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Splittings of Relatively Hyperbolic Groups and Classifications of 1-dimensional Boundaries

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SPLITTINGS OF RELATIVELY HYPERBOLIC GROUPS AND CLASSIFICATIONS OF 1-DIMENSIONAL BOUNDARIES

by

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ABSTRACT

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In the first part of this dissertation, we show that the existence of non-parabolic local cut point in the relative (or Bowditch) boundary, $\partial(\Gamma, \mathbb{P})$, of a relatively hyperbolic group $(\Gamma, \mathbb{P})$ implies that Γ splits over a 2-ended subgroup. As a consequence we classify the homeomorphism type of the Bowditch boundary for the special case when the Bowditch boundary $\partial(\Gamma, \mathbb{P})$ is one-dimensional and has no global cut points.

In the second part of this dissertation, We study local cut points in the boundary of CAT(0) groups with isolated flats. In particular the relationship between local cut points in ∂X and splittings of Γ over 2-ended subgroups. We generalize a theorem of Bowditch by showing that the existence of a local point in ∂X implies that Γ splits over a 2-ended subgroup. The first chapter can be thought of as an key step in the proof of this result. Additionally, we provide a classification theorem for 1-dimensional boundaries of groups with isolated flats. Namely, given a group Γ acting geometrically on a CAT(0) space X with isolated flats and 1-dimensional boundary, we show that if Γ does not split over a virtually cyclic subgroup, then ∂X is homeomorphic to a circle, a Sierpinski carpet, or a Menger curve. This theorem generalizes a theorem of Kapovich-Kleiner, and resolves a question due to Kim Ruane.

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Introduction

As a field, geometric group theory studies the interplay between the algebraic properties of finitely generated groups and the geometric and topological properties of spaces on which they act. When a group G acts discretely on a geometric space we can often compactify that space by attaching a "boundary at infinity." Topological classifications of this boundary can prove useful in classifying groups upto quasi-isometry, which is one of the major goals of geometric group theory. Directly determining topological properties of the boundary can often prove difficult; however, there are strong connections between the topological properties of the boundary and the algebra of G. In this chapter we investigate the relationship between local cut points in the boundary and 2-ended splittings of the group.

There are many types of boundaries that one may associate to a group. In Chapter 1 we will be interested in the Bowditch boundary of a relatively hyperbolic group and in Chapter 2 we study the visual boundary of a CAT(0) group. As mentioned above, in both settings we study the connection between local cut points in the boundary and 2-ended splittings of the group. These splitting results are then applied to prove a pair of classification theorems, which under certain hypotheses classify the homeomorphism type of the Bowditch and CAT(0) boundaries. Each of these classification theorems generalizes a well known result of Kapovich and Kleiner [KK00] to a different setting.

Chapter 1

Local cut points and splittings of relatively hyperbolic groups

1.1 Introduction

The notion of a relatively hyperbolic group was introduced by Gromov [Gro87] to generalize both word hyperbolic and geometrically finite Kleinian groups. Introduced by Bowditch [Bow12] there is a boundary for relatively hyperbolic groups. The Bowditch boundary generalizes the Gromov boundary of a word hyperbolic group and the limit set of a geometrically finite Kleinian group. The homeomorphism type of the Bowditch boundary is known to to be a quasi-isometry invariant of the group [Gro13] under modest hypotheses on the peripheral subgroups. Consequently, it is desirable to describe the topological features of the Bowditch boundary. Topological features of the boundary are closely related to algebraic properties of the group; in particular they are often related to as splittings of the group as a fundamental group of a graph of groups [Ser03].

For hyperbolic groups Bowditch [Bow98a] shows that the existence of a splitting over a 2-ended subgroup is equivalent to the existence of a local cut point in the Bowditch boundary. As evidenced by the work of Kapovich and Kleiner [KK00], this result has proved useful in classifying the homeomorphism type of 1-dimensional boundaries of hyperbolic groups. Because the existence or non-existence of 2-ended splittings can be verified directly in many natural settings, Kapovich and Kleiner's results provide techniques for constructing examples of hyperbolic groups with Menger curve or Sierpinski carpet boundary. Obstructions to 2-ended splittings are well understood for hyperbolic 3-manifold groups [Mye93], Coxeter groups [MT09], and random groups [DGP11].

Papasoglu-Swenson [PS06, PS09], and Groff [Gro13] have extended Bowditch's results [Bow98a] from hyperbolic groups to CAT(0) and relatively hyperbolic groups respectively. Their results describe the relationship between 2-ended splittings and cut pairs in the boundary. In particular, their results make no mention of local cut points. Guralnik [Gur05] and Groves-Manning [GM] have observed that many of Bowditch's local cut point results extend to relatively hyperbolic groups, provided that the Bowditch boundary has no global cut points and the peripheral subgroups are 1-ended. However, the former assumption is quite restrictive. Bowditch has shown [Bow01] that the Bowditch boundary often has many global cut points. Thus a general theorem relating local cut points in the Bowditch boundary to 2-ended splittings was still missing from the literature. The primary result of this chapter addresses the general setting with the following theorem that makes no assumption about the existence or non-existence of global cut points in the Bowditch boundary.

Theorem 1.1.1. *(Splitting Theorem) Let $(G, \mathbb{P})$ be a relatively hyperbolic group with tame peripherals. Assume that $\partial(G, \mathbb{P})$ is connected and not homeomorphic to a circle. If G does not split over a 2-ended subgroup, then $\partial(G, \mathbb{P})$ does not contain a non-parabolic local cut point. Moreover, if G splits over a non-parabolic 2-ended subgroup, then $\partial(G, \mathbb{P})$ contains a non-parabolic local cut point.*

A relatively hyperbolic group $(G, \mathbb{P})$ has *tame peripherals*, if every $P \in \mathbb{P}$ is finitely presented, one- or two-ended, and contains no infinite torsion subgroup. Bowditch has shown [Bow01] that if $(G, \mathbb{P})$ has tame peripherals and the Bowditch boundary $\partial(G, \mathbb{P})$ is connected, then $\partial(G, \mathbb{P})$ is locally connected. In this chapter we will always assume that

$\partial(G, \mathbb{P})$ is connected and that $(G, \mathbb{P})$ has tame peripherals.

Theorem 1.1.1 is used by the author in Chapter 2 to determine the homeomorphism type of 1-dimensional visual boundaries of CAT(0) groups with isolated flats which do not split over 2-ended subgroups. (Note that visual boundary and the Bowditch boundary are not the same in general [Tra13].) The application of Theorem 1.1.1 in Chapter 2 requires an understanding of the general case where the Bowditch boundary has global cut points, which is a case not addressed by the earlier result of Groves-Manning [GM].

The other major result of this chapter, Theorem 1.1.2, is a boundary classification result that generalizes a well known classification result of Kapovich and Kleiner [KK00] for boundaries of hyperbolic groups. In the hyperbolic setting $\partial(G, \mathbb{P})$ has no global cut points and no parabolic points. In the general relatively hyperbolic case $\partial(G, \mathbb{P})$ contains many global cut points and parabolic points. However, the spaces in the conclusion of Theorem 1.1.2 have no global cut points. So, we are necessarily in the restricted case previously studied by Guralnik [Gur05] and Groves-Manning [GM]. The Sierpinski carpet and Menger curve also have no local cut points. Thus we need to understand local cut points in $\partial(G, \mathbb{P})$ and find group theoretic conditions to rule out their existence (see Section 1.1.1 for more discussion).

Theorem 1.1.2. *(Classification Theorem) Let $(G, \mathbb{P})$ be a 1-ended relatively hyperbolic group with tame peripherals and let $\mathcal{P}$ be the set of all subgroups of elements of $\mathbb{P}$. Assume that G does not split over a virtually cyclic subgroup and does not split over any subgroup in $\mathcal{P}$. If every $P \in \mathbb{P}$ is one-ended and $\partial(G, \mathbb{P})$ is 1-dimensional, then one of the following holds:*

1. *$\partial(G, \mathbb{P})$ is a circle*
2. *$\partial(G, \mathbb{P})$ is a Sierpinski carpet*
3. *$\partial(G, \mathbb{P})$ is a Menger curve.*

1.1.1 Method of Proof

The proof of Theorem 1.1.1 utilizes results of Bowditch [Bow98a] for hyperbolic groups; however, because we are interested in the relatively hyperbolic setting and Bowditch's results depend on hperbolicity in an essential way additional techniques are required. In particular, the existence of global cut points in the Bowditch boundary $\partial(G,\mathbb{P})$ needs to be delt with. For hyperbolic groups the Bowditch boundary $\partial(G,\mathbb{P})$ is known to be a Peano continuum (i.e. a compact connected and locally connected metric space) without global cut points. As previously mentioned, for a relatively hyperbolic group $(G,\mathbb{P})$ with tame peripherals $\partial(G,\mathbb{P})$ is locally connected if it is connected [Bow01]. Thus $\partial(G,\mathbb{P})$ is Peano continuum, but in general it may have many global cut points [Bow01]. Our strategy involves demonstrating that it suffices to consider only the case when $\partial(G,\mathbb{P})$ has no global cut points. In particular, using the theory of peripheral splittings [Bow01] and basic decomposition theory we are able to restrict our attention to "blocks" of $\partial(G,\mathbb{P})$, where a *block* of $\partial(G,\mathbb{P})$ is a subcontinuum consisting of points which cannot be separated from each other by global cut points. Blocks have two key features. The first is that a block of $\partial(G,\mathbb{P})$ is the limit set of a relatively hyperbolic subgroup $(H,\mathbb{Q})$ of $(G,\mathbb{P})$ (see Theorem 1.3.1). The second is that there is a retraction of $\partial(G,\mathbb{P})$ onto a given block; moreover, the retraction map has nice decomposition theoretic properties. This combination of Bowditch's theory of peripheral splittings with decomposition theory techniques is one of the major contributions of this paper, and it is the focus of Section 1.3. Using these techniques allows us to reduce the proof of Theorem 1.1.1 to proving Theorem 1.4.5, which describes non-parabolic local cut points in a boundary without global cut points.

One would like to obtain Theorem 1.4.5 directly from the results of Bowditch [Bow98a]. However, there is one key step where Bowditch uses techniques which do not apply to the relatively hyperbolic setting (see Lemma 5.1 and Lemma 5.2 of [Bow98a]). Guralnik [Gur05] observed that when the Bowditch boundary has no local cut points Bowditch's results carry over to the relatively hyperbolic setting if you have a key technical result, which may be found as Lemma 1.4.1 in this exposition. Guralnik proved Lemma 1.4.1 using the work of Tukia [Tuk98]. For completeness, in Section 1.4 we include a new self-contained proof of Lemma 1.4.1. The short proof uses techniques different than those of [Gur05] that my prove useful in other settings.

The other main result of the paper is Theorem 1.1.2. Two key tools used in the proof of Theorem 1.1.2 are the topological characterization of the Menger curve due to R.D. Anderson [And58a, And58b], and the topological characterization of the Sierpinksi carpet due to Whyburn [Why58]. Anderson's theorem states that a compact metric space M is a Menger curve provided M is 1-dimensional, M is connected, M is locally connected, M has no local cut points, and no non-empty open subset of M is planar. We note that if the last condition is replaced with "M is planar," then we have the topological characterization of the Sierpinski carpet (see Whyburn [Why58]).

In order to apply Anderson and Whyburn's theorems we must rule out the existence of local cut points. Theorem 1.1.1 can be used to rule out non-parabolic local cut points, but we also need to rule out the existence of parabolic local cut points. A point p in $\partial(G, \mathbb{P})$ is a *local cut point* if $\partial(G, \mathbb{P}) \setminus \{p\}$ is disconnected, or $\partial(G, \mathbb{P}) \setminus \{p\}$ connected and has more than one end. In Theorem 1.1.2 we are in a setting where $\partial(G, \mathbb{P})$ contains no global cut points, so $\partial(G, \mathbb{P}) \setminus \{p\}$ is connected. Thus we need only know that $\partial(G, \mathbb{P}) \setminus \{p\}$ is 1-ended. Because the group $P = \mathrm{Stab}(p)$ is 1-ended, and Bowditch [Bow12] has shown that P acts properly and cocompactly on $\partial(G, \mathbb{P}) \setminus \{p\}$, a reader familiar with geometric group theory may think that we are done. However, the author was unable to find sufficiently general results in the literature. It would appear that known results stating that ends of a group is independent of the space on which the group acts are only found for groups acting on less general spaces, such as CW-complexes [Geo08, Gui16]. In this paper we require an understanding of the ends of a group acting on a connected open subset of Peano continuum. The study of ends occurs naturally in the setting of connected, locally compact, locally path connected, Hausdorff spaces (see [Gui16]). The natural question to ask is, what happens when a group acts proper and cocompactly on such spaces? If G acts properly and cocompactly on two connected, locally compact, locally path connected, Hausdorff spaces X and Y, is $Ends(X)$ homeomorphic to $Ends(Y)$?

Theorem 1.1.3, which may be considered a folk theorem, extends known ends results to this larger class of spaces where the study of ends occurs naturally. It is worth noting that this general class of spaces includes open connected subspaces of Peano continua. Theorem 1.1.3 has already proved useful outside of this paper, as this fact is used by Groves and Manning in their proof of a special case of Theorem 1.1.1 (see Section 7 of [GM]).

Theorem 1.1.3. *Let X be a connected, locally compact, and locally path connected, Hausdorff space, and assume that G is a with finite generating set S acting properly and cocompactly on X. Then* $\text{Ends}\left(\Upsilon(G,S)\right)$ *is homeomorphic to* $\text{Ends}(X)$.

Here $\Upsilon(G,S)$ is the Cayley graph of G with respect to S. Again, our interest in spaces which meet the hypotheses of Theorem 1.1.3 lies in the fact that if $(G,\mathbb{P})$ is a relatively hyperbolic group with tame peripherals, then $\partial(G,\mathbb{P})$ minus a parabolic point satisfies these hypotheses.

1.2 Preliminaries

1.2.1 Relatively Hyperbolic Groups and Their Boundaries

Let G be a group and $\mathbb{P}$ a collection of infinite subgroups that is closed under conjugation, called *peripheral subgroups.*

Definition: We say that G is *hyperbolic relative to* $\mathbb{P}$ and write $(G,\mathbb{P})$ if G admits a proper isometric action on a proper δ-hyperbolic space X such that:

1. $\mathbb{P}$ is the set of all maximal parabolic subgroups of G

2. There exists a G-invariant system of disjoint open horoballs based at the parabolic

points of G, such that if $\mathcal{B}$ is the union of these horoballs, then G acts cocompactly on $X \setminus \mathcal{B}$.

In [Bow12] Bowditch shows:

Theorem 1.2.1. *If G is hyperbolic relative to $\mathbb{P}$, then $\mathbb{P}$ consists of only finitely many conjugacy classes.*

The Bowditch boundary $\partial(G, \mathbb{P})$ is defined to be the visual boundary of X, i.e the set of equivalence classes of geodesic rays of X, where two geodesic rays are equivalent if their Hausdorff distance is bounded. It is a result of Bowditch [Bow12] that $\partial(G, \mathbb{P})$ is well defined for $(G, \mathbb{P})$.

We say that a relatively hyperbolic group $(G, \mathbb{P})$ has *tame peripherals* if every $P \in \mathbb{P}$ is finitely presented, one- or two-ended, and contains no infinite torsion subgroup. Under the assumption of tame peripherals Bowditch has shown the following two results in [Bow99b] and [Bow01], respectively.

Theorem 1.2.2. *Suppose that $(G, \mathbb{P})$ is relatively hyperbolic with tame peripherals and that $\partial(G, \mathbb{P})$ is connected, then every global cut point of $\partial(G, \mathbb{P})$ is a parabolic point.*

A *global cut point* is a point whose removal disconnects $\partial(G, \mathbb{P})$ and a *parabolic point* is point which is stabilized by a parabolic subgroup (see Section 1.2.2).

Theorem 1.2.3. *If $(G, \mathbb{P})$ is relatively hyperbolic with tame peripherals and $\partial(G, \mathbb{P})$ is connected, then $\partial(G, \mathbb{P})$ is locally connected.*

In this paper we are interested in the case where $\partial(G, \mathbb{P})$ is locally connected, so we will generally assume that $(G, \mathbb{P})$ has tame peripherals and that $\partial(G, \mathbb{P})$ is connected.

1.2.2 Convergence Group Actions

Let M be a compact metrizable space. Let G be a group acting by homeomorphisms on M. A group G is called a *convergence group* if for every sequence of distinct group elements (g_k)

there exist points $\alpha, \beta \in M$ (not necessarily distinct) and a subsequence $(g_n) \subset (g_k)$ such that $g_n(x) \to \alpha$ locally uniformly on $M \setminus \{\beta\}$, and $g_n^{-1}(x) \to \beta$ converges locally uniformly on $M \setminus \{\alpha\}$. By *locally uniformly* we mean, if C is a compact subset of $M \setminus \{\beta\}$ and U is any open neighborhood of α, then there is an $N \in \mathbb{N}$ such that $g_n C \subset U$ for all $n > N$.

Elements of convergence groups can be classified into three types: elliptic, loxodromic, and parabolic. A group element is *elliptic* if it has finite order. An element g of G is *loxodromic* if has infinite order and fixes exactly two points of M. If $g \in G$ has infinite order and fixes a single point of M then g is *parabolic*. A subgroup P of G is *parabolic* if it contains no loxodromic elements and stabilizes a single point p of M. The point p is uniquely determined by P, and the point p is called a *parabolic* point. We call p a *bounded parabolic* point if P acts cocompactly on $M \setminus \{p\}$.

For the purpose of this chapter we are interested in the case where $M = \partial(G, \mathbb{P})$. A point $x \in \partial(G, \mathbb{P})$ is a *conical limit* point if there exists a sequence of group elements $(g_n) \in G$ and distinct points $\alpha, \beta \in M$ such that $g_n x \to \alpha$ and $g_n y \to \beta$ for every $y \in M \setminus \{x\}$. Tukia has shown (see [Tuk98]) that:

Proposition 1.2.4. *A conical limit point cannot be a parabolic point*

A convergence group G acting on M is called *uniform* if every point of M is a conical limit point, and G is called *geometrically finite* if every point of M is a conical limit point or a bounded parabolic point. Bowditch has shown [Bow98b] G is a uniform convergence group if and only if it is hyperbolic. A generalization of this result was completed by Bowditch [Bow12] and Yaman [Yam04]. Bowdtich [Bow12] shows that a relatively hyperbolic group with finitely generated peripheral subgroups acts on it's Bowditch boundary as a geometrically finite convergence group, and Yaman [Yam04] proves a strong converse. We remark that in general geometrically finite convergence group actions are not uniform.

