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# Dynamic Pricing with Variable Order Sizes for a Model with Constant Demand Elasticity

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# DYNAMIC PRICING WITH VARIABLE ORDER SIZES FOR A MODEL WITH CONSTANT DEMAND ELASTICITY

by

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# ABSTRACT

# DYNAMIC PRICING WITH VARIABLE ORDER SIZES FOR A MODEL WITH CONSTANT DEMAND ELASTICITY

by

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We investigate a dynamic pricing model under constant demand elasticity which accounts for customers ordering multiple items at once. A closed form expression for the optimal expected revenue and pricing strategy is found. Models with the same demand are shown to have asymptotically similar expected revenue and pricing strategies, even if the order size distributions of the customers are different. Surprisingly, the relative difference between comparable models is shown to be independent of time and the magnitude of demand. Variations of the model are considered, including different low inventory behavior as well as the effect of advertising. Some numerical simulations are presented to provide better insight on the model.

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# LIST OF SYMBOLS

$p$ Price. 1

$t$ Current time, note $0 \leq t \leq T$. 1

$\lambda(p,t)$ Customer arrival rate of a Poisson process dependent on price $p$ and current time $t$. 1

$\varepsilon$ Demand elasticity. 1, 9

$a(t)$ Customer arrival rate scaling factor dependent on time $t$. 1

$T$ Ending time of the sales period. 6

$\mathbf{q}$ Probability distribution for the order size of customers. 6

$q_i$ Probability that a customer orders $i$ items, note $\mathbf{q} = (q_1, \ldots, q_M)$. 6

$M$ Maximum order size. 6

$v_n(t; \mathbf{q}, \lambda)$ Optimal expected revenue with $n$ inventory at current time $t$ for a model with customer arrival rate $\lambda$ and customers who have order size distribution $\mathbf{q}$. 6

$\mu(\mathbf{q})$ Average order size of the distribution $\mathbf{q}$, $\mu(\mathbf{q}) := \sum_{i=1}^{M} q_i i$. 7

$p_n(t)$ A pricing strategy which defines a price at inventory level $n$ at current time $t$. 7

$A(t)$ Number of future sales at a price equal to 1, $A(t) := \int_t^T a(s)ds$. 10

$\beta_n(\mathbf{q})$ A recursively defined sequence, see reference page for definition. 11

$p^*_n(t;\mathbf{q})$ Price which maximizes expected revenue with $n$ inventory at current time $t$ under order size distribution $\mathbf{q}$. 11

$\gamma_n(\mathbf{q})$ The ratio defined as $\gamma_n(\mathbf{q}) := \frac{\beta_n(\mathbf{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}}$. 15

$\mathscr{M}_{\mathbf{q},\lambda}$ A Poisson based model where customers have order size distribution $\mathbf{q}$ and arrival rate is $\lambda$. 31

$g_{n,t}$ Relative difference between optimal expected revenue of two models. 35

$\alpha_j$ Price dependent cost for selling the $j$-th item ($j < 0$ represents overselling costs). 44

$r_j$ Price independent cost for selling the $j$-th item ($j < 0$ represents overselling costs). 44

$v^C_n(t;\mathbf{q})$ Optimal expected revenue when customers who want more items than are available buy all the remaining inventory. 44

$m$ Minimum inventory level. 45

$v^G_n(t;\mathbf{q})$ Optimal expected revenue when customers who want more items than are available buy no items instead. 45

$\mu^C_n(\mathbf{q})$ Modified average order size for $v^G_n(t;\mathbf{q})$, $\mu^C_n(\mathbf{q}) := \sum_{i=1}^{M}\left(q_i i - \sum_{j=n-i+1}^{n} q_i\alpha_j\right)$. 48

$\mu^G_n(\mathbf{q})$ Modified average order size for $v^G_n(t;\mathbf{q})$, $\mu^C_n(\mathbf{q}) := \sum_{i=1}^{M\wedge(n-m)}\left(q_i i - \sum_{j=n-i+1}^{n} q_i\alpha_j\right)$. 48

$p^{C*}_n(t;\mathbf{q})$ Optimal pricing strategy for $v^C_n(t;\mathbf{q})$. 49

$p^{G*}_n(t;\mathbf{q})$ Optimal pricing strategy for $v^G_n(t;\mathbf{q})$. 49

$\beta_n(\mathbf{q},B)$ A $\beta$-type sequence with finite sequence $B$ as its base cases. 51

$S_n(t;\mathbf{q})$ Social value. 56

$w$ Advertising rate. 58

$k(t)$ Represents a tax or subsidy on advertising such that $f(t)w$ is the money spent on advertising which obtains an effective advertising rate $w$. 58

$r(t)$ Instantaneous discount rate at time $t$. 59

$R(t)$ Accumulated discount rate, $R(t) := \int_0^{\delta t} r(s)ds$. 59

$v_n^A$ Optimal expected revenue when advertising and discounting effects are included. 60

$\delta$ Advertising elasticity. 61

$\gamma$ Joint advertising and demand elasticity, $\gamma := \frac{\varepsilon-1}{1-\delta}$. 63

$\eta$ The term $\eta := \left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{1-\delta}} \left(\frac{\varepsilon-\delta}{\varepsilon}\right)^{\gamma}$. 63

$g(t)$ The term $g(t) := \left(\frac{a(t)f(t)^{\varepsilon}}{k(t)^{\delta}}\right)^{\frac{1}{1-\delta}}$. 63

$\zeta(t)$ The term $\zeta(t) := g(t)\left(\frac{\eta}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1}$. 63

$A^{Ad}(t)$ Defined as $A^{Ad}(t) := e^{\gamma R(t)} \int_t^T e^{-\gamma R(s)} g(s)ds$. 64

$\beta_n^A(\mathbf{q})$ The $\beta$-type sequence which is useful when considering advertising and discounting. 64

$\mathscr{M}_{\mathbf{q},\lambda}^A$ A Poisson based which has arrival rate $\lambda$, customer order size distribution $\mathbf{q}$, and accounts for advertising and discounting. 67

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# 1.

# Introduction

## 1.1 Overview

Dynamic pricing concerns sellers who attempt to maximize their profits by adjusting prices over time. The company develops a rule, or pricing strategy, which takes into account market conditions. Take as example the airline industry, where ticket prices are frequently updated based on factors like how much time is left until a flight and how many seats have already been sold. There are many other factors that could influence pricing, although this dissertation focuses on limited time to sell and limited inventory, as these two factors are enough to create interesting models. Some other industries which care deeply about these factors include event ticket sales, fashion, and hotels.

By setting prices, sellers influence the demand for their goods, and more importantly, the amount of revenue they earn. In order to maximize this revenue, a model is needed for sales. Customer arrivals are stochastic and independent of one another, so a Poisson based process is the natural choice for this model. The arrivals are also dependent on the price $p$ and time $t$. Thus we let $\lambda(p,t)$ be the intensity of the Poisson based process to reflect these dependencies. We will focus on the case of constant demand elasticity, which necessitates $\lambda(p,t) = a(t)p^{-\varepsilon}$, where $\varepsilon$ is the demand elasticity and $a(t)$ is an arrival rate scale factor. Demand elasticity is a measure of how much the relative change in quantity demanded changes with respect to a relative change in price. It is an important measure in economics to understand how prices affect sales and helps indicate if items are necessities or luxuries.

Optimal prices are found numerically in many contexts by solving the Hamilton-Jacobi-Bellman (HJB) equations; however, this equation can be difficult to solve in general. The setting of constant demand elasticity offers a tractable example with analytic results, which then lead to deeper insights on the problem itself. This setting has been explored by (McAfee and te Velde 2008) and expanded upon by (Helmes and Schlosser 2013). Compared to these previous works, the key difference for our work is that customers can order multiple items at a time. In terms of the model, that means a compound Poisson process is used instead of a regular Poisson process. To distinguish between these ideas, the compound Poisson setting will be referred to as "variable order sizes," while the regular Poisson setting will be referred to as "unit order sizes."

We now outline the structure of this dissertation. Section 1.2 provides a literature review, and highlights papers related to various aspects of our model. Section 1.3 explains the dynamic programming formulation which is used to determine optimal prices. Here, the formulation is presented in a general context and is not tied to constant demand elasticity.

In Chapter 2 we focus on the basic model of constant demand elasticity and variable order sizes, and this section contains the heart of our results. We find closed form expressions for the optimal expected revenue and pricing strategy. A key insight is that this term involved the average order size $\mu$, which is not observed under the unit order case where $\mu = 1$.

In Section 2.2, we turn to comparing constant elasticity models which have the same demand (average rate of sales over time), yet different order size distributions, calling such models comparable. Under constant demand elasticity, we show that as the size of the inventory tends to $\infty$, comparable models have asymptotically equivalent optimal expected revenue and optimal pricing strategies. This means, that to some degree, unit order models may be used to approximate variable order models. For low inventory, these differences can be quite large from a revenue management standpoint. Section 2.3 provides numerical results to greater analyze aspects like the convergence rate. We also show the surprising result that the relative difference between comparable models is independent of both time and magnitude of demand; the relative difference is completely determined by the inventory level and the two order size distributions between the comparable models.

Chapter 3 discusses important extensions to the basic problem, which improve the flexibility of the model. Section 3.1 discusses the idea of overselling, like an airline overbooking their seats. Determining how to handle low inventory is a problem which must be addressed for variable order sizes, and this section presents a general method for handling these cases. Section 3.2 confirms the result of the unit order case that the monopolist pricing scheme is socially efficient. Section 3.3 introduces advertising and other factors, such as subsidies, into the model. This translates to more involved formulas, but does not fundamentally alter any methods used for the basic model. Lastly, Section 3.4 examines how variable order sizes affect the optimal pricing problem for exponential and linear customer arrival rates.

Lastly, an Appendix includes the Mathematica code which was used in the numerical parts of the dissertation. In particular, functions are provided which can numerically compute optimal expected pricing strategies and revenue for any type of customer arrival rate function. These algorithms may be helpful to anyone wishing to do any further work on the subject of variable order sizes.

## 1.2 Literature Review

Our work is most closely related to that of (McAfee and te Velde 2008) and (Helmes and Schlosser 2013), who both explore the problem of optimal pricing in continuous time with constant demand elasticity. (McAfee and te Velde 2008) explored the basic model and (Helmes and Schlosser 2013) expanded on their work by including advertising and other factors like discounts and subsidies. In both papers, analytic formulas for the optimal pricing strategy are found. These papers also discuss monopolist pricing strategies. With no advertising, a monopolist's pricing is shown to be efficient, and with advertising a parameter adjustment (like a subsidy) can also ensure this. These papers also show that, in general, waiting as a customer is not a beneficial strategy.

A key feature of these papers and our study is that the specific arrival rate $\lambda(p,t) = a(t)p^{-\varepsilon}$ allows for analytic solutions to be found. Indeed, finding analytic solutions for general $\lambda$ is difficult primarily because $\lambda$ may be complicated in general, and even if $\lambda$ is known, there is not a general analytic solution technique which can be applied in all cases. At the very least, for $\lambda(p,t) = e^{-p}$, an analytic solution for the optimal pricing strategy is shown. This was shown in the seminal paper (Gallego and van Ryzin 1994), which is a widely cited article that created general theory and foundations for dynamic pricing with Poisson models. A wider overview of dynamic pricing models is provided by (Talluri and van Ryzin 2004). This book also contains useful applications of dynamic pricing in the context of specific industries.

The paper by (Monahan, Petruzzi, and Zhao 2004) also considers a dynamic pricing model with constant demand elasticity; however, their model evolves in discrete time. Under the discrete setup, what matters is how many items are sold per time period. This shifts the focus away from customer arrivals, making it significantly different relative to the considerations in our study. (Chung and Flynn 2011) expands upon the discrete time model by introducing holding costs.

At this point we make a note about terminology in the literature. "Demand" is often used interchangeably to refer to the customer arrival rate $\lambda$ and the rate of sales $\lambda\mu$, where $\mu$ is the average order size. This is not a problem for unit order sizes, where $\mu = 1$ and so demand and the arrival rate are equal; however, it is important to retain this distinction when working with variable order sizes. We have elected to refer to our model as a "variable order size" model to keep the focus on the behavior of customers. Similar generalizations have been made in other contexts and are sometimes referred to as "batch demand," "random order rates," or "compound Poisson demand." See (Lin, Lu, and Yao 2008), (Elmaghraby and Keskinocak 2008), and (Xu, Yao, and Zheng 2011) for some examples. It is worth noting that in these scenarios when variable order sizes are applied, generally only numerical results are obtained.

Variable order size models also have connections with multiproduct dynamic pricing models. (Gallego and van Ryzin 1997) provides a good reference for this model, and the later paper (Maglaras and Meissner 2006) expands upon their model. Both models consider multiple products which require the same resource. An interesting observation from these papers, although treated briefly, is that a variable order size model can be made equivalent to a multiproduct model by considering each order size level as a different type of product. This similarity is not completely surprising when comparing the optimization equations used: both have a similar form due to a summation term required to handle different types of sales.

For the classical models in operations research, the probability distribution of customer order sizes is unaffected by the price. This is a reasonable assumption for industries like airlines or hotels, where a group's size is essentially fixed before they make a purchase. Therefore, changes in price do not affect the distribution of the number of items purchased. In other scenarios, this assumption may not hold. For example, a particularly good deal on food is likely to induce people to buy more items at a time than they normally would. Discounts for bulk purchasing is also a very common pricing strategy. This practice is called nonlinear pricing, see (Wilson 1993).

The next section shows that low inventory behavior must be addressed to work with a variable order size problem. One approach to handling low inventory is to allow overselling or overbooking. For example, an airline can sell more seats than are actually available on a flight. This is a common practice in the airline industries, as overbooking is a great way for companies to earn extra revenue in spite of cancellations or no sales. Empty spaces on flights essentially represent lost revenue. Of course, overbooking needs to be balanced with the risk of having too many people for a flight, in which case the airline can offer compensation in the form of upgraded seats or flight discounts. There are many papers which explore the topic of overbooking, see (Kunnumkal and Topaloglu 2009) and (Bertsimas and Popescu 2003) for a couple.

## 1.3 Dynamic Programming Formulation

We formulate the basic version of the problem. Later, in Chapter 3 we examine extensions. A seller has $n$ items to sell over the time period $[0, T]$, with maximum time $0 < T < \infty$. Items have no value after the sales period. The customer arrival rate $\lambda(p, t)$ is assumed to be known and depends on price $p \geq 0$ and time $0 \leq t \leq T$. In practice, $\lambda$ is obtained either from vast amounts of historical data, or more recent data that sellers have gathered. The seller wants to determine the optimal price to use for every time and inventory level which maximizes their total expected revenue. This is known as a pricing strategy.

For now, the specific form of $\lambda(p, t)$ is unspecified, as the general model formulation does not depend on the specific $\lambda$. Customer arrivals are modeled by a random time change of a compound Poisson process with intensity $\lambda(p, t)$. Random time change refers to the fact that price $p$ is stochastic in time, as $p$ is set by the seller and will be adjusted based on the specific realization of the process. When a customer arrives, the number of items they order is governed by a probability distribution $\mathbf{q} = (q_1, \ldots, q_M)$, where $q_i$, $1 \leq i \leq M$, is the probability that the customer orders $i$ items and $M$ is the maximum order size (which can be arbitrarily large, if desired). $\mathbf{q}$ is assumed to not depend on price, meaning we are focused on models for industries like airlines and hotels where groups tend to stay together.

Let $v_n(t; \mathbf{q}, \lambda)$ be the optimal expected revenue with $n$ items to sell, at current time $0 \leq t \leq T$, under customer order size distribution $\mathbf{q}$ and customer arrival function $\lambda$. The terms $\mathbf{q}$ and $\lambda$ are after a semicolon to indicate that these features are part of the model setup. Such notation will be useful for comparing models in the future.

The optimal expected revenue has natural boundary conditions: $v_n(T; \mathbf{q}, \lambda) = 0$ (time for selling is done) and $v_0(t; \mathbf{q}, \lambda) = 0$ (no inventory left to sell). One can write a formula for $v_n(t; \mathbf{q}, \lambda)$ by using dynamic programming, also known as the Principle of Optimality. The principle states that a problem can be divided into several subproblems: at any moment make an optimal choice, and then proceed optimally from that point on.

Formally, let $\delta t > 0$ be a small time interval. Since the sales process is Poisson based, $0 < i \leq M$ items are sold with probability $q_i \lambda(p, t)\delta t$ over the period $\delta t$, with a profit of $ip$ earned. 0 items are sold with probability $1 - \lambda(p, t)\delta t$. In either case, to find an optimal policy, we then proceed optimally with the new inventory and at the new time. Thus the Principle of Optimality states

$$v_n(t; \mathbf{q}, \lambda) = \sup_p \Bigg[ \underbrace{(1 - \lambda(p, t)\delta t) v_n(t + \delta t; \mathbf{q}, \lambda)}_{\text{From selling no items}} + \sum_{i=1}^{M} \underbrace{q_i \lambda(p, t)\delta t (ip + v_{n-i}(t + \delta t; \mathbf{q}, \lambda))}_{\text{From selling } i \text{ items for } p \text{ each}} \Bigg]. \tag{1.3.1}$$

Next, we heuristically derive the Hamilton-Jacobi-Bellman equations. More details to validate the heuristic derivation of (1.3.2) are provided for more general counting processes in (Bremaud 1981). We rearrange (1.3.1) so that the left hand side is $(v_n(t + \delta t; \mathbf{q}, \lambda) - v_n(t; \mathbf{q}, \lambda))/\delta t$ and then take the limit of the equation as $\delta t \to 0$ to get:

$$\begin{aligned} v_n'(t; \mathbf{q}, \lambda) &= -\sup_p \lambda(p, t) \left[ -v_n(t; \mathbf{q}, \lambda) + \sum_{i=1}^{M} q_i (ip + v_{n-i}(t; \mathbf{q}, \lambda)) \right]. \\ &= -\sup_p \lambda(p, t) \left[ \mu(\mathbf{q}) p - \left( v_n(t; \mathbf{q}, \lambda) - \sum_{i=1}^{M} q_i v_{n-i}(t; \mathbf{q}, \lambda) \right) \right], \end{aligned} \tag{1.3.2}$$

where $\mu(\mathbf{q}) := \sum_{i=1}^{M} q_i i$ is the average order size of the order size distribution $\mathbf{q}$. The Principle of Optimality states that a pricing policy $p_n(t)$ which satisfies the supremum in (1.3.2) for all valid inventory $n > 0$ and time $0 \leq t < T$ is an optimal pricing policy.

The recursive elements in (1.3.1) and (1.3.2) show that $M$ base case terms of $v_n$ are needed for these equations to be well defined. The simplest choice is to define $v_n(t; \mathbf{q}, \lambda) = 0$ for $n < 0$. Under this assumption, (1.3.1) models overselling with no penalty costs, since items can be sold to go into negative inventory. The practice of overselling actually common in the airline industry, where some passengers are expected to miss their flights. Of course, there is a risk to do so, as a plane might be overfull and compensation needs to be given out.

Without penalty costs, the current behavior of the model may feel a bit limited. We note that the results in Chapter 2 are primarily recursive in nature, so the behavior at low inventory often plays an unimportant role, as the eventually recursion drives the results. This will formally be explored in Section 3.1. So for now we will assume $v_n(q; \mathbf{q}, \lambda) = 0$ when $n < 0$ for ease of exposition.

# 2.

# Main Results

In this chapter, constant demand elasticity $\varepsilon$ is now assumed. As will be shown, this property requires that $\lambda(p,t) = a(t)p^{-\varepsilon}$. This chapter has three parts. Section 2.1 proves a closed form solution for the optimal expected revenue. Its asymptotic behavior is then analyzed, the results of which are used in Section 2.2, which focuses on comparable models. These are models with the same demand, but different order size distributions. Comparable models are found to have the same asymptotic behavior in the inventory size. A surprising result is that the relative difference between comparable models does not depend on time or the magnitude of demand. Finally, Section 2.3 shows some numerical calculations in order to obtain a better understanding of the model as a whole.

## 2.1 Analytic Results for the Basic Model

Demand $Q$ is the rate of item sales, which is calculated by multiplying the rate of customer arrivals $\lambda(p,t)$ by a customer's average order size $\mu$, that is $Q(p,t) := \lambda(p,t)\mu(\mathbf{q})$. Demand elasticity $\varepsilon$ measures how sensitive that demand is to price fluctuations. $\varepsilon$ is a measure of the relative change in demand compared the relative change in price, taken in absolute value, thus

$$\varepsilon := \left|\frac{dQ/Q}{dp/p}\right| = -\frac{p\,(dQ/dp)}{Q} = -\frac{p\,(d\lambda/dp)}{\lambda}. \tag{2.1.1}$$

Note that the evaluation of the absolute value is justified because for all practical purposes, quantity demanded and price are inversely proportional, making $(dQ/Q)/(dp/p) \leq 0$. Since constant $\varepsilon$ is assumed, (2.1.1) can be solved for $\lambda$ through separation of variables. This yields $\lambda(p,t) = a(t)p^{-\varepsilon}$ for some time-dependent function $a(t) > 0$, which is an arrival rate scaling factor. An interesting feature of this arrival rate $\lambda$ form is that it can be made arbitrarily large by taking $p \to 0$, thus ensuring all inventory can always be sold.

For this chapter, $\lambda$ is always of the form $a(t)p^{-\varepsilon}$, so write $v_n(t;\mathbf{q}) = v_n(t;\mathbf{q},\lambda)$ for clarity. Substituting $\lambda(p,t) = a(t)p^{-\varepsilon}$ into (1.3.2) yields

$$\begin{aligned} v_n'(t;\mathbf{q}) &= -\sup_p a(t)p^{-\varepsilon}\left[-v_n(t;\mathbf{q}) + \sum_{i=1}^{M} q_i(ip + v_{n-i}(t;\mathbf{q}))\right] \\ &= -\sup_p a(t)p^{-\varepsilon}\left[\mu(\mathbf{q})p - \left(v_n(t;\mathbf{q}) - \sum_{i=1}^{M} q_i v_{n-i}(t;\mathbf{q})\right)\right]. \end{aligned} \tag{2.1.2}$$

Using standard calculus, the price $p^*$ which attains the supremum in (2.1.2) is found to be

$$p_n^*(t;\mathbf{q}) = \frac{\varepsilon}{\varepsilon - 1}\mu(\mathbf{q})^{-1}\left(v_n(t;\mathbf{q}) - \sum_{i=1}^{M} q_i v_{n-i}(t;\mathbf{q})\right). \tag{2.1.3}$$

An immediate consequence of this formula is that $\varepsilon > 1$, since the solutions for $\varepsilon \leq 1$ would make no practical sense. From an economic standpoint this also makes sense, since goods with $\varepsilon \leq 1$ are not considered elastic goods. That means that price has little influence on their demand, meaning that dynamic pricing is of little use. With pricing strategy (2.1.3),

$$v_n'(t;\mathbf{q}) = -a(t)\mu(\mathbf{q})^{\varepsilon}\frac{(\varepsilon - 1)^{\varepsilon - 1}}{\varepsilon^{\varepsilon}}\left(v_n(t;\mathbf{q}) - \sum_{i=1}^{M} q_i v_{n-i}(t;\mathbf{q})\right)^{1-\varepsilon}. \tag{2.1.4}$$

Before proceeding, some notation is needed. Their meaning and usefulness will become more apparent through the proofs. Let $A(t) := \int_t^T a(s)ds$, which can be thought of as the expected number of future customer arrivals at a price of 1. Also let $(\beta_n(\mathbf{q}))_n$ be the sequence given by $\beta_n(\mathbf{q}) = 0$ for $n \leq 0$ and such that for $n > 0$, $\beta_n(\mathbf{q})$ is the non-negative solution to

$$\frac{\varepsilon - 1}{\varepsilon} = \beta_n(\mathbf{q})^{\frac{1}{\varepsilon-1}} \left( \beta_n(\mathbf{q}) - \sum_{i=1}^{M} q_i \beta_{n-i}(\mathbf{q}) \right). \tag{2.1.5}$$

Equation (2.1.5) will be helpful because it mirrors some structure in (2.1.4). Note that analytically finding $\beta_n(\mathbf{q})$ would be cumbersome if not impossible, but can be computed numerically rather simply. The existence and uniqueness of a solution to (2.1.5) is given by the following lemma:

**Lemma 2.1.1.** *Let $f_{b,c}(x) := x^{\frac{1}{\varepsilon-1}}(bx - c) = bx^{\frac{\varepsilon}{\varepsilon-1}} - cx^{\frac{1}{\varepsilon-1}}$ for any constants $b > 0$ and $c$. Then there exists a unique positive solution to $f_{b,c}(x) = \frac{\varepsilon-1}{\varepsilon}$.*

*Proof.* Let $b > 0$ and $c$ be constants. Then $f_{b,c}(x) < 0$ for $0 < x < c/b$, $f_{b,c}(x) = 0$ at $x = c/b$, and $f_{b,c}(x) > 0$ and increasing for $x > c/b$. Therefore $f_{b,c}(x) = \frac{\varepsilon-1}{\varepsilon}$ has a unique positive solution. □

The lemma applies to (2.1.5) because when computing $\beta_{n+1}(\mathbf{q})$, all previous $\beta_i(\mathbf{q})$ for $i < n + 1$ are known. Therefore we can think of (2.1.5) as the equation $\frac{\varepsilon-1}{\varepsilon} = f_{b,c}(\beta_{n+1}(\mathbf{q}))$, where $b = 1$ and $c = \sum_{i=1}^{M} q_i \beta_{n-i}(\mathbf{q})$. The lemma is simple, but will be a useful reference in Section 3.1, where new types of sequences like $\beta_n(\mathbf{q})$ are examined. Now we present the optimal pricing policy.

