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Local Connectedness of Bowditch Boundary of Relatively Hyperbolic Groups

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by

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ABSTRACT

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by

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If the Bowditch boundary of a finitely generated relatively hyperbolic group is connected, then, we show that it is locally connected. Bowditch showed that this is true provided the peripheral subgroups obey certain tameness condition. In this paper, we show that these tameness conditions are not necessary.

To

L.V. Tarasov,

for that incredible piece on Dialectics, Socratic dialogue and Calculus,

Rabindranath Tagore,

for the magic of decoupling ‘structure’ and ‘form’

and Dr. Chris Hruska

my advisor, who transformed the way I do mathematics

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1. Introduction

1.1 Background

One of the central themes of geometric group theory is to understand algebraic structures of a group by analyzing the group's action on suitable topological spaces. In that spirit, one often constructs a topological space called 'boundary' of the group on which the group acts by homeomorphisms. We must caution the reader that there are multiple ways of constructing spaces which are referred to as 'boundary' in literature. These boundary spaces can be very different from each other, with different utilities.

In [Gro87], Mikhail Gromov introduced the idea of hyperbolic groups and defined the Gromov boundary of hyperbolic groups. Several group theoretic properties of hyperbolic groups are encoded in the topological information of their boundary. Gromov also introduced relatively hyperbolic groups, a generalization of the notion of hyperbolic groups. Brian Bowditch defined the boundary of relatively hyperbolic groups. In this paper, we are interested in the boundary of relatively hyperbolic groups. For a detailed account of relatively hyperbolic groups and Bowditch boundary, refer to [Bow12] and [Gro87].

The algebraic content of a word-hyperbolic group G has deep connections with the topology of its Gromov boundary ∂G. For example, in [BM91], Bestvina–Mess shows that, if a word hyperbolic group G has one end then ∂G is connected. Furthermore, if ∂G contains no cut points, then ∂G is locally connected. Subsequent investigations by Levitt in [Lev98], Bowditch in [Bow99b] and Swarup in [Swa96] shows that ∂G indeed contains no cut points, thereby proving ∂G is locally connected. In this paper, we generalize Bowditch and Swarup's argument and prove that the Bowditch boundary of a relatively one-ended, finitely generated relatively hyperbolic group is locally connected. This result was proved earlier by Bowditch in [Bow99b], under more restrictive hypothesis.

Local connectedness is a strong characterization of a topological space. For example, Hahn–Mazurkiewicz theorem shows that a necessary and sufficient condition for a compact, connected, locally connected metric space P is that P is the image of the unit interval under a continuous mapping into a Hausdorff space [HY88, Theorem 3-30]. Apart from topological significance, the property of local connectedness is also important from a group theoretic point of view. For example in [Bow98b], Bowditch constructs a JSJ tree of two ended splittings for hyperbolic groups using only the topology of the boundary assuming local connectedness of the boundary. In [HH19], Haulmark and Hruska prove an analogous theorem in the setting of elementary splitting of relatively hyperbolic groups.

Next we will briefly describe Bowditch and Swarup's argument to show the local connectedness of Gromov boundary of one-ended word hyperbolic groups. In [Bow99c], assuming one endedness of the group G and existence of a cut point in ∂G, Bowditch obtains a two ended splitting of G. He uses the technology of $\mathbb{R}$–tree and a construction similar to [Lev98] due to Levitt, in the argument. In [Swa96], Swarup uses the Rips machine due to [BF91] to obtain a splitting of G over a finite subgroup. This leads to a contradiction, as G is assumed to be one-ended in the first place. Hence the Bowditch and Swarup's argument shows that ∂G has no cut points, implying ∂G is locally connected.

In [Bow99b], Bowditch extends the investigation on local connectedness of boundary of hyperbolic groups to the local connectedness of the boundary of relatively hyperbolic groups. The notion of relatively hyperbolic group is a generalization of fundamental groups of complete noncompact hyperbolic manifolds of finite volume. There are several equivalent definitions of relatively hyperbolic groups. For our purposes, we use the following characterization due to Yaman [Yam04]. A relatively hyperbolic group is a group that admits a geometrically finite convergence group action on a compact metrizable space M (see Definition 2.0.10 for details). To every relatively hyperbolic group G and a given collection of its subgroups $\mathbb{P}$, one can associate a compact metrizable space M, on which G acts as a convergence group (as in Definition 2.0.4) such that each member of $\mathbb{P}$ fixes a point in M. In fact M is uniquely determined by the choice of the pair $(G, \mathbb{P})$. We say M is the *Bowditch Boundary* of $(G, \mathbb{P})$.

Bowditch, in [Bow99b], shows that, if $(G, \mathbb{P})$ is a finitely generated group and one-ended relative to $\mathbb{P}$ then every global cut point in the Bowditch boundary is a parabolic fixed point provided the peripheral subgroups are finitely presented, one- or two-ended and without infinite torsion subgroups (these three properties are collectively referred to as *tame* from now on). Under these tameness assumptions on peripheral subgroups, in [Bow99b] the Bowditch boundary is proved to be locally connected. In this paper, we show that the tameness conditions on the peripheral subgroups, are not necessary to obtain local connectedness of the boundary.

Subsequent work of Guirardel–Levitt and Osin improved some of the tools used by Bowditch and Swarup. In [GL15], Guirardel–Levitt, created a relative version of the Rips machine which is a tool similar to Bass Serre Theory used to analyze splittings of a group using the group's action on $\mathbb{R}$–tree. The work of Guirardel–Levitt helps us to analyze the action of finitely generated relatively hyperbolic groups on $\mathbb{R}$–trees (relative to the peripheral subgrups). In [Osi06], Osin prescribes a presentation complex on which a relatively hyperbolic group acts in a suitable manner. In [GL17], Guirardal–Levitt, further extends the accessibility theorems and examines the JSJ decomposition of relatively hyperbolic groups.

1.2 Main Theorem

In this paper we adapt the Bowditch - Swarup argument and also use the later developments due to Guirardel, Levitt and Osin, to prove the following theorem:

Theorem 1.2.1. *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group. Suppose the Bowditch boundary $M = \partial(G, \mathbb{P})$ is connected. Then M is locally connected.*

Bowditch shows in [Bow99a, Corollary 0.2] that if every global cut point of the connected Bowditch Boundary of a relatively hyperbolic group, is a parabolic fixed point, then the boundary is also locally connected. Hence to prove Theorem 1.2.1 it is sufficient to show the following:

Theorem 1.2.2. *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group. Suppose the Bowditch boundary $M = \partial(G, \mathbb{P})$ is connected. Then every global cut point of M is a parabolic fixed point.*

Recall that Bowditch has proved a restricted version of Theorem 1.2.2, under the additional assumption of peripheral subgroups being tame. We adopt a similar strategy of proof, though in our case, the particular steps require additional tools as the peripheral subgroups are no longer assumed to be tame.

1.3 Original Results

In this paper, we have relativized several theorems that are known to be true for hyperbolic groups. 'Relativization' is, informally speaking, the act of generalization of a theorem that is known to be true for hyperbolic case, to the relatively hyperbolic case.