1.2.3 Splittings

A *splitting* of a group G over a given class of subgroups is a finite graph of groups representation of G, where each edge group belongs to the given class. The group G is said to split *relative* to another class of subgroups, $\mathbb{P}$, if each element of $\mathbb{P}$ is conjugate into one of the vertex groups. A splitting is called *trivial* if there exists a vertex group equal to G. Assume that G is hyperbolic relative to a collection $\mathbb{P}$. A *peripheral splitting* of $(G, \mathbb{P})$ is a finite bipartite graph of groups representation of G, where $\mathbb{P}$ is the set of conjugacy classes of vertex groups of one color of the partition called peripheral vertices. Non-peripheral vertex groups will be referred to as *components*. This terminology stems from the correspondence between the cut point tree of $\partial(G, \mathbb{P})$ and the peripheral splitting of $(G, \mathbb{P})$, where elements of $\mathbb{P}$ correspond to stabilizers of cut point vertices and the components correspond to stabilizers of blocks in the boundary (see Theorem 1.3.1).

A peripheral splitting $\mathcal{G}$ is a refinement of another peripheral splitting $\mathcal{G}'$ if $\mathcal{G}'$ can be obtained from $\mathcal{G}$ via a finite sequence of foldings that preserve the vertex coloring. In [Bow01] Bowditch proves the following accessibility result:

Theorem 1.2.5. *Suppose that* $(G, \mathbb{P})$ *is relatively hyperbolic with tame peripherals and connected boundary. Then* $(G, \mathbb{P})$ *admits a (possibly trivial) peripheral splitting which is maximal in the sense that it is not a refinement of any other peripheral splitting.*

Combining Proposition 5.1 and Theorem 1.2 of [Bow99a] Bowditch also shows:

Theorem 1.2.6. *If* $(G, \mathbb{P})$ *is a relatively hyperbolic with tame peripherals,* $\partial(G, \mathbb{P})$ *is connected, and* $\partial(G, \mathbb{P})$ *has a global cut point, then there exists a non-trivial peripheral splitting of* $(G, \mathbb{P})$.

The following theorem was communicated to the author by Chris Hruska and relies on Theorem 1.3.1 (4) and known results about the action of the G on $\partial(G, \mathbb{P})$. In particular, Bowditch has shown [Bow12] that the action of G on $\partial(G, \mathbb{P})$ is *minimal*, i.e. $\partial(G, \mathbb{P})$ does not properly contain a closed G-invariant subset. Because it will be of use in Section 1.7, it is worth noting that the action of G on $\partial(G, \mathbb{P})$ is minimal if and only if $\mathrm{Orb}_G(m)$ is dense for every $m \in \partial(G, \mathbb{P})$.

Theorem 1.2.7. *If $(G, \mathbb{P})$ is relatively hyperbolic with tame peripherals, $\partial(G, \mathbb{P})$ is connected, and $\partial(G, \mathbb{P})$ contains a global cut point. Then $(G, \mathbb{P})$ splits non-trivially over every edge group in the maximal peripheral splitting of $(G, \mathbb{P})$ that corresponds to an edge connecting a component vertex to a peripheral cut point vertex.*

Proof. Assume that T is the Bass-Serre tree for the maximal peripheral splitting of G. Assume there exists an edge e in T such that G does not split over the edge group G_e non-trivially. Then there is a G-invariant subtree B in T which does not contain e (see [HR] Lemma 12.8). Thus, $B \neq T$. By Theorem 1.3.1 (4) there is a closed G-invariant proper subspace of $\partial(G, \mathbb{P})$. Thus the action of G on $\partial(G, \mathbb{P})$ is not minimal, a contradiction. □

The immediate corollary is:

Corollary 1.2.8. *If $(G, \mathbb{P})$ is relatively hyperbolic with tame peripherals, $\partial(G, \mathbb{P})$ is connected, and $\partial(G, \mathbb{P})$ contains a global cut point p. Then G splits non-trivially over a subgroup of the maximal parabolic group stabilizing p.*

1.2.4 Cut Point Structures In Metric Spaces

Recall that a *continuum* is a compact connected metric space and that a *Peano continuum* is a locally connected continuum. Though many of the following definitions are valid for general continua we are only interested in the locally connected case. Let M be a Peano continuum. A *global cut point* of M is a point $x \in M$ such that $M \setminus \{x\}$ is disconnected. A *cut pair* is a set of two distinct points $\{a, b\} \subset M$ which contains no global cut points, and such that $M \setminus \{a, b\}$ is disconnected. The set of components of $M \setminus \{a, b\}$ will be denoted by $\mathcal{U}(a, b)$ and $\mathcal{N}(a, b)$ will denote the cardinality of $\mathcal{U}(a, b)$ We leave it as an exercise to show if x is a global cut point and $\{a, b\}$ is a cut pair, then a and b cannot lay in different components of $M \setminus \{x\}$. Two cut pairs $\{a, b\}$ and $\{c, d\}$ are said to *mutually separate* M if c and d lie in different components of $M \setminus \{a, b\}$ and vice versa. A cut pair is called *inseparable* if it does not mutually separate with any other cut pair. If $M = \partial(G, \mathbb{P})$ then a cut pair $\{a, b\}$ will be called *loxodromic* if it is stabilized by a loxodromic element $g \in G$.

Let Δ be a subset of M. We say that Δ is *cyclically separating* if for every finite subset F of Δ with $|F| > 3$ there is an embedding of $i\colon F \to S^1$ such that given any four $a, b, c, d \in F$ the set $\big\{i(a), i(b), i(c), i(d)\big\}$ can be partitioned into pairs which mutually separate S^1, and i induces a map from the components of $M \setminus \{a, b, c, d\}$ onto the components of $S^1 \setminus \big\{i(a), i(b), i(c), i(d)\big\}$, where a component C of $M \setminus \{a, b, c, d\}$ with $Fr(C) = \{x, y\} \subset \{a, b, c, d\}$ is mapped to the component of $S^1 \setminus \big\{i(a), i(b), i(c), i(d)\big\}$ with frontier $\{i(x), i(y)\}$. Two points a and b in a cyclically separating set Δ are called *adjacent* if $\big\{i(a), i(b)\big\}$ cannot be mutually separated by $\big\{i(c), i(d)\big\}$ for any $c, d \in \Delta$. An unordered pair of adjacent points in cyclically separating set will be referred to as a *jump*.

A point $x \in M$ is a *local cut point* if $M \setminus \{x\}$ is disconnected or has more than one end. If $M \setminus \{x\}$ is connected the *valence*, $val(x)$, of a local cut point is the number of ends of $M \setminus \{x\}$. A detailed discussion of ends of spaces can be found in Section 1.2.5, but we remark that saying a point $x \in M$ is a local cut point is equivalent to saying that there exists a neighborhood U of x such that for every neighborhood V of x with $V \subset U$, there exist points $z, y \in V \setminus \{x\}$ such that there does not exist a connected subset of $U \setminus \{x\}$ containing z and y. Alternatively, to check that x is not a local cut point it suffices to show that given a neighborhood U of x there exists a neighborhood $V \ni x$ with $V \subset U$ and $V \setminus \{x\}$ connected. We wish to "collect" all the local cut points and that end we introduce notation similar to that of Bowditch [Bow98a] to describe the various "local cut point structures" in M. Let $M(n) = \big\{\, x \in M \;\big|\; val(x) = n \,\big\}$ and $M(n+) = \big\{\, x \in M \;\big|\; val(x) \geq n \,\big\}$.

Now assume that a group G acts on M with a geometrically finite convergence group action. Then G is relatively hyperbolic and M is homeomorphic to $\partial(G, \mathbb{P})$ [Bow98b] [Yam04].

If $M = \partial(G, \mathbb{P})$, then M consists entirely of conical limit points and parabolic points; moreover, global cut points in M correspond to parabolic points (See Section 1.2.2). Because parabolic points cannot be conical limit points (Proposition 1.2.4), the goal is to understand local cut points which are conical limit points to ensure that the points we are considering do not separate M globally. Define $\mathcal{C}$ to be the collection of conical limit points in M. We will denote by $M^*(n)$ and $M^*(n+)$ the intersections of $M(n)$ and $M(n+)$ with $\mathcal{C}$. We define relations on $M^*(2)$ and $M^*(3+)$. Let $x, y \in M^*(2)$. We write $x \sim y$ if and only if $x = y$ or $\mathcal{N}(x, y) = 2$. For two elements $a, b \in M^*(3+)$ we write $a \approx b$ if $a \neq b$ and $\mathcal{N}(a, b) = val(a) = val(b) \geq 3$. From the definitions above we immediately obtain a partition of the set of conical limit points which are local cut points. In other words:

Lemma 1.2.9. *Let $x \in M$ be a conical limit point which is a local cut point. Then $x \in M^*(2) \cup M^*(3+)$*

The following results are proved using the same arguments as those of Bowditch in [Bow98a]:

Lemma 1.2.10. *The collection of $\approx$-classes in $M^*(3+)$ is partitioned into pairs, which do not mutually separate.*

Lemma 1.2.11. *The relation $\sim$ is an equivalence relation on $M^*(2)$.*

We say that a cut pair $\{c, d\}$ in M *separates* a subset $C \subset M$ if C is contained in at least two distinct components of $M \setminus \{c, d\}$.

Lemma 1.2.12. *Let $a, b, c, d \in M^*(2)$. If $a \sim b$ and $\{c, d\}$ separates $\{a, b\}$, then $c \sim d \sim a \sim b$, and the pairs $\{a, b\}$ and $\{c, d\}$ mutually separate.*

In the case $M = \partial(G, \mathbb{P})$ an argument similar to that of Bowditch [Bow98a] shows that there are no singleton $\sim$-classes in $M^*(2)$; consequently, a $\sim$-class in $M^*(2)$ consists of either a cut pair or a cyclically separating collection of cut pairs. The closure of a $\sim$-class ν

containing at least three elements will be called a *necklace*. Notice that if ν is infinite, then $\overline{\nu}$ may contain parabolic points. Lastly remark that, because cut pairs cannot be separated by global cut points neither can $\sim$-classes or their closure.

1.2.5 Ends of Spaces

In this section we review ends of spaces. Roughly speaking the number of ends of a connected space X counts the number of components at infinity in X. A more detailed discussion about ends of spaces may be found in Section 3 of [Gui16].

A nested sequence $C_1 \subseteq C_2 \subseteq C_3 \subseteq ...$ of compact sets in X is called an *exhaustion of* X if $X = \cup_{i=1}^{\infty} C_i$. An exhaustion is said to be *efficient* if for every $i \in \{0, 1, 2, ..\}$ C_i is connected with $C_i \subseteq int(C_{i+1})$, and $U_i = X \setminus C_i$ consists entirely of components with non-compact closure. A connected, locally compact, and locally path connected, Hausdorff space will be called a *fatigued space*. It is an exercise to show that a fatigued space has an efficient exhaustion. We remark that it follows from the definition that any connected open subset of a fatigued space is fatigued. Moreover, the context of this chapter makes it worth noting that a connected open subset of a Peano continuum is fatigued.

In the remainder of this section we shall assume that X is fatigued. Let $C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots$ be an efficient exhaustion of X, and let $U_i = X \setminus C_i$ for every i. The set Ends(X) of *ends of* X is the set of all sequences $(E_1, E_2, E_3, ...)$ where E_i is a component of U_i and such that for each i, $E_i \supseteq E_{i+1}$. We shall see later that Ends(X) is independent of the choice of efficient exhaustion. The cardinality of Ends(X) is the *number of ends* of the space X. Let G be a finitely generated group with generating set S. When we speak of the ends of G we are referring to Ends(X), where X is an fatigued space on which G acts properly and cocompactly. We show in Theorem 1.1.3 that Ends(G) is well defined.

The *Freudenthal Compactification* of X is $X \cup \text{Ends}(X)$ with the topology generated by the basis consisting of all open subsets of X and all sets $\overline{E}_i$ where

$$\overline{E}_i = E_i \cup \left\{ (F_1, F_2, F_3, ...) \in \text{Ends}(X) \;\middle|\; F_i = E_i \right\}.$$

It is well known that the Freudenthal compactification is compact, separable and metrizable. The space Ends(X) is given the subspace topology.

Recall that a map between two spaces $f\colon X \to Y$ is called *proper* if for every compact subset C of Y we have $f^{-1}(C)$ is compact. The following well known result can be found in [Gui16] as an exercise. We include the proof for completeness.

Proposition 1.2.13. *Let $f\colon X \to Y$ be a proper map between fatigued spaces, then f can be uniquely extended to a continuous map f^* from $X \cup \text{Ends}(X)$ and $Y \cup \text{Ends}(Y)$.*

Proof. Let $\{C_i\}_{i=1}^{\infty}$ and $\{D_i\}_{i=1}^{\infty}$ be efficient exhaustions of X and Y respectively. As f is proper we have that $f^{-1}(D_i)$ is compact and thus for each i there exists an n_i such that $f^{-1}(D_i) \subseteq C_{n_i}$, which implies that $f(X \backslash C_{n_i}) \subseteq f\big(X \setminus f^{-1}(D_i)\big) \subseteq Y \backslash D_i$.

Let $E = (E_1, E_2, E_3, ...)$ be an end of X. The continuous image of a connected set is connected. Since E_{n_i} is a connected component of $X \setminus C_{n_i}$ it must be mapped into a connected component F_i of $Y \backslash D_i$. Let f^* be equal to f on X and define $f^*(E_{n_i}) = F_i$ for all i. If $j \geq i$ we have that $E_{n_i} \supseteq E_{n_j}$, which implies that $F_i \supseteq F_j$. Thus we have found a compatible sequence $\{F_i\}$ which represents an end F of Y. Continuity of f^* follows, because for any neighborhood $\overline{F}_i$ of F there is a neighborhood $\overline{E}_{n_i}$ of E such that $f^*(\overline{E}_{n_i}) \subset \overline{F}i$. $\square$

Corollary 1.2.14. Ends(X) *is independent of choice of efficient exhaustion.*

A useful and more geometric way to describe the ends of a fatigued space X is by proper rays. By *proper ray* we mean any proper map $\alpha\colon [0, \infty) \to X$. Two rays α and β are equivalent if there is a proper map h of the *infinite ladder* (or simply *ladder*)

$$L_{[0,\infty)} = \big([0, \infty) \times \{0, 1\}\big) \cup \big(\mathbb{N} \times [0, 1]\big)$$

such that α and β are the *sides*, i.e. $\alpha = h|_{[0,\infty)\times\{0\}}$ and $\beta = h|_{[0,\infty)\times\{1\}}$. The image under h of $n \times [0,1]$ is called a *rung*. The set of ends can be identified with the collection of equivalence classes of proper rays.

1.3 Reduction

Let $(G, \mathbb{P})$ be a relatively hyperbolic group with tame peripherals. The results in this section can be considered the first step in the proof of Theorem 1.1.1. In particular, we show that the proof of Theorem 1.1.1 can be reduced to the case where the Bowditch $\partial(G, \mathbb{P})$ has no global cut points.

1.3.1 Blocks and Branches

In this subsection we look at cut point decompositions of $\partial(G, \mathbb{P})$. For a more in depth overview see [Bow99c] and [Swe00].

Let M be a Peano continuum, and let Π be the set of global cut points of M. We define a relation R on M by xRy if x and y cannot be separated by an element of Π. In other words, xRy means x and y lie in the same component of $M \setminus \{z\}$ for every z in $\Pi \setminus \{x, y\}$. Assume x is not a global cut point, then the *block* containing x is the collection of points $y \in M$ such that xRy, and will be denoted $[x]$. If two blocks $[u]$ and $[v]$ intersect, then they intersect in an element of P or $[u] = [v]$ (see [Swe00]).

If M is the boundary of a relatively hyperbolic group with tame peripherals, then M is a Peano continuum and the relation R naturally associates to M a simplicial bipartite tree T [Bow01]. The vertices of T correspond to elements of Π and the set of blocks B. Additionally, two vertices $b \in B$ and $p \in \Pi$ are adjacent if $p \subset b$.

Now, let T be the Bass-Serre tree for the maximal peripheral splitting $\mathcal{G}$ of G (see Theorem 1.2.5), and assume that $\mathcal{R}$ and $\mathcal{P}$ are the collections of component and peripheral vertices respectively. Then Bowditch [Bow01] has shown the following:

Theorem 1.3.1. *Let $(G, \mathbb{P})$ be relatively hyperbolic with tame peripherals and connected Bowditch boundary. Assume that T, $\mathcal{R}$, and $\mathcal{P}$ are as above. There exists an injective map $\beta\colon \mathcal{P} \cup \partial T \to \partial(G, \mathbb{P})$ and for every $v \in \mathcal{R}$ there exists a set $B(v) \subset \partial(G, \mathbb{P})$ satisfying the following:*

1. *$B(v)$ is closed for every $v \in \mathcal{R}$*

2. *If $x \in \mathcal{P}$ then $\beta(x)$ is a parabolic point.*

3. *If $(x_n) \subset \mathcal{P}$ is a sequence of points converging to $i \in \partial T$, then the sequence $\beta(x_n)$ converges to a point $\iota = \beta(i)$ in $\partial(G, \mathbb{P})$. Such a point will be referred to as an ideal point.*

4. *If v is a vertex in $\mathcal{R}$, then $\beta(v)$ is a block in $\partial(G, \mathbb{P})$ stabilized by a relatively hyperbolic group $(H, \mathbb{Q})$ where $\mathbb{Q} = \{ Q \mid Q = stab_G(v) \cap P \text{ with } Q \text{ infinite and } P \in \mathbb{P} \}$. Moreover, $B(v)$ is homeomorphic to $\partial(H, \mathbb{Q})$ and has no global cut points.*

5. *Given a subtree S in T and let $\mathcal{P}(S)$ and $\mathcal{R}(S)$ be $\mathcal{P} \cap S$ and $\mathcal{R} \cap S$, respectively. Then the set $\Psi^0(S) = \beta(\mathcal{P}(S)) \cup \bigcup_{v \in \mathcal{R}(S)} B(v)$ is connected and its closure is the set $\Psi(S) = \beta(\mathcal{P}(S) \cup \partial S) \cup \bigcup_{v \in \mathcal{R}(S)} B(v)$. If S is a branch in T then $\Psi(S)$ is called a branch of $\partial(G, \mathbb{P})$.*

6. *$\Psi(T) = \partial(G, \mathbb{P})$*

7. *If v is a vertex in $\mathcal{R}$, then $B(v)$ does not contain any ideal points.*

8. *Every ideal point ι has a neighborhood base consisting of branches, and any branch containing ι is a neighborhood of ι.*

Corollary 1.3.2. *A local cut point in $\partial(G, \mathbb{P})$ must be in a block, i.e. ideal points are not local cut points.*

Proof. We first show that a branch in $\partial(G, \mathbb{P})$ minus an ideal point is connected. Let ι be an ideal point in $\partial(G, \mathbb{P})$. Then ι is contained in some branch, $\Psi(B)$. As $\Psi^0(B) \subset \Psi(B) \setminus \{\iota\} \subset \Psi(B)$, $\Psi^0(B)$ is connected, and $\Psi(B)$ is the closure of $\Psi^0(B)$, we have that $\Psi(B) \setminus \{\iota\}$ is connected. Thus $\partial(G, \mathbb{P}) \setminus \{\iota\}$ is connected.

