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God's Number in the Simultaneously-Possible Turn Metric

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GOD'S NUMBER IN THE SIMULTANEOUSLY-POSSIBLE TURN METRIC

by

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ABSTRACT

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by

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In 2010 it was found that God's number is 20 in the face turn metric. That is, if the Rubik's cube hasn't been disassembled, it can always be solved in 20 twists or fewer, but sometimes requires 20 twists. However, the face turn metric only allows one face to be turned at a time for a total of 18 generators, or 18 possible twists at any time. This dissertation allows opposing, parallel faces to be twisted independent amounts at the same time and still get counted as 1 twist for a total of 45 generators. A new optimal-solving program was constructed, and the results so far show that God's number is at least 16 for the simultaneously-possible turn metric.

I note that in 3 dimensions the simultaneously-possible turn metric is the same as the axial turn metric (or robot turn metric), but not in 4 dimensions nor higher (e.g. 2×2×2×2, 3×3×3×3, 4×4×4×4, etc.--not to be confused with the 3-dimensional 4×4×4 cube). This difference is also described.

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LIST OF NOMENCLATURE

- 0 orientation: a cubie has 0 orientation if its highest priority tile is located on the highest priority face available for that location. (If needed, i.e. 4D and higher, additionally its secondary tile is on the secondary face, etc.) In 3D, a corner cubie has orientation 1 if its priority tile is one tile clockwise from a priority tile location (see Priority at right and Figures 0.1 and 1.1)
- 2D twist: a 2D twist on an M^N Rubik's cube is a twist of a subset with size such that two of the dimensions are M and the rest are 1.
- 3D face: see Cube.
- Compatible subsets: two twistable subsets, A and B are compatible for a simultaneous twist if $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$.
- Cube: Considering the M^N Rubik's cube,
  It has 2^N total vertices.
  It has $N*2^{N-1}$ total edges.
  It has $\binom{N}{k}*2^{N-k}$ total kD faces for any integer k with $2 \leq k \leq N-1$. When k=N-1, these are faces.
  The ND volume is the whole puzzle.
- Cubie: Each mini-cube of the Rubik's cube is called a cubie. There are M^N cubies for an M^N Rubik's cube (including the central cubie(s) where applicable).
- Distance: $|g|$ = the minimum twists necessary to obtain state g from the solved state.
- Face: see Cube.
- God's number = max({|g|:all states, g possible via twists alone}). (see Distance)
- Inversion symmetry class: Two sequences of twists, A and B are in the same inversion symmetry class if there exists a symmetry, s such that sAs^{-1} has the same effect as B or B^{-1}. Alternatively, {symmetry class} ∪ {symmetry class of inverse}. (compare: Symmetry class)
- Optimal solution: A solution is optimal if it cannot be solved in fewer twists.
- Priority: The priority order of faces is as follows: Up Down Face Back Right Left. (see Figure 0.1)
- Slice: A twistable subset of an M^N Rubik's cube of size 1×…×1×M×…×M (at least one 1 and at least 2 M's).
- Support: The support of a series of twists is the set of cubies that don't return to the locations and orientations they were in at the start of the sequence of twists.
- Symmetry: A rotation and/or a reflection of the whole puzzle that maps all vertices to vertices, edges to edges, etc. Unless a stated otherwise, a symmetry is not a twist and counts as 0 in the turn metric. Ex. $|F_{all}(g)| = |g|$. There are 24 rotations of a cube—48 symmetries.
- Symmetry class: Two sequences of twists, A and B are in the same symmetry class if there exists a symmetry, s such that sAs^{-1} has the same effect as B.
- Tiles: the new version of stickers.
- Turn Metrics: ATM: A twist is of twistable subset(s) of the puzzle about a common axis.
  - FTM: Face Turn Metric. A twist is of a face. 90-, 180-, and 270-degree twists count as 1.
  - SPTM: A twist is actually a set of twists applied to proper subsets of the puzzle simultaneously.
- U D … or U1 D1 … – 90-degree clockwise twist
- U2 D2 F2 B2 R2 L2 – 180-degree twist
- U3 or U' – 90-degree counterclockwise twist

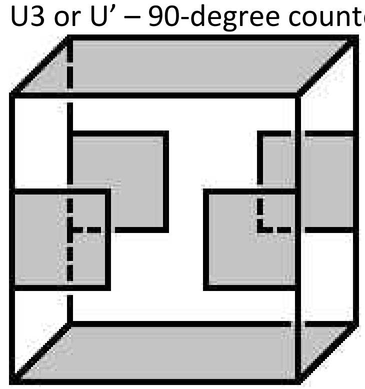

Figure 0.1 – Priority Tile Locations

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I want to thank everybody: Mom, Dad, and the rest of my family including my mom's parents in whose house around 1993 I memorized my first solution method, layer-by-layer; my friends; my teachers in both school and extracurricular activities; Tomas Rokicki, whose general advice on using lookup tables was very helpful to me who, before this, had only written a few small programs outside of two and one third programming courses; Bruce Wade who suggested I move over to C++ from MATLAB (it's much faster now); and last, but not least, my advisor whose hands-off approach was good for my exact situation.

Chapter 1
Introduction

Each face of the Rubik's cube has 9 tiles, but many solvers focus on the cubic pieces. The 1-tile pieces, one in the center of each of the 6 faces, only spin--they never move. Additionally, there are 20 pieces that move: 12 edge pieces or 2-tile pieces which move from edge to edge, as well as 8 corner pieces or 3-tile pieces which move from corner to corner. Knowing this alone does not help novice solvers much. The main difficulty in solving is moving only a few pieces without messing up others. This is done with memorized twist sequences which move the pieces you want while only temporarily messing up other pieces before moving them back to exactly how they were. Many professional human solvers have a whole collection of memorized solving sequences as well as twist sequences that can create pretty patterns, but many solvers from beginner to professional are stunned when they find out that the Rubik's cube can always be solved in 20 face turns (FTM) or less.

One famous pretty pattern is the superflip, a state of the Rubik's cube for which all cubic pieces are in their solved locations, but where the corner pieces are completely solved, all edge pieces are flipped. For example, the orange and yellow edge cubie is correctly between the orange and yellow faces, but its yellow tile is on the orange face and its orange tile is on the yellow face. In 1995, Michael Reid used about 210 cpu hours [1] to prove that the superflip requires 20 face turns to be solved. It was the first state of the cube for which this property was found. It meant that God's number is at least 20 (FTM).

The main result of Cube20 [2] is that in 2010 Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge used 35 years of CPU time donated by Google to determine that 20 face turns suffice to solve any position [3]. That is, God's number is no more than 20. Along with the lower bound found by Michael Reid, this means that God's number is 20 (FTM). This also gives an upper bound of 20 for God's number in SPTM and the slice turn metric (because face turns are allowed in STM and SPTM and count as 1 twist) where a lower bound of 18 is known. (STM allows a single slice at a time to get twisted any amount and count as 1 twist.)

FTM requires only one face can be twisted at a time, but I've enjoyed simultaneous twists since my early cube solving days in the 1990's. If you're good at them (finger placement changes back and forth from 'on the cracks' to 'next to the cracks'), they help mix up the cube faster and solve the cube faster. Thus, the question arose, is it possible to solve every state in fewer than 20 twists if simultaneous turns were allowed?

I used the simple counting method around 2011 to find that God's number for the simultaneously-possible turn metric was at least 14 (the fewest number of twists before a cumulative total of upper bounds of states is greater than the total number of states of the Rubik's Cube, $4.3252*10^{19}$—see Tables 3.2.01 and 3.2.02), and I embarked on programming a new optimal solver.

In the meantime, Tomas Rokicki and Morley Davidson used about 29 years of CPU time at the Ohio Supercomputing Center (ending Aug. 2014) to essentially solve every state of the Rubik's cube in 26 moves or less in the quarter turn metric [4]. (Here twisting faces counts as 1 twist unless you are twisting by 180 degrees in which case it counts as 2 twists.) The lower bound of the superflip + 4-spot requiring 26 twists (QTM) to be solved implies that God's number in the quarter turn metric is 26.

Originally my program was in MATLAB and compatible with both FTM and SPTM. This helped with bug checking and verification. After my proposal hearing followed by my first correspondence with Tomas Rokicki in 2015 I switched to C++. I actually rewrote the program from scratch and used more lookup tables per Tomas Rokicki's advice. The changes made it faster. It was then only compatible with SPTM, but I verified with my MATLAB numbers.

About the start of June 2017, I found states that were at least distance 15 (SPTM) on my computer. By the start of July, I had tacked on a solver and had found a 14-twist solution (SPTM) for the superflip. It solves the superflip in a little over 7 seconds—7.009 seconds which I couldn't believe at first, but my program takes advantage of symmetry, and I verified the twists by hand. In the first 10 days of August, 2017 I had the program running on UWM's Peregrine system and discovered the first 4 states that were precisely distance 16 (SPTM). This was while solving states in 20moves.txt from the symmetric2 page of the Kociemba website [5] (a representative from each inversion symmetry class that exhibits symmetry at distance 20 FTM). I found 5th and 6th states that were distance 16 (SPTM) (distance 18 and 20 FTM) in late September while solving C2va1.txt. Thus, my results have set a lower bound of 16 for God's number in the simultaneously-possible turn metric which is what I set out to do.

1.1 Proofs of What is not Possible

According to [6] the three things that are not possible are well-known [7, 8], but I insert my proofs of here of Theorems 4, 5, and 6 for convenience.

Lemma 1. Any possible twist of an M^N Rubik's cube can be composed with 90-degree twists of 2D twistable subsets of the Rubik's cube.

Note. If M is an odd integer greater than 2, every 90-degree 2D face twist of the M^3 cube performs an odd permutation (4-cycle) of middle-edge cubies and an odd permutation (4-cycle) of corner cubies.

Definition. If N = 3 and a corner cubie's priority tile is one location clockwise from a priority tile location, the cubie is said to have orientation 1. If its priority tile is one location counter-clockwise from a priority tile location, the cubie is said to have orientation 2. (Recall if the priority tile is located on a priority face it has orientation 0, or solved orientation.)

Lemma 2. Let N=3. Every 90-degree face twist preserves the summation of orientation of all the corner cubies (mod 3).

Proof. Note that U, U', D, D', and a 180-degree twist of any face preserve the orientation of each cubie. The remaining possibilities are in Table 1.1.01. Note that each of these possibilities has 2 cubies in each, the middle and right columns.

Table 1.1.01 – Change in Corner Cubie Orientation

	Cubies that stay on the U (or D) face	Cubies that go from the U to the D face (or vice versa)
F, B, R, L	+1 (mod 3)	-1 (mod 3)
F3, B3, R3, L3	-1 (mod 3)	+1 (mod 3)

Lemma 3. For N=3 and an odd integer, M greater than 2, every twist performs an even permutation of middle-edge tiles.

Proof. An arbitrary 90-degree twist performs two four-cycles of middle-edge tiles. Lemma 1 implies an arbitrary twist will perform an even permutation.

Theorem 4. Let M be an odd integer greater than 2. For the M^3 Rubik's cube, any sequence of twists performs an odd permutation of middle-edge cubies if and only if it performs an odd permutation of corner cubies.

Proof. Let A1 be a sequence of twists on a M^3 Rubik's cube. By Lemma 1, there exists a sequence of twists, A2 of 90-degree twists that has the same effect as A1.

The Note implies that each of the 90-degree twists of A2 performs an odd permutation of middle-edge cubies and an odd permutation of corner cubies. Suppose A2 performs n many 90-degree twists. Then A2 performs an odd permutation of middle-edge cubies if and only if n is odd, which happens if and only if A2 performs an odd permutation of corner cubies.

Theorem 5. For the 3^3 Rubik's cube, every sequence of twists preserves the summation of orientation of corner cubies.

Proof. Let A1 be a sequence of twists on a 3^3 Rubik's cube. Each of these twists is either a 90-degree twist or preserves the orientation of each corner cubie, thus preserving the summation. Lemma 2 implies that each of the 90-degree twists of A1 also preserves the summation of orientation of corner cubies.

Theorem 6. For N=3 and an odd integer M greater than 2, every sequence of twists of an M^N Rubik's cube performs an even permutation of middle-edge tiles.

Proof. Let A1 be a sequence of twists of such a M^N Rubik's cube. By Lemma 1, there exists a sequence of twists, A2 of 90-degree twists that has the same effect as A1. Lemma 3 implies that each of the twists of A2 performs an even permutation of middle-edge tiles. Therefore, A2 performs an even permutation of middle-edge tiles.

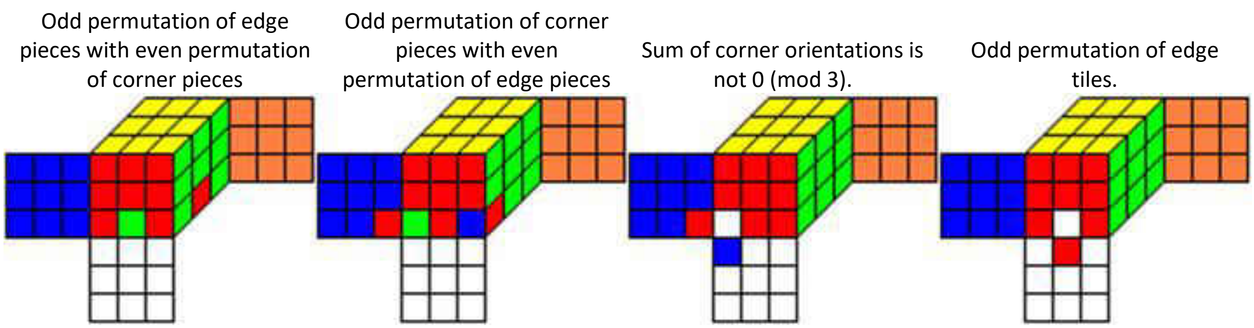

Figure 1.1 – Impossible States

Although one can get from the left-most picture to the left-of-center picture and back, it is not possible to go between any of the other possible pairings of these pictures even if the solved state were included.

The left 2 pictures feature states with all cubies having 0 orientation, but one pair of cubies has been swapped. The right 2 pictures feature states with all cubies having solved location, but the right-most picture has one edge piece with orientation 1 while the right-of-center picture has one corner piece with orientation 2.

Pictures created using [9].

Chapter 2
What is Possible

In 3 dimensions, the axial turn metric (or robot turn metric) is the same as the simultaneously-possible turn metric. However, the term “simultaneously-possible” doesn’t require all movement to be parallel to one plane of rotation.

In 4 dimensions there is room for two proper, twistable subsets to have distinct planes of rotation if the subsets are compatible (either non-intersecting or nested, i.e. one is contained in the other). For example, two non-intersecting 2D faces could be { (w,x,y,z) | $-1 < w < 0$ and $0 < y < 1$ } and { (x,y,z,w) | $0 < w < 1$ and $0 < z < 1$ } as in Figure 2.0. The first set has an x-z plane of rotation and the second has an x-y plane of rotation. These subsets don’t intersect because they’re on opposing 3D W faces.

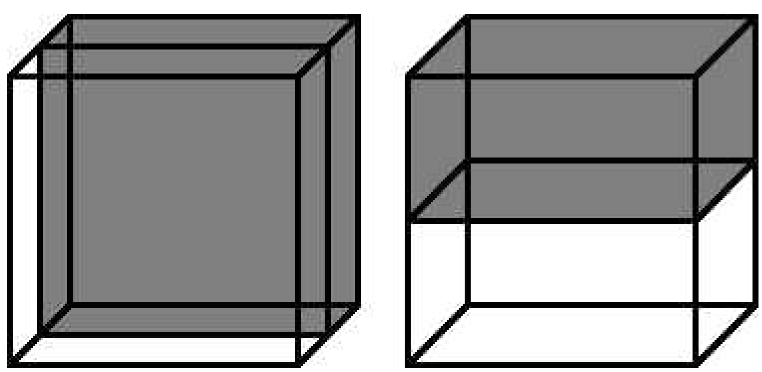

Figure 2.0 – Two Compatible Subsets
At left is the region $-1 < w < 0$. At right is the region $0 < w < 1$.
The top half of each is $0 < z < 1$, and the back half of each is $0 < y < 1$.

An example of two nested subsets in the 2×2×2×2 puzzle would be a 2D face inside a 3D face: { (w,x,y,z) | $0 < w < 1$ and $0 < z < 1$ } ⊂ { (w,x,y,z) | $0 < w < 1$ }. Here the x-y plane of rotation for the 2D subset may not necessarily stay so as it moves while the 3D subset turns.

2.1 Some Turn Metrics on the 2×2×2×2

The 4-dimensional 2×2×2×2 Rubik's cube has twenty-four 2-dimensional faces and eight 3-dimensional faces. I made a spreadsheet to tally the number of states just 1 twist from solved. As you will see, the tallies grow very large quickly for a few metrics. To verify the spreadsheet, I made a MATLAB program that first makes a table to keep track of which pairs of twistable, proper subsets are compatible. It then constructs all possible unions of subsets that are pairwise compatible and performs all possible combinations of twists on them. Finally, it removes duplicates, and the results are shown in Table 2.1.01.

I will lead you in with, in the 2D quarter turn metric, each of the twenty-four 2D faces can move 90-degrees in either direction. Also, all motion is relative to one corner.

Table 2.1.01 – Possible Twists in Some Different Turn Metrics on the 2x2x2x2
These numbers can also represent the number of states that are 1 twist from solved.

Twist Metric	Possible Twists
2D QTM	48
2D FTM	72
2D Axial	378
3D QTM	24
3D Atomic	80
3D FTM	92
QTM	72
Atomic	128
FTM	164
Atomic, Same Angle	140
Same Angle	182
Axial	434
0-intersection	6,194
SPTM	10,082

- FTM stands for face turn metric.
- QTM stands for quarter turn metric (only 90-degree twists).
- Axial turn metric means any motion is about a common axis (The planes of rotation are parallel, but the twist angles can differ for different subsets). The 3D Axial turn metric is not listed because it's the same as the 3D FTM.
- Atomic means only the smallest angle is allowed in each direction (this allows 270-degree twists but restricts the 180-degree twists which are twice a 90-degree twist).
- Same Angle means all subsets being twisted at that time will be twisted by the same angle.

Partial breakdowns of twist counts for the simultaneously-possible turn metric and the 0-intersection turn metric are in the following two tables:

Table 2.1.02 – Counts of possible twists for the 0-intersection turn metric on the 2×2×2×2

First note, the union of two opposing 3D faces is the whole puzzle (also the union of four distinct, parallel 2D faces). Often a category won't appear because it's counted in another category (for example, 'Two 3D faces' is counted in 'One 3D face' because motion is relative to one corner cubie).

Sometimes only some of the twists will appear in a category because the rest were counted in another category. For example, in 'Three 2D faces,' (all) three adjacent: if two of these faces have the same twist, apply the inverse of that twist to the whole puzzle to arrive at a 0-intersection twist performed on two 2D faces.

Such redundancies can also occur when a twist of a 3D face is the same as a twist of an external 2D face (or the inverse of a twist of a nested 2D face).

Also note, any twists involving a 2D face nested inside (outside of) a 3D face can be generated by the internal (external) complement of said 2D face relative to the 3D face.

0-intersection	Unions /sets	Twists per	Twists	
One 2D face:	24	3	**72**	
One 3D face:	4	23	**92**	- 4 not 8 because all motion is relative to one corner cubie.
One 3D face and one 2D face:	24	66	**1584**	- For the 2D faces, no need to count their still-external complement, so 8 3D faces * 3 2D faces = 24 ways. - 66 and not 69: when the 3D subset's twist = twist of 2D face the result can also be generated by a single 2D face twist.
Two 2D faces:				- Two adjacent 2D faces were counted in 'One 3D face and one 2D face.'
opposing	12	7.5	**90**	- "1/2" is in the formula because: same twist applied to opposing 2D faces = inverse twist applied to their complement, another pair of opposing 2D faces.
else	96	9	**864**	- Each 2D face has 8 non-parallel 2D faces to union with, but then each face gets counted twice, so 24*8/2 = 96.
Three 2D faces:				
3 adjacent	6	6	**36**	- Due to complements, there is only one unique union for the four ways to union the xy slices. Six unique twists for this union.
2 adjacent	96	18	**1728**	- Twenty-four possible adjacent pairs * 4 non-adjacent = 96.
0 adjacent	32	27	**864**	- The complement of any union is 2 opposing edges, and each pair of opposing edges (16 pairs) has 2 unique unions.
Four 2D faces:				
4 adjacent			**0**	- Already counted in 'Union of three 2D faces--3 adjacent'
2 pair 2 adjacent	24	36	**864**	- 4 ways to choose two 3D bricks, the first of which has 3 possible 2D faces, the second has 2 remaining choices.
Total			**6194**	

Table 2.1.03 – Counts of possible twists for the SPTM turn metric on the 2×2×2×2

SPTM	Unions /sets	Twists per	Twists	
2D faces:	24	3	**72**	
3D faces:	4	23	**92**	
One 3D face and one 2D face:	24	66	**1584**	Here, you can either count 8 3D faces * 3 2D faces = 24 or 4 3D faces * 6 2D faces = 24 ways. - 66 and not 69: when the 3D subset's twist = twist of external 2D subset (or inverse of nested) the result can also be generated by a single 2D face twist.
Two 2D faces:				- Two adjacent 2D faces were counted in 'One 3D face and one 2D face.'
opposing	12	7.5	**90**	- "1/2" is in the formula because: same twist applied to opposing 2D faces = inverse twist applied to their complement, another pair of opposing 2D faces.
not parallel	96	9	**864**	- Each 2D face has 8 non-parallel 2D faces to union with, but then each face gets counted twice, so 24*8/2 = 96.
One 3D face and two 2D faces, one nested, one not but they are:				- No need for subsection, '2D faces are opposing.' All of those are counted here in 'adjacent.'
adjacent	12	180	**2160**	- 180 not 207: 3D twists parallel to the nested subset's twist are counted in 'Three 2D faces.'
not parallel	24	162	**3888**	- 162 not 180 because of two situations: 3D twist = twist of external 2D face, and 3D twist = twist of external 2D face applied to inverse twist of nested 2D face.
Three 2D faces:				
3 adjacent	6	6	**36**	- If 2 of 3 were the same twist, you could apply the inverse of that twist to the whole puzzle to get two 2D faces
2 adjacent	24	18	**432**	- Alternate count for 0 adjacent: complement of any union here is 2 opposing edges, and each pair of opposing edges has 2 unique unions in its complement (there are 32 edges).
0 adjacent	32	27	**864**	
Total			**10082**	

If you're skeptical of what twists are possible in 4 dimensions and higher, I also include one possible design for a 4D hinge in Figure 2.1. As a visual aid, it also contains 2D slices of what a 3D version of this hinge would look like. The classic design of corner cubies of the Rubik's Cube also extends to 4D and higher.