In Chapter 3, we begin by relativizing the proof of [Bow99c, Theorem 6.1]. Bowditch proves this theorem for finitely generated groups that are one ended. We modify the proof to obtain a similar theorem that is true for relatively one ended groups.

Theorem 1.3.1. *Suppose that $(G, \mathbb{P})$ is a finitely generated group, finitely presented relative to $\mathbb{P}$ and one-ended relative to $\mathbb{P}$, that admits a minimal convergence action on a continuum, M relative to $\mathbb{P}$. If M has a cut point that is not a parabolic fixed point, then there exists a G–equivariant quotient $D(M)$ that is a dendrite. Moreover each member of $\mathbb{P}$ fixes some* *point in $D(M)$, the induced action of G on $D(M)$ is a convergence action and without global fixed point.*

Next, in Chapter 4, we generalize [Lev98, Theorem 1]. Levitt's theorem is true for the finitely presented groups. We improve it to work for relatively finitely presented groups.

Theorem 1.3.2. *If a finitely generated, relatively finitely presented group $(G, \mathbb{P})$ admits a non-trivial non-nesting action by homeomorphisms on a real tree T' relative to $\mathbb{P}$, then it admits a non-trivial isometric action on some $\mathbb{R}$–tree T_0 relative to $\mathbb{P}$. Every subgroup fixing an arc in T_0 fixes an arc in T'. Moreover given a finite collection of finitely generated subgroups $G_j \subset G$, each fixing a point of T, one may require that each G_j fixes a point of T_0.*

Finally, we consider a particular JSJ decomposition T_{CAN} of a finitely generated relatively hyperbolic group $(G, \mathbb{P})$. We construct an $\mathbb{R}$–tree T_0, on which $(G, \mathbb{P})$ acts by isometries, without global fixed point and such that each of the peripheral subgroups fix some point in T_0. In Chapter 6 we show that each of the vertex groups of T_{CAN} fixes some point in T_0.

We also use the following theorem which is the same as [GL15, Corollary 9.10] to obtain a splitting of a particular variety of vertex groups of the JSJ decomposition of G (in Chapter 4).

Theorem 1.3.3. *Let G be hyperbolic relative to finitely generated subgroups $\mathbb{P} = \{P_1, ..., P_n\}$, with $P_i \neq G$. Let $\mathcal{H} = \{H_1, ..., H_q\}$ be another family of finitely generated subgroups. If G acts non-trivially on an R-tree T relative to $\mathbb{P} \cup \mathcal{H}$ with elementary arc stabilizers, then G splits over an elementary subgroup relative to $\mathbb{P} \cup \mathcal{H}$.*

2. Preliminaries

In this chapter we will provide some definitions. We begin by defining the notion of relatively hyperbolic groups. There are several equivalent definitions of relatively hyperbolic groups (see [Hru10] for a detailed account). We will use a definition that suits our purpose. First we need the definition of *convergence group* to state this definition of relatively hyperbolic group.

Definition 2.0.4 (Convergence group)**.** Let G be a group acting by homeomorphisms on a compact metrizable space M. The group G is called a convergence group if for every sequence of distinct group elements (g_k) there exist points $a, b \in M$ (not necessarily distinct) and a subsequence $(g_{k_i}) \subset (g_k)$ such that $g_{k_i}(x) \to a$ locally uniformly on $M - \{b\}$, and ${g_{k_i}}^{-1}(x) \to b$ converges locally uniformly on $M - \{a\}$. By *locally uniformly* we mean, if C is a compact subset of $M - \{b\}$ and U is any open neighborhood of a, then there is an $N \in \mathbb{N}$ such that $g_{k_i}C \subset U$ for all $i > N$.

Next, we will classify the elements of a convergence group.

Definition 2.0.5. Suppose G acts as a convergence group on the compact hausdorff space M. Given $g \in G$ let $\text{fix}(g)$ be the set of fixed points of g in M. An element $g \in G$ is *elliptic* if it has finite order. It is *parabolic* if it has infinite order and $\text{fix}(g)$ consists of a single point. It is *loxodromic* if it has infinite order and $\text{fix}(g)$ consists of a pair of points.

Definition 2.0.6. Suppose G acts as a convergence group on the compact hausdorff space M. A subgroup P of G is *parabolic* if it is infinite and contains no loxodromic element.

A subgroup Q of G is *loxodromic* if its a maximal virtually cyclic subgroup of G, and not parabolic.

The following result provides a classification of the elements of G.

Theorem 2.0.7 ([GM87], [Tuk94]). *Suppose that G acts as a convergence group on a compact hausdorff space, M, with at least three points. Then every element of G is elliptic, parabolic or loxodromic.*

Definition 2.0.8 (Cut point). A point $a \in M$ is a *cut point* if $M - \{a\}$ is not connected.

Definition 2.0.9 (Parabolic point and Parabolic subgroup). A *parabolic point* (in M) is one whose stabilizer is infinite and contains no loxodromic elements. The stabilizer of a parabolic point is a *parabolic subgroup*. [Bow99c]

Definition 2.0.10 (Relatively Hyperbolic Group and Bowditch Boundary). Suppose a group G acts on a compact metrizable space M as a convergence group. Also suppose that $\mathbb{P}$ is the collection of representatives of conjugacy classes of stabilizers of the parabolic fixed points. Then we say G is hyperbolic relative to $\mathbb{P}$ and M is the *Bowditch Boundary* of $(G, \mathbb{P})$

Definition 2.0.11 (Relative group action, Relative splitting). A group G is said to *act relative* to a class of subgroups $\mathbb{P}$ on a topological space T if each member of $\mathbb{P}$ fixes a point in T. A group G is said to *split relative* to a class of subgroups $\mathbb{P}$, if G has a graph of groups splitting and each member of $\mathbb{P}$ is a conjugate into one of the vertex groups.

Bowditch characterizes the connectedness property of the Bowditch boundary of relatively hyperbolic groups in the following result analogous to Stallings theorem for hyperbolic groups.

Theorem 2.0.12 ([Bow12]). *The boundary $M = \partial(G, \mathbb{P})$ of a relatively hyperbolic group $(G, \mathbb{P})$ is connected if and only if $(G, \mathbb{P})$ does not split nontrivially over any finite subgroup relative to the peripheral subgroups.*

We will define the terms real tree, $\mathbb{R}$–tree and dendrite next. A detailed account of these concepts can be found in [Bow99c].

Definition 2.0.13. An *arc* is a subset (of a topological space) homeomorphic to a closed real interval. A *uniquely arc-connected space*, T, is a Hausdorff topological space in which every pair of distinct points are joined by a unique arc.

Definition 2.0.14 (real tree)**.** A *real tree* is a locally connected, uniquely arc-connected Hausdorff space T.

Definition 2.0.15 (dendrite)**.** A *dendrite* is a compact separable real tree.

Definition 2.0.16. Suppose T is a real tree. Given $x \in T$, the *degree* of $x = deg(x)$ is the cardinality of the set of components of $T - \{x\}$. A point $x \in T$ is *terminal* if $deg(x) = 1$.

Definition 2.0.17. Suppose a group G acts as a convergence group on a compact, metrizable space M. A subgroup of G is *elementary* if it is parabolic, finite or loxodromic.

