theta_1^{a+1}(q-\theta_2)^{-a} - \theta_2^{a+1}(q-\theta_1)^{-a}\Big] \tag{3.10}$$

Proof. In Proposition 3.10 we calculated the cdf of X_1+X_2. With the definition of the Value at Risk it holds that:

$$\begin{aligned}
& VaR_\alpha(X_1+X_2) = q \Leftrightarrow F_{X_1+X_2}(q) = \alpha \\
& \Rightarrow 1 + \frac{1}{\theta_2-\theta_1}\Big[\theta_1^{a+1}(q-\theta_2)^{-a} - \theta_2^{a+1}(q-\theta_1)^{-a}\Big] = \alpha
\end{aligned}$$

Since $X_1 + X_2$ is a continuous random variable and $F_{X_1+X_2}(x)$ is strictly increasing, the solution q is unique. □

Proposition 3.12 *Let $X_1 \sim Par(a, \theta_1)$ and $X_2 \sim Par(a, \theta_2)$ with $a > 2$, $\theta_1 > 0$, $\theta_2 > 0$*

and $\theta_1 \neq \theta_2$. *Then the Tail Value at Risk of* $X_1 + X_2$ *is:*

$$\begin{aligned} TVaR_\alpha(X_1+X_2) =& \frac{1}{(1-\alpha)(\theta_2-\theta_1)}\Big[\theta_2^{a+1}\Big(\frac{1}{a-1}(q-\theta_1)^{-(a-1)} + \theta_1(q-\theta_1)^{-a}\Big) \\ & - \theta_1^{a+1}\Big(\frac{1}{a-1}(q-\theta_2)^{-(a-1)} + \theta_2(q-\theta_2)^{-a}\Big)\Big] \end{aligned} \tag{3.11}$$

in which q *is given in Proposition 3.11.*

Proof. With q as described in Proposition 3.11 and the definition of $TVaR_\alpha(X)$ we get:

$$\begin{aligned} TVaR_\alpha(X_1+X_2) &= \frac{1}{1-\alpha}\int_q^\infty v f_{X_1+X_2}(v)\,dv \\ &= \frac{a(\theta_1\theta_2)^{a+1}}{(\theta_2-\theta_1)(1-\alpha)}\int_q^\infty v\Big[\big(\theta_1 v - \theta_1^2\big)^{-(a+1)} - \big(\theta_2 v - \theta_2^2\big)^{-(a+1)}\Big]\mathbb{1}_{\{v\geq\theta_1+\theta_2\}}\,dv \\ &= \frac{a}{(\theta_2-\theta_1)(1-\alpha)}\Big[\int_q^\infty \theta_2^{a+1} v\big(v-\theta_1\big)^{-(a+1)}\,dv - \int_q^\infty \theta_1^{a+1} v\big(v-\theta_2\big)^{-(a+1)}\,dv\Big] \end{aligned}$$

For the first integral we substitute $u = v - \theta_1$, $du = dv$ and for the second integral we substitute $u = v - \theta_2$, $du = dv$:

$$\begin{aligned} &\frac{a}{(\theta_2-\theta_1)(1-\alpha)}\Big[\int_{q-\theta_1}^\infty \theta_2^{a+1}(u-\theta_1)(u)^{-(a+1)}\,dv - \int_{q-\theta_2}^\infty \theta_1^{a+1}(u+\theta_2)u^{-(a+1)}\,dv\Big] \\ =&\frac{a}{(\theta_2-\theta_1)(1-\alpha)}\Big[\Big[\theta_2^{a+1}(\frac{u^{-a+1}}{-a+1} + \frac{\theta_1 u^{-a}}{-a})\Big]_{u=q-\theta_1}^{q=\infty} - \Big[\theta_1^{a+1}(\frac{u^{-a+1}}{-a+1} + \frac{\theta_2 u^{-a}}{-a})\Big]_{u=q-\theta_2}^{q=\infty}\Big] \\ =&\frac{1}{(1-\alpha)(\theta_2-\theta_1)}\Big[\theta_2^{a+1}\Big(\frac{1}{a-1}(q-\theta_1)^{-(a-1)} + \theta_1(q-\theta_1)^{-a}\Big) \\ &- \theta_1^{a+1}\Big(\frac{1}{a-1}(q-\theta_2)^{-(a-1)} + \theta_2(q-\theta_2)^{-a}\Big)\Big] \end{aligned}$$