**Theorem 2.1.2.** *With arrival rate $\lambda(p, t) = a(t)p^{-\varepsilon}$ and customer order sizes $\boldsymbol{q}$, the optimal expected revenue is*

$$v_n(t; \boldsymbol{q}) = \mu(\boldsymbol{q})\beta_n(\boldsymbol{q})A(t)^{1/\varepsilon}. \tag{2.1.6}$$

*Furthermore, the optimal pricing strategy is*

$$p_n^*(t; \boldsymbol{q}) = \beta_n(\boldsymbol{q})^{-1/(\varepsilon-1)}A(t)^{1/\varepsilon}.$$

*These two equations hold for all integer inventory levels $n$, and for all times $0 \leq t < T$.*

*Proof.* Proceed by induction to show (2.1.6) holds. For $n \le 0$, by definition, $\beta_n(\mathbf{q}) = 0$ and $v_n(t; \mathbf{q}) = 0$, showing that (2.1.6) holds. Recall Equation (2.1.4),

$$v'_{n+1}(t;\mathbf{q}) = -a(t)\mu(\mathbf{q})^{\varepsilon}\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\left(v_{n+1}(t;\mathbf{q}) - \sum_{i=1}^{M} q_i v_{n+1-i}(t;\mathbf{q}))\right)^{1-\varepsilon}. \tag{2.1.7}$$

This is an ordinary differential equation, so verifying that the induction hypothesis holds for this equation will complete the induction. Note that throughout this proof $t$ and $\mathbf{q}$ dependencies are often suppressed for readability. Assume Equation (2.1.6) holds up to $n$. Then (2.1.7) becomes

$$v'_{n+1} = -a\mu^{\varepsilon}\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\left(v_{n+1} - \sum_{i=1}^{M} q_i \mu \beta_{n+1-i} A^{1/\varepsilon}\right)^{1-\varepsilon}. \tag{2.1.8}$$

To prove that the induction assumption holds for $n+1$, we substitute the induction assumption for $n+1$ into (2.1.8) and show that the equality remains true. Substituting the induction assumption for $n+1$ into the left-hand side of (2.1.8) yields

$$\begin{aligned} v'_{n+1} &= \mu\beta_{n+1}\frac{1}{\varepsilon}A^{(1/\varepsilon)-1}\frac{dA}{dt} \\ &= -\frac{a\mu}{\varepsilon}A^{(1-\varepsilon)/\varepsilon}\beta_{n+1}. \end{aligned}$$

Substituting the induction assumption (2.1.6) into the right-hand side of (2.1.8) gives

$$\begin{aligned} &-a\mu^{\varepsilon}\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\left(\mu\beta_{n+1}A^{1/\varepsilon} - \sum_{i=1}^{M} q_i\mu\beta_{n+1-i}A^{1/\varepsilon}\right)^{1-\varepsilon} \\ &= -a\mu\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}A^{(1-\varepsilon)/\varepsilon}\left(\beta_{n+1} - \sum_{i=1}^{M} q_i\beta_{n+1-i}\right)^{1-\varepsilon} \\ &= -a\mu A^{(1-\varepsilon)/\varepsilon}\frac{1}{\varepsilon}\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon-1}\left(\beta_{n+1} - \sum_{i=1}^{M} q_i\beta_{n+1-i}\right)^{1-\varepsilon} \\ &= -\frac{a\mu}{\varepsilon}A^{(1-\varepsilon)/\varepsilon}\beta_{n+1}, \end{aligned}$$

where the last equality is justified by the definition of $\beta_n(\mathbf{q})$, equation (2.1.5). This shows that the left- and right-hand sides of (2.1.8) are equal for the induction assumption at $n+1$, thus verifying the equation

$$v_n(t;\mathbf{q}) = \mu(\mathbf{q})\beta_n(\mathbf{q})A(t)^{1/\varepsilon}$$

holds for all $n$. Furthermore, by substituting that equation into (2.1.3) gives

$$\begin{aligned}
p_n^*(t;\mathbf{q}) &= \frac{\varepsilon}{\varepsilon-1}\mu(\mathbf{q})^{-1}\left(\mu(\mathbf{q})\beta_n(\mathbf{q})A(t)^{1/\varepsilon} - \sum_{i=1}^{M} q_i\mu(\mathbf{q})\beta_{n-i}(\mathbf{q})A(t)^{1/\varepsilon}\right) \\
&= A(t)^{1/\varepsilon}\frac{\varepsilon}{\varepsilon-1}\left(\beta_n(\mathbf{q}) - \sum_{i=1}^{M} q_i\beta_{n-i}(\mathbf{q})\right) \\
&= A(t)^{1/\varepsilon}\left(\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon-1}\left(\beta_n(\mathbf{q}) - \sum_{i=1}^{M} q_i\beta_{n-i}(\mathbf{q})\right)^{1-\varepsilon}\right)^{-1/(\varepsilon-1)} \\
&= A(t)^{1/\varepsilon}\beta_n(\mathbf{q})^{-1/(\varepsilon-1)},
\end{aligned}$$

finishing the proof. □

Equation (2.1.6) reveals the term $\mu(\mathbf{q})$, which was not observed for the unit order case shown in (McAfee and te Velde 2008), where $\mu = 1$. The equation appears to match our intuition that as the average order size $\mu(\mathbf{q})$ increases, so does our optimal expected revenue; however, it is important to note that as $\mathbf{q}$ changes, so does $\beta_n(\mathbf{q})$. Therefore, more analysis of $\beta_n(\mathbf{q})$ is necessary to truly understand (2.1.6).

**Lemma 2.1.3.** *$\beta_n(\boldsymbol{q})$ is a non-decreasing sequence in $n$.*

*Proof.* For $n \leq 0$, $\beta_n(\mathbf{q}) = 0$. Proceed by induction and assume $\beta_{k-1}(\mathbf{q}) \leq \beta_k(\mathbf{q})$ for all $k < n-1$. Recall (2.1.5), the recursion equation for $\beta_n(\mathbf{q})$:

$$\frac{\varepsilon-1}{\varepsilon} = \beta_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}\left(\beta_n(\mathbf{q}) - \sum_{i=1}^{M} q_i\beta_{n-i}(\mathbf{q})\right),$$

or multiplied out as

$$\frac{\varepsilon - 1}{\varepsilon} = \beta_n(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-1}} - \sum_{i=1}^{M} q_i \beta_{n-i}(\mathbf{q}) \beta_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}. \tag{2.1.9}$$

Solving for $\beta_n(\mathbf{q})$ happens recursively. So in solving (2.1.5) for $\beta_n(\mathbf{q})$, it is assumed for any $i < n$ that $\beta_i(\mathbf{q})$ would be known. Thus we treat the right hand side of (2.1.9) as a function of the variable $\beta_n(\mathbf{q})$. That is, we can think of the function

$$f(x) = x^{\frac{\varepsilon}{\varepsilon-1}} - \sum_{i=1}^{M} q_i \beta_{n-i}(\mathbf{q}) x^{\frac{1}{\varepsilon-1}},$$

where finding $x > 0$ such that $\frac{\varepsilon-1}{\varepsilon} = f(x)$ is equivalent to solving for $\beta_n(\mathbf{q})$. Similarly, writing

$$g(x) = x^{\frac{\varepsilon}{\varepsilon-1}} - \sum_{i=1}^{M} q_i \beta_{(n-1)-i}(\mathbf{q}) x^{\frac{1}{\varepsilon-1}}$$

and solving for $x > 0$ such that $\frac{\varepsilon-1}{\varepsilon} = g(x)$ would give the value of $\beta_{n-1}(\mathbf{q})$. Lemma 2.1.1 shows that the positive solutions for $\frac{\varepsilon-1}{\varepsilon} = f(x)$ and $\frac{\varepsilon-1}{\varepsilon} = g(x)$ exist and are unique.

Next we show the solution to $\frac{\varepsilon-1}{\varepsilon} = f(x)$ is greater than or equal to the solution of $\frac{\varepsilon-1}{\varepsilon} = g(x)$. Compare the coefficients of $f(x)$ and $g(x)$. The induction hypothesis shows that the coefficients of $f(x)$ are less than or equal to those in $g(x)$. Therefore, for positive $x$, $f(x) \leq g(x)$. Thus the positive solution to $f(x) = \frac{\varepsilon-1}{\varepsilon}$ must be greater than or equal to the positive solution to $g(x) = \frac{\varepsilon-1}{\varepsilon}$, or equivalently, $\beta_n(\mathbf{q}) \geq \beta_{n-1}(\mathbf{q})$. This completes the induction proof, proving that $\beta_n(\mathbf{q})$ is a non-decreasing sequence in $n$. $\square$

This lemma confirms the intuition that the optimal expected revenue $v_n(t; \mathbf{q})$ should increase as the inventory $n$ increases. For an example why $\beta_n(\mathbf{q})$ is not always strictly increasing, consider the order size distribution $\mathbf{q} = (0, 1)$. That is, customers buy 1 item with probability 0 and 2 items with probability 1. In this case, the recursion equation (2.1.5) is the same for $n = 1$ and $n = 2$, $\beta_1(\mathbf{q}) = \beta_2(\mathbf{q})$. The lack of a strictly increasing property for $\beta_n(\mathbf{q})$ is one of the reasons that analysis of $\beta_n(\mathbf{q})$ is more difficult for variable orders compared to the unit order case.

The next step is to look at $\beta_n(\mathbf{q})$ as $n \to \infty$. As it turns out, $\beta_n(\mathbf{q})$ is asymptotically equivalent to $n^{\frac{\varepsilon-1}{\varepsilon}}$, up to a scale factor based on $\mu(\mathbf{q})$. So we define $\gamma_n(\mathbf{q}) = \frac{\beta_n(\mathbf{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}}$ to help simplify notation. For the unit order case, $\gamma_n(\mathbf{q})$ has monotonicity and a simple upper bound which makes finding its limit relatively straightforward. Unfortunately, when generalizing to variable order sizes, these properties are not true for all $\mathbf{q}$ and so the proof of the limiting behavior of $\gamma_n(\mathbf{q})$ is more difficult. Before finding $\lim_{n\to\infty} \gamma_n(\mathbf{q})$, several lemmas are needed to help identify the asymptotic behavior of $\gamma_n(\mathbf{q})$.

The first lemma contains a function whose recursive structure is similar to that of $\beta_n(\mathbf{q})$. The asymptotic properties of this function are therefore useful in examining the asymptotic properties of $\beta_n(\mathbf{q})$.

**Lemma 2.1.4.** *The function*

$$f(n; \boldsymbol{q}) := n \left( 1 - \sum_{i=1}^{M} q_i \left( \frac{n-i}{n} \right)^{\frac{\varepsilon-1}{\varepsilon}} \right)$$

*is decreasing for $n > M$, $n \in \mathbb{R}$. Moreover,*

$$\lim_{n\to\infty} f(n; \boldsymbol{q}) = \mu(\boldsymbol{q}) \left( \frac{\varepsilon - 1}{\varepsilon} \right).$$

*Proof.* Let $n > M$, $n \in \mathbb{R}$. First we show that $f$ has a limiting value:

$$\begin{aligned}
\lim_{n\to\infty} f(n;\mathbf{q}) &= \lim_{n\to\infty} \frac{1-\sum_{i=1}^{M} q_i \left(\frac{n-i}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}}}{1/n} \\
&\overset{H}{=} \lim_{n\to\infty} \frac{-\sum_{i=1}^{M} q_i \left(\frac{\varepsilon-1}{\varepsilon}\right)\left(\frac{n-i}{n}\right)^{-1/\varepsilon}\left(\frac{i}{n^2}\right)}{-1/n^2} \\
&= \lim_{n\to\infty} \sum_{i=1}^{M} i q_i \left(\frac{\varepsilon-1}{\varepsilon}\right)\left(\frac{n-i}{n}\right)^{-1/\varepsilon} \\
&= \sum_{i=1}^{M} i q_i \left(\frac{\varepsilon-1}{\varepsilon}\right) \\
&= \mu(\mathbf{q})\left(\frac{\varepsilon-1}{\varepsilon}\right)
\end{aligned}$$

Note that $f$ is continuous for $n > M$, so if $f$ is concave up ($f'' > 0$), then $f$ must decrease to the limit value just shown. In other words, to show that $f$ is decreasing for $n > M$, we will show that it is concave up. Start with $f'$, which is

$$\begin{aligned}
f'(n;\mathbf{q}) =& n\left(-\sum_{i=1}^{M} q_i \left(\frac{\varepsilon-1}{\varepsilon}\right)\left(\frac{n-i}{n}\right)^{-1/\varepsilon}\left(\frac{i}{n^2}\right)\right) + \left(1-\sum_{i=1}^{M} q_i \left(\frac{n-i}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right) \\
=& 1-\sum_{i=1}^{M} q_i \left(\left(\frac{\varepsilon-1}{\varepsilon}\right)\left(\frac{n-i}{n}\right)^{-1/\varepsilon}\left(\frac{i}{n}\right) + \left(\frac{n-i}{n}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right) \\
=& 1-\sum_{i=1}^{M} q_i \left(\frac{n-i}{n}\right)^{-1/\varepsilon}\left(\frac{i\varepsilon-i}{n\varepsilon} + \frac{n-i}{n}\right) \\
=& 1-\sum_{i=1}^{M} q_i \left(\frac{n}{n-i}\right)^{1/\varepsilon}\left(1-\frac{i}{n\varepsilon}\right).
\end{aligned}$$

Then

$$
\begin{aligned}
f''(n;\mathbf{q}) &= -\sum_{i=1}^{M} q_i \left[\left(\frac{n}{n-i}\right)^{1/\varepsilon}\left(\frac{i}{n^2\varepsilon}\right)+\left(\frac{1}{\varepsilon}\right)\left(\frac{n}{n-i}\right)^{\frac{1}{\varepsilon}-1}\left(\frac{-i}{(n-i)^2}\right)\left(\frac{n\varepsilon-i}{n\varepsilon}\right)\right] \\
&= -\sum_{i=1}^{M} \frac{iq_i}{\varepsilon}\left(\frac{n}{n-i}\right)^{1/\varepsilon}\left[\frac{1}{n^2}-\left(\frac{n}{n-i}\right)^{-1}\left(\frac{1}{(n-i)^2}\right)\left(\frac{n\varepsilon-i}{n\varepsilon}\right)\right] \\
&= -\sum_{i=1}^{M} \frac{iq_i}{\varepsilon n^2}\left(\frac{n}{n-i}\right)^{1/\varepsilon}\left[1-\frac{n\varepsilon-i}{n\varepsilon-i\varepsilon}\right] \\
&> 0.
\end{aligned}
$$

Note that the inequality is justified since $\varepsilon > 1$ implies that $\left[1-\frac{n\varepsilon-i}{n\varepsilon-i\varepsilon}\right] < 0$. Thus $f$ is concave up, completing the proof of Lemma 2.1.4. $\square$

The next lemma gives conditional bounds for $\gamma_n(\mathbf{q})$, based on properties $\gamma_n(\mathbf{q})$ might satisfy. These properties involve checking whether $\gamma_n(\mathbf{q})$ is the minimum or maximum of itself and the previous $M$ terms in the $\gamma_n$ sequence. It seems that, numerically, $\gamma_n(\mathbf{q})$ satisfies the given properties for large enough inventory $n$. That said, there are challenges with proving this analytically for any $\mathbf{q}$. Fortunately, the proof of $\lim_{n\to\infty}\gamma_n(\mathbf{q})$ in Theorem 2.1.7 will sidestep this issue by considering all cases for $\gamma_n(\mathbf{q})$, whether the term meets has the given properties or not.

**Lemma 2.1.5.** *(a)* $\liminf_{n\to\infty}\gamma_n(\boldsymbol{q}) > 0.$

*(b) If there exists a strictly increasing sequence* $(N_k)_k \subset \mathbb{N}$ *such that* $\gamma_{N_k}(\boldsymbol{q}) = \min_{0\le i\le M}\gamma_{N_k-i}(\boldsymbol{q})$ *for all* $k$*, then* $\liminf_{n\to\infty}\gamma_n(\boldsymbol{q}) \ge \mu(\boldsymbol{q})^{\frac{1-\varepsilon}{\varepsilon}}$.

*(c) If there exists an* $N \ge M$ *such that* $\gamma_N(\boldsymbol{q}) = \max_{0\le i\le M}\gamma_{N-i}(\boldsymbol{q})$*, then* $\limsup_{n\to\infty}\gamma_n(\boldsymbol{q}) \le \mu(\boldsymbol{q})^{\frac{1-\varepsilon}{\varepsilon}}$.

*Proof.* **Proof of part (a):** Recall $\gamma_n(\mathbf{q}) = \beta_n(\mathbf{q})/n^{\frac{\varepsilon-1}{\varepsilon}}$. By definition, $\beta_n(\mathbf{q}) \ge 0$, and therefore $\gamma_n(\mathbf{q}) \ge 0$ and $\liminf_{n\to\infty}\gamma_n(\mathbf{q}) \ge 0$. Assume to the contrary that $\liminf_{n\to\infty}\gamma_n(\mathbf{q}) = 0$. Then we can construct the decreasing subsequence $(\gamma_{N_k}(\mathbf{q}))_k$ of $\gamma_n(\mathbf{q})$ such that $N_1 > M$

and the subsequence contains all $\gamma_{N_k}(\mathbf{q})$ such that $\gamma_{N_k}(\mathbf{q}) = \min_{1 \le i \le N_k} \gamma_i(\mathbf{q})$. In other words, the subsequence contains all terms $\gamma_n$ which are the smallest value seen up to that term. This also gives that $\lim_{k\to\infty} \gamma_{N_k}(\mathbf{q}) = 0$.

Let $\delta > 0$. Then there exists a $k > 0$ such that $\gamma_{N_k}(\mathbf{q}) \le \delta$. Substitute $\gamma_{N_k}(\mathbf{q}) N_k^{\frac{\varepsilon-1}{\varepsilon}} = \beta_{N_k}(\mathbf{q})$ into (2.1.5), noting that the inequality is justified by the construction of $\gamma_{N_k}(\mathbf{q})$,

$$\begin{aligned}
\frac{\varepsilon-1}{\varepsilon} &= \gamma_{N_k}(\mathbf{q})^{\frac{1}{\varepsilon-1}} N_k \left( \gamma_{N_k}(\mathbf{q}) - \sum_{i=1}^{M} q_i \gamma_{N_k - i}(\mathbf{q}) \left( \frac{N_k - i}{N_k} \right)^{\frac{\varepsilon-1}{\varepsilon}} \right) \\
&\le \gamma_{N_k}(\mathbf{q})^{\frac{1}{\varepsilon-1}} N_k \left( \gamma_{N_k}(\mathbf{q}) - \sum_{i=1}^{M} q_i \gamma_{N_k}(\mathbf{q}) \left( \frac{N_k - i}{N_k} \right)^{\frac{\varepsilon-1}{\varepsilon}} \right) \qquad (2.1.10) \\
&= \gamma_{N_k}(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-1}} N_k \left( 1 - \sum_{i=1}^{M} q_i \left( \frac{N_k - i}{N_k} \right)^{\frac{\varepsilon-1}{\varepsilon}} \right) \\
&=: \gamma_{N_k}(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-1}} f(N_k; \mathbf{q}) \qquad (2.1.11)
\end{aligned}$$

By Lemma 2.1.4, $f(n;\mathbf{q})$ is decreasing for $n > M$ and therefore $f(N_k;\mathbf{q}) \le f(M;\mathbf{q})$. Thus from (2.1.11) we get

$$\frac{\varepsilon-1}{\varepsilon} \le \gamma_{N_k}(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-1}} f(M;\mathbf{q}) \le \delta^{\frac{\varepsilon}{\varepsilon-1}} f(M;\mathbf{q}),$$

but this is a contradiction since $f(M;\mathbf{q})$ is a constant while $\delta$ is arbitrary. Therefore $\liminf_{n\to\infty} \gamma_n(\mathbf{q}) \ne 0$, completing the proof of part (a).

**Proof of part (b):** Suppose there exists an increasing sequence $(N_k)_k$ such that $N_1 > M$ and $\gamma_{N_k}(\mathbf{q}) = \min_{0 \le i \le M} \gamma_{N_k - i}(\mathbf{q})$ for all $k$. Assume that $(N_k)_k$ contains every integer $n > M$ such that $\gamma_n(\mathbf{q}) = \min_{0 \le i \le M} \gamma_{n-i}(\mathbf{q})$. First we get a bound for $\liminf_{k\to\infty} \gamma_{N_k}(\mathbf{q})$, then we show

$$\liminf_{k\to\infty} \gamma_{N_k}(\mathbf{q}) = \liminf_{n\to\infty} \gamma_n(\mathbf{q}).$$

Note that $\gamma_{N_k}(\mathbf{q}) = \min_{0 \le i \le M} \gamma_{N_k - i}(\mathbf{q})$ implies that the inequality (2.1.10) holds for all $k$, and thus (2.1.11) holds for all $k$ as well:

$$\frac{\varepsilon-1}{\varepsilon} \leq \gamma_{N_k}(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-1}} f(N_k;\mathbf{q}),$$

or written another way,

$$\gamma_{N_k}(\mathbf{q}) \geq \left(\frac{\varepsilon-1}{\varepsilon f(N_k;\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon}}.$$

Let $\delta > 0$. By Lemma 2.1.4, there exists a $K \geq M$ such that for all $k \geq K$, $f(N_k;\mathbf{q}) \leq \mu(\mathbf{q})\left(\frac{\varepsilon-1}{\varepsilon}\right)(1+\delta)$ and thus

$$\gamma_{N_k}(\mathbf{q}) \geq \left(\frac{1}{\mu(\mathbf{q})(1+\delta)}\right)^{\frac{\varepsilon-1}{\varepsilon}}.$$

Therefore

$$\liminf_{k\to\infty} \gamma_{N_k}(\mathbf{q}) \geq \liminf_{k\to\infty} \left(\frac{1}{\mu(\mathbf{q})(1+\delta)}\right)^{\frac{\varepsilon-1}{\varepsilon}} = \left(\frac{1}{\mu(\mathbf{q})(1+\delta)}\right)^{\frac{\varepsilon-1}{\varepsilon}},$$

but $\delta$ was arbitrary and so

$$\liminf_{k\to\infty} \gamma_{N_k}(\mathbf{q}) \geq \left(\frac{1}{\mu(\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon}} = \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}.$$

With a bound established, we now show that $\liminf\limits_{k\to\infty} \gamma_{N_k}(\mathbf{q}) = \liminf\limits_{n\to\infty} \gamma_n(\mathbf{q})$.

For each $j \geq N_1$, define $a_j$ such that $N_{a_j} \leq j < N_{a_j+1}$. We claim that

$$\gamma_j(\mathbf{q}) \geq \gamma_{N_{a_j}}(\mathbf{q}). \tag{2.1.12}$$

To prove the claim, consider if $j = N_{a_j}$. Then $\gamma_j(\mathbf{q}) = \gamma_{N_{a_j}}(\mathbf{q})$ and the claim is true. If $j > N_{a_j}$, suppose towards contradiction that $\gamma_j(\mathbf{q}) \leq \gamma_{N_{a_j}}(\mathbf{q})$. Without loss of generality assume that $j$ is the smallest value greater than $N_{a_j}$ such that $\gamma_j(\mathbf{q}) \leq \gamma_{N_{a_j}}(\mathbf{q})$ holds. If not, we could just find some value between $N_{a_j}$ and $j$ which is. Then $\gamma_j(\mathbf{q}) = \min\limits_{N_{a_j}-M\leq i\leq j} \gamma_i(\mathbf{q}) =$

$\min_{0 \le i \le M} \gamma_{j-i}(\mathbf{q})$, and so $j \in (N_j)_j$. But also $(N_j)_j$ is an increasing sequence and $j > N_{a_j}$, so $j \ge N_{a_j+1}$. This contradicts the fact that $N_{a_j} \le j < N_{a_j+1}$. Hence $\gamma_j(\mathbf{q}) \ge \gamma_{N_{a_j}}(\mathbf{q})$, proving claim (2.1.12). This equation then shows

$$\liminf_{j \to \infty} \gamma_j(\mathbf{q}) \ge \liminf_{j \to \infty} \gamma_{N_{a_j}}(\mathbf{q}).$$

Also $(N_j)_j$ is a subsequence of $(N_{a_j})_j$, and thus $\liminf_{j \to \infty} \gamma_{N_{a_j}}(\mathbf{q}) \ge \liminf_{j \to \infty} \gamma_{N_j}(\mathbf{q})$. Combining everything together yields

$$\liminf_{j \to \infty} \gamma_j(\mathbf{q}) \ge \liminf_{j \to \infty} \gamma_{N_{a_j}}(\mathbf{q}) \ge \liminf_{j \to \infty} \gamma_{N_j}(\mathbf{q}) \ge \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}},$$

completing the proof of part (b).

**Proof of part (c)** Assume there exists an $N_1$ such that $N_1 > M$ and $\gamma_{N_1}(\mathbf{q}) = \max_{0 \le i \le M} \gamma_{N_1 - i}(\mathbf{q})$. We follow the same development in part (a) for equation (2.1.11), except our assumption now reverses inequality (2.1.10), thus

$$\frac{\varepsilon - 1}{\varepsilon} \ge \gamma_{N_1}(\mathbf{q})^{\frac{\varepsilon}{\varepsilon - 1}} f(N_1; \mathbf{q}).$$

Use the fact from Lemma 2.1.4 that $f$ is decreasing and $\lim_{n \to \infty} f(n; \mathbf{q}) = \mu(\mathbf{q}) \left(\frac{\varepsilon - 1}{\varepsilon}\right)$ to get

$$\frac{\varepsilon - 1}{\varepsilon} \ge \gamma_{N_1}(\mathbf{q})^{\frac{\varepsilon}{\varepsilon - 1}} \mu(\mathbf{q}) \left(\frac{\varepsilon - 1}{\varepsilon}\right),$$

or

$$\mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}} \ge \gamma_{N_1}(\mathbf{q}).$$

Now suppose $N_2$ is the smallest value greater than $N_1$ such that $\gamma_{N_2}(\mathbf{q}) > \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}$. Then $\gamma_{N_2}(\mathbf{q}) = \max_{N_1 - M \le i \le N_2} \gamma_i(\mathbf{q}) = \max_{0 \le i \le M} \gamma_{N_2 - i}(\mathbf{q})$. But by the same argument for $N_1$, $\mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}} \ge \gamma_{N_1}(\mathbf{q})$, a contradiction. Hence no such $N_2$ exists and $\gamma_n(\mathbf{q}) \le \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}$ for all $n > N_1$, completing the proof. $\square$

The next lemma relates $\limsup \gamma_n(\mathbf{q})$ and $\liminf \gamma_n(\mathbf{q})$. These relations are helpful with the conditional bounds from Lemma 2.1.5, as we will be able to obtain equalities instead of inequalities. This proof is the most involved proof of the dissertation, and illustrates how variable order sizes can complicate the basic problem. In this case, the complication comes from not being able to prove that $\gamma_n(\mathbf{q})$ actually has specific properties for all $n$.