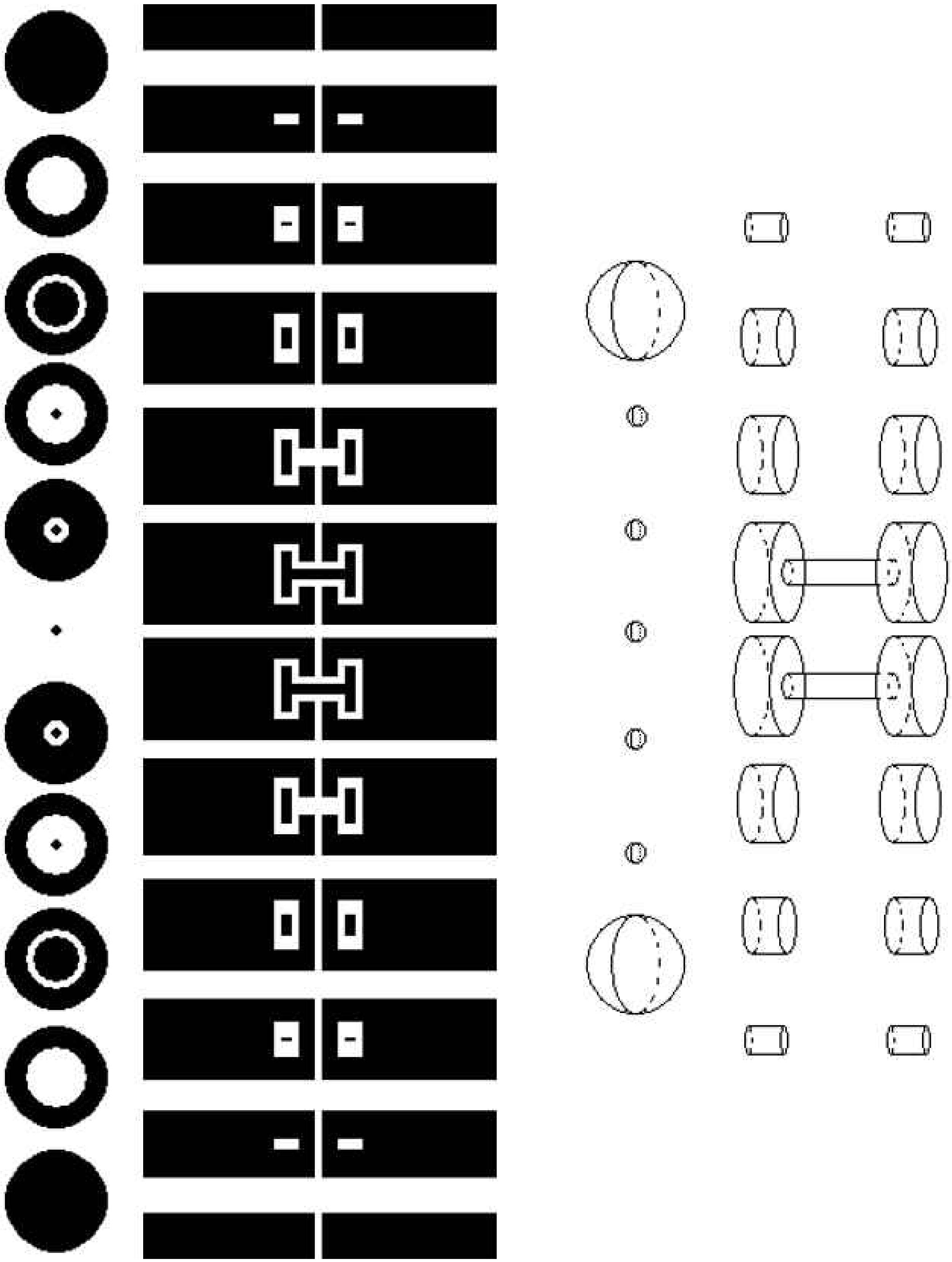

Figure 2.1 – 4D Hinge. yz and xz 2D slices of a 3D hinge (left). wyz and xyz 3D slices of a 4D hinge without enclosure (right).

Chapter 3
The Optimal Solver

3.1 The Parts Common to Other Solvers

I focus on two properties of the 20 moving pieces: location and orientation. The corner locations and corner pieces are numbered 0 through 7 while the edge locations and edge pieces are numbered 0 through 11. Corner piece 0's location is solved when it's in location 0, corner piece 1's location is solved when it's in location 1, etc. This is the same for the edges.

I can now describe applying one set of cubie locations to another. If the solved state is eight corner pieces ordered via 0 1 2 3 4 5 6 7, consider the two states: a = 7 2 3 5 0 6 4 1 and b = 2 0 1 3 7 4 5 6. Backwards notation is used. This comes from my MATLAB version of the program where backwards notation was more convenient for batch application. Thus 7 2 3 5 0 6 4 1 means that piece 7 is in location 0, piece 2 is in location 1, …, and piece 1 is in location 7. Furthermore, b[0] accesses address 0 of b, which is 2. Therefore, b applied to a is

a[b[0]] a[b[1]] a[b[2]] a[b[3]] a[b[4]] a[b[5]] a[b[6]] a[b[7]] =

a[2] a[0] a[1] a[3] a[7] a[4] a[5] a[6] =

3 7 2 5 1 0 6 4

In C++, each is computed separately via

```
*c=a[*b], c[1]=a[b[1]], c[2]=a[b[2]], c[3]=a[b[3]],
c[4]=a[b[4]], c[5]=a[b[5]], c[6]=a[b[6]], c[7]=a[b[7]];
```

If this happens millions of times, consider using constant pointers to b and c via

```
uint8_t b[8], *const b1=b+1, *const b2=b+2, *const b3=b+3, …, *const b7=b+7,
        c[8], *const c1=c+1, *const c2=c+2, *const c3=c+3, …, *const c7=c+7;
//used via
*c=a[*b],*c1=a[*b1],*c2=a[*b2],*c3=a[*b3],*c4=a[*b4],*c5=a[*b5],*c6=a[*b6],*c7=a[*b7];
```

Yet another alternate method that seemed to be fastest for me is defining a uint64_t pointer to the 8 bytes of the uint8_t c.

```
uint64_t *const c_8B = (uint64_t*) c;
//used via
*c_8B = ((uint64_t)((a[*b7]<<24)|(a[*b6]<<16)|(a[*b5]<<8)|a[*b4])<<32)|
        ((a[*b3]<<24)|(a[*b2]<<16)|(a[*b1]<<8)|a[*b]);
```

Now suppose the state corresponding to a has orientations d1 = 0 1 0 2 0 1 0 2 and the state corresponding to b has orientations d2 = 1 2 1 2 2 1 2 1. Reading left to right, d1 means that the piece in location 0 has orientation 0, the piece in location 1 has orientation 1, the piece in location 2 has orientation 0, etc. Still applying b to a, b will move each piece from a and add the orientations together (mod 3). Hence, the resulting orientation is

d1[b[0]]+d2[0] (mod3) d1[b[1]]+d2[1] (mod3) d1[b[2]]+d2[2] (mod3) … d1[b[7]]+d2[7] (mod3)

Two similar stages exist for the edges, although edges only have 2 possible orientations, 0 and 1 so addition mod 2 can be performed with the xor function. Cube20 combines location and orientation into one set of numbers, but in different ways. For each corner, they use the first 3 bits for the corner location and the next 2 bits for orientation. For the edges, they use the first bit for orientation and the next 4 bits for location. The difference between my method and theirs is minimal, at least for corners as this is only used to make lookup tables which are much faster than computing 8 permutations and worth it when the computation is to be performed millions of times. If you have single numbers, A and B representing a and b, usage of lookup tables may look like C = Lookup_Table[A*40320+B] where 8! = 40320.

My optimal solver program uses tables to check if the state in question is the solved state, if it's distance 1 from solved, if it's distance 2 from solved, …, if it's distance 8 from solved, then it checks if it's distance 1 from distance 8, distance 2 from distance 8, etc.

3.2 Testing and Verifying

Along the way to constructing an optimal solver, anything that could be verified needed to be verified. This is because there are a limited number of ways to double-check that a result is optimal. When I originally made my MATLAB program it was compatible with FTM in part so that the number of states at each distance could be verified with known FTM numbers. My program can create a list of all inversion symmetry class representatives at small to medium distances. This was performed, and via computing the total number of states in each representative's inversion symmetry class, the total number of states at each distance was calculated and verified with Cube20's numbers (see Table 3.2.03).

When I switched to C++ the analogous SPTM numbers were verified with my MATLAB SPTM numbers (Table 3.2.04).

When construction of the optimal solver was completed, every state of a list of all 348,938 distance-5 inversion symmetry class representatives was put into the solver. The program solved them in a way that was similar to how it solves high-distance states. Each one was correctly found to be distance 5.

Similarly, every state of a list of all 9,602,778 distance-6, and every 40th state of a list of all 266,133,337 distance-7 inversion symmetry class representatives were put into the solver. Each one was correctly found to be their respective distance.

Table 3.2.01 – FTM Simple Counting Method
18 generators, and then each bound below that is 13.5 times the one above it: 12 twists on perpendicular faces + 3 on the opposing, parallel face get counted twice. From this table we can conclude that God's number for the FTM is at least 18.

FTM	Upper bound For States	Cumulative
0	1	1
1	18	19
2	243	262
3	3280.5	3542.5
4	44286.75	47829.25
5	597871.125	645700.375
6	8071260.188	8716960.56
7	108962012.5	117678973
8	1470987169	1588666142
9	19858326784	2.1447E+10
10	2.6809E+11	2.8953E+11
11	3.6192E+12	3.9087E+12
12	4.8859E+13	5.2768E+13
13	6.5960E+14	7.1236E+14
14	8.9045E+15	9.6169E+15
15	1.2021E+17	1.2983E+17
16	1.6229E+18	1.7527E+18
17	2.1909E+19	2.3661E+19
18	**2.9577E+20**	**3.1943E+20**
19	3.9928E+21	4.3123E+21

Table 3.2.02 – SPTM Simple Counting Method
45 generators, and then each bound below that is 30 times the one above it since twisting the same pair of faces twice in a row has already been counted. It stops when **4.32E+19** is passed. From this table we can conclude that God's number for the SPTM is at least 14.

SPTM	Upper bound For States	Cumulative
0	1	1
1	45	46
2	1350	1396
3	40500	41896
4	1215000	1256896
5	36450000	37706896
6	1093500000	1131206896
7	32805000000	33936206896
8	9.8400E+11	1.0200E+12
9	2.9500E+13	3.0500E+13
10	8.8600E+14	9.1600E+14
11	2.6600E+16	2.7500E+16
12	7.9700E+17	8.2500E+17
13	2.3900E+19	2.4700E+19
14	**7.1700E+20**	**7.4200E+20**
15	2.1500E+22	2.2300E+22

Table 3.2.03 - FTM States
The 'States' column is from Cube20's website [2]. My MATLAB program agreed as far as it reached (4 FTM and 7 FTM with Inversion Symmetry).

FTM	States	Symmetry Classes
0	1	1
1	18	2
2	243	9
3	3240	75
4	43239	934
5	574908	12077
6	7618438	159131
7	100803036	2101575
8	1332343288	

Table 3.2.04 - SPTM States
This is the number of states that can be solved in this many twists (and cannot be solved in fewer). My C++ program has gathered a representative from each inversion symmetry class at SPTM 8.

SPTM	States	Symmetry Classes	Inversion Symmetry Classes
0	1	1	1
1	45	6	6
2	1347	45	27
3	39631	994	578
4	1152290	25382	12912
5	32717804	692436	348938
6	917301226	19189952	9602778
7	25514695958		266133337
8	7.05325E+11		7348986607

Additionally, a verifier program in C++ was created to make sure input states and resultant optimal generators have the same effect on the puzzle. It was used on all resultant states in Section 4.2 and the Appendix, where the input was most of the lists of the symmetric2 page of the Kociemba website [5]. It was also used to verify that the original sequences of twists to get to the 'neighbors' early in Chapter 4 have the same effect on the cube as the resultant optimal generators for the neighbors.

The verifier program itself was tested via changing one small twist (e.g. a U to an F). It correctly reported 1 row failed each time.

Chapter 4
Results

Optimal solutions for two types of states were found: random states, and symmetric states. Of all the states tested, 6 were found to be distance 16 (SPTM), all the rest were distance 15 or less.

Table 4.0.01 – Distance 16 (SPTM) states so far

All six are distance 20 (FTM) except for the fifth which is distance 18 (FTM). They are all symmetric states.
Note, Theorem 5 (Section 1.1) implies: only 3 disoriented corners means they must have the same orientation, and only 2 disoriented corners means they must have differing orientation.

• U1F1U2F2L2U2R3B2U3L3F3R2F2U1R1D3F2B1U1R1	(Superflip + 2 opposing corners disoriented)
• U1F1U1F2D2F2R3D1L3F2D2F2B3L2U2F3D1R3U1B3	(Superflip + 2 adjacent corners disoriented)
• U1F1U1R2U1D1R3F1U3R1L1U3B1R3U1D1R2U1B1D3	(Superflip + 2 same-face [but not adjacent] corners disoriented)
• U1B1U1B2U1L3U2B3F1L3R1B3D3L2D2L1F2L2B1D3	(Superflip + 3 non-adjacent corners disoriented—all 3 adjacent to a common, oriented corner)
• R2U3F2B3U1D3F3R3D1B1D3R3B1U2R3U2F3R2	
• B2R1B1L3F3U1B1R3U1L2D1L3D3F2L1F3R3F1R3U3	

All neighbors of these 6 states (that is, all states that are one twist from these states) were also tested and all were found to be distance 15 (see Table 4.0.02).

Table 4.0.02 – Distance 16 (SPTM) neighbors

The odd lines with a common neighbor start the same except the last face twist or two.
The even, indented lines are optimal generators (all distance 15 SPTM) for the preceding line.
The six chunks here, surrounded by mid-page horizontal lines, correspond with the six states of Table 4.0.01.

F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3R1
D1U2F1R3D3U1L1R2F1B1R3F3D2U3R3F3R1F3B3D2F2B3
F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3R2
D3U2B3U1L2F2B3D1U3L3R2D3U1L3R2U2F2B1L3R2F1B3L2R1F3B2
F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3R3
B1D3U3L2F1B3L2D2U1R3F1L1R2F1D1U1R3D3U1R1D3
F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L1
F3D1U1R2F1B3R2D3U2L1B3L2R3B3D3U3L1D3U1L3U1
F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L1R1
F2B1L3R2D3U3B1D2U1F3B1D2F3B2U3L2R3F1B2L1R3F1R2D1
F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L1R2
D1F2D3L1R2D1F2B1L3F2B2D3U3F1B1D3U3F2B2R3F2B3D3
F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L1R3
F2B1D2U1R2F3L2R1D2U2F1D3L2D1U3F1B2L3R1F3B2D3F2B3
F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L2
D2U1F1D3R2F1B2D1U3L2R1D3U1L2R1D2F3B2L2R1F1B3L3R2F2B1
F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L2R1
F2D2U1L2R2B3L1R2F1B2R3F3B3D2U1F2B3L2R3D3L3R3U1F3B2

F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L2R2 B1D3U3L2R2F3L1R3U1B3D1U1F1B3U1L2R1F1B1D2U3B2D3 F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L2R3 U3B2U1L2R3U3F3B2R1F2B2D1U1F3B3D1U1F2B2L1F1B2U1 F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L3 D2U3B3L1D3U1L2R3F3B3L1B1D1U2L1B1L3F1B1U2F1B2 F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L3R1 F3L2R1F1B1L1R3F3B2D3F1B1L2R3F2R3F2B2D3U2L3R3F2B2D1 F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L3R2 B2D3U2L2R2F1L2R3F2B3L1F1B1D3U2F1B2L1R2U1L1R1D3F2B1 F1B3L2R1U1L1R2F1B3L3R2D1U3F2U1L3D2U2F2B3L1B2D3U2L2R3L3R3 F3B2L2R1D1U1F3D3U2F3B1U2F2B1D1L1R2F2B3L1R3B3L2U3
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3R1 U1F2B1U3L2U1F2B3U3R1F1D1U1F2B2L3R3D3U3L2R2B1 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3R2 D3F2B3L3R2D2L3R3F3B2R2D1U2F3D1U3L2R1D3U3L1F3U3 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3R3 D3F2B3D1L2D3F2B1D1R3F3D3U3F2B2L1R1D1U1L2R2B3 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L1 U1F2B1U3L2U1F2B3U3L1D1F1B1D2U2L3R3F3B3L2R2U1 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L1R1 D2U3R1F2B1L1R1D2L3R2D1U3L1R1F1B1D3U2R1F1B1L1F1B3D2U3 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L1R2 D3F2B3D1L2D3F2B1D1L1R2D3F3B3D2U2L1R1F1B1L2R2U3 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L1R3 F3B2L1R3D3U2F2R1D3L3R2B2D2U3F2B1D3U1F3B2D1U2R2F3B2 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L2 F3B3L1D3F1L3R3F1U3L1F3B3L2F3U3F3B1D3F3 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L2R1 D3F2B3D1L2D3F2B1D1L2R1B3D3U3F2B2L1R1D1U1L2R2F3 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L2R2 F3R3F2B3R2F1U1F1B3D1U1L2F3B2L1R1D1U2R2F1B2D2 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L2R3 U1F2B1U3L2U1F2B3U3L2R3B1D1U1F2B2L3R3D3U3L2R2F1 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L3 D3F2B3D1L2D3F2B1D1L3U3F3B3D2U2L1R1F1B1L2R2D3 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L3R1 F1B2L3R1D2U1F2R3U1L1R2B2D1U2F2B3D3U1F1B2D2U3R2F1B2 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L3R2 U1F2B1U3L2U1F2B3U3L3R2U1F1B1D2U2L3R3F3B3L2R2D1 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3L3R3 D1U2R3F2B3L3R3U2L1R2D1U3L3R3F3B3D2U1R3F3B3L3F3B1D1U2 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3B1 D3L3R3D2U2F1B1L1R1F2B2U3B1L3D2U3L1F2L3D2U1L1 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3B2 U1R1F3D1U1F2B3D3U1R1D3U2B2L2R1F1B1D2F1B2L1R2D1 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3B3 U1L1R1D2U2F3B3L3R3F2B2D1B3L1D1U2L3F2L1D3U2L3 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F1 D3L3R3D2U2F1B1L1R1F2B2U3F1U3L2R3U1F2U3L2R1U1 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F1B1 L3R2D1U3F1D1U1B1L2R3D1U1F1B1L3R1F3B2R2F1B1D2U1B1L3R2 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F1B2 D3L3R3D2U2F1B1L1R1F2B2U3F1B2U1L1R2U3F2U1L3R2U3 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F1B3 D2U3B2L1R2D2U3L3R1D1U2L2R3D2F3B2L3B1U2L3R2F1B3D2U3 D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F2 L1U1L1R3D1L1F2L1R1F3D1L3F1B1L3U1F3L1R1

D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F2B1
D3L3R3D2U2F1B1L1R1F2B2U3F2B1R1D2U1R3F2R1D2U3R3
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F2B2
D2L2R3B2D3U2F3B3L2R1F2D3U3L1R3U3R3B2L1R2B1R1
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F2B3
U1L1R1D2U2F3B3L3R3F2B2D1F2B3R3D3U2R1F2R3D1U2R1
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F3
U1L1R1D2U2F3B3L3R3F2B2D1F3D1L2R1D3F2D1L2R3D3
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F3B1
D1U2B2L3R2D1U2L1R3D2U3L2R1U2F1B2L1B3D2L1R2F3B1D1U2
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F3B2
U1L1R1D2U2F3B3L3R3F2B2D1F3B2D3L3R2D1F2D3L1R2D1
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3F3B3
L1R2D1U3F3D3U3B3L2R1D3U3F3B3L1R3F1B2R2F3B3D3U2B3L1R2
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3D1
B1L1D1U1L2R2F3B3D3U3F2B2R1B3D1L2R1D3F2D1L2R3
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3D2
F3B2D1U2F1B3D2U3F2B1L2D2U3L1R3D2U3L3R1F2B3U2L1B3L2R3
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3D3
L1R1U2F3B2R2B2L3R3F2B1D3U1F1D2U1L3R2D1U1R3F2B2L3
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3U1
L3R3D2F1B2R2B2L1R1F2B3D3U1F3D3U2L1R2D3U3R1F2B2L1
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3U1D1
F1D2L1D3U1L1R2D3U2B3L3R2F2L3R1F2B1R1F3B3D2U3L2R1
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3U1D2
F3B3D2U3L1R3F1L1R1B1D3U2L1R1F1B1D1U3F3B2D2F1B1L2R1B1
D2U3F3B3D2U1F2B3D1U2F1B1L2R3F1B3U1R3D1U3R3F2B1D1U1F2B3L3R3U1D3
L2R3F1B3D2B1R3F2B2D2U3R1U2F3B2L3R2D1U3L1R2B1L2R1
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R2U3F2B3D3U1F3R3D1B1D3R3B1U2R3U2F3R2R1 R3B1L2D3R2B3L2F2B3D3U2B3D2U2L1F1D3B2
R2U3F2B3D3U1F3R3D1B1D3R3B1U2R3U2F3R2R2 L1F1B2D1R3D3U2B2L2D3U1B3L1R1F3D2U1L2R3U3F3B2
R2U3F2B3D3U1F3R3D1B1D3R3B1U2R3U2F3R2R3 F1B3U1B3R3D2B3R2U2L2R3F2B1L3R1F3D3U3L2R1F3B1
R2U3F2B3D3U1F3R3D1B1D3R3B1U2R3U2F3R2L1R1 U3L2R1D2U3L1F1B2L2R1B1D2U3F2B1U2F1D1U1L3R1D1U2F1B1
R2U3F2B3D3U1F3R3D1B1D3R3B1U2R3U2F3R2L1R2 U2R3F1B1D2U1R1D2U1F3B3D1U1L1R3U1B1L2R1F2L2R2D3U2
R2U3F2B3D3U1F3R3D1B1D3R3B1U2R3U2F3R2L1R3 B3D2U3R2D1L2U1L3R2F2B3R3F2B1L2B3L2F3U1
R2U3F2B3D3U1F3R3D1B1D3R3B1U2R3U2F3R2L2R2 F1B2D3L2R1D3U3F1D2U2L3R3F3B3L2R1F3U1F1B2L1R1B2D3U2
R2U3F2B3D3U1F3R3D1B1D3R3B1U2R3U2F3R2L2R3 D3U1F1L2R3B2L2F1L3D1U1R3F1B1L2R3D3F3B2R1F1
R2U3F2B3D3U1F3R3D1B1D3R3B1U2R3U2F3R2L3R3 L3F1L3R3F2B3D1F1L1D2F1B1U1L1R2F1D2U1F3D3U3

F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1U1 L2R3F3B3D1U1L3R2D2U3L2U2F3B3R1D1R2D1U1F3B2L1R1F3B1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1U2 F2D1U3F1U3L2R2F2B1D3R2F1B1R3F3B2L3R2D2U1R3F3B3
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1D1 F1B3R2U1F3B3L2F2B3L2R3D2R2F3B3U1F1B2D3U3L1R1F2B1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1D1U1 F2B3U3B2L3R2U2L1R2F3B2D2B3U1L2R3F1B1L1R2D3U2L2R3
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1D1U2 F1B2L2R1B3D1U3F2B1D3U3L3D3U2B1L1D3U3F1B1D1U2F3B2L1R3
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1D1U3 F2B2R1F2B1D2U1L2F3B2D1L3D3U2L3R2F3D2L3R1D1B1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1D2 F1B3R3F1B2D1U1L3R3F2B1L3R2F3B3D2R2D1F3U2F3B3L2R1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1D2U1 B2D3U2L1R1D2U1B3R3F2B3D2U1F2R1D1F3B1L1R3D1F1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1D2U2 L3R2U3R1D2U2B1R3F1B2L3B1D1B1U3L2D1U2R1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1R1 R3F3B3D3L3R3B1U1L3R1D3U2F3B1U3F3U3B1L1R1D2U1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1R2 R1F1L3R1F2D2U1F3L1R1D2U1L3R2D2U3L3R3D1B3L3R1D1U1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1R3 F2B2D2U3F2B2L1F1D2L2R1F3B2U3L2D1U2F2B3L1R2U2B1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1L1R1 D3U3B1L3U3L3R2F2B3D2F2L1R1F1B1D3L3R1D1L1D1U2
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1L1R2 L1R1F1B1L3R1U3F1L3R3F2B3L3R2F2B1L1R1D3F2B1D2L3R1D1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1L1R3 D1U3F1L1B1L1U2F2B1D3U2L3D3F2B1L1R1F3B2D2F2B1
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1L2R2 F3L2R3D3U2B2L1R2U3B1D2U1F2B1L1U2F3B1U3R3D3U1

```
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1L2R3
          D3U1L3D2U3L1R1F1B2U3L3D1L1R2D2U1R3F3B3L1R1F3B2D2
F2B3L2R3D1B3U1F3U3F2B2D1U3L2F2B1R3B3U3F3B2L2R1L3R3
          B1U2L3R2F3D3U2B1R2B2D2U1F1B2L1R1F2R2F2B3U3
```

4.1 Random State Results

2000 random states were gathered in C++ using a time seed in srand() and then rand(). They were solved on 2 computers using message passing with an average time of 83 seconds per solve (or an estimated 166 seconds/solve had only one computer been solving it). The resulting distances are in Table 4.1.01.