□

Proposition 3.13 *Let* $X_1 \sim Par(a, \theta_1)$ *and* $X_2 \sim Par(a, \theta_2)$ *with* $a > 0$, $\theta_1 > 0$, $\theta_2 > 0$

and $\theta_1 \neq \theta_2$. Then it holds that:

$$\bullet\, \mathrm{E}\big[X_1|X_1+X_2>VaR_\alpha(X_1+X_2)\big] = \frac{1}{(1-\alpha)(\theta_2-\theta_1)^2}\Bigg[\frac{\theta_1\theta_2^{a+1}}{a-1}\big(q-\theta_1\big)^{-(a-1)}$$
$$-\frac{a+1}{a-1}\theta_2\theta_1^{a+1}\big(q-\theta_2\big)^{-(a-1)} + \frac{a\theta_1^{a+2}}{a-1}\big(q-\theta_2\big)^{-(a-1)} - \big(\theta_1-\theta_2\big)\theta_1\theta_2^{a+1}\big(q-\theta_1\big)^{-a}\Bigg] \quad (3.12)$$

$$\bullet\, \mathrm{E}\big[X_2|X_1+X_2>VaR_\alpha(X_1+X_2)\big]$$
$$= TVaR_\alpha(X_1+X_2) - \mathrm{E}\big[X_1|X_1+X_2>VaR_\alpha(X_1+X_2)\big]$$

Proof. With the defintion of TVaR and because the expectation is additive it holds that:

$$TVaR_\alpha(X_1+X_2) = \mathrm{E}\big[X_1+X_2|X_1+X_2>VaR_\alpha(X_1+X_2)\big]$$
$$= \mathrm{E}\big[X_1|X_1+X_2>VaR_\alpha(X_1+X_2)\big] + \mathrm{E}\big[X_2|X_1+X_2>VaR_\alpha(X_1+X_2)\big]$$

Rearranging this equation proves the second point. Now we only need to show the first point. The joint density of X_1 and X_1+X_2 is given in (3.9). We then get:

$$\mathrm{E}\big[X_1|X_1+X_2>VaR_\alpha(X_1+X_2)\big] = \frac{1}{1-\alpha}\int_{-\infty}^{\infty} x \int_q^{\infty} f_{X,X+Y}(x,v)\,dv\,dx$$
$$= \frac{1}{1-\alpha}\int_{-\infty}^{\infty} x \int_q^{\infty} a(a+1)(\theta_1\theta_2)^{a+1}(\theta_1 v + (\theta_2-\theta_1)x - \theta_1\theta_2)^{-(a+2)})\mathbb{1}_{\{v-\theta_2\geq x, x\geq\theta_1\}}\,dv\,dx$$

After exchanging the integrals we get:

$$\frac{a(a+1)(\theta_1\theta_2)^{a+1}}{1-\alpha}\int_q^{\infty}\Big(\int_{\theta_1}^{v-\theta_2} x(\theta_1 v + (\theta_2-\theta_1)x - \theta_1\theta_2)^{-(a+2)})\,dx\Big)\,dv$$

Substituting $u = (\theta_2-\theta_1)x - \theta_1\theta_2 + \theta_1 v$, $dx = \frac{du}{\theta_2-\theta_1}$ and integrating gives us:

$$\frac{a(a+1)(\theta_1\theta_2)^{a+1}}{(1-\alpha)(\theta_2-\theta_1)^2}\int_q^{\infty}\frac{1}{a}\Big(\theta_1^{-a}(v-\theta_1)^{-a} - \theta_2^{-a}(v-\theta_2)^{-a}\Big)$$
$$+\frac{1}{a+1}\Big(\theta_1\theta_2^{-(a+1)}(v-\theta_2)^{-a}\Big)\,dv - \int_q^{\infty}\frac{\theta_1^{-a}}{a+1}(v-\theta_2)(v-\theta_1)^{-(a+1)}\,dv$$

After substituting $w = v - \theta_1$, $dw = dv$ in the second integral and integrating the integral we get:

$$\frac{1}{(1-\alpha)(\theta_2-\theta_1)^2}\Bigg[\frac{\theta_1\theta_2^{a+1}}{a-1}\big(q-\theta_1\big)^{-(a-1)} - \frac{a+1}{a-1}\theta_2\theta_1^{a+1}\big(q-\theta_2\big)^{-(a-1)}$$
$$+\frac{a}{a-1}\theta_1^{a+2}\big(q-\theta_2\big)^{-(a-1)} - \big(\theta_1-\theta_2\big)\theta_1\theta_2^{a+1}\big(q-\theta_1\big)^{-a}\Bigg]$$

□

3.2.1 Premium calculation for non-identical Pareto risks

In this subsection, we analyze the impact θ_1 has on the premium of X_1 and X_2 for each proposed premium principle.