**Lemma 2.1.6.** *(a)*

$$\frac{1}{\limsup_{n\to\infty} \gamma_n(\boldsymbol{q})^{\frac{1}{\varepsilon-1}}} \le \mu(\boldsymbol{q}) \liminf_{n\to\infty} \gamma_n(\boldsymbol{q}).$$

*(b)*

$$\frac{1}{\liminf_{n\to\infty} \gamma_n(\boldsymbol{q})^{\frac{1}{\varepsilon-1}}} \ge \mu(\boldsymbol{q}) \limsup_{n\to\infty} \gamma_n(\boldsymbol{q}).$$

*Proof.* **Proof of part (a)** The idea of this proof is to start with the recursive equation for $\beta_n(\mathbf{q})$ in (2.1.5). From there we significantly alter the look of the equation in order to write it using the successive difference terms $\Delta\beta_n(\mathbf{q}) := \beta_n(\mathbf{q}) - \beta_{n-1}(\mathbf{q})$. These successive differences will prove to be useful due to the fact that they have a convenient bound which we now show. Since the $q_i$ form a probability distribution, $\sum_{i=1}^M q_i = 1$ and thus

$$\Delta\beta_n(\mathbf{q}) = \beta_n(\mathbf{q}) - \beta_{n-1}(\mathbf{q}) = \beta_n(\mathbf{q}) - \sum_{i=1}^{M} q_i\beta_{n-1}(\mathbf{q}).$$

Lemma 2.1.3 states that $\beta_n(\mathbf{q})$ is non-decreasing in $n$, thus from the previous equation we get

$$\Delta\beta_n(\mathbf{q}) \le \beta_n(\mathbf{q}) - \sum_{i=1}^{M} q_i\beta_{n-i}(\mathbf{q}).$$

Notice that the right hand side of this equation is one of the terms in the recursion equation for $\beta_n(\mathbf{q})$, equation (2.1.5). Thus we can substitute to get

$$\Delta\beta_n(\mathbf{q}) \leq \left(\frac{\varepsilon - 1}{\varepsilon}\right)\left(\frac{1}{\beta_n(\mathbf{q})^{1/(\varepsilon-1)}}\right), \tag{2.1.13}$$

giving a bound for $\Delta\beta_n(\mathbf{q})$. Differences between non-successive $\beta_n(\mathbf{q})$ can be written as a telescoping sum of successive differences, that is $\beta_n(\mathbf{q}) - \beta_{n-k}(\mathbf{q}) = \sum_{j=0}^{k-1} \Delta\beta_{n-j}(\mathbf{q})$. This fact is also useful for finding bounds. We now reformulate (2.1.5) by using successive differences. Recall also that the average order size is given by $\mu(\mathbf{q}) = \sum_{i=1}^{M} iq_i$. Equation (2.1.5) is now reformulated:

$$\begin{aligned}
\left(\frac{\varepsilon - 1}{\varepsilon}\right)\frac{1}{\beta_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} &= \beta_n(\mathbf{q}) - \sum_{i=1}^{M} q_i\beta_{n-i}(\mathbf{q}) \\
&= \sum_{i=1}^{M} q_i(\beta_n(\mathbf{q}) - \beta_{n-i}(\mathbf{q})) \\
&= \sum_{i=1}^{M}\sum_{j=0}^{i-1} q_i\Delta\beta_{n-j}(\mathbf{q}) \\
&= \sum_{i=1}^{M} q_i\left[\Delta\beta_n(\mathbf{q}) + \sum_{j=1}^{i-1}\Delta\beta_{n-j}(\mathbf{q})\right] \\
&= \sum_{i=1}^{M} q_i\Delta\beta_n(\mathbf{q}) + \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i\Delta\beta_{n-j}(\mathbf{q}) \\
&= \sum_{i=1}^{M} iq_i\Delta\beta_n(\mathbf{q}) - \sum_{i=1}^{M}(i-1)q_i\Delta\beta_n(\mathbf{q}) + \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i\Delta\beta_{n-j}(\mathbf{q}) \\
&= \mu(\mathbf{q})\Delta\beta_n(\mathbf{q}) - \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i\Delta\beta_n(\mathbf{q}) + \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i\Delta\beta_{n-j}(\mathbf{q}) \\
&= \mu(\mathbf{q})\Delta\beta_n(\mathbf{q}) - \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i\left[\Delta\beta_n(\mathbf{q}) - \Delta\beta_{n-j}(\mathbf{q})\right]
\end{aligned}$$

Substitute $\beta_n(\mathbf{q}) = n^{\frac{\varepsilon-1}{\varepsilon}}\gamma_n(\mathbf{q})$ into the left hand side of the previous equation to get

$$\left(\frac{\varepsilon - 1}{\varepsilon}\right)\frac{1}{n^{\frac{1}{\varepsilon}}\gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} = \mu(\mathbf{q})\Delta\beta_n(\mathbf{q}) - \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i\left[\Delta\beta_n(\mathbf{q}) - \Delta\beta_{n-j}(\mathbf{q})\right]. \tag{2.1.14}$$

It may look odd to have both $\gamma_n(\mathbf{q})$ and $\beta$ terms in the equation. Eventually (2.1.13) will be used to eliminate any $\beta$ terms from the equation, while keeping only the one $\gamma_n(\mathbf{q})$ term.

Let $\delta > 0$. Then choose $N_\delta$ large enough such that for all $n \geq N_\delta$.

$$\frac{1}{\gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} \geq \frac{1}{\limsup_{k\to\infty} \gamma_k(\mathbf{q})^{\frac{1}{\varepsilon-1}}} - \delta \tag{2.1.15}$$

We now explain why such an $N_\delta$ exists. First note that Lemma 2.1.5(a) states that $\liminf_{n\to\infty} \gamma_n(\mathbf{q}) > 0$, and so the right hand side of (2.1.15) is not $\infty$. Then $N_\delta$ can be chosen large enough to ensure that whenever $\gamma_n(\mathbf{q}) > \limsup_{k\to\infty} \gamma_k(\mathbf{q})$, that $\gamma_n(\mathbf{q})$ is close enough $\limsup_{k\to\infty} \gamma_k(\mathbf{q})$ in order to satisfy (2.1.15). Whenever $\gamma_n(\mathbf{q}) \leq \limsup_{k\to\infty} \gamma_k(\mathbf{q})$, (2.1.15) is already satisfied.

Let $N > N_\delta$. Eventually we will take the limit as $N \to \infty$, but for now we sum equation (2.1.14) from $n = N_\delta$ to $N$ to obtain

$$\sum_{n=N_\delta}^{N} \left(\frac{\varepsilon-1}{\varepsilon}\right) \frac{1}{n^{\frac{1}{\varepsilon}} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} = \sum_{n=N_\delta}^{N} \mu(\mathbf{q}) \Delta\beta_n(\mathbf{q}) - \sum_{n=N_\delta}^{N} \sum_{i=1}^{M} \sum_{j=1}^{i-1} q_i \left[\Delta\beta_n(\mathbf{q}) - \Delta\beta_{n-j}(\mathbf{q})\right].$$

Note the telescoping sum $\sum_{n=N_\delta}^{N} \Delta\beta_n(\mathbf{q}) = \beta_N(\mathbf{q}) - \beta_{N_\delta-1}(\mathbf{q})$, making the previous equation

$$\sum_{n=N_\delta}^{N} \left(\frac{\varepsilon-1}{\varepsilon}\right) \frac{1}{n^{\frac{1}{\varepsilon}} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} = \mu(\mathbf{q})(\beta_N(\mathbf{q}) - \beta_{N_\delta-1}(\mathbf{q})) - S, \tag{2.1.16}$$

where the telescoping sum also is used for writing

$$S = \sum_{i=1}^{M} \sum_{j=1}^{i-1} q_i \left[(\beta_N(\mathbf{q}) - \beta_{N_\delta-1}(\mathbf{q})) - (\beta_{N-j}(\mathbf{q}) - \beta_{N_\delta-j-1}(\mathbf{q}))\right].$$

Why is this new form useful? Note that $\Delta\beta_N(\mathbf{q}) = \beta_N(\mathbf{q}) - \beta_{N-1}(\mathbf{q})$ depends on the $N$ for two terms. However, the telescoped sum $\sum_{n=N_\delta}^{N} \Delta\beta_n(\mathbf{q}) = \beta_N(\mathbf{q}) - \beta_{N_\delta-1}(\mathbf{q})$ only depends on $N$ for one term. So as $N \to \infty$, $\beta_{N_\delta-1}(\mathbf{q})$ remains fixed while $\beta_N(\mathbf{q})$ increases.

Next we eliminate the summation on the left-hand side of (2.1.16). Observe the the following sum can be thought of as a Riemann sum estimate of an integral, that is,

$$\sum_{n=N_\delta}^{N} \frac{1}{n^{\frac{1}{\varepsilon}}} \geq \int_{N_\delta}^{N+1} x^{-1/\varepsilon} dx \geq \int_{N_\delta}^{N} x^{-1/\varepsilon} dx = \left(\frac{\varepsilon}{\varepsilon-1}\right)\left(N^{\frac{\varepsilon-1}{\varepsilon}} - N_\delta^{\frac{\varepsilon-1}{\varepsilon}}\right).$$

Applying the previous inequality along with (2.1.15) to equation (2.1.16) yields

$$\left(N^{\frac{\varepsilon-1}{\varepsilon}} - N_\delta^{\frac{\varepsilon-1}{\varepsilon}}\right)\left(\frac{1}{\limsup\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} - \delta\right) \leq \mu(\mathbf{q})(\beta_N(\mathbf{q}) - \beta_{N_\delta-1}(\mathbf{q})) - S,$$

and dividing by $N^{\frac{\varepsilon-1}{\varepsilon}}$,

$$\left(1 - \left(\frac{N_\delta}{N}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)\left(\frac{1}{\limsup\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} - \delta\right) \leq \mu(\mathbf{q})\left(\gamma_N(\mathbf{q}) - \frac{\beta_{N_\delta-1}(\mathbf{q})}{N^{\frac{\varepsilon-1}{\varepsilon}}}\right) - \frac{S}{N^{\frac{\varepsilon-1}{\varepsilon}}}.$$

Now take the liminf as $N \to \infty$ to get

$$\frac{1}{\limsup\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} - \delta \leq \liminf_{N\to\infty}\left(\mu(\mathbf{q})\gamma_N(\mathbf{q}) - \frac{S}{N^{\frac{\varepsilon-1}{\varepsilon}}}\right). \tag{2.1.17}$$

In order to simplify the liminf further, we now show that $\lim\limits_{N\to\infty} \frac{S}{N^{\frac{\varepsilon-1}{\varepsilon}}} = 0$. Begin by writing

$$\begin{aligned}
S &= \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i \left[[\beta_N(\mathbf{q}) - \beta_{N_\delta-1}(\mathbf{q})] - [\beta_{N-j}(\mathbf{q}) - \beta_{N_\delta-j-1}(\mathbf{q})]\right] \\
&= \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i \left[[\beta_N(\mathbf{q}) - \beta_{N-j}(\mathbf{q})] - [\beta_{N_\delta-1}(\mathbf{q}) - \beta_{N_\delta-j-1}(\mathbf{q})]\right] \\
&= \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i \left[\sum_{k=0}^{j-1} \Delta\beta_{N-k}(\mathbf{q}) - [\beta_{N_\delta-1}(\mathbf{q}) - \beta_{N_\delta-j-1}(\mathbf{q})]\right].
\end{aligned} \tag{2.1.18}$$

Then

$$\left|\frac{S}{N^{\frac{\varepsilon-1}{\varepsilon}}}\right| \leq \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i \left[\sum_{k=0}^{j-1}\left|\frac{\Delta\beta_{N-k}(\mathbf{q})}{N^{\frac{\varepsilon-1}{\varepsilon}}}\right| + \left|\frac{\beta_{N_\delta-1}(\mathbf{q}) - \beta_{N_\delta-j-1}(\mathbf{q})}{N^{\frac{\varepsilon-1}{\varepsilon}}}\right|\right]$$

Using this inequality along with with the bound (2.1.13) yields

$$\left|\frac{S}{N^{\frac{\varepsilon-1}{\varepsilon}}}\right| \leq \sum_{i=1}^{M}\sum_{j=1}^{i-1} q_i \left[\sum_{k=0}^{j-1}\left(\frac{\varepsilon-1}{\varepsilon}\right)\left|\frac{1}{\beta_{N-k}(\mathbf{q})^{1/(\varepsilon-1)}N^{\frac{\varepsilon-1}{\varepsilon}}}\right| + \left|\frac{\beta_{N_\delta-1}(\mathbf{q})-\beta_{N_\delta-j-1}(\mathbf{q})}{N^{\frac{\varepsilon-1}{\varepsilon}}}\right|\right]$$

Then since $N_\delta$ is fixed, the right hand side of this inequality goes to 0 as $N \to \infty$ (recall that Lemma 2.1.3 stated that $\beta_n(\mathbf{q})$ was non-decreasing in $n$). Therefore

$$0 = \lim_{N\to\infty} \frac{S}{N^{\frac{\varepsilon-1}{\varepsilon}}}.$$

Thus (2.1.17) becomes

$$\frac{1}{\limsup\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} - \delta \leq \mu(\mathbf{q}) \liminf_{N\to\infty} \gamma_N(\mathbf{q}). \tag{2.1.19}$$

Since this equation is true for all $\delta > 0$, we get

$$\frac{1}{\limsup\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} \leq \mu(\mathbf{q}) \liminf_{n\to\infty} \gamma_n(\mathbf{q}), \tag{2.1.20}$$

completing the proof of part (a).

**Proof of part (b)** The proof of part (b) is done in a nearly identical way to part (a). The adjustments are mostly in notation: limsup is exchanged with liminf, the inequalities are reversed, and a $+\delta$ is needed instead of $-\delta$. For instance, the analog of (2.1.15) is

$$\frac{1}{\gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} \leq \frac{1}{\liminf\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} + \delta. \tag{2.1.21}$$

□

To summarize these lemmas, Lemma 2.1.5 gives conditional bounds for $\gamma_n(\mathbf{q})$ and Lemma 2.1.6 gives useful relations relating $\liminf \gamma_n(\mathbf{q})$ to $\limsup \gamma_n(\mathbf{q})$. We now use these results together to prove the following theorem:

**Theorem 2.1.7.** *For any choice of $\varepsilon > 1$ and order size distribution* $\boldsymbol{q}$*,*

$$\lim_{n\to\infty} \gamma_n(\boldsymbol{q}) = \lim_{n\to\infty} \frac{\beta_n(\boldsymbol{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\boldsymbol{q})^{\frac{1-\varepsilon}{\varepsilon}}.$$

*Proof.* Suppose the following claim is true (it will be proved later in the proof):

$$\limsup_{n\to\infty} \gamma_n(\mathbf{q}) = \liminf_{n\to\infty} \gamma_n(\mathbf{q}) = \lim_{n\to\infty} \gamma_n(\mathbf{q}). \tag{2.1.22}$$

Then substituting this equation into Lemma 2.1.6(a) and Lemma 2.1.6(b) gives

$$\begin{aligned}
\frac{1}{\lim\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} &\leq \mu(\mathbf{q}) \lim_{n\to\infty} \gamma_n(\mathbf{q}) \\
\frac{1}{\lim\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} &\geq \mu(\mathbf{q}) \lim_{n\to\infty} \gamma_n(\mathbf{q}),
\end{aligned}$$

meaning equality holds,

$$\frac{1}{\lim\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} = \mu(\mathbf{q}) \lim_{n\to\infty} \gamma_n(\mathbf{q}).$$

Solving for the limit gives

$$\lim_{n\to\infty} \gamma_n(\mathbf{q}) = \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}},$$

which proves the theorem.

Now we prove the claim. This is accomplished by considering three cases, based on whether the following properties hold or not:

**Property 1:** There exists an infinite, increasing sequence $(N_k)_k \subset \mathbb{N}$ such that $\gamma_{N_k}(\mathbf{q}) = \min\limits_{0\leq i\leq M} \gamma_{N_k-i}(\mathbf{q})$ for all $k$.

**Property 2:** There exists an $N \geq M$ such that $\gamma_N(\mathbf{q}) = \max\limits_{0\leq i\leq M} \gamma_{N-i}(\mathbf{q})$.

Note that these properties are related to the conditions present in Lemma 2.1.5.

**Proof of claim, case 1:** Suppose Property 1 holds.

Then the conditions of Lemma 2.1.5($b$) are met and thus

$$\liminf_{n\to\infty} \gamma_n(\mathbf{q}) \geq \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}.$$

Combining this with inequality from Lemma 2.1.6($b$) gives

$$\frac{1}{(\mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}})^{\frac{1}{\varepsilon-1}}} \geq \frac{1}{\liminf\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} \geq \mu(\mathbf{q}) \limsup_{n\to\infty} \gamma_n(\mathbf{q}),$$

and simplifying $\mu(\mathbf{q})$ terms yields

$$\mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}} \geq \limsup_{n\to\infty} \gamma_n(\mathbf{q}).$$

Therefore

$$\mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}} \geq \limsup_{n\to\infty} \gamma_n(\mathbf{q}) \geq \liminf_{n\to\infty} \gamma_n(\mathbf{q}) \geq \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}},$$

and equality holds throughout, proving claim (2.1.22) for case 1.

**Proof of claim, case 2:** Suppose Property 2 holds.

Then by Lemma 2.1.5($c$),

$$\limsup_{n\to\infty} \gamma_n(\mathbf{q}) \leq \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}.$$

Combining this with Lemma 2.1.6($a$) we get

$$\frac{1}{(\mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}})^{\frac{1}{\varepsilon-1}}} \leq \frac{1}{\limsup\limits_{n\to\infty} \gamma_n(\mathbf{q})^{\frac{1}{\varepsilon-1}}} \leq \mu(\mathbf{q}) \liminf_{n\to\infty} \gamma_n(\mathbf{q}),$$

and simplifying the $\mu(\mathbf{q})$ terms yields

$$\mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}} \leq \liminf_{n\to\infty} \gamma_n(\mathbf{q}).$$

Therefore

$$\mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}} \leq \liminf_{n\to\infty} \gamma_n(\mathbf{q}) \leq \limsup_{n\to\infty} \gamma_n(\mathbf{q}) \leq \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}},$$

and equality holds throughout, proving the claim for case 2.

**Proof of claim, case 3:** Suppose neither Properties 1 nor 2 hold. The idea of this case is to find a sequence of upper and lower bounds for $\gamma_n(\mathbf{q})$, then to show that the difference between these upper and lower bound sequences goes to 0.

Since Property 1 is false, there exists an $N_1 > M$ such that for all $n \geq N_1$, $\gamma_n(\mathbf{q}) \neq \min\limits_{0\leq i\leq M} \gamma_{n-i}(\mathbf{q})$. Let $(N_k)_k \subset \mathbb{N}$ be a strictly increasing sequence starting with $N_1$ such that for all $k$ the condition

$$\gamma_{N_k-M}(\mathbf{q}) = \min_{0\leq i\leq M} \gamma_{N_k-i}(\mathbf{q}) \tag{2.1.23}$$

holds. That is, $\gamma_{N_k-M}(\mathbf{q})$ is the minimum of itself and the next $M$ terms in the $\gamma$ sequence.

Why does such a sequence exist? Suppose instead $(N_k)_k \subset \mathbb{N}$ is any strictly increasing sequence starting with $N_1$. We can use this sequence to construct one which also satisfies (2.1.23). Let $0 < l_k \leq M$ be such that $\gamma_{N_k-l_k}(\mathbf{q}) = \min\limits_{0\leq i\leq M} \gamma_{N_k-i}(\mathbf{q})$. In other words, $l_k$ is the value of $i$ which achieves the minimum $\min\limits_{0\leq i\leq M} \gamma_{N_k-i}(\mathbf{q})$. If $l_k = M$, then condition (2.1.23) holds for $N_k$. Note that $l_k \neq 0$ since for all $n > N_1$, $\gamma_n(\mathbf{q}) \neq \min\limits_{0\leq i\leq M} \gamma_{n-i}(\mathbf{q})$. By the same reasoning, $\gamma_{N_k+1}(\mathbf{q}) \neq \min\limits_{0\leq i\leq M} \gamma_{N_k+1-i}(\mathbf{q})$, and thus $\gamma_{N_k-l_k}(\mathbf{q}) = \min\limits_{0\leq i\leq M} \gamma_{N_k+1-i}(\mathbf{q})$. Repeat this argument to see $\gamma_{N_k-l_k}(\mathbf{q}) = \min\limits_{0\leq i\leq M} \gamma_{N_k+(M-l_k)-i}(\mathbf{q})$, or written another way, $\gamma_{(N_k+M-l_k)-M}(\mathbf{q}) = \min\limits_{0\leq i\leq M} \gamma_{(N_k+M-l_k)-i}(\mathbf{q})$. This shows condition (2.1.23) holds for $N_k + M - l_k$. Therefore the sequence $(N_k + M - l_k)_k$ satisfies condition (2.1.23) for all $k$. Lastly, $(N_k+M-l_k)_k$ may not be strictly increasing, but since $N_k$ is strictly increasing and $M-l_k \geq 0$, a strictly increasing subsequence of $(N_k+M-l_k)_k$ exists, thereby showing the appropriate sequence exists.

We now aim to find lower and upper bounds for $\gamma_n(\mathbf{q})$. For all $k \geq 1$, let $a_k = N_k - M$. Then $\gamma_{a_k}(\mathbf{q}) = \min\limits_{0\leq i\leq M} \gamma_{N_k-i}(\mathbf{q})$ by the properties of $(N_k)_k$. Suppose some $n$ is the first $n > a_k$ such that $\gamma_n(\mathbf{q}) < \gamma_{a_k}(\mathbf{q})$. Then $\gamma_n(\mathbf{q}) = \min\limits_{N_k-M\leq i\leq n} \gamma_i(\mathbf{q}) = \min\limits_{0\leq i\leq M} \gamma_{n-i}(\mathbf{q})$. But this is a contradiction since all $\gamma_n(\mathbf{q})$ for $n > N_1$ lack this property. Therefore there is no $n > a_k$ such that $\gamma_n(\mathbf{q}) < \gamma_{a_k}(\mathbf{q})$, giving for all $k$ the lower bound

$$\gamma_{a_k}(\mathbf{q}) \leq \liminf_{n\to\infty} \gamma_n(\mathbf{q}). \tag{2.1.24}$$

For all $k \geq 1$, let $b_k$ be such that $N_k - M < b_k \leq N_k$ and $\gamma_{b_k}(\mathbf{q}) = \max_{0\leq i\leq M} \gamma_{N_k-i}(\mathbf{q})$ (note that $b_k \neq N_k - M = a_k$, because $\gamma_{a_k}(\mathbf{q}) = \min_{0\leq i\leq M} \gamma_{N_k-i}(\mathbf{q})$). Suppose $n$ is the first $n > b_k$ such that $\gamma_n(\mathbf{q}) > \gamma_{b_k}(\mathbf{q})$. Then $\gamma_n(\mathbf{q}) = \max_{N_k-M\leq i\leq n} \gamma_i(\mathbf{q}) = \max_{0\leq i\leq M} \gamma_{n-i}(\mathbf{q})$. But this is a contradiction since Property 2 is false. Therefore there is no $n > b_k$ such that $\gamma_n(\mathbf{q}) > \gamma_{b_k}(\mathbf{q})$, giving the upper bound for all $k$

$$\limsup_{n\to\infty} \gamma_n(\mathbf{q}) \leq \gamma_{b_k}(\mathbf{q}). \tag{2.1.25}$$

Equations (2.1.24) and (2.1.25) together imply that

$$\limsup_{n\to\infty} \gamma_n(\mathbf{q}) - \liminf_{n\to\infty} \gamma_n(\mathbf{q}) \leq \gamma_{b_k}(\mathbf{q}) - \gamma_{a_k}(\mathbf{q}). \tag{2.1.26}$$

Thus a bound for $\gamma_{b_k}(\mathbf{q}) - \gamma_{a_k}(\mathbf{q})$ is also a bound for $\limsup_{n\to\infty} \gamma_n(\mathbf{q}) - \liminf_{n\to\infty} \gamma_n(\mathbf{q})$.

To help towards finding a bound, recall equation (2.1.13) of Lemma 2.1.6:

$$\beta_n(\mathbf{q}) - \beta_{n-1}(\mathbf{q}) \leq \left(\frac{\varepsilon - 1}{\varepsilon}\right)\left(\frac{1}{\beta_n(\mathbf{q})^{1/(\varepsilon-1)}}\right). \tag{2.1.27}$$

Thus the following is true (notes have been added to the left-hand side in the calculation to provide specific justification for some steps):

$$\begin{aligned}
\gamma_{b_k}(\mathbf{q}) - \gamma_{a_k}(\mathbf{q}) &= \frac{\beta_{b_k}(\mathbf{q})}{b_k^{\frac{\varepsilon-1}{\varepsilon}}} - \frac{\beta_{a_k}(\mathbf{q})}{a_k^{\frac{\varepsilon-1}{\varepsilon}}} \\
(a_k < b_k \text{ by construction}) &\leq \frac{\beta_{b_k}(\mathbf{q})}{b_k^{\frac{\varepsilon-1}{\varepsilon}}} - \frac{\beta_{a_k}(\mathbf{q})}{b_k^{\frac{\varepsilon-1}{\varepsilon}}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b_k^{\frac{\varepsilon-1}{\varepsilon}}} \left(\beta_{b_k}(\mathbf{q}) - \beta_{a_k}(\mathbf{q})\right) \\
&= \frac{1}{b_k^{\frac{\varepsilon-1}{\varepsilon}}} \sum_{i=a_k+1}^{b_k} \beta_i(\mathbf{q}) - \beta_{i-1}(\mathbf{q}) \\
\text{(Equation (2.1.27))} &\leq \frac{1}{b_k^{\frac{\varepsilon-1}{\varepsilon}}} \sum_{i=a_k+1}^{b_k} \left(\frac{\varepsilon-1}{\varepsilon}\right)\left(\frac{1}{\beta_i(\mathbf{q})^{1/(\varepsilon-1)}}\right) \\
\text{(Lemma 2.1.3 states } \beta_n(\mathbf{q}) \text{ is non-decreasing)} &\leq \frac{1}{b_k^{\frac{\varepsilon-1}{\varepsilon}}} \sum_{i=a_k+1}^{b_k} \left(\frac{\varepsilon-1}{\varepsilon}\right)\left(\frac{1}{\beta_{a_k+1}(\mathbf{q})^{1/(\varepsilon-1)}}\right) \\
\left(b_k - a_k \leq M \text{ by construction, } \frac{\varepsilon-1}{\varepsilon} < 1\right) &\leq \frac{1}{b_k^{\frac{\varepsilon-1}{\varepsilon}}} \left(\frac{M}{\beta_{a_k+1}(\mathbf{q})^{1/(\varepsilon-1)}}\right). \qquad (2.1.28)
\end{aligned}
$$

Now combining (2.1.26) and (2.1.28) we get, for each $k$,

$$
\limsup_{n\to\infty} \gamma_n(\mathbf{q}) - \liminf_{n\to\infty} \gamma_n(\mathbf{q}) \leq \gamma_{b_k}(\mathbf{q}) - \gamma_{a_k}(\mathbf{q}) \leq \frac{1}{b_k^{\frac{\varepsilon-1}{\varepsilon}}} \left(\frac{M}{\beta_{a_k+1}(\mathbf{q})^{1/(\varepsilon-1)}}\right). \qquad (2.1.29)
$$

Note that $b_k$ is a sequence of increasing integers, since $N_k - M < b_k \leq N_k$ and $N_k$ was an increasing sequence of integers. Also, $\beta_k(\mathbf{q})$ is at worst non-decreasing in $k$. So taking the limit of (2.1.29) as $k \to \infty$ shows that

$$
0 \leq \limsup_{n\to\infty} \gamma_n(\mathbf{q}) - \liminf_{n\to\infty} \gamma_n(\mathbf{q}) \leq 0,
$$

or in other words,

$$
\limsup_{n\to\infty} \gamma_n(\mathbf{q}) = \liminf_{n\to\infty} \gamma_n(\mathbf{q}) = \lim_{n\to\infty} \gamma_n(\mathbf{q}),
$$

proving the claim for case 3. □

It is quite surprising that $\beta_n(\mathbf{q})$ has such a simple comparison in the limit, given that (2.1.5), the recursion equation for $\beta_n(\mathbf{q})$, looks quite complex. Theorem 2.1.7 shows again the importance of the average order size term $\mu(\mathbf{q})$, which is not observed in the unit order case where $\mu(\mathbf{q}) = 1$. The term $\mu(\mathbf{q})$ was also seen in Theorem 2.1.2, which said that the optimal expected revenue was given by $v_n(t;\mathbf{q}) = \mu(\mathbf{q})\beta_n(\mathbf{q})A(t)^{1/\varepsilon}$.