3. Construction of a nontrivial dendrite

In this chapter we prove Theorem 3.0.18 in which we construct a nontrivial dendrite $D(M)$ assuming the existence of a non parabolic cut point in the Bowditch boundary M of a finitely generated, relatively one ended, relatively hyperbolic group $(G, \mathbb{P})$. Bowditch proves a similar result in [Bow99c, Theorem 6.1] with a more restrictive hypothesis that G is one-ended. We on the other hand assume G to be one-ended relative to $\mathbb{P}$.

Suppose M has a cut point p that is not a parabolic fixed point. Let T be the set of all G-translates of p. We will adapt the argument of [Bow99c, Proposition 6.1] to obtain the following relative version of Theorem [Bow99c, Proposition 6.1].

Theorem 3.0.18. *Suppose that $(G, \mathbb{P})$ is a finitely generated group, finitely presented relative to $\mathbb{P}$ and one-ended relative to $\mathbb{P}$, that admits a minimal convergence action on a continuum, M relative to $\mathbb{P}$. If M has a cut point that is not a parabolic fixed point, then there exists a G–equivariant quotient $D(M)$ that is a dendrite. Moreover each member of $\mathbb{P}$ fixes some point in $D(M)$, the induced action of G on $D(M)$ is a convergence action and without global fixed point.*

We will need the following definitions from [Bow99c]

Definition 3.0.19 (pretree)**.** The set T with a ternary relation of 'betweenness' is a *pretree* if the following axioms are satisfied:

- (T0) For all $(x, y)(\neg xyx)$
- (T1) $xzy \iff yzx$

- (T2) For all x, y, z then we cannot have xyz and xzy simulataneously.

- (T3) If xzy and $z \neq w$ then we either have xzw or yzw

Definition 3.0.20 (interval in pretree)**.** Given distinct points x, y in a pretree T, we shall write

- $(x, y) = \{z \in T | xzy\}$ is an *open interval*

- $[x, y) = (y, x] = (x, y) \cup \{x\}$ is a *half open interval*

- $[x, y] = (x, y) \cup \{x, y\}$ is a *closed interval*

- Without reference to the points x, y, open, half-open, closed might be ambiguous.

- $(x, x) = [x, x) = \varnothing$

- $[x, x] = \{x\}$

Definition 3.0.21 (adjacent point)**.** Suppose T is a pretree. Two distinct points $x, y \in T$ are *adjacent* if $(x, y) = \varnothing$.

Definition 3.0.22 (full subset of pretree)**.** A subset A of a pretree T is *full* if $[x, y] \subseteq A$ for all $x, y \in A$.

Definition 3.0.23 (linear subset of pretree)**.** A subset A of a pretree T is *linear* if for all distinct $x, y, z \in A$, we have xyz or yzx or zxy.

Definition 3.0.24 (arc of pretree)**.** An *arc* of a pretree is a non empty full linear subset.

Definition 3.0.25 (direction in pretree)**.** If A is a linear subset of a pretree T, a *direction* on A is a linear (i.e. total) order $<$ on A such that xyz implies either $(x < y < z)$ or $(z < x < y)$. We refer to $(A, <)$ as a directed linear set.

If $(A, <)$ is a directed linear set, then so also is $(A, >)$ where $x > y$ implies $y < x$

Remark 3.0.26. Bowditch observes in the remark after [Bow99c, Lemma 2.7] that every interval in a pretree is an arc and hence every interval is by definition a linear set.

Lemma 3.0.27. *[Bow99c, Lemma 2.7] A linear set with at least two elements admits precisely two directions.*

Definition 3.0.28 (median of pretree)**.** Suppose T is a pretree. Given $x, y, z \in T$, we shall say that $c \in T$ is a *median* of x, y, z if $c \in [x, y] \cap [y, z] \cap [z, x]$. Applying Lemma 3.0.27, Bowditch observes that if a median exists, then it must be unique. In this case we write $c = med(x, y, z)$.

Definition 3.0.29 (median pretree)**.** A *median pretree* is a pretree in which every set of three points has median.

Definition 3.0.30 (complete pretree)**.** A pretree is *complete* if every arc is an interval.

3.1 Betweenness and embedding of the cut point set

Let M be a connected Hausdorff topological space. A point $a \in M$ is a cut point if $M - \{a\}$ is not connected. Thus we can write $M - \{a\} = U \sqcup V$, where U and V are nonempty open subsets of M. We shall write UaV to represent this situation. Given $x, y, z \in M$ we shall write xzy to mean that there are open sets U and V of M with UzV, $x \in U$ and $y \in V$. We say z is *between* x, y.

By [Bow99c, Lemma 5.3], with the ternary relation of 'betweenness' thus defined M is a pretree. Suppose T is a set of cut points in M. Notice that T is also a pretree. In [Bow99c, Section 3] Bowditch uses the concept of 'flows' to embed any pretree, T in a complete median pretree Φ. This embedding will have the property that for any distinct pair of points $x, y \in \Phi - T$, there is some $z \in T$ with xzy. Also for any pair of distinct points $x, y \in T$ there is some $z \in \Phi$ with xzy.

Definition 3.1.1 (full relation)**.** A *full relation* on a pretree is an equivalence relations for which every equivalence class is full.

Suppose $\sim$ is a full equivalence relation on a pretree Φ. Let $\Phi/\sim$ be the quotient. For all $x \in \Phi$ we write the $\sim$ equivalence class containing x as $[x]_\sim \in \Phi/\sim$. If $X, Y, Z \in \Phi/\sim$ write XYZ to means that there is some $y \in Y$ such that XyZ. This is a ternary 'betweenness' relation on $\Phi/\sim$. In fact Bowditch shows that, with the betweenness relation thus defined, $\Phi/\sim$ is a pretree. Moreover if Φ is a median pretree then so is $\Phi/\sim$. Also if Φ is a complete pretree, then so is $\Phi/\sim$ [Bow99c, Lemma 4.2].

Definition 3.1.2 (dense pretree)**.** A pretree Φ is *dense* if for all distinct $x, y \in \Phi$ there exists $z \in \Phi$ with xzy. In other words, no two points in Φ are adjacent.

Definition 3.1.3 (codense relation)**.** A *codense relation* on Φ is a full relation, $\sim$, such that $\Phi/\sim$ is dense.

Suppose $\sim, \sim'$ are two binary relations on Φ. We define a partial order $\leq$ on the set of relations as follows: $\sim \leq \sim'$ means $x \sim y$ implies $x \sim' y$.

Let $\mathcal{R}$ be the set of all codense relations on Φ. Then there exists unique minimal codense relation in $\mathcal{R}$ [Bow99c, Section 4]. Bowditch shows that this minimal codense relation can be obtained in an inductive manner as described in Section 3.2. In the proof of Theorem 3.0.18, we modify the steps of this induction process.

Definition 3.1.4 (Finite Interval Relation)**.** Let Φ be a pretree. Two points $x, y \in \Phi$ are said to be equivalent by *finite interval relation* ($x \sim y$), if and only if $[x, y]$ is finite.