Now if U is any neighborhood of ι, we have from Theorem 1.3.1 (7) that there is branch $B \subset U$ containing ι. By the argument in the preceding paragraph $B \setminus \{\iota\}$ is connected and ι cannot be a local cut point (see Section 1.2.4). $\square$

1.3.2 Decompositions and Reduction

A *decomposition* $\mathcal{D}$ of a topological space X is a partition of X. Associated to $\mathcal{D}$ is the *decomposition space* whose underlying point set is $\mathcal{D}$, but denoted $X/\mathcal{D}$. The topology of $X/\mathcal{D}$ is given by the *decomposition map* $\pi \colon X \to X/\mathcal{D}$, with $x \mapsto D$, and where $D \in \mathcal{D}$ is the unique element of the decomposition containing x. A set U in $X/\mathcal{D}$ is deemed open if and only if $\pi^{-1}(U)$ is open in X. A subset A of X is called *saturated* (or $\mathcal{D}$-saturated) if $\pi^{-1}\big(\pi(A)\big) = A$. The *saturation* of A, $Sat(A)$, is the union of A with all $D \in \mathcal{D}$ that intersect A. The decomposition $\mathcal{D}$ is said to be *upper semi-continuous* if every $D \in \mathcal{D}$ is closed and for every open set U containing D there exists and open set $V \subset U$ such that $D \subset V$ and $sat(V)$ is contained in U. An upper semi-continuous decomposition $\mathcal{D}$ is called *monotone* if the elements of $\mathcal{D}$ are compact and connected.

A collection of subsets $\mathcal{S}$ of a metric space is called a *null family* if for every $\epsilon > 0$ there are only finitely $S \in \mathcal{S}$ with $diam(S) > \epsilon$. The following proposition can be found as Propositions I.2.3 in [Dav07].

Proposition 1.3.3. *Let $\mathcal{S}$ be a null family of closed disjoint subsets of a compact metric space X. Then the associated decomposition of X is upper semi-continuous.*

Lemma 1.3.4. *If $\mathcal{D}$ is an upper semi-continuous decomposition of a space X, then the saturation of a closed set is closed.*

Now suppose that X is fatigued (see 1.2.5).

Lemma 1.3.5. *If $\mathcal{D}$ be an upper semi-continuous monotone decomposition of a fatigued space X, then $X/\mathcal{D}$ is fatigued.*

Lemma 1.3.4 can be found in [Dav07] and 1.3.5 follows from standard point set topology results [Wil04] and Section 2 Proposition 1 of [Dav07].

Proposition 1.3.6. *Assume that $\mathcal{D}$ is an upper semi-continuous decomposition, and let $f\colon X \to X/\mathcal{D}$ be the decomposition map. If $x \in X$ is a local cut point and $\{x\} \in \mathcal{D}$ then $f(x)$ is a local cut point.*

Proof. Assume the hypothesis. There exists an open neighborhood U of x such that for every $V \subset U$ with $x \in V$, there exist $w, v \in V \setminus \{x\}$ such that w and v are not contained in any connected subset of $U \setminus \{x\}$. Define $U^* = X \setminus Sat(X \setminus U)$. Notice that U^* is open in X by Lemma 1.3.4, $U^* \subset U$, $Sat(U^*) = U^*$, and $f(U^*)$ is open in $X/\mathcal{D}$. Notice that $\{x\} \in \mathcal{D}$, and let $A \subset f(U^*)$ be an open neighborhood of $x = f(x)$. The claim is that there exist two points in A which are not in the same connected subset of $f(U^*) \setminus \{x\}$.

The preimage $f^{-1}(A)$ is an open subset of $U^* \subset U$ and must contain two points a' and b' which are not contained in the same connected subset of $U \setminus \{x\}$ and thus not contained in the same connected subset of $U^* \setminus \{x\}$. So, $f^{-1}(A) \setminus \{x\}$ is disconnected in $U^* \setminus \{x\}$ and meets at least two components of $f^{-1}(A) \setminus \{x\}$ call them C_1 and C_2. Since U^* is saturated and the elements of the decomposition $\mathcal{D}$ are connected, we know that there does not exist an element of the decomposition inside of U^* which intersects both C_1 and C_2. Thus $f(C_1)$ and $f(C_2)$ are disjoint in $f(U^*) \setminus \{x\}$. Choose $a \in C_1$ and $b \in C_2$, then $f(a)$ and $f(b)$ are not both contained in any connected subset of $X/\mathcal{D}$. $\square$

Returning to the setting of Bowditch boundaries we will use the notation introduced in Section 1.3.1. Bowditch has shown in section 8 of [Bow01] that the set of all branches attached to a component of $\partial(G, \mathbb{P})$ forms a null family. Consequently, we have the following lemma:

Lemma 1.3.7. *Let $R = \beta(v)$ for some $v \in \mathcal{R}$, and define $f\colon \partial(G, \mathbb{P}) \to R$ to be the quotient map obtained by identifying all branches rooted in R with their roots. Then f is upper semi-continuous monotone retraction onto R.*

Corollary 1.3.8. *If a cut pair $\{a, b\}$ separates R, then is separates $\partial(G, \mathbb{P})$.*

Proof. Let R be a block of $\partial(G, \mathbb{P})$. The decomposition of $\partial(G, \mathbb{P})$ associated the quotient map $f\colon \partial(G, \mathbb{P}) \to R$ given in Lemma 1.3.7 is upper semi-continuous and monotone. If C_1 and C_2 are two components of $R \setminus \{a, b\}$ then their preimages under f must be connected and disjoint. Otherwise, there would exist a branch with root in C_1 and C_2. □

Lastly if R is a block and $f\colon \partial(G, \mathbb{P}) \to R$ is as in Lemma 1.3.7, then we have:

Lemma 1.3.9. *Let x be a point contained in a block R. If x is a local cut point and a conical limit point, then $f(x)$ is a local cut point of R.*

Proof. This follows immediately from Proposition 1.3.6 and Lemma 1.3.7. □

1.4 Local Cut Points in $\partial(G, \mathbb{P})$

The goal of this section is to prove Theorem 1.4.7. In the hyperbolic setting Bowditch [Bow98a] showed that a local cut point must be contained in an inseparable loxodromic cut pair or a necklace. As first observed by Guralnik [Gur05], a careful examination of [Bow98a] reveals that many of Bowditch's argument could directly translate to $\partial(G, \mathbb{P})$ if one restricts their attention only to local cut points which are conical limit points. However, there is on key step where Bowditch uses hyperbolicity. Namely, in section 5 of [Bow98a] his argument requires that G act as a uniform convergence group on its boundary, i.e that the action on the triple space is proper and cocompact. As mentioned in section 1.2 in the relatively hyperbolic setting the action of G on $\partial(G,\mathbb{P})$ is not uniform. The following lemma generalizes Lemma 5.2 of [Bow98a] to the relatively hyperbolic setting and allows us to plug directly in to Bowditch's results. Lemma 1.4.1 can also be found in [Gur05], but for completeness we include an alternate more self-contained proof, which uses different techniques.

Lemma 1.4.1. *There exist finite collections $(U_i)_{i=1}^p$ and $(V_i)_{i=1}^p$ of open connected sets of $\partial(G,\mathbb{P})$ with disjoint closures, $\overline{U}_i \cap \overline{V}_i = \emptyset$, such that if $K \subseteq \partial(G,\mathbb{P})$ is closed and $x \in \partial(G,\mathbb{P})\backslash K$ is a conical limit point then there exists $g \in G$ and $i \in \{1,...,p\}$ such that $gx \in U_i$ and $gK \subseteq V_i$.*

We postpone the proof of 1.4.1, as it will require a few lemmas. Let X be the proper $\delta - hyperbolic$ space on which G acts as given by the definition of relatively hyperbolic. We know from Theorem 1.2.1 that there finitely many orbits of horoballs in $\mathcal{B}$. Let $B_1, B_2, ..., B_n$ be representatives from each orbit and $p_1, p_2, ..., p_n$ the associated parabolic points for each representative horoball. In [Bow12] it is shown that $C_i = fr(B_i)/Stab_G(p_i)$ is compact for every $i \in \{1,2,...,n\}$ and from the definition of relatively hyperbolic we know $(X \setminus \mathcal{B})/G$ is compact. Define

$$C = \big((X \setminus \mathcal{B})/G\big) \cup C_1 \cup C_2 \cup ... \cup C_n.$$

Then C is a compact subset of X and $\mathrm{Orb}_G(C) \supseteq X$.

Let $\Theta_2\partial(G,\mathbb{P})$ the space of distinct pairs in $\partial(G,\mathbb{P})$ and define $E(C) \subseteq \Theta_2\partial X$ to be the collection of pairs (x,y) such that $x = c(\infty)$ and $y = c(-\infty)$ for some line $c\colon \mathbb{R} \to X$ with $im(c) \cap C \neq \emptyset$.

Lemma 1.4.2. *The set $E(C)$ is compact in $\Theta_2\partial(G,\mathbb{P})$.*

The proof of Lemma 1.4.2 follows from sequential compactness using a standard diagonal argument to see that a sequence of lines each meeting C converges to a line meeting C. We leave the details as an exercise.

For any pair $(x, y) \in \Theta_2\partial(G, \mathbb{P})$ we may find a line whose ends are x and y(see Chapter III). Such a line must be a translate of a line which passes through C; consequently we obtain:

Corollary 1.4.3. *G acts cocompactly on $\Theta_2\partial(G, \mathbb{P})$.*

Lemma 1.4.4. *There exist finite collections $(U_i)_{i=1}^p$ and $(V_i)_{i=1}^p$ such that $\overline{U}_i \cap \overline{V}_i = \emptyset$ for every $i \in \{1, ..., p\}$, and such that if $x, y \in \partial(G, \mathbb{P})$ are then there exists $g \in G$ and $i \in \{1, ..., p\}$ such that $gx \in U_i$ and $gy \in V_i$.*

Proof. Let d be the visual metric on $\partial(G, \mathbb{P})$. Let K be a compact set whose G translates cover $\Theta_2(\partial(G, \mathbb{P}))$. Clearly, $K \cap D = \emptyset$, where D is the diagonal. For every $(x, y) \in K$ define $r(x, y) = \frac{1}{4}d(x, y)$ and define $U_x = B(x, r(x, y))$ and $V_y = B(y, r(x, y))$. Then $\bigcup_{(x,y)\in C}(U_x \times U_y)$ covers K. By compactness there exist finitely many $(x_i, y_i) \in K$ such that $U_{x_i} \times V_{x_i}$ cover K. Notice that by construction $\overline{U}_{x_i} \cap \overline{V}_{y_i} = \emptyset$. Thus by the cocompactness of the action we are done. □

Proof of 1.4.1. Let x be a conical limit point. By the definition of conical limit point there exists $(g_n) \in G$ and distinct points $\alpha, \beta \in \partial(G, \mathbb{P})$ such that $g_n x \to \alpha$ and $g_n x \to \beta$ for every $y \in \partial(G, \mathbb{P}) \setminus \{x\}$; moreover, by passing to a subsequence we may assume that the (g_n) are distinct.

G acts on $\partial(G, \mathbb{P})$ as a convergence group implies that every sequence (g_n) of distinct group elements has a subsequence (g_i) such that if $K \subset \partial(G, \mathbb{P}) \setminus \{x\}$ then for any neighborhood $V \ni \beta$ there exists $g_{i_0} \in (g_i)$ such that $g_{i_0} \in V$.

Let $(U_i')_{i=1}^p$ and $(V_i')_{i=1}^p$ be the neighborhoods found in Lemma 1.4.4. As $(\alpha, \beta) \in \Theta_2\partial(G, \mathbb{P})$ there exists $g \in G$ and $i \in \{1, ..., p\}$ such that $g\alpha \in U_i'$ and $g\beta \in V_i'$. Set $U_i = g^{-1}U_i'$ and $V_i = g^{-1}V_i'$. Then for large enough n we have $g_n x \in U_i$ and $g_n K \subseteq V_i$. □

1.4.1 Collection Theorem

As we mentioned at the beginning of this section, now that we have proved Theorem 1.4.1 we may plug into the arguments of Bowditch [Bow98a] in the case when $\partial(G,\mathbb{P})$ has no global cut points. In particular we refer the reader to section 5 of [Bow98a] to obtain:

Theorem 1.4.5. *Let $(G,\mathbb{P})$ and $M = \partial(G,\mathbb{P})$. If the $\partial(G,\mathbb{P})$ is connected and locally connected, without global cut points, and not homeomorphic to S^1, then we have the following:*

1. *A point $m \in M^*(2)$ is either in a necklace or an inseparable loxodromic cut pair.*

2. *$M^*(3+)$ consists of equivalences classes of inseparable loxodromic cut pairs.*

3. *A necklace ν in $\partial(G,\mathbb{P})$ is cyclically separating and homeomorphic to a S^1 or a Cantor set. Moreover, if ν is a Cantor set the jumps are inseparable loxodromic cut pairs.*

As an immediate corollary we have:

Corollary 1.4.6. *Assume $\partial(G,\mathbb{P})$ is connected with no global cut points and not homeomorphic to S^1. If $\partial(G,\mathbb{P})$ has a non-parabolic local cut point, then $\partial(G,\mathbb{P})$ contains a loxodromic cut pair.*

We remark that Lemma 1.4.5 (iii) may also be obtained from the work of Groff (see Proposition 7.2 and the definition of relatively-QH in [Gro13]). Also note that cut pairs are not separated by global cut points, hence a necklace ν will be contained in contained in some block of the form $\partial(H,\mathbb{Q})$. This means we may now invoke the results of Section 1.3 to remove the hypothesis the $\partial(G,\mathbb{P})$ has global cut points.

Theorem 1.4.7. *Let $(G,\mathbb{P})$ be a relatively hyperbolic group with tame peripherals and assume $\partial(G,\mathbb{P})$ is connected. If $p \in \partial(G,\mathbb{P})$ is a local cut point, then one of the following holds:*

1. *p is parabolic point*

2. *p is contained in a loxodromic cut pair*

3. p is in a necklace

Proof. Let p be a local cut point. By Lemma 1.3.2 we have p must be either a parabolic point or a conical limit point contained in a block. By Theorem 1.3.1 the block is stabilized by some $(H, \mathbb{Q})$. Theorem 1.3.1 also implies that $\partial(H, \mathbb{Q})$ has no global cut points, so we may apply 1.4.5 to $\partial(H, \mathbb{Q})$. Thus if p is not a parabolic point, then $\partial(H, \mathbb{Q})$ contains a necklace or a loxodromic cut pair which contains p. Now, Corollary 1.3.8 implies that loxodromic cut pairs and necklaces in $\partial(H, \mathbb{Q})$ also separate $\partial(G, \mathbb{P})$, so we are done. □

1.5 Splitting Theorem

Having developed the appropriate tools, we now wish to prove Theorem 1.1.1. We start with a few lemmas.

Lemma 1.5.1. *Assume that $\partial(G, \mathbb{P})$ is not homeomorphic to a circle. If $\partial(H, \mathbb{Q})$ homeomorphic to a circle, then there exists a non-trivial peripheral splitting over a 2-ended subgroup.*

Proof. If $\partial(H, \mathbb{Q})$ is a circle, then a result of Tukia ([Tuk88] Theorem 6B) implies that H is virtually a surface group, and the peripheral subgroups are boundaries of that surface. Because the $\partial(G, \mathbb{P})$ is not a circle, there must be a global cut point in $\partial(H, \mathbb{Q})$ stabilized by a 2-ended subgroup. By Corollary 1.2.8 we are done. □

Lemma 1.5.2. *Let $\{a, b\}$ be an inseparable cut pair in $\partial(G, \mathbb{P})$ and Q the quotient space obtained by identifying ga to gb for every $g \in G$. Then Q contains a cut point for each pair in $\mathrm{Orb}_G\left(\{a, b\}\right)$.*

Proof. Let $M = \partial(G, \mathbb{P})$ and assume $\{a, b\}$ is an inseparable cut pair. We first need to know that any two pairs in $\mathrm{Orb}_G\left(\{a, b\}\right)$ do not mutually separate. Now, if $\{a, b\}$ is an $\approx$-class in $M^*(3+)$, then we are done by Lemma 1.2.10. If $a, b \in M^*(2)$ and there existed some pair $\{ga, gb\}$ which is separated by $\{a, b\}$, then Lemma 1.2.12 implies that the pairs $\{a, b\}$ and $\{ga, gb\}$ mutually separate, a contradiction.