Table 4.1.01 – Distances of 2000 Random States

SPTM distance	Number of States
11	1
12	15
13	414
14	1546
15	24

4.2 Symmetric State Results

The main type of state solved was symmetric states. This was because Tomas Rokicki pointed out, my solver seemed to solve symmetric states faster. Also, he knows they are at larger distances than random states [10]. All available lists of symmetry class representatives from the symmetric2 page of the Kociemba website [5] were solved except the largest, C_3 (note that the 5 lists of symmetric states larger than C_3 were not available for download and were not tested). I use light parentheses for only Table 4.2.01. For the 9 shortest lists, see Appendix for optimal generators to these class representatives in each, FTM [5] and SPTM side by side.

Table 4.2.00 – SPTM Distances of 20moves.txt

All 32,625 inversion symmetry classes, 46,514 symmetry classes, and 1,091,994 states that are distance 20 (FTM) and exhibit symmetry.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
				<11	0	0	0
				11	7	8	192
				12	176	197	4510
				13	1592	2078	46408
				14	17656	26117	615820
<20	0	0	0	15	13189	18109	424980
20	32625	46514	1091994	16	5	5	84

Table 4.2.01 – Symmetry Type O_h

All 4 inversion symmetry classes, 4 symmetry classes, and 4 states.

In FTM the solved state is 0 twists from solved. Pons Asinorum is 6 twists: B2 F2 L2 R2 D2 U2 Pons Asinorum + superflip is 19 twists: R B L F D L R' B2 F D' U L D' U R' D' F' R' U' The superflip is 20 twists: D' R2 F' D2 F2 U2 L' R D' R2 B F R' U2 L' F2 R' U2 R' U'
In SPTM the solved state is still 0 twists from solved. Pons Asinorum is 3 twists: (D2U2)(F2B2)(L2R2) Pons Asinorum + superflip is 13: B2(L3R2)(F1B3)(L1R3)B1(L1R1)U3B2(L1R3)U2B3(D1U1)(L1R1) The superflip is 14 twists: D2(F2B1)(L3R3)(D1U1)F2D1(L3R1)(D3U1)R1(F3B3)(L2R2)D3B2(L3R1)

Table 4.2.02 – Symmetry Type T_h

All 8 inversion symmetry classes, 10 symmetry classes, and 20 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<8	0	0	0	<6	0	0	0
8	1	1	2	6	1	1	2
9	0	0	0	7	0	0	0
10	0	0	0	8	0	0	0
11	0	0	0	9	0	0	0
12	0	0	0	10	1	1	2
13	0	0	0	11	2	3	6
14	0	0	0	12	0	0	0
15	0	0	0	13	3	4	8
16	3	4	8	14	1	1	2
17	1	1	2				
18	0	0	0				
19	0	0	0				
20	3	4	8				

Table 4.2.03 – Symmetry Type T

All 12 inversion symmetry classes, 12 symmetry classes, and 48 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<12	0	0	0	<6	0	0	0
12	2	2	8	6	2	2	8
13	0	0	0	7	0	0	0
14	0	0	0	8	0	0	0
15	0	0	0	9	0	0	0
16	0	0	0	10	0	0	0
17	2	2	8	11	2	2	8
18	5	5	20	12	3	3	12
19	3	3	12	13	5	5	20

Table 4.2.04 – Symmetry Type D_{3d}

All 12 inversion symmetry classes, 12 symmetry classes, and 48 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<14	0	0	0	<11	0	0	0
14	1	1	4	11	1	1	4
15	0	0	0	12	1	1	4
16	0	0	0	13	3	3	12
17	3	3	12	14	7	7	28
18	4	4	16				
19	4	4	16				

Table 4.2.05 – Symmetry Type C_{3v}

All 16 inversion symmetry classes, 16 symmetry classes, and 128 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<13	0	0	0	<10	0	0	0
13	2	2	16	10	1	1	8
14	1	1	8	11	2	2	16
15	3	3	24	12	3	3	24
16	2	2	16	13	7	7	56
17	2	2	16	14	3	3	24
18	4	4	32				
19	2	2	16				

Table 4.2.06 – Symmetry Type D_3

All 160 inversion symmetry classes, 208 symmetry classes, and 1,664 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<11	0	0	0	<8	0	0	0
11	1	1	8	8	1	1	8
12	2	2	16	9	1	1	8
13	0	0	0	10	2	2	16
14	5	6	48	11	5	8	64
15	5	5	40	12	10	11	88
16	11	14	112	13	34	43	344
17	18	26	208	14	78	108	864
18	62	85	680	15	29	34	272
19	52	65	520				
20	4	4	32				

Table 4.2.07 – Symmetry Type S_6

All 2,610 inversion symmetry classes, 3,870 symmetry classes, and 30,960 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<8	0	0	0	<4	0	0	0
8	1	1	8	4	1	1	8
9	2	3	24	5	1	1	8
10	2	2	16	6	0	0	0
11	1	2	16	7	2	3	24
12	8	11	88	8	8	11	88
13	0	0	0	9	0	0	0
14	10	12	96	10	7	7	56
15	16	22	176	11	15	22	176
16	106	158	1264	12	114	152	1216
17	261	397	3176	13	379	557	4456
18	979	1476	11808	14	1198	1830	14640
19	1143	1698	13584	15	884	1285	10280
20	81	88	704	16	1	1	8

Table 4.2.08 – Symmetry Type C_3
All 497,882 inversion symmetry classes,
942,716 symmetry classes, and 15,083,456 states.

FTM distance	I. s. classes	Symmetry classes	States
<9	0	0	0
9	3	3	48
10	5	6	96
11	3	5	80
12	28	32	512
13	46	71	1136
14	260	418	6688
15	1238	2216	35456
16	9105	16705	267280
17	62806	120050	1920800
18	271003	520076	8321216
19	152809	282475	4519600
20	576	659	10544

A vast majority of the SPTM results for the 942,716 symmetry classes of C_3 remain unchecked so only FTM results appear here.

Table 4.2.09 – Symmetry Type D_{4h}
All 124 inversion symmetry classes, 124 symmetry classes, and 372 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<2	0	0	0	0	0	0	0
2	1	1	3	1	1	1	3
3	0	0	0	2	1	1	3
4	1	1	3	3	0	0	0
5	0	0	0	4	6	6	18
6	3	3	9	5	4	4	12
7	0	0	0	6	6	6	18
8	6	6	18	7	4	4	12
9	1	1	3	8	14	14	42
10	2	2	6	9	10	10	30
11	0	0	0	10	8	8	24
12	6	6	18	11	16	16	48
13	14	14	42	12	28	28	84
14	18	18	54	13	25	25	75
15	3	3	9	14	1	1	3
16	19	19	57				
17	20	20	60				
18	19	19	57				
19	10	10	30				
20	1	1	3				

Table 4.2.10 – Symmetry Type D_4

All 176 inversion symmetry classes, 192 symmetry classes, and 1,152 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<2	0	0	0	0	0	0	0
2	1	1	6	1	1	1	6
3	0	0	0	2	0	0	0
4	0	0	0	3	1	2	12
5	0	0	0	4	6	7	42
6	3	4	24	5	5	5	30
7	4	5	30	6	11	11	66
8	5	5	30	7	18	19	114
9	1	1	6	8	24	27	162
10	3	3	18	9	15	17	102
11	5	5	30	10	6	6	36
12	27	30	180	11	1	1	6
13	21	24	144	12	56	64	384
14	8	8	48	13	32	32	192
15	12	12	72				
16	16	20	120				
17	34	38	228				
18	22	22	132				
19	14	14	84				

Table 4.2.11 – Symmetry Type C_{4v}

All 400 inversion symmetry classes, 448 symmetry classes, and 2,688 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
0	0	0	0	0	0	0	0
1	1	1	6	1	1	1	6
2	0	0	0	2	0	0	0
3	0	0	0	3	1	2	12
4	0	0	0	4	2	3	18
5	1	2	12	5	2	2	12
6	0	0	0	6	8	8	48
7	1	1	6	7	44	46	276
8	2	3	18	8	44	54	324
9	1	1	6	9	38	48	288
10	6	6	36	10	91	103	618
11	10	10	60	11	81	85	510
12	32	37	222	12	42	50	300
13	57	70	420	13	46	46	276
14	63	67	402				
15	39	49	294				
16	57	61	366				
17	75	83	498				
18	51	53	318				
19	4	4	24				

Table 4.2.12 – Symmetry Type C_{4h}

All 608 inversion symmetry classes, 704 symmetry classes, and 4,224 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<2	0	0	0	0	0	0	0
2	1	1	6	1	1	1	6
3	0	0	0	2	0	0	0
4	0	0	0	3	3	3	18
5	0	0	0	4	1	1	6
6	3	3	18	5	1	1	6
7	1	1	6	6	2	2	12
8	3	3	18	7	14	14	84
9	0	0	0	8	38	38	228
10	0	0	0	9	22	22	132
11	10	10	60	10	15	16	96
12	27	28	168	11	14	22	132
13	35	35	210	12	148	159	954
14	15	20	120	13	247	283	1698
15	29	34	204	14	94	126	756
16	76	87	522	15	8	16	96
17	86	95	570				
18	162	184	1104				
19	132	171	1026				
20	28	32	192				

Table 4.2.13 – Symmetry Type C_4

All 21,664 inversion symmetry classes, 36,160 symmetry classes, and 433,920 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
0	0	0	0	0	0	0	0
1	1	1	12	1	2	2	24
2	1	1	12	2	0	0	0
3	0	0	0	3	2	4	48
4	0	0	0	4	8	12	144
5	1	2	24	5	6	8	96
6	5	8	96	6	16	17	204
7	5	7	84	7	280	377	4524
8	5	7	84	8	685	1115	13380
9	7	7	84	9	1407	2300	27600
10	41	42	504	10	3173	5325	63900
11	131	177	2124	11	6413	11303	135636
12	267	381	4572	12	6750	11564	138768
13	389	583	6996	13	2569	3684	44208
14	674	958	11496	14	341	431	5172
15	1507	2302	27624	15	12	18	216
16	3148	5362	64344				
17	6475	11461	137532				
18	7118	12066	144792				
19	1852	2756	33072				
20	37	39	468				

Table 4.2.14 – Symmetry Type S_4

All 61,840 inversion symmetry classes, 109,376 symmetry classes, and 1,312,512 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<7	0	0	0	<5	0	0	0
7	7	7	84	5	22	22	264
8	16	16	192	6	34	45	540
9	19	25	300	7	141	180	2160
10	40	48	576	8	580	838	10056
11	90	110	1320	9	2016	3231	38772
12	210	256	3072	10	6214	10879	130548
13	697	937	11244	11	18509	33296	399552
14	1283	1944	23328	12	23682	43595	523140
15	3011	4965	59580	13	9441	15803	189636
16	7588	12900	154800	14	1189	1474	17688
17	17201	31218	374616	15	12	13	156
18	24049	43854	526248				
19	7554	13014	156168				
20	75	82	984				

Table 4.2.15 – Symmetry Type D_{2d}(edge)

All 1,152 inversion symmetry classes, 1,472 symmetry classes, and 8,832 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<7	0	0	0	<4	0	0	0
7	3	3	18	4	1	1	6
8	8	8	48	5	13	14	84
9	11	13	78	6	15	20	120
10	6	10	60	7	41	45	270
11	18	22	132	8	62	66	396
12	39	41	246	9	72	82	492
13	52	57	342	10	195	237	1422
14	98	103	618	11	190	239	1434
15	93	103	618	12	219	310	1860
16	161	208	1248	13	280	367	2202
17	192	242	1452	14	64	91	546
18	288	389	2334				
19	172	256	1536				
20	11	17	102				

Table 4.2.16 – Symmetry Type D_{2d}(face)

All 176 inversion symmetry classes, 192 symmetry classes, and 1,152 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<8	0	0	0	<4	0	0	0
8	1	1	6	4	1	1	6
9	0	0	0	5	1	2	12
10	9	11	66	6	13	14	84
11	9	9	54	7	17	17	102
12	27	28	168	8	32	35	210
13	19	22	132	9	17	20	120
14	16	18	108	10	6	6	36
15	5	5	30	11	5	7	42
16	12	14	84	12	39	45	270
17	24	28	168	13	39	39	234
18	42	44	264	14	6	6	36
19	12	12	72				

Table 4.2.17 – Symmetry Type D_{2h}(edge)

All 960 inversion symmetry classes, 960 symmetry classes, and 5,760 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<7	0	0	0	<4	0	0	0
7	2	2	12	4	1	1	6
8	4	4	24	5	7	7	42
9	1	1	6	6	1	1	6
10	11	11	66	7	35	35	210
11	28	28	168	8	60	60	360
12	42	42	252	9	94	94	564
13	57	57	342	10	30	30	180
14	54	54	324	11	52	52	312
15	71	71	426	12	201	201	1206
16	106	106	636	13	305	305	1830
17	254	254	1524	14	169	169	1014
18	218	218	1308	15	5	5	30
19	107	107	642				
20	5	5	30				

Table 4.2.18 – Symmetry Type D_{2h}(face)

All 1,672 inversion symmetry classes, 1,982 symmetry classes, and 11,892 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<4	0	0	0	<3	0	0	0
4	2	2	12	3	6	6	36
5	0	0	0	4	6	6	36
6	11	11	66	5	15	15	90
7	0	0	0	6	13	13	78
8	20	20	120	7	63	63	378
9	0	0	0	8	220	220	1320
10	2	2	12	9	148	148	888
11	0	0	0	10	187	200	1200
12	45	45	270	11	196	279	1674
13	106	106	636	12	400	459	2754
14	375	378	2268	13	341	435	2610
15	164	172	1032	14	77	138	828
16	137	161	966				
17	244	252	1512				
18	406	568	3408				
19	143	240	1440				
20	17	25	150				

Table 4.2.19 – Symmetry Type D_2(edge)

All 16,720 inversion symmetry classes, 23,232 symmetry classes, and 278,784 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<6	0	0	0	<5	0	0	0
6	4	4	48	5	28	32	384
7	6	6	72	6	71	89	1068
8	18	22	264	7	271	347	4164
9	9	10	120	8	470	608	7296
10	91	111	1332	9	1156	1439	17268
11	172	230	2760	10	1310	1727	20724
12	304	403	4836	11	1610	2394	28728
13	373	461	5532	12	2812	3864	46368
14	861	1064	12768	13	5662	7717	92604
15	1111	1396	16752	14	3231	4854	58248
16	2051	2667	32004	15	99	161	1932
17	3980	5397	64764				
18	5922	8736	104832				
19	1807	2713	32556				
20	11	12	144				

Table 4.2.20 – Symmetry Type D_2(face)

All 13,752 inversion symmetry classes, 23,356 symmetry classes, and 280,272 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<4	0	0	0	<2	0	0	0
4	3	4	48	2	3	4	48
5	1	2	24	3	15	22	264
6	18	27	324	4	65	102	1224
7	10	18	216	5	204	336	4032
8	98	152	1824	6	477	773	9276
9	79	132	1584	7	980	1683	20196
10	321	515	6180	8	1598	2686	32232
11	387	641	7692	9	1385	2236	26832
12	881	1428	17136	10	1040	1798	21576
13	897	1608	19296	11	2262	4056	48672
14	1257	2185	26220	12	3657	6263	75156
15	1135	1850	22200	13	1994	3276	39312
16	2207	3708	44496	14	72	121	1452
17	3104	5370	64440				
18	2601	4456	53472				
19	749	1256	15072				
20	4	4	48				

Table 4.2.21 – Symmetry Type C_{2v}(a1)

All 12,352 inversion symmetry classes, 15,552 symmetry classes, and 186,624 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<7	0	0	0	<5	0	0	0
7	1	1	12	5	2	2	24
8	1	1	12	6	2	2	24
9	1	1	12	7	65	73	876
10	20	22	264	8	205	255	3060
11	73	92	1104	9	826	1092	13104
12	107	134	1608	10	890	1086	13032
13	102	130	1560	11	998	1258	15096
14	554	704	8448	12	1272	1593	19116
15	993	1249	14988	13	1913	2390	28680
16	1824	2347	28164	14	4372	5595	67140
17	2933	3735	44820	15	1805	2204	26448
18	4174	5340	64080	16	2	2	24
19	1530	1757	21084				
20	39	39	468				

Table 4.2.22 – Symmetry Type C_{2v}(a2)

All 175,376 inversion symmetry classes, 290,880 symmetry classes, and 3,490,560 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<3	0	0	0	<2	0	0	0
3	1	2	24	2	1	2	24
4	0	0	0	3	3	4	48
5	6	8	96	4	13	21	252
6	3	4	48	5	21	30	360
7	21	33	396	6	58	93	1116
8	13	17	204	7	365	503	6036
9	26	34	408	8	980	1362	16344
10	86	111	1332	9	2261	2957	35484
11	203	280	3360	10	4694	6551	78612
12	715	960	11520	11	9946	15378	184536
13	1551	2080	24960	12	29270	48006	576072
14	3313	4549	54588	13	45967	74514	894168
15	5542	7712	92544	14	57690	100192	1202304
16	14838	22012	264144	15	24107	41267	495204
17	34551	55305	663660				
18	72962	125363	1504356				
19	41100	71764	861168				
20	445	646	7752				

Table 4.2.23 – Symmetry Type C_{2v}(b)

All 36,864 inversion symmetry classes, 48,128 symmetry classes, and 577,536 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<6	0	0	0	<5	0	0	0
6	3	3	36	5	6	6	72
7	1	1	12	6	4	4	48
8	3	3	36	7	66	66	792
9	18	18	216	8	131	131	1572
10	34	34	408	9	468	468	5616
11	58	58	696	10	881	916	10992
12	180	183	2196	11	2104	2427	29124
13	289	293	3516	12	3646	4824	57888
14	622	671	8052	13	8723	11964	143568
15	1292	1491	17892	14	16171	21668	260016
16	3625	4381	52572	15	4664	5654	67848
17	7840	10329	123948				
18	15168	20873	250476				
19	7638	9696	116352				
20	93	94	1128				

The symmetry classes in the C_{2v}(b) list took 19 days, 11 hours, 10 minutes to solve. Subtracting the 10-minute startup time, this averages to 34.9 seconds/solve. Had it had no symmetry instead of a symmetry factor of 4 and only had one computer working on it instead of 2, it would have been an estimated 280 seconds/solve. Their average distance was 17.59 (FTM), 13.36 (SPTM), or the breakdown at the end of the previous page.

Table 4.2.24 – Symmetry Type C_{2h}(a)

All 90,864 inversion symmetry classes, 143,552 symmetry classes, and 1,722,624 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<4	0	0	0	<2	0	0	0
4	1	2	24	2	1	2	24
5	1	2	24	3	1	2	24
6	4	5	60	4	14	17	204
7	7	7	84	5	31	35	420
8	35	42	504	6	78	103	1236
9	29	33	396	7	377	457	5484
10	89	115	1380	8	749	922	11064
11	327	423	5076	9	1325	1614	19368
12	521	622	7464	10	2855	3691	44292
13	1043	1285	15420	11	7118	10357	124284
14	1997	2489	29868	12	14940	22618	271416
15	3566	4636	55632	13	16812	25741	308892
16	8057	11153	133836	14	33660	55953	671436
17	17388	26636	319632	15	12903	22040	264480
18	35586	58874	706488				
19	21920	36878	442536				
20	293	350	4200				

Table 4.2.25 – Symmetry Type C_{2h}(b)

All 36,852 inversion symmetry classes, 48,116 symmetry classes, and 577,392 states.

FTM distance	I. s. classes	Symmetry classes	States	SPTM distance	I. s. classes	Symmetry classes	States
<6	0	0	0	<4	0	0	0
6	1	1	12	4	2	2	24
7	1	1	12	5	8	8	96
8	7	7	84	6	2	2	24
9	10	10	120	7	92	92	1104
10	33	33	396	8	212	212	2544
11	72	72	864	9	473	473	5676
12	191	192	2304	10	902	918	11016
13	323	329	3948	11	1844	2215	26580
14	634	671	8052	12	4069	5422	65064
15	1471	1720	20640	13	9982	13643	163716
16	3496	4360	52320	14	15563	20708	248496
17	7953	10513	126156	15	3703	4421	53052
18	15207	20776	249312				
19	7384	9360	112320				
20	69	71	852				

The symmetry classes in the C_{2h}(b) list took 15 days, 16 hours 58 minutes. Subtracting the 10-minute startup time, this averages to 28.2 seconds/solve. Had it had no symmetry instead of a symmetry factor of 4 and only had one computer working on it instead of 2, this would have been an estimated 226 seconds/solve. Their average distance was 17.56 (FTM), 13.28 (SPTM), or the breakdown above.

God's number in the simultaneously-possible turn metric is at least 16.