3.2 Obtaining a minimal codense relation

In this section we generalize Bowditch's construction of a dendrite from the convergence group action of a finitely generated group $(G, \mathbb{P})$ on a compact, metrizable space M, relative to $\mathbb{P}$. We also prove a few lemmas to keep track of the parabolic fixed points in the construction of the dendrite from the boundary M.

Suppose M is a compact, metrizable space on which a finitely generated relatively hyperbolic group $(G, \mathbb{P})$ acts as a convergence group. Let T be a set of cut points of M that are not parabolic fixed points and invariant under the action of G. Finally suppose Φ is a complete median pretree in which T embeds as in Section 3.1. For an ordinal α, we associate a full relation $\sim_\alpha$ on Φ by a process of transfinite induction as follows. Suppose $\sim_0$ is the trivial relation on Φ (equality). Suppose now that $\sim$ is any full relation on Φ. Let $\approx$ be the finite interval relation on $\Phi/\sim$. Define a relation $\sim'$ on Φ by $x \sim' y$ if and only if $[x]_\sim \approx [y]_\sim$.

Suppose α, β, are ordinals. If α is successor ordinal and $\alpha = \beta + 1$, define $\sim_\alpha = (\sim_\beta)'$. If α is a limit ordinal, define $\sim_\alpha = \vee\{\sim_\beta |\beta < \alpha\}$, that is $x \sim_\alpha y$ if $x \sim_\beta y$ for some $\beta < \alpha$. This gives us a full relation, since, by transfinite induction, all the relations $\sim_\beta$ for $\beta < \alpha$ are full relations. Moreover, if $\gamma \leq \beta \leq \alpha$, then $\sim_\gamma \leq \sim_\beta$, so that $\{\sim_\beta |\beta < \alpha\}$ is a chain. Bowditch proves that these relations must eventually stabilise, i.e. for some α, we have $\sim_{\alpha+1} = \sim_\alpha$, so that, in fact, $\sim_\alpha = \sim_\beta$ for all $\beta \geq \alpha$. [Bow99c, Lemma 4.4]

All through the rest of this chapter the following notation is assumed. We use M to denote a compact, metrizable space on which a finitely generated relatively hyperbolic group $(G, \mathbb{P})$ acts as a convergence group. Let T be the G translates of p, where p is a cut point of M that is not a parabolic fixed point. Finally, suppose Φ is a complete median pretree that is a completion of T as in Section 3.1.

Remark 3.2.1. In [Bow99c, Section 5], Bowditch defines a G-equivariant map $\phi \colon M \to \Phi$. Since Φ is a completion of T, we regard T as a subset of both M and Φ. By Bowditch's definition, the map ϕ restricts to identity on T. Moreover in [Bow99c, Lemma 5.14], Bowditch shows that if $p \in \Phi$ then either $p \in \phi(M)$, or else p is adjacent (in Φ) to some element of T.

Definition 3.2.2. Let Φ' be the set of those elements of $\Phi - T$ which are adjacent of precisely one element of T. Define $\Phi_0 = \Phi - \Phi'$.

The following lemma is similar to [Bow99c, Lemma 6.10] which has a single parabolic element in its hypothesis instead of an entire parabolic subgroup.

Lemma 3.2.3. *Suppose P is an infinite parabolic subgroup whose fixed point, $a \in M$, does not lie in T. Let $F(P)$ be the collection of points fixed by P in Φ. Then $F(P)$ consists of a single point of $\Phi_0 - T$.*

Proof. We know that $F(P) \cap T$ is empty. By Remark 3.2.1, there is a G-equivariant map $\phi\colon M \to \Phi$. As $a \in M$ is fixed by P, hence $\phi(a) \in \Phi$ is also fixed by P. If $\phi(a) \notin \Phi_0$ then it must be adjacent to a single point of T which is also fixed by P. But no point in T is fixed by P by hypothesis. Hence $\phi(a) \in \Phi_0$. Thus $F(P) \cap \Phi_0$ is non empty.

Suppose $r, s \in F(P)$ are distinct. Then $[r, s]$ is infinite. For, if $[r, s]$ is finite, consider the element z, adjacent to r in $[r, s]$. It must be fixed by P as under the action of P nothing should come in between $\gamma(r) = r, \gamma(z)$ for all $\gamma \in P$. But that implies $F(P) \cap T$ is non empty which is not true by hypothesis. Hence $[r, s]$ is infinite.

By the observation in the proof of [Bow99c, Lemma 6.9], there exists disjoint closed P-invariant sets $R_M(r)$ and $R_M(s)$ as defined in [Bow99c, Section 3]. But closed subsets of compact space M are compact. The convergence action of P on M is locally uniform and P is a parabolic subgroup. Let p be the unique fixed point of P in M. Without loss of generality suppose $p \notin R_M(s)$. Hence there is $g \in P$ such that $g \cdot R_M(s)$ is contained in the open neighborhood $M - R_M(s)$ of p. But that means $g \cdot R_M(s)$ and $R_M(s)$ are disjoint giving us a contradiction. Therefore $F(P)$ contains just one point. □

Notice that by Remark 3.2.1 there is an action of G on Φ induced by the action of G on M. We now show that each infinite torsion subgroup P of G fixes some point in a P–invariant full subset S of Φ. First we need a lemma about finite order elements.

Lemma 3.2.4. *Suppose Φ is any pretree and g is an action on Φ by pretree automorphisms. Suppose g is finite order and there is an element $x \in \Phi$ such that $gx = x$. Let y be different from x. Then $gy \notin (x, y)$*

Proof. Assume by way of contradiction $gy \in (x, y)$. As gy is in between x and y, then, g^2y is in between x and gy. In fact g^ny is in between x and gy for all $n \in \mathbb{N}$. But that implies that $g^n y \neq y$ for all $n \in \mathbb{N}$. This contradicts the fact that g is finite order. $\square$

Lemma 3.2.5. *Suppose Φ is a pretree and g is a finite order automorphism of Φ. Then the fixed point set of g in Φ is full.*

Proof. Suppose $x, y \in \Phi$ are fixed by g. If x, y are adjacent then we are done. Otherwise assume that there exists $z \in [x, y]$ implying gz is in between x and y. Hence $gz \in [gx, gy] = [x, y]$. If $gz \neq z$, then $gz \in [x, z)$ or $gz \in (z, y]$. Without loss of generality, assume $gz \in [x, z)$. But by Lemma 3.2.4 this is a contradiction as g is finite order. $\square$

Definition 3.2.6. Let $\mathrm{fix}_\Phi(g)$ be the fixed point set of g in Φ.

Lemma 3.2.7. *Suppose Φ is a median pretree and a group G is acting on Φ by pretree automorphisms. Let g be a torsion element of G such that g fixes a point x in Φ. If $y \in \Phi$ and $m = med(x, y, gy)$, then $gm = m$.*

Proof. Since g maps $[x, y]$ to $[x, gy]$, if $gm \neq m$ then we have $gm \in (m, gy)$ or $gm \in (x, m)$. Replacing g with g^{-1} if necessary, we assume that $gm \in (x, m)$. But by Lemma 3.2.4 this is not possible as g is finite order. Hence $gm = m$. $\square$

Lemma 3.2.8. *Suppose Φ is a median pretree and a group G is acting on Φ by pretree automorphisms. Let g be a torsion element of G such that g fixes a point in Φ. If S is a g–invariant full subset of Φ, then g fixes some point in S and $\mathrm{fix}_\Phi(g) \cap S$ is full.*

Proof. Suppose $x \in \mathrm{fix}_\Phi(g)$. Consider $m = med(x, y, gy)$. Clearly $m \in S$ as S is full.