Define $q\colon \partial(G,\mathbb{P}) \to Q$ to be the quotient map described in the statement of the lemma. Let C_1 and C_2 be components of $\partial(G,\mathbb{P}) \setminus \{c,d\}$ for some pair $\{c,d\}$ in $\mathrm{Orb}_G\big(\{a,b\}\big)$. Because we are identifying inseparable pairs and every pair in $\mathrm{Orb}_G\big(\{a,b\}\big) \setminus \big\{\{c,d\}\big\}$ is contained in C_1 or C_2, we have that $q(C_1)$ and $q(C_2)$ are disjoint connected components of $Q \setminus \big\{q(c) = q(d)\big\}$. □

Lemma 1.5.3. *If $\partial(G,\mathbb{P})$ contains a loxodromic cut pair, then $(G,\mathbb{P})$ splits over a two-ended group.*

Proof. Assume the hypothesis. Then there exists a loxodromic group element $g \in G$, and $\langle g\rangle$ is contained in a maximal 2-ended subgroup, H. By Theorem 1.1 of [Yan14] we may extend $\mathbb{P}$ to a new peripheral structure $\mathbb{P}'$, by adding H and all of its conjugates to $\mathbb{P}$; moreover, $(G,\mathbb{P}')$ is relatively hyperbolic. By Corollary 1.5.2 there is a cut point $\partial(G,\mathbb{P}')$ stabilized by $\langle g\rangle$, which by Corollary 1.2.8 implies that $(G,\mathbb{P}')$ has a non-trivial peripheral splitting. As every subgroup of $\langle g\rangle$ is 2-ended, we are done. □

Proof of the Splitting Theorem 1.1.1 If G splits over a non-parabolic 2-ended subgroup, then the proof that $\partial(G,\mathbb{P})$ contains a non-parabolic local cut point is the same as in the proof of Theorem 7.8 of [GM].

Now, assume that $x \in \partial(G,\mathbb{P})$ is a non-parabolic local cut point. By Theorem 1.4.7 we know that x is contained in either a loxodromic cut pair or a necklace. If x is in a loxodromic cut pair we are done by Lemma 1.5.3.

Assume x is in a necklace ν. Then ν is either a circle or it is not. If ν is homeomorphic to S^1 we are done by Lemma 1.5.1. If ν is not a circle, then ν contains a loxodromic cut pair by Lemma 1.4.5, and again we are done by Lemma 1.5.3.□

1.6 Ends of Fatigued Spaces Admitting Proper and Cocompact Group Actions

Let G be a group with finite generating set S, and let $\Upsilon(G,S)$ denote the Cayley graph of (G,S). The goal of this section is to prove Theorem 1.1.3, which is a result about the ends of $\Upsilon(G,S)$ and the ends of any sufficiently nice space on which G acts in a nice way. An analogous result is well known for CW-complexes [Geo08], and is one way of showing that Ends(G) is well defined. In Theorem 1.1.3 we provide a generalization to fatigued spaces, a classes of spaces which need not be CW-complexes. This result appears to be a folk theorem, but it is not readily found in the literature. We remark that the techniques used to prove Theorem 1.1.3 differ from those found in [Geo08].

One consequence of Theorem 1.1.3 for $\partial(G,\mathbb{P})$ is that if the peripherals are one-ended then a parabolic point can only be a local cut point if it is a global cut point (see Corollary 1.6.4). This particular fact will be required for the proof of the Classification Theorem 1.1.2.

Let G be a finitely generated discrete group acting properly and cocompactly on a fatigued space X. We want to use Proposition 1.2.13 to prove Theorem 1.1.3. To do so we must first construct a proper map $\Phi\colon \Upsilon(G,S) \to X$ from the Cayley graph of G to S.

Let S be a finite generating set for G and fix a base point x_0 in the fundamental domain of the action of G on X and for every vertex v_g in $\Upsilon(G,S)$ define $\Phi(v_g) = g.x_0$. For every $s \in S \cup S^{-1}$ fix a path, p_s, in X with $p_s(0) = x_0$ and $p_s(1) = s.x_0$. We will denote $P(S)$ the collection of paths found in this way, i.e. $P(S) = \{p_s | s \in S\}$. Now, for any edge $e_s \in \Upsilon(G,S)$ with end points v_g and v_{gs} define $\Phi(e_s)$ to be $g.(p_s)$. Notice that Φ well defined because $g.p_s$ is a path with end points $g.x_0$ and $gs.x_0$ for every g and s. Also, note that by the pasting lemma Φ is continuous.

Lemma 1.6.1. *The map $\Phi\colon \Upsilon(G,S) \to X$ is proper.*

Proof. Let $A \subseteq X$ be compact. As X is Hausdorff A is closed, therefore $\Phi^{-1}(A)$ is closed. We show that $\Phi^{-1}(A)$ consists of finitely many vertices and edges.

First, assume that $\Phi^{-1}(A)$ meets infinitely vertices. This implies that A contains $g_n x_0$ for infinitely many $g_n \in G$. As A is compact we have that $(g_n x_0) \to a$ for some $a \in A$. By local compactness there exists a compact set C containing a neighborhood U of a. U contains infinitely many members of the sequence $(g_n x_0)$ After passing to a subsequence if necessary, for large enough $i \in \mathbb{N}$ and any $j \geq i$ we have that $g_j g_i^{-1} C \cap C \neq \emptyset$, contradicting properness of the action.

Now assume that infinitely many edges meet $\Phi^{-1}(A)$. As there are finitely many orbits of edges there must be infinitely many edges with the same label, say s, meeting $\Phi^{-1}(A)$. Thus we may find an infinite sequence of group elements, $(g_i)_{i=1}^{\infty}$ such that $g_i p_s \cap A \neq \emptyset$ for every i. Set $C = p_s \cup A$, then C is compact and $C \cap g_i C \neq \emptyset$ for every i, again a contradiction. □

Define $\Phi^*\colon \operatorname{Ends}(\Upsilon(G,S)) \to \operatorname{Ends}(X)$ be the ends map induced by Φ.

Lemma 1.6.2. *Then Φ^* is a surjection.*

Proof. Let $K \subset X$ be a compact connected set whose G-translates cover X, let $\{C_i\}_{i=1}^{\infty}$ be an efficient exhaustion of X, and let $E = (E_1, E_2, E_3, ...) \in \operatorname{Ends}(X)$.

Let $x_i \in E_i$ for some i. The translates of K cover X, so there exists some $g_i \in G$ such that $x_i \in g_i K$. As $g_i K$ is compact there exists some $j \in \mathbb{N}$ such that $gK \subseteq C_j$. Let $x_j \in E_j \subset X \backslash C_j$ as before there exists some $g_j \in G$ such that $x_j \in g_j K$ and some C_k containing $g_j K$. So we may pass to a subsequence $(E_{i_1}, E_{i_2}, E_{i_3}, ...)$ of E corresponding to a sequence of distinct group elements $(g_{i_1}, g_{i_2}, g_{i_3}, ...)$ of G found in the manner just described.

The sequence $(g_{i_1}, g_{i_2}, g_{i_3}, ...)$ corresponds to an infinite sequence, $(v_{g_{i_j}})_{j=1}^{\infty}$, of distinct vertices in $\Upsilon(G,S)$. Because the map Φ is proper, compactness of $\Upsilon(G,S) \cup \operatorname{Ends}(\Upsilon(G,S))$ we have that some subsequence $(v_{g_{i_{j_k}}})_{k=1}^{\infty}$of $(v_{g_{i_j}})_{j=1}^{\infty}$ must converge to an end of $\Upsilon(G,S)$. Thus we may find a proper ray, r, in $\Upsilon(G,S)$ containing the vertices $(v_{g_{i_{j_k}}})_{k=1}^{\infty}$. The ray r

determines an end of $\Upsilon(G,S)$, which by construction Φ maps to the end E under Φ^*. Thus Φ^* is surjective. $\square$

Proof of Theorem 1.1.3. By lemma 1.6.2 we need only show that Φ^* is injective. To do this we will make use of the ladder definition of ends found in Section 1.2.5.

By hypothesis there exists a compact set K whose G translates cover X. We may assume that K is connected. Define S to be $\{ s \in G \mid K \cap sK \neq \emptyset \}$. It is a standard result that S generates G, because Ends(G) is independent of choice of generating set it suffices to consider S.

Let α and β be proper rays in $\Upsilon(G,S)$ and (a_i) and (b_i) the corresponding sequences of vertices. Note that, if necessary, α and β may be homotoped combinatorial proper rays, so we may assume that no vertex in (a_i) or (b_i) occurs infinitely many times. Assume that Φ maps α and β to the same end in X. Then we may find a proper map of the infinite ladder into X such that $\Phi(\alpha)$ and $\Phi(\beta)$ form the sides; moreover, by concatenating paths if necessary we may assume that the rungs, r_i, of the ladder have end points $\Phi(a_i)$ and $\Phi(b_i)$. Call this ladder L. Note that the rungs r_i of L may not pull back to paths in $\Upsilon(G,S)$ under Φ^{-1}. We show that we can find an alternate sequence of rungs ρ_i connecting $\Phi(a_i)$ to $\Phi(b_i)$ and such that each ρ_i pulls back to an edge path in $\Upsilon(G,S)$.

For any rung r_i we may find a finite number of translates of K that cover r_i. Let $\{g_1, g_2, ... g_n\}$ be such that $im(r_i) \subset \bigcup_{j=1}^{n} g_j K$. Notice that by connectedness of the rung r_i we may assume that $\{g_1, g_2, ... g_n\}$ is enumerated in such a way that $g_j K \cap g_{j+1} K \neq \emptyset$. Consequently, the $g_j K$ form a chain of connected compact neighborhoods such that the points $g_i x_o$ in the translates of K can be connected by paths which are translates of paths in $P(S)$ (see the construction of Φ); in other words, because of the specific choice of generating set they are the images of edges in $\Upsilon(G,S)$. By concatenating paths in $\mathrm{Orb}_G(P(S))$ we may find a path, ρ_i, which pulls back to an edge path in $\Upsilon(G,S)$ connecting (a_i) and (b_i).

Lastly, we need to check that some sub-ladder of the ladder L pulls back to a ladder in

$\Upsilon(G,S)$ under Φ. Let $C \subset \Upsilon(G,S)$ be a compact. We find a ρ_i such that $\Phi^{-1}(\rho_i)$ is in $\Upsilon(G,S) \setminus C$.

Set $C' = \Phi(C)$ and $K' = \big(\bigcup_{s\in S} sK) \cup P(S)\big)$. Assume that there does not exist a subsequence of rungs $\{\rho_i\}$ entirely outside of C'. Then we may find a compact set $N = \bigcup_{g\in I} gK'$ where $I = \{g \in G | K' \cap gK' \neq \emptyset\}$ such that every rung, r_i of L meets N. As the ladder L was proper this is a contradiction. Thus there must exist a ρ_i outside of $\Phi(C)$, which implies that $\Phi^{-1}(\rho_i) \subset \Upsilon(G,S) \setminus C$. Therefore as C was chosen to be arbitrary we have that α and β represent the same end of $\Upsilon(G,S)$. □

As an immediate corollary we obtain:

Corollary 1.6.3. *Let G be a one-ended finitely generated group acting properly and cocompactly on a fatigued space X. Then X is* 1*-ended.*

In particular, we have:

Corollary 1.6.4. *Let $(G,\mathbb{P})$ be relatively hyperbolic with tame peripherals and every $P \in \mathbb{P}$* 1*-ended. If p is parabolic point in $\partial(G,\mathbb{P})$ which is not a global cut point, then p cannot be a local cut point.*

Proof. Assume the hypotheses and let P be the maximal parabolic subgroup which stabilizes p. Bowditch [Bow12] has shown that P acts properly and cocompactly on $\partial(G,\mathbb{P}) \setminus \{p\}$. Because p is not a global cut point, we know that $\partial(G,\mathbb{P}) \setminus \{p\}$ is connected. We are assuming that $(G,\mathbb{P})$ has tame peripherals, so $\partial(G,\mathbb{P})$ is locally connected. Thus, $\partial(G,\mathbb{P}) \setminus \{p\}$ is an open connected subset of a Peano continuum; consequently, $\partial(G,\mathbb{P}) \setminus \{p\}$ is fatigued. □

1.7 Classification Theorem

In this section we prove Theorem 1.1.2. This theorem is a generalization of a theorem due to Kapovich and Kleiner [KK00] concerning the boundaries of hyperbolic groups. A key fact used by Kapovich and Kleiner is the topological characterization of the Menger curve due to R.D. Anderson (see [And58a, And58b]). Anderson's theorem states that a compact metric space M is a Menger curve provided: M is 1-dimensional, M is connected, M is locally connected, M has no local cut points, and no non-empty open subset of M is planar. If the last condition is replaced with, "M is planar," then we have the topological characterization of the Sierpinski carpet, due to Whyburn [Why58]. The Proof provided below was inspired by that of Kapovich and Kleiner [KK00].

Proof of Theorem 1.1.2: Assume the hypotheses and assume that $\partial(G, \mathbb{P})$ is not homeomorphic to a circle. Then $\partial(G, \mathbb{P})$ is a compact and 1-dimensional metric space. Because we are assuming that G is one-ended, $\partial(G, \mathbb{P})$ is connected. Since we are assuming the $(G, \mathbb{P})$ has tame peripherals, connectedness of $\partial(G, \mathbb{P})$ implies that is must also be locally connected (see Theorem 1.2.3).

There are two types of local cut points, those a that separate $\partial(G, \mathbb{P})$ globally and those that do not. By Theorem 1.2.6 the no peripheral splitting hypothesis implies that $\partial(G, \mathbb{P})$ is without global cut points. Additionally, the peripheral subgroups are assumed to be 1-ended, so by Theorem 1.1.3 we have that there are no parabolic local cut points. Thus any local cut point must be a conical limit point. If there was a conical limit local cut point, then The Splitting Theorem 1.1.1 would imply that G splits over a 2-ended subgroup, a contradiction.

Now, $\partial(G, \mathbb{P})$ is planar or it is not. If it is planar then it is a Sierpinski carpet. Assume $\partial(G, \mathbb{P})$ is not planar, then by the Claytor embedding theorem it must contain a topological embedding of a non-planar graph, K. We need to find a homeomorphic copy of K inside any open neighborhood V in $\partial(G, \mathbb{P})$.

As conical limit points are dense, let x be a conical limit point in $\partial(G, \mathbb{P}) \setminus \{K\}$. By definition of conical limit point there exists $a, b \in \partial(G, \mathbb{P})$ and a sequence of group elements $(G_i) \subset G$ such that $G_i x \to a$ and $G_i z \to b \neq a$ for every $z \in \partial(G, \mathbb{P}) \setminus \{x\}$. Now, G acts on $\partial(G, \mathbb{P})$ as a convergence group. Thus we have that $G_i z \to b$ converges locally uniformly on compact sets and we may find a homeomorphic copy of K inside any neighborhood U of b.

Let V be any neighborhood in $\partial(G, \mathbb{P})$. The action of G on $\partial(G, \mathbb{P})$ is minimal (see [Bow12]), so we have that there exists some group element g such that $gb \in V$. Let W be a neighborhood of gb inside V and set U from the previous paragraph equal to $g^{-1}(W)$. Then we may find a homeomorphic copy of K inside of V. $\square$

Chapter 2

Boundary classification and 2-ended splittings of groups with isolated flats

2.1 Introduction

When a group Γ acts discretely on a geometric space X, we can often compactify X by attaching a "boundary at infinity" ∂X to X. In the presence of non-positive curvature, Γ has an induced action by homeomorphisms on the boundary. There are strong connections between the topological properties of ∂X and the algebraic properties of Γ. A natural question posed by Kapovich and Kleiner [KK00] is: which topological spaces occur as boundaries of groups?

In [KK00] Kapovich and Kleiner prove a classification theorem for boundaries of one-ended hyperbolic groups. They show that if the boundary is 1-dimensional and the group does not split over a virtually cyclic subgroup then the boundary of the group is either a circle, a Sierpinski carpet, or a Menger curve.

Problem 2.1.1 (K. Ruane). *Can the Kapovich-Kleiner result be extended to some natural family of* CAT(0) *groups?*

Kapovich and Kleiner's result relies heavily on JSJ results due to Bowditch [Bow98a].

Bowditch's results relate the existence of local cut points in the boundary to the existence of cut pairs, which is further related to two-ended splittings of the group. For CAT(0) groups Papasoglu and Swenson [PS09] extend the connection between cut pairs and two-ended splittings, but leave the issue of local cut points completely unresolved.

In this article we resolve this issue for groups acting *geometrically* (i.e. properly, cocompactly, and by isometries) on a CAT(0) space with isolated flats (see [HK05]) and obtain the following result:

Theorem 2.1.2 (Main Theorem)**.** *Let Γ be a group acting geometrically on a* CAT(0) *space X with isolated flats. Assume ∂X is one-dimensional. If Γ does not split over a virtually cyclic subgroup then one of the following holds:*

1. *∂X is a circle*
2. *∂X is a Sierpinski carpet*
3. *∂X is a Menger curve.*

A key tool used by Kapovich and Kleiner is the topological characterization of the Menger curve due to R.D. Anderson [And58a, And58b]. Anderson's theorem states that a compact metric space M is a Menger curve provided: M is 1-dimensional, M is connected, M is locally connected, M has no local cut points, and no non-empty open subset of M is planar. We note that if the last condition is replaced with "M is planar," then we have the topological characterization of the Sierpinski carpet (see Whyburn [Why58]).

Prior to Kapovich and Kleiner's theorem [KK00], results of Bestvina and Mess [BM91], Swarup [Swa96], and Bowditch [Bow99a] had shown that the boundary of a one-ended hyperbolic group Γ is connected and locally connected. The planarity issue is easily dealt with using the dynamics of the action of the group on its boundary, leaving only the local cut point issue. However, Bowditch has shown [Bow98a] that if ∂G is not homeomorphic to a circle, then ∂G has a local cut point if and only if Γ splits over a two-ended subgroup.

We follow a similar outline to prove Theorem 2.1.2. The question of which groups with isolated flats have locally connected boundary has been completely determined by Hruska and Ruane [HR]. So we will begin by assuming, for now, that ∂X is locally connected. In the isolated flats setting the planarity is again easily dealt with using an argument similar to that of Kapovich and Kleiner, leaving only the local cut point issue. So in order to complete the proof of Theorem 2.1.2, the remaining difficulty is understanding the connection between local cut points in ∂X and splittings of Γ.