We assume that $X_1 \sim Par(a, \theta_1)$, $X_2 \sim Par(a, \theta_2)$, $\theta_1 > 0$, $\theta_2 > 0$, $\theta_1 \neq \theta_2$ and $a > 2$. We can calculate the premium for each proposed principle. For that we need (3.1) and (3.10) for the VaR premium, (3.2) and (3.11) for the TVaR premium and (3.12) for the conditional premium.

Let's consider $X_1 \sim Par(a, \theta_1)$ and $X_2 \sim Par(a, \theta_2)$ with $a = 3$, $\alpha = 0.99$, $\theta_2 = 1$ and θ_1 is increasing from 2 to 10. We are saying that X_1 is the bigger risk since the expectation and the variance of it is greater than that of risk X_2.

In Figure 3.1 we can see the premium of X_1 for different θ_1 and for each proposed principle. The solid line represents the TVaR premium, the dashed line represents the conditional premium and the dotted line represents the VaR premium. The VaR premium is for all θ_1 less than the other two premiums. This makes sense because the TVaR and conditional premium is more conservative than the VaR premium. Furthermore, the distance between the VaR premium and the other two premiums is increasing when θ_1 is increasing. This is true since X_1 is positively correlated with X_2 which means if one of the risks is high it is more likely that the other risk is high too. Also, θ_1 is a scale parameter which means that if θ_1 is increasing the distribution is more spread out. Likewise, the TVaR is increasing faster than the VaR.

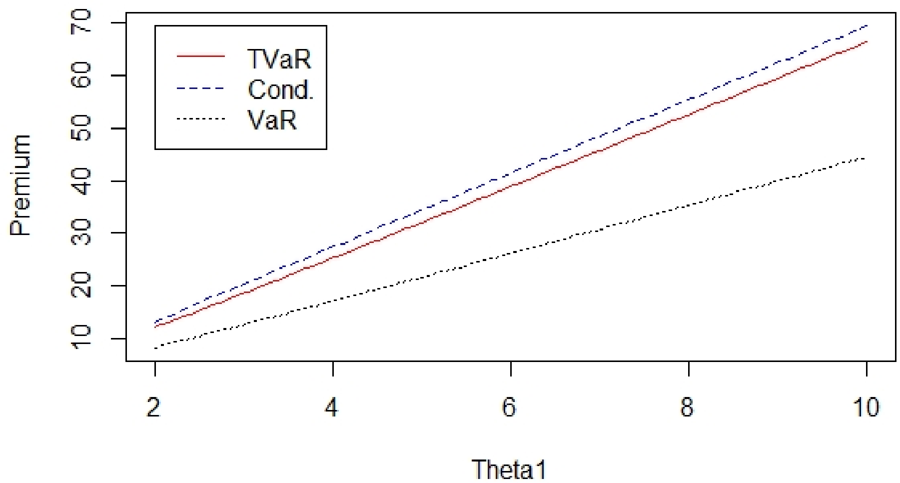

Figure 3.1: The premium for each principle of risk X_1 while θ_1 is increasing

Additionally, we can see that the conditional premium is greater than the TVaR premium, meaning the conditional premium requires a higher premium for overtaking risk X_1 than the TVaR principle. This also means that the conditional premium principle assigns X_1 a higher fault than the TVaR premium principle when $X_1 + X_2$ exceeds the $VaR_{0.99}(X_1 + X_2)$. We can say this because $P_{Cond} = \mathrm{E}\big[X_1|X_1 + X_2 > VaR_0.99(X_1 + X_2)\big] > P_{TVaR}$.

In Figure 3.2 we can see the premium for X_2 for an increasing θ_1 while $\theta_2 = 1$ is fixed. The solid line represents the TVaR premium and the dashed line the conditional premium. The TVaR premium is greater than the conditional premium and it is increasing for an increasing θ_1. On the other hand, the conditional premium is decreasing for an increasing θ_1. Which means that the conditional premium demands a smaller premium if the risk is much smaller than the other risk in the portfolio. However, the TVaR premium demands a higher premium for the smaller risk if the other risk in the portfolio is a lot bigger than the smaller risk.