The limit in Theorem 2.1.2 along with the formula for $v_n(t;\mathbf{q})$ imply the intuitive result that, since $\frac{\varepsilon-1}{\varepsilon} < 1$, having more inventory has diminishing returns. Also consider from Theorem 2.1.2 the formula for the optimal price, $p_n^*(t;\mathbf{q}) = \beta_n(\mathbf{q})^{-1/(\varepsilon-1)}A(t)^{1/\varepsilon}$. The limit of Theorem 2.1.2 shows that $p^*$ changes less and less as $n \to \infty$. This is in line with the idea that dynamic pricing becomes less relevant with large inventory, and will have the best benefits for small to moderate inventory.

From a practical or numerical standpoint, the simpler term $(n/\mu(\mathbf{q}))^{\frac{\varepsilon-1}{\varepsilon}}$ can be used to approximate $\beta_n(\mathbf{q})$, and therefore $v_n(t;\mathbf{q}) = \mu(\mathbf{q})\beta_n(\mathbf{q})A(t)^{1/\varepsilon} \approx n^{\frac{\varepsilon-1}{\varepsilon}}(\mu(\mathbf{q})A(t))^{\frac{1}{\varepsilon}}$. This approximation can be important to save computation time if $n$ is very large or there are a lot of order sizes. However, as the next section will show, there is very useful application of the approximation when looking at comparable models.

## 2.2 Comparable Models

We now turn to the analysis of comparable models, that is, models which have the same demand yet have different order size distributions. Let $\mathscr{M}_{\mathbf{q},\lambda}$ denote a model with order size distribution $\mathbf{q}$ and arrival rate $\lambda(p,t)$. Define two models $\mathscr{M}_{\mathbf{q},\lambda}$ and $\mathscr{M}_{\bar{\mathbf{q}},\bar{\lambda}}$ to be comparable if their demands are equal for all $p > 0$ and $0 < t < T$, i.e.

$$\lambda(p,t)\mu(\mathbf{q}) = \bar{\lambda}(p,t)\mu(\bar{\mathbf{q}}).$$

For constant demand elasticity, arrival rates are of the form $\lambda(p,t) = a(t)p^{-\varepsilon}$ and $\bar{\lambda}(p,t) = \bar{a}(t)p^{-\varepsilon}$. Thus we see an equivalent condition to (2.2) is

$$a(t)\mu(\mathbf{q}) = \bar{a}(t)\mu(\bar{\mathbf{q}}). \tag{2.2.1}$$

An important observation from (2.2.1) is that the arrival rate scaling factor $\bar{a}(t)$ is just a scalar multiple of $a(t)$. Observe that if $\mathscr{M}_{\mathbf{q},\lambda}$ and $\mathscr{M}_{\bar{\mathbf{q}},\bar{\lambda}}$ are comparable, then so are $\mathscr{M}_{\mathbf{q},c\lambda}$

and $\mathscr{M}_{\bar{\mathbf{q}},c\bar{\lambda}}$ for any constant $c > 0$. Thus, to help distinguish between these types of cases, it is helpful to think about the term $a(t)\mu(\mathbf{q})$ as a demand rate scaling factor (or demand magnitude), similar to how $a(t)$ is the arrival rate scaling factor. The second theorem in this chapter will examine sets of comparable models which have the same order size distributions, but different demand magnitudes.

Since we are now considering how multiple models compare, $\lambda$ can no longer be suppressed in notation for terms like $v_n$. For example, $v_n(t;\mathbf{q},\lambda)$ is the optimal expected revenue and $p_n^*(t;\mathbf{q},\lambda)$ is the optimal pricing strategy for a model with order size distribution $\mathbf{q}$ and customer arrival rate $\lambda$. The next result shows some interesting asymptotic behavior of the optimal expected revenue and price for comparable models.

**Theorem 2.2.1.** *Let $\mathscr{M}_{\boldsymbol{q},\lambda}$ and $\mathscr{M}_{\bar{\boldsymbol{q}},\bar{\lambda}}$ be comparable models which both have constant demand elasticity $\varepsilon$. Then,*

$$\lim_{n\to\infty}\frac{v_n(t;\boldsymbol{q},\lambda)}{v_n(t;\bar{\boldsymbol{q}},\bar{\lambda})} = 1 \text{ and } \lim_{n\to\infty}\frac{p_n^*(t;\boldsymbol{q},\lambda)}{p_n^*(t;\bar{\boldsymbol{q}},\bar{\lambda})} = 1.$$

*In other words, $\mathscr{M}_{\boldsymbol{q},\lambda}$ and $\mathscr{M}_{\bar{\boldsymbol{q}},\bar{\lambda}}$ have asymptotically equivalent optimal expected revenue and pricing strategies.*

*Proof.* Let $\mathscr{M}_{\mathbf{q},\lambda}$ and $\mathscr{M}_{\bar{\mathbf{q}},\bar{\lambda}}$ be comparable models with $\lambda(p,t) = a(t)p^{-\varepsilon}$ and $\bar{\lambda}(p,t) = \bar{a}(t)p^{-\varepsilon}$. Recall Theorem 2.1.2 which stated that $v_n(t;\mathbf{q}) = \mu(\mathbf{q})\beta_n(\mathbf{q})A(t)^{1/\varepsilon}$. This was for a model with $\lambda(p,t) = a(t)p^{-\varepsilon}$ but now $\lambda$ dependencies mater, so the notation of the formula must be updated. Define $A(t;a) = \int_t^T a(s)ds$ and $A(t;\bar{a}) = \int_t^T \bar{a}(s)ds$. Then Theorem 2.1.2 says

$$v_n(t;\mathbf{q},\lambda) = \mu(\mathbf{q})\beta_n(\mathbf{q})A(t;a)^{1/\varepsilon}. \tag{2.2.2}$$

Also recall Theorem 2.1.7, which stated $\lim\limits_{n\to\infty}\frac{\beta_n(\mathbf{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}$. Let $\delta > 0$. By combining (2.2.2) with Theorem 2.1.7 we see that there exists $N_a > 0$ such that for all $n > N_a$,

$$\left| v_n(t;\mathbf{q},\lambda) - \mu(\mathbf{q})\left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon}} A(t;a)^{\frac{1}{\varepsilon}} \right| < \delta.$$

Similarly, there exists $N_{\bar{a}} > 0$ such that for all $n > N_{\bar{a}}$,

$$\left| v_n(t;\bar{\mathbf{q}},\bar{\lambda}) - \mu(\bar{\mathbf{q}}) \left(\frac{n}{\mu(\bar{\mathbf{q}})}\right)^{\frac{\varepsilon-1}{\varepsilon}} A(t;\bar{a})^{\frac{1}{\varepsilon}} \right| < \delta.$$

Note that for a constant $c$, $A(t;ca) = \int_t^T ca(t)dt = c\int_t^T a(t)dt = cA(t;a)$. We also have from the condition for comparable models (2.2.1) that $a(t) = \frac{\bar{a}(t)\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}$. Thus

$$\begin{aligned}
\mu(\mathbf{q}) \left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon}} A(t;a)^{\frac{1}{\varepsilon}} &= \mu(\mathbf{q}) \left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon}} A\left(t;\frac{\bar{a}\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}\right)^{\frac{1}{\varepsilon}} \\
&= \mu(\mathbf{q}) \left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon}} \left(\frac{\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}\right)^{\frac{1}{\varepsilon}} A\left(t;\bar{a}\right)^{\frac{1}{\varepsilon}} \\
&= \mu(\bar{\mathbf{q}}) \left(\frac{n}{\mu(\bar{\mathbf{q}})}\right)^{\frac{\varepsilon-1}{\varepsilon}} A(t;\bar{a})^{\frac{1}{\varepsilon}}.
\end{aligned}$$

Then for $n > \max(N_a, N_{\bar{a}})$,

$$\begin{aligned}
\left|v_n(t;\mathbf{q},\lambda) - v_n(t;\bar{\mathbf{q}},\bar{\lambda})\right| \leq{}& \left| v_n(t;\mathbf{q},\lambda) - \mu(\mathbf{q}) \left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon}} A(t;a)^{\frac{1}{\varepsilon}} \right| \\
&+ \left| \mu(\bar{\mathbf{q}}) \left(\frac{n}{\mu(\bar{\mathbf{q}})}\right)^{\frac{\varepsilon-1}{\varepsilon}} A(t;\bar{a})^{\frac{1}{\varepsilon}} - v_n(t;\bar{\mathbf{q}},\bar{\lambda}) \right| \\
<{}& 2\delta.
\end{aligned}$$

Since $\delta$ was arbitrary, this proves that two comparable models have asymptotically equivalent optimal expected revenues as $n \to \infty$.

Theorem 2.1.2 also gives the optimal expected revenue as $p_n^*(t;\mathbf{q}) = \beta_n(\mathbf{q})^{-1/(\varepsilon-1)} A(t)^{1/\varepsilon}$. We follow a similar argument as was used for $v_n$. Let $\delta > 0$. Then by Theorems 2.1.2 and

2.1.7 there exists an $N$ such that for all $n > N$,

$$\left|p_n^*(t;\mathbf{q},\lambda) - \left(\left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{-\frac{1}{\varepsilon-1}} A(t;a)^{\frac{1}{\varepsilon}}\right| = \left|p_n^*(t;\mathbf{q},\lambda) - \left(\frac{n}{\mu(\mathbf{q})}\right)^{-\frac{1}{\varepsilon}} A(t;a)^{\frac{1}{\varepsilon}}\right| < \delta$$

$$\left|p_n^*(t;\bar{\mathbf{q}},\bar{\lambda}) - \left(\left(\frac{n}{\mu(\bar{\mathbf{q}})}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{-\frac{1}{\varepsilon-1}} A(t;\bar{a})^{\frac{1}{\varepsilon}}\right| = \left|p_n^*(t;\bar{\mathbf{q}},\bar{\lambda}) - \left(\frac{n}{\mu(\bar{\mathbf{q}})}\right)^{-\frac{1}{\varepsilon}} A(t;\bar{a})^{\frac{1}{\varepsilon}}\right| < \delta.$$

Also,

$$\left(\frac{n}{\mu(\mathbf{q})}\right)^{-\frac{1}{\varepsilon}} A(t;a)^{\frac{1}{\varepsilon}} = \left(\frac{n}{\mu(\mathbf{q})}\right)^{-\frac{1}{\varepsilon}} \left(\frac{\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}\right)^{\frac{1}{\varepsilon}} A\left(t;\bar{a}\right)^{\frac{1}{\varepsilon}} = \left(\frac{n}{\mu(\bar{\mathbf{q}})}\right)^{-\frac{1}{\varepsilon}} A(t;\bar{a})^{\frac{1}{\varepsilon}}$$

Therefore

$$\begin{aligned}\left|p_n^*(t;\mathbf{q},\lambda) - p_n^*(t;\bar{\mathbf{q}},\bar{\lambda})\right| \leq & \left|p_n^*(t;\mathbf{q},\lambda) - \left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{-1}{\varepsilon}} A(t;a)^{\frac{1}{\varepsilon}}\right| \\ & + \left|\left(\frac{n}{\mu(\bar{\mathbf{q}})}\right)^{\frac{-1}{\varepsilon}} A(t;\bar{a})^{\frac{1}{\varepsilon}} - p_n^*(t;\bar{\mathbf{q}},\bar{\lambda})\right| \\ < & 2\delta\end{aligned}$$

Hence the optimal pricing strategies for the comparable models are asymptotically equivalent as $n \to \infty$. □

Observe that given a particular model $\mathscr{M}_{\mathbf{q},\lambda}$, we can always find a comparable model $\mathscr{M}_{\bar{\mathbf{q}},\bar{\lambda}}$ of any order size distribution $\bar{\mathbf{q}}$. Comparable models have equal demands, that is $\lambda(p,t)\mu(\mathbf{q}) = \bar{\lambda}(p,t)\mu(\bar{\mathbf{q}})$. So set $\bar{\lambda}(p,t) = \mu(\mathbf{q})/\mu(\bar{\mathbf{q}})\lambda(p,t)$ in order to make $\mathscr{M}_{\mathbf{q},\lambda}$ and $\mathscr{M}_{\bar{\mathbf{q}},\bar{\lambda}}$ comparable. In particular, by choosing $\bar{\mathbf{q}} = (1)$, a variable order size model always has a comparable unit order size model.

Theorem 2.2.1 proves the intuitive idea that models where inventory is being sold at similar rates will have similar behavior. The next natural question to ask is how similar are comparable models? One measure of the difference between two comparable models is their relative difference. Define the relative difference of optimal expected revenue between comparable models $\mathscr{M}_{q,\lambda}$ and $\mathscr{M}_{\bar{\mathbf{q}},\bar{\lambda}}$ by

$$g_{n,t}(\mathscr{M}_{q,\lambda}, \mathscr{M}_{\bar{\mathbf{q}},\bar{\lambda}}) := \frac{v_n(t;\mathbf{q},\lambda) - v_n(t;\bar{\mathbf{q}},\bar{\lambda})}{v_n(t;\bar{\mathbf{q}},\bar{\lambda})},$$

for $0 \leq t < T$ and $n > 0$.

**Theorem 2.2.2.** *Let $\mathscr{M}_{\boldsymbol{q},\lambda}$ and $\mathscr{M}_{\bar{\boldsymbol{q}},\bar{\lambda}}$ be comparable models which both have constant demand elasticity $\varepsilon$. For any $0 \leq t < T$ and $n > 0$,*

$$g_{n,t}(\mathscr{M}_{\boldsymbol{q},\lambda}, \mathscr{M}_{\bar{\boldsymbol{q}}\bar{\lambda}}) = \frac{\left(\frac{\mu(\boldsymbol{q})}{\mu(\bar{\boldsymbol{q}})}\right)^{(\varepsilon-1)/\varepsilon}\beta_n(\boldsymbol{q}) - \beta_n(\bar{\boldsymbol{q}})}{\beta_n(\bar{\boldsymbol{q}})}.$$

*Notably, the relative difference $g$ does not depend on time nor the demand magnitude of the models.*

*Proof.* Let $\mathscr{M}_{\mathbf{q},\lambda}$ and $\mathscr{M}_{\bar{\mathbf{q}},\bar{\lambda}}$ be comparable models with $\lambda(p,t) = a(t)p^{-\varepsilon}$ and $\bar{\lambda}(p,t) = \bar{a}(t)p^{-\varepsilon}$. Since the models are comparable, (2.2.1) holds and so $a(t) = \frac{\bar{a}(t)\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}$. Recall the optimal expected revenue from Theorem 2.1.2, modified to include all relevant dependencies,

$$v_n(t;\mathbf{q},\lambda) = \mu(\mathbf{q})\beta_n(\mathbf{q})A(t;a)^{1/\varepsilon}.$$

Recall also that for a constant $c$, $A(t;ca) = cA(t;a)$. Then

$$
\begin{aligned}
g_{n,t}(\mathscr{M}_{\mathbf{q},\lambda}, \mathscr{M}_{\bar{\mathbf{q}},\bar{\lambda}}) &= \frac{v_n\left(t; \mathbf{q}, \frac{\bar{\lambda}\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}\right) - v_n(t; \bar{\mathbf{q}}, \bar{\lambda})}{v_n(t; \bar{\mathbf{q}}, \bar{\lambda})} \\
&= \frac{\mu(\mathbf{q})\beta_n(\mathbf{q}) \left(A\left(t; \frac{\bar{a}\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}\right)\right)^{1/\varepsilon} - \mu(\bar{\mathbf{q}})\beta_n(\bar{\mathbf{q}})A(t;\bar{a})^{1/\varepsilon}}{\mu(\bar{\mathbf{q}})\beta_n(\bar{\mathbf{q}})A(t;\bar{a})^{1/\varepsilon}} \\
&= \frac{\mu(\mathbf{q})\beta_n(\mathbf{q}) \left(\frac{\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}\right)^{1/\varepsilon} A(t;\bar{a})^{1/\varepsilon} - \mu(\bar{\mathbf{q}})\beta_n(\bar{\mathbf{q}})A(t;\bar{a})^{1/\varepsilon}}{\mu(\bar{\mathbf{q}})\beta_n(\bar{\mathbf{q}})A(t;\bar{a})^{1/\varepsilon}} \\
&= \frac{\left(\frac{\mu(\mathbf{q})}{\mu(\bar{\mathbf{q}})}\right)^{(\varepsilon-1)/\varepsilon} \beta_n(\mathbf{q}) - \beta_n(\bar{\mathbf{q}})}{\beta_n(\bar{\mathbf{q}})}.
\end{aligned}
$$

□

Consider two comparable models and then consider those models with their demand magnitude $a(t)\mu(q)$ doubled. For the demand magnitute expression, $\mu(q)$ does not change in this scenario, doubling demand magnitude is equivalent to doubling $a(t)$. Theorem 2.2.2 states that their relative difference will not change. This is surprising, since it might intuitively feel like increased demand should lead to increased differences between the models. The key feature which enables the proof of Theorem 2.2.2 is that the formula for the optimal expected revenue is separable. This property allows the time terms $a(t)$ and $\bar{a}(t)$ to drop out. Since demand magnitude is given by $a(t)\mu(\mathbf{q})$, but in this scenario $\mu(\mathbf{q})$ is fixed, changing the demand magnitude equates to changing $a(t)$. Thus changing the demand magnitude does not change the relative difference.

One of the main goals of this dissertation is to identify the changes generated by generalizing unit order models to variable order models. In the previous section, we saw how analytic results changed and the new techniques needed to handle these changes. This section provided a structure that can be used to compare models and presented results related to these ideas. Of particular note is that variable order models have a comparable model which only has unit orders. In the next section we will use numerical observations to get a more concrete idea of what variable order size models look like, especially relative to their unit order counterparts.

## 2.3 Numerical Observations

This section will provide graphs from numerical calculations to help visualize the results proven thus far in the dissertation. Note that many of the graphs will show continuous plots even though the graphs are technically discrete plots. This makes the graphs easier to parse. Additionally, all of the the code used in this section is presented in the appendix.

We begin with Theorem 2.1.7, which stated that

$$\lim_{n\to\infty}\gamma_n(\mathbf{q}) = \lim_{n\to\infty}\frac{\beta_n(\mathbf{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}.$$

One of the crucial observations from this result is that $\left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon}}$ is an approximation for $\beta_n(\mathbf{q})$, a fact seen in Theorem 2.2.1. In a practical context, such an approximation would be useful in the case where $\beta_n(\mathbf{q})$ may be slow to compute. Such might be the case if dynamic prices are needed frequently. Thus having an idea of how quickly $\beta_n(\mathbf{q})$ converges to $\left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon}}$ is useful.

An algorithm in Mathematica was created to numerically calculate $\beta_n(\mathbf{q})$. Figure 2.1 uses the algorithm to compute $\beta_n(\mathbf{q})$ and then compare convergence of $\beta_n(\mathbf{q})$ and $(n/\mu(\mathbf{q}))^{\frac{\varepsilon-1}{\varepsilon}}$ for four different order size distributions. Note that for these distributions, full weight was not put on $q_M$ because then the graph shows multiple lines per distribution and is hard to interpret. This is due to $\beta_n(\mathbf{q})$ only changing every $M$ terms if $q_M = 1$.

Models with larger $\mu(\mathbf{q})$ appear to have slower convergence rates. We observe from the graph that order size distributions with larger $\mu(\mathbf{q})$ converge more slowly. That said, convergence is within 1% by $n = 250$, even for the distribution with the highest $\mu$. This appears like relatively quick convergence considering large aircraft can easily carry over 500 passengers and concert venues often have thousands of seats.

Figure 2.1: Relative difference between $\beta_n(\mathbf{q})$ and $(n/\mu(\mathbf{q}))^{\frac{\varepsilon-1}{\varepsilon}}$ and for several order size distributions $\mathbf{q}$.

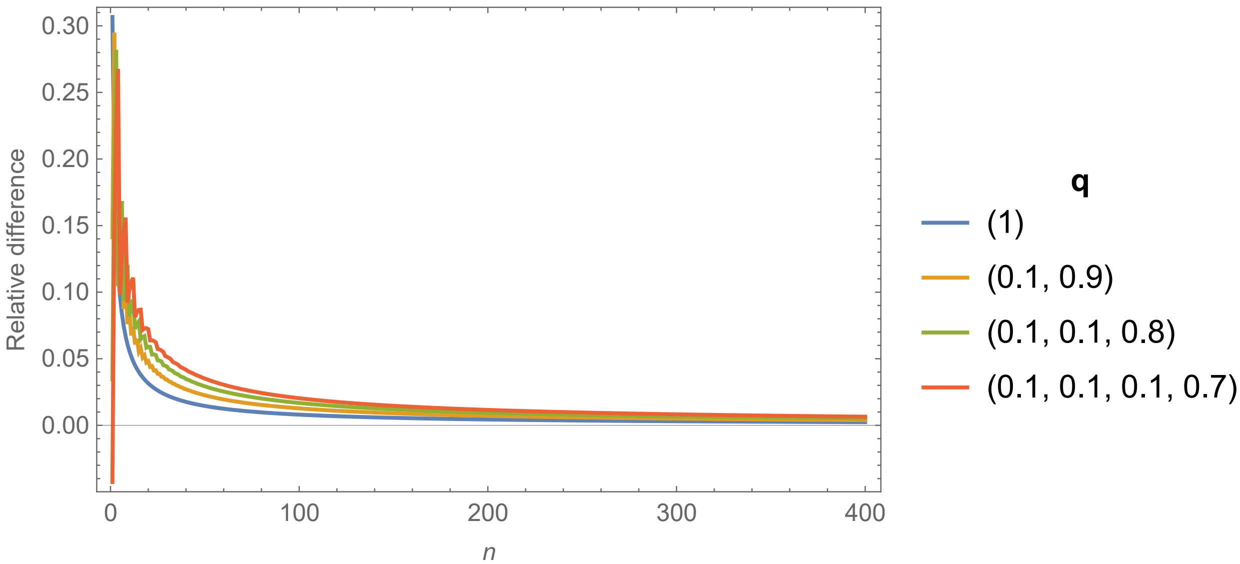
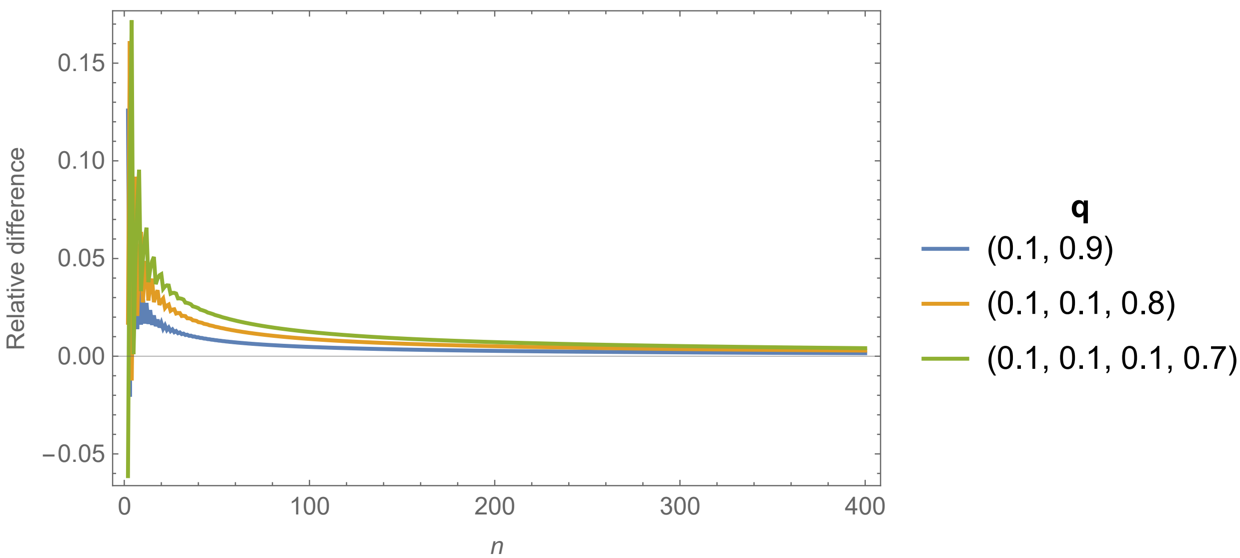



Figure 2.2 shows similarly quick convergence for the optimal expected revenue for a variable order model to its comparable unit order model. This is an important comparison to consider because it indicates the numerical differences between variable order and unit order models. Again, larger average order sizes lead to slower convergence. In this graph we observe that there appears to be more noise for low inventory. This is due to overselling, which allows variable order models to have higher potential revenue at low inventory. Once the inventory size gets large enough, this benefit is largely mitigated. This is one of the reasons why we have waited to address alternative low inventory behavior until Section 3.1.

Figure 2.2: Relative difference of $v_n$ between a variable order model and its comparable unit order model.




Now we consider more general comparable models. The main result for comparable models is Theorem 2.2.1, which states that the expected total revenue of comparable models have the same asymptotic behavior in the inventory $n$. Figure 2.2 gave one application of this idea by comparing the optimal expected revenue of variable order and unit order models. But there is more to consider than just the expected value. Since the model is a compound Poisson type process, any realization of the model will involve randomness.

To examine this randomness, a Mathematica program was created to simulate the Poisson based process defined in section 1.3. One property of Poisson processes with intensity $\lambda$ is that over a small interval of time $\delta t$ there is a $\lambda\delta t$ chance of a customer arriving. By choosing a specific value for $\delta t$ the problem is discretized, which allows the digital implementation of the model. That said, based on testing different $\delta t$, the discretization does not need to be super fine to obtain precise results. We choose $\delta t$ to be around 0.2% of the total time scale for our results.

In each time step, the program randomly determines if a customer arrives, based on the arrival rate $\lambda(p, t)$. Any pricing policy can be used to then update $\lambda(p, t)$ for the next time step. At the end of the simulation, the amount of money earned from sales is recorded.

Figure 2.3 shows probability distributions of this simulated revenue. To generate each distribution, 20,000 trials were run. Each density curve corresponds to a model which uses a different order size distribution $\mathbf{q}$; however, all models used are comparable with demand magnitude 3. The figure confirms the intuition that as the average order size increases, so does the variance of the revenue.

Quantifying the exact amount of variance for each density curve is tricky though. For one, the expected revenue can never go below 0, giving a 1 sided lower limit. But also, multiple order size distributions can have the same average order size. This idea is explored in Figure 2.4, which shows several comparable models which have the same $\mu(\mathbf{q})$, but different order size distributions. Note that these models presented in that plot are all comparable to the models used in Figure 2.3 too. Figure 2.4 reveals that the maximum order size also plays a role in how spread the simulated revenue is.