We prove the following splitting theorem which is independent of the dimension of ∂X and thus more general than is required for the proof of Theorem 2.1.2:

Theorem 2.1.3. *Let Γ be a group acting geometrically on a* CAT(0) *space X with isolated flats. Suppose ∂X is locally connected and not homeomorphic to S^1. If Γ does not split over a virtually cyclic subgroup, then ∂X has no local cut points.*

Techniques developed by the author for the proof of Theorem 2.1.3 have already been used by Hruska and Ruane [HR] in the proof of their local connectedness theorem. In the special case when the boundary is one-dimensional Hruska and Ruane's [HR] theorem shows reduces to the statement that ∂X is locally connected if Γ does not split over a two-ended subgroup (see Theorem 2.2.3). So, we obtain a simplified version of Theorem 2.1.3, which is used in the proof of Theorem 2.1.2.

Corollary 2.1.4. *Let Γ be a one-ended group acting geometrically on a* CAT(0) *space X with isolated flats and assume that ∂X is 1-dimensional and not homeomorphic to S^1. If Γ does not split over a virtually cyclic subgroup, then ∂X has no local cut points.*

Theorem 2.1.3 fills a gap in the JSJ theory literature on CAT(0) groups and most of this chapter is spent on the proof. We mention that the significance of this gap in the literature has also been observed of observed by Świątkowski [Ś16].

In [Bow98a] Bowditch studied local cut points in boundaries of hyperbolic groups and their relation to so called JSJ-splittings. As mentioned above, Bowditch showed that for a one-ended hyperbolic group Γ which is not cocompact Fuchsian the existence of a local cut point is equivalent to a splitting of Γ over a 2-ended subgroup. Generalizations of Bowditch's results have been studied by Papasoglu-Swenson [PS06] [PS09], and Groff [Gro13] for CAT(0) and relatively hyperbolic groups, respectively.

Groups with isolated flats have a natural relatively hyperbolic structure [HK05], and there is a strong relationship between ∂X and the Bowditch boundary $\partial(\Gamma, \mathbb{P})$. Analogous to the limit set of a Kleinian group, the Bowditch boundary $\partial(\Gamma, \mathbb{P})$ was introduced by Bowditch [Bow12] and used to study splittings of hyperbolic groups. In general $\partial(\Gamma, \mathbb{P})$ may have infintely many global cut points. In fact, the global cut point structure of $\partial(\Gamma, \mathbb{P})$ is key to a general theory of splittings [Bow01].

Using cut pairs instead of local cut points Groff [Gro13] obtains a partial extension of Bowditch's JSJ tree construction [Bow98a] for relatively hyperbolic groups, and Guralnik [Gur05] observed that in the special case that the relative boundary $\partial(\Gamma, \mathbb{P})$ has no global cut points, then many of Bowditch's results [Bow98a] about the valence of local cut points in the boundary of a hyperbolic group translate directly to the relatively hyperbolic setting. Their results were subsequently used by Groves and Manning [GM] to show that if $\partial(\Gamma, \mathbb{P})$ has no global cut points and all the peripheral subgroups are one-ended, then the existence of a local cut point in $\partial(\Gamma, \mathbb{P})$ is equivalent to the existence of a splitting of Γ relative to $\mathbb{P}$ over a non-parabolic 2-ended subgroup. The relative boundary (or Bowditch boundary) $\partial(\Gamma, \mathbb{P})$ is different from the CAT(0) boundary mentioned above.

In Chapter 1 the author investigates local cut points in $\partial(\Gamma, \mathbb{P})$ and provides a splitting theorem for relatively hyperbolic groups without making any assumptions about global cut points. Namely, he shows that under some very modest conditions on the peripheral subgroups, the existence of a non-parabolic local cut point in $\partial(\Gamma, \mathbb{P})$ implies that Γ splits over a 2-ended subgroup (see Theorem 2.2.5). Because of the close relationship between ∂X and $\partial(\Gamma, \mathbb{P})$, this splitting theorem will be used in Section 2.6 to show that the existence of a local cut point of ∂X which is not in the boundary of flat implies that Γ splits over a 2-ended subgroup.

We conclude the chapter by discussing applications of Theorem 2.1.2. In particular, in Section 2.10 we discuss groups with Menger curve boundary and in Section 2.9 we generalize the work of Świątkowski [Ś16] to obtain the following result:

Theorem 2.1.5. *Let (W, S) be a Coxeter system such that W has isolated flats. Assume that the nerve L of the system is planar, distinct from simplex, and distinct from a triangulation of S^1. If the labeled nerve $L^{\bullet}$ of (W, S) is distinct from a labeled wheel and inseparable, then $\partial\Sigma$ is homeomorphic to the Sierpinski carpet.*

Definitions of the terms used in Theorem 2.1.5 can be found in Section 2.9.

In [DO01] Davis and Okun show that if W is a Coxeter group whose nerve L is planar, then W acts properly on a 3-manifold. Consequently, Theorem 2.1.5 is in line with the following extension of a conjecture due to Kapovich and Kleiner [KK00]:

Conjecture 2.1.6. *Let Γ be a* CAT(0) *group with isolated flats and Sierpinski carpet boundary. Then Γ acts properly on a contractible 3-manifold.*

2.1.1 Methods of Proof

The strong connection between $\partial(\Gamma, \mathbb{P})$ and the CAT(0) boundary ∂X is given by Hung Cong Tran [Tra13]. For spaces with isolated flats Tran's result implies that $\partial(\Gamma, \mathbb{P})$ is the quotient space obtained from ∂X by identifying points which are in the boundary of the same flat. Using basic decomposition theory (see Section 2.6.1), we are able to show that if there exists a local cut point $\xi \in \partial X$ that is not in the boundary of a flat, then it must push forward under this quotient map to a local cut point of $\partial(\Gamma, \mathbb{P})$. This allows us to apply Theorem 2.2.5 mentioned above, and prove that the existence of a local cut point that is not in the boundary of a flat implies the existence of a 2-ended splitting (see Proposition 2.6.1).

Assuming that our group Γ does not split over a two ended group, we are left with the remaining question: Can a point which lies in the boundary of a flat be a local cut point?

Much of this chapter is spent answering that question in the negative when ∂X is locally connected. Let X be a CAT(0) space which has isolated flats with respect to $\mathcal{F}$ and let $F \in \mathcal{F}$ be an n-dimensional flat in X, then it can be shown that $\mathrm{Stab}_\Gamma(F)$ has a finite index subgroup H isomorphic to $\mathbb{Z}^n$. In Section 2.3 we show that H acts properly and cocompactly on $\partial X \setminus \partial F$. This is done by means of a relation on $F \times (\partial X \setminus \partial F)$, which uses orthogonal rays to associate points in F with points in the boundary. In Sections 2.4 and 2.5 we assume that ∂X is locally connected and show that we may put an H-equivariant metric on $\partial X \setminus \partial F$. Then in Section 2.7 we use the properties of this action to deduce that a point in the boundary of a flat cannot be a local cut point. This combined with Proposition 2.6.1 allow us to obtain Theorem 2.1.3.

Once we have completed the proof of Theorem 2.1.3 we are ready to prove Theorem 2.1.2. This is accomplished in Section 2.8 with an argument inspired by Kapovich and Kleiner [KK00]. Using the dynamics of the action of Γ on the boundary we show that if ∂X contains a non-planar graph K, then every open subset of must contain a homeomorphic copy of K. This will be enough to complete the proof.

2.2 Preliminaries

2.2.1 The Bordification of a CAT(0) Space

Throughout this chapter we will assume that X is a proper CAT(0) metric space, unless otherwise stated. We refer the reader to [BH99] for definitions and basic results about CAT(0) spaces.

The *boundary* of X, denoted ∂X, is the set of equivalence classes of geodesic rays. Where two rays $c_1, c_2 \colon [0, \infty) \to X$ are equivalent if there exists a constant $D \geq 0$ such that $d\big(c_1(t), c_2(t)\big) \leq D$ for all $t \in [0, \infty)$. The *bordification* of X is the set $\overline{X} = X \cup \partial X$.

The bordification $\overline{X}$ comes equipped with a natural topology called the *cone topology*, where one considers rays based at some fixed point. A basis for the cone topology consists of open balls in X together with "neighborhoods of points at infinity." Given a geodesic ray c and positive numbers $t > 0$, $\epsilon > 0$, define

$$V(c,t,\epsilon) = \left\{ \, x \in \overline{X} \;\middle|\; t < d\big(x, c(0)\big) \text{ and } d\big(\pi_t(x), c(t)\big) < \epsilon \, \right\}$$

Where π_t is the orthogonal projection onto the closed ball $\overline{B}\big(c(0), t\big)$. For fixed c and $\epsilon > 0$ the sets $V(c,t,\epsilon)$ form a neighborhood base at infinity about c. Intuitively, this means that two points in ∂X will be close if they are represented by rays which are ϵ close at for large values of t. We will denote by $V_\partial(c,t,\epsilon)$ the set $V(c,t,\epsilon) \cap \partial X$.

2.2.2 The Bowditch Boundary and Splittings

Let Γ be a group and $\mathbb{P}$ a collection of infinite subgroups which is closed under conjugation, called *peripheral subgroups.*

We say that Γ is *hyperbolic relative to* $\mathbb{P}$ if Γ admits a proper isometric action on a proper δ-hyperbolic space Y such that:

1. $\mathbb{P}$ is the set of all maximal parabolic subgroups
2. There exists a Γ-invariant system of disjoint open horoballs based at the parabolic points of Γ, such that if $\mathcal{B}$ is the union of these horoballs, then Γ acts cocompactly on $Y \setminus \mathcal{B}$.

The *Bowditch boundary* $\partial(\Gamma, \mathbb{P})$ of $(\Gamma, \mathbb{P})$ is defined to be the boundary of the space Y. If $(\Gamma, \mathbb{P})$ is relatively hyperbolic and acts geometrically on a CAT(0) space X there is a close relationship between the $\partial(\Gamma, \mathbb{P})$ and the visual boundary ∂X.

Theorem 2.2.1 (Tran)**.** *$\partial(\Gamma, \mathbb{P})$ is Γ-equivariantly homeomorphic to the quotient of ∂X obtained by identifying points which are in the boundary of the same flat.*

A *splitting* of a group Γ over a given class of subgroups is a finite graph of groups $\mathcal{G}$ of Γ, where each edge group belongs to the given class. A splitting is called *trivial* if there exists a vertex group equal to Γ. Assume that Γ is hyperbolic relative to a collection $\mathbb{P}$. A *peripheral splitting* of $(\Gamma, \mathbb{P})$ is a finite bipartite graph of groups representation of Γ, where $\mathbb{P}$ is the set of conjugacy classes of vertex groups of one color of the partition called peripheral vertices. Nonperipheral vertex groups will be referred to as *components*. This terminology stems from the correspondence between the cut point tree of $\partial(\Gamma, \mathbb{P})$ and the peripheral splitting of $(\Gamma, \mathbb{P})$, where elements of $\mathbb{P}$ correspond to cut point vertices and the components correspond to components of the boundary (i.e. equivalence classes of points not separated by cut points).

A peripheral splitting $\mathcal{G}$ is a refinement of another peripheral splitting $\mathcal{G}'$ if $\mathcal{G}'$ can be obtained from $\mathcal{G}$ via a finite sequence of foldings that preserve the vertex coloring. In [Bow01] Bowditch proved the following accessibility result:

Theorem 2.2.2. *Suppose that $(\Gamma, \mathbb{P})$ is relatively hyperbolic and that $\partial(\Gamma, \mathbb{P})$ is connected. Then $(\Gamma, \mathbb{P})$ admits a (possibly trivial) peripheral splitting which is maximal in the sense that it is not a refinement of any other peripheral splitting.*

2.2.3 Isolated Flats

Here we introduce basic definitions and pertinent results regarding spaces with isolated flats. We refer the reader to [HK05] for a more detailed account. Let X be a CAT(0) space with Γ acting geometrically on X. A *k-flat* in X is an isometrically embedded copy of Euclidean space, $\mathbb{E}^k$. A 1-flat will also be referred to as a *line* and a 2-flat may be referred to as a *flat plane*.

The space X is said to have *isolated flats* if there is a Γ-invariant collection of flats, $\mathcal{F}$, of dimension 2 or greater and such that the following hold:

1. (capturing condition) There exists a constant $D < \infty$ such that each flat in X lies in the D-tubular neighborhood of some $F \in \mathcal{F}$

2. (isolating condition) For every $\rho < \infty$ there exists $\kappa(\rho) < \infty$ such that for any two distinct $F, F' \in (F)$ we have $\operatorname{diam}\left(\mathcal{N}_\rho(F) \cap \mathcal{N}_\rho(F')\right) < \kappa(\rho)$

Hruska and Kleiner have shown in [HK05] that if Γ is a group acting geometrically on a CAT(0) space with isolated flats, then Γ is hyperbolic relative relative to a collection of virtually abelian subgroups of rank at least 2 (Theorem 1.2.1 of [HK05]). Hruska and Kleiner have also shown that for isolated flats ∂X is an invariant of the group Γ up to quasi-isometry (Theorem 1.2.2 of [HK05]). The following recent result concerning groups with isolated flats is due to Hruska and Ruane [HR], and is particularly relevant to this project:

Theorem 2.2.3 (Hruska-Ruane)**.** *Let Γ be a one-ended group acting geometrically on a* CAT(0) *space with isolated flats. Let $\mathcal{G}$ be the maximal peripheral splitting of Γ. Then each vertex group of $\mathcal{G}$ acts geometrically on a* CAT(0) *space with locally connected boundary.*

Furthermore ∂X is locally connected if and only if the following condition holds: Each edge group of $\mathcal{G}$ has finite index in the adjacent peripheral vertex group.

In the case where ∂X is 1-dimensional we have the following corollary:

Corollary 2.2.4. *Assume Γ is acting geometrically on a* CAT(0) *space X with isolated flats, and assume ∂X is* 1*-dimensional. Then ∂X is locally connected if and only if Γ does not have a peripheral splitting over a* 2*-ended subgroup.*

Remark. As "no splitting over a two-ended subgroup" is a hypothesis in both Theorem 2.1.2 and Theorem 2.1.3, we may assume that ∂X is locally connected when required. Also, notice that for the proof of Theorem 2.1.2 we are concerned with 1-dimensional boundaries, so in that case the dimension of the flats we are interested in is 2. However, for many of the result we will not need to make any assumption about the dimension of flats.

2.2.4 Local Cut Points

Recall that a *continuum* is a non-empty, connected, compact, metric space, and let M be such a space. A *cut point* of M is a point $x \in M$ such that $M \setminus \{x\}$ is disconnected. A point $x \in M$ is a *local cut point* if x is a cut point or $M \setminus \{x\}$ has more than one end. A detailed discussion of ends of spaces can be found in Section 3 of [Gui16]. In this chapter we are often interested in whether a given point is a local cut point or not. Thus we remark that saying a point $x \in M$ is a local cut point is equivalent to saying that there exists a neighborhood U of x such that for every neighborhood V of x with $V \subset U$, there exist points $z, y \in V \setminus \{x\}$ which cannot be connected inside $U \setminus \{x\}$, i.e. z and y are not contained in the same connected subset of $U \setminus \{x\}$. In Section 2.7 we will be interested in showing that a point cannot be a local cut point, so it is worth noting the negation of the above. In other words, to check that x is not a local cut point it suffices to show that given a neighborhood U of x there exists a neighborhood $V \ni x$ with $V \subset U$ and $V \setminus \{x\}$ connected.

In his study of JSJ splittings of hyperbolic groups Bowditch investigated the local cut point structure of the boundary. In that setting Bowditch shows that the existence of a local cut point implies that group splits over a 2-ended subgroup. In Chapter 1 the author studies local cut points in the relative boundary (or Bowditch Boundary) $\partial(\Gamma, \mathbb{P})$ and has generalized Bowditch's result to show:

Theorem 2.2.5. *Let $(\Gamma, \mathbb{P})$ be a relatively hyperbolic group and suppose each $P \in \mathbb{P}$ is finitely presented, one- or two-ended, and contains no infinite torsion subgroup. Assume that $\partial(\Gamma, \mathbb{P})$ is connected and not homeomorphic to a circle. If $\partial(\Gamma, \mathbb{P})$ contains a non-parabolic local cut point, then Γ splits over a 2-ended subgroup.*

The majority of this chapter is concerned with determining the existence or non-existence of local cut points ∂X. Theorem 2.2.5 will be used in Section 2.6 to show that the existence of a local cut point which is not in the boundary of a flat implies the existence of a splitting over 2-ended subgroup.

2.2.5 Limit Sets

We will need a few basic results about limit sets sporadically through this chapter, consequently, we conclude the preliminary section with a terse discussion of limit sets. In this section X will be a CAT(0) space and Γ some group of isometries of X.

Recall, that a for a sequence $(\gamma_n) \subset G$ we write $\gamma_n \to \xi \in \partial X$ if $\gamma_n x \to \xi$ for some $x \in X$. It is clear that if $\gamma_n x \to \xi$ for some x, then $\gamma_n x' \to \xi$ for any $x' \in X$. The *limit set*, $\Lambda(\Gamma)$, of Γ is the subset of ∂X consisting of all such limits. The set $\Lambda(\Gamma)$ is a closed and Γ-invariant. Given that the action of Γ is geometric we have the following:

Lemma 2.2.6. $\Lambda(\Gamma) = \partial X$

We leave the proof of this result as an exercise.

A subset M of $\Lambda(\Gamma)$ is said to be *minimal* if M is closed, non-empty, Γ-invariant, and does not properly contain a closed Γ-invariant subset. A useful fact about minimal sets is that $M \subset \Lambda(\Gamma)$ is minimal if and only if $\mathrm{Orb}_\Gamma(m)$ is dense in M for every $m \in M$. The action of Γ on $\Lambda(\Gamma)$ is called *minimal* if $\Lambda(\Gamma)$ is minimal.