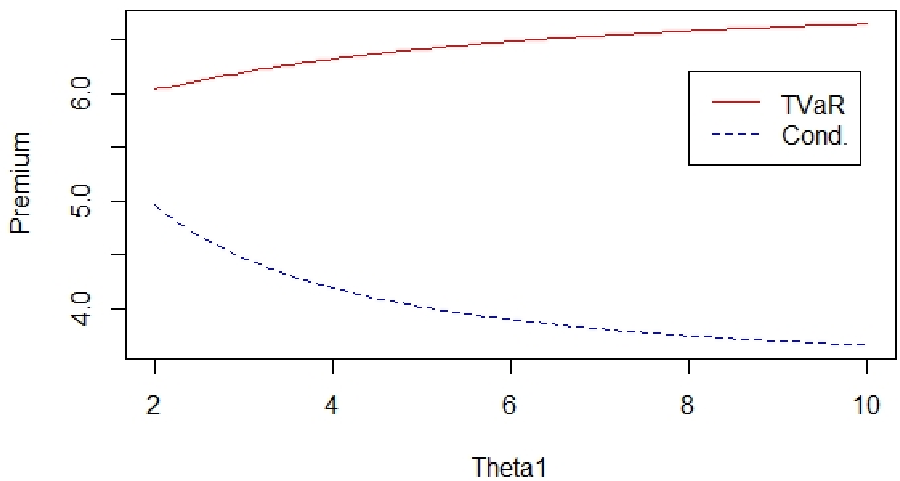

Figure 3.2: The TVaR and conditional premium for X_2 while θ_1 is increasing

Figure 3.3 shows the VaR premium for X_2 for an increasing θ_1 while $\theta_2 = 1$ is fixed. We can see that the premium is increasing for an increasing θ_1. The increase in the premium is 0.3 or 9%. This means that VaR principle demands a higher premium if there is a bigger risk in the portfolio of the insurance company.

In conclusion, we see that the VaR and TVaR premium principles are similar in the sense that they demand a higher premium for both risks if the risks are pareto distributed with a postive correlation but have very different values for θ. On the other hand, the conditional premium rewards the risk with the lower θ with a smaller premium if the difference in θ_1 and θ_2 is big. Since the total premium of the conditional premium has to be equal to the $TVaR_\alpha(X_1 + X_2)$, the premium for the bigger risk is increasing disproportionately. This means that the smaller risk would prefer the conditional premium principle and the bigger risk would prefer the TVaR premium principle or the VaR premium principle, since the TVaR principle and VaR principle behave similiarly.

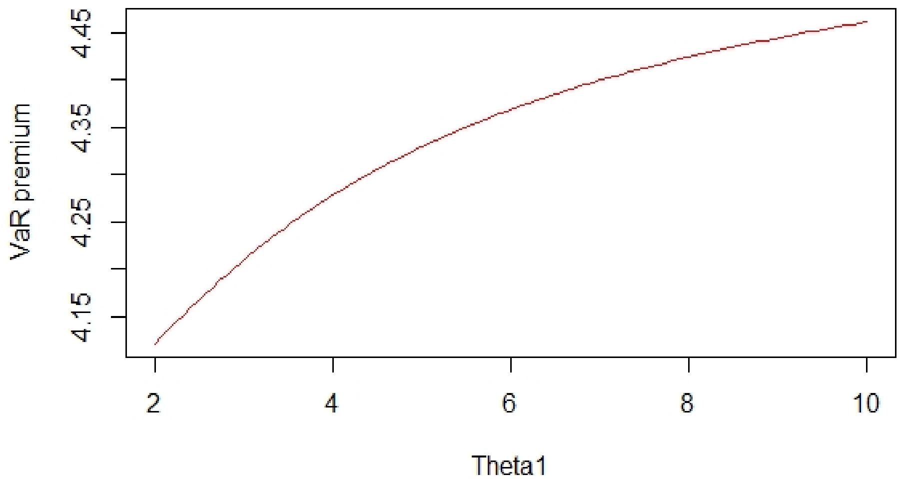

Figure 3.3: The VaR premium for X_2 while θ_1 is increasing

4 Aggregate loss model

The aggregate loss model is commonly used in insurance because we can model the amount of claims an insurance company has to cover with a random variable. This is useful because the insurance company does not know how many claims will occur in the next time period. Therefore, we can model the total loss as $TL = \sum_{j=1}^{N} Y_j$ where N is a discrete random variable that models the claim frequency and Y_j is a random variable that models the claim size. Usually it is assumed that N and Y_j are mutually independent for all j.