Figure 2.3: Probability distributions of simulated revenue for comparable models while using policy $p_n^*(t; \mathbf{q})$. Per distribution: trials = 20,000, $n = 100$, $T = 30$, $\varepsilon = 1.5$, demand magnitude = 3.

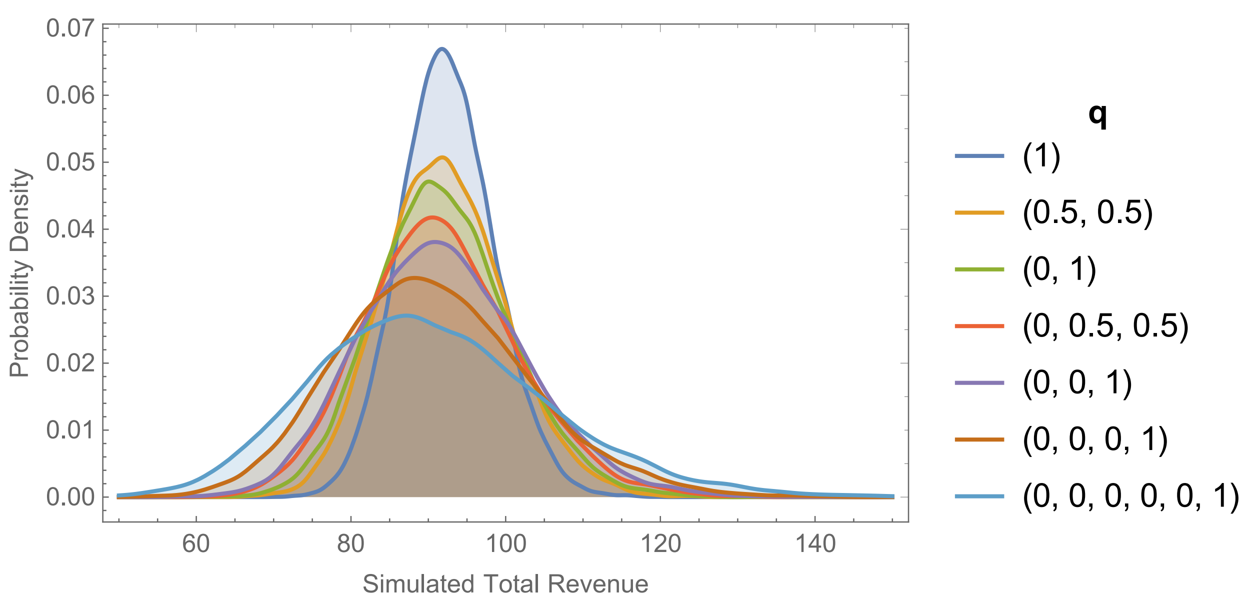
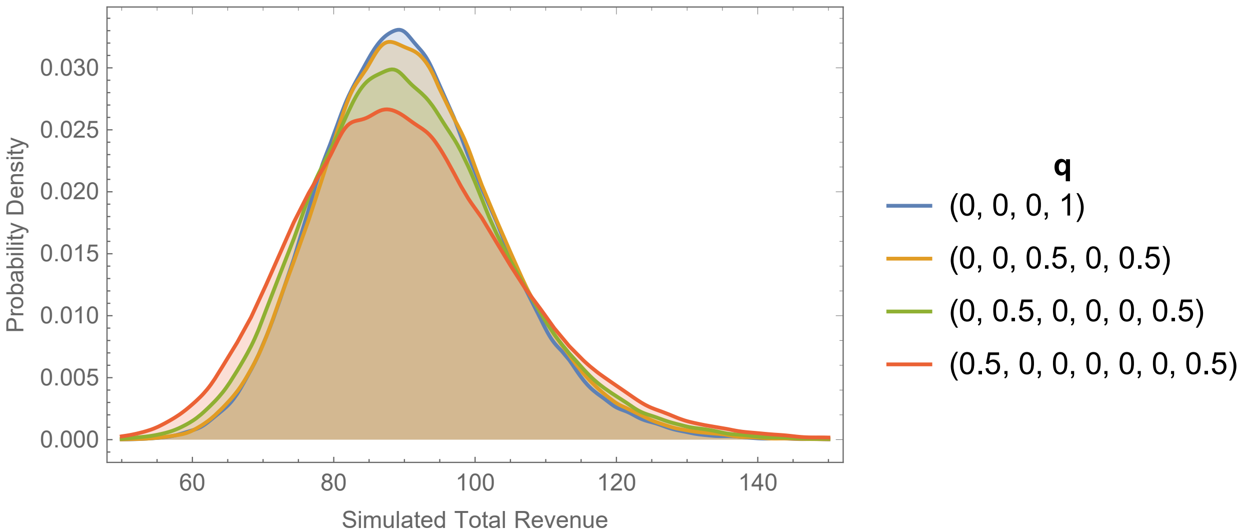


Figure 2.4: Probability distributions of simulated revenue for comparable models with the same average order size while using policy $p_n^*(t; \mathbf{q})$. Per distribution: trials = 100,000, max inventory = 100, $T = 30$, $\varepsilon = 1.5$, demand rate = 3.




The observations in this section highlight interesting aspects of our problem, and the figures provide a more concrete context through which the model can be understood. Analytic results can inform the numerical results, and vice versa. So understanding both gives better understanding of the model. In fact, the numerical algorithms were the original indicator for the comparison results in Section 2.3. We hope that the programs used in this section and presented in the appendix are useful to anyone who wishes to do any future work regarding the variable order size model.

# 3.

# Extensions

The previous sections have examined a model for dynamic pricing with variable order sizes for the specific demand function $\lambda(p) = a(t)p^{-\varepsilon}$. The variable order size model was presented in the most basic setting in order to illustrate the main proof methods and demonstrate the new results variable order sizes creates. In the following sections, we provide adaptations to the basic model.

Section 3.1 discusses how to incorporate different low inventory behavior into the model. One adaptation allows the model to account for penalties for overselling to better model real world situations. Another adaptation lets the seller control the minimum inventory level which they are willing to put their inventory at. This inventory level can be negative, which even allows the seller to control how much overselling they are willing to do. Since most of our results rely on the asymptotic nature of the problem, we are also able to show that different low inventory behavior retains these results.

Section 3.2 explores the idea of social efficiency. This is the idea of maximizing the value to the consumer, rather than the revenue earned. While a straightforward adaptation of our results, social efficiency has important economic implications: a monopolist's pricing under constant demand elasticity is socially efficient. These ideas translate the important economic insights presented in the papers (McAfee and te Velde 2008) to the case of variable order sizes.

Section 3.3 shows how to incorporate additional effects into the model. Some effects are small, like subsidies or tax, but some are large, like advertising. This section expands upon the work in (Helmes and Schlosser 2013). We are able to extend their results in order to find an optimal pricing strategy for constant demand elasticity with advertising and variable order sizes.

Section 3.4 does not assume constant demand elasticity and explores new arrival functions $\lambda$ with variable order sizes. This follows the general dynamic programming development presented in Section 1.3. We are unable to find analytic results like we did in Section 2.1, but we provide a Mathematica program which is able to numerically compute optimal pricing strategies for any $\lambda$. We use it to examine two specific types of arrival rate functions: one which is exponential and another which is linear. The numerical calculations reveal that comparable models with these new arrival rate functions have similar asymptotic behavior, but their relative differences are not independent of the demand magnitude.

## 3.1 Low Inventory Behavior

So far our model has only accounted for one specific type of behavior at low inventory: overselling 0 to $M-1$ items based on how many items the last buyer purchases. This behavior is not very realistic from a practical standpoint; however, this simpler model allowed us to present the results in a more understandable fashion compared to a more complicated model. By establishing the results for a simpler case, it will also be easier to see how modifications to the basic model propagate through the results.

Any change in the model to low inventory behavior changes how much revenue is earned from a sale. Therefore we must go back to the original optimal revenue equation, which was developed through dynamic programming in Section 1.3. Recall (1.3.1):

$$v_n(t;\mathbf{q}) = \sup_p \left[ (1-\lambda(p,t)\delta t) v_n(t+\delta t;\mathbf{q}) + \sum_{i=1}^{M} q_i \lambda(p,t)\delta t(ip + v_{n-i}(t+\delta t;\mathbf{q})) \right]. \quad (3.1.1)$$

Observe that this equation demonstrates why overselling was an issue in the first place: it is recursive with $M$ base case terms required. The initial solution was to define $v_n(t;\mathbf{q}) = 0$ for $n \leq 0$ . Although this choice works mathematically, it is not practical for a couple reasons. First, the optimal expected revenue equation does not account for any expected costs associated with overselling. We would expect there to be some risk for this practice, for example a flight which has too many passengers show up for it requiring compensation to be given out. Second is that there is no control over the number of items which are oversold. The number could be anywhere from 0 to $M-1$ items, based on the order size of the last customer. These two issues are what we seek to overcome in this section.

We first discuss how to model some risk associated with overselling. Note that any costs used are expected costs, because it is uncertain if the cost will be incurred or not. These probabilities could be determined from sales data. The type of costs may take a couple forms: those depending on the price of the oversold item, and those not. Price dependent costs could include things like full or partial refunds of the ticket price. Price independent costs could include renting a hotel for a customer who was unable to take a flight.

It is also important to observe that the potential risk is likely to increase for each extra oversold item. Thinking about a flight, the more total tickets that are sold, the more likely too many passengers show up to take the flight. With this in mind, let $c(n,i,p)$ be the expected costs of selling $0 < i \leq M$ items for a price of $p$ while at inventory level $n$ ($n$ can be negative, allowing us to keep overselling items). Since price dependent expected costs are likely to be some percentage of the sale price, we assume $c$ depends linearly on $p$. Then $c(n,i,p) = \sum_{j=n-i+1}^{n} (\alpha_j p + r_j)$, where $\alpha_j$ and $r_j$ are such that $\alpha_j p + r_j$ denotes the expected cost with selling one item at price $p$ while at inventory level $j$ ($j$ can be negative, indicating overselling risks).

In the dynamic programming equation, $ip$ is the revenue earned from a sale, so we subtract the cost from this value to account for the risks of overselling. Let $v_n^C(t;\mathbf{q})$ denote the optimal expected revenue while accounting for overselling costs. Then adapting (3.1.1) gives

$$v_n^C(t;\mathbf{q}) = \sup_p \Bigg[(1-\lambda(p,t)\delta t)v_n^C(t+\delta t;\mathbf{q}) \\ + \sum_{i=1}^{M} q_i\lambda(p,t)\delta t \left(ip - \sum_{j=n-i+1}^{n}(\alpha_j p + r_j) + v_{n-i}^C(t+\delta t;\mathbf{q})\right)\Bigg] \tag{3.1.2}$$

for $n \geq 0$. We still define the same boundary conditions as before: $v_n^C(t;\mathbf{q}) = 0$ for $n \leq 0$ and $v_n^C(0;\mathbf{q}) = 0$. We now assume $\alpha_j = 1$ and $r_j = 0$ for $j \leq 0$. Observe that this negates any profit from oversold items. This equates to customer buying only available inventory, even if they want more. This choice finally gives control over the minimum inventory level–no items are sold which would take the inventory below 0. In fact, this threshold need not be 0. Let $m$ be the minimum inventory level. This means that items are never sold if it it would bring the inventory below $m$.

By making $m$ negative, one controls the maximum amount of oversold items to be $|m|$. With a minimum inventory level, equation (3.1.2) does not actually change. Rather, the indices of when the equation is valid change. Equation (3.1.2) is now valid for all $n > m$, with $v_n^C(t;\mathbf{q}) = 0$ for $n \leq m$, which we also now assume going forward.

This method for controlling the minimum inventory assumes that customers will buy as much as they are able to. However, for something like ticket sales, this does not always model costumer behavior. For example, a family would not want to buy plane tickets for only part of their family. This provides another method through which the minimum inventory can be controlled. In terms of the dynamic programming equation, this shifts any probability from cases where selling past the minimum inventory level would occur and places this probability with no sales instead. The most items which can be sold is the minimum of $M$ and $n - m$, the difference between our current inventory level $n$ and the minimum inventory level $m$. Let $v_n^G(t;\mathbf{q})$ be the optimal expected revenue with overselling costs and where groups stay together. By adapting (3.1.2) for $v_n^G(t;\mathbf{q})$, it becomes

$$
\begin{aligned}
v_n^G(t;\mathbf{q}) = \sup_p \Bigg[ &\left(1-\lambda(p,t)\delta t + \sum_{i=n-m+1}^{M} q_i\lambda(p,t)\delta t\right) v_n^G(t+\delta t;\mathbf{q}) \\
&+ \sum_{i=1}^{M\wedge(n-m)} q_i\lambda(p,t)\delta t \left(ip - \sum_{j=n-i+1}^{n}(\alpha_j p + r_j) + v_{n-i}^G(t+\delta t;\mathbf{q})\right)\Bigg] \qquad (3.1.3)
\end{aligned}
$$

$n \geq m$. We still define the same boundary conditions as before: $v_n^G(t;\mathbf{q}) = 0$ for $n \leq m$ and $v_n^G(0;\mathbf{q}) = 0$. Now use the dynamic programming equations (3.1.2) and (3.1.3) to heuristically derive their HJB equations, as we did for (1.3.2). Move all terms not multiplied by $\lambda$ to the left-hand side, divide by $\delta t$, and then take the limit as $\delta t \to 0$ to get

$$
\begin{aligned}
\dot{v}_n^C(t;\mathbf{q}) &= -\sup_p \lambda(p,t)\left[-v_n^C(t;\mathbf{q}) + \sum_{i=1}^{M} q_i\left(ip - \sum_{j=n-i+1}^{n}(\alpha_j p + r_j) + v_{n-i}^C(t;\mathbf{q})\right)\right] \\
&= -\sup_p \lambda(p,t)\Bigg[\sum_{i=1}^{M}\left(q_i i - \sum_{j=n-i+1}^{n} q_i\alpha_j\right)p \\
&\qquad - \left(v_n^C(t;\mathbf{q}) - \sum_{i=1}^{M} q_i\left(v_{n-i}^C(t;\mathbf{q}) - \sum_{j=n-i+1}^{n} r_j\right)\right)\Bigg] \qquad (3.1.4)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{v}_n^G(t;\mathbf{q}) = -\sup_p \lambda(p,t)\Bigg[&\left(-1 + \sum_{i=n-m+1}^{M} q_i\right) v_n^G(t;\mathbf{q}) \\
&+ \sum_{i=1}^{M\wedge(n-m)} q_i\left(ip - \sum_{j=n-i+1}^{n}(\alpha_j p + r_j) + v_{n-i}^G(t;\mathbf{q})\right)\Bigg]. \\
= -\sup_p \lambda(p,t)\Bigg[&\sum_{i=1}^{M\wedge(n-m)}\left(q_i i - \sum_{j=n-i+1}^{n} q_i\alpha_j\right)p \\
&- \left(\sum_{i=1}^{M\wedge(n-m)} q_i v_n^G(t;\mathbf{q}) - \sum_{i=1}^{M\wedge(n-m)} q_i\left(v_{n-i}^G(t;\mathbf{q}) - \sum_{j=n-i+1}^{n} r_j\right)\right)\Bigg]. \qquad (3.1.5)
\end{aligned}
$$

Note that the second equality is justified because

$$-1+\sum_{i=n-m+1}^{M} q_i = -\sum_{i=1}^{M} q_i + \sum_{i=n-m+1}^{M} q_i = -\sum_{i=1}^{M\wedge(n-m)} q_i.$$

For constant demand elasticity, $\lambda(p,t) = a(t)p^{-\varepsilon}$. To find the values $p*$ which attain the supremum in (3.1.4) and (3.1.5), notice the general form of these equations: $-\sup_p a(t)p^{-\varepsilon}\left(K_1 p - K_2\right)$, where $K_1$ and $K_2$ are independent of price. This supremum expression is maximized when

$$p^* = \frac{\varepsilon}{\varepsilon-1}K_1^{-1}K_2 \tag{3.1.6}$$

and evaluates to $-a(t)K_1^{\varepsilon}\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}K_2^{1-\varepsilon}$. Thus the HJB equations for $n > m$ become

$$\begin{aligned}\dot{v}_n^C(t;\mathbf{q}) = &-a(t)\left(\sum_{i=1}^{M}\left(q_i i - \sum_{j=n-i+1}^{n} q_i\alpha_j\right)\right)^{\varepsilon}\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\\ &\times\left(v_n^C(t;\mathbf{q}) - \sum_{i=1}^{M} q_i\left(v_{n-i}^C(t;\mathbf{q}) - \sum_{j=n-i+1}^{n} r_j\right)\right)^{1-\varepsilon}\end{aligned} \tag{3.1.7}$$

and

$$\begin{aligned}\dot{v}_n^G(t;\mathbf{q}) = &-a(t)\left(\sum_{i=1}^{M\wedge(n-m)}\left(q_i i - \sum_{j=n-i+1}^{n} q_i\alpha_j\right)\right)^{\varepsilon}\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\\ &\times\left(\sum_{i=1}^{M\wedge(n-m)} q_i v_n^G(t;\mathbf{q}) - \sum_{i=1}^{M\wedge(n-m)} q_i\left(v_{n-i}^G(t;\mathbf{q}) - \sum_{j=n-i+1}^{n} r_j\right)\right)^{1-\varepsilon}\end{aligned} \tag{3.1.8}$$

We place a few restrictions on the expected costs. First, assume that there exists an $N$ such that $\alpha_n = 0$ and $r_n = 0$ for all $n > N$. This way, there are eventually no costs (or more appropriately, risks) with inventory sales, and the previous equations are in harmony with with the original dynamic programming equation (1.3.1) for large enough $n$. Also, we will assume the simpler model with $r_j = 0$ for all $j$. For this model we will show the optimal pricing strategy.

To find the closed form expression for $v_n^C(t;\mathbf{q})$ and $v_n^G(t;\mathbf{q})$, we need to define some analogs of terms used with the basic model. Define modified average order size $\mu$ terms

$$\mu_n^C(\mathbf{q}) := \sum_{i=1}^{M}\left(q_i i - \sum_{j=n-i+1}^{n} q_i\alpha_j\right)$$
$$\mu_n^G(\mathbf{q}) := \sum_{i=1}^{M\wedge(n-m)}\left(q_i i - \sum_{j=n-i+1}^{n} q_i\alpha_j\right).$$

Note that for $n \geq N$, $\mu(\mathbf{q}) = \mu_n^C(\mathbf{q}) = \mu_n^G(\mathbf{q})$.

We also define some $\beta$-like sequences. By this, we mean sequences which have the same recursion structure as $\beta_n(\mathbf{q})$ for large enough $n$. Define the sequence $(\beta_n^C(\mathbf{q}))_n$ by $\beta_n^C(\mathbf{q}) = 0$ for $n \leq m$ and such that for $n > m$, $\beta_n^C(\mathbf{q})$ is the non-negative solution to

$$\frac{\varepsilon-1}{\varepsilon} = \beta_n^C(\mathbf{q})^{\frac{1}{\varepsilon-1}}\left(\beta_n^C(\mathbf{q}) - \sum_{i=1}^{M} q_i\left(\frac{\mu_{n-i}^C(\mathbf{q})}{\mu_n^C(\mathbf{q})}\right)\beta_{n-i}^C(\mathbf{q})\right).$$

Similarly define $(\beta_n^G(\mathbf{q}))_n$ by $\beta_n^G(\mathbf{q}) = 0$ for $n \leq m$, and such that for $n > m$, $\beta_n^G(\mathbf{q})$ is the non-negative solution to

$$\frac{\varepsilon-1}{\varepsilon} = \beta_n^G(\mathbf{q})^{\frac{1}{\varepsilon-1}}\left(\sum_{i=1}^{M\wedge(n-m)} q_i\beta_n^G(\mathbf{q}) - \sum_{i=1}^{M\wedge(n-m)} q_i\left(\frac{\mu_{n-i}^G(\mathbf{q})}{\mu_n^G(\mathbf{q})}\right)\beta_{n-i}^C(\mathbf{q})\right). \tag{3.1.9}$$

Existence and uniqueness of the two previous $\beta$ sequences follows from (2.1.1).

**Corollary 3.1.1.** *(Of Theorem 2.1.2) Assume constant demand elasticity $\varepsilon$ and a model with order size distribution $\boldsymbol{q}$. For all $t \geq 0$ and $n$, the optimal expected revenue for two different low inventory behaviors is given by*

$$v_n^C(t;\boldsymbol{q}) = \mu_n^C(\boldsymbol{q})\beta_n^C(\boldsymbol{q})A(t)^{1/\varepsilon}, \qquad v_n^G(t;\boldsymbol{q}) = \mu_n^G(\boldsymbol{q})\beta_n^G(\boldsymbol{q})A(t)^{1/\varepsilon},$$

*with optimal pricing strategies*

$$p_n^{C*}(t;\boldsymbol{q}) = \beta_n^C(\boldsymbol{q})^{-1/(\varepsilon-1)}A(t)^{1/\varepsilon}, \qquad\qquad p_n^{G*}(t;\boldsymbol{q}) = \beta_n^G(\boldsymbol{q})^{-1/(\varepsilon-1)}A(t)^{1/\varepsilon}.$$

*Proof.* The proof follows the same technique as that of Theorem 2.1.2. The main aspect we wish to highlight is how the new coefficients in the equations for $\beta_n^C(t;\mathbf{q})$ and $\beta_n^G(t;\mathbf{q})$ come into play. The proof will be completed for the more complex term $v_n^G(q;\mathbf{q})$, but note that the proof is essentially the same for $v_n^C(t;\mathbf{q})$.

Proceed by induction to show $v_n^G(t;\mathbf{q}) = \mu_n^G n\beta_n^G(\mathbf{q})A(t)^{1/\varepsilon}$. For $n \leq m$, $v_n^G(t;\mathbf{q}) = 0$ and $\beta_n^G(\mathbf{q}) = 0$, showing the base cases hold. Assume that the induction assumption holds up to $n-1$. Equation (3.1.8), is an ordinary differential equation, so verifying the induction holds for this equation will prove the induction. We will suppress dependencies for readability. Take the derivative of the induction assumption to get the left-hand side of (3.1.8) (also recall $A(t) = \int_t^T a(s)ds$):

$$\begin{aligned}\dot{v}_n^G &= \mu_n^G\beta_n^G\left(\frac{1}{\varepsilon}\right)A^{(1-\varepsilon)/\varepsilon}(-a)\\ &= -a\mu_n^G A^{(1-\varepsilon)/\varepsilon}\left(\frac{1}{\varepsilon}\right)\beta_n^G.\end{aligned}$$

Substituting the induction assumption into the right-hand side of (3.1.8) gives

$$\begin{aligned}&-a\left(\mu_n^G\right)^{\varepsilon}\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\left(\sum_{i=1}^{M\wedge(n-m)} q_i\mu_n^G\beta_n^G A^{1/\varepsilon} - \sum_{i=1}^{M\wedge(n-m)} q_i\mu_{n-i}^G\beta_{n-i}^G A^{1/\varepsilon}\right)^{1-\varepsilon}\\ &\quad= -a\mu_n^G A^{(1-\varepsilon)/\varepsilon}\left(\frac{1}{\varepsilon}\right)\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon-1}\left(\sum_{i=1}^{M\wedge(n-m)} q_i\beta_n^G - \sum_{i=1}^{M\wedge(n-m)} q_i\left(\frac{\mu_{n-i}^G}{\mu_n^G}\right)\beta_{n-i}^G\right)^{1-\varepsilon}\\ &\quad= -a\mu_n^G A^{(1-\varepsilon)/\varepsilon}\left(\frac{1}{\varepsilon}\right)\beta_n^G,\end{aligned}$$

where the last equality is justified by the definition of $\beta_n^G(\mathbf{q})$, equation (3.1.9). Observe that the first equality, where $A^{1/\varepsilon}$ was factored, indicates why there would be problems if $r_j \neq 0$ for all $j$. The cost terms $r_j$ have no time dependency, so $A^{1/\varepsilon}$ could not be factored out. This cannot be compensated in the equation for $\beta_n^G(t;\mathbf{q})$, as this term needs to be independent of time in order to proceed with our prior analysis.

Thus we have shown that the left-hand and right-hand sides of (3.1.8) are equal for the induction assumption at $n$, verifying that the induction assumption

$$v_n^G(t;\mathbf{q}) = \mu_n^G(\mathbf{q})\beta_n^G(\mathbf{q})A(t)^{1/\varepsilon}$$

holds for all $n$. Furthermore, by substituting this equation into (3.1.6) gives

$$\begin{aligned}
p_n^*(t;\mathbf{q}) &= \frac{\varepsilon}{\varepsilon-1}\left(\mu_n^G\right)^{-1}\left(\sum_{i=1}^{M\wedge(n-m)} q_i v_n^G(t;\mathbf{q}) - \sum_{i=1}^{M\wedge(n-m)} q_i v_{n-i}^G(t;\mathbf{q})\right)\\
&= \frac{\varepsilon}{\varepsilon-1}\left(\mu_n^G\right)^{-1}\left(\sum_{i=1}^{M\wedge(n-m)} q_i \mu_n^G(\mathbf{q})\beta_n^G(\mathbf{q})A(t)^{1/\varepsilon} - \sum_{i=1}^{M\wedge(n-m)} q_i \mu_{n-i}^G(\mathbf{q})\beta_n^G(\mathbf{q})A(t)^{1/\varepsilon}\right)\\
&= A(t)^{1/\varepsilon}\frac{\varepsilon}{\varepsilon-1}\left(\sum_{i=1}^{M\wedge(n-m)} q_i \beta_n^G(\mathbf{q}) - \sum_{i=1}^{M\wedge(n-m)} q_i\left(\frac{\mu_{n-i}^G(\mathbf{q})}{\mu_n^G(\mathbf{q})}\right)\beta_n^G(\mathbf{q})\right)\\
&= A(t)^{1/\varepsilon}\beta_n^G(t;\mathbf{q})^{1/(\varepsilon-1)},
\end{aligned}$$

finishing the proof. □

The previous corollaries in this section have established that the closed form expression for optimal expected revenue remains structurally consistent across different low inventory properties. It is worth observing that for $n \geq M$, both $\mu_n^C(\mathbf{q}) = \mu_n^G(\mathbf{q}) = \mu(\mathbf{q})$ and the structure of the recursion equations for $\beta_n^C(\mathbf{q})$, $\beta_n^G(\mathbf{q})$, and $\beta_n(\mathbf{q})$ is the same. This feature will play an important role in showing asymptotic results for the new low inventory terms.

We now wish to show that the result from Theorem 2.1.7,

$$\lim_{n\to\infty}\frac{\beta_n(\mathbf{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}, \tag{3.1.10}$$

holds for $\beta_n^C(\mathbf{q})$ and $\beta_n^G(\mathbf{q})$ as well. Examining the proof of Theorem 2.1.7 reveals two properties of $\beta_n(\mathbf{q})$ that were necessary: the recursive structure of $\beta_n(\mathbf{q})$ and that $\beta_n(\mathbf{q})$ was non-decreasing. $\beta_n^C(\mathbf{q})$ and $\beta_n^G(\mathbf{q})$ both have the proper recursive structure for large enough $n$, but numerical calculations show that the non-decreasing property does not always hold for them. However, we can find upper and lower bounds for $\beta_n^C(\mathbf{q})$ and $\beta_n^G(\mathbf{q})$ using other $\beta$-like sequences which are non-decreasing and have the limiting behavior of (3.1.10).