2.3 A Proper and Cocompact Action on $\partial X \backslash \partial F$

Let X be a CAT(0) space with isolated flats and let $F \in \mathcal{F}$ be a flat in X. Set $Y = \partial X \setminus \partial F$. In this section we a follow a strategy similar to that of Bowditch in Lemma 6.3 of [Bow12] to show that $\mathrm{Stab}_\Gamma(F)$ acts properly and cocompactly on Y. The key observation made by Bowditch is as follows:

Lemma 2.3.1. *Let G be a group acting on topological spaces A and B. Define the action of G on $A \times B$ to be the diagonal action and let $\mathcal{R} \subset A \times B$. If $\mathcal{R}$ is G-invariant and the projections pr_A and pr_B from $\mathcal{R}$ onto the factors are both proper and surjective, then the following are equivalent:*

1. *G acts properly and cocompactly on A*
2. *G acts properly and cocompactly on $\mathcal{R}$*
3. *G acts properly and cocompactly on B*

Set $G = \text{Stab}_\Gamma(F)$. Define $\perp(F)$ be the set of all geodesic rays orthogonal to F. Recall that a geodsic ray $r\colon [0,\infty) \to X$ is orthogonal to a convex set $C \subset X$ if for every $t > 0$ and for any $y \in C$ the Alexandrov angle, $\angle_{r(0)}\big(r(t), y\big)$, is greater than or equal to $\pi/2$. It is well known that G acts cocompactly on F (see [HK05] Lemma 3.1.2). Let $A > 0$ be the diameter of the fundamental domain of this action. We define $\mathcal{R} = \big\{(x,q)\big|\text{there exists } q \in \perp(F) \text{ with } d\big(x, q(0)\big) \leq A\big\}$. Unless otherwise stated, we will assume that our base point is in the flat F.

We want that $\mathcal{R}$ satisfies the hypotheses of Lemma 2.3.1, with the roles of A and B played by F and $Y = \partial X \setminus F$. We begin with the following observation:

Lemma 2.3.2. *$\mathcal{R}$ is G-invariant.*

Proof. $G \leq \Gamma$ acts on X by isometries, so if $(x,q) \in \mathcal{R}$ and $h \in G$ then $d\big(h.x, h.q(0)\big) < A$. If π_F is the orthogonal projection onto F, then by $d\big(h.q(t), h.q(0)\big) = d\big(q(t), q(0)\big)$ and the uniqueness of the projection point π_F (see [BH99] Proposition II.2.4) we must have that $\pi_F\big(h.q(t)\big) = h.\pi_F\big(q(t)\big)$ for every $t \in [0,\infty)$. □

To continue our study of $\mathcal{R}$, we require the following useful lemma, which allows one to construct a new orthogonal ray from a sequence of orthogonal rays with convergent base points. The proof relies on a standard diagonal argument and will not be presented here; however, it is not dissimilar to the proof presented in Lemma 5.31 of [BH99].

Lemma 2.3.3. *If (Y,ρ) is a separable metric space, (X,d) is proper metric space, $y_0 \in Y$, and K a compact subset of X, then any sequence of isometric embeddings, $c_n\colon Y \to X$, with $c_n(y_0) \in K$ has a subsequence which converges point-wise to an isometric embedding $c\colon Y \to X$.*

Next, we check that $\mathcal{R}$ projects surjectively onto the factors Y and F.

Lemma 2.3.4. *Let $\xi \in Y$. If r is a ray representing ξ, then there exists a geodesic ray $q \in \perp(F)$ asymptotic r.*

Proof. Let $x_n = r(n)$ and $y_n = pr_F(x_n)$ for all $n \in \mathbb{N}$. Our first claim is that the sequence (y_n) is bounded as $n \to \infty$. Assume not, then $y_n \to \eta \in \partial F$. By Corollary 7 of [HK09] there exists some constant $M > 0$ such that $d\big(x_0, [x_n, y_n]\big) < M$ for all n. Then by the triangle inequality and the definition of y_n as the orthogonal projection we have that $d(x_0, y_n) \leq 2M$. Then (y_n) converges in $\overline{B}(x_0, 2M)$ and we may apply Lemma 2.3.3 to construct the orthogonal ray q. □

Corollary 2.3.5. *The projections $pr_Y(\mathcal{R})$ and $pr_F(\mathcal{R})$ are surjective.*

Proof. The surjectivity of pr_Y is immediate. For pr_F we need only that each point in the flat is within a bounded distance of an element of $\perp(F)$. Let $A > 0$ be the constant used in the definition of $\mathcal{R}$ above. We know from the previous lemma that there exists some $q \in \perp(F)$. The result follows as $\mathrm{Orb}_G(q) \subset \perp(F)$ and $\mathrm{Orb}_G(q) \cap F$ is A-dense. □

In order to check the properness of the projections, we need to know that as a sequence (r_n) of orthogonal rays moves the corresponding sequence of asymptotic rays based at x_0 travel within a bounded distance of the points $\big(r_n(0)\big)$. We provide a quasiconvexity result below, which is a corollary of the following theorem presented by Hruska and Ruane in 4.14 of Theorem [HR].

Theorem 2.3.6. *Let X be a* CAT(0) *space with isolated flats with respect to $\mathcal{F}$. There exists a constant $L > 0$ such that the following hold:*

1. *Given two flats $F_1, F_2 \in \mathcal{F}$ with c the shortest length geodesic from F_1 to F_2, we have that $F_1 \cup F_2 \cup c$ is L-quasiconvex in X.*

2. *Given a point p and a flat $F \in \mathcal{F}$, with c the shortest path from F to p, then $F \cup c$ is L-quasiconvex in X.*

Lemma 2.3.7. *Let $F \in \mathcal{F}$ and $q \in \perp(F)$ then there exists a constant L such that $q \cup F$ is L-quasiconvex in X.*

Proof. The proof follows from the Theorem 2.3.6 by passing to the limit as $n \to \infty$ of the geodesic segments $\big[q(n), q(0)\big]$. □

To prove properness we will also need to know that convergence of orthogonal rays in the space X corresponds to convergence of points in Y.

Lemma 2.3.8. *If (r_n) is a sequence of rays in $\perp(F)$ which converge to an element of $\perp(F)$, then the corresponding points at infinity converge in topology on ∂X.*

Proof. Let x_0 be the base point for the cone topology on ∂X and let $r \in \perp(F)$ be limit ray. For each ray r_n there exists a an asymptotic ray c_n based at x_0 (see [BH99] Chapter II.8 Proposition 8.2). Define $D = d\big(x_0, r(0)\big)$, then $d\big(r_n(t), c_n(t)\big) \leq D$ for every $t \in \mathbb{R}$. Thus we may apply Lemma 2.3.3 with $K = \overline{B}(r(0), D)$ to find the limiting based ray c. Then c is asymptotic to r. The claim is that $c_n(\infty) \to c(\infty)$ in the cone topology. Fix $\epsilon > 0$ and let $s > 0$. Then $U(c, s, \epsilon) = \big\{ c' \in \partial_{x_0} X \;\big|\; d\big(c'(s), c(s)\big) < \epsilon \big\}$ is a basic neighborhood of c. As $c_n \to c$ pointwise we have that there exists $N \in \mathbb{N}$ $d\big(c_m(s), c(s)\big) < \epsilon$ for every $m > N$. Thus we have the claim. □

We now wish to prove that the projections pr_Y and pr_F are proper. In order to do so we will need several lemmas concerning the relationship between base points of orthogonal rays and the based rays which represent them in the boundary (See Figure 1 for an intuitive picture).

Lemma 2.3.9. *Let $\omega \in \perp(F)$ and c the ray based at x_0 representing $\omega(\infty)$. Assume that $d\big(x_0, \omega(0)\big) = t$. Then there exists a constant $M > 0$ such that $d\big(\omega(0), c(t)\big) < M$.*

Proof. Let $\beta\colon [0, t] \to F$ be the geodesic with $\beta(0) = x_0$ and $\beta(t) = \omega(0)$. By Corollary 2.3.7 there exists a constant L such that r is contained within the L-tubular neighborhood of $\omega \cup F$. This implies that there exists an $s \in [0, \infty)$, $x \in im(\omega)$, and $y \in F$ with $d\big(c(s), x\big) \leq L$ and $d\big(c(s), y\big) \leq L$. By orthogonal projection we know that $d\big(x, \omega(0)\big) \leq 2L$, which implies

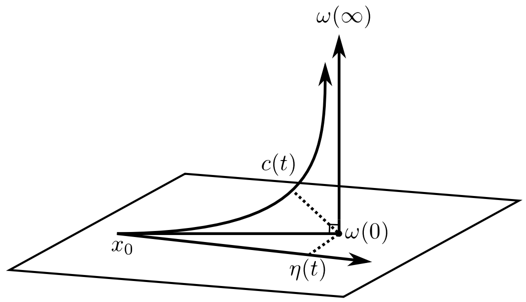

Figure 2.1: This figure illustrates the relationship between an orthogonal ray ω, its based representative c, and the neighborhood $U(\eta, t, \epsilon)$.

that $d\big(c(s), \omega(0)\big) \leq 3L$. Because, β and c are geodesics the triangle inequality gives us that $t \in \big[s - 3L, s + 3L\big]$. Thus setting $M = 6L$ we are done. $\square$

Lemma 2.3.10. *Let $\omega \in \perp(F)$ and c the ray based at x_0 representing $\omega(\infty)$. Assume that $\omega(0) \in U(\eta, t, \epsilon,)$ for some ϵ and some $\eta \in \partial F$. Then there exists some δ such that $c(t) \in U(\eta, t, \delta)$.*

Proof. Let $\beta\colon [0, a] \to F$ be the geodesic with $\beta(0) = x_0$ and $\beta(a) = \omega(0)$. If M is the constant from Lemma 2.3.9, then we know that $d\big(\beta(a), (a)\big) \leq M$. So, if $a = t$ we are done.

Assume that $t < a$. Then by convexity $d\big(c(t), \beta(t)\big) \leq M$ and $d\big(\beta(t), \eta(t)\big) \leq \epsilon$, which implies that $d\big(c(t), \eta(t)\big) \leq M + \epsilon$.

If $a < t$, then $\beta(a) = \beta(t)$. By hypothesis $\beta(a) \in U(\eta, t, \epsilon,)$, which implies that $d\big(\beta(a), \eta(t)\big) < \epsilon$. So, $d\big(c(a), \eta(t)\big) \leq M + \epsilon$, but c and η are geodesics so we have that $d\big(c(t), \eta(t)\big) \leq 2(M + \epsilon)$. Set $\delta = M + \epsilon$. $\square$

Lemma 2.3.11. *Let $W \subset Y$ be compact. The C set of all points $x \in F$ with $d(x, w) \leq A$ for some $w \in W$ is bounded.*

Proof. Assume not, then there exists a sequence (c_n) in $\perp(F)$ with $c_n(0) \in C$ for every $n \in \mathbb{N}$ such that $c_n(0) \to \eta$ as $n \to \infty$ for some $\eta \in \partial F$. For every n let r_n be the ray based at the base point $x_0 \in F$ and asymptotic to c_n.

Recall that for any D the sets $U(\eta, t, D)$ form a neighborhood base at η. Fix $\epsilon > 0$. Then $c_n(0) \to \eta$ implies that for any $t \in [0, \infty)$ we have $c_n(0)$ lies in $U(\eta, t, \epsilon)$ for all but finitely many n. Lemma 2.3.10 then gives us that for any $t \in [0, \infty)$ we have $r_n(0)$ lies in $U(\eta, t, \delta)$ for all but finitely many n, which by Lemma 2.3.8 implies that $r_n(\infty) \to \eta$, a contradiction. $\square$

Lemma 2.3.12. *The projections $pr_Y(\mathcal{R})$ and $pr_F(\mathcal{R})$ are proper.*

Proof. Let $K \subset F$ be compact. We want that $pr_F^{-1}(K)$ is compact. Let (r_n) be a sequence of rays in $\perp(F)$ with base points in K. As K is compact using Lemma 2.3.3 we know that the sequence (r_n) has a subsequence which converges to a ray r based in F. Set $x_n = r_n(1)$, $p_n = r_n(0)$, and let y be some fixed point in F. We know that $\angle_{p_n}(y, x_n) \geq \pi/2$ and that the function $(p, x, y) \mapsto \angle_p(x, y)$ is upper semi-continuous for all $p, x, y \in X$ (see [BH99] Proposition II.3.3(1)), thus r must be a ray orthogonal to K. By the previous lemma we have that the sequence of points at infinity converges, which implies that $pr_F^{-1}(K)$ is compact.

Now, let W be a compact subset of Y and C the set of all points $x \in F$ with $d(x, w) \leq A$ for some $w \in W$ is compact. We need that C is compact. By Lemma 2.3.11 the set C is bounded. We only need that C is closed.

Assume that c is a limit point of C. Then there exists a sequence $(c_i)_{i=0}^{\infty}$ of points in C which converge to C, and there exists a sequence of rays $(w_i)_{i=0}^{\infty}$ in $\perp(F)$ with $d\big(c_i, w_i(0)\big) \leq A$ and $w_i(\infty) \in W$ for every i. The sequence $\big(w_i(0)\big)_{i=0}^{\infty}$ converges in $\overline{\mathcal{N}}_A(C)$, so by a diagonal argument we have that (w_i) converges to a ray $w \in \perp(F)$. Lemma 2.3.8 and compactness of W give that $w(\infty) \in W$. It is now easy to see that $d(c, w) \leq A$. $\square$

Combining the previous results we may apply Lemma 2.3.1 to conclude:

Theorem 2.3.13. *Let $G = \mathrm{Stab}_\Gamma(F)$. Then G acts properly and cocompactly on $\partial X \backslash \partial F$.*

As mentioned above in Lemma 3.1.2 of [HK05] Hruska and Kleiner showed that $G = \mathrm{Stab}_\Gamma(F)$ acts cocompactly on F. The Beiberbach theorem then gives that G contains a subgroup of finite index H isomorphic to $\mathbb{Z}^n$, where n is the rank of the flat F.

We then obtain the following corollary:

Corollary 2.3.14. *The subgroup H acts properly and cocompactly on $\partial X \backslash \partial F$.*

2.4 Additional Properties of $\partial X \setminus \partial F$

Throughout this section we will assume that ∂X is locally connected. Let $\mathcal{C}$ be the collection of connected components of $Y = \partial X \backslash \partial F$. The goal of this section is to show that $\mathcal{C}$ has finitely many orbits and that each $C \in \mathcal{C}$ has stabilizer isomorphic to $\mathbb{Z}^n$, where n is the dimension of the flat. This fact will play a crucial role in Sections 2.5 and 2.7.

Let Z be a closed convex subset of a metric space M, and let G be any subgroup of $\mathrm{Isom}(M)$. We say Z is *G-periodic* if $\mathrm{Stab}_G(Z)$ acts cocompactly on Z. As in the previous section let $H \leq \mathrm{Stab}_\Gamma(F)$ be a finite index subgroup isomorphic to $\mathbb{Z}^n$, where n is the dimension of F. We begin with two results concerning the H-periodicity of elements of $\mathcal{C}$ that will be needed to prove the main result of this section.

2.4.1 H-periodicity

Lemma 2.4.1. *The collection $\mathcal{C}$ is locally finite, i.e only finitely many $C \in \mathcal{C}$ intersect any compact set $K \subset Y$.*

Proof. This simply follows from the local connectedness of Y. Assume that $\mathcal{C}$ is not locally finite. Then there exists $K \subset Y$ such that K meets infinitely elements of $\mathcal{C}$. We may then find a sequence $(x_C)_{C \in \mathcal{C}}$ of points from distinct elements of $\mathcal{C}$ which meet K. This sequence must converge to a point x in K. Thus any neighborhood of x meets infinitely many members of $\mathcal{C}$. Y is an open subset of a locally connected space and thus must be locally connected, a contradiction. □

Lemma 2.4.2. *Let $\mathcal{C}$ be as above. Then we have the following:*

1. *The elements of $\mathcal{C}$ lie in only finitely many H-orbits.*

2. *Each $C \in \mathcal{C}$ is H-periodic, i.e $\mathrm{Stab}_H(C)$ acts cocompactly on C.*

Proof. From Lemma 2.4.1 we know that $\mathcal{C}$ is locally finite, and we saw in Corollary 2.3.14 that H acts properly and cocompactly on on Y. We may now follow word for word the proof of Lemma 3.1.2 of [HK05]. $\square$

2.4.2 Full Rank Components

Lemma 2.4.3. *Assume we have a sequence of rays in $(r_i) \subset \perp(F)$ with base points, $r_i(0)$, converging to a point ξ in ∂F, then the sequence $\big(r_i(\infty)\big)$ converges to ξ in ∂F.*

Proof. By Corollary 2.3.7 each point $r_i(\infty)$ is represented by based rays c_i that stay within the L-neighborhood of $F \cup \mathrm{im}(r_i)$. Thus if $r_i(0) \to \xi \in \partial F$, then for any $n \in \mathbb{N}$ all but finitely many members of the sequence $\big(r_i(0)\big)$ are inside $U(\xi, n, \epsilon)$, which implies that for any n all but finitely many c_i lie in $U(\xi, n, \epsilon + L)$. $\square$

Corollary 2.4.4. *Every point in ∂F is a limit point of points in $\partial X \backslash \partial F$.*

Proof. Let ξ be in the boundary of a F and $c\colon [0, \infty) \to X$ a based ray representing ξ. Then by Corollary 2.3.5 for each $c(n)$ there is an $r_n \in \perp(F)$ such that $d\big(c(n), r_n(0)\big) < A$. So, the sequence $r_n(0)$ converges to ξ and we may apply the proceeding lemma. $\square$

The proof of the following lemma essentially amounts to checking Bestvina's nullity condition [Bes96] for the action of H on $\partial X \setminus \partial F$.

Lemma 2.4.5. *Let $C \in \mathcal{C}$ then elements of $\mathrm{Orb}_H(C)$ are asymptotic in the sense that two components meet in $\Lambda\big(\mathrm{Stab}_H(C)\big) \subset \partial F$.*

Proof. Let $C' \in \mathrm{Orb}_H(C)$ and let c'_n a sequence of points in C' converging to a point of ∂F. We show that there is a sequence of points in C which converge to the same point of ∂F.

Each c'_n is a translate of some point some point c_n in C. Notice that $\mathrm{Stab}_H(C') = \mathrm{Stab}_H(C)$ and by Lemma 2.4.2 exists a compacts set K' and K whose $\mathrm{Stab}_h(C)$-translates cover C' and C, respectively. So there exists a sequence of group elements (h_n) in $\mathrm{Stab}_H(C)$ such that c'_n is contained in $h_n K'$ and $c_n \in h_n K$ for every n.