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APPENDIX
The 9 Shortest Lists of Symmetric Types

(Optimal FTM solutions on left [5]) (An optimal SPTM solution to the same state on right)

Symmetry Type O_h

(0f*)	(0sp*)
B2F2L2R2D2U2 (6f*)	B2F2L2R2D2U2 (3sp*)
RBLFDLR'B2FD'ULD'UR'D'F'R'U' (19f*)	B2L3R2F1B3L1R3B1L1R1U3B2L1R3U2B3D1U1L1R1 (13sp*)
D'R2F'D2F2U2L'RD'R2BFR'U2L'F2R'U2R'U' (20f*)	D2F2B1L3R3D1U1F2D1L3R1D3U1R1F3B3L2R2D3B2L3R1 (14sp*)

Symmetry Type T_h

U2L2F2D2U2F2R2U2	D1U1B2D2U2L2R2F2D3U3
UL'B'L'RUFL'RU'R'D'UB'F2RU'	D3U1L1R2U2L3R3F1B1D3U2B2L3R1D2F2B3L3R3D2U1L3R1
B2LR2B2F2D2U2R'F2DU2B2F2L2R2U'	F1B2L2R2D2U2B3U2L2R1F2B2D2U2L3D2
B2U'R2BFR2DF2D2BR2DUR2F'U2	F2L2R1D2U2F2B2L3F2D3F2B2L2R2U3L2R2
BU'RD2B'D2B'RB'D2FURFLB2D'U'R'U'	L3R3F1B1L1R1F1D1L2R3D1U3B3D3U1F1B2D1U3R1D2U1F3B2
B'FULU2BL2B2R2D'L'RDL2F2R2F'U2R'U'	L1R2F1B1R1D1U1L3R2D1U1L3R2F1B1D2U2R1D1U1R3F3B3
R2D2RB2DUF2RU2R2D'F2L'R'F2U'	D1U2L2R2F2B2U3L3R3F2B2D2U3F2B2L2R2D1L1R1
LF2DUF2RU2L2U'F2L'R'B2U'R2U2	F3B3D2U1L2R2F2B2D3L2R2F1B1U1F2B2L2R2D3U2
B2R'B2FUL2F2LF'DBF2L2F2RUL2B'R'U'	R3D3U3R3D2U2F3B3L3R2D3U3L3D3U3L2R3F3B3L3R2F1B1
UF2R2F2DU2BU'L'RF'U'F2L'D2B'R2DF2R'	L3R3F2B1L1R1F3D1U1F2B1D1U1B1L1R1D2U2F3D1U1F3B2

Symmetry Type T

BFL'R'BFD'U'LRD'U'	D1U1F3B3L1R1F3B3L1R1D3U3
R2FLR'BFD'L'R'D'F2LD'U'L'F2U'	F1L3R2D3U1L3R1D3F2B1D1U3F3B1D3U2L2R2F1B2
DULB'FL'RB2F'R2DULU2B'FR2U'	D1B2L3R1D2F3D3U3B2L1R2F1B3L3R1F3D3U3
B2L'R2D2B'L'R'BD2L'B'F'L'DUBF'U'	L3D2U2F2B2L1F1B1D1U1F3B3D3U3L2R3F2B2D2U2R3
R'ULR'B2D2U2R'BLFRUFRU'F'R'U'	L2R1F3B3L3D3U3L3R2D2U2L1F1B1L2R1F2B2D3U3F2B2L1R2
F'D'F2L2B2R2UL'D'U'L'D'U2B2R2U'F'D'U'	L3D3U3R3F3B3R3F1B1D3U3L3F3B3L2R3D3U3L2R1F2B2
BFLRB'F'D'U'LRDU	D1U1F1B1L3R3F3B3L1R1D1U1
DBLDLURFLUF2RF'DU'RU	D2U3F2B2L3R3U3L1R1D1F1B1D2U3L1R1D2U1F3B3D1F1B1
UB2F2L2R2UBFU2LRU'B2F2U2L2R2U'	D3U3L3R3D3U3F2B1D2U2L2R2B1L3R3F2B1L2R2D2U2B1
D'B'F'D'LRD'B2F2L2R2U'BFU'L'R'U'	D3U3L3R2F2B2D2U2L3F1B2D1U3F2B2D3U1B1D3U3
D'U'L'D'UR2FU2BD2FU2L'RUR2F2U'	B2R2U2F1B1D2U3F2B2L1R1F3D3U1F3B1D2U3L1R1F3
BDRFRD2URB2U'F'LUB'R2B'U'R'U	D1L3R3D1F1B1D1L1R1U1L2R2F1B1U3F3B3D2U3L3R3

Symmetry Type D_{3d}

ULDUL'D'U'RB2U2B2L'R'U'	L1R2F1B1U2R2U2F3L3R3B1L1R1B3L3R2
U2L2R2F'R'UBR'DB2F2L'B'DRB'U'	B1L3R2F2L2R1B3U3L3D2L1R3D1U1F1B2D2R2D2U1
L2B'UB'D'BFD2F'U'BL2U'F2U2R2U'	L1R2F1B1U2R2U2F3L3R3B1L1R1B3L1F2B2D2U2
D2B'U2LB2D2UF2DB2R'B2UBD'F2U'	L1R2U1L1R3B3D3U2L2R3D2U3L3R1U2R1F3B3L2R3U3
DFR'U2B2U2R'U'LR'BU'R2F2R2U'R'U'	F1B3D1U1R3D2U1F2B1D2U1F1B2D3U2F2B3D2U3F2B3L1F3B3D3U1
ULUB'U'LR2U2BU'L'U'BF2R2B'F2U'	L2R1D3U2F3B3L2R2D1U2R1D2U2F1B1L3R3U2F3B2L1R1D1L2R3
F'D'RU2BFL2B2R2D2LD'UR2U2RB'U'	F1B2D2U2R1D3U3F2B2L1F2B1L3R1F1B3R3D2U3F2B3
DBD'LUBF'R'F'LBLR'D'F'UR'U'	F2B3D2U3F2B1L3R1F2B1D3U2B1U3F1B2D2U3L3R1D2U3F2B1D2U1
R'B'U'B2R'B'D'ULUB2FD2U'LUFR'U'	L1R3F3D1U2F1L1R3D2U1F2B1L3R2F3B1D3F3B2D2U3F1B3D3U2
LR2D'BF2L2R'B'D2R2D'U'R'D2BDU'F'U'	D3U2L1F2B3D2U1B1L3R1U1L3R1B2L3R3B1L3R1F1R1
D'U2B2L2R2B'RDU'LR'DB2F2RB'F'RU'	F3B2D1U2L1R1D2U3F2B2D1U1B3D1U3F1B3D3L2R1D2U2B3
FL'B2F2UB2F'DRD2B2UF2LBFD'R'U'	D2U2L3R2F1B2D1U1F3L2F3B1L1R2F1B2L2R1U1F1B3L3B1

Symmetry Type C_{3v}

DUR2BD2U2L'D2U2RB'R2D'U'	L3R1D3F3B1D2U1L1R3D1U3L3R2F3B1R1D3U1
URU2BFL'B2D2B2RF2RD2F'D'	D2U2L2R2F1D1U1L2F2B1D2L1U2F1B2R2D1U1F1B2
L2R2F2L2U'BF'LD'LDFD'BD'U2	D2U1R3U1L3U3F3U1F3L1R3D1F2L2F2B2
U2R'FDF2R'DU'BR2U'B'D2U2RU2	L2R2F3B2U1B1L2R1F2B1L3R1D2U3B1L3U3F2B1
D'RDB'L'D'R'D'F'LUB2F2D'L'R'U'	U2B3D3U1R1D3U2F3U1L1R3F3B2R3U2B3U1
U2L'D'FR2U2R2BD2L2F'L'DB2F'D2RU'	D3U2L1R1B1D3U2L2R2U3R2D3U3B2L3R2F3B3R2U3
RU'RDBD'L2D2L'D2U2L'R'B'FL'F'D'	R1B1L3R3D3U3F1B3L3R3U3F2B3D2U2F1B2D3L3R1D3L3
D2FR2U'RUL2UF'DLR'DF2R'B'D'R'U'	D1U3B3R3D1U2L2R2F1B2L3R3F3B1D3U2B3L3F3B1D3U1
D'BL2UF2LU2BR2UB2LD'	F1B3L2R3D1U3L3F1B3L3R1F1D1U3B3D2U2L1R3
UL'R'B2U'R2BL2D'F2L'R'U'	F3B2D1U1L2F2B1D2L1U2F1B2R2D1U1F1B2
DFL2DR'B2F2D2U2L2R'D'L2F'D'	L2R3F1D1U2L3R2U2F2B2L2R2D2L1R2D3U2F3L2R1
LD'FL2R2B2U'R'B2LBF'U'B'U'	D1U2B3D1F3D3L3D1L3F1B3U1L2B2D2U2
L2R2F'L'F'LB2R'BDRD'B'F'RD2U2	D2U2F1U1B1L2R1F2B1L3R1D2U3B1L3U3F2B1
LU2BD'LB'R'U'F2L'D'U'F2RFL2D2U'	D2U2L1R1B3U3R3U3R3F3B1D1R1D1U3L2R1F1B2
LD'F'D'L2R'D'UFL'RB2L2F2D'R2BU'	L2R2F2B3L3R2F2L2R2U1F3B1R3D3U3F2B3R1F3B2R1U1
L'DF2RU2FD'R'B'U2L2R2U'L'RF'D'R'U'	F1B2D1U1L2R1U1F3B1L2D1U2F1B3L2R3F2B3L1F1B1L3R1F3B2

Symmetry Type D_3

DBDU2B2F2L2R2U'FU	U1R1D2U1L2R2F2B2D3L1D1
D'LDUR'D'U'L'DURU'	D1U2R1D1U1L3D3U3R3D1U1L1D2U1
B2L2D2L2U'F2UB2U2F2R2U'R2U'	L2R1D2F1B1D1L1R1F3B3L1R1U1F1B1U2L1R2
U'L2D'L2DB2F2L2R2DF2U'F2U'	L2R1B2L1B2L3D2U2F2B2L3D2R1D2R3
RD'LDU2B2R'U2L'D'U2L'DU2	D1R3U1L2R1F2D3L2U3L2R3U3L2R1
UR2DF'LD2U2R'D2U2FD'R2U'	D3U3F2R1D2U2F3D2U2B1R3F2D1U1
UF2UBU'B'FDBUB'D'U'FU'	F2B3R1D3U3L1D3U3F2B2L1F3B3D3U3L2R1D3U3F3B2
D2L2DL'DB2F2L2R2D2U'R'UR2U2	F1B1U2F2B3R3F3B2L2R2D2U2B1L3F3B2D2F1B1
U'F'L2D'R'F'L'FRB'R'F'D'B2R'U'	F2D3U1F2B2U3L2R3F1B3D3U3F2B3R1D2B1D3L1R3D1B3
D2UB'R2D'B2R'D'U'L'F2U'L2F'DU2	D1U2L2R1F2D3U2R2F1B2D3U3F2B1L2D2U3B2L1R2D2U1
DBLU2L'D'LD'U'RU'R'D2RFU	F2B3L2R1U2F3L1R1D1U2F1B2L3R2D1L2R1F3U1F3B2D3U1L3
RFR2BU2B2RB2LB2DB2L'UB'U'	L1U1B2L1F1B2L2R1B3D3U1F1B3D1F3B2L1R3F1R1
U'L'F'D'R'U'F'D'U'B'R'ULB'L'U'	B1L1B3D1F2B3D2U2L2R2B3D1U2L2R2F1B2R1F1
B2F2R'D'L2R'D'L2U'F2L2RDR'D2U'	D2U3F3D1F1B2R2U3B2D3F3B2D3F3L2R2
LUL2R2U'LULR'B'U'L'BLR'U'	F1B2L1R1D3U2L2R3F1B3D1U2F3L3R1U1L2R1D3U3F2B3
U2L2URU'LRU'RU2RU'LRU'RU'	F2B1U1F2B1L2F1B2U1F3B3D1F2B1R2F1B2D1F1B2
D'FLR2F2D'LR2FD'U2R'FL2RBU'	D3F1L1R2B1L3D2U3B1L2R1U3B2L2R1B1U3
D2R'D'F'L'R2F'D'UF'U'R'B'F2R'D'U'	U2R3F3B2R3U3B3D3U1B3L3R2B3D3R3D1U1
D'L'B2L2BDU2B2F2L2R2U'FR2F2R'U'	D2U1L3R3B3L3R3U2L2R1F3B3R1F3B3L3R3F3B3L2R3D1
U'FD'L'F2L2D'L'D2F'U'F2R2F'U'RU'	D1L1F3B2R2F2B1L3U3L1R1D2B2D3U2L1R1F1B2
FD2F'R2UF2D'LUBL2D2BR2F2LU'	F3B1L1F1D2U1L1B2D2R2F1D2U1L1F2B3D1U3L1R3
U'FU'L2B2L2DRD2FDL2B2L2U'RU'	L1B2L1R2D3U3F1B1R3U2F2B1L3R3B1R1F3B1U3B2L1
L2UL'UB'LB'UF'D'BU2BL2DFU'	D3U1L3R3B2R1F3B1R1F3B3L3R2F1B1D3U1F3B3D3L3R3F1B1
FUL2B'R2D2BDF'R'FD2B2DL2R'U'	F3B2D1L1R1F2B3L1R3D1U1L2R2D3L1R1F1D2R1D3U3F3B2
BFL2D2U'L2RD'L2U'R'U'B2R2D2B'FU2	L3R3D2U1L2R2F2B1D2U1B2D1U2L2R2F2L2R2B1D1U2F2L3R3
B2R2U'F2U2RB'U'R2BL2B2FL2UB'LU'	D3U3F2L2F2L3R2D2U2F2B1U3B2D3F2B1D2U2L3R2D1U1
F'LUR2D2LFU'LF'D'ULR2F2UF'U'	R2F1B1L2R3F3B2L1D1U3L1R1B3L3R2F3B2L3R3B2D3U2F1B2
U2B2FDRB2RU2B'LBLBR'UL2D2U'	U3F2B3L2R3F2B3L3F3B1L2R3D3F1B1L1R1D1R2B1L2R3B2
L2BF'D2L'D'U'F'R2U2F2R2D2B'U2L'R'U'	F1B3L1R3F2B1D3U3L2R3D3U3L1R3U2F2L3R3F1B2D3U3R3
LUB'L'F'U'L'F2D'B'R2DB'F2U'B'FU2	D3U2R3B1D3L3R1F1D1U2F3B2L1D1U2L1R2F1L2R3F3D1
D'L'U'FU'L2D'U2L2D2B2UF2U'BD'R'U'	D2U2L3F1B1D2R2F1D3U3L1F1B2L1R2D3F2B1D2U2F3B1U3
UR'F2L'RB2FU'F2R2B2DB2F'R2B2L'U'	D3U1F1B1U2B1L2R1F3B3R3F1B1L2R2B3U2F3B3D1U3
BFR2B2R'F2LBD'L'UB'R'B2D'URU	D2U3L3R1F1D3U3R1F1B2L1R3F3B3L3R1F1B2L2R3F1B3R2U2
BLDFLBUFDBURURFUBU'	L1R2D1L3R1D3L1R3F1L1R2D3U3B1D3U3B1D2U2L1
L'B'LD'L'U'B2FL2R2DB2F2L2R2U2R'U'	U3F1B2L2R2D1U2L1D1F1U1F3R1F3B2L2R2D2U2

UL'F2UL'F'L'U'R2D'L'F'LFD2LR2U'	D3U1L3R2D1U3L3F1R1D3U3L1B1R3D3U1L2R3D1U3
RFLB'F2L'D2UB'L'F2D'F2D'UB'R'U'	B1L1F1B1U3B3D3U2F2B3R1F2B1L3R1D3U2B1L2R3D3
U'B'L2D2R'F2U'R2B2F2R2FLDB'R'F2U'	L1R1U1L3R2F1B1D3U1R3U1L3R1F1B3R3D2U3L1R1B3D1U2
UL'F2D'B'F'DL2R2B'F2UL2R2D'B'LU'	L3R2B3D1U3F2B3L1F2B2D3U3F2B2R1F3B2D3U1F3L2R3
LUFL2U2LBF'U2FRBU2LBDRU2	B1D3U3R1D1U1L1R2D1U1L2R2F3B2R2D1U1F1B1D2U1F1B1R1F1
D2RUR'BFR2U'B2LRF'LBL'B'FU'	D3U3F1B2D3L2R1U2F1B1D2L3R2D3U2F1B2D1U2R3F1B1L3R2
BD2L2U2R'B2U2L'FURBDB'UF2L2U'	L1R1D1F2D1L2F1B2U3L2R1F1D3F1B1L1R3D2U1F2B1
R2U2B'RFDB'R2BRU2R2U2F2ULFU'	L3R3F2B3D2U3L3R2D3U2R3F1B1R3B1L2D1U1F3B1L3D2U3
DR2D'B'F'L'U'L2F2U2BR2B2U'L'F'D2U'	D2U3R3D2L1R1D1U1L3D2U3L2R3D2B1D3U2L1B2D2U3
FRB2U'L2F'D2LBDB'RU2LDFL2U'	D2F1D3F3B1L3R3F2B3D3U2L3R2D3U2R1F3U1L1R2F3
BUB'D'F'DBLDLD'UBDR2BF'U'	D3U1L2R1F3B1D1B3U1F2B3D3U2L3R1U1F1B1L1R3B1R1
L2R2F'D2F2RF2D'B2RD2B2D2LR'F'D2U2	F1B2L2R2D3L1R2D2U1B3R3D2U2F3B2U3L2R2D1B3D1U2
B'DU2L2R2B2FRFDB'F2L2R2D2U'R'U'	U3F1B2L2R2D1U2L1D1F1D2U3L2R2F2B3L3B3
LUB2UB'F2RB'L2F2LU'L2DFDL2D2U'	D3L3R3F3B1R3U2F2L3R1F3B2L3R2U3L2D3F2B3L1R2
B2FD2L'F2U'BD2UB'FL2UL'F'LB2R'U2	D2U1L3R3F1B3L3R2F1R3F3B3L3R1D1L2R3F3B2D2U1F1B3D2
L'RFR2B'U'F2L2RF'L2B'UB2F'R2B2F'U'	D2U2B2D1U3F3B2D3F3B3L2R1D2U1L2R1F1B1L3R3F3B2D1U1F1B3
L2BD2L2B'DF2D2B'D2L'D'R'B'L'DR'D2U'	L1R3F2D3U2L3R3F2B3D3U3B2D1L3D3U3L3R1U3F3B2L3R3
UFL2R2F2D2R2U'BDFRFLF2UB2R'U'	L2F2B3D1U2R3F1B1D1U2F1B2L3D2U2B2L2D3U2L2R2F1B3L3R2
BRUB'LFU2B2L2UL'FD2L2R2B'D2F'U'	R3D2U1F3B3U1F1B1D1U1F3B3U1L1R1U3F3B3L2R1D1U3F1B3D3U1
D2L2F2D'UB2L'F2L'F'L'F2L'F'R2F2R2D'U'	R3D1U1F3R2D1U1B3U3L1R1D1U2L2R1D3U2F1D3U3F3B1
D2FU'L'F'D'U'RB'L'D'F2D'R'U2B'U'F'U'	D3L2R1D1U1L2R3D1U1F2D2U1L3R3U1F1B3D3U2L1R3F1U3
B'F'LF2U2LU2L2F2D'UFU2FL2FLR'U'	F3B1L1R3F1B2D3U3L3R2D3U3L3R1D2F2L3R3F1B2D3U3R3
FU'L2U2RB2UB2DFUL'R2UR'UBDU'	L3F2L1B2R1D3U2L1R1D1U1R3F1B2U1R3F3L3F1B1
LR2BU2FDB'D'R'FL'F2DF2U2B2D'RU2	L1F3B2U1F1B1L3R2D2U2F3L1R2B1L1R2B3U2F2D2F3
L2F2U'L2DF2D2F'L'BRF2DRUL2FRF	F1B1R3F3D2U1L1R1F3D2L1R2D2U1R1F2B1L2R1F1D3U3
B2DU2B'L'RBU'B2FU2RD'FR'B2D2R2U'	D2U2L3R2D1F2B3U2F1B2D3R3D1U1L2F2L3R2D1U1B3
R2UBR'U'L2U2F2R'B'F'U2R'D2BRF'R2U'	R1U1B1R2F1B1L1U3L2F2L2D2U1F2B3L2R1U1F1
U'F2L2U'BR2DB2L'U2B'L2DF2LU'B2R2U'	B3L2R1D2U3F3B1L2R1D2U2F3L2R3F1B1D2F2B2L3R3F3B2R2F1B2
D2UR'U2B2L'B2FRB2U'B'DLB'F'R2B2U'	L3U2F3B3L2R1F2L2F2B1D1U1F2B3L1R1D3U3R2F2B2D1R1
D'B2DBR'D2R2U'F2D'L2U2L'B'D2L2U'F2U'	D2F1B2D3U1L1R2F1B2D1U3R1D3U2L3D3U3L3D1F2B2L2D3U3
L2D2F'DB'LB'R'U'B'F2D2B2FD2L2DR'U'	F2B3L1R2F1B1L1D3U3L1F3B3L1D2U2R1D3U3L1R2D1U1F3B2
FDB2LB'FUFD2LD'B2U'B2F2RB'F2U2	L1D1U3L1R2F1B3D3L2R3F1B1L3D2F2B1L3R3F3B2L3R3F1B3L1
B2LR2B'F2U'LU2F'LU'R2U2L2B'F'UF'U'	L2R3D2U2B3L3R3B1D1U2L2F1B1D1U1F2B3D1U2F3B3L1R1D1U2L3
B'RD2L2D'R'FDB'F2D2B'LR'FD'F2RU'	F2D2U3F2B2R2F3B2L2R3F3L2R3F2B3L1D1U2L1R2D3U3L3R3D2U1
L'F'L'FR2U2L'R2D'F2R2DRB2L'R2U'R'U'	F2L2R3F1B3D3U3L1R3B3L3R3D1U2R3F3B3R3D2U3F3B3L2F2B2
D2B2D'U2F2R2D2F'L2D'L'RB'D'FR2UB'U2	L3R2U2B2D1L3R1B1L2R1F3B3D1U3F2B3L1R1F2B2D3U3L1
DF'U2L2U2FR'U'B2U2L'F2D'F'D2B2LB2U'	L1R2D3F3L1R2U1R3F3B3L3R2U2F2B1D1U3L1R1F1B2U1
B2RDBLFR'B2U'F2DL'B2L2F2DLF'U2	F1B2D2U2L2R1D1U2F3B2L1R3B3D1U2L2D2U2L1D2U2F1B2D1U1
UFL2DU'B2R2D2U'RUB'FLUFL2RU	D3U2F1D2L2F2B2D1U3F3D2U1L2R1F3B1D1L1R3D3U3
UL2UFD2L'DB'D2B'DR'D2FDB2U2L2U'	U1L2R1F1B1R1D1U1L1D3U1F3B2L1R1F3L3R3D3U3F3U1
UR2D'LBL2F'D'B'D'FR'D'LDUB2R'U'	L2R2F2B2D3U2B1D3F1B3U1F3D3U1L1D3L1R2F2B3U1
F'RD'B'FL'F2D2F2R2D'BULUB2L'RU'	D3F2B3L1R3U3L3R2D3U3F3B1U1L3R3U1F2L3U3
BU'RU2R2F'D2B'L'U'BD'B'RB2FLF'U'	L2R3D1U1F2B1D3U3F2B1D1U1F1B3D3U3F2B2L3F1B3L3R1F1B2D2U1R1
ULR2B2D'L'URU2L'D'RF'D'FL2D2R2U'	D2B1L1D1U2L1R1F3B2R3F3B1D1U3F3B1L3D3L1F1D1U1
BR'B2R'D2B2D2LD'BF'URD'L'DF2R'U2	L1R3F2B3D2U1B3D3U2L1R1F3B2D1U2F2U2R3F1D1U2F1B3
F2UR2FLBD'U'FD'U'B'R'FUL2DR2U'	R2F3B3L1R3F2B1R3B1L1F3L2F2B3D3U2B2D1U2B3
U2FU'B'D'L2D'R2D2F'LRDFL'D2B2R'U'	L1F1B3U2R2D3F2B1L2B1L1R2D2F3L1R3F2B3D3U3
FLD'L2R2DU2RULBL'R2BLR'D'F'U'	F2B1L1R1D3U2F1B1D1L3R1B1D3U1L3R2F1B1L3F1B1L1R1F1
L2R'D2BF'R'UR'U'L'DBRD2U'L'R'D2U'	D1U3B3L3D1U2L2R2F2B1L3R3F1B3D3U2F3R3D3U1
F'DR2F'L'D'FURUL2FL'U'BU2F2U2R2U'	R2F3B3U2F1B3U1L1R2F3B3D2B1L2R2D3U3F3L3R3F2B3D2U3
RF'DF'R'DL2U'L2DB'F2RD'LF2U'L2R'U'	U1F3B3L2R1U2F2B1R2F1B1R2D1F3B1L3D3R1F1B3
RU2BF'U2LFUF'D'RBD2BUL2RF2R'U'	L2R3B1D2U1L1R3D3U1L2R1D1U3L3R3F3B3U1L1R1D1F1B1R1
D'LR2BF2DF2L2U2L2FLRD2U2F'RD'R'U'	R3D3F3B3D2U1R1F1B1D3U1F3B1D2L1R2F3B2D1U1F2B3L3R2
D'F'UBF'R'BLR'D'LU	B3R3F1L1R3U3L1D1U3B3D1F1
F'D2L2D2FR'B2D2B2RU'L2B2L2U	B1U3F3D3U2B3D1U3F1B3D2U3L1R2F1L1F1B2
URU'FD2U'B2F2L2R2U'F'UR'U'	R2D1U2L2R3D3F1B2L2R2D2U2F1D1L2R1D3U2R2
U'B2L'F2LD'U'BR2FD2F'R2DU2	L3R2F2U1L2U3F2D3L1R1B3U2B1D2R1
DR2D'B'R2BUF'U'FUF'UF'R2U'	L3U2L2R3U2F2B3L3R3B1L1R1F2B3D2L2R1B2R1