The particular $\beta$-type sequence we want has base cases of any value (not just 0), and also these base cases may be for any $M$ successive inventory levels (not just for $n = -M+1$ to $n = 0$). Let $N$ be an integer and let $B = (b_n)_{n=N-M+1}^{N}$ be a finite sequence of $M$ terms. Then define the sequence $(\beta_n(\mathbf{q}, B))_{n=N-M+1}^{\infty}$ by $\beta_n(\mathbf{q}; B) = b_n$ for $N - M + 1 \leq n \leq N$ (in other words, $B$ is the base case sequence for $\beta_n(\mathbf{q}; B)$) and such that for $n > N$, $\beta_n(\mathbf{q}; B)$ is the non-negative solution to

$$\frac{\varepsilon - 1}{\varepsilon} = \beta_n(\mathbf{q}; B)^{\frac{1}{\varepsilon-1}} \left( \beta_n(\mathbf{q}; B) - \sum_{i=1}^{M} q_i \beta_{n-i}(\mathbf{q}; B) \right). \tag{3.1.11}$$

Existence and uniqueness of this solution follows from Lemma 2.1.1.

**Theorem 3.1.2.** *For any probability distribution $\boldsymbol{q}$ and finite base case sequence $B = (b_n)_{n=N-M+1}^{N}$, where $N$ is an integer,*

$$\lim_{n\to\infty} \frac{\beta_n(\boldsymbol{q}; B)}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\boldsymbol{q})^{\frac{1-\varepsilon}{\varepsilon}}.$$

*Proof.* First we show the result if the base case sequence $B$ is non-decreasing. Then to prove for a general $B$, we find upper and lower bounds using sequences whose base cases are non-decreasing.

Suppose $B$ is non-decreasing. Recall Theorem 2.1.7, which stated that

$$\lim_{n\to\infty} \frac{\beta_n(\mathbf{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}. \tag{3.1.12}$$

The proof of that theorem relied on two properties of $\beta_n(\mathbf{q})$: the recursion equation for $\beta_n(\mathbf{q})$ (equation (2.1.5)) and the non-decreasing nature of $\beta_n(\mathbf{q})$ (Theorem 2.1.3). So if it is shown that $\beta_n(\mathbf{q}; B)$ has these two properties, (3.1.12) will also hold with $\beta_n(\mathbf{q}; B)$ in place of $\beta_n(\mathbf{q})$.

It is readily seen that the recursion equation for $\beta_n(\mathbf{q}; B)$, equation (3.1.11), matches the structure of the recursion relation for $\beta_n(\mathbf{q})$, equation (2.1.5). To show $\beta_n(\mathbf{q}; B)$ is non-decreasing, recall Theorem 2.1.3, which stated that $\beta_n(\mathbf{q})$ was non-decreasing. The proof of that theorem used induction and relied on the recursive equation for $\beta_n(\mathbf{q})$ along with the fact that the base cases of $\beta_n(\mathbf{q})$ also satisfied the induction. Since $B$, the base case sequence for $\beta_n(\mathbf{q}; B)$, is non-decreasing, the proof of Theorem 2.1.3 also shows $\beta_n(\mathbf{q}; B)$ is non-decreasing. Therefore the limit (3.1.12) holds for $\beta_n(\mathbf{q}; B)$:

$$\lim_{n\to\infty} \frac{\beta_n(\mathbf{q}, B)}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}. \tag{3.1.13}$$

Now consider any finite base case sequence $B = (b_n)_{n=N-M+1}^{N}$. Define the constant base case sequences $B_{max} = (\max(B))_{n=N-M+1}^{N}$ and $B_{min} = (\min(B))_{n=N-M+1}^{N}$. These constant sequences are useful because they are non-decreasing, so by what was shown earlier in the proof, $\lim_{n\to\infty} \frac{\beta_n(\mathbf{q};B_{min})}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}} = \lim_{n\to\infty} \frac{\beta_n(\mathbf{q};B_{max})}{n^{\frac{\varepsilon-1}{\varepsilon}}}$.

We now follow an argument similar to the one for Theorem 2.1.3. Proceed by induction to show that $\beta_n(\mathbf{q}; B_{min}) \le \beta_n(\mathbf{q}; B) \le \beta_n(\mathbf{q}; B_{max})$ for all $n \ge N-M+1$. By definition, the induction assumption holds for $N-M+1 \le n \le N$. Now suppose the induction assumption holds up to $n-1$. The $\beta_n(\mathbf{q}; B)$ terms are calculated recursively, so to compute $\beta_n(\mathbf{q}; B)$, the terms $\beta_k(\mathbf{q}; B)$ for any $k < n$ are known and treated as constants when solving the recursion equation (3.1.11) for $\beta_n(\mathbf{q}; B)$. Thus computing $\beta_n(\mathbf{q}; B)$ is equivalent to solving the equation $g(x) = \frac{\varepsilon-1}{\varepsilon}$ for $x$ where

$$g(x) := x^{\frac{\varepsilon}{\varepsilon-1}} - \sum_{i=1}^{M} q_i \beta_{n-i}(\mathbf{q}, B) x^{\frac{1}{\varepsilon-1}}.$$

Similarly, consider the functions $f$ corresponding to $\beta_n(\mathbf{q}; B_{max})$ and $h$ corresponding to $\beta_n(\mathbf{q}; B_{min})$,

$$f(x) := x^{\frac{\varepsilon}{\varepsilon-1}} - \sum_{i=1}^{M} q_i \beta_{n-i}(\mathbf{q}, B_{max}) x^{\frac{1}{\varepsilon-1}},$$

$$h(x) := x^{\frac{\varepsilon}{\varepsilon-1}} - \sum_{i=1}^{M} q_i \beta_{n-i}(\mathbf{q}, B_{min}) x^{\frac{1}{\varepsilon-1}}.$$

The inequality

$$f(x) \leq g(x) \leq h(x) \tag{3.1.14}$$

holds by comparing the coefficients of $f$, $g$, and $h$ and applying the induction assumption. That means the solution to $f(x) = \frac{\varepsilon-1}{\varepsilon}$ is greater than or equal to the solution for $g(x) = \frac{\varepsilon-1}{\varepsilon}$, which is greater than or equal to the solution for $h(x) = \frac{\varepsilon-1}{\varepsilon}$. In other words,

$$\beta_n(\mathbf{q}; B_{min}) \leq \beta_n(\mathbf{q}; B) \leq \beta_n(\mathbf{q}; B_{max}), \tag{3.1.15}$$

which completes the induction.

Note that the solutions exist and are unique for $x \geq 0$ by Lemma 2.1.1. Take inequality (3.1.15), divide by $n^{\frac{\varepsilon-1}{\varepsilon}}$ and take the limit as $n \to \infty$ to get

$$\lim_{n\to\infty} \frac{\beta_n(\mathbf{q}; B_{min})}{n^{\frac{\varepsilon-1}{\varepsilon}}} \leq \lim_{n\to\infty} \frac{\beta_n(\mathbf{q}; B)}{n^{\frac{\varepsilon-1}{\varepsilon}}} \leq \lim_{n\to\infty} \frac{\beta_n(\mathbf{q}; B_{max})}{n^{\frac{\varepsilon-1}{\varepsilon}}}.$$

Then applying the limit (3.1.13) shows

$$\mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}} \leq \lim_{n\to\infty} \frac{\beta_n(\mathbf{q}; B)}{n^{\frac{\varepsilon-1}{\varepsilon}}} \leq \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}},$$

and equality holds throughout, completing the proof. □

**Corollary 3.1.3.**

$$\lim_{n\to\infty} \frac{\beta_n^G(\boldsymbol{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\boldsymbol{q})^{\frac{1-\varepsilon}{\varepsilon}}.$$

$$\lim_{n\to\infty} \frac{\beta_n^C(\boldsymbol{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\boldsymbol{q})^{\frac{1-\varepsilon}{\varepsilon}},$$

*Proof.* Observe that $\beta_n^G(\mathbf{q})$ differs from a typical $\beta$-like sequence because for $m < n < M$ its recursion equation (3.1.9), differs from the typical $\beta$-like recursion; however, for $n \geq M$ the recursion structure is the same. The idea is to treat the values of $\beta_n^G(\mathbf{q})$ for $m < n < m+M$ from (3.1.9) as the base cases of a new $\beta$-like sequence. Then this new $\beta$-like sequence and $\beta_n^G(\mathbf{q})$ would actually be one and the same, but the $\beta$-like sequence would allow Theorem 3.1.2 to be applied to it.

To show this formally, let $B = (\beta_n^G(\mathbf{q}))_{n=m}^{m+M-1}$. Then $\beta_n^G(\mathbf{q}) = \beta_n(\mathbf{q}, B)$ for all $n \geq m$ and

$$\lim_{n\to\infty} \frac{\beta_n^C(\mathbf{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \lim_{n\to\infty} \frac{\beta_n(\mathbf{q}, B)}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}},$$

completing the proof. A similar argument shows the result for $\beta_n^C(\mathbf{q})$ as well. □

We have shown two ways to control the minimum inventory level when a customer's demand would place the inventory below that level. Either the customer buys as many items as they can, down to the minimum inventory level, or the customer buys no items. A particularly useful application of defining the minimum inventory level is trying to determine the optimal amount of overselling to do, which is an interesting optimization problem with its own challenges. We also showed how to implement a penalty cost dependent linearly on the price for selling an item while at a particular inventory level. This allows us to implement costs which might be incurred while engaging in overselling.

With these adaptations to the model for low inventory, we were able to keep the structure of our original results by creating analogs of $\mu(\mathbf{q})$ and $\beta_n(\mathbf{q})$. The key feature that made this work was that these terms matched with the original terms for a large enough inventory level. These results agree with the intuition that the behavior when the inventory is low should have little influence on the behavior when inventory is high. It also confirms that dynamic pricing will be most powerful the smaller the inventory.

## 3.2 Social Efficiency

For a unit order model with constant demand elasticity, the surprising result that a monopolist sets socially efficient prices was shown in (McAfee and te Velde 2008). We find this also holds for variable order sizes. The mathematical techniques presented are minor adaptations from the main results, but they are important to show some of the economic connections with constant demand elasticity.

When thinking about our arrival rate function $\lambda(p,t)$, it is the compound rate of customers who arrive and also are willing to order some number of items at the current price. Suppose $\lambda(p,t)$ can be written to highlight both of those factors, that is $\lambda(p,t) = \alpha(t)(1-F(p))$, where $\alpha(t)$ is the arrival rate of customers at time $t$ and $1-F(p)$ is the probability that a customer is willing to pay at least $p$ for each item they order. It is reasonable that $\alpha(t)$ is independent of $p$, since with no advertising, customers will not know the price before they arrive.

Let $V$ be the social value of an item to a customer, where $V$ can be thought of as the maximum price a customer would pay for an item. So when a customer buys an item at price $p$, the social value of the item to them is at least $p$. Then how much is the expected value of $V$, given that $V \geq p$? Letting $f(v)$ be the probability density function of $V$, we get

$$\mathbb{E}[V|V \geq p] = \int_p^\infty \frac{vf(v)}{1-F(p)}dv = \int_p^\infty \frac{\alpha(t)vf(v)}{\alpha(t)(1-F(p))}dv = \int_p^\infty \frac{-v\frac{d\lambda}{dp}(v,t)}{\lambda(p,t)}dv,$$

and under constant demand elasticity where $\lambda(p,t) = a(t)p^{-\varepsilon}$,

$$\mathbb{E}[V|V \geq p] = \int_p^\infty \frac{\varepsilon v^{-\varepsilon}}{p^{-\varepsilon}} dv = \frac{\varepsilon}{\varepsilon - 1} p.$$

Let $S_n(t;\mathbf{q})$ be the optimal expected total social value of a product when there are $n$ items for sale at current time $0 \leq t \leq T$, given constant demand elasticity $\varepsilon$ and customer order distribution $\mathbf{q}$. Then there is a dynamic programming equation similar to (1.3.1), except when a sale of $i$ items occurs, instead of earning $ip$ revenue, $i\mathbb{E}[V|V \geq p] = i\left(\frac{\varepsilon}{\varepsilon-1}\right)p$ social value is earned. That is,

$$\begin{aligned} S_n(t;\mathbf{q}) = \sup_p \Bigg[ & \underbrace{(1-\lambda(p,t)\delta t)S_n(t+\delta t;\mathbf{q})}_{\text{From selling no items}} \\ & + \sum_{i=1}^M \underbrace{q_i \lambda(p,t)\delta t \left(\frac{i\varepsilon}{\varepsilon-1}p + S_{n-i}(t+\delta t;\mathbf{q})\right)}_{\text{From selling } i \text{ items for } p \text{ each}} \Bigg]. \end{aligned}$$

As we have seen several times now, this yields an HJB equation similar to (1.3.2),

$$\begin{aligned} S_n'(t;\mathbf{q}) &= -\sup_p a(t)p^{-\varepsilon}\left[-S_n(t;\mathbf{q}) + \sum_{i=1}^M q_i\left(\frac{i\varepsilon}{\varepsilon-1}p + S_{n-i}(t;\mathbf{q})\right)\right] \\ &= -\sup_p a(t)p^{-\varepsilon}\left[\frac{\mu(\mathbf{q})\varepsilon}{\varepsilon-1}p - \left(S_n(t;\mathbf{q}) - \sum_{i=1}^M q_i S_{n-i}(t;\mathbf{q})\right)\right] \end{aligned}$$

The price $p^{E*}$ which attains the supremum is

$$p_n^{E*}(t;\mathbf{q}) = \mu(\mathbf{q})^{-1}\left(S_n(t;\mathbf{q}) - \sum_{i=1}^M q_i S_{n-i}(t;\mathbf{q})\right),$$

and so the supremum evaluates to

$$
\begin{aligned}
S_n'(t;\mathbf{q}) =& -a(t)\mu(\mathbf{q})^{\varepsilon}\left(\frac{1}{\varepsilon-1}\right)\left(S_n(t;\mathbf{q})-\sum_{i=1}^{M} q_i S_{n-i}(t;\mathbf{q})\right)^{1-\varepsilon}.\\
=& -a(t)\left(\frac{\mu(\mathbf{q})\varepsilon}{\varepsilon-1}\right)^{\varepsilon}\frac{(\varepsilon-1)^{\varepsilon-1}}{\varepsilon^{\varepsilon}}\left(S_n(t;\mathbf{q})-\sum_{i=1}^{M} q_i S_{n-i}(t;\mathbf{q})\right)^{1-\varepsilon}.
\end{aligned}
$$

We see this equation is now the same as (2.1.4), except with $\mu(\mathbf{q})$ replaced by $\left(\frac{\mu(\mathbf{q})\varepsilon}{\varepsilon-1}\right)^{\varepsilon}$. Therefore Theorem 2.1.2 holds with the same substitution. Hence the optimal revenue equation (2.1.6) now becomes

$$
S_n(t;\mathbf{q}) = \left(\frac{\mu(\mathbf{q})\varepsilon}{\varepsilon-1}\right)^{\varepsilon}\beta_n(\mathbf{q})A(t)^{1/\varepsilon}
$$

and the optimal expected price remains unchanged,

$$
p_n^{E*} = \beta_n(\mathbf{q})^{-1/(\varepsilon-1)}A(t)^{1/\varepsilon}.
$$

This proves the following result.

**Corollary 3.2.1.** *With constant demand elasticity and variable order sizes, the monopolist optimal pricing strategy is also the strategy which maximizes social efficiency.*

We can draw some parallels between the social efficiency and the low inventory behavior in the previous section. In both cases, one of the modifications to the dynamic programming equation was what was earned when a sale occurred. The structure of the dynamic programming equation is such that terms which modify earnings based on price do not influence the optimal pricing strategy. Rather, just the total expected revenue (or social value) is influenced. See Corollary 3.1.1).

## 3.3 Advertising and Infinite Time Horizon

This section discusses how to introduce other features into the variable order size dynamic pricing model. The main adaptations are sale earning modifiers, discounting, and advertising. Sale earning modifiers include the effect of taxes, subsides, and costs, which influence the profit earned from one sale. Discounting allows the model to support an infinite time horizon. Advertising changes the basic problem, since there are now two features which a seller can control, and so the seller is trying to find a pair of pricing and advertising policies which optimizes revenue. The work in this section adapts the work of (Helmes and Schlosser 2013) to include variable order sizes.

Recall the dynamic programming equation (1.3.1),

$$v_n(t;\mathbf{q}) = \sup_p \left[ \underbrace{(1-\lambda(p,t)\delta t)v_n(t+\delta t;\mathbf{q})}_{\text{From selling no items}} + \sum_{i=1}^{M} \underbrace{q_i\lambda(p,t)\delta t(ip + v_{n-i}(t+\delta t;\mathbf{q}))}_{\text{From selling } i \text{ items for } p \text{ each}} \right].$$

As we proceed, we will discuss how new model features influence this equation. First, consider effects which modify the amount of revenue earned on a sale. Suppose there are taxes or subsidies, based on the sale price of an item. This introduces a multiplicative factor $f(t)$ such that a sale at price $p$ earns $f(t)p$ instead of $p$. Thus the dynamic programming equation becomes

$$\begin{aligned} v_n(t;\mathbf{q}) = \sup_p \Bigg[ &(1-\lambda(p,t)\delta t)v_n(t+\delta t;\mathbf{q}) \\ &+ \sum_{i=1}^{M} q_i\lambda(p,t)\delta t(if(t)p + v_{n-i}(t+\delta t;\mathbf{q})) \Bigg]. \end{aligned}$$

Now consider advertising. Since the dynamic programming equation considers a small interval of time $\delta t$, it makes the most sense to think about a fixed advertising rate $w > 0$ (later we let $w$ depend on time). Suppose advertising has a tax or subsidy factor $k(t)$. Then to acheive an effective advertising rate $w$, a rate of $k(t)w$ needs to be spent. Over the time span $\delta t$, the total spent on advertising is $(\delta t)k(t)w$. This new cost is subtracted from all terms in the dynamic programming equation, since it is incurred regardless if a sale occurs or not.

Moreover, advertising makes other fundamental changes. The supremum is now taken over both price and advertising, since both terms can be controlled. Advertising also affects the arrival rate of customers; more advertising implies a larger arrival rate. Thus, the arrival rate $\lambda(p, t)$ must now be $\lambda(p, w, t)$ to account for this dependency. With advertising, the new dynamic programming equation is

$$
\begin{aligned}
v_n(t; \mathbf{q}) = \sup_{p>0, w\geq 0} \Bigg[ &(1 - \lambda(p, w, t)\delta t) v_n(t + \delta t; \mathbf{q}) \\
&+ \sum_{i=1}^{M} q_i \lambda(p, w, t)\delta t (if(t)p + v_{n-i}(t + \delta t; \mathbf{q})) - (\delta t)k(t)w \Bigg].
\end{aligned}
$$

Finally consider discounting. If $r(t) > 0$ is the instantaneous time-dependent discount rate, then over a time period $\delta t$ there is an accumulated discount rate $R(\delta t) = \int_0^{\delta t} r(s)ds$. Any future expenditures or earnings are adjusted to their present time value, and it is widely known in the literature that the appropriate multiplicative factor is $e^{-R(\delta t)}$. In other words, every term on the right hand side of the previous equation is multiplied by $e^{-R(\delta t)}$. Note that $R(t) + R(\delta t) = R(t + \delta t)$, and so

$$
\begin{aligned}
e^{-R(t)} v_n(t; \mathbf{q}) = e^{-R(t+\delta t)} \sup_{p>0, w\geq 0} \Bigg[ &(1 - \lambda(p, w, t)\delta t) v_n(t + \delta t; \mathbf{q}) \\
&+ \sum_{i=1}^{M} q_i \lambda(p, w, t)\delta t (if(t)p + v_{n-i}(t + \delta t; \mathbf{q})) - (\delta t)k(t)w \Bigg].
\end{aligned}
$$

As seen previously, we work towards developing the HJB equations by rearranging the previous equation as

$$\frac{e^{-R(t+\delta t)}v_n(t;\mathbf{q}) - e^{-R(t)}v_n(t;\mathbf{q})}{\delta t} = -\, e^{-R(t+\delta t)} \sup_{p>0, w\geq 0} \Bigg[ -\lambda(p,w,t)v_n(t+\delta t;\mathbf{q}) + \sum_{i=1}^{M} q_i \lambda(p,w,t)(if(t)p + v_{n-i}(t+\delta t;\mathbf{q})) - k(t)w \Bigg].$$

Take the limit as $\delta t \to 0$. Observe that the left-hand side of the previous equation is the limit definition of a derivative, making the left-hand side $\frac{d}{dt}\left[e^{-R(t)}v_n(t;\mathbf{q})\right]$ and yields

$$\begin{aligned} &- r(t)e^{-R(t)}v_n(t;\mathbf{q}) + e^{-R(t)}\dot{v}_n(t;\mathbf{q}) \\ &\quad = -e^{-R(t)} \sup_{p>0,w\geq 0} \left[ -\lambda(p,w,t)v_n(t;\mathbf{q}) + \sum_{i=1}^{M} q_i\lambda(p,w,t)(if(t)p + v_{n-i}(t;\mathbf{q})) - k(t)w \right]. \end{aligned}$$

So far we have kept the same notation as Section 2.1 while demonstrating how new features are added to the model. While working through this section, many terms will be analogs to terms from the model without advertising. Therefore we will use superscripts "A" to help distinguish these types of terms in their new context. So in the previous equation we will write $v_n^A$ instead of $v_n$. Additionally, dependencies on $t$ and $\mathbf{q}$ are generally suppressed, unless the dependencies improve understanding. With these details in mind, the previous equation can be rearranged to get the HJB equations while including advertising (recalling that $\sum_{i=1}^{M} iq_i = \mu$):

$$\begin{aligned} \dot{v}_n^A &= rv_n^A - \sup_{p>0,w\geq 0} \left[ -\lambda(p,w)v_n^A + \sum_{i=1}^{M} q_i\lambda(p,w)(ifp + v_{n-i}^A) - kw \right] \\ &= rv_n^A - \sup_{p>0,w\geq 0} \left[ \lambda(p,w)\left(\mu fp - \left(v_n^A - \sum_{i=1}^{M} q_i v_{n-i}^A\right)\right) - kw \right] \\ &= rv_n^A - \sup_{p>0,w\geq 0} \lambda(p,w)\mu fp \left[ 1 - \frac{1}{\mu fp}\left(v_n^A - \sum_{i=1}^{M} q_i v_{n-i}^A\right) - \frac{kw}{\lambda(p,w)\mu fp} \right] \end{aligned} \tag{3.3.1}$$

Equation (3.3.1) has been developed in generality. Now we assume constant demand elasticity and constant advertising elasticity. Under this scenario, $\lambda(p, w, t) = a(t)w^{\delta}p^{-\varepsilon}$, where $0 \leq \delta < 1$ is the advertising elasticity. Note how $\lambda$ depends on $w^{\delta}$ is similar to how $\lambda$ depends on $p^{-\varepsilon}$. This type of dependency is due to the constant elasticity assumptions for advertising and price. For more details on how advertising is defined, see (MacDonald and Rasmussen 2009).

This new arrival rate $\lambda$ can be thought of as a composite arrival rate. First, $a(t)w^{\delta}$ models the rate of customer arrivals. More advertising expenditure $w$ equates to more arrivals, but since $0 \leq \delta < 1$, there are diminishing returns. Once a customer arrives, $p^{-\varepsilon}$ models the effect price has on turning them into a buyer.

Let $w_n^*(t; \mathbf{q})$ and $p_n^{A*}(t; \mathbf{q})$ denote the optimal advertising rate and pricing policy, respectively, at time $0 \leq t < T$ and inventory $n$. To determine $w_n^*$ and $p_n^{A*}$ which achieve the supremum in (3.3.1), one could employ normal maximization techniques by setting the partial derivatives to 0. However, the problem of determining the optimal relationship between pricing and advertising policies has been known for many years as the Dorfman-Steiner Identity (from (Dorfman and Steiner 1954)).

The Dorfman-Steiner Identity states that at any moment the relative cost of optimal advertising $w_n^*(t; \mathbf{q})$ compared to optimal revenue $p_n^{A*}(t; \mathbf{q})\lambda(p, w, t)\mu(\mathbf{q})$ is equal to the ratio of advertising elasticity $\delta$ and price elasticity $\varepsilon$. In our case, we actually use the effective advertising $k(t)w_n^*(t; \mathbf{q})$ and the effective revenue $f(t)p_n^{A*}(t; \mathbf{q})\lambda(p, w, t)\mu(\mathbf{q})$, since the terms $k(t)$ and $f(t)$ indicate a tax or subsidy to their original terms. Therefore we obtain a Dorfman-Steiner Identity of

$$\frac{k(t)w_n^*(t; \mathbf{q})}{f(t)p_n^{A*}(t; \mathbf{q})\lambda(p, w, t)\mu(\mathbf{q})} = \frac{\delta}{\varepsilon}. \tag{3.3.2}$$

If we substitute $\lambda(p, w, t) = a(t)\left(w_n^*\right)^{\delta}\left(p_n^{A*}\right)^{-\varepsilon}$ into this equation, then we can write the optimal advertising in terms of optimal price as

$$w_n^*(t;\mathbf{q}) = \left(\left(\frac{a(t)f(t)\mu(\mathbf{q})}{k(t)}\right)\left(\frac{\delta}{\varepsilon}\right)\left(\frac{1}{p_n^{A*}(t;\mathbf{q})}\right)^{\varepsilon-1}\right)^{\frac{1}{1-\delta}}. \tag{3.3.3}$$

This shows that not only are advertising and pricing related to each other, but they are inversely related since $\varepsilon > 1$. Equation (3.3.3) can now be substituted into the supremum of (3.3.1), which allows the maximal $p$ to be calculated. To find $p_n^{A*}$, the derivative with respect to $p$ of the term in the supremum is taken and set to 0. (This is assumed to be a maximum since the expected optimal value is finite.)