Now, as in Section 2.3 consider the projection of K and K' to the the flat F and choose two points $k' \in K'$ and $k \in K$. For every n the point $h_n k'$ is within a bounded distance of the base of an orthogonal representative for c'_n. Similarly, each $h_n k$ is within a bounded distance of an orthogonal representative of c_n; moreover, the distance $d(k, k')$ is bounded. So, $(h_n k)$ and $(h_n k')$ converge to the same point ξ in ∂F, which implies that the bases of the orthogonal representatives of the (c_n) and (c'_n) also converge to ξ. We may apply Lemma 2.4.3 to complete the proof. □

Proposition 2.4.6. *Let $C \in \mathcal{C}$, then C is connected with stabilizer isomorphic to $\mathbb{Z}^n$.*

Proof. By way of contradiction suppose $C \in \mathcal{C}$ and assume that $H' = \mathrm{Stab}_H(C) \cong \mathbb{Z}^k$ for some $k < n$. As $k < n$ we may find an $h \in H \setminus H'$ and an axis $\ell \colon \mathbb{R} \to F$ passing through $x_0 \in F$ for h with $\ell(+\infty)$ and $\ell(-\infty)$ not in the limit set $\Lambda(H')$. Let ξ be the point of ∂F represented by the ray $\ell|[0, \infty)$. As $\Lambda(H')$ is a closed subsphere of ∂F, we may find a $U = U(\xi, n, \epsilon)$ neighborhood of ξ in $\partial X \setminus \Lambda(H')$.

Fix $\eta \in C$ and let r be an orthogonal representative of η. Then $h^n(r)$ is an orthogonal ray for every n and the sequence $h^n\big(r(0)\big)$ converges to ξ, which by Lemma 2.4.3 implies $h^n(\eta) \to \xi$. As each $h^n(\eta)$ lies in a different element of $\mathrm{Orb}_{\langle h \rangle}(C)$ we have that infinitely members of $\mathrm{Orb}_{\langle h \rangle}(C)$ intersect U. As $\langle h \rangle$ stabilizes ∂F and the orbit under H' of points in C converges to points in $\Lambda(H')$, Lemma 2.4.5 implies that no element of $\mathrm{Orb}_{\langle h \rangle}(C)$ is contained in U. Thus ∂X is not locally connected, a contradiction. □

Combining this result with the H-periodicity result Lemma 2.4.2 we obtain:

Corollary 2.4.7. *There are only finitely many components of $\partial X \setminus \partial F$.*

2.5 An Equivariant Metric on $\partial X \backslash \partial F$

In this section we assume that $Y = \partial X \setminus \partial F$ is locally connected. Let H be the maximal free abelian subgroup of $\mathrm{Stab}_\Gamma(F)$. We will put an H-equivariant metric on Y. First, we remind the reader of a standard result about covering spaces that will be used several times throughout this section:

Lemma 2.5.1. *Let G be a torsion free group acting properly on a locally compact Hausdorff space X, then X together with the quotient map $q\colon X \to X/G$ form a normal covering space of X/G.*

We begin with a review of how one defines the *pull-back length metric* of a length space. We refer the reader to [Pap14] for a more detailed account. Recall that a length metric is one where the distance between two points is given by taking the infimum of the lengths of all rectifiable curves between x and y. Suppose that X is a length space and $\tilde{X}$ is topological space, and $p\colon \tilde{X} \to X$ is a surjective local homeomorphism. Define a pseudometric on $\tilde{X}$ by:

$$\tilde{d}(\tilde{x}, \tilde{y}) = \inf\left\{ L(p \circ \tilde{\gamma}) \;\middle|\; \tilde{\gamma}\colon [0,1] \to \tilde{X} \text{ a curve from } \tilde{x} \text{ to } \tilde{y} \right\}$$

Where $L(p \circ \tilde{\gamma})$ is the length of the path $p \circ \tilde{\gamma}$. If $\tilde{X}$ is Hausdorff then $\tilde{d}$ is a length metric (see [Pap14] Proposition 3.4.7). Also, it is easy to show that:

Lemma 2.5.2. *If X is obtained as the quotient of a free and proper action by a group G then the metric $\tilde{d}$ is G-equivariant.*

Proof. Let $P(\tilde{x}, \tilde{y})$ be the set of all paths between $\tilde{x}$ and $\tilde{y} \in \tilde{X}$ and $Q(x', y') = \left\{ p \circ \sigma \;\middle|\; \sigma \in P(x', y') \right\}$. To prove that $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}(g\tilde{x}, g\tilde{x})$ it suffices to show that $Q(\tilde{x}, \tilde{y}) = Q(g\tilde{x}, g\tilde{y})$. But this is clear, as X is obtained as the quotient of the group action, i.e. if γ is a path in $\tilde{X}$, then γ and $g\gamma$ are identified. $\square$

Let G be a torsion free group, acting, properly and cocompactly on a connected component C of $\partial X \backslash \partial F$, set $Q = C/G$, and define $q\colon C \to C/G$ to be the associated quotient map. In order to apply the above construction to our setting we need that Q is a length space. I would like to thank Ric Ancel for pointing out the following theorem due to R.H. Bing (see [Bin52]), which we will use to show that Q is a length space:

Theorem 2.5.3 (Bing)**.** *Every Peano continuum admits a convex metric.*

Recall that that a *Peano continuum* is a compact, connected, locally connected metrizable space. The notion of convexity used by Bing is that of Menger convexity. For proper metric spaces Menger convexity is known to be equivalent to being geodesic [Pap14]. Recall that a *geodesic*, between two points x and y in a metric space X is an isometric embedding of an interval $\gamma\colon [0, D] \to X$ such that $\gamma(0) = x$, $\gamma(D) = y$, and $D = d_X(x, y)$. By a *geodesic metric space* we mean that there is a geodesic joining any two points of the space. Compact metric spaces are proper, so we may replace the word "convex" with "geodesic" in Bing's result. Note that by default a geodesic metric space a length space. Therefore, we need only show that Q is a Peano continuum to obtain that Q is a length space.

To show that Q is metrizable we use Urysohn's metrization theorem:

Theorem 2.5.4. *Let X be a T_1 space. If X is regular and second countable, then X is separable and metrizable.*

This theorem and all general topology results used in this section can be found in [Wil04].

Lemma 2.5.5. *Q is second countable.*

Proof. First note that ∂X is a compact metric space, which implies that ∂X is separable. Subspaces of separable metric spaces are separable. So, C is separable. For pseudometric spaces separability and second countability are equivalent (see [Wil04] Theorem 16.11), so C is second countable. Q is the continuous open image of a second countable space, therefore Q is second countable (see [Wil04] Theorem 16.2(a)). □

Lemma 2.5.6. *Q is T_1.*

Proof. A topological space is T_1 iff each one point set is closed [Wil04]. Let $[x] \in Q$. As Q is the quotient of a proper group action $q^{-1}\big([x]\big)$ is a discrete set of points, this implies that $q^{-1}\big([x]\big)$ is closed in C. Quotients by group actions are open maps, so q is a surjective open map. Therefore $q(C \setminus q^{-1}\big([x]\big) = Q \setminus \{[x]\}$ is open, which implies that $[x]$ is closed. □

Lemma 2.5.7. *Q is regular.*

Proof. It suffices to show that for each open set $U \in Q$ and $x \in U$ that there exists an open set $V \subset U$ such that $x \in V$ and $\overline{V} \subset U$ (see [Wil04] Theorem 14.3). As Y is locally compact and C is a component of Y, C must be locally compact. The continuous open image of locally compact is locally compact, so we have that Q is locally compact. Let U be a neighborhood of x in Q. Then by local compactness for any U neighborhood of x in Q there exists an open set $V \subset U$ such that $x \in V$ and $\overline{V} \subset U$. □

Theorem 2.5.8. *Q is a Peano continuum.*

Proof. We have shown that Q is metrizable and Q is compact by definition. C is connected. So, by continuity of q, we have that Q is connected. C locally connected and q is a local homeomorphism, so Q is locally connected. □

Thus, by Bing's theorem we have that Q is a geodesic metric space. Defining H as in the previous two sections we may use the construction mentioned at the beginning of this section to obtain:

Proposition 2.5.9. *There exists an H-equivariant metric on C.*

From Corollary 2.4.7 we know that $\mathcal{C}$ consists of only finitely many components each stabilized by H. Thus by defining distance to be the same in each component and the distance between points in different components to be infinite we may prove the following corollary:

Corollary 2.5.10. *There exists an H-equivariant metric on Y.*

We conclude this section with an important corollary that will prove very useful in Section 2.7. Let $\mathcal{R}$ be the relation defined in Section 2.3.

Corollary 2.5.11. *The relation $\mathcal{R}$ is a quasi-isometry relation, i.e. if $(x_1, y_1), (x_2, y_2) \in \mathcal{R}$ then there exists constants $L > 0$ and $C \geq 0$ such that*

$$\frac{1}{L} d(x_1, x_2) - C \leq d(y_1, y_2) \leq L d(x_1, x_2) + C$$

.

Proof. We have H acting geometrically on F and Y. So we may find a quasi-isometry $\Phi\colon F \to Y$. If $(x_1, y_1), (x_2, y_2) \in \mathcal{R}$, then we know that there exists $L > 0$ and $C \geq 0$ such that:

$$\frac{1}{L} d(x_1, x_2) - C \leq d\big(\Phi(x_1), \Phi(x_2)\big) \leq L d(x_1, x_2) + C$$

.

If we can find a constant $D \geq 0$ such that $d\big(\Phi(x_1), y_1\big) < D$ and $d(\Phi(x_2), y_2) < D$, the we will be done. Let $K \subset F$ be a compact set whose H-translates cover F. We saw in Section 3 that the projection pr_F and pr_Y are proper and equivariant. So, if $h_i \in K$ is such that $x_i \in h_i K$, then $y_i \in h_i K_\infty$ for $i \in \{1, 2\}$, where $K_\infty = pr_Y(pr_F^{-1})(K)$. We need only that $\Phi(x_i) \in h_i K_\infty$ for $i \in \{1, 2\}$. But, this follows from the fact that Φ is the composition of an orbit map and the inverse of an orbit map. □

2.6 Local cut points which are not in the boundary of a flat

In this section we wish to prove the following:

Proposition 2.6.1. *Let Γ be a one-ended group acting geometrically on a* CAT(0) *space X with isolated flats. Suppose ∂X is not homeomorphic to S^1 and let $\xi \in \partial X$ be such that ξ is not in ∂F for any $F \in \mathcal{F}$. If ξ is a local cut point, then Γ splits over a 2-ended subgroup.*

The proof of this proposition relies on Theorem 2.2.5 and a result of Hung Cong Tran (Theorem 2.2.1), which provides a strong connection between ∂X and $\partial(\Gamma, \mathbb{P})$ via a quotient map. Let $f \colon \partial X \to \partial(\Gamma, \mathbb{P})$ be this quotient map. To prove Proposition 2.6.1 we need more information about the behavior of the map f. The particular question that needs to be addressed is as follows: Let $\xi \in \partial X$ which is not in the boundary of a flat. If ξ is a local cut point can its image, $f(\xi)$, fail to be a local cut point in $\partial(\Gamma, \mathbb{P})$?

To answer this question in the negative we will first need to recall some basic decomposition theory. We refer the reader to [Dav07] for more information on decomposition theory.

2.6.1 Decompositions

A *decomposition*, $\mathcal{D}$, of a topological space X is a partition of X. Associated to $\mathcal{D}$ is the *decomposition space* whose underlying point set is $\mathcal{D}$, but denoted $X/\mathcal{D}$. The topology of $X/\mathcal{D}$ is given by the *decomposition map* $\pi \colon X \to X/\mathcal{D}$, $x \mapsto D$, where $D \in \mathcal{D}$ is the unique element of the decomposition containing x. A set U in $X/\mathcal{D}$ is deemed open if and only if $\pi^{-1}(U)$ is open in X. A subset A of X is called *saturated* (or $\mathcal{D}$-saturated) if $\pi^{-1}\big(\pi(A)\big) = A$. The *saturation* of A, $Sat(A)$, is the union of A with all $D \in \mathcal{D}$ that intersect A. The decomposition $\mathcal{D}$ is said to be *upper semi-continuous* if every $D \in \mathcal{D}$ is closed and for every open set U containing D there exists and open set $V \subset U$ such that $Sat(V)$ is contained in U. $\mathcal{D}$ is called *monotone* if the elements of $\mathcal{D}$ are compact and connected.

A collection of subsets $\mathcal{S}$ of a metric space is called a *null family* if for every $\epsilon > 0$ there are only finitely many $S \in \mathcal{S}$ with $\operatorname{diam}(S) > \epsilon$. The following proposition can be found as Proposition I.2.3 in [Dav07].

Proposition 2.6.2. *Let $\mathcal{S}$ be a null family of closed disjoint subsets of a compact metric space X. Then the associated decomposition of X is upper semi-continuous.*

In the isolated flats setting a theorem of Hruska and Ruane [HR] shows:

Proposition 2.6.3. *The collection $\partial F_{F \in \mathcal{F}}$ forms a null family in ∂X*

Let $f\colon \partial X \to \partial(\Gamma, \mathbb{P})$ be as above. Note that f is the decomposition map of the monotone and upper semi-continuous decomposition $\mathcal{D}$ of ∂X where $\mathcal{D} = \left\{ \partial F \;\middle|\; F \in \mathcal{F} \right\} \cup \left\{ \{x\} \;\middle|\; x \notin \partial F \text{ for all } F \in \mathcal{F} \right\}$. By Proposition 1.3.6 of Chapter 1 we have:

Lemma 2.6.4. *Let $\xi \in \partial X$ and assume that $\xi \notin \partial F$ for any $F \in \mathcal{F}$. If ξ is a local cut point, then $f(\xi)$ is a local cut point.*

Now that we know that non-parabolic local cut points in ∂X get mapped to non-parabolic local cut points in $\partial(\Gamma, \mathbb{P})$, the proof of Proposition 2.6.1 follows almost immediately from Theorem 2.2.5.

Proof of Proposition 6.1. Let ξ be a point in ∂X which is a local cut point which is not in that boundary of a flat. As CAT(0) groups with isolated flats are relatively hyperbolic, Proposition 2.6.4 implies that there is a non-parabolic local cut point in $\partial(\Gamma, \mathbb{P})$. Therefore we are done by Theorem 2.2.5. □

2.7 Local Cut Points in the Boundary of a Flat

The goal of this section is to complete the proof of Theorem 2.1.3 by showing that a point ξ in the boundary of a flat cannot be a local cut point. We begin this section by defining basic neighborhoods "of infinity" in $Y = \partial X \setminus \partial F$ and provide a useful lemma. Then in Section 2.7.2 we develop machinery required to prove that ξ cannot be a local cut point. Throughout this section we will assume that ∂X is locally connected.

2.7.1 Basic Neighborhoods in Y

Let ξ be an element of ∂F. Given a neighborhood $V(\xi, n, \epsilon)$ in the bordification of X, recall that $V_\partial(\xi, n, \epsilon)$ is the restriction of $V(\xi, n, \epsilon)$ to points of ∂X. Given a boundary neighborhood $V_\partial(\xi, n, \epsilon)$ we define $V_Y(\xi, n, \epsilon)$ to be the subset $V_\partial(\xi, n, \epsilon) \backslash \partial F$. Then $V_Y(\xi, n, \epsilon)$ is open in Y with the subspace topology. Although it is somewhat of a misnomer $V_Y(\xi, n, \epsilon)$, will refer to $V_Y(\xi, n, \epsilon)$ as *a basic neighborhood of* ξ *in* Y. Notice that these sets $V_Y(\xi, n, \epsilon)$ form a basis in the sense that given any open set A consisting of an open neighborhood of ξ in ∂X intersected with Y we may find a $k > 0$ large enough so that $V_Y(\xi, k, \epsilon) \subset A$. Lastly, the set $V(\xi, n, \epsilon) \cap F$ will be referred to as a *flat neighborhood* of ξ and denoted $V_F(\xi, n, \epsilon)$. When there is no ambiguity about the parameters n we will simply write V_∂, V_Y, and V_F. The following is a consequence of Lemmas 2.3.9 and 2.3.10:

Lemma 2.7.1. *Let* $\eta \in V_Y(\xi, n, \epsilon)$ *and* $r \in \perp(F)$ *the orthogonal representative of* η. *Then there exists a* δ *such that* $\delta > \epsilon$ *and* $r(0) \in V_F(\xi, n, \delta)$.

2.7.2 $\xi \in \partial F$ cannot be a local cut point

Recall that a point $\xi \in \partial X$ is a *local cut point* if $X \setminus \{\xi\}$ is not one-ended. A path connected metric space is one-ended if for each compact K there exists a compact K' such that points outside of K' can be connected by paths outside of K. In other words, to show that ξ is not a local cut point we need to show that for any neighborhood U_∂ of ξ, there exists a neighborhood V_∂ of ξ such that all points of $V \backslash \{\xi\}$ can be connected by paths in $U_\partial \backslash \{\xi\}$. Intuitively the idea is to show that we may connect two points close to ξ up by a path which does not travel "too far" into Y.

In Section 2.5 we saw that Y admits a geometric action by $H = \mathbb{Z}^n$; moreover, by Proposition 2.4.6 and 2.4.2 we know that Y consists of finitely many components whose stabilizers are $\mathbb{Z}^n$ subgroups of full rank. So the components of Y coarsely look like $\mathbb{Z}^n$ and this particularly nice structure will help us control the length of paths in ∂X near ξ.

The majority of the arguments in this section only concern the action of $\mathbb{Z}^n$ on a single connected component; therefore, we may assume for now that Y consists of a single connected component. A reader only interested in the proof of Theorem 2.1.2 may wish to focus on the simple case when $n = 2$, as this is an intuitively simpler case.

Also, recall that our main concern is 1-dimensional boundaries so the reader may wish to think of n as being equal to 2 for intuitive purposes; however, the following arguments do not require that assumption.