L'BF2DU2L2UBR'B2R'D2L'R'B2F'U	R3D3U2L2D3U3F2B1R2F3B2U3R3F2L1D3U2B1
U2F'U2B'R'B2RUR'B'UF2DU'R'D'R2U'	U3R2F3B1R3U1F2B1R3F2B2D2F3D1R2D2L1R3D3U1
F'R'D'F2D2LFDB2D2B'U2LB'U'RBU'	U1F1B3U3L3R2F2B1D1L2R1F3B2L1R2F2B1R3F3B3D1F3B3
U2L'BD'L2D'RF'LD2U'B'F'DR2D2LU'	D2U1F1L1R1D1F3B1D1U3F1L2R1D1U1L3R3D3U3R1F2B2D1
F2DL'ULUL2F2D'R'DL'D'R'D2LR2U'	D1U2F1B1D2U1L1R3D1U3L1R2F2B3D1U1F2L3R3B1L2R2F3B2U1
R2U'B'U2R2B'D'R2D2RF2RB2UB2R'F'U2	D3L1F1B1D2U3L3D3U3F3U3R2U1F3B2D1U3R1U1L2R3
LD2BRB2L2U2BL'B2UBFD'U'L'R'U'	U1F1B1L2R3D2U1L3R1D1U3L3R2D1U1L3R3F3B2L3R3D3U3F1D2U3
L'D'L'D2RFLR2D2B'F'UR2U2RB'F'U'	F1B3L1D3F3B2D1F3D3U2F1B3R1D2U1F1B1R1D2U1F2B1
UFLF2RB2U2B2R'DULD'U'F2L'F'U'	R2F2B2U2L3R3D1U2L2D2F2B1D1U3B2L2D3U2L1R3
D'F'L2R2D2FL'R'DBF'U2RU2L'D2R2U'	F3B2L3R3F1B1D3U1F1B1U1L2D1U1L3R3F1B2L3R1U3F3
U2L2D'R2B'R2BLDL'F2L2B'F2U'L2F2U'	L2R2U3B1L2R2F3B2D2L3R3D3L3F1B1L3R3D2U3L3R1F2B3
D'RD'U'LB'UF'U'LUL'DB'LR2FU'	R3D1U2L2D3U3L3R3F1L1R1F3D1U1R2D3F1B1D1L3
B'D'B'D2BR2F'L2U'FDLD2B2D2L'R2U'	L3R2F3B3D1U2R2F2B3L2U1F1B1D2L1R2F3B2L2R2F2B1
LF2L2D'L'F2R2B'FL'DF2U'F2LU2R'U'	L2R1F3B1D2U3F2B3D1U3L1R2B1L1F2B2U2F1L1R1F2B3D3
L'D'L'B2F2L2R'F2D'U2LULBU2L'R2U'	D3U3F3B2L3R1U2L1R2F2B1R1F3B1D2R1B3R3D3U1
DRBDBF'L'D2R'U'F'D'UB'L2RF2U'	D3U1L1B1D3U2F2B2L3R2F1B1L1R3D1U2R1F3B2L2R2D3U1
L2DL2B'F'RBR'DL'BURB2L2B'R2F'U'	D1U2B1L2R2U1F1B3L1R2U3L3R1F1B1L3F3B1D3U2F2D3U2
D2F2R'F'D2U'LR2D'F2URFU2FDU2L2U'	L1R2F2B2L1R1F3L2R3D3B1D3U2B2D3U3L2R3D2U1L2R1D2
RU2FR'DU'F'U'R2U'L2R'U2BDFD'R'U'	F1B1L1R3F3B3L1R1B1D2U1F3B3D2F2B3L3R3F1B1L1R3F3B3
RFR'URDBDU2B2F2L2RU'	L1D1L3B1L1F1U1F1B2D2U2L1R2B3
R'D2UB2F2LR2BLURU'FU	L3F2B1D2U2L2R1U1R1B1L1B3D1B1
D'U2B'L2D'R2BDUB'R2DU2B'FU'	L1R1D3U1L3R1D2U1F3L3R3D3U3F1L3R3F3L1
U'FR2U'B'R2D'B2F2U'L2F'U'B'FU'	F2B1D2U3F3B3U3R2D3U3F3B2D1U1F3B1U2B3
D'U2LR'U'F2LD'U'L'R2F2UB2L'U'	L3F1L1R1F3D1U1L1R1F1D2U3L1R3D1U3L3R3
B'D'B'F2D2B'L2BFDBDBU'F'U	B1U2F1B3D3U3F1B2D1U1R2U1F1B1D2U1F2B3
DB2F'L2RU'RDU'B'DBF'U'FU	D1F3B2L2R1U3R1D1U3B3D1F3B1U3F1U1
DBD'B'FUF'D'ULD'LR2B'F2U	U1R1U3L1R3D1L3D1U3F1U3F1B2L2R3D1
L2D'L2F'R2U2B'D'R2BD'BF2L'U'L'U'	D2U3L2R1B2D3B2L1U2B3L1D2U2R1D1U2F2B1L3R1
DB'R2U'L2R2D2U'FR2D2R2B'R'D2B'U	F2B1D1U2L3R1D1U3F1U1L2R1F1B2L3R2F2L3R1U3L2R1F2B2
URD'U2R'B'D'RD2U2R'F2LF'D'F'U'	R2D1U2F2B2L2R1D3L2R1U1F1L2R3U2L1R3D1B3D2U2
U'F'D2L2UB'D'B'D2FLF'L2DURU'	F1B2L3U2F3B2R2F1B1D3B2D1U3B1D1U3L1F3B2
LULUB2F'RF'D2RFL2D2BU2RU2	L1R3F2B3D3U2L2R1D2U2L1R2B1U2L3B2D1B2L2R3D2U1
D'RU2BRB2U2B2L'D'B2F2D'U2B2RU'	F2B2L2R3U1L1R3F2L1R2F3B2L2R3U3F3D3U1L1R3D3U2F2B3
UFDFL'F2RD2U2R'DBRDU2R'U'	D2U2B1D3L3R1U2L2R1F3U3L2R3D1L2R3F2B2D3U2R2
R'D'R2U2F2R2D'LB2D'FR'B2D2UF2U2	L3F2B1D3U3F3D3U3F1B1R3B2U2B2L2R1D3L2F2B3R3
R2F2L2RB2U'FL'B2DL'U2F2R2U2L'U'	F3L3R2B2D3F1B2L2U2L2F3L1R1D3U3R3D3U3L1R2B3
URD2L2R2B'FR'B'FLDU'B'RBU	U1R1U2R3D1U3F2B3L1R3B3L1R3F1B3L1B1U1
UR'DURB2R2BD2L'R'F2U'BFR2U	F1U1F1B1D2U1F1B1L1R3B1R2D1U1L2R2F2R1B3
ULFL'DU'BL'RF'L'RB2F2D2FU	U1L1B1L3R1F3B1L3F3B1L3R2D1U3F3U2F1U1
UF2LRU'R2B'F'D2LF2L2FDUF'U	L3F1R2F2B2D1U1F2L1F3B1L1R1D2U1L1R1U1R1
BF'R'B'U2BDB'L'R'BDL2RU'LU'	L3R1B3R3D2R1U1R3F3B3R1U1F2B1D3F1D3
U'RB'D2L2R'B2FR'U'L'BU'B2UF'U'	F1B2L3D3U1B3D3U1B2D1F3B3R2F1B2U2L1F3B2
F'LF'L2RBDL'R'D'BDF2D'R'DU'	D3R1D3L1R2U1F1L3R3F3U1F1D2F3L3F1B3
BR'DR2UB'DBL2U2R'DU'F'R2U2B2U'	L3R1U3L3R1F2R1U3R1F1B2D2F1B2L3R2D3R1D2U3L3R1
BL'R2DF'L2UF2U'R2B'DR'D2BUF'U2	L1R3D3U3L3R2F3B3L1R1F3D2U1F1B2L3R3D3L2R3D1U3L2D2U2
DBLRDU2B2L2DU'L2UB'F2L'R'FU	U1R1F1B1D2U1R2F2D3U1F2D1L2R3F3B3L1D1
LF'R'U'B2DRFU2F2DL'R'B'L'FR'U	B1D3U3F1B1L3D1U1L2R2F3B3L2R3F1B2L1R3F3B1L3D2U1B1
DBD2LD2LD2U2FLD2BUBD2U'F2U'	U1F1B3D1U2L2R3D1U3B1U2F1R1F3B1U1L3B1L3R1
U'RD'BLR2B2L'DR'D2U'B2D'F2L2R2U'	F2B3L1R2F2B2D1U1F3B3R1F1B1L3D2B3L1R3U1F3
F2D'B'U'LBU'L2U2F2U'FR'UL'B2F'U	R3D2U1F3B1L2R3F2D1U2F1B1D3U3F1B1D1U2F1B1D2U3F2B2L2R3
DB'D'F2D2U2F2L2R'D2U'LD2R'U'RF'U	D2U3F2B2L2R1D3U3F3B3L3F2B3L2D1U1F1B1D3U2F3B3L2R1D1
URDL'D'UFU'R'D2F'U2R'BRU2LU'	L3R2D1L3R3F3B3R2D2U2B1U2L3F3B3L2R3B2D3U2L1R2
D'U'R2F2R2BR2B'R'B'R2B'L2R'DU'L2U2	F2B1L2R3D3L1R1F3L2R3D2U3R2D1U1B1L1R2F3B2L1R2
B2D'L'BRD'LDB2URU2B'LB2U2L'U	F3R3F1B2D2U2F1L3B3D3U2R1B3D1U2B1D3U2F3B2
ULB2DL'B'L'D2UF2D'R'F'RD2L'R2U'	L3F1B2D2U3F1B2L1R2F1B3D3U2F1B2D1U3L1R1D3U2L2R1D2U1F2B3
D2B'URF'DB2F2RD'B'L'RFRDFU2	D3U2L3R2D2L1R2B2D2U3L1D2U3L1R1F3B2D1U3F1B1L3R1F3
D'L'BFL2RD'F2DU'F2R2D2U'B'F'R'U'	D3L3F1B1L2R1D3F2D1U3F2R2D2U3F3B3R3U3
L'DB'UFDUR'B2L2B'D'R'F2LDBU'	B3D2U3L1F1B3L3R1F3B2L2R1F3B3L2R2D3U3L2R3F3B3D3U3B3

DR'DF'DU'L2B2DL'B'F2DBF2U'BU'	F1B3D2U3F1D3F2B3L2R1D2L2R1F1U3F1R2F1B3U3F1B3
D'LB'DBU2F'DU2BF2R2D2U2R2URU'	D3L2R3F1B1D1U2F3B3D3U3L2F2B1L1F1B1D1U1L2R3F2B2D2U1
UL'U2R'B'RU2FD2RUF'DU'LD'R'U'	L3R2D1U2B2L2R1F1B1L1U2B3D2U2R2F1B1L1R1D3L1R2
B'RB2U2R'UB2L'B'U2F'R'FL'U'RFU2	F1B2D1U2B3D3U2B1R3D1U2B1L1F1B2D2U2F1R1F1
DL2DU2R'D'R'U2F'L'D2U2F'U2F'U2R'U'	L1R3B3L1U3F1B3R3F3U2B3D3U1L2R1D3U2F3B1U3
L2D'LUR2B'RU'B2L'D2LF2D'RB2F'U	D2U2B2D1U3F3B2D3F3B3L2R1D2U1R3F1B1L3R3F2B3D3U3F3B1
D2U'L2R2B2DF2U'RD'LB2L'R2B'DR'U	F1U3L3R1B1D2L1F3B3R3F1B1D3U3F2B2L3R2F2B1
R'D2UBR'FU'RU2R2B2RD'B'RDLU2	L2R1F2B2D2U1F3B3D3U2F3B3D1U1F3B3D3U2F2L2R1F1B3D2U3R1
D2U2LD2U'F'R'U2B2RU2FU2R'U2L2RU'	F2B1D2U3L2R3D1U2L3R3D3U1F3B2D1U2F3B1L3R2F3B2D2U1F3B2L1
F2RBF'D'U2R2BLFU2L'B'FU'L2F'U'	D1U2F3B1L2R1F1B1D3U3B1L1F1B1L2D2U1F2B2D3L2R3D2U2
LF2L'R2F'DBL2DB'D2B2F2L'F'D2F2U2	L1R3F2B2D1U2L2B2D1U3F1B2L2F3B3L1R3F2L1R1U3L1R3
U2B2DU'B2F'L'F2L'F'L'F2LF2R2F2D'U'	F2B1L2R3F2B1L1D1U1F2D2U3F3B2R3F1B1D3F3B2L1R2
RUB2RDU'BR2U'B'D'F2LR2D'UF'U2	D2U2L2R1D1F2B2D2U3L2F3B3L3B3D1U1F3B3L2R3F1B3D3U2
L2B2R2BUB2F2R2FR'U2F'R'BD2UB2U'	L3R1U1L3R3F2L3R1F1B1L2F3B2D3U1B2L2D3U2F2B2L3R1
F2U2R2F2U'F2D2B2LF2UBU2B'UB2RU'	B1D2F3B2L2R1F1L2R1F1B2L2R2D3U3F1D2U3F1L2F2B1D3U1
FUL2BR2U2F'D2BR2B2L'B'U'F'URU'	F1D2U3F3B3D2U2L3R3B2R1F3B3R3D3U3F2D1F3
R'U'F'D'L'D2L2R2F2L'D2FLFU2R'F'U'	U1L3F1B1L1R1F3B3L3R3F2D2U1L3R3D1F1B1L2F2B1D3
L2F'D'BU'F'D2URF'D'BL2D2RBRU'	D3U1L1R2B2R1D2U3R1D3U3F2B2L2R1F1B2R1F1B2L2R3D2L1
F'RFU'F'D'L'D2R2DB2U'F2R2DL2FU	R3D1R2D3U3F3L3R3F1L2F3B3D2U2L3R3D2U3R1
D2BULBLR'F'U'LB2F2UB'LDF'U2	R3F1B3L1R1D1U3L2R3F1B1D2U3B1D2U3L2F2B1D2F2B3D3U2
R'F'U'R2FDFL2D'F2D2U2L2D'L'F'R'U'	D3L1R2B2L1R1D1F3B3D2U1R2F3B3L3R3F1B1L1R1B3U1
DB2D2L2F2D2F2U'BL2BU2R'F'L2D2FR	D1U1F3B3L1D1U2L2R2F3B3L1R3F1B2L1D2L3R2D3U1B3L1R3D3U3
RB2UL2RFR2UB'FR2BURD2R'B2U	F2B3L2R3D2U1F3B2L3R3D1U1L2R2D3L3U1F2B3D2U1F2B1L3R3
ULU2BL'R'FU'R2D'UL2F2U'FLRU	L2R1B1L3R3F3B2U3B1D1U1L1R2U1L1R2B3L2R1U3L3R2
LBU2R2B'L'D2FR2FD'B2U2B2R2U2F2U	D3U3F3B1L3D3U1F2B3D2B1L2R1F3B1L3R3F2B2D1U2B1L3R3D1U1
RB2U'L2URBU2B'FRU2FD2URB2U	F3B3L1R2D2U1L3R2U1B3D3F2B2D1U1F3B3L2R3D2U1F3B2L3R2
UBFRU'R2B2D'UF2U'RB'F'LU2BU	F2B3U3F1B2L3F2B1U1F2B1D1U1L1U3L2R3F3B3L1F1B2
UR'F2D2L'R'D2UBFDU2L2BL'R'FU	B1L3D2F2L3R3F2B1D1U1F1B2R2U1L3R3D1B1
FLR'FD'U'F2LF'UL'D'F'L'RU2FU2	D2U3L2R3D1U1F2B3L3D3U2L1R2F3L2R1F2L2U3R2D1U1
URB'F'LB2DU2LRD2UB'F'D2R2F'U	L1D1F3B3U1F2L2R1D1U1L1R2F3B3R2D2B3L1
F2UF2L'RU'B'L'FU'LU2B'F'ULR'U	D1U1F2U3B2R2F1B2R3F2B1D3U2B3L3R2D1U1F3B2D2U3
FL'RBUL'D'ULFU2LU2L'B2F2U2F2U'	L3F3B3L2R1F2B2D3U1B1L1D2U3F3D2U3L1B3U3B1
BLDBD2F'LD2L'D2F2R2BD'R'U2FLU'	B1D1U1L1U3F1B3D3U1F2B1D1U2L1R1F1B1U1L1R1D3U3F2B3
BLULR'B2FD2R'B'R'B'LUF'L'RF'U'	D3U2F3B2U1F3L2R2D3F2B2L2R3D3U1R1D1U2B3R3D1U2
U2F'U2L'DF2R2D2R'B'F'D'B2L'B2RF'R2U'	R1D1U3R1F2U1L3R3D2U1F1B3L3R3F3B3L3R2B2D3U2L3
F'RDLU'F2U2R'DRBRB2FL2U'FR2U2	D2U2B1U3F3B1D1U3L3R3D1U1F2B3U2L3F3B2D1F3B1L2R1
DB'L2DUBD2R2D2BL'D2BU2LD'U'RU'	B1D3U3F3B3R1F2B2D3U3L3R2D2F1L2D3U2L1R1U1F3
F2D2R'F'UF'LB'L2F'DLR'FD2F'D2R2U'	R1F2D3U3F3B2L1R1F3D3U3L1R1F3D2L1R2D1U3F2B3L1
DUR2B2DRU'B'R2D2B2L'B'UR'BL2R'U'	L3R2F1B1L2R2D2U3F1B1D3F2D3U3R1F1B1L1R2D1U3B3L1
DB'R2UL'U'RB'F'LR2U'FU2R2F2L'D2U'	F2B3L1R2D1U1R1F2B2D3U1F1D3U3L2R2F2B1D2R1F1
B2UBL2R'UF2U'BL2D2BD'U'B'D2LR'U'	F1U3L3R3D1U2L2F3D2L1R2D1U1F2B2R3F1B1D1U1B3
U'R'B'DU'LD'B'U'B'L'RB'R'F'D2L2RU'	L3F2B1D3U1L3R2D2F1L3R3D1U1F1L3R3F1B2D1U1F2R3
BR'D'B2LFU'L2D2F2RF2R'DF2U'F'R'U'	F2B1D1U1L3R3U3F3B3L3R3D3U2F2B3D1U3F3B1U1L3D3U3B3
BDL'RDB'R'ULULF2D'U2LR'B'R'U'	D3U2R1B1D3U2R3D1U3L2R1F2B2D1L2R2F1U3F1B2D1U2
F2D2BL2U'B2RDF'RU2RDLD2L'U2FU2	L1D1U2B2L1R2F1B1L1R1F3B1D2U3L1R1U3F2R3D3U1R3
B2R2DF'DU'BDFLURB2F'D'L2U'LU'	L2R3F1B3D3F1B2L1U2F2B1D3U3L1R1D3U1F1B3U1B3D2U2
F'L2U'RFLU'F'RUF2L'F'RDU'F'R'U'	B3U1B1L3D2U1F1D2U1L3B3D1U3F2B2L2R3F1B1L1
D2F'DR'F2R'FL'D'F'D2B'LR'F'LD2F'U'	L3B1D3U1L3R2F3B3R3D1U1F2D1F3B3D2U1L2R2F3B3L1R2
DU2FL2B2D2L'DB2F'LRB'DFD'B2RU'	F3R3D2F2B3L2R2D1U1F3D1U3F2B2R3D3U3L3R2F2B1
R2F2D2UBL2B2ULFRDUB'R'UL2DU'	L1R3D3U1L3D3U3F1B3U3F3L2D1U2L2R3F3B3R2D3R3B2
L2U2B2FL'U2FR'DU2R'U2LU'L2R2B'FU'	D2U3L2R1F1B1L3R2B2D2U1B2D1U1F1B1L2R3D1U1L2R3F1B1D1
L'R2D'RD'BL2R'D2F2R2BL'U'LR2B'F'U'	B2R1D1R2F1B1L2R1D3U2L2F1U1F3B1D1U1L1D1U3L3R1
ULU2F2L'U'FR'D2LD2L2UB'L'F'U'B2U'	D3F3B3L2R1D3U3L2R1F3B3D3U3B2D2U3B2L1R2F3B3L2R3D2U1
UL2R'F2D'B'D'F2R2FR2U2R2U'F2R2FR'U'	B1R3F1B3L2R2D1U2L2R2F2D1U1L2R1F3B3L3R2F3B3D1U1F3B2
D2LB'D'L2F2R'D'FR2DBDFU'B'FD2U'	L2R3D1U1L3R3F2B3L3R3F1B2D1U1R2F2B2D3L3R1D2U2F1B2L1
L2F2L2B'U'B2RD'R2UR2B2LFU2R'DFU'	L3R3D1F3B1U2R3D2U3F1D2U1R1D2U1L3D3U3F3B2L3R1
D2U'R2U'LB2U'B'D'U2F'R'D'U'LD'FRU'	F1B3L2R3D3U3B3D2U1F1D2U1R1D2U3F3U2L1R3D1F3B3

Symmetry Type D_{4h}

D2U2	D2U2
B2F2L2R2	L2R2F2B2
D2B2F2L2R2U2	U2L2R2F2B2D2
L2F2L2R2F2R2	B2L2F2B2L2F2
U2B2F2L2R2U2	D2L2R2F2B2D2
DB2F2D'UL2R2U'	L2R2D3U1F2B2D1U3
L2F2R2B2F2R2F2R2	D1U1L2R2D3U1L2R2D2
L2R2DU'L2R2DU'	D1U3F2B2D1U3F2B2
DB2F2DU'L2R2U'	L2R2D1U3F2B2D1U3
L2R2D'UL2R2DU'	F2B2D3U1F2B2D1U3
D2L2F2L2R2F2R2U2	F2B2D2F3B3D2U2F3B3D2
U2L2R2DU'L2R2D'U'	D1U1F2B2D1U3F2B2D2
DF2R2F2D'UR2F2R2U'	F2B2L2R2U2L2R2F2B2D2
DB2L2B2DU'R2F2R2U'	F3B3D1U1L2R2D1U1F3B3
BFDULRB'F'D'U'L'R'	L3R3D1U1F3B3L3R3D1U1F3B3
B'F'D'U'L'R'B'F'D'U'L'R'	L3R3D3U3F3B3L3R3D3U3F3B3
BFDUL'R'BFD'U'L'R'	L3R3D1U1F1B1L1R1D1U1F3B3
B'F'D'U'LRBFD'U'L'R'	L3R3D3U3F1B1L1R1D3U3F3B3
UR2BFLRBFDUR2U'	D1F2D3U3L3R3F3B3L3R3F2D3
UR2B'F'L'R'B'F'D'U'R2U'	D1F2D1U1L1R1F1B1L1R1F2D3
U'BFD2R2D2R2B'F'R2D2R2U'	D2U3F3B3L2D3U3F1B3D1U1L2F1B1D3
U'BFD2L2U2L2B'F'R2D2R2U'	D1L3R3F1B1L3R3F1B1L3R3F1B1D3
UL2U2L'R'B'F'L'R'DU'R2U'	D1B2D1U3F1B1L1R1F1B1D2F2D3
D2U'R2BFLRBFDUR2U'	D3U2F2D3U3L3R3F3B3L3R3F2D3
U'BFU2R2U2R2B'F'R2U2R2U'	D2U3F3B3L2D3U3F3B1D1U1L2F1B1D3
U'BFU2L2D2L2B'F'R2U2R2U'	D1L3R3F3B3L3R3F1B1L1R1F1B1D3
DB2L'RBF'D2F2LRU2R2U'	D3U2B2D1U3F1B1L1R1F1B1D2F2D3
U'F2D2F2LRF2D2F2D2L'R'U'	D2U3F1B1L2D1U1F3B1D3U3L2F3B3D3
U'B2D2B2LRF2U2F2D2L'R'U'	D1L1R1F3B3L1R1F3B3L1R1F3B3D3
UL2D2LRBFLRDU'R2U'	D1B2D3U1F3B3L3R3F3B3D2F2D3
D2U'R2B'F'L'R'B'F'D'U'R2U'	D3U2F2D1U1L1R1F1B1L1R1F2D3
U'F2U2F2LRF2U2F2U2L'R'U'	D2U3F1B1R2D1U1F3B1D3U3R2F3B3D3
U'B2U2B2LRF2D2F2U2L'R'U'	D3U2L1R1F3B3L1R1F3B3L1R1F3B3D3
DB2U2BFL2D2L'RBF'R2U'	D3U2B2D3U1F3B3L3R3F3B3D2F2D3
D'L'RU2L2U2BFU2R2U2LR'U'	L2R2D2L3R3D1U1F1B1L3R3D1U1F3B3D2
UL'R'B'D2U2B'FD2U2FLRU'	L1R1D3U3F3B3L3R3D3U1F2B2U2F3B3
RDUF'L2U2R2D2R2U2FD'U'R'	B1L1R1U3B2L3R3U2L3R1U3L3R3U2B3
UL'R'F'R2B2F2R2D2U2FLRU'	L3R3D3U3F1B1D2U2L1R1D3U3F3B3
LU2BF'R2UL2R2U'L2BF'U2R'	R1B2D1U3R2F2B3L2R2B3L2D1U3B2L3
LB2F2L'U2BF'R2DL2B'FU2R'	L3R2F2B2L1R2U2F3B1R2D1L2F1B3U2R3
ULRBD2U2BF'D2U2F'L'R'U'	L1R1D1U1F3B3L3R3D1U3F2B2U2F3B3
ULRFR2B2F2R2D2U2F'L'R'U'	L1R1D1U1F3B3D2U2L1R1D3U3F3B3
D'L'RD2L2U2BFD2R2U2LR'U'	L3R3D1U1F3B3L3R3D1U1F3B3D2U2
D'L'RU2R2U2B'F'U2L2U2LR'U'	L1R1D1U1F1B1L1R1D3U1F2B2D2F3B3
LUB2L2UB'D2U2FU'R2F2U'R'	F1L1R1D3L2F3B3U2F3B1D3L3R3U2B3
D'L'RU2L2D2BFD2R2U2LR'U'	L2R2D2L1R1D3U3F3B3L3R3D1U1F3B3D2
D'B2L2B2L'RBFU2R2U2LR'U'	L3R3D3U3F3B3L3R3D3U3F3B3D2U2
D'B2R2F2L'RB'F'D2R2U2LR'U'	L1R1D3U3F1B1L1R1D1U3F2B2D2F3B3
L'D2U2LD2BF'L2UR2BF'U2R'	L3R2D2U2L3U2F1B3L2D1L2F1B3U2R3
LR'U2L2U2BFD2L2U2LR'D'U'	L1R1D1U1F3B3U2L1R1D1U1F3B3D2
F2L2B2LR'BFU2R2U2LR'D'U'	L3R3D3U3F1B1U2L1R1D1U1F3B3D2
L2D'L2B'R2B2F2D2U2R2F'R2U'R2	F2D2U3L3R3U2F3B3U2F1B1L3R3F2B2D3B2
R2D'F2U2LRB2D2R2F2D2L'R'U'R2	F3B3L3R3F3B3L1R1F1B1L1R1U3L2R2F2B2D3
RU2B'FL2DB2F2DU2L2BF'U2R'	L2R1U2F3B1R2U1F2B2U1R2F1B3U2R3
B2L'R'BL2B2R2U2B2R2F2U2F'L'R'	L3R3F3B3L3R3F1B1L1R1F1B1D2U1L2R2F2B2D3
DBR'DF2L'BD'UR'BL2D'FR'U'	F1R2D3U1F2L3B2D1U3L2D2U2B3D2U2F1
U'B2UB2L2D2B2F2L2F2D2F2R2U'R2U'	D2U3F3B3L2D3U3F1B3D1U1L2F1B1D1U2L2R2F2B2