By Lemma 3.2.7 Therefore $\mathrm{fix}_\Phi(g) \cap S$ is nonempty. Moreover as by Lemma 3.2.5, $\mathrm{fix}_\Phi(g)$ is full, and by hypothesis S is full, hence $\mathrm{fix}_\Phi(g) \cap S$ is full. $\square$

Lemma 3.2.9 (Helly's Theorem for Median Pretree)**.** *[Rol98, Theorem 2.2] Suppose $S = S_1, ..., S_n$ is a finite collection of full subsets of a median pretree S, such that $S_i \cap S_j$ is nonempty for all $1 \leq i, j \leq n$. Then $S_1 \cap S_2 \cap ... \cap S_n$ is nonempty.*

Lemma 3.2.10. *Suppose a finitely generated infinite torsion group P is acting on a compact, connected, metrizable space M as a convergence group relative to $\mathbb{P}$. Suppose Φ be a complete median pretree obtained from M as in Section 3.1. If S is a $P-$ invariant full subset of Φ, and if every element of P fix some point in S, then P will fix some point in S.*

Proof. Choose a finite generating set S_P of P. Suppose $x, y \in \Phi$ are fixed by $g_x, g_y \in S_P$ respectively. We show that $fix_\Phi(g_x) \cap fix_\Phi(g_y)$ is nonempty.

If y is not a fixed point of g_x, let m_x be the $med(x, y, g_x y)$. By [Bow99c, Lemma 2.5], m_x is unique. By Lemma 3.2.7, we have $g_x m_x = m_x$.

Note that $m_x \neq y$ as we assumed that y is not fixed by g_x. Therefore g_x fixes m_x and sends $(m_x, y]$ to $(m_x, g_x y]$. In fact $g_x g_y$ sends y to $g_x g_y y = g_x y$.

The following section of the proof is inspired by Serre's argument in [Ser77, Prop. I.26]. Suppose m_{xy} is a fixed point of $g_x g_y$. By Lemma 3.2.7, $m' = med(m_{xy}, y, g_x g_y y)$ is fixed by $g_x g_y$. Therefore we have m' between y and $g_x y = g_x g_y y$. If $m_x = m'$ then $m_x = m' = g_x g_y m' = g_x g_y m_x$ implying $g_y m_x = g_x^{-1} m_x = m_x$ implying m_x is a fixed point of g_y. Hence we have $m_x \in fix(g_x) \cap fix(g_y)$ and we will be done.

Hence assume by way of contradiction that $m_x \neq m'$.

Choose one of the two possible directions possible in the interval $[g_x y, y]$ which by Remark 3.0.26 is a linear set. Denote the chosen direction by $<$. Without loss of generality we can assume $g_x g_y y < m' < y$. In this convention $g_x g_y y = g_x y$ is the 'least' element of $[g_x y, y]$. Since $m_x \neq m'$, by (T3) axiom of pretree, we have the following cases.

Case 1: m_x is in between m' and y.

Since m' is between $g_x g_y y$ and m_x. hence Hence $(g_x)^{-1}(m')$ is between $(g_x)^{-1}(g_x g_y)(y) = g_y y = y$ and $(g_x)^{-1}(m_x) = m_x$. Therefore $g_y(m') = (g_x)^{-1}(g_x g_y)(m') = (g_x)^{-1}(m')$ lies between y and m_x. Then $g_y(m')$ is between m' and y but this contradicts Lemma 3.2.4.

Case 2: m' is in between m_x and y is similar

Hence $m_x = m'$ that is $m_x = g_x g_y m_x$ or $m_x \in fix(g_y)$. Moreover $m_x \in S$ as S is full. Hence the fixed point sets of every pair of members in S_P (and also in P) has nonempty intersection.

Since the fixed point sets in S are full and nonemempty by Lemma 3.2.8, hence by Lemma 3.2.9, intersection of all the (finitely many) fixed point sets of members of S_P has a nonempty intersection with S. Hence we found point in S fixed by all elements of P. $\square$

Definition 3.2.11. Suppose $\bar{T}$ is any pretree. $x, y \in \bar{T}$ and Q is a full subset of $\bar{T}$. Then xyQ means that for all $q \in Q$, we have xyq.

Definition 3.2.12 (Preclosed subset of a pretree)**.** Suppose $\bar{T}$ is a general pretree. A full subset $Q \subseteq \bar{T}$ is *preclosed* if for all $x \in \bar{T} - Q$ there exists $y \in \bar{T}$ such that xyQ.

The following lemma is proved by Bowditch.

Lemma 3.2.13. *[Bow99c, Lemma 5.19] If $Q \subseteq \Phi$ is preclosed, then $\phi^{-1}(Q) \subseteq M$ is closed and connected.*

Lemma 3.2.14. *Suppose finitely generated group $(G, \mathbb{P})$ is acting on a compact, connected, metrizable space M as a convergence group, relative to $\mathbb{P}$. Let $P \in \mathbb{P}$ be a parabolic subgroup of G. Suppose Φ be a complete median pretree obtained as in Section 3.1. Let $g \in P$ is infinite order and x', m be points in Φ such that $(x', m] = \cup_{i=0}^{\infty}[g^i m, m]$ is an interval in Φ and $x' < ... < g^n m < ... < gm < m$ for all $n \in \mathbb{N}$. Then $gx' = x'$.*

Proof. Consider the map $\phi \colon M \to \Phi$ as in Remark 3.2.1. Clearly each closed interval of Φ is preclosed. Then by Lemma 3.2.13 $\phi^{-1}([x', m])$ is closed and connected in the continuum M. Suppose $t \in \phi^{-1}(m)$. Since ϕ is G-equivariant, we have $g(\phi(t)) = \phi(gt) = gm$ implying $gt \in \phi^{-1}(gm)$. Similarly $g^n t \in \phi^{-1}(g^n m)$ for all $n \in \mathbb{N}$.

Notice that $g^n t \in \phi^{-1}([g^n m, m])$ and $g^n t \notin \phi^{-1}([g^{n-1}m, m])$. Clearly $\cup_{i=1}^{\infty}\phi^{-1}([g^n m, m])$ is a compact exhaustion of $\phi^{-1}((x', m])$.

Since M is compact and metrizable, M is sequentially compact. Hence the sequence $(g^n t)$ has a convergent subsequence (g^{k_n}). Suppose the $\lim_{n\to\infty} g^{k_n} t = t'$. Notice that $t' \notin \phi^{-1}([g^n m, m])$ for all $n \in \mathbb{N}$. Hence $t' \notin \phi^{-1}\{(x', m]\}$.