Now begin optimizing from the supremum of (3.3.1):

$$\begin{aligned}
0 =& \frac{d}{dp}\left[\lambda(p,w)\mu f p\left[1 - \frac{1}{\mu f p}\left(v_n^A - \sum_{i=1}^M q_i v_{n-i}^A\right) - \frac{kw}{\lambda(p,w)\mu f p}\right]\right] \\
=& \frac{d}{dp}\left[a w^\delta p^{-\varepsilon}\mu f p\left[1 - \frac{1}{\mu f p}\left(v_n^A - \sum_{i=1}^M q_i v_{n-i}^A\right) - \frac{\delta}{\varepsilon}\right]\right] \\
=& \frac{d}{dp}\left[\left(\left(\frac{af\mu}{k}\right)\left(\frac{\delta}{\varepsilon}\right)\left(\frac{1}{p}\right)^{\varepsilon-1}\right)^{\frac{\delta}{1-\delta}} p^{1-\varepsilon}\left[\left(\frac{\varepsilon-\delta}{\varepsilon}\right) - \frac{1}{\mu f p}\left(v_n^A - \sum_{i=1}^M q_i v_{n-i}^A\right)\right]\right] \\
=& \frac{d}{dp}\left[p^{\frac{1-\varepsilon}{1-\delta}}\left(\frac{\varepsilon-\delta}{\varepsilon}\right) - p^{-\frac{\varepsilon-\delta}{1-\delta}}\left(\frac{1}{\mu f}\right)\left(v_n^A - \sum_{i=1}^M q_i v_{n-i}^A\right)\right] \\
=& \left(\frac{1-\varepsilon}{1-\delta}\right)p^{\frac{1-\varepsilon}{1-\delta}-1}\left(\frac{\varepsilon-\delta}{\varepsilon}\right) + \left(\frac{\varepsilon-\delta}{1-\delta}\right)p^{-\frac{\varepsilon-\delta}{1-\delta}-1}\left(\frac{1}{\mu f}\right)\left(v_n^A - \sum_{i=1}^M q_i v_{n-i}^A\right) \\
=& \left(\frac{1-\varepsilon}{\varepsilon}\right)p + \left(\frac{1}{\mu f}\right)\left(v_n^A - \sum_{i=1}^M q_i v_{n-i}^A\right),
\end{aligned}$$

and from this computation we find the price which obtains the supremum in (3.3.3) is

$$p_n^{A*}(t;\mathbf{q}) = \left(\frac{\varepsilon}{\varepsilon-1}\right)\left(\frac{1}{f(t)\mu(\mathbf{q})}\right)\left(v_n^A(t;\mathbf{q}) - \sum_{i=1}^M q_i v_{n-i}^A(t;\mathbf{q})\right). \tag{3.3.4}$$

At first glance it seems that this price does not depend on advertising $w$ or advertising elasticity $\delta$; however, $v_n^A(t;\mathbf{q})$ has dependency on $\delta$, so price does in fact depend on the advertising.

Before proceeding a few terms will be defined in order to greatly simplify future calculations:

$$\begin{aligned}
\gamma &:= \frac{\varepsilon-\delta}{1-\delta} \\
\eta &:= \left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{1-\delta}} \left(\frac{\varepsilon-\delta}{\varepsilon}\right)^{\gamma}, \\
g(t) &:= \left(\frac{a(t)f(t)^{\varepsilon}}{k(t)^{\delta}}\right)^{\frac{1}{1-\delta}}, \\
\zeta(t) &:= g(t)\left(\frac{\eta}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1}.
\end{aligned} \tag{3.3.5}$$

One useful relation of the first term is $1-\gamma=\frac{1-\varepsilon}{1-\delta}$. Note that $\gamma$ is chosen to match the notation of (Helmes and Schlosser 2013), but is unrelated to the $\gamma_n$ used in Section 2.1. $\gamma$ can be thought of as a joint elasticity that relates the elasticities $\delta$ and $\varepsilon$. The other three terms do not have particular interpretations, but will come up in future computations.

Begin with $\dot{v}_n^A$ from (3.3.1), with the optimal pricing and advertising strategies to eliminate the supremum,

$$\dot{v}_n^A = rv_n^A - \lambda(p_n^{A*}, w_n^*)\mu f p_n^{A*}\left[1-\frac{1}{\mu f p_n^{A*}}\left(v_n^A-\sum_{i=1}^{M} q_i v_{n-i}^A\right)-\frac{kw_n^*}{\lambda(p_n^{A*}, w_n^*)\mu f p_n^{A*}}\right].$$

Simplify the bracketed term using (3.3.4) and (3.3.2) to get

$$\begin{aligned}
\dot{v}_n^A &= rv_n^A - \lambda(p_n^{A*}, w_n^*)\mu f p_n^{A*}\left[1-\frac{\varepsilon-1}{\varepsilon}-\frac{\delta}{\varepsilon}\right] \\
&= rv_n^A - a\left(\left(\frac{af\mu}{k}\right)\left(\frac{\delta}{\varepsilon}\right)\left(\frac{1}{p_n^{A*}}\right)^{\varepsilon-1}\right)^{\frac{\delta}{1-\delta}} \left(p_n^{A*}\right)^{-\varepsilon} \mu f p_n^{A*}\left(\frac{1-\delta}{\varepsilon}\right) \\
&= rv_n^A - (a\mu f)^{\frac{1}{1-\delta}}\left(\frac{\delta}{k\varepsilon}\right)^{\frac{\delta}{1-\delta}} \left(p_n^{A*}\right)^{\frac{1-\varepsilon}{1-\delta}}\left(\frac{1-\delta}{\varepsilon}\right) \\
&= rv_n^A - (a\mu f)^{\frac{1}{1-\delta}}\left(\frac{\delta}{k\varepsilon}\right)^{\frac{\delta}{1-\delta}} \left(\left(\frac{\varepsilon}{\varepsilon-1}\right)\left(\frac{1}{f\mu}\right)\left(v_n^A-\sum_{i=1}^{M} q_i v_{n-i}^A\right)\right)^{\frac{1-\varepsilon}{1-\delta}}\left(\frac{1-\delta}{\varepsilon}\right)
\end{aligned}$$

$$
\begin{aligned}
&= r v_n^A - \mu^{\frac{\varepsilon}{1-\delta}} \left[ a^{\frac{1}{1-\delta}} f^{\frac{\varepsilon}{1-\delta}} \left(\frac{\delta}{k\varepsilon}\right)^{\frac{\delta}{1-\delta}} \left(\frac{1-\delta}{\varepsilon}\right) \left(\frac{\varepsilon}{\varepsilon-1}\right)^{1-\gamma} \right] \left( v_n^A - \sum_{i=1}^{M} q_i v_{n-i}^A \right)^{1-\gamma} \\
&= r v_n^A - \mu^{\frac{\varepsilon}{1-\delta}} \cdot \zeta \cdot \left( v_n^A - \sum_{i=1}^{M} q_i v_{n-i}^A \right)^{1-\gamma}. \qquad (3.3.6)
\end{aligned}
$$

The last equality is justified by the following work starting with the definition for $\zeta$, equation (3.3.5):

$$
\begin{aligned}
\zeta &= \left(\frac{af^{\varepsilon}}{k^{\delta}}\right)^{\frac{1}{1-\delta}} \frac{\left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{1-\delta}} \left(\frac{\varepsilon-\delta}{\varepsilon}\right)^{\gamma}}{\gamma} \left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1} \\
&= \left(\frac{af^{\varepsilon}}{k^{\delta}}\right)^{\frac{1}{1-\delta}} \left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{1-\delta}} \left(\frac{\varepsilon-\delta}{\varepsilon}\right)^{\gamma} \left(\frac{\varepsilon-\delta}{1-\delta}\right)^{-1} \left(\frac{\varepsilon-1}{\varepsilon-\delta}\right)^{\gamma-1} \\
&= \left(\frac{af^{\varepsilon}}{k^{\delta}}\right)^{\frac{1}{1-\delta}} \left(\frac{\delta}{\varepsilon}\right)^{\frac{\delta}{1-\delta}} \left(\frac{1}{\varepsilon}\right)^{\gamma} (1-\delta)(\varepsilon-1)^{\gamma-1} \\
&= a^{\frac{1}{1-\delta}} f^{\frac{\varepsilon}{1-\delta}} \left(\frac{\delta}{k\varepsilon}\right)^{\frac{\delta}{1-\delta}} \left(\frac{1-\delta}{\varepsilon}\right) \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\gamma-1}.
\end{aligned}
$$

The next theorem will give the optimal expected revenue and pricing strategy while considering advertising and variable order sizes. It was obtained by synthesizing the result from Theorem 2.1.2 and the work in (Helmes and Schlosser 2013). The proofs themselves are largely about multiplying the terms out and verifying that they achieve the desired results. The more important observation is that the idea to find a solution is the same as the solution for variable order sizes without advertising: separate the solution by a term dependent on $n$, a term dependent on $t$, and a term dependent on the average order size.

A few analogs to definitions from Section 2.1 are needed. First define

$$
A^{Ad}(t) := e^{\gamma R(t)} \int_t^T e^{-\gamma R(s)} g(s) ds.
$$

Next define the sequence $(\beta_n^A(\mathbf{q}))_n$ by $\beta_n^A(\mathbf{q}) = 0$ for $n \leq 0$, and for $n > 0$, $\beta_n^A(\mathbf{q})$ satisfies

$$
\left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1} = \beta_n^A \left( \beta_n^A - \sum_{i=1}^{M} q_i \beta_{n-i}^A \right)^{\gamma-1}. \qquad (3.3.7)
$$

The uniqueness and existence follows from Lemma 2.1.1.

**Theorem 3.3.1.** *Assume a model with constant demand elasticity $\varepsilon$, advertising elasticity $\delta$, and variable order sizes. For all $0 \leq t \leq T$ and $n$, the following holds:*

$$v_n^A(t;\boldsymbol{q}) = \mu(\boldsymbol{q})^{\frac{\varepsilon}{\varepsilon-\delta}} \beta_n^A(\boldsymbol{q}) \left(\eta A^{Ad}(t)\right)^{\frac{1}{\gamma}}. \tag{3.3.8}$$

$$p_n^{A*}(t;\boldsymbol{q}) = \left(\frac{\delta\mu(\boldsymbol{q})}{\varepsilon}\right)^{\frac{\delta}{\varepsilon-\delta}} \left(\frac{1}{f(t)}\right) \left(\beta_n^A\right)^{\frac{-1}{\gamma-1}} \left(A^{Ad}(t)\right)^{\frac{1}{\gamma}} \tag{3.3.9}$$

$$w_n^*(t;\boldsymbol{q}) = \left(\frac{\delta\mu(\boldsymbol{q})}{\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-\delta}} \left(\frac{g(t)}{k(t)}\right) \beta_n^A \left(A^{Ad}(t)\right)^{\frac{1-\gamma}{\gamma}} \tag{3.3.10}$$

*Proof.* Proceed by induction to prove (3.3.8). For $n \leq 0$, $v_n^A(t;\mathbf{q}) = 0$ and $\beta_n^A(t;\mathbf{q}) = 0$, which verifies the base cases. Suppose (3.3.8) holds through $n-1$. We verify that the induction holds in the differential equation (3.3.6) in order to prove the induction. Recall $R(t) = \int_0^t r(s)ds$. Also let $\theta = \mu^{\frac{\varepsilon}{\varepsilon-\delta}}$ to make computations more understandable.

Then the left-hand side (LHS) of (3.3.6) is

$$\begin{aligned}
\text{LHS} &= \dot{v}_n^A \\
&= \frac{\theta\beta_n^A\eta^{\frac{1}{\gamma}}}{\gamma}\left(A^{Ad}(t)\right)^{\frac{1}{\gamma}-1}\dot{A}^A(t) \\
&= \frac{\theta\eta^{\frac{1}{\gamma}}\beta_n^A}{\gamma}\left(A^{Ad}(t)\right)^{\frac{1-\gamma}{\gamma}}\left(e^{\gamma R(t)}\gamma r(t)\int_t^T e^{-\gamma R(s)}g(s)ds + e^{\gamma R(t)}(-e^{-\gamma R(t)}g(t))\right) \\
&= \frac{\theta\eta^{\frac{1}{\gamma}}\beta_n^A}{\gamma}\left(A^{Ad}(t)\right)^{\frac{1-\gamma}{\gamma}}\left(\gamma r(t)A^{Ad}(t) - g(t)\right)
\end{aligned}$$

The right-hand side (RHS) of (3.3.6) is

$$
\begin{aligned}
\text{RHS} &= rv_n^A - \mu^{\frac{\varepsilon}{1-\delta}} \cdot \zeta \cdot \left( v_n^A - \sum_{i=1}^{M} q_i v_{n-i}^A \right)^{1-\gamma} \\
&= r\theta \left(\eta A^{Ad}\right)^{\frac{1}{\gamma}} \beta_n^A - \mu^{\frac{\varepsilon}{1-\delta}} g \left(\frac{\eta}{\gamma}\right) \left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1} \left( \theta \left(\eta A^{Ad}\right)^{\frac{1}{\gamma}} \beta_n^A - \sum_{i=1}^{M} q_i \theta \left(\eta A^{Ad}\right)^{\frac{1}{\gamma}} \beta_{n-i}^A \right)^{1-\gamma} \\
&= r\theta \left(\eta A^{Ad}\right)^{\frac{1}{\gamma}} \beta_n^A - \mu^{\frac{\varepsilon}{1-\delta}} g \left(\frac{\eta}{\gamma}\right) \eta^{\frac{1-\gamma}{\gamma}} \theta^{1-\gamma} \left(A^{Ad}\right)^{\frac{1-\gamma}{\gamma}} \left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1} \left( \beta_n^A - \sum_{i=1}^{M} q_i \beta_{n-i}^A \right)^{1-\gamma} \\
&= r\theta \left(\eta A^{Ad}\right)^{\frac{1}{\gamma}} \beta_n^A - \mu^{\frac{\varepsilon}{1-\delta}} g \left(\frac{1}{\gamma}\right) \eta^{\frac{1}{\gamma}} \theta^{1-\gamma} \left(A^{Ad}\right)^{\frac{1-\gamma}{\gamma}} \beta_n^A \\
&= \frac{\theta \eta^{\frac{1}{\gamma}} \beta_n^A}{\gamma} \left(A^{Ad}\right)^{\frac{1-\gamma}{\gamma}} \left(\gamma r A^{Ad} - \mu^{\frac{\varepsilon}{1-\delta}} \theta^{-\gamma} g\right) \\
&= \frac{\theta \eta^{\frac{1}{\gamma}} \beta_n^A}{\gamma} \left(A^{Ad}\right)^{\frac{1-\gamma}{\gamma}} \left(\gamma r A^{Ad} - g\right).
\end{aligned}
$$

Where the last equality is justified by

$$
\mu^{\frac{\varepsilon}{1-\delta}} \theta^{-\gamma} = \mu^{\frac{\varepsilon}{1-\delta}} \left(\mu^{\frac{\varepsilon}{\varepsilon-\delta}}\right)^{-\frac{\varepsilon-\delta}{1-\delta}} = \mu^{\frac{\varepsilon}{1-\delta}} \mu^{\frac{-\varepsilon}{1-\delta}} = 1.
$$

Therefore the LHS and RHS are equal, proving the induction. Now find the optimal pricing policy by substituting (3.3.8) into (3.3.4):

$$
\begin{aligned}
p_n^{A*}(t;\mathbf{q}) &= \left(\frac{\varepsilon}{\varepsilon-1}\right) \left(\frac{1}{f(t)\mu(\mathbf{q})}\right) \left( v_n^A(t;\mathbf{q}) - \sum_{i=1}^{M} q_i v_{n-i}^A(t;\mathbf{q}) \right) \\
&= \left(\frac{\varepsilon}{\varepsilon-1}\right) \left(\frac{1}{f(t)\mu(\mathbf{q})}\right) \left( \mu^{\frac{\varepsilon}{\varepsilon-\delta}} \beta_n^A(\mathbf{q}) \left(\eta A^{Ad}(t)\right)^{\frac{1}{\gamma}} - \sum_{i=1}^{M} q_i \mu^{\frac{\varepsilon}{\varepsilon-\delta}} \beta_{n-i}^A(\mathbf{q}) \left(\eta A^{Ad}(t)\right)^{\frac{1}{\gamma}} \right) \\
&= \left(\frac{\varepsilon}{\varepsilon-1}\right) \left(\frac{\mu(\mathbf{q})^{\frac{\delta}{\varepsilon-\delta}}}{f(t)}\right) \left(\eta A^{Ad}(t)\right)^{\frac{1}{\gamma}} \left( \beta_n^A(\mathbf{q}) - \sum_{i=1}^{M} q_i \beta_{n-i}^A(\mathbf{q}) \right) \\
&= \left(\frac{\varepsilon}{\varepsilon-1}\right) \left(\frac{\mu(\mathbf{q})^{\frac{\delta}{\varepsilon-\delta}}}{f(t)}\right) \left(\eta A^{Ad}(t)\right)^{\frac{1}{\gamma}} \left(\frac{\gamma-1}{\gamma}\right) \left(\beta_n^A\right)^{\frac{-1}{\gamma-1}} \\
&= \left(\frac{\delta\mu(\mathbf{q})}{\varepsilon}\right)^{\frac{\delta}{\varepsilon-\delta}} \left(\frac{1}{f(t)}\right) \left(A^{Ad}(t)\right)^{\frac{1}{\gamma}} \left(\beta_n^A\right)^{\frac{-1}{\gamma-1}}.
\end{aligned}
$$

To get the optimal advertising, substitute the previous result into (3.3.3):

$$
\begin{aligned}
w_n^*(t;\mathbf{q}) &= \left(\left(\frac{a(t)f(t)\mu(\mathbf{q})}{k(t)}\right)\left(\frac{\delta}{\varepsilon}\right)\left(\frac{1}{p_n^{A*}(t;\mathbf{q})}\right)^{\varepsilon-1}\right)^{\frac{1}{1-\delta}}. \\
&= \left(\left(\frac{a(t)f(t)\mu(\mathbf{q})}{k(t)}\right)\left(\frac{\delta}{\varepsilon}\right)\left(\left(\frac{\delta\mu(\mathbf{q})}{\varepsilon}\right)^{\frac{\delta}{\varepsilon-\delta}}\left(\frac{1}{f(t)}\right)\left(A^{Ad}(t)\right)^{\frac{1}{\gamma}}\left(\beta_n^A\right)^{\frac{-1}{\gamma-1}}\right)^{1-\varepsilon}\right)^{\frac{1}{1-\delta}} \\
&= \left(\left(\frac{a(t)f(t)^{\varepsilon}}{k(t)}\right)\left(\frac{\delta\mu(\mathbf{q})}{\varepsilon}\right)^{\frac{\varepsilon(1-\delta)}{\varepsilon-\delta}}\left(A^{Ad}(t)\right)^{\frac{1-\varepsilon}{\gamma}}\left(\beta_n^A\right)^{\frac{\varepsilon-1}{\gamma-1}}\right)^{\frac{1}{1-\delta}} \\
&= \left(\frac{\delta\mu(\mathbf{q})}{\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-\delta}}\left(\frac{g(t)}{k(t)}\right)\left(A^{Ad}(t)\right)^{\frac{1-\gamma}{\gamma}}\beta_n^A
\end{aligned}
$$

Verifying the last equality is a matter of multiplying out the exponents. □

Note that this Theorem is consistent with Theorem 2.1.2 when the various factors are defined to align with the no advertising model: $\delta = 0$, $f(t) = 0$, and $k(t) = 0$. We also observe that $\beta_n^A(\mathbf{q})$ has the same structure as $\beta_n(\mathbf{q})$ (compare (3.3.7) and (2.1.5)), except $\varepsilon$ is replaced with $\gamma$. Since $\gamma > 1$ also, this means that Theorem 2.1.7 holds for $\beta_n^A(\mathbf{q})$, with $\gamma$ in place of $\varepsilon$. In other words,

$$
\lim_{n\to\infty}\frac{\beta_n^A(\mathbf{q})}{n^{\frac{\gamma-1}{\gamma}}} = \mu(\mathbf{q})^{\frac{1-\gamma}{\gamma}}. \tag{3.3.11}
$$

Surprisingly, the average order size plays a role in the optimal pricing and optimal advertising strategies; whereas, it was not seen in the formulas without advertising. That said, the next corollary shows that comparable models still have the same asymptotic behavior as $n \to \infty$.

**Corollary 3.3.2.** *(of Theorem 2.2.1) Let $\mathscr{M}^A_{\boldsymbol{q},\lambda}$ and $\mathscr{M}^A_{\bar{\boldsymbol{q}},\bar{\lambda}}$ be comparable models under constant demand elasticity $\varepsilon$ and advertising elasticity $\delta$. Then,*

$$
\lim_{n\to\infty}\frac{v_n^A(t;\boldsymbol{q},\lambda)}{v_n^A(t;\bar{\boldsymbol{q}},\bar{\lambda})} = 1, \qquad \lim_{n\to\infty}\frac{p_n^{A*}(t;\boldsymbol{q},\lambda)}{p_n^{A*}(t;\bar{\boldsymbol{q}},\bar{\lambda})} = 1, \qquad \lim_{n\to\infty}\frac{w_n^*(t;\boldsymbol{q},\lambda)}{w_n^*(t;\bar{\boldsymbol{q}},\bar{\lambda})} = 1.
$$

*In other words, $\mathscr{M}^A_{\mathbf{q},\lambda}$ and $\mathscr{M}^A_{\bar{\mathbf{q}},\bar{\lambda}}$ have asymptotically equivalent optimal expected revenues, optimal pricing strategies, and optimal advertising strategies. and pricing strategies.*

*Proof.* The proof will follow the original proof of Theorem 2.2.1, just with notation adjusted for the advertising models being worked with. Let $\mathscr{M}^A_{\mathbf{q},\lambda}$ and $\mathscr{M}^A_{\bar{\mathbf{q}},\bar{\lambda}}$ be comparable models with $\lambda(p,t) = a(t)w^\delta p^{-\varepsilon}$ and $\bar{\lambda}(p,t) = \bar{a}(t)w^{-\delta}p^{-\varepsilon}$. Let

$$A^{Ad}(t;a) := e^{\gamma R(t)} \int_t^T e^{-\gamma R(s)} \left( \frac{a(s)f(s)^\varepsilon}{k(s)^\delta} \right)^{\frac{1}{1-\delta}} ds.$$

Then Theorem 3.3.1 says

$$v_n^A(t;\mathbf{q},\lambda) = \mu(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-\delta}} \beta_n^A(\mathbf{q}) \left(\eta A^{Ad}(t;a)\right)^{\frac{1}{\gamma}}. \tag{3.3.12}$$

Also recall (3.3.11): $\lim\limits_{n\to\infty} \frac{\beta_n(\mathbf{q})}{n^{\frac{\varepsilon-1}{\varepsilon}}} = \mu(\mathbf{q})^{\frac{1-\varepsilon}{\varepsilon}}$.

Let $\delta > 0$. By combining (3.3.12) with (3.3.11) we see that there exists an $N > 0$ such that for all $n > N$,

$$\left| v_n^A(t;\mathbf{q},\lambda) - \mu(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-\delta}} \left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\gamma-1}{\gamma}} \left(\eta A^{Ad}(t;a)\right)^{\frac{1}{\gamma}} \right| < \delta,$$
$$\left| v_n^A(t;\bar{\mathbf{q}},\bar{\lambda}) - \mu(\bar{\mathbf{q}})^{\frac{\varepsilon}{\varepsilon-\delta}} \left(\frac{n}{\mu(\bar{\mathbf{q}})}\right)^{\frac{\gamma-1}{\gamma}} \left(\eta A^{Ad}(t;\bar{a})\right)^{\frac{1}{\gamma}} \right| < \delta.$$

Note that for a constant $c$, $A^{Ad}(t;ca) = e^{\gamma R(t)} \int_t^T e^{-\gamma R(s)} \left(\frac{ca(s)f(s)^\varepsilon}{k(s)^\delta}\right)^{\frac{1}{1-\delta}} ds = c^{\frac{1}{1-\delta}} A^{Ad}(t;a)$. The fact that the constant $c$ is raised to the $\frac{1}{1-\delta}$ power is the main difference between this proof with advertising and the original proof. However, the next computation will show that this does not create any problems. First recall the condition for comparable models (2.2.1), which gives $a(t) = \frac{\bar{a}(t)\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}$. Thus

$$\begin{aligned}
\mu(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-\delta}} \left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\gamma-1}{\gamma}} \left(\eta A^{Ad}(t;a)\right)^{\frac{1}{\gamma}} &= \mu(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-\delta}} \left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\gamma-1}{\gamma}} \left(\eta A\left(t; \frac{\bar{a}\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}\right)\right)^{\frac{1-\delta}{\varepsilon-\delta}} \\
&= \mu(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-\delta}} \left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\varepsilon-1}{\varepsilon-\delta}} \left(\frac{\mu(\bar{\mathbf{q}})}{\mu(\mathbf{q})}\right)^{\frac{1}{\varepsilon-\delta}} \left(\eta A\left(t;\bar{a}\right)\right)^{\frac{1-\delta}{\varepsilon-\delta}} \\
&= \mu(\bar{\mathbf{q}})^{\frac{\varepsilon}{\varepsilon-\delta}} \left(\frac{n}{\mu(\bar{\mathbf{q}})}\right)^{\frac{\gamma-1}{\gamma}} \left(\eta A^{Ad}(t;\bar{a})\right)^{\frac{1}{\gamma}}.
\end{aligned}$$

Then for $n > N$,

$$\begin{aligned}
\left|v_n^A(t;\mathbf{q},\lambda) - v_n^A(t;\bar{\mathbf{q}},\bar{\lambda})\right| &\leq \left|v_n^A(t;\mathbf{q},\lambda) - \mu(\mathbf{q})^{\frac{\varepsilon}{\varepsilon-\delta}} \left(\frac{n}{\mu(\mathbf{q})}\right)^{\frac{\gamma-1}{\gamma}} \left(\eta A^{Ad}(t;a)\right)^{\frac{1}{\gamma}}\right| \\
&\quad + \left|\mu(\bar{\mathbf{q}})^{\frac{\varepsilon}{\varepsilon-\delta}} \left(\frac{n}{\mu(\bar{\mathbf{q}})}\right)^{\frac{\gamma-1}{\gamma}} \left(\eta A^{Ad}(t;\bar{a})\right)^{\frac{1}{\gamma}} - v_n^A(t;\bar{\mathbf{q}},\bar{\lambda})\right| \\
&< 2\delta.
\end{aligned}$$

Since $\delta$ was arbitrary, this proves that two comparable models have asymptotically equivalent optimal expected revenues as $n \to \infty$. By similar arguments, the optimal pricing strategies and optimal advertising strategies are also asymptotically equivalent as $n \to \infty$. $\square$

We have shown how variable order sizes change the advertising and discounting model. The structure of the formulas are the same, but with variable order sizes the average order size $\mu(\mathbf{q})$ now plays a role. There are many more topics in (Helmes and Schlosser 2013), including economic insights and infinite horizon results. These results are largely unaffected by variable order sizes, so we do not include a discussion of them. With this, we conclude the section.

## 3.4 Other Arrival Rate Functions

Using constant demand elasticity has allowed us to get analytic solutions for our model. This, in turn, has allowed us to get a good understand of how variable order sizes changes the structure of an optimal pricing problem and what considerations must be taken. Constant demand elasticity is a useful economic model for a specific type of arrival rate $\lambda$, but it would also be useful to understand how models with different arrival rate structures are affected with variable order sizes.

This section aims to explore some similarities and differences between problems with different arrival rates $\lambda$. In particular, we examine how the comparison results of Section 2.2 change in the context of different $\lambda$. Two special cases of $\lambda$ will be considered: when $\lambda$ is exponential and when it is linear. A couple notes on the graphs. First, the graphs are technically discrete plots, but are presented with connected lines for readability. Second, the legends indicate the order size distributions $\mathbf{q}$ used for that line.