B2F2L2R2UR2BFLRBFDUR2U'	L2R2F2B2D3F2D1U1L1R1F1B1L1R1F2D1
L2U2L2U'R2B'F'LR'B2R2BF'U'R2U2	D3U1F2B2U3B2D1U1L1R1F1B1L1R1B2D1F2B2
F2D'B2F2L2R2U'F2DUL'R'B'F'L'R'	F2B2L2R2D1U2F2D3U3L3R3F3B3L3R3F2D1
D2L'D2U2LD2BF'L2UR2BF'U2R'U2	D2U2F2B3L1R1D2U3F1B3D3U1F3B2D1U3L2F1B3L3R3F3
U2LB2F2L'U2BF'R2DL2B'FU2R'U2	B3D3U3F1B3D2L1R3F1L1R3F1B3L1D1U1B1
R'B2F2R'U2BF'R2U'L2BF'L2R2U2R'	F2B1R2D1U3B2L3F2D1U3F2B2R2F3D2U2B3
R'D2B'FR2U'L2R2U'R2BF'L2R2U2R'	R1D2U2F1B1R2D1U2F1B3L1R3F3B1U1L2F1B1L1R2
DFU2R'URF'DU'RB'U'BU2R'U'	F1L1R1D3F2L3R3D2L3R1D3L3R3D2F3D2U2
U2L'D2U2LD2BF'L2UR2BF'U2R'U2	L1R2F3B3L1R3B2D3U1L3R2D1U3L1R3D3U2F1B1R1
D2LB2F2L'U2BF'R2DL2B'FU2R'U2	D2U2F3B2D1U1R1F3B1L3R1B1L3R1U2F3B1D3U3F1B2
L'D2BF'L2D'B2F2D'R2BF'L2R2U2R'	F3D1U1L3R2F3B1D3U1F3B1L3D1U1R2D2U2R2F3B2
BF'R'DUF'U2R2F'LR'DF2D'R2U'	R2F3B1U2R3F3B3U3F3B1D3U1B1D2L1R1U3
BF'L'F2R'B2LD'UF'R2U2F'LRU'	F2L3R1U2B3D3U3L3F1B3L3R1B1D2L1R1D1U2
F2D'B2F2L2R2U'F2D'U'L'R'B'F'L'R'	L2R2F2B2U3F2D3U3L3R3F3B3L3R3F2U1
L2D2L2UF2D2B'FL'R'BFR2U'R2U2	F2B2L2R2D2U1F2D1U1L1R1F1B1L1R1F2U1
R'D2B'FL2U'L2BF'U2R'U2L2R2U2R'	F1B2R2D3U1F2L3F2D1U3F2B2R2F3D2U2B3
L2U2R2UB2U2B'FL'R'BFR2U'R2U2	D1U3F2B2U3B2D1U1L1R1F1B1L1R1B2D1F2B2
U2B'FLD2LU2L'DU'FL2U2FL'R'U'	B2D2U1L1R1F1L1R3F3B1L2R1F2D3U3B3D2L3R1
F2L'RU2F'DURB'FL'RF'U2L'R'U'	R2U3F1B1L3R2F1B3L3R1F2B1L2D3U3R3D2F1B3
R'D2R2DFD2B2L2F2R2F2U2FUR2U2R'	D1L1R1F1D3U1F1B3U3F2B2L2F1B1L1U2F1B3R2
F2D'LRFL'RBF'L'B2D'U'F'U2LR'	F3B1D2L3F1B1U1F1B3D1U3B3D2L3R3U3L2
U'B2U2L2R2D'F2R2U2L2F2U2F2R2U'R2U'	F1B1L1R1F1B1L3R3F3B3L3R3D2U1L2R2F2B2D3
U'B2D2L2R2D'B2L2D2L2F2U2F2R2U'R2U'	D2U3F1B1R2D1U1F3B1D3U3R2F3B3D1U2L2R2F2B2
LR'D'UR'D2B'F'UB2LR'U2F'L'R'U'	D3U2R2F1B1R3U2F3B1L2U1L1R1F2B3D3U1F3B1
D2R2DB2F2L2R2UR2DU'L'R'B'F'L'R'	D3U1L2R2D3R2D3U3F3B3L3R3F3B3R2U1L2R2
RBD2U'LR'B'L'BF'DLBF'U'F'R'	F3B1L1R2B2L1R1D1U1B1U2L3R1B2U1L3R3F1B2L1R3
D2B'FLD2LU2L'DU'FL2U2FL'R'U'	R2F3B1U2L1F3B3D3F3B1D1U3B1U2L1R1D1
D2B'FRF2LB2R'DU'FL2U2FL'R'U'	B2L1R3U2F1D1U1B2L3R2F3B1L1R3F2B1L3R3D3U2
LR'D'UR'U2B'F'DB2L'RD2F'L'R'U'	D2U3B2L1R1B3U2L1R3F2U1F1B1L2R3D1U3L3R1
UL2R2U2F2DF2R2U2L2F2U2F2R2U'R2U'	F2B2L2R2D2U1L3R3F1B1L3R3F1B1L3R3F1B1U1
F2LR'D2B'DULB'FLR'F'U2L'R'U'	L1R3D1U3R3F3B3D1U2B2L3R1D2F3B2L3R3F2D3
UL2R2U2F2DB2L2U2L2F2D2F2R2U'R2U'	F2B2L2R2U1F1B1L2D1U1F3B1D3U3L2F3B3D3
U2R2DB2F2L2R2UR2DU'L'R'B'F'L'R'	D3U1F2B2U3B2D3U3L3R3F3B3L3R3B2D1F2B2
B2U'L'R'U2B'L'RBF'RDUB'U2LR'	L1R3D2F1B2D3U3B2L1R2F1B3L3R1F2B3L1R1D2U1B2
BF'DU'B'L2D'U'LF2DU'R2F'L'R'U'	L3R1D1U3L3F3B3U3F2L3R1U2B1L3R3B2D3
B'FLR2DUB'L2U2B'LR'UL2U'R2U'	F2L1R3D2B3D3U3L2R1F3B1L1R3F3B2U2L1R1D2U1
L2BF'D2R'BFR2UBF'DU'F'L'R'U'	R2F1B3D2L3D3U3B3L3R1F1B3L1U2F1B1U3
U2LR'DU'LU2BFDF2L'RU2F'LRU'	D1B2L3R3B3D2L1R3B2U1F3B3L1R2D3U1L3R1D1U1
B'F2UR'F'L2U2B'D2FDLU'FUF2R2U'	D3U2R2F3B3R3U2F1B3R2D1L3R3B3D3U1F1B3D3U3
F2U'L'R'U2BF2LR'BF'LDUB'U2LR'	F3B1D2L1R2F1B1D2U3F3B1D1U3F3D2L3R3U3L2
F2LR'D2BF2DULB'FLR'F'U2L'R'U'	F3B1D3U1F2B1R2D3U3R1B2D1U3L2B3L3R3D3
R2UL2D'B'FRU2B2RD'U'BD2U2LR'U'	F2B2D1L1R1F1D1U3F1B3D3L2F3B3L3R2U2F1B3L2
B2DF2D'BF'RD2F2RD'U'FD2U2LR'U'	L2R2D3U2F1B1L3R2D3U1L1R3D2U1B2L3R3F2B3U2L1R3B2
B'F'L'D'F'UF2D2L'D'U'R'U2B2DB'U'R'	F1B3D3U1B3U2F3B3L1R1D1L2F3B1D2L3R2F3B3D2U1
LU'B'F'R'F2DU'R2FU2B2DU'R'DU'R'	L1R3D3U1R3D2L1R1F1B1D3U2F2L1R3D2F3B2L3R3D2U1
RU'B'F'L'F2D'UL2FU2B2DU'R'DU'R'	L1F3B2L1R3F1B3L3F3B3U3R2F1B3D2L1D3U3R3
D2LR'DU'LU2BFDF2L'RU2F'LRU'	F3B1R1F3B3L2R2U3F1B1D1U1L3R3U3F1B1R1F1B3D3U1
U2BF'DU'BLRDL2BF'D2R'BFR2U'	D1L2F3B3D2U2L1U2F3B1L2D3L1R1F2B3D3U1F3B1
LR'D'UL2RD2B'F'UB2LR'U2F'L'R'U'	F1B3D1U3F1B2D3U3L3F2D1U3R2F1L3R3B2U3
B2LR'U2BF2DUF2LB'FLR'F'L'R'U'	F2L3R1U2F2B3D1U1F2R1F1B3L3R1F3L3R3D1
BF'R2U2L'D'UBR2BL2R2F'D2B'LR'U'	R2F3B1D2L3F3B3D2U1F3B1D3U1F2B3U2L1R1D3F2B2
B2L2DB'F'LDU'L2R2F'U2B'D2FLR'U'	L3R1F2B1L3R3D2U1F2L3R1D2L2R2F2B3D3U3F2R3F1B3
LD2L2DBR2BF2L2D2L2F'L2FU'R2U2R'	L2F2R2D3U3F3B1L1F1B1D2U3F1B3D1U3F2B1L3R3D1
F2R2F2U2BF'R'BFDBF'D'UF'L'R'U'	F3B3D1U1L1R1D3F1B1D2U2L3R2F1B3D3U1F1B3L3F3B3D2U3
B2U'L'R'U2F'LR'B'FL'R2DUB'U2LR'	L3R1D2F3B2D3U3L1R2F1B3L3R1B1D2L1R1D3U2F2
BF'D'UB2FL'R'UR2B'FD2R'B'F'R2U'	D1R2F1B1R3D2F1B3L2D1L1R1F2B3D3U1F3B1
L2B2L2D'U'BF'L'BFUBF'DU'F'L'R'U'	D2U3L1R1F3B2D3U1F1B3D2U1F2B2R2F1B1R1D2F1B3L2
B'FDU'F'R2B'F'DUR'F2DU'R2F'L'R'U'	F1B3L1R2B2L3R3D1U1F1U2L1R3F2D2U3L3R3F2B1L1R3

D'BFR'F2DU'R2F'D'U'L'DU'LR'U'R2U'	D2R1D3U1L3R1D3U2L1R1B3U2L3R1B2U3F1B1L1R1
D'B'LR'DF2L2UR2U'BF'R'U2B2R2U2R'U'	F3B3R2F1D3U1F3B1U1L1R1B3R2D3U1F2L3D3U3
L'RD'UL'U2L'R'BFD'B2LR'U2F'L'R'U'	F3B3R2F1D3U1F3B1U1L1R1B3L2D1U3F2R3D1U1
R'DLD'B'R'F2DU'BRFU'BR2B2F2U2R'	B2L3R2F3B1L1R3F3B2L1R1D2U1B2L1R3U2B3D1U1L1R1
LFRBDL'RB'DU'L'R2D'UR'D'F'R'U'	F2R1F1B3L3R1F2B3L1R1U3B2L1R3U2B3D1U1L1R1
F2R2B2R'F2R'BF'DB2R2DB2L'R'B'DU'R'	F2R1F3B1L3R1F1L1R1D2U1B2L1R3U2B3D1U1L1R1
D'BFL'F2D'UL2F'D'U'L'DU'LR'U'R2U'	B2D3L1R3D1U3L2R1F3B3D1U2F2L1R3D2F3L3R3D1U1
D'F2DB'FDU'BDURF2DU'R2FL'R'U'	F1B2L3R2D2U1F1B3L3R2F3L1R3D3U2B3L3R1D2U3L1R2F2B3
DL2F2R2B'D2LRU'R2U'F2D'R2U'B'F'D2R'U'	L3R1B2D1U2L1R1F3B3L2R2D2F3B2L1R3F3B1L1D1U1F2B1U2

Symmetry Type D_4

D'U'	D3U3
DB2F2L2R2U	U1L2R2F2B2D1
B2F2D'UL2R2U2	L2R2D1U3F2B2D2
DU2B2F2L2R2U'	D2U1L2R2F2B2D3
DL2F2L2R2F2R2U	B2R2F2B2R2F2D1U1
UL2R2DU'L2R2U	D1F2B2D3U1F2B2D1
U'L2R2DU'L2R2U'	D1U2F2B2D1U3F2B2D3
B2L2B2DU'R2F2R2U2	L2R2D3U3F1B1D2U2F1B1
B2F2L2R2UB2F2L2R2U'	U3F2B2L2R2U1L2R2F2B2
B2F2L2R2D'B2F2L2R2U'	L2R2F2B2U3L2R2F2B2D3
L2R2UB2F2L2R2UL2R2U2	F2B2D1F2B2L2R2D1F2B2D2
D2B2F2L2R2DB2F2L2R2U'	F2B2L2R2D2U3L2R2F2B2D1
D2B2F2L2R2U'B2F2L2R2U'	F2B2L2R2D1U2L2R2F2B2D1
L2R2D'B2F2L2R2D'L2R2U2	L2R2D3U1L2R2D2U1L2R2F2B2D1
B2L'RBF'D2F2LRU2R2U2	D1U1B2D3U1F3B3L3R3F3B3D2F2
R2BFLRBFDUR2D'U'	D3U3F2D1U1L1R1F1B1L1R1F2
B2U2BFL2D2L'RBF'R2U2	D1U1B2D3U1F1B1L1R1F1B1U2F2
R2B'F'L'R'B'F'D'U'R2D'U'	F3B3L3R3D1U1F3B3D1U1F1B1D1U1
B2F2L2R2U'L2R2D'UL2R2U'	F2B2D1U3F2B2D3L2R2F2B2D3
B2L2B2L'RBFU2R2U2LR'	L1R1D1U1F1B1L1R1D1U1F1B1D3U3
B2R2F2L'RB'F'D2R2U2LR'	F3B3U2F3B3L3R1F3B3D2L1R1F2
DL2R2DB2F2L2R2DL2R2U'	L2R2D3U1L2R2D3U2L2R2F2B2D1
L'RD2L2U2BFD2R2U2LR'	L1R1D3U3F1B1L1R1D3U3F1B1D3U3
L'RU2R2U2B'F'U2L2U2LR'	L2R2U2L1R1D3U3F3B3L1R1D3U3F1B1D3U1
L2F2U2L2F2R2B2U2L2DU'R2U2	D2F3B3L2D3U3F1B3D1U1L2F1B1
R2F2U2L2B2R2F2U2R2DU'R2U2	L1R1F3B3L1R1F3B3L1R1F3B3D1U1
L2B2F2LR'F'L2R2D2U2FL'R'	F3B3R1F2B2D2U2R3F1B3L2R2F2
R2U'B2F2L2R2ULRBFLR'	U1L3R3D3U3F1B1L3R3D1U1F1B1D1
LRB2L2R2FL2R2D2U2FL'R'	F3B3L1R3F2B2L1F2B2D2U2R3F1B1
L2F2D2L2F2R2B2D2L2DU'R2U2	D2F1B1R2D1U1F1B3D3U3R2F3B3
R2F2D2L2B2R2F2D2R2DU'R2U2	D1U1L1R1F3B3L1R1F3B3L1R1F3B3
ULR'U2L2U2BFD2L2U2LR'U'	F3B3L1R2F2B2L1R3F2B2R1F1B1D3U3
UF2L2B2LR'BFU2R2U2LR'U'	F1B1L3R2F2B2L3R1F2B2R3F3B3D3U3
UF2R2F2LR'B'F'D2R2U2LR'U'	U1L1R1D1U1F3B3L3R3D1U1F1B1D1
LRF'R2B2F2R2D2U2FD2U2L'R'	F1B1L3F2B2D2U2L1F3B1L2R2F2D2U2
D2LRB'L2R2D2U2B'L2R2F2L'R'U2	F1B1L2R3D2U2F2B2R3F1B1R2D2U2R2
D'LR'D2U2R2FL2R2D2U2F'L'R'U'	F1B1R3F2B2D2U2R1F3B1L2R2F2D3U3
DLRB2L2R2FL2R2D2U2FL'R'U	F3B3L1R3F2B2L1F2B2D2U2R3F1B1D1U1
BR'DF2L'BD'UR'BL2D'FR'U2	L3R2D3U3L1R3U2F1B3L2R3F1B3L3R1F3B2D1U1L2R1D1U1
BU2L'ULB'D'URB'U'BU2R'U2	L1D3U3L1R3D2F3B1R1F1B3L3R1F1D1U1L3D3U3
D'L2B2F2LR'F'L2R2D2U2FL'R'U'	F3B3R1F2B2D2U2R3F1B3L2R2F2D3U3
DLRB'L2R2D2U2B'L2R2F2L'R'U	F1B1L1R3F2B2R3F2B2D2U2L1F3B3D1U1
BFR'D2B2R'DU'BD2B'U2B'LR'U	D1F2L1R3D2B3L1R1B2D3U2L1R3D3U1L2R1F3B3
BFR'D2B2R'DU'FL2B'R2F'LR'U	R2B3D3U1F3B1D1U2L3R3F3L2D3U1F2D2U2L1
DL'D2U2LD2BF'L2UR2BF'U2R'U	U3L3F3B3L1R3B2D3U1L3R2D1U3L1R3D3U2F1B1L2R3D1U2
D'LU2BF'R2UL2R2U'L2BF'U2R'U'	B2U2L1R3D3U3L1R2D3U3F3L3R1F3B1L2R3D1U1F3L1R1