Lemma 2.7.2. *Let $D > 0$ and $y \in Y$. Then there exists an $M > 0$ such that points of the ball $B(y, D)$ may be connected by paths in $B(y, M)$.*

If $\overline{B}(y, D)$ is connected, then $D = D'$ and we are done. So, assume $\overline{B}(y, D)$ is not connected. Then $\overline{B}(y, D)$ must contain only finitely many path components, contradicting local connectedness of ∂X. Let $\mathcal{A}$ be the set of components of $\overline{B}(y, D)$ and assume $|\mathcal{A}| = n$. Then for any pair of components $A_1, A_2 \in \mathcal{A}$ with $A_1 \neq A_2$ we may find a path γ in Y with $\gamma(0) \in A_1$ and $\gamma(1) \in A_2$. Let P be the collection of all such paths. Then $|P| = \binom{n}{2} < \infty$. For every $\gamma \in P$ we have that $\operatorname{diam}(\gamma) < \infty$, so set $N = \max\{ \operatorname{diam}(p) \mid p \in P \}$. Then $\overline{B}(y, D) \cup P$ is connected and has diameter $\leq D + N$.

Corollary 2.7.3. *Let $D > 0$, and $y_0 \in Y$. Then there is a $M > 0$ such that for any $y \in \operatorname{Orb}_H(y_0)$ $B(y, D)$ is path connected inside of $B(y, M)$.*

Proof. This follows immediately from the lemma and the geometric action of H on Y. □

Lemma 2.7.4. *Let $V_Y = V_Y(\xi, n, \epsilon)$ be a basic neighborhood of ξ in Y. Then There exists a metric neighborhood $\mathcal{N}(V_Y)$ of V_Y in Y such that points in V_Y can be connected by paths inside $\mathcal{N}(V_Y)$.*

Proof. Let $\mu, \nu \in V_\partial$ and $u(0), v(0)$ be the base points of elements u and v of $\perp(F)$ representing μ and ν respectively. By Lemma 2.7.1 there exists a $\delta > 0$ such that $u(0)$ and $v(0)$ are in $V_F(\xi, n, \delta)$. As $V_F(\xi, n, \delta)$ is a sector of an embedded Euclidean plane, there exists a path γ in $V_F(\xi, n, \delta)$ connecting $u(0)$ and $v(0)$. Define $A > 0$ to be the constant from the definition of the relation $\mathcal{R}$ in Section 2.3. We may find a finite sequence of points $(a_i)_{i=0}^n$ contained in the image of γ with $a_0 = u(0)$, $a_n = v(0)$, and such that γ is contained in $\bigcup_{i=0}^n \overline{B}(a_i, A)$. By choice of A, for each a_i we may find $c_i \in \perp(F)$ with $c_0 = a_0$ and $c_n = a_n$, and such that $d_X\big(c_i(0), a_i\big) < A$. This implies that $d_X\big(c_i(0), c_{i+1}(0)\big) < 4A$.

The boundary points $c_i(\infty)$ need not be in V_Y, but using an argument similar to that of Lemma 2.3.10 one sees that they are in $V'_Y = V_Y(\xi, n, \delta + A + K)$ for some K. From Lemma 2.7.1 we have that $\delta > \epsilon$, so we see that $V_Y \subset V'_Y$.

Recall that Corollary 2.5.11 gives an (L, C)-quasi-isometry associated to $\mathcal{R}$, which implies that $d_Y\big(c_i(\infty), c_{i+1}(\infty)\big) < L(4A)+C$. Fixing a point a base point y_0 in Y we know that $H \cong \mathbb{Z}^n$ acts cocompactly on Y, so there exists a constant J and points $\{y_1, ..., y_2\} \subset \mathrm{Orb}_H(y_0)$ such that for every i we have $d_Y\big(y_i, c_i(\infty)\big) \leq J$. Setting $D = L(4A) + C + J$, we see that the neighborhoods $\overline{B}(y_i, D)$ form a chain from $\mu = c_1(\infty)$ to $\nu = c_n(\infty)$. Corollary 2.7.3 tells us that we may find a constant $M > 0$ such that $\bigcup \overline{B}(y_i, D)$ is connected by paths in $N_M\big(\bigcup \overline{B}(y_i, D)\big)$. Therefore, μ and ν are connected by a path in $\mathcal{N}(V_Y) = N_M(V'_Y)$. $\square$

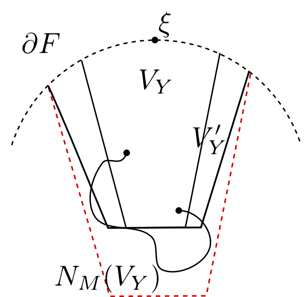

Figure 2.2: This figure depicts the neighborhood $\mathcal{N}(V_Y) = N_M(V'_Y)$.

Although it is not truly a neighborhood (in the sense that it is not open in ∂X), we will use $\mathcal{N}(V_\partial)$ to denote $\mathcal{N}(V_Y) \cup V_\partial$. In other words, $\mathcal{N}(V_\partial)$ is $N(V_Y)$ with $V_\partial \cap \partial F$ attached.

Corollary 2.7.5. *Let V_Y be a basic neighborhood of ξ in Y. Then any two points in $V_\partial \setminus \{\xi\}$ can be connected by paths in $\mathcal{N}(V_\partial) \setminus \{\xi\}$.*

Proof. We have three 3 cases to check. First assume that $\mu, \nu \in V_Y$. Then by 2.7.4 we have that μ and ν can be connected by a path in $\mathcal{N}(V_Y)$.

Second, suppose that $\mu, \in V_Y$ and $\nu \in \partial(F \cap V_\partial) \setminus \{\xi\}$. We know that ∂X is locally path connected. As V_∂ is a basic neighborhood of ξ in ∂X, we may find a path connected basic ∂X-neighborhood U of ν in $V_\partial \setminus \{\xi\}$. By Corollary 2.4.4 and choice of U we have that $U \cap V$

is non-empty, we may find a point $\rho \in U \cap V_Y$ which is connected to ν by a path in U. Thus we may apply the first case to connect μ to ρ by a path in $\mathcal{N}(V_Y)$. By concatenating these paths we complete case two.

Lastly, if μ and ν are both in $\partial(F \cap V_\partial) \setminus \{\xi\}$ we may pick a point in V_Y and apply the second case twice. □

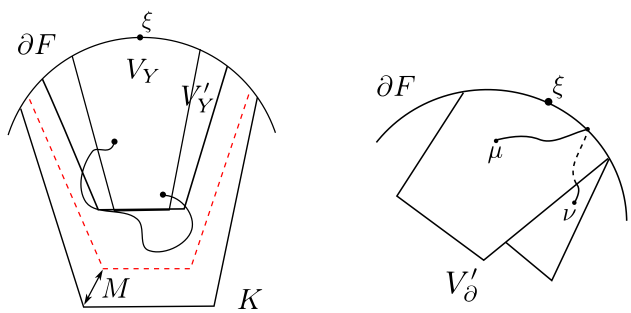

Figure 2.3: The figure on the left illustrates depicts the M-neighborhood of the compact set K and a path contained in the compliment of its closure connecting two points of V_Y. The figure on the right depicts two points near ξ in different components connected by a path in ∂X.

We now disregard the hypothesis that $Y = \partial X \setminus \partial F$ consists of a single connected component and prove the main result of this section.

Theorem 2.7.6. *A point ξ in the boundary of a flat cannot be a local cut point.*

Proof. Let K be a compact subset of $\partial X \setminus \{\xi\}$. Recalling the discussion at the beginning of this subsection, we need to find a compact set L such that points outside of L can be connected by paths outside of K. Let ϵ, δ, A and K be as in Lemma 2.7.4 and set $\kappa = \delta + A + K$. The sets $V_\partial(\xi, n, \kappa)$ form a neighborhood base so we may find an $n > 0$ large enough so that

$$V_Y(\xi, n, \kappa) \subset V_\partial(\xi, n, \kappa) \subset Y \setminus \overline{N}_M(K),$$

where $M > 0$ is the constant found in 2.7.4. Let $V'_\partial = V_\partial(\xi, n, \kappa)$. Notice that $\epsilon < \kappa$ implies that $V_\partial = V_\partial(\xi, n, \epsilon)$ is contained in V'_∂ and consequently $Y \setminus \overline{N}_M(K)$ (see Figure 2.3). Define $L = \partial X \setminus V_\partial$.

We need that that points in $V_\partial \setminus \{\xi\}$ can be connected by paths outside of K. Let $\mu, \nu \in V_\partial \setminus \{\xi\}$.

If μ and ν are in $(\partial F \cap V_\partial) \setminus \{\xi\}$ or in the same component of $V_Y = V_Y(\xi, n, \epsilon)$, then Corollary 2.7.5 tells us that they can be connected a path which misses K. So, if μ and ν are in different components of V_Y we may connect them by passing through a point of $(\partial F \cap V_\partial) \setminus \{\xi\}$ (see Figure 2.3). Thus, μ and ν can be connected by path outside of K. □

Combining this theorem with Proposition 2.6.1 we have completed the proof of Theorem 2.1.3.

2.8 Proof of the Main Theorem

The goal of this section is to prove Theorem 2.1.2, but first we review a few facts about the dynamics of the action of Γ on ∂X.

2.8.1 Tits Distance and the Dynamical Properties of $\Gamma \curvearrowright \partial X$

Recall that a *line* in X is a 1-flat. A *flat halfplane* is a subspace of X isometric to the Euclidean half-plane, i.e $\{y \geq 0\}$ in the (x, y)-plane. A line, L, is called *rank one* if it does not bound a flat half-plane. By $L(\xi, \eta)$ we denoted the line with endpoints ξ and η in ∂X. L is said to be Γ-*periodic* if there is an arc-length parametrization of σ of L, an element $\gamma \in \Gamma$, and a constant $\alpha > 0$ such that $\gamma\sigma(t) = \sigma(t + \alpha)$ for all $t \in \mathbb{R}$.

We will need the following two results regarding rank one geodesics and dynamics of the action on the boundary. The first can be found in Section III.3 of [Bal95] and the second may be found as Proposition 1.10 of [BB08].

Lemma 2.8.1. *Suppose L is an oriented rank one line shifted by an axial isometry g. Let ξ and η be the end points of L. Then for all neighborhoods U of ξ and V of η in $\overline{X}$ there exists $n \geq 0$ such that $g^k(\overline{X} \setminus V) \subset U$ and $g^{-k}(\overline{X} \setminus U) \subset V$ for every $k \geq n$.*

Proposition 2.8.2. *Suppose that the limit set $\Lambda \subset \partial X$ is non-empty. Then the following are equivalent:*

1. *X contains a Γ-periodic rank one line.*

2. *For each $\xi \in \Lambda$ there is a $\eta \in \Lambda$ with $d_T(\xi, \eta) > \pi$, where d_T is the Tits metric.*

3. *There are points $\xi, \eta \in \Lambda$ with $d_T(\xi, \eta) > \pi$, where η is contained in some minimal subset of ∂X.*

Recall that the *Tits metric* is the length metric associated to the angular metric on ∂X. We refer the reader to [BH99] Chapter II.9 for the required background. Though not stated in Proposition 2.8.2, it is clear from the proof provided Ballmann and Buyalo [BB08] that the end points of the Γ-periodic rank one line can be found arbitrarily close to the points ξ and η. This is precisely the way in which Proposition 2.8.2 will be used below. We will also need the following:

Lemma 2.8.3. *Let Γ be a one-ended group acting geometrically on a* CAT(0) *space with isolated flats. The action of Γ on ∂X is minimal.*

Proof. Assume not, then ∂X contains a closed Γ-invariant set M. If $f \colon \partial X \to \partial(\Gamma, \mathbb{P})$ is the equivariant quotient map defined in Theorem 2.2.1, then f is a closed map by Proposition 1 of [Dav07]. Thus $f(M)$ is closed and Γ-invariant. From [Bow12] we have that the action of Γ on $\partial(\Gamma, \mathbb{P})$ is minimal. Thus, if $f(M)$ is a proper subset of $\partial(\Gamma, \mathbb{P})$ we will obtain a contradiction.

As M is properly contained in ∂X we may find a neighborhood U of M which is properly contained in ∂X. By upper semi-continuity of the decomposition (see Proposition 2.6.2) we have the $Sat(M) \subseteq U$. Thus $\partial X \setminus Sat(M) \neq \emptyset$, which implies that $f(M)$ is properly contained in $\partial(\Gamma, \mathbb{P})$. ☐

Proposition 2.8.4. *Let K be a proper closed subset of ∂X, then for any U open set in ∂X we may find a homeomorphic copy K' of K such that $K' \subset U$.*

Proof. Let U be and open subset of the boundary and let $\rho \in \partial X \setminus K$ be a conical limit point. In Theorem 5.2.5 of [HK05] Hruska and Kleiner show that components of $\partial_T X$ are boundary spheres $\partial F_{F \in \mathcal{F}}$ and isolated points. So let η be another point in $\partial X \setminus K$, then $d_T(\rho, \eta) > \pi$. Choose any neighborhoods V of ρ and W of η in $\partial X \setminus K$. Then Proposition 2.8.2 implies that we may find a periodic rank one line L such that the ends $L(\infty)$ and $L(-\infty)$ are in V and W respectively. We may then apply Lemma 2.8.1 to find a homeomorphic copy of K in W (or V).

By Lemma 2.8.3 we have that the action of Γ on the boundary is minimal, which implies we have that $\mathrm{Orb}_\Gamma(\eta)$ is dense in ∂X. Thus there exists a $\gamma \in \Gamma$ such that $\gamma\eta \in U$. Choosing W small enough we have that $\gamma(W) \subset U$. As γ is a homeomorphism U contains a copy of K. □

We now prove the main theorem.

Proof of Theorem 2.1.2: Using the toplogical characterizations of the Menger curve and Sierpinski carpet we provide a proof similar to that of Kapovich and Kleiner in Section 3 of [KK00].

By hypothesis and Corollary 2.2.4, we have that ∂X is connected, locally connected, and 1-dimensional. Theorem 2.1.3 gives that if ∂X has a local cut point, then ∂X is homeomorphic to S^1 or Γ splits over a 2-ended subgroup. Assume that ∂X does not have a local cut point.

The boundary of X is planar, or it is not. If ∂X is planar, then it is a Sierpinski carpet by the characterization of Whyburn [Why58]. So, assume that ∂X is non-planar. Claytor's embedding theorem [Cla34] then implies that ∂X contains a non-planar graph. We may now use Proposition 2.8.4 to show that no non-empty open subset of ∂X is planar. Thus ∂X must be a Menger curve. □

2.9 Non-hyperbolic Coxeter groups with Sierpinski carpet boundary

In this section we give sufficient conditions for the boundary of a Coxeter group with isolated flats to have a Sierpinski carpet boundary. This result is an easy consequence of Theorem 2.1.2 and results of Świątkowski [Ś16].

A *Coxeter system* is a pair (W, S) such that W is a finitely presented group with presentation $\langle S \mid R \rangle$ with

$$R = \left\{ s^2 \mid s \in S \right\} \cup \left\{ (s,t)^{m_{st}} \mid s, t \in S,\, m_{st} \in \{2, 3, ...\infty\} \, \text{and} \, m_{st} = m_{ts} \right\},$$

and $m_{st} = \infty$ means that there is no relation between s and t.

The *nerve* $L = L(W, S)$ of the Coxeter system (W, S) is a simplicial complex whose 0-skeleton is S and a simplex is spanned by a subset $T \subset S$ if and only if the subgroup generated by T is finite. The *labeled nerve* $L^\bullet$ of (W, S) is the nerve L with edges (s, t) in the 1-skeleton of L labeled by the number m_{st}. A *labeled suspension* in $L^\bullet$ is a full subcomplex K of L isomorphic to the simplicial suspension of a simplex, $K = \{s, t\} * \sigma$, such that any edge in K adjacent to t or s has edge label 2. The labeled nerve is called *inseparable* if it is connected, has no separating simplex, no separating vertex pair, and no separating labeled suspension. The labeled nerve $L^\bullet$ is called a *labeled wheel* if L is the cone over a triangulation of S^1 with cone edges labeled by 2.

Associated to any Coxeter system (W, S) is a piecewise Euclidean CAT(0) space called the *Davis complex* $\Sigma = \Sigma(W, S)$. The group W acts geometrically on Σ by reflections. Caprace [Cap09] has completely determined when the Davis complex has isolated flats.

Proof of Theorem 2.1.5: Assume the hypotheses. In Lemmas 2.3, 2.4, and 2.5 of [Ś16] Świątkowski shows that $\partial\Sigma$ is connected, planar, and 1-dimensional. Lastly inseparability of $L^\bullet$ implies that W does not split over a virtually cyclic subgroup [MT09, Ś16], thus

Theorem 2.1.2 implies that $\partial\Sigma$ must be a circle or a Sierpinski carpet.

If W is hyperbolic, then Świątkowski [Ś16] shows that $\partial\Sigma$ cannot be homeomorphic to S^1. Assume that $\partial\Sigma$ is homeomorphic to S^1 and W is not hyperbolic. Then Σ contains a flat F; moreover, F must be the only flat. Thus W is a 2-dimensional Euclidean group. Because the nerve L is planar, L must be a wheel or a triangulation of S^1, a contradiction.

□

2.10 Non-hyperbolic Groups with Menger Curve Boundary

In the hyperbolic setting groups with Menger curve boundary are quite ubiquitous. It is a well known result of Gromov [Gro87] that with overwhelming probability random groups are hypeberbolic; subsequently, Dhamani, Guirardel, and Przytycki [DGP11] have shown that with overwhelming probability random groups also have Menger curve boundary. In stark contrast no example of a non-hyperbolic group with Menger curve boundary can presently be found in the literature, leading Kim Ruane to pose the challenge of finding finding a non-hyperbolic group with Menger curve boundary.

Prior to Theorem 2.1.2 there were no known techniques for developing examples of such a group. The author claims that one example is the fundamental group of the space obtained by gluing three copies of a finite volume hyperbolic 3-manifold with totally geodesic boundary together along a torus corresponding to a cusp. This particular example was suggested to the author by Jason Mannning, and a detailed proof is to be provided in [HG]. The author believes that many examples of non-hyperbolic groups with Menger curve boundary may now be constructed in a similar fashion.

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