We begin with the general HJB equations (1.3.2), which were developed independently of $\lambda(p,t)$:

$$v_n'(t;\mathbf{q},\lambda) = -\sup_p \lambda(p,t)\left[-v_n(t;\mathbf{q}) + \sum_{i=1}^{M} q_i(ip + v_{n-i}(t;\mathbf{q}))\right].$$

For "nice" $\lambda(p,t)$, one can find the $p^*$ which attains the supremum using normal maximization techniques; however, that $p^*$ will be defined using $v_n$ terms. This means some other technique to determine the optimal expected revenue is needed. For constant demand elasticity, the key property was that the optimal expected revenue solution was separable in $n$ and $t$. This may not be the case in general.

For a general customer arrival rate function $\lambda(p,t)$, let $v_n(t;\lambda)$ denote the optimal expected revenue and $p_n^*(t;\lambda)$ denote the optimal pricing strategy. Exponential arrival rates are given of the form $\lambda(p,t) = a(t)e^{-\alpha p}$, where $a(t) > 0$ is the same type of arrival rate scaling factor as seen before, and $\alpha > 0$ is another scaling factor. When $\alpha = 1$, (Gallego and van Ryzin 1994) found analytic solutions to the optimal revenue and pricing strategies:

$$v_n(t; ae^{-p}) = \log\left(\sum_{i=0}^{n} \frac{(a(t)t/e)^i}{i!}\right) \tag{3.4.1}$$

$$p_n^*(t; ae^{-p}) = v_n(t; ae^{-p}) - v_{n-1}(t; ae^{-p}) + 1. \tag{3.4.2}$$

Notably, the optimal expected revenue is not separable in $n$ and $t$. As such, generalizing this result to include variable order sizes has been challenging.

While using a max order size of $M = 2$, we obtained analytic solutions for the optimal expected revenue for $n = 1, 2, 3$. For $n > 3$ the problem became unwieldy as a hypergeometric function appears as a term in $v_3$, which then becomes difficult to use for $n = 4$ since solutions are solved via recursion. but there are challenges.

Compared to constant demand elasticity, there are some fundamental differences when using exponential arrivals. By looking at (3.4.2) we see that for exponential arrivals, there is actually a minimum price of 1 (since $v_n \geq v_{n-1}$); whereas for constant demand elasticity, there is no minimum price. This is likely related to the related $\lambda$ functions. For constant demand elasticity $\lambda(p, t) = p^{-\varepsilon}$, which is unbounded as $p \to 0$. When $\lambda(p, t) = e^{-p}$, $\lambda \to 1$ as $p \to 0$. Thus it makes sense that for exponential arrivals the price has a minimum threshold.

Despite a lack of analytic results, we have created a Mathematica program to compute the numerically optimal expected revenue and pricing strategies. see the appendix for the code. The program implements the dynamic programming equation (1.3.1), and allows for a user defined $\lambda(p, t)$ to be used. Of course, we have a continuous problem and the program must be implemented on a discrete time scale by picking a value for $\delta t$. The presented graphs use $\delta t = 0.1$, which is about 2% of the total time scale. Little accuracy seemed to be gained by using a finer time scale.

Figure 3.1 shows the optimal expected revenue for exponential arrivals given different comparable models. They all converge as $n$ increases, but not quite how we may have expected. Not only do the terms converge to each other, they converge to a specific value.

This makes sense considering that $e^{-p}$ is bounded. Without a larger time scale there is an upper limit to the amount of inventory which can be sold on average.

Note also that some of the graphs look like step functions. This is due to the overselling assumption that we have made when the dynamic programming equation was initially defined. If all the orders are of only one size, say 4, then the expected revenue won't change between inventory sizes 1 through 4. In each case, just 4 items are sold. This idea extends for larger inventory too (if there are 5 through 8 items, 8 items are still sold in 2 batches of 4).

Figure 3.1: Expected revenue with $\lambda(p,t) = 2e^{-2p}$, $T = 5$

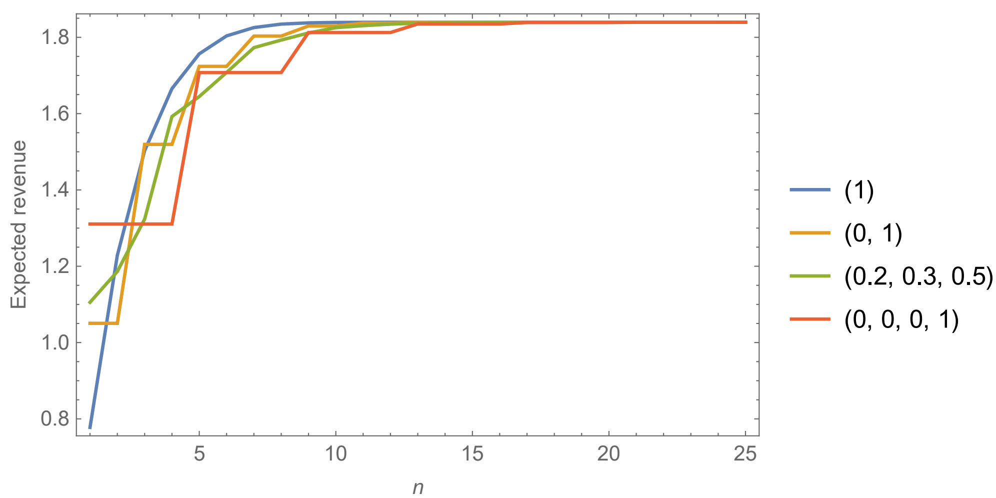
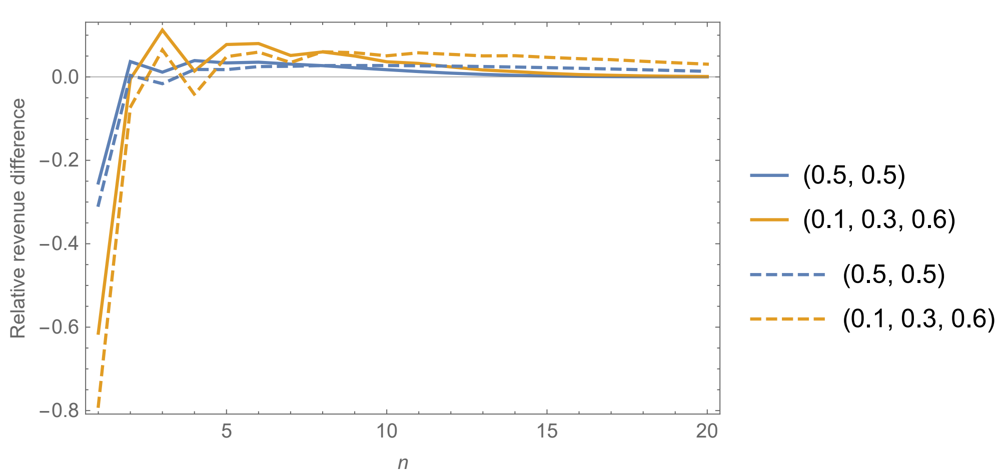
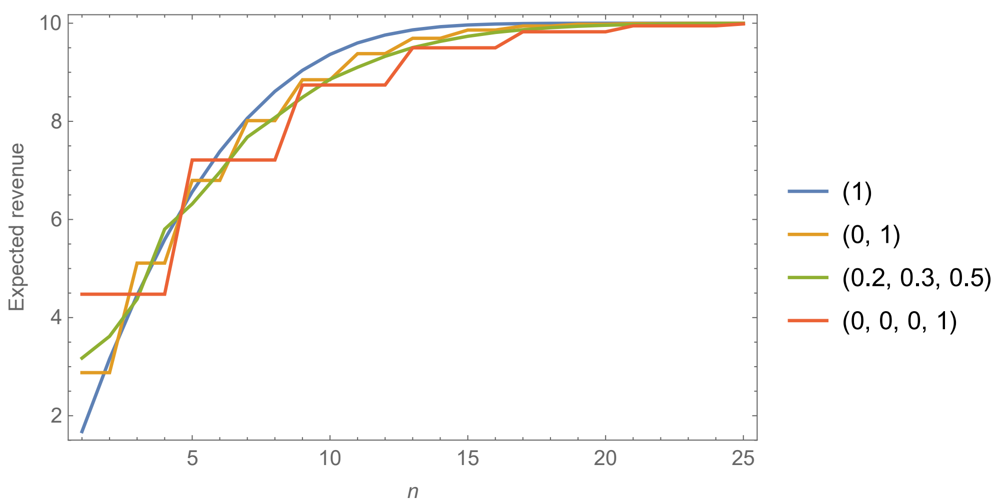



Figure 3.2: Relative difference between a variable order model compared to unit order model. Solid: $\lambda(p,t) = 2e^{-2p}$, $T = 5$; Dashed: $\lambda(p,t) = 5e^{-2p}$.



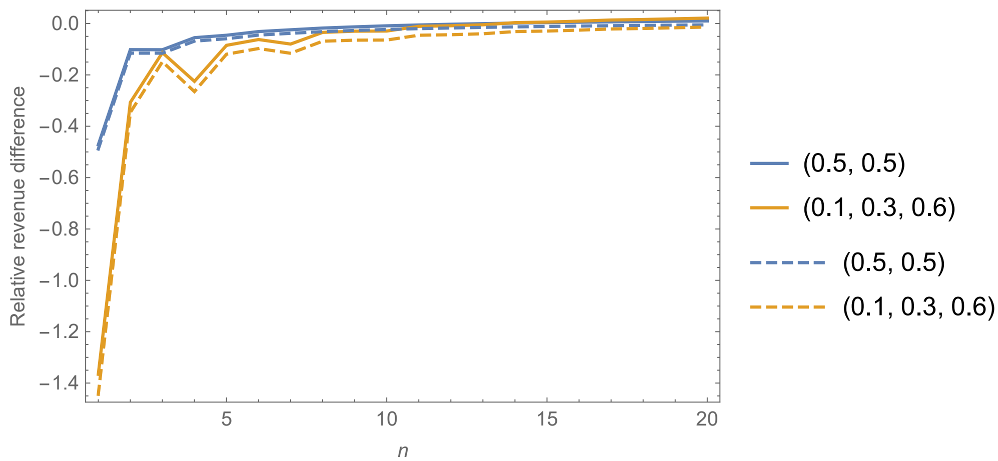
Next, Figure 3.2 shows the relative difference between the unit order model and a comparable variable order size model for $\mathbf{q} = (0, 1)$ at different demand magnitudes. Recall Theorem 2.2.2 which stated that for constant demand elasticity, this relative difference of comparable models was independent of the demand magnitude. However, this is not the case for exponential arrivals. Why might this be? Looking back to Theorem 2.2.2, the proof of this theorem relied on the fact that $v_n$ was separable in $n$ and $t$. For exponential arrivals, (3.4.1) indicates that the solution for $v_n$ is not separable in $n$ and $t$.

Figure 3.3: Expected value with $\lambda(p, t) = 2(2 - p)$




Another specific $\lambda$ explored in the literature is linear arrival rates. That is, $\lambda(p, t) = a(t)(\alpha - p)$, where $a(t)$ is an arrival rate scaling factor and $a(t)\alpha$ is the maximum arrivals when $p = 0$. Again we see that arrivals are bounded, so we expect optimal expected revenue to be bounded as well. Using the numerical computations again we plot the numerical optimal expected revenue in Figure 3.3. Indeed the graph matches our intuition that there is an upper bound on $v_n$. Similar to exponential arrivals, Figure 3.4 shows that the relative difference between comparable models is not independent of the demand magnitude.

Figure 3.4: Relative difference between a variable order model compared to unit order model. Solid: $\lambda(p,t) = 2(2-p)$, $T = 5$; Dashed: $\lambda(p,t) = 5(2-p)$ , $T = 5$.




The numerical computations in this section reveal some important insights. First is that the comparable models, regardless of the $\lambda$ chosen, appear to have the same asymptotic behavior in the inventory size. It is pleasing that the numerical calculations support this idea, as it is one which makes intuitive sense. We conjecture that the result holds for general $\lambda$. We have also seen that the relative difference between comparable models is not independent of the demand magnitude for general $\lambda$, showing that constant demand elasticity is a special case. We have also presented useful code which can be utilized to explore more questions about how variable order sizes influences dynamic pricing.

# 4.

# Conclusion

We have explored the problem of how variable order sizes influence the optimal pricing strategies for a model with constant demand elasticity. We obtained analytic results for the optimal revenue and pricing strategies and found that the average order size was important, a term not seen in the unit order case. Comparisons between models with the same demand were also examined, and we showed that comparable models have the same asymptotic pricing behavior in the inventory size.

In a practical sense, the comparison results give the insight that a variable order size model may be approximated with a unit order size model. Numerical results indicated that comparable models converge relatively quickly. This amounts to extra modeling flexibility, since unit order models are easier to work with. Moreover, the relative difference between the optimal expected revenue and pricing strategy for two comparable models is not dependent on the scale of demand.

Variable order size models require the problem of low inventory to be addressed, due to the structure of the dynamic programming equation which is developed. Of particular interest was defining different behavior at low inventory. We showed two ways of approaching the problem of what to do when there is not enough inventory let to meet a customers demand: turn them away or sell them all remaining inventory.

This was further extended to allow the minimum inventory to be any value, where negative inventory indicated overselling inventory (such as overbooking tickets on a flight). Costs based on the price of an item were also introduced in order to account for risks associated with overselling. Even with the low inventory features, our asymptotic and comparison results still held.

From an economic standpoint, constant demand elasticity is interesting because the monopolist pricing strategy is also socially efficient. We also discussed how to introduce advertising and discounting into the variable order size model. These were two very different features that greatly improved the generality of the model.

There is still more to be explored related to variable order size models. While variable order sizes are not new to the literature, they are often treated in terms of numerical calculations. This is fine from a practical application of the theory, but as shown in this dissertation there are also useful analytic insights. Given the importance of comparable models, and the comparison results, we also think examining in more detail the converge rate of comparable models would also be an interesting topic. Other arrival rate functions were briefly explored, and more work could be done to analyze these models as well.

# 5.

# Appendix

Figure 5.1: Mathematica code which implements basic definitions from the paper.

```
(*beta[n_,ep_,qvec_] creates the beta function as defined in the paper. Make sure to run before using any beta terms!
  returns: nothing;
 n := the index of beta;
 ep := the demand elasticity;
 qvec := the order size distribution, given as list {q1,...,qM};*)
defineBetas[n_, ep_, qvec_] :=
  Module[{RHS = ((ep - 1.)/ep)^(ep-1.), i, lengthq, current = 0., prev, q = qvec, j},
   If[Total[q] ≠ 1.0, q = Input["q sum error"]];
   If[ValueQ[beta[n, ep, qvec]], Return["Already Done!"]];
   lengthq = Length[q];
   prev = ConstantArray[0., lengthq];
   For[i = 1, i ≤ n, i++,
    current = NSolve[x (x - Sum[q[[j]] prev[[j]], {j, 1, lengthq}])^(ep-1.) == RHS && x ≥ 0, x][[1, 1, 2]];
    beta[i, ep, q] = current;
    For[j = lengthq, j > 1, j--,
     prev[[j]] = prev[[j - 1]];
    ];
    prev[[1]] = current;
   ];];

(*Helpful functions*)
(*mu[...] is the Average order size of a probability distribution*)
mu[qlist_] := Sum[i * qlist[[i]], {i, 1, Length[qlist]}];
(*A[...] as defined in the paper*)
A[t_, T_, a_] := Integrate[a, {s, t, T}];
(*V[...] is the optimal expected revenue for a constant elasticity model, with inputs matching those from the paper*)
V[n_, t_, T_, a_, ep_, q_] := mu[q] beta[n, ep, q] A[t, T, a]^(1/ep);
(*PStarCE[...] is the optimal pricing policy for a constant elasticity model, with inputs matching those from the paper*)
PStarCE[n_, t_, T_, a_, ep_, q_] := beta[n, ep, q]^(-1/(ep - 1)) A[t, T, a]^(1/ep);
```

Figure 5.2: Mathematica code which implements a Poisson model to simulate the revenue earned while using the optimal pricing strategy.

```
(*randomOrderSize[q_]
 returns: a random order size using probability distribution q;
 q := a probability distribution given as a list {q1,...,qM}*)
randomOrderSize[q_] := Module[{pick = RandomReal[], count = q[[1]], i},
   For[i = 1, i ≤ Length[q], i++,
    If[pick <= count, Return[i], count += q[[i + 1]]]]];

(*runPricingSimulation[parameters_] simulates the Poisson random arrival of customers under constant demand elasticity. The
 optimal pricing policy PStarCE[...] is used to control the arrival rate of customers Make sure beta is defined before running;
 returns: the revenue earned during the simulation;
 parameters := a list of parameters for the simulation, see the module variables for each term;*)
runPricingSimulation[parameters_] := Module[{inventory = parameters[[1]],
    T = parameters[[2]],
    a = parameters[[3]],
    ep = parameters[[4]],
    dt = parameters[[5]],
    q = parameters[[6]],
    revenue = 0,
    prices = {},
    jumpTimes = {},
    precision = 7,
    t, successRate, jumpSize = 0},

   For[t = 0, t < T, t += dt,
    If[inventory > 0,
     AppendTo[prices, {t, PStarCE[inventory, t, T, a, ep, q]}];
     successRate = LambdaCE[prices[[-1, -1]], a, ep] * dt;
     If[RandomReal[WorkingPrecision → precision] < successRate,
      AppendTo[jumpTimes, t];
      jumpSize = randomOrderSize[q];
      inventory -= jumpSize;
      revenue += jumpSize * prices[[-1, -1]]],
     (*Else, if all sold*)
     AppendTo[prices, {t, 0}];]
   ];
   revenue
  ];

(*runManyPricingSimulations[parameterSets_] sequentially runs Poisson pricing simulations,
allowing results for several different parameter sets to be run, as well as several trials per parameter set;
 returns: a list containing elements of the form {{parameter list}, {list of revenue earned}};
 parameterSets := a list where each element is a list of parameters. See runPricingSimulation for how to format parameters;
 trials := how many trials to run for each parameter set;*)
runManyPricingSimulations[parameterSets_, trials_] := Module[{
    masterResults = {},
    i = 0,
    j = 0,
    trialResults},
   Print["Trial progress:"];
   Print[ProgressIndicator[Dynamic[i / trials]]];
   Print["Parameter progress:"];
   Print[ProgressIndicator[Dynamic[j / Length[parameterSets]]]];
   For[j = 1, j ≤ Length[parameterSets], j++,
    trialResults = {};
    For[i = 1, i ≤ trials, i++,
     AppendTo[trialResults, runPricingSimulation[parameterSets[[j]]]];
    ];
    AppendTo[masterResults, {parameterSets[[j]], trialResults}];
   ];
   masterResults
  ];
```

Figure 5.3: Mathematica code which implements an algorithm to numerically calculate the optimal expected revenue and pricing strategy for any arrival rate function.

```
(*The following functions define different types of customer arrival functions. Used for optimalRevenueNumeric[...];
 params := a list which corresponds to different scaling factors of that particular function*)
LambdaExp[p_, params_] := params[[1]] * Exp[-params[[2]] * p];
LambdaCE[p_, params_] := params[[1]] p^(-params[[2]]);
LambdaLinear[p_, params_] := params[[1]] (params[[2]] - p);

(*optimalRevenueNumeric[n_,T_,q_,Lambda_,lambdaParams_,dt_] is an algorithm which uses the HJB equations to numerically
 calculate and define the optimal expected revenue JStarN[...] and pricing strategy PStarN[...] functions for all inventory levels n,
and discrete time values 0≤t≤T with steps dt. IMPORTANT! The functions which are generated are evaluated at integer time steps,
not at specific times. To calculate the time step you must use IntegerPart[t/dt] (This avoids floating point number issues.);
 returns: nothing;
 n := the maximum inventory;
 T := the maximum time;
 q := order size distribution given as a list {q1,...,qM};
 lambda := the name of the function which defines the arrival rate;
 lambdaParams := the parameters for the chosen lambda function;
 dt := the size of timesteps for the numeric calculations*)
optimalRevenueNumeric[n_, T_, q_, Lambda_, LambdaParams_, dt_] := Module[{inv = 0, time, timeSteps = 1, result, M = Length[q]},
  (*Display Computation progress*)
  Print["Inventory progress"];
  Print[ProgressIndicator[Dynamic[inv / n]]];
  Print["Time progress"];
  Print[ProgressIndicator[Dynamic[time / T]]];

  (*Declare initial values*)
  JStarN[0, anyt_, anyq_, anyLambda_, anyLambdaParams_, anydt_] := 0; (*No inventory*)
  JStarN[anyn_, 0, anyq_, anyLambda_, anyLambdaParams_, anydt_] := 0; (*No Time*)

  (*Use dynamic programming to calculate values*)
  For[inv = 1, inv ≤ n, inv++,
   timeSteps = 0;
   For[time = dt, time ≤ T, time += dt,
    timeSteps += 1; (*Integer timesteps are used to avoid numerical imprecision of defining PStarN and JStarN at discrete decimal t values*)
    result =
     NMaximize[{Lambda[p, LambdaParams] * dt * Sum[q[[k]] (k * p + JStarN[Max[0, inv - k], timeSteps - 1, q, Lambda, LambdaParams, dt]), {k, 1, M}] +
        (1 - Lambda[p, LambdaParams] * dt) JStarN[inv, timeSteps - 1, q, Lambda, LambdaParams, dt],
       p ≥ 0, 0 <= Lambda[p, LambdaParams] * dt ≤ 1}, (*Constraints*)
      p, MaxIterations → 250];
    PStarN[inv, timeSteps, q, Lambda, LambdaParams, dt] = result[[2, 1, 2]];
    JStarN[inv, timeSteps, q, Lambda, LambdaParams, dt] = result[[1]];
   ]
  ]
 ]

(*constDemand[dem_,q_];
 returns: the arrival rate scaling factor a which makes a*mu[q]=dem;
 dem := the target demand value;
 q := desired order size distribution;*)
constDemand[dem_, q_] := If[Total[q] == 1., dem / mu[q], Print["Faulty q"]];

(*runManyOptimalRevenueNumeric[...] will run sequential computations to find the optimal revenue and optimal pricing strategy
 for several comparable models, only the order size distributions must be provided;
 returns: nothing;
 n := max inventory;
 T := max time;
 qList := a list of order size distributions, {{dist 1}, {dist 2},...{last dist}};
 Lambda := the arrival rate function;
 demMag := the demand magnitude all the models should have;
 alpha := *)
findRevenueForComparableModels[n_, T_, qList_, Lambda_, demMag_, alpha_, dt_] := Module[{i},
  For[i = 1, i ≤ Length[qList], i++,
   optimalRevenueNumeric[n, T, qList[[i]], Lambda, {constDemand[demMag, qList[[i]]], alpha}, dt];
  ]
 ]
```

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**Mathematics PhD** Jan 2014 - Dec 2018
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## Teaching Experience

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- Fully responsible for teaching and grading of students
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    - Applications of integration, techniques of integration; infinite sequences and series; parametric equations, conic sections, and polar coordinates.
- Math 231: Calculus and Analytic Geometry I Spring, Fall 2017
    - Limits, derivatives, and graphs of algebraic, trigonometric, exponential, and logarithmic functions; antiderivatives, the definite integral, and the fundamental theorem of calculus, with
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    - Arithmetic, geometry, and beginning algebra; development of mathematical reasoning, problem solving, and mathematical object manipulation. Individualized instruction via adaptive learning
- Math 98: Algebraic Literacy I Fall 2014
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- Math 211 (Online): Survey in Calculus and Analytic Geometry Spring 2014
    - Applications to business administration, economics, and non-physical sciences. Topics include coordinate systems, equations of curves, limits, differentiation, integration, applications.
- Math 116: College Algebra Fall 2013
    - Function concepts: Polynomial, rational, exponential, logarithmic. Systems of equations and inequalities. Matrices and determinants. Sequences and series. Analytic geometry and conic
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**Mathematics Tutor** Sep 2009 - May 2012
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## Work Experience

**Technical Services Worker** Sep 2008 - May 2012
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- Applied sensitive attention to detail while processing books and helping with other library projects

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- Led games with authority and quick decision making

## Published Works

**"Spatial Structures in a Simple Model of Population Dynamics for Parasite-Host Interactions" (Article)** June 2015
*Europhysics Letters, v111 (2015)*
- Used C++ to create simulations and utilized them for numerical data analysis

## Awards

**University of Wisconsin-Milwaukee** - *Milwaukee, WI*
- Levine Science Fellowship Spring 2018
- Ernst Schwandt Teaching Award Spring 2018
- Research Excellence Award 2017 - 2018
- Chancellor's Graduate Student Award 2013 - 2014, 2016 - 2017
- GAANN Fellowship Sum 2015 - Sum 2016

**Hamline University** - *St. Paul, MN*
- Fulford-Karp Physics Scholarship 2008 - 2012
- Presidential Scholarship 2008 - 2012
- Lindgren Mathematics Scholarship 2010 - 2012
- Hoel Scholarship 2011 - 2012
- Carl Malmstrom Research Scholarship Summer 2010
- First Year in Mathematics Award 2008 - 2009

## Presentations

**"Dyanmic Pricing with Variable Order Sizes for a Model with Constant Demand Elasticity"** June 2018
*A Symposium on Optimal Stopping - Houston, TX*

• Invited speaker

**"Dyanmic Pricing with Multiple Order Sizes"** July 2017
*SIAM Conference on Control and Applications - Pittsburgh, PA*

• Info

**"Exploring a Parasite-Host Model with Monte Carlo Simulations"** April 2011
*National Conference of Undergraduate Research (NCUR) - Ithaca, NY*

• Included results, methods, and theory of parasite-host research from Sum 2010

**"Exploring a Parasite-Host Model with Monte Carlo Simulations"** March 2011
*American Physical Society (APS) March meeting - Dallas, TX*

• Recognized by Society of Physics Students (SPS) for outstanding presentation

## Leadership

**Game Design and Development (President)** April 2014 - Dec 2017
*University of Wisconsin-Milwaukee - Milwaukee WI* Joined Nov 2012

• Work with others to hone game development skills.
• Lead discussions about game design theory and implementation.

**Theta Chi Fraternity (President & Treasurer)** Dec 2009 - Nov 2011
*Beta Kappa Chapter at Hamline University - St. Paul, MN* Nov 2008 - May 2012

• Provided vision for chapter and served as point of contact for alumni and administration.
• Improved financial accountability within chapter.
• Chaired committee responsible for financial decisions of the chapter and management of the chapter's house.

**Hamline University Swing Dancers (Treasurer)** Feb 2009 - Feb 2010
*Hamline University - St. Paul, MN* Nov 2008 - May 2012

• Integral to budgeting and planning of PR events

**New Student Mentor** Feb 2009 - Feb 2010
*Hamline University - St. Paul, MN*

• Served as role model and resource to incoming students
• Developed leadership skills while assisting Freshman orientation

## Skills

Global Game Jam (Project manager, programmer, and game designer) Jan 2014, 2015, 2016, 2017
French (5 years taken)
**Programming:** C++, Python, Lisp