D'LB2F2L'U2BF'R2DL2B'FU2R'U'	B2D2L3R1D3U3R3D3U3F1B2L3R1F3B1L2R3D1U1F3L1R1
BFLB2DU'L2F'DUF2RDU'LR'U'	L1R3F3L3R3D3L3R3D3U3F1B1D3L1R1B1L3R1D2
F2DB2D'B'FLD2B2LD'U'B'F2LR'U'	D3U2B2L1R3U2B3L3R3B2D1U2L1R3D1U3L3R2F1B1
U2BFR'D2B2R'DU'FL2B'R2F'LR'U'	D1R2F1B3D2L3D3U3R2F2B1L1R3F1B3L2R3F1B1
BFRF2DU'R2FDURD'ULR'UR2	F1B1R1F2D1U3R2F1D1U1R1D3U1L1R3U1R2
B2U'BF'DU'B'D'U'R'F2DU'R2F'L'R'	F3B3R3F2D1U3R2F3D3U3L3D1U3L1R3U3L2
B'LR'DF2L2UR2U'BF'R'U2B2R2U2R'	L3R2D2F3B1R2U1F1B1L2R3F3B1L1R3F2B3L2F3B3
BDLDB'L'R'DF'L'B'D'UF'D'U'R2	L1R2B2D1U3L2B3L3R3D3F1B3D3U1F3L2F1B1
DLB2F2L'U2BF'R2DL2B'FU2R'D2U'	D2U1R3F3B3L3R1B2D1U3L1D1U3L1R3D3U2F1B1L2R3D1U2
BFL'F2D'UL2F'D'U'L'DU'LR'U'R2	F2U3L1R3D3U1R3F3B3D2U3F2L3R1U2F1L1R1
F2DB'FDU'BDURF2DU'R2FL'R'	B2D3U2L3R1D3U1L2R3F1B1D2U1F2L3R1U2F3L3R3
L'R'FD2L'RF2U'B'F'D2R'DU'LR'U	U2L3R1F2B1L3R3D1U2F3B3D3U3L1R1D1U2L1R1F2B3L3R1
U2L'R'FD2L'RF2U'B'F'D2R'DU'LR'U'	F1B3L1R2F3B3U3L3R3D3U3F1B1U3F1B1L3R2F1B3D2
LD'U2B2R'UBDB'F'UFDL'F2D2U'R	R2F3B1U2R3F3B3D3U3R2U1F3B1D1U3F2B3L1R1U3
B2U2R2F2ULRBR2DU'B2R'B'F'R2U'R2	F1B1R3F2D3U1L2B3D1U1L2R3D3U1L1R3U1R2
R2B'D2F2U2L'RU'R2B'F'R'F2R'U2L'F2R'	L2R2F2D1L1R3D3U1L1F1B1D2U1F2L1R3D2B3L1R1
F2D'LR'DU'L2RB'F'UF2L'RU2FL'R'	F3B3R1F2D3U1L2B1D3U3L1R2D1U3L1R3U3L2
D'L'D2BF'L2D'B2F2D'R2BF'L2R2U2R'U'	B3D2L1R3F2L2R2D2U3F3B3L3F3B1L1R3B3D1U3R2
D'R'D2B'FL2U'L2BF'U2R'U2L2R2U2R'U'	B2D3U1L1D1U3L1R3D1F1B1R3F3B3L1R3D3U3F2B2L2R3
BFLF2DU'R2BD'U'LR2DU'LR'U'R2	F2D2U1L3R1D3U1L3F3B3U1F2L3R1U2F1L1R1
B2DLR'D'UL'R2BFU'F2L'RU2F'L'R'	B2D1L1R3D3U1L3R2F1B1U3F2L3R1U2F3L3R3
B2L'R'BU2LR'F2U'B'F'R'DU'LR'D2U'	L1R2U2F1B3L2F2B2D1U2L1R1F3B2D1U3F1B3U1L3R1
R2B'FD2L'BFDB'FD'UFD2BFL'R'U'	F3B2D3U3F1D3U3F2B3L1R1D2U2F2B1L1R1B3D3U3F3L3R3
D'B2D'L'RDU'L'B'F'D'B2LR'U2F'L'R'U'	D3U3R2D3F1B3D1U3B3L3R3D3R2F3B1U2L3F3B3
F2R2U2F2D'B'F'L'DU'BR2D2FU2F'LR'U'	L3F1B1L1D3U3L3R2D1U1F2B2L2R3F3B3R3F3B3L2R1F3B3
B2U2F2D2LR'D2BLRDBF'DU'F'L'R'U'	F2U2L1R3D3U3R1D1U1F1L3R1F1B3L1R2D3U3F1L3R3
B2LR'U2FDUL'R'F2R'B'FLR'F'L'R'U'	F2D2L3R1D3U3L3R2D1U1F3B2L3R1F1B3L1R2D3U3F1L3R3
L2B2R2F'U2LRD'R2D'B2U'R2D'B'F'U2R'U2	L2R2F2D3U2L3R1D3U1R1F1B1U3F2L1R3D2B3L1R1
B2U2B2U2LR'U2BLRDBF'DU'F'L'R'U'	F2L3R1U2F2B3L3R3D1U1F2U1L3R1D1U3R1F1B1D2U1
R2B2U'B2D2L'R'U2B'R2D'UF2R'BFUR2U2	F2L3R1U2F2B3L3R3D3U3B2D2U3L3R1D1U3R1F1B1D2U1
F2L'RF2D2F2U2B'LRUBF'D'UF'L'R'U'	D1U2B2L1R3U2F3L1R1D3U3B2U3L1R3D3U1R3F3B3
F2L'RU2F'DUL'R'F2L'B'FL'RF'L'R'U'	D2U1F2L3R1U2F3D1U1L1R1B2L3F3B1L3R1F2B1L3R3
R2B2D'BFRDU'F'L2U2FR2F2R2F'LR'U'	B2U1L1R3D1U3L3R2F1B1D1F2L3R1U2F2B3L1R1D1U1
B2D'F2L2U'BF'RDUFD2B2L2D2F'LR'U'	B2D2U3L3R1D1U3L2R3F1B1D3U2F2L3R1U2F2B3L1R1D1U1
UB2F2L2R2U'	D1L2R2F2B2D3
U'B2F2L2R2D2U'	D1U2L2R2F2B2D1
D'L2R2D'UL2R2U	D2U1L2R2D3U1L2R2D1
DL2R2D'UL2R2U'	U1F2B2D1U3F2B2D3
R2B'F'L'R'B'F'D'U'R2	F2D1U1L1R1F1B1L1R1F2
L2U2L'R'B'F'L'R'DU'R2	B2D1U3F1B1L1R1F1B1D2F2
LRBD2U2BF'D2U2F'L'R'	F1B1L2R3F2B2L1R3F2B2L3F3B3
LRFR2B2F2R2D2U2F'L'R'	F1B1R1D2U2F2L2R2F2R3F3B3
L'R'B'L'RB2F2LR'FLR	F3B3R3F1B3L2R2F3B1L1F1B1
F2R2DUR2F2R2F2D'U'F2R2	L1R1D1U3L3R3F2D3U3L1R3D1U1F2
F2R2D'U'L2B2L2B2DUF2R2	L1R1F1B1L1R1F3B3L3R3F3B3
F2R2DUL2F2R2B2D'U'F2R2	D3U1F3B3R2D3U3F1B3D1U1R2F1B1
F2R2D'U'R2B2L2F2DUF2R2	L3R3F1B1L3R3F1B1L3R3F1B1
D2R2B'F'L'R'B'F'D'U'R2U2	F1B1L1R1D3U3F1B1D3U3F3B3
B2L2DUR2B2R2B2D'U'F2R2	L1R1F3B3L3R3F1B1L1R1F3B3
B2L2D'U'L2F2L2F2DUF2R2	L1R1D3U1L1R1B2D1U1L3R1D3U3F2
B2F2L2BFLRBFDUR2	L2R2B2D3U3L3R3F3B3L3R3F2
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R2F2D2R2F2R2F2U2R2DU'R2U2	D3U3L3R3F3B3L1R1F3B3L1R1F1B1
L2F2D2R2B2R2B2U2L2DU'R2U2	F1B3D3U3F1B3D1U1F1B2R2D2U2L2F3
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L2U2LRD2BF'U2L'R'U2R2U2	B2L1R1F1B3L3R3F2B3R2D2U2L2F3

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B2D2B'F'U2F2LR'BFD'U'R2	F2D2L2R2D1U3L3R3F1B1L1R1F2
L2BFL'R'F2D2U2BF'D'U'R2	L2R2F2D3U1F3B3L3R3F3B3D2F2
R2F2U2R2F2R2F2D2R2DU'R2U2	D3U3L1R1F1B1L3R3F1B1L3R3F3B3
L2F2U2R2B2R2B2D2L2DU'R2U2	D1U1L1R1D1U3L1R1B2D1U1L1R3D3U3F2
B2LRF2L2B'FD2LRU2R2U2	D1U1L2R2B2D3U3L3R3F3B3L3R3F2
L2U2L'R'U2BF'D2LRU2R2U2	D3U3L2R2F2D1U3F1B1L1R1F1B1D2F2
D'F2L2B2LR'BFU2R2U2LR'U'	F3B3L2R1F2B2L3R1F2B2L1F1B1D2U2
B'F'L'R'B'F'R2DB2F2L2R2UR2	L3R3F3B3L3R3F2D1F2B2L2R2U1F2
F2D'U'R2D'BFD2BFL2R2UR2	L2D3U3F2D2U3L2R2F3B3U2F3B3D3F2
B2DUL2DB2F2L'R'D2L'R'U'R2	L2D1U1B2U1L2R2F3B3U2F3B3D3F2
L2R2D2F2DF2R2U2L2F2U2F2R2U'R2	L2R2F2B2D3U2F1B1L3R3F1B1L3R3F1B1L3R3D2U3
L2R2D2F2DB2L2U2L2F2D2F2R2U'R2	F3B3R3F2B2D2U2R3F3B3D1U1F3B3L3R1F1B1
LRB2R2BD2U2L2BF'U2FU2L'R'	L1R1F1B1L1R1B2D1U2F2B2D1U3F2B2D3F2
F2U'F2L'R'BFLU2R2DU2R2U2R'	L2R2F2B2U1F2D3U3L3R3F3B3L3R3F2D1
B2U2B2DR2BFL'R'BF'U2F2U'R2	R2D1U1F1B1L1R2U2F3B3L2D1U1F1B3L3F1B1
U'LU2BF'R2UL2R2U'L2BF'U2R'U	D1L3F3B3L1R3B2D3U1L3R2D1U3L1R3D3U2F1B1L2R3D1U2
U'LB2F2L'U2BF'R2DL2B'FU2R'U	D3U2R3F3B3L3R1B2D1U3L1D1U3L1R3D3U2F1B1L2R3D1U2
UL'D2U2LD2BF'L2UR2BF'U2R'U'	D2U3R1D2U2L3B2D1U3R2F1B2L2D1U3F2R1D2U1
B'U2F2DU'R'B'FU'B2D'F2D'BF'R'	L2D3U1F2B1D3U1F1B3U1L1R1F3L1R1F1B3D3U3F2B1
F2D'R2DBF'R'F2U2R'DUF'LR'U'	D1U2B2L3R1D2B3L3R3B2D2U1L3R1D3U1L2R3F1B1
BFR'U2F2R'D'UBL2F'R2B'LR'U'	L1R1D2U1L2F1B3U2L3F3B3D1U2F1B3D1U3F3B2D2
B'D'UR'B2L'F2L'DU'B'D'UR2U2R'	F1B3L1F1B1U3L1R1D3U3F3B3U3F3B3R3F3B1U2
URU2B'FL2DB2F2DU2L2BF'U2R'U'	U1L1D2U2R3B2D3U1L2F1B2L2D1U3F2R1D2U1
B2L2UB2U2BF'R'F2D'UL2F'L'R'U'R2	D3U2F3B3R3F3B3L3R3D1U1R3F1B1D1F1B3L3R1F3B1
F2R'U2BF'L2D'F2D2L'RB'L2R2F'U2R2	F1B1U1L2F3B1D2R1D1U1F2B3L1R3F3B1L3D3U1
B'DULU2L2F2U2L'D'U'BL2FR2F'R2	F1B3L1R2F3B3U3L3R3D3U3F1B1U3F1B1L3R2F1B3D3U1
F2LRFU2LR'F2D'BFLDU'LR'U	F3B1R3F3B3U3F3B3D3U3L1R1U3F1B1L1F1B3D1U3
F2L'FLR'URB'FL'D'R'U'L'F'R'U'	D1U2F2L3R1U2B3L1R1B2D2U3L3R1D1U3L1R2F3B3
B'F'LU2F2LDU'B'R2FL2BLR'D2U'	D3U2F2L1R3D2F1D3U3R3F3B1L1R3F1D2L1R1
B'F'LB2D'UR2B'D'U'B2L'DU'LR'U'	F3B1R1F1B1U3F1B1D3U3L3R3U3F3B3R3F3B1D3U1
L'B'FDB'F'DU2BDRF'DB'D'F'R'	F1B3L3R2F1B1U3L1R1D3U3F3B3U3F3B3L2R1F3B1D1U3
R'F2D2BFD'R2U'L2F2R2UB2D'BF'R'	F1B3L3R2F1B1U1F1B1D1U1L3R3U1F3B3L2R1F3B1D1U3
L2BF'D2LBFL2UBF'DU'F'L'R'U'	F3B1D3U1B3D3U3L1B2D1U3L2F3L3R3B2D2U3
L2BF'D2L'BFUBF'DU'F'U2L'R'U	R2F1B3D2L2R1F1B1U1F1B3D1U3B3U2L3R3D2U3
R2U'BF'LF2LD2U2R'U2L'D'UF'LR'	D1U2F3B3D2U3L1R1U1F2B2L1R1U3F3B3D3F3B3D1L1R1
LDU'FR2B2FL2R2F'L2FL2R2DU'R'	F1B3L1R2F3B3U1F3B3D1U1L1R1U1F1B1L3R2F1B3D3U1
B2FU2L2F'D'URB2L'F2R'BF'U'L'R'	L1R1U1R2F3B1U2R1F1B1U1F3B1D1U3F1B2U2
B2L2B2D'L'R'B'LR'BF'LDUF'LR'	L2B2R2D3U2F3B3L1R2F3B1L1R3F2B3D1U1L1F1B3
F2R2B2UL'R'B'L'RBF'RDUFLR'	R2B2R2D2U3F1B1L2R3F3B1L1R3F1B2D3U3L2R3F3B1
B2R2U'B2U2B'FL'F2D'UL2B'LRUR2	F3B1L3R2F1B1D3F1B1D3U3L3R3D3F3B3L1R2F3B1U2
B2L'D'U'FD'UR'B2R'F2RB'FU'LR'	R3D1U3L1R3D2U3L1R1F2B1U2L1R3F2D3F1B1D3U1
B2D'L'F'D'BL2F2L2U2L2U2F2RU'L'F'R'	F2L3R1U2F3B2L1R1D1U1F2D2U1L3R1D3U1L3F3B3D2U3
RUFU'LR'BLR'BF'LBF'U'F'U'R'	F2L3R1U2F3B2L1R1D3U3B2U3L3R1D3U1L3F3B3D2U3
L2FL'RB'FLBFD'U2B2F2L2BF'U2R'	R2F1D3U1F3B1U1L1R1F3B2R2D1U3F2D2U2R3
UB'L'RD'L2D'R2U'LR'B'DU'L2U2R'U'	F3L2R2F3D1U1L3R3F3B1L3R3B1D1U1F1B3L3R3F2B1
B'F'D'B2L'RD2B'L'R'D'L'RDU'L2R'U2	L1F3B1D1U3F3B1L3F3B3D1U2L1R1D1U1F3B3U1F3B3
B'F'D'L2B'FU2R'D'U'B'LR'B'FL2R'U2	F2B1L3R1F3B1R1D1U1B3U2L3R1B2D2U1L1R1U2
B2D'L2R2UL2UB'FR'F2R2DU'FLR'U'	D1F3B3D2U1L1R1D2U1F2B2L1R1U3F3B3D3F3B3D1L1R1
R'D'B'L'U2LBFDU'R'DBU'L2F2R2U'	U3F3B3L1D3U3L1R1F1B1L1F1B1D1F1B3L3R1F3B1
BFDF2LR'D2FLRD2U'LR'DU'R'U2	F2B1L3R1F3B1R1B2D1U1F2B1D2L3R1F2U3L1R1
L2U'LRBD'UBF'D'B2L'R'B'U2LR'U2	D1F3B3D2U3L1R1U1F2B2L1R1D2U3F3B3D3F3B3D1L1R1
L2F2D'LRFDU'B'FU'L'R'FL2F2LR'	D1U2F3B3D2U1L1R1D2U1F2B2L1R1D2U3F3B3D3F3B3D1L1R1
L2DR2U'B'FLD2B2LDUL2R2F'LR'U	L2F3D1U3F1B3U3L3R3F3R2D1U3F2D2U2L2R3
B2LR'F2U2F2U2F'LRDBF'DU'F'L'R'U'	F2D1U2L1R3D1U3R3F3B3D1F2L1R3D2F1L1R1D3U3
D'BLR'BF'RD2LRB'F'U'R2BF'U2R'U'	F2D3L3R1D1U3L3F3B3D3U2F2L1R3D2F1L1R1D3U3
B2F2L2R2DU	L2R2F2B2D1U1
DUB2F2L2R2	D1U1L2R2F2B2

U2L2R2DU'L2R2	U2F2B2D1U3F2B2
L2R2DU'L2R2U2	F2B2D3U1F2B2D2
LR'U2L2U2BFD2L2U2LR'	F3B3D3U3L1R1D3U1L3R3D1U1F1B1D2
LR'D2R2U2B'F'U2R2U2LR'	L1R1D3U3F3B3L3R3D3U3F1B1D1U1
D'R2B'F'L'R'B'F'D'U'R2U'	U3F2D1U1L1R1F1B1L1R1F2D3
DR2B'F'L'R'B'F'D'U'R2U	D2U1F2D3U3L3R3F3B3L3R3F2D3
F2L2B2LR'BFU2R2U2LR'	F3B3D3U3L1R1D3U1L1R1D3U3F3B3D2
F2R2F2LR'B'F'D2R2U2LR'	L3R3D3U3F1B1L1R1D3U3F3B3D1U1
DF2D2F2LRF2D2F2D2L'R'U'	D1F3B3L2D3U3F1B3D1U1L2F1B1D1
DB2D2B2LRF2U2F2D2L'R'U'	U3L1R1F1B1L1R1F3B3L3R3F3B3D3
U'B2U2BFL2D2L'RBF'R2U'	D1B2D1U3F1B1L1R1F1B1D2F2U1
DBFU2R2U2R2B'F'R2U2R2U'	D1U2F3B3L2D3U3F1B3D1U1L2F1B1D3
DBFU2L2D2L2B'F'R2U2R2U'	U3L3R3F1B1L3R3F1B1L3R3F1B1D3
D'L2U2L'R'B'F'L'R'DU'R2U'	U3B2D3U1F1B1L1R1F1B1U2F2D3
DUL'D2U2LD2BF'L2UR2BF'U2R'	F1L1R1F3B1L2D3U1F1B2D1U3F3B1D2U1L3R3F2B1D1U1
LB2F2L'U2BF'R2DL2B'FU2R'D'U'	F2B3L2R2F2B1D1U1F3B1L1R1F3D3U3F1B3L1R1B1D3U3
DF2D'R2DBF'R'F2U2R'DUF'LR'	D1U2L3R3D2B3D1U3F1B3U1F1B1R1U2F1B3R2
UBFR'D2B2R'DU'FL2B'R2F'LR'	D3L3R3U2B3L3R1F3B1R1D1U1B1D2L3R1F2
B'FRU2RD2R'D'UFR2D2FL'R'U'	R2F3B1U2R3F3B3U3F3B1D3U1B1D2L1R1D3U2
D'U'LB2F2L'U2BF'R2DL2B'FU2R'	D3U3B1L1R1F1B3D3U3F3L1R1F3B1D1U1B3L2R2B1
B'FLF2RB2L'D'UFR2D2FL'R'U'	F2L1R3D2B3D3U3R3F1B3L1R3B1U2L1R1D1
LU2BF'R2UL2R2U'L2BF'U2R'D'U'	D1U1F2B3L1R1D2U3F1B3D3U1F3B2D1U3L2F1B3L3R3F3
B2L'F'R'D2U'L2FD2LDBD'F2RU2R2	B2D2U3L1R3D1U3L3R2F1B1U1F2L3R1U2F1L1R1
F2D'L'RDU'L'B'F'U'F2L'RU2F'L'R'	B2U1L3R1D1U3L2R3F1B1D2U3F2L3R1U2F1L1R1
UL'R'FD2L'RF2U'B'F'D2R'DU'LR'	U1F3B3R3U2F3B1L2U1L3R3D2B3D1U3F1B3
U'BFLB2DU'L2F'DUF2RDU'LR'	D2U1B2L1R1F1R2D1U3B2R3D1U1F3B2D1U3F3B1
BF'DU'FR2DURB2D'UR2F'LRU'	F3B1D3U1B1D2L1R1U3L2F1B3U2R1F1B1U3
BF'D'UB'R2D'U'RF2D'UL2F'L'R'U	F1B3D3U1F1B2D3U3R1B2D3U1R2F3L3R3B2D2U3
B2U2BL2D'RDFD2LF2D2U'B'L'F'R2	L1R1F1U2L1R3F2U1F1B1L2R3D1U3L3R1D2U3B2
BLR'BF'RD2LRB'F'U'R2BF'U2R'	L1R1F1U2L1R3F2D2U3F1B1L3R2D1U3L1R3U1B2

Symmetry Type D_{2d}(face)

U'LRB2F2L'R'U'	D1U1F3B3L2R2F1B1
U'LRD2B2F2U2L'R'U'	D3U3F2L3R1F2B2L3R1F2
U'B2L2U2B2F2U2R2F2U'	L2R2D1U1B2D2U2L2R2F2
U'B2L2R2U2B2F2U2F2U'	L2R2D1U1B2L2D2U2L2F2
B2F2U'B2F2L2R2UL2R2	L2R2U1L2R2F2B2U3F2B2
L2R2U2L2R2U'B2F2L2R2U'	L2R2D2U3F2B2D3U1F2B2D3F2B2
B2F2D2U'B2F2L2R2U'L2R2	L2R2D2U1F2B2L2R2U1F2B2
B2F2D2B2F2U'B2F2L2R2U'	F2B2U2F2B2D3F2B2L2R2D3
UL'R'U2B'F'U2B'F'LRU'	D3U1F3B3L2D3U3F2D1U1R2F1B1
UL'R'U2BFD2BFLRU'	D3U3L1R1F1B1L1R1F1B1L3R3F1B1
U'B'F'R2U2BFR2U2F2R2U'	D2U3F3B3L2R2U2F3B3L3R3U2L3R3U1
DB2R2BFLR'D2B'F'R2U'	F1B1L1F2B2D2U2L3F3B1L2R2F2
L2B2F2R2D'B2F2R2D2U2R2U'	U1F2B2L2R2D1F2L2R2D2U2F2
L2B2F2R2D'F2D2U2F2L2R2U'	F2L2R2B2D1F2B2L2D2U2L2U1
DB2R2B'F'LR'U2B'F'R2U'	D2U1L3R3D1U3F1B1L1R1D1U3F3B3U1
ULRU2BFU2BFL'R'U'	D1U3F1B1L2D1U1F2D3U3R2F3B3
ULRU2B'F'D2B'F'L'R'U'	D1U1L3R3F1B1L3R3F3B3L3R3F3B3
D'B2L'R'U2BFR2D2L'R'U'	D3U2L3R3D3U1L3R3D1U3F1B1L1R1D3
D'F2LRD2B'F'R2U2L'R'U'	D1L3R3D3U1L1R1D3U1F3B3L1R1D3
F2L'R'B2R2B'FU2LRU2R2U2	R2F1B1L3R3F2L3R3U2F2B2L3R3F3B1
R2D2LRD2BF'U2LRU2R2U2	F2B2D3U3B2D1U1L1R1F1B1L1R1F2
B2L2R2F2DU2F2D2U2F2L2R2U'	D1U2L2R2F2B2U3F2D2U2L2R2F2
U'F2R2D'ULR'B'FL'R'F2U'	D2U3L1R1F3B3D3U1L3R3D1U3L3R3U1
U'F2L2DU'L'RB'FL'R'F2U'	D2U3L3R3D3U1L3R3D1U3F3B3L1R1U1
R2B'F'L2F2L'RB2R2B'F'R2U2	B2L3R3F3B1L3R3F2B1D2L2R2U2F1