We assume that each risk X_i $i = 1, \dots, n$ is an aggregate loss: $X_i = \sum_{j=1}^{N_i} Y_{ij}$. Also, we assume that $N_1, \dots, N_n$ are dependent. Therefore, we can assume that $\{Y_{ij},\ i = 1, 2, \dots, j = 1, 2, \dots\}$ are independent and we can focus on modeling the dependence of $N_1, \dots, N_n$. To simplify, we assume that $Y_{ij} \sim Y\ \forall i, j$. We will consider two different cases of dependence structures:

- Strong dependence $N_1 = \cdots = N_n = N$
- Weak dependence $N_i = N_0 + M_i$ where $N_0, M_1, \dots, M_n$ are mutually independent

In the next section we analyze the impact the dependence structure has on the different premium principles in the poisson-exponential model. We consider the VaR (1.3) and the TVaR premium principle (1.4). But, before we can calculate the premium for the different principles we need the following results.

Proposition 4.1 *Let S be an aggregate loss, $S = \sum_{j=1}^{N} Y_j$ and $Y_j \overset{i.i.d.}{\sim} Y$. Then the characteristic function of S is given by:*

$$\varphi_S(z) = P_N\big(\varphi_Y(z)\big) \tag{4.1}$$

where P_N is the probability generating function of N.

Proof. With the definition of the characteristic function we get:

$$\varphi_S(z) = \mathrm{E}\big[e^{izS}\big] = \mathrm{E}\big[e^{iz0}\big]\, P(N=0) + \sum_{n=1}^{\infty} \mathrm{E}\big[e^{iz(Y_1+\cdots+Y_n)}|N=n)\big]\, P(N=n)$$

Since all Y_j are i.i.d. and with the defintion of the probability generating function we get:

$$= 1\; P(N=0) + \sum_{n=1}^{\infty} \mathrm{E}\Big[\prod_{j=1}^{n} e^{izY_j}\Big] P(N=n)$$

$$= \big(e^{izY}\big)^0\; P(N=0) + \sum_{n=1}^{\infty} \mathrm{E}\Big[\big(e^{izY}\big)^n\Big] P(N=n) \quad = \sum_{n=0}^{\infty} \big(\varphi_Y(z)\big)^n P(N=n) = P_N\big(\varphi_Y(z)\big)$$

□

4.1 Poisson-exponential model

The Poisson-exponential model is an aggregate loss model where the claim frequency N follows a poisson distribution with mean λ and the claim severity Y_j follows an exponential distribution with mean θ. We assume that each risk $X_i = \sum_{j=1}^{N_i} Y_{ij}\; i = 1, \ldots, n$ is an aggregate loss.

To calculate the VaR premium and the TVaR premium for sums of aggregate loss risks we need to be able to calculate the Value at Risk for aggregate loss risks by obtaining the distribution of the aggregate loss. But, finding the exact distribution of an aggregate loss is very difficult. Therefore, it is common to use the following technique to find the distribution of an aggregate loss given in Klugman et al.(2012) [1] (p. 254f):

The first step is to discretize the continous severity function. We will use the method of rounding, described in Defintion 3.

Definition 3 (The method of rounding) *Let f_j denote the probability placed at jh, $j =$*

$1,\ldots,n$. Then set

$$
\begin{aligned}
f_0 &= P\Big(X < \frac{h}{2}\Big) = F_X\Big(\frac{h}{2}\Big)\\
f_j &= P\Big(jh - \frac{h}{2} \leq X < jh + \frac{h}{2}\Big)\\
&= F_X\Big(jh + \frac{h}{2}\Big) - F_X\Big(jh - \frac{h}{2}\Big),\ j = 1,\ldots,m-1\\
f_n &= P\Big(X > mh - \frac{h}{2}\Big) = 1 - F_X\Big(mh - \frac{h}{2}\Big)
\end{aligned}
$$

where j denotes the step size, n the number of steps and m the highest possible value that the discrete probability could become. Lastly, $m = jn$.

The second step is to calculate the characteristic function of the discrete severity function with the discrete fast fourier transformation. The third step is to get the characteristic function of the aggregate loss with the help of Proposition 4.1. In the case of the poisson-exponential model we can calculate the characteristic function of the aggregate loss S with $\varphi_S(z) = exp\big(\lambda(\varphi_Y(z) - 1)\big)$. The last step is to get the distribution function of the aggregate loss by applying the inverse fast fourier transformation. After following these four steps we have a discrete distribution function of the aggregate loss and can calculate the Value at Risk and Tail Value at Risk.

4.1.1 Premium calculation for strong dependence

In this subsection, we will consider the Poisson-exponential aggregate loss model with strong dependence. This means that $N_1 = \cdots = N_n \sim Poi(\lambda)$. For the premium calculation we need the following result.