Now consider the infinite sequence of group elements $g^{k_1}, g^{k_2}, \ldots$ They are all contained in the parabolic subgroup P by hypothesis. Hence all of them have a unique common fixed point t'' in M, which is also a fixed point of g. Moreover by the property of convergence group we have $\lim_{n\to\infty} g^{k_n}t = t''$, therefore $t' = t''$. Hence $gt' = t'$

Case 1: $x' \in \phi(M)$

Note that $\phi^{-1}([x', m])$ is closed and connected. Moreover $\phi^{-1}([x', m]) = \phi^{-1}((x', m]) \cup \phi^{-1}\{x'\} \supset cl(\phi^{-1}((x', m]))$. Hence $t' \in \phi^{-1}\{x'\}$ as we observed $t' \notin \phi^{-1}\{(x', m]\}$. By the above discussion $gt' = t'$.

Now by equivariance we have $gx' = g\phi(t') = \phi(gt') = \phi(t') = x'$.

Case 2: $x' \notin \phi(M)$

By Remark 3.2.1, x' is adjacent to $x'' \in T$. Clearly $x'' \notin (x', m]$, otherwise we have $\cup_{i=1}^{\infty}\phi^{-1}([g^n m, m]) = (x'', m] = (x', m]$ which implies $x' = x''$. Hence $x'' \in [x, x')$. Again note that $\phi^{-1}([x'', m]) = \phi^{-1}([x', m]) \cup \phi^{-1}\{x''\} = cl(\phi^{-1}((x', m]))$ is closed and connected. As in the first case $t' \in \phi^{-1}(x'')$. By the above discussion $gt' = t'$. Now by equivariance we have $gx'' = g\phi(t') = \phi(gt') = \phi(t') = x''$. But $x'' \in T$, hence it cannot be fixed by g (as T is the collection of cut points in M that are not parabolic.)

Hence Case 2 cannot happen. Therefore $x' \in \phi(M)$ and $gx' = x'$. □

Lemma 3.2.15. *Suppose finitely generated group $(G, \mathbb{P})$ is acting on a compact, connected, metrizable space M as a convergence group, relative to $\mathbb{P}$. Let $P \in \mathbb{P}$ be a parabolic subgroup of G. Suppose Φ be a complete median pretree obtained as in Section 3.1. If S is a P invariant full subset of Φ, then P fixes some point in S.*

Proof. By Lemma 3.2.3 P fixes a unique point x in Φ_0. If x is in S then we are done. Hence assume $x \notin S$.

Choose $g \in P, s \in S$. There exists $m = med(x, s, gs) \in S$ as Φ is median pretree and S is full. Moreover this m is unique by Lemma 2.5 of [Bow99c].

Clearly g sends $[x, s] \to [x, gs]$. Hence if $gm \neq m$, then $gm \in [x, m)$ or $gm \in (m, gs]$.

Case 1: $gm \in [x, m)$.

Clearly, gm is in between x and m (and not equal to x or m). We write $x < gm < m$ to mean gm is in between x and m in the linear set $[x, m] \cap S$. As g is acting by median pretree automorphism, it preserves the betweenness property. Hence we have $gx < g^2m < gm$ which implies $x < g^2m < gm < m$ as $gx = x$.

Notice that $g^n m \neq g^{n-1}m$ as that would imply $gm = m$ and we are assuming otherwise.

Thus we have $x < ... < g^n m < g^{n-1}m < ... < m$ in the arc $[x, m] \cap S$. Since Φ is complete, every arc is an interval. Hence there exists $x' \in [x, m]$ such that $[x', m] = \cup_{i=0}^{\infty}[g^i m, m]$.

Hence by Lemma 3.2.14, x' is a fixed point of g. Since $x < g^n m < gm < m$ hence $g^n m \neq m$ for all $n \in \mathbb{N}$. Hence g is not torsion. Therefore g must be a parabolic element. But by Lemma 6.10 of [Bow99c] the fixed point set of g contains only one element. Hence we have $x = x'$.

Now, consider the sequence of points $g^{-1}m, g^{-2}m....$ These are distinct points inside S as we have assumed $gm \neq m$. We next show that set $\cup_{i=1}^{\infty}[m, g^{-i}m]$ is also an arc. Notice that $g^i m < g^{i-1}m < ... < gm < m$ are contained in the arc $[x, m] \cap S$ in that linear order. Therefore g^{-i} translate of this linear set of points is also linear hence $m < g^{-1}m < ... < g^{-(i-1)}m < g^{-i}m$ is a linear set of points in S. Hence by induction $m < g^{-1}m < ... < g^{-(i-1)}m < g^{-i}m < ...$ is linear. Finally as S is full, so is $\cup_{i=1}^{\infty}[m, g^{-i}m]$.

As Φ is complete, the arc $\cup_{i=1}^{\infty}[m, g^{-i}m]$ is an interval. Therefore there exists x'' such that $g^{-1}m < g^{-2}m < ...g^{-n}m < ... < x''$ for all $n > 0$ and $\cup_{i=0}^{\infty}[m, g^{-i}m] = [m, x'']$. The point x'' must be fixed point of g by Lemma 3.2.14. Again applying Lemma 6.10 (uniqueness of fixed point) we see $x = x' = x''$

Notice that we have $x < m < x$ contradicting the first pretree axiom (T0) in [Bow99c]

Case 2: $gm \in (m, gs]$.

Notice that g^{-1} maps $[x, gs] \to [x, s]$. This implies $g^{-1}(m) \in [x, s]$. Since $m < gm < gs$, hence $g^{-1}(m) < m < s$ implying $x < g^{-1}(m) < m$. Therefore for all $n \in \mathbb{N}$, we have $x < g^{-n}m < g^{-1}(m) < m$ implying g^{-1} is not finite order implying g is not finite order.

Now we can apply the same argument as in Case 1.

Notice that if g is a torsion element, then $gm = m$ where $m = med(x, s, gs)$ and x is a fixed point of g, possibly $x \notin S$.

We have now established that all elements of P has fixed points inside S.

Suppose P contains at least one parabolic element g. By [Bow99c, Lemma 6.10] each parabolic element has a unique fixed point in Φ. Suppose x_g is the fixed point of g. By the above discussion $x_g \in S$. By Lemma 3.2.3, the group P has a unique fixed point in Φ. Note that $fix(P) \subset \mathrm{fix}(g)$. Hence $fix(P) = x_g$. But $x_g \in S$. Hence P fixes a point $x_g \in S$.

Suppose P is infinite torsion, then by Lemma 3.2.10 we are done. □

Theorem 3.2.16. *Suppose P is a group that acts on a pretree T via pretree automorphism. Let $\sim$ be a finite interval relation on T. If P fixes a point $x \in T$, then P fixes set-wise, the $\sim$ equivalence class of T containing x.*

Proof. If y and z are adjacent in T, and $\gamma \in P$, then $\gamma(y)$ and $\gamma(z)$ are also adjacent. Otherwise, by way of contradiction, assume that r is between $\gamma(y)$ and $\gamma(z)$. Then $\gamma^{-1}(r)$ is in between y and z. As y and z are adjacent, this gives a contradiction.

In particular if there are finitely many points between y and z, then there are finitely many points between $\gamma(y)$ and $\gamma(z)$.