R2D2L'R'U2BF'D2L'R'U2R2U2	D3U3B2L3R3F1B1D2U2L1R1D1U1F2
DF2LRB2F2D2B'F'R2U2L'R'U'	U1L1R1F1B1D1U3L1R1D1U3L3R3U1
DB2L'R'U2BFR2B2F2U2L'R'U'	U1L3R3D3U1L1R1D3U1F1B1L1R1U1
L2U2B2F2U2R2U'L'RD2U2LR'U'	F2B2U2L2R2U3F2B2L1R1F2B2L1R1U1
L'R'BD2U2B'FD2U2F'LRD'U'	D1U1F3B3R1F2B2L1R3F2B2L3F1B1
R2D2U'B2F2L2R2U'LRBFLR'	F1B3L3R3F3B3D3U2F2B2L2R2D3F2
LR'U2R2B2F2D2BFU2R2U2LR'	D1U1L1R1D1U1F1B1L1R1D1U1L2R2F1B1
LR'U2B2F2L2D2BFU2R2U2LR'	F3B3U2F3B3L3R1F3B3D2F2B2L1R1F2
F2L2R2F2L2R2D'B2F2R2D2U2R2U'	L2R2U2F2B2U3F2B2L3R3F2B2L3R3U1
F2L2R2F2L2R2D'F2D2U2F2L2R2U'	L2R2D1U1L2R2D3F2B2L3R3F2B2L3R3U1
LRB'D2U2BF'D2U2FL'R'D'U'	D1U1F1B1R3D2U2L3R1D2U2L1F3B3
RD2B2F2U2R'BFRB2D2U2F2R'	F1B1D1U1L1R1D3U3L1R1D1U1L2R2F1B1
B2L2D2U2F2LR'B'F'U2R2U2LR'	F1B1D2L2R2F1B1L3R1F3B3D2L1R1F2
L2R2U2B2F2UR2B2F2U2B2F2U2R2U'	F2B2D1U1F2B2D3F2B2L1R1F2B2L1R1U1
LRF'R2B2F2R2D2U2B2F'L'R'D'U'	D3U3F1B1L2R1F1B3L2R2F3B1L1F3B3
R2UB'FR'DUB'R2D2FLR'D'R2U'	U2F3B2L3R1D3L3R3B1D3U1L1F3B3U3F3B1L3
R2F'LFLF2R'U'F'DU'L'B'FUR'	D3U1L2R3F3B1L1R3F3D3U3L2R1U2F3B1L2D3F3B3
U'LD2L2BF'UB2F2D'B'FR2U2R'U'	L3R1D2U3L3R1D3U1L2R1D3U3B3L2D3U1B2R1
U'LDL2F'URB2F2L'D'FR2U'R'U'	L1R3D1U2L1R3D3U1L2R3D1U1B1R2D1U3B2L3
F'R'BL'D'UBR'F2DRF2L'U'F'R	R2D3U2L3R1D1U3L2R1D3U3F3L2D1U3F2R1F3B3
U'LB2F2L'U2L2U2B'FUR2BF'U2R'U'	L3R1D1U2L1R3D3U1L3D3U3F1L2D3U1F2D2U2L2R3
B'D'RFLU'BRF2RFLF'L'R'F'U'	L1R1D2U1L3R3D2U3F3B3D3U2L2R2D1L1R1D1U2F1B1D1F3B3
D'R'B'DU2L2DUR'BFR'D'U2B'R'U'	U3B1R1D2U1B3L1R1F3B3D2U2F1D1U2R1B1D1U2
F2L2RU2FUL'FD'UL'BU'LU2F'R2	L2R1D3U3F1B2D3U1L1R3D1U3B3L2R2F2D3U3R1D1U1
B2LR2D2BD'RB'DU'LB'DLD2F'R2	R3D3U3F3D1U3L1R3D1U3F3L2R2B2D3U3L3R2D3U3
L'U'B2DU'B'FUF'L'D2F'LR2B'U'R'	F1B2D2L3R1F2D3L1R1F1L3R1F3B1L2R3F2L1R1
B'F'R2FL'RB'FLBFU'R2BF'U2R'	F2B1D2L3R1F2U3L3R3F3L1R3F1B3R3B2L1R1
U'LR2D'UFD2BL2R2FU2BDU'R'U'	L1R3D3U2F3B1D3U1B1L1R1D3R2F1B3U2F2B2L2R1
L2BL2U2FL'R'DBF'U2RU2F2L'U2R'	L2B3D3U1F3B1D1U2L3R3F3R2D1U3F2D2U2L2R3
RFRD'B'FLD'F'U'L'RU'F2L2F2U'	L2R3D3U1L3R1U3L3R3B1D2L3R1F2U1F3B3D3U1
D'RBDU2RB'F'RD'U'L2D'U2BRU'	D2U1F1R1D1U2F3L1R1F1B1D2U2F2B3D2U1R1F1D1U2
LFLU'B'FRU'F'D'LR'D2UR2F2R2U'	L2B1D3U1F1B3D2U3L1R1F3B2L2D3U1F2D2U2L3
R'D2B'FR2ULB2F2D2U2RDR2B'FU2R'	L2D2U1F1B3D3U1B3L3R3D1F2B2L2F1B3D2L2R3F3B3
D'B2F'D'UR'BFUL'RF'L2FD'UR'U'	L1R1U1L3R3U3F3B3U3F2B2U1L1R1D1U2F1B1D1F3B3
D'BR2B'F'L'RD'BF'D2B2U2LF2U2R'U'	L2D2F3B1D3U3F1B2D3U3L3F1B3L3R1F1D1U1L1R2F3B3
DF'LR'DUR2D'R2B2R2B'FR'F2U2R'U'	B2L1R3U2F1B2L3R3D3U3F2D3U2L1R3D3U1L1F1B1D1U2
DF2R'F2U2B2R'U'FUR'BF'UL'F'R'U'	L1R1U1L3R3U3F3B3D3U2L2R2D3L1R1D1U2F1B1D1F3B3
L2D'F2L'FRDRBU'F'L'RU2LBR'U2	L3R3D2U3F1B1D1L1R1D1U2L2R2D1U2F3B3U3L3R3D3F1B1
B2U2R2F2ULRFR2D'UF2R'B'F'R2U'R2	D1F1B1D1L3R3D1U2F2B2L3R3D1U2F1B1U1F1B1U3L3R3
LU'R'D'F'LR2F2U'R2FD2UB'R'F'R2U2	D3F1B1D3L3R3D1F2B2L3R3D1U2F1B1U1F1B1U3L3R3
R'B2FUR'F2R2D2B'F'U2L2F2R'UBF2R'	F2B2L2D3F3B1D1U3F3L3R3U1L2F3B1D2R1F3B3
F2LB2D2L2B2R'B'F'DLR'F'R2F'D2F'R2	F2B2L2D1U2F1B3D1U3B3L3R3D2U3L2F3B1D2R1F3B3
LD'F'R2U'FD2L'DU2LR2B2U'F'U2F'U'	L3R3D2U3F1B1D1L1R1U1F2B2U1F3B3U3L3R3D3F1B1
LDUF2UL2U2BD2L2DB'UR'UB2F'U'	L3R3D2U1F1B1D1U2L1R1D2U3F2B2D2U1F3B3U3L3R3D3F1B1
B2D2F2D'R2B'F'LB2DU'L2B'U2L'R'U'R2	R2D2F3B1D3U3F3D3U3L3F1B3L3R1F3B2D1U1L2R1F1B1
L2BLFD2U'B'L2UB2L2R'BDLUR'U2	R2U2F1B3D3U3F2B1D3U3L1R2F1B3L3R1F3B2D1U1L2R1F1B1
L'U'R2F'L2U'R2F2R2FD'L2RD'RU'R'U'	L1R3D1U1L3D3U3F1B1L2B1L1R3D1U3L1R3F3B2D1U1R3F1B1
L'U2B2U'B'D'U'B'UF2R2D2RUR2FU'R'	F1B3D1U1F1B2L1R1D3U3B2D3F1B3L3R1F3B1D1U2L3R3F1B2L1R1
D'L'B'D'U2L'B2D2BFU2F2R'D2U'F'R'U'	L2D1U1F3B3L2R1U2F1B3R2U3L3R3F3D3U1F3B1U3
U'R'B2D'UL2D2U2B'L2B2L2DU'R'D2U2R'U'	F1B2D3U3F1B2D3U3B3L1R1D2U2B1L1R1B1D3U3F3B2L3R3
D'F'D2L'R'U2F2DF2L'RU2BD2R2DU'R'U'	D1U1R3F2B2D2U3F1L1R3D1U2F3B1U3L1R3F3B2D2U1L2R3
DL2U2BU2F2LRB2U'B2LR'U2B'DUR'U'	R2D2F3B1D3U3F1B2D3U3R3F3B1L1R3B1D1U1L3F3B3
URB2F2LD2L2D2BF'U'B2F2R2B'FU2R'U'	R2U2F1B3D3U3B3D3U3L2R1F3B1L1R3B1D1U1L3F3B3
B2U2F2D2L'RU2BLRDBF'DU'F'L'R'U'	L2D2F3B1D3U3F3D3U3R3F3B1L1R3F2B3D1U1R3F1B1
D'L2D2FDUF2R'B2D'UR2F'D'U'R2U2R'U'	L2R3D3U3L2R1F3B3L3R2D2U2F1B1D2U2L3R2F3B3L2R1D3U3R1D1U1
UL2B'L2D2U2R2F'R2U'	D3F3B3U2L3R3U2L1R1F1B1D3
R2B'F'LRBFD'U'R2	L2D1U1F3B3L3R3F1B1L2
L2R2U'B2F2L2R2UL2R2	L2R2D3L2R2F2B2D1L2R2

L2R2D2B2F2U'B2F2L2R2U'	D1U2F2B2L2R2D3L2R2D2F2B2
D2U'L2F'L2D2U2R2B'R2U'	D3F3B3D2L3R3D2L1R1F1B1D3
UR2FL2R2F2L2R2F'R2U'	F3B3L1R1F3B3L3R3F1B1L3R3
L2R2D2U'B2F2L2R2U'L2R2	L2R2D3U2F2B2L2R2D3L2R2
R2F2D'U'BFLRBF'R2	L2F3B1L3R3F3B3D1U1B2L2
B2F2U2L2R2U'B2F2L2R2U'	F2B2D2L2R2D3F2B2L2R2D3
R2B'F'L'R'BFD'U'R2D'U'	D3U3F2L1R1F1B1L3R3D1U1F2
F2U2BFR2U2L'RBF'R2U2	D2L1R1F1B3L1R1D2F3B3L2F3B3
R2BFLRB'F'DUR2D'U'	D3U3F2D1U1L3R3F1B1L1R1F2
F2LR'B'FU2F2LRU2R2U2	F3B3L1R3F3B3D2L1R1B2L1R1D2
L2BFL'R'BFL2R2DUR2	R2F1B1L3R3F1B1L2R2D1U1L2
D'L2F2U2B2R2U2L2DU'R2U'	F1B1L1R1F1B1L1R1F1B1L1R1
B2F2L2BFL'R'B'F'DUR2	F2B2R2F1B1L3R3F3B3D1U1L2
LRB'D2U2B'FD2U2FL'R'	F3B3L3R2F2B2L3R1F2B2R3F1B1
F2R2D'U'L2F2R2B2D'U'F2R2	F3B3L3R3F1B1L1R1F3B3L3R3
F2R2DUR2B2L2F2DUF2R2	F3B3U2F3B3R2D3U3F2D1U1L2
U2R2B'F'LRBFD'U'R2U2	L1R1F1B1D1U1L3R3D3U3L3R3
LRF'R2B2F2R2D2U2FL'R'	F3B3L1F1B3L2R2F3B1R3F1B1D2U2
F2R2D'U'R2F2R2F2D'U'F2R2	F1B1L3R3F3B3L3R3F3B3L1R1
F2R2DUL2B2L2B2DUF2R2	L1R1F2L2R2D1U1R2D1U1F2L1R1D2
L'R'BL'RB2F2LR'F'LR	L3R3F1L3R1F2B2L1R3B3L1R1
R2B2D2L2F2R2F2U2L2DU'R2U2	D1U1L1R1F3B3L3R3F3B3L1R1F1B1
L2B2D2L2B2R2B2U2R2DU'R2U2	B3L2R2D2U2F3D3U3R3F2B2D2U2L3
L2BFLRB2D2U2BF'D'U'R2	R2F1B3L1R1D2U2F3B3D3U3B2L2
B2D2B'F'U2F2L'RB'F'D'U'R2	L2F3B3L1R1F3B3D3U1F2B2D2L2
F2DB2R2U2R2F2U2B2R2U'R2U2	F1B1D1U1F3B3D1U1L2R1F3B1L2R2F1B3R3F3B3
F2DF2L2U2R2F2D2B2R2U'R2U2	L2D3U3F3B3L1F2B2D3U3R2D3U3R3F3B3
F2LR'B'FU2B2LRU2R2DU'	L1R1D3U1L3R3D1U1F1B1D2L3R3
R2B2U2L2F2R2F2D2L2DU'R2U2	D1U1L3R3F1B1L1R1F1B1L3R3F3B3
L2B2U2L2B2R2B2D2R2DU'R2U2	F3B1D3U3F1B3D1U1F3B2R2D2U2L2F1
BFL'R'BFR2D'B2F2L2R2U'R2	L3R3F1B1L3R3B2U1L2R2F2B2D1B2
D2L'R'BD2U2BF'D2U2F'LRU2	L3R3F2B3L2R2F1B3L2R2F3L1R1D2U2
LRF2R2BD2U2L2BF'U2FU2L'R'	L1R1F3B3L1R1F2D1U2F2B2D1U3F2B2D3B2
F2D'F2L'R'B'F'LU2R2DU2R2U2R'	L2R2F2B2D1B2D3U3L3R3F1B1L3R3B2D1
B2D2B2U'R2B'F'L'RB2R2BF'U'R2	D1U2B2L3R1F1B1L3R3D3U3L2B2D1U2F2B2L2R2
L'U'R'D'F2UBLBF'DBF'R'F'R2	F3B3U1L2F3B1D2R1D1U1B1L3R1F3B1R3D1U3
FU'FR'B'FDBF2UBD'LR'F'R2	U3F3B3D3U3L1R1U3F1B1R1F1B3D1U3F1B3L2R1F3B3
B2L2R2D'U'L2DB'F'D2B'F'L2R2U'R2	F1B1D3U3F3B3D3U3L2R1F1B3L2R2F3B1R3F3B3
L2R2F2DUR2D'B2F2L'R'D2L'R'UR2	F1B1L1R3F3B3D1U1F3B3R3D2U2F2B2R3F3B3
BFL'DU'BD2FU2B'LR'DF2R2U'	B1D2U2F1L3R3D3U3F1B1D1U1B1D1U1F1B1L3R3F3B2
F2L2FL'RU'B2D'B2F2D'L2D'BF'R'U2	D1U2L1R1D3U3F3B3D3F3B3R3F1B3D3U1F1B3L2R3F1B1
L2U'BF'LD'U'BL2D2B2FL'RUR2U'	L1R1F3B2L1R1F3B3D3U3F1L1R1D2U3F1B3L1R3F1B3D1F2B2
R2B'F'RF2D'UL2F'DURDU'LR'U'	D3F1B3D3U1B1D1U1R3F2D1U3R2B1L3R3B2
F'L'RU'B2D'B2F2D'L2D'BF'RF2R2U2	D2U3F3B3D1U1L1R1U1F1B1L1F3B1D1U3F3B1L1R2F3B3
URU'BDBL2UB2U'L2B'D'B'UR'U'	L1R1F3B2L1R1F3B3D3U3F1L1R1D2U3L1R3F1B3L1R3D1F2B2
R'U'B'F2D'L2R'U2B2F2U2R'DBF2UR	B2L1R3U2F1L1R1D3U3B2D1U2L1R3D1U3L2R1F3B3D1
B2D'LR'BD'U'RD2B2L'B'FD2U'R2U'	F1B1L2R3F1B1L3R3D3U3R1F1B1U1L1R3F3B1L1R3D1L2R2
L2U2LBF'UL'R'FU2R2FDU'R'U2R'	U1F3B3D3U3L1R1D3F1B1L2R3F1B3D1U3F1B3L2R1F3B3
B2L'R'BL2DU'F2L'DUFDU'BF'U'	U3L1R3D1U3R1D1U1B3R2D1U3B2L1F3B3L2
BD'F2L'R'FD'LRU'BL'R'B2U'FD'U'	L2F3B1D2R3F3B3D3U3R2U1F3B1D1U3F2B3L1R1U3
U2R2BFRB2D'UR2F'D'U'L'D'ULR'U'	L1R3U3F3B1D3U1F3L3R3U1R2F3B1D2F2B2L2R3
BRDF'DFUF'L'B'R'D'LD'U'R2F2U'	L3R1D1U3R1B2D2F1B3U3L2F3B1D2L3R2F3B3D2U1
BFLU2LD2RDU'F'LRU'R2F2UR2U2	D2U1L2B2L1R3F1U2L3R1B2D2U1L1R1B1D1U3F3B1
B2L'RD2FDUB2L2R'B'FL'RF'L'R'U'	L1R3D3U1R1D1U1F2B1R2D1U3B2L1F1B1R2D2U1
L2BF'D2RBFD2U'BF'DU'F'U2L'R'U'	B2L3R1D2B1L3R3D1U2L3R1D1U3R1D2F1B1D1U2
R2U'B2F2LR'FD'U'LF2D2LBF'U'R2U'	F1B3D1U3F3B2D2L1R1D2U1F2B2R2F3B1D2L1F1B1U3
F2D2LR'FD'ULB2RD2U2L'U2L'BF'U'	L3R1D1U3R1U2F1B1D1U2L2R2B2L3R1U2F1L1R1D2U1
B2D'L2B2DB'F'LDU'B'U2F'D2F'L'R'U2	B2L3R1D2F1D2U2L1R1D2U1L3R1D3U1R1D2F1B1D2U1

F2L2B2D'BFL2R'D'UL'RU'L'R'F'LR'	L2F2R2D2U3F3B3L1R2F1B3L1R3F1D1U1R1F3B1
BFL'DU'L2F'L2B'D2B'U2LR'D'F2R2U'	L2B2R2D2U3L3R3F3D1U3F3B1D2U3F1B1L1F1B3
BD2L'R2F2U'L2B'R'B'U'B'U'B'RFL'R'	F3B3D3U3F2B3R2D1U3B2L3F3B3U3L1R3D1U3L1R2F3B1
FD2L2BU'R2F2U2RUL2F'DL'DL'F'U'	F1B2L2R2B3D1U1F3B3D1U1L3R3B1L3R3F3B3D1U1F1
L2B2RU2B2R'D2F2U2BF'U'L2D2BF'R'U2	F3B3L2R3D1U1L3R3F3B3R1F1B1D2U3L1R3F1B3L3R1D2U3
L2D'B'FD2R'DUF'D2L2BU2LR'UR2U'	U3F3B3D1U1L1R1D1F1B1L2R3F1B3D1U3F1B3L2R1F3B3
BF'D2R'D2R'U2R'B2DU'F'D2R2FL'R'U'	D1F3B1L3R1F3B1U1F1B1L2R1F3B3L1R1D1U1R3F3B3
L2BFL'B2D'UR2B'F2D'U'L'DU'LR'U'	L3R3F2B3D1U1F3B3L3R3B1L1R1U1F3B1L1R3F1B3D2U3
B'F'U2L'B'FLR'B2F'DUR'U2B'FR2U'	D3U2L1R1D1U1F3B3U3F1B1L2R3F1B3D1U3F1B3L2R1F3B3
LDFDB'F2RB'L2R2BL'BF2U'F'U'R'	L1R2B3D3R3D1U3F3B2L3R1F3L3R1D1B1L3R1B1L3R2
BFL2R'B'FLR'B'R2D'U'R'U2B'FR2U'	D1U2L1R1D3U3F3B3U1F1B1L2R3F1B3D1U3F1B3L2R1F3B3
B'DF2LRF'DL'R'UB'LRB2UF'D'U'	D2U3L2F1B3U2L1D1U1F1B1L2B3L3R1F1B3L2R1F3B3
U2LRFU2L'RB2U'B'F'U2R'DU'LR'U'	L1R3D2U3L1R3D3U1L2R3D1U1F3R2D1U3F2D2U2L1R2
UR2U2F2LB'F'ULR'B2R2U2R2B'D'U'R'U'	D3U1L2F2B3D2U2L1R1B2D2U1F1B3D1U3B1D1U1L1R2B2
L2D2RB'R'B'R2F'L'D'BUL2F'L2F'LR'U'	L3R3D3U3L2R3B2D3U1R2F3L3R3U3F1B3D1U3F1B2L1R3
B2DR2UB'FLD'U'L2F2D2F'L2U2FLR'U'	F1D3U3B1L3R3F1B2L2R2D3U3F1B1L3R3B3L1R1F2B3D1U1F1B2
R2F'R2F'D2R'U'B'DU'RU'L'D2U'RF'R2U'	D1U2R2F3B1U2L3F3B3D1U1L2U1F3B1D1U3B1L1R1
B2R2DF2D2R2BFR'D'UB'L2F'U2FLR'U'	L1R1F1B2L3R3F1B2L3R3F1B2D1U1F1B2D3U3L2R2B3L1R1B3
R2U2R2DR2F2U'LR'BDUL'U2F2R'BF'U'	L2R3D2U2F1B1R1D3L1R3F3B1L1R3D3U2L2F2B2R1F3B3L1R2
D'U'R2B2F2U2B2F2U2R2	U2F2B2D3U1F1B1L2R2F1B1
B2D2R2B2F2R2U2F2D'U'	D3U3L2R2F1B1L2R2F1B1
R2B2F2U2B2F2U2R2D'U'	F2B2D3U1F3B3L2R2F3B3D2
R2F2U2B2F2U2F2R2D'U'	L2R2F3B3L2R2F3B3D3U3
U'F2L'R'BFL'R'D'U'F2U'	D3B2D1U1L1R1F3B3L1R1B2D3
U'R2BFL'R'BFDUR2U'	D1U2B2D3U3L3R3F1B1L3R3B2D3
UF2U2F2LRB2D2F2D2L'R'U'	D1F3B3L1R1F3B3U2F3B3L3R3F3B3D3
UB2U2B2LRB2U2F2D2L'R'U'	D3L1R1F3B3L1R1F3B3L3R3F3B3D3
U'L2D2LRB'F'LRDU'R2U'	D1F2D1U3F1B1L3R3F1B1D2B2D1
UBFU2R2U2L2B'F'R2D2R2U'	D2U1F3B3R2D3U3F3B1D1U1L2F1B1D3
UBFU2L2D2R2B'F'R2D2R2U'	D3L3R3F3B3L3R3F1B1L3R3F1B1D3
U'R2B2DUBFL'R'BF'R2U'	D1U2B2L3R1F1B1L3R3D3U3L2B2D3
U'LRBD2U2B'FD2U2F'L'R'U'	D2U3L3R3F2B1D2U2F1B3D2U2F1L1R1U1
R2U2R2F2L'RBFL2F2R2U2LR'	F1B1L2R2D1U1L3R3D1U1L3R3D3U3F1B1
U'L'R'B'D2U2BF'D2U2FLRU'	D1L1R1B1D2U2F1B3D2U2F3L3R3D3U2
L'RU2L2B2F2U2B'F'U2R2U2LR'	F3B3D1U1L1R1D1U1L1R1D3U3L2R2F3B3
D'U'RB2F2R'U2BF'R2UR2BF'U2R'	B3U2L3R1B2D1F1B1L1D1U3L1R3D1U2F1B3
D'U'RDU'F'D2FD2U2B'U2BDU'R'	B1U2L3R1B2D3F3B3L1R2D1U3L3R1U1F1B3
RB2F2R'U2BF'R2UR2BF'U2R'D'U'	F1B3D2U1L1R3D3U1L1F1B1U3F2L3R1U2B1
RDU'F'D2FD2U2B'U2BDU'R'D'U'	F1B3D1L3R1D3U1L1R2F3B3U1F2L3R1U2B3
R2F'U2R2F'LR'UL2DF2D2F2DBF'R	F3B2L1R3F1B3L2R3U2L1R1F3B3D2U3R2F3B1U2R1
L2FD2R2B'L'RU'R2U2R2U'L2UBF'R'	B3L3R1F1B3R3D2L3R3F1B1D1U2R2F1B3D2R3
RBF'UF2U2F2UL2DLR'F'R2D2F'R2	R1D2F3B1R2D1F3B3L3R3U2L3R2F1B3L3R1F1
F'D'B'L'B2L'ULFD2U2LBDU'RU'	R3U2F1B3R2D2U1F1B1L3R3U2R3F1B3L3R1B3
U'B2UL2U'BF'LB2D2L'R2D'U'FLR'	D1U2F3B3D2R3D1U3L1R3D2U3L1R1B1D2L1R3B2
DR2DU2R2UB'FL'D2F2L'DUF'LR'	D3F3B3U2R3F1B3L3R1F3B2D1U1L1U2F1B3R2
B'FLD2LU2L'DU'FL2U2B2F'L'R'U'	B2L3R1D2B3L3R3D2U1L3R1D3U1R1D2F1B1D3U2
B'FRF2LB2R'DU'FL2U2B2F'L'R'U'	R2F3B1U2L3D3U3F1B2L1R3F3B1R1U2F1B1D1
DL'R'B2F'D2L'RF2U'B'F'D2R'DU'LR'	D2U3F2L3R3B3R2D3U1F2L2R3D3U3F1B2D1U3F1B3
U'R2U2B'D'B'R'U'RBR'B2D2F'R'DFR'	U3L3R1F3B1L1R3U3L3R3F1B2L3R3F3B3D1U1F3L1R1
BF'D'UFDULU2F2U2R2DU'F'LRU'	F3B1D3U1F3B2D1U1L2R1F2D1U3R2B1L1R1F2D2U1
R2F2DB'F'RDU'R2FD2BU2FD2LR'U'	L3R3F3B2D3U3F1B1L1R1F1L1R1U1F3B1L1R3F1B3D2U3

Curriculum Vitae

Education

PhD, Mathematics
UW-Milwaukee — Milwaukee, WI, USA — December 2017
Advisor: Hans Volkmer

MS, Mathematics
UW-Milwaukee — Milwaukee, WI, USA — May 2007

BS, Applied Mathematics, Engineering, and Physics
UW-Madison — Madison, WI, USA — May 2004

Assistant Teaching at UW-Milwaukee 2007-2012

Math 095, Introductory Algebra
3 semesters with 2 sections each (students never saw course coordinator)

Math 105, Intermediate Algebra
2 semesters with 2 sections each (students never saw course coordinator)

Math 211, Survey in Calculus and Analytic Geometry
1 semester with 3 discussion sections

Research

MATLAB programming:

Sudoku solver

Slow and limited Rubik's Cube optimal solver compatible with both FTM and SPTM

Twist counter for the 2×2×2×2 Cube (partly compatible with higher dimensions)

C++ programming:

Good-speed Rubik's Cube optimal solver in SPTM