Proposition 4.2 *Let X_i $i = 1,\ldots,n$ be each a poisson-exponential aggregate loss: $X_i = \sum_{j=1}^{N_i} Y_{ij}$ with $N_1 = \cdots = N_n = N \sim Poi(\lambda)$ and $Y_{ij} \overset{i.i.d.}{\sim} Exp(\theta)$. Then the total loss is*

Poisson over Gamma distributed:

$$TL = \sum_{j=1}^{N} Y_j$$

where $N \sim Poi(\lambda)$ *and* $Y_j \sim \Gamma(n, \theta)$

Proof. Since $N_1 = \cdots = N_n = N$, Y_{ij} are i.i.d. for all i, j and from the defintion of the total loss we get:

$$TL = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n}\sum_{j=1}^{N_i} Y_{ij} = \sum_{j=1}^{N}\sum_{i=1}^{n} Y = \sum_{j=1}^{N} Z$$

where $Z \sim \Gamma(n, \theta)$. This holds because the sum of n independent and identical exponential random variable $Y \sim Exp(\theta)$ with mean θ is gamma distributed with parameter n and θ. □

Let's consider $n = 2$ aggregate losses, X_1 and X_2, where $X_i = \sum_{j=1}^{N_i} Y_{ij}$ for $i = 1, 2$. Also, let $Y_{ij} \overset{i.i.d.}{\sim} Exp(3)$ and $N_1 = N_2 \sim Poi(\lambda)$ where λ is increasing from 1 to 8. To calculate the VaR and TVaR of the total loss we use the technique described in Section 4.1 and the method of rounding with $m = 121.38$, $n = 2^{16} = 65,536$ and $j = \frac{m}{n}$. Since all risks are identical the premium for each risk has to be the same.

In Figure 4.1 we can see the premium for the VaR premium and TVaR premium when λ is increasing. The solid line represents the VaR premium and the dashed line represents the TVaR premium. The first thing we notice is that the VaR premium is smaller than the TVaR premium but this makes sense because the TVaR premium is more conservative. Moreover, both premiums are increasing when λ is increasing. This makes sense because a larger λ means the probability of more claims happening is increased. The consequence is that this is riskier and more expensive for the insurance company. That is why the insurance company demands a higher premium.

In Table 4.1 we consider that the amount of contracts n is increasing from 2 to 100. Each risk X_i is an aggregate loss which means that $X_i = \sum_{j=1}^{N_i} Y_{ij}$ for $i = 1, \ldots, n$, where $Y_{ij} \overset{i.i.d.}{\sim}$

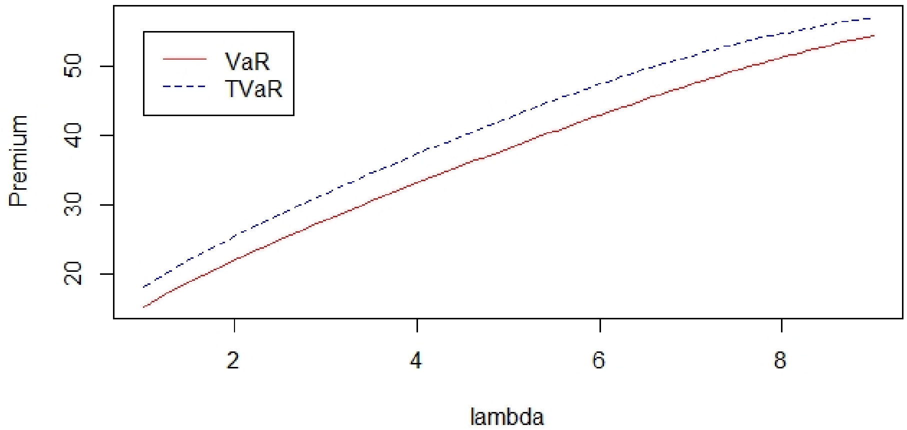

Figure 4.1: Premium for different principles when λ is increasing

$Exp(3)$ and $N_1 = \cdots = N_n \sim Poi(2)$. To calculate each premium we use the technique described in Section 4.1. We use the method of rounding with $n = 2^{16} = 65,536$, $j = \frac{m}{n}$ and m is given in the table and is equal to the 99.9999% quantile of $Z \sim \Gamma(n, 3)$.

n	2	3	5	10	25	100
m	121.38	130.92	147.68	183.31	271.03	616.33
VaR premium	22.06	20.71	19.54	16.67	10.57	6.13
TVaR premium	25.49	23.72	22.03	17.42	10.70	6.15

Table 4.1: The premium for an aggregate loss X_i for different principles depending on the amount of contracts in the portfolio

In Table 4.1 we can see the VaR premium and the TVaR premium for a risk X_i while the amount of contracts n in the insurance portfolio is increasing. Because all risks are identical the premium for each risk is the same. We can see that both premiums are decreasing if n is increasing. Moreover, the difference between both premium principles is decreasing from 3.43 to 0.02. This shows that for a large insurance portfolio the premium is decreasing for each principle and the difference between the VaR and TVaR principle is getting smaller.