By hypothesis P fixes $x \in T$. Suppose $y \sim x$. If $\gamma \in P$, we show that $\gamma(y) \sim x$. By definition of finite interval relation, there are finitely many points between x and y. Hence there are finitely many points between $\gamma(x)$ and $\gamma(y)$. But $\gamma(x) = x$. Hence there are finitely many points between x and $\gamma(y)$ thus implying $\gamma(y) \sim x$.

Therefore P fixes the equivalence class containing x setwise. □

Corollary 3.2.17. *Suppose G is a group, P is a subgroup of G, β is an ordinal, $\alpha = \beta + 1$, and $\sim$ is a finite interval relation on a pretree. Let $\sim_\alpha = \sim$ for $\alpha = 1$. Also suppose, whenever $\sim_\beta$ is a full relation on Φ, we have a full relation $\sim_\alpha$ on Φ such that $\Phi/\sim_\alpha = (\Phi/\sim_\beta)/\sim$. If a group G acts on $\Phi_\beta = \Phi/\sim_\beta$ relative to P then it acts on $\Phi/\sim_\alpha$ relative to P.*

Definition 3.2.18. (relatively one-ended group) Suppose G is a finitely generated group and $\mathbb{P}$ is a collection of subgroups of G. Then we say G is *relatively one-ended* if, whenever G acts without edge inversions on a simplicial tree relative to $\mathbb{P}$ such that all the edge stabilisers are finite, G fixes some vertex of the tree.

Lemma 3.2.19. *P fixes a point of $\Phi/\sim_1$.*

Proof. Let $F(P)$ denote the set of fixed points of P in Φ. By Lemma 3.2.3, $F(P)$ consists of a single point of $\Phi_0 - T$. Also by the remark after [Bow99c, Lemma 6.8], $\Phi/\sim$ can be identified with $\Phi_0/\sim$, where $\sim=\sim_1$ is the finite interval relation.

Suppose P fixes an equivalence class X of $\Phi_0/\sim$ setwise. Then X is a maximal full discrete subset of Φ. By Lemma 3.2.15, if P fixes X setwise, then it must fix some point $x \in X$.

Clearly, if another equivalence class X_1 is fixed setwise by P, then some $x_1 \in X_1$ is fixed P by Lemma 3.2.15. Then $x_1 \in F(P)$.

But by Lemma 3.2.3, $F(P)$ consists of a single point of $\Phi_0 - T$. Hence $x = x_1$ implying $X = X_1$ by maximality.

Hence $\mathbb{P}$ fixes a unique equivalence class in $\Phi/\sim_1$. □

Lemma 3.2.20. *$P \in \mathbb{P}$ fixes an equivalence class of $\Phi/\sim_\alpha$ for all ordinal α.*

Proof. We proceed by induction. By Lemma 3.2.19 $\Phi/\sim_1$ has a unique fixed point for P.

Suppose α is a successor ordinal, and $\alpha = \beta + 1$. By inductive hypothesis, assume that $\Phi/\sim_\beta$ has a unique fixed point of P. Since each equivalence class of $\Phi/\sim_\alpha$ is a maximal discrete full subset of $\Phi/\sim_\beta$ there fore the argument of Lemma 3.2.19 works.

Suppose α is a limit ordinal. By inductive hypothesis, assume that $\Phi/\sim_\beta$ has a fixed point of P for all $\beta < \alpha$. The fixed point (equivalence class) in each $\Phi/\sim_\beta$ is a nested sequence of subsets of points in Φ as each fixed equivalence class must contain a fixed point which is a fixed equivalence class of the previous ordinal and so on. Consider the union of this nested sequence of subsets of Φ. This is the equivalence class that is fixed by P in $\Phi/\sim_\alpha$. □

Lemma 3.2.21. *G does not fix any point of $\Phi/\sim$, where $\sim$ is the finite interval relation on Φ.*

Proof. Let $X = \sim$ equivalence class. X, is a (maximal) discrete full subset [Bow99c, Lemma 4.4]. Since X is full, it must itself be a median pretree, and so by [Bow99c, Lemma 3.34], it can be thought of as a simplicial tree.

Assume by way of contradiction, that G preserves setwise $X \subseteq \Phi$. Suppose $\Phi_0 = \Phi-$ (the set of those elements of $\Phi - T$ which are adjacent of precisely one element of T).

Let $S = X \cap \Phi_0$. S is a simplicial tree by [Bow99c, Lemma 3.34]. Let $S_0 = S \cap T$ and $S_1 = S - T$, thus $S = S_0 \sqcup S_1$. By [Bow99c, Lemma 3.28], each edge of S has one endpoint in each of S_0 (a cut point) and S_1 (not a cut point).

By Lemma 3.2.15, the action of G on S is relative to $\mathbb{P}$. As $(G, \mathbb{P})$ is relatively one-ended, if all the edge stabilizers of S is finite, by Definition 3.2.18 G would fix a point in S which contradicts [Bow99c, Lemma 6.11].

Suppose $a \in S_0$ (a cut point of M) and $p, q \in S_1$ adjacent to a. Let stabilizer of a be $G(a)$. It is either finite or loxodromic (as it is not a parabolic fixed point). Suppose the edge stabilizers $G(a) \cap G(p)$ and $G(a) \cap G(q)$ are both infinite. This implies $G(a)$ is infinite hence loxodromic. Thus $G(a) \cap G(p)$ and $G(a) \cap G(q)$ are finite index subgroups of $G(a)$, implying there is an infinite order element $\gamma \in G(p) \cap G(q)$. Therefore γ is a loxodromic that fixes p, a, q. But $a \in T, p, q \in \Phi_0$ contradicting [Bow99c, Lemma 6.9]. This shows that at least one of $G(a) \cap G(p)$ and $G(a) \cap G(q)$ must be finite. Without loss of generality, suppose $G(p) \cap G(a)$ is finite.

This implies that for every point $a \in S_0$, stabilizers of all but possibly one of the edges incident at a are finite. Collapsing all edges with infinite stabilizers, we have a tree on which G acts with finite edge stabilizers relative to $\mathbb{P}$. By relative one endedness, G fixes one of the vertices of this tree. But each vertex contains the star of at most one point in S_1. Therefore G fixes one of the points in S leading to a contradiction of [Bow99c, Lemma 6.11]. $\square$

Lemma 3.2.22. *Suppose a group G acts by pretree automorphisms on a pretree Φ. If a subpretree S of Φ is stabilized by G, then $S' = S - \{terminal\ points\}$ is also stabilized by G.*

Proof. Suppose x is terminal in S. Then $g \cdot x$ is also terminal in S. If not, suppose $y, z \in S$ such that $g \cdot x$ is between y and z. Then x is between $g^{-1}y$ and $g^{-1}z$ which is a contradiction.

Similarly points which are not terminal, are sent to points which are not terminal by the action of G. Hence G stabilizes S'. □

Corollary 3.2.23. *Let Φ be a complete median pretree and S be a full subset of Φ. Let G act on Φ by median pretree automorphism and suppose G stabilizes S as a set. If a subgroup P (of G) fixes a point in Φ then P fixes some point in $S' = S - \{terminal\ points\}$.*

Proof. It is easy to see that S is also a median pretree. By Lemma 3.2.23, S' is stabilized by G. By Lemma 3.2.15, P fixes some point in S'. □

3.3 Proof of the existence of nontrivial dendrite

In this section we will generalize Bowditch's construction of nontrivial dendrite (as in Section 3.2) and create a nontrivial dendrite in the relative case. Our construction differs from Bowditch's construction in one essential aspect. Bowditch assumes the group G to be one-ended. We, on the other hand, assume G to be one-ended relative to $\mathbb{P}$.