4.1.2 Premium calculation for weak dependence

In this subsection we assume that the risks $X_i = \sum_{j=1}^{N_i} Y_{ij}$, $i = 1, \ldots, n$ have a weak dependence structure. This means that $N_i = N_0 + M_i$, where $N_0, M_1, \ldots, M_n$ are mutually independent. Because $N_0, M_1, \ldots, M_n$ are mutually independent and $N_0 \sim Poi(\lambda_0)$, $M_i \sim Poi(\lambda_i)$ for all i, $N_0 + M_i \sim Poi(\lambda_0 + \lambda_i)$.

Let's consider $n = 2$ aggregate loss risks with $M_1 \sim Poi(1)$, $M_2 \sim Poi(3)$, $Y_{ij} \sim Exp(3)$ for all i, j and $N_0 \sim Poi(\lambda_0)$ with λ_0 increasing from 0.5 to 10. We use the method described in Section 4.1 to calculate VaR and TVaR for X_1 and X_2 with $\alpha = 0.99$ and the method of rounding with $m = 75$, $n = 2^{16} = 65536$ and $j = \frac{m}{n}$. For the VaR and TVaR of the total loss we use the Monte-Carlo method with $100,000$ simulations.

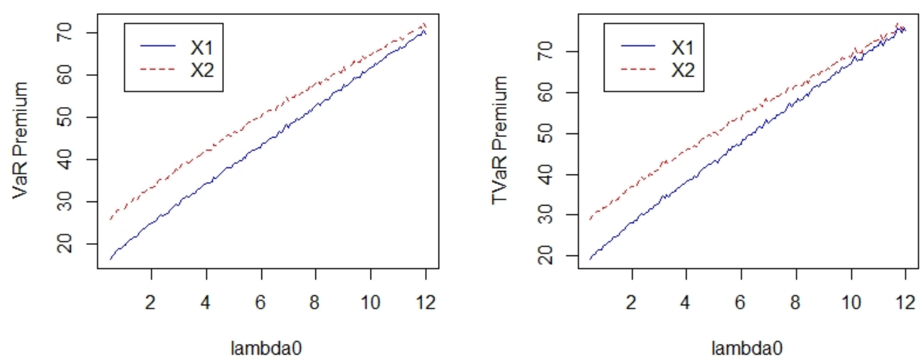

Figure 4.2: Premium for different principles and risks when λ_0 is increasing

In Figure 4.2 we can see how the premium changes when λ_0 is increasing. On the left side it shows the VaR premium for X_1 and X_2 and on the right side the TVaR premium for X_1 and X_2. In both graphs the solid line represents the premium for X_1 and the dashed line represents the premium for X_2. We notice that all lines are not smooth, because we used Monte-Carlo simulations to find the VaR and TVaR of the total loss and Monte-Carlo simulations does not give a precise results. Furthermore, we can see that the premium for X_1 is less than the premium for X_2 for all λ_0. This makes sense because λ_1 is smaller than

λ_2 which means that it is more likely that X_1 has less claims than X_2. Additionally, we can see that the difference between the premium for each risk is decreasing. This holds for both premium principles. This is happening because the dependence between both risks is increasing, which is described by λ_0. Moreover, for an increasing parameter λ of a poisson distribution, we saw in Figure 4.1 that both premium principles are concave. That means that the original difference in λ_1 and λ_2 resulted in a rather large difference in the premium because both parameters are small. But, with an increasing λ_0 the difference between the parameters stays the same. Both parameters are bigger, however, which means the difference between the premium is smaller because of the concavity of the premium principles.

In this example we assume a similiar setup as in the previous example, $N_0 \sim Poi(1)$, $M_2 \sim Poi(1)$, $Y_{ij} \sim Exp(3)$ for all i, j and $M_1 \sim Poi(\lambda_1)$ where λ_1 is increasing from 1 to 10. We use the method described in Section 4.1 to calculate the VaR and TVaR for X_1 and X_2 with $\alpha = 0.99$. For the method of rounding we use $m = 75$, $n = 2^{16} = 65536$ and $j = \frac{m}{n}$. To calculate the VaR and TVaR of the total loss we use the Monte-Carlo method with $100,000$ simulations.

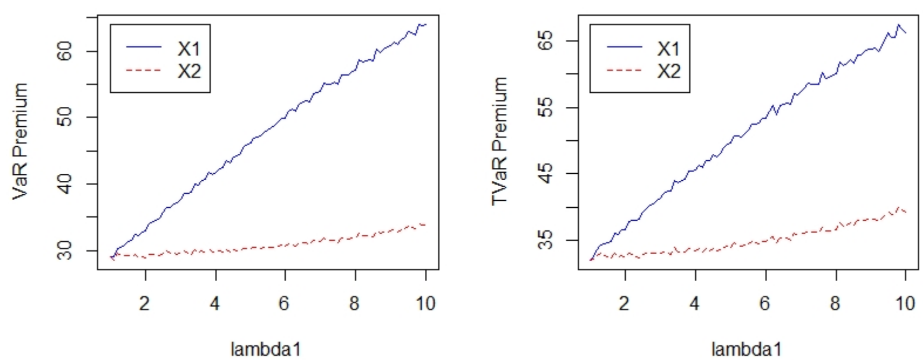

Figure 4.3: Premium for different principles and risks when λ_1 is increasing

Figure 4.3 shows the VaR premium and TVaR premium in dependence of λ_1. We can see on the left side the VaR premium for X_1 and X_2 and on the right side the TVaR premium for X_1 and X_2. In both graphs the solid line represents the premium for X_1 and the dashed line represents the premium for X_2. As in Figure 4.1, we do not see a smooth line because we used a Monte-Carlo simulation to obtain the VaR and TVaR of the total loss. We notice that both premiums of X_1 are increasing, which makes sense because λ_1 is the parameter of the claim frequency of X_1 and if this parameter is increasing it is more likely that more claims will happen. This results in a higher risk for the insurance company and they will demand a higher premium. Furthermore, we see that the premium for X_2 is also increasing when λ_1 is increasing.

In conclusion we can say that in the poisson-exponential model and in the case of a strong dependence structure the premium for the VaR and TVaR principle is increasing if λ is increasing. Also, the premium is decreasing for an increase in the amount of risks in the insurance portfolio. This means that the important concept of pooling risks is still functioning. In the case of a weak dependence structure the premium strongly depends on the parameter λ_0. When λ_0 is high the premium for the risks are mostly equal but if λ_0 is small the premium for each risk depends on their poisson parameter λ_i. Furthermore, the premium for both risks are increasing if λ_1 is increasing even if λ_2 and λ_0 is fixed. This means that the premium principles assign a comparatively smaller premium when the risks are similiar.

5 Conclusion

The goal of this master's thesis was to price depedent risks. Therefore, we proposed three premium principles. We assumed that the risks follow a normal distribution, pareto distribution and that each risk is an aggregate loss. We derived formulas for the first two distributions and for each premium principle. For the case that each risk is an aggregate loss we explained a method to calculate the VaR premium and the TVaR premium.
We observed that when the dependence between the insured risks is increasing that the premium is increasing as well. This means that strongly correlated risks represent a higher total risk for the insurance company than weakly correlated risks. Further, we observed that an increase in the portfolio size of the insurance company results in a decrease in the individual premium for a risk. This holds for the normally distributed risks and poisson-exponential distributed risks but only if the risks are not perfectly correlated. Also, we found out that if the risks in an insurance portfolio are very different, that is, their parameters of the distributions are very different, then the VaR and the TVaR premium principle demand a higher premium for the smaller risk compared to the case when both risks are equal. On the other hand, the conditional premium principle demands a smaller premium for the smaller risk when the risks are very different compared to the case that both risks are equal.

Although we derived formulas for the proposed premium principle it is interesting to extend the bivariate pareto model into the multivariate pareto model. Moreover it would be interesting to derive the distribution function of the total loss in the poisson-exponential model for the case of a weak dependence structure. Then we would not have to rely on a simulation procedure to get the Value at Risk and Tail Value at Risk of the total loss. Of course, it is also possible to extend this study to other distributions.

References

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[2] Mardia, K. V. (1962) *Multivariate Pareto distributions*. Ann. Math. Statist., Volume 33, Number 3, 1008-1015.