Proof of Theorem 3.0.18. By hypothesis, there is at least one cut point $p \in M$ that is not a parabolic fixed point. Suppose T is the set of the G translates of p. Hence the stabilizer of each point of T is either finite or loxodromic.

Bowditch proves in [Bow99c, Section 3], there exists a complete median pretree Φ, such that T embeds in Φ. This embedding has the property that for any distinct pair of points $x, y \in \Phi - T$, there is some $z \in T$ with xzy. Also for any pair of distinct points $x, y \in T$, there is some $z \in \Phi$ with xzy. Moreover Bowditch shows in [Bow99c, Section 3], the action of G on M induces an action of G on Φ via median pretree automorphism.

By Lemma 3.2.3, the induced action of G on Φ_0 is relative to $\mathbb{P}$ and each member of $\mathbb{P}$ has a unique fixed point in Φ_0.

Let $\sim_0$ be the trivial relation (equality) on Φ. Suppose $\sim_1 = \simeq$ be the finite interval relation on Φ. If α, β are ordinals such that α is a successor ordinal and $\alpha = \beta + 1$, then define $\Phi/\sim_\alpha = (\Phi/\sim_\beta)/\simeq$. If α is a limit ordinal, then define $\sim_\alpha = \vee\{\sim_\beta \mid \beta < \alpha\}$. Bowditch proves that these relations must eventually stabilise, i.e. for some α, we have $\sim_{\alpha+1} = \sim_\alpha$, so that, in fact, $\sim_\alpha = \sim_\beta$ for all $\beta \geq \alpha$ [Bow99c, Lemma 4.4]. If γ is the minimal ordinal for which these relations stabilize, then Bowditch proved that $\Phi/\sim_\gamma$ is a dendrite in [Bow99c, Theorem 23]. We write $\Phi/\sim_\gamma$ as $D(M)$.

We prove, by transfinite induction on the ordinal α, that G cannot fix any element of $\Phi/\sim_\alpha$. Bowditch proves that G does not fix any point in Φ without assuming one-endedness [Bow99c, Lemma 6.11]. Lemma 3.2.21 has proved that G does not fix any point for the case $\simeq = \sim_1$, so we can assume that $\alpha > 1$.

Suppose α is a limit ordinal such that $\sim_\alpha = \vee\{\sim_\beta \mid \beta < \alpha\}$. If G does not fix any point in $\Phi/\sim_\beta$ for all $\beta < \alpha$, then it does not fix any point in $\Phi/\sim_\alpha$ as in the proof of [Bow99c, Theorem 6.1]. In this part of the proof Bowditch does not use the one endedness of G.

Suppose α is a successor ordinal such that $\alpha = \beta + 1$. By induction hypothesis, we can assume that G acts on $\Phi/\sim_\beta$ without a global fixed point, relative to P. Hence by Corollary 3.2.17, it acts on $\Phi/\sim_\alpha$ relative to $\mathbb{P}$. We will show that G does not fix any point in $\Phi/\sim_\alpha$.

Assume by way of contradiction, that it fixes some equivalence class $\Xi \subseteq \Phi$ in $\Phi/\sim_\alpha$. Let $\Sigma = \Xi/\sim_\beta$. Therefore by definition, Σ is a finite interval equivalence class of $\Phi/\sim_\beta$, hence by [Bow99c, Lemma 3.34] Σ is a simplicial tree. Thus Σ admits a G-action, which by the inductive hypotheses has no G-invariant vertex. Moreover G acts on Σ without edge inversions as in the proof of [Bow99c, Theorem 6.1]. In this part of the proof Bowditch does not use the one endedness of G.

We will show that if the stabiliser of an edge of Σ is infinite, then one of the incident vertices will be terminal in Σ.

Suppose then that $X, Y \in \Sigma$ are adjacent. We can suppose that $p = p(Y, X) \in Y$ exists as in the definition before [Bow99c, Theorem 6.1]. Now if the edge stabilizer, $G(X) \cap G(Y)$ were infinite, it is either a subgroup of a parabolic subgroup or it would have to contain an infinite order element γ that must fix p. After all, if all elements are of finite order then, the edge stabilizer must be parabolic. But two adjacent vertices of Σ cannot be stabilized by a parabolic group as by Lemma 3.2.20, a parabolic subsgroup fixes a unique point in Σ. Hence we have the other case: $G(X) \cap G(Y)$ contains an infinite order element γ that fixes p.

Let W be the set of points $x \in \Phi$ such that $\neg xpX$. Thus $X \subseteq W$, and $Y \cap W = \phi$. Moreover if $x, y \in W$, then $[x, p)$ and $[y, p)$ are cofinal (since $med(x, y, p) \neq p$). Since X and p are γ- invariant, so is W. By [Bow99c, Theorem 6.1], W contains a fixed point q of γ. Since $q \notin Y$, we have $p \not\sim_\beta q$ and so $p \not\sim_1 q$. In other words, $[p, q]$ is infinite. By [Bow99c, Lemma 6.9, Lemma 6.10], we see that γ must be loxodromic. In fact, we are in case (2) of [Bow99c, Lemma 6.9], and so, in particular, p is terminal in Φ. By [Bow99c, Theorem 6.1] it follows that, in fact, $Y = p$. Hence Y must be terminal in $\Phi/\sim_\beta$ and so in particular, in Σ.

In summary, we have shown that if an edge of Σ is stabilised by an infinite group, then one its endpoints must be terminal. Now if we delete from Σ each such edge together with its terminal endpoint, we obtain a simplicial tree $S \subseteq \Sigma$ all of whose edge stabilisers are finite. Since as in the remark after [Bow98a, Theorem 7.1], none of the parabolic fixed points are terminal, hence G acts on $\Sigma - \{\text{terminal points}\}$ without edge inversion, relative to P. By definition 3.2.18 of relative one-endedness, we see that G must fix some vertex of S, i.e. some element of $\Phi/\sim_\beta$, contrary to the inductive hypothesis.

In summary, we conclude that for each ordinal α, no vertex of $\Phi/\sim_\alpha$ is fixed by G. In particular, $\Phi/\sim_\alpha$ is non-trivial. Now, by [Bow99c, Lemma 4.4], the minimal codense relation on Φ has the form $\Phi/\sim_\alpha$ for some ordinal α. We deduce that the quotient by the minimal codense relation is non-trivial. $\square$

4. Construction of the $\mathbb{R}$–tree

In this chapter we will prove Theorem 4.1.1 which uses the action of a relative one-ended, relatively hyperbolic group $(G, \mathbb{P})$ on its boundary M, such that (if possible) M has a cut point that is not a parabolic fixed point, and constructs an isometric action of G on an $\mathbb{R}$–tree T_0 relative to $\mathbb{P}$, without a global fixed point. Earlier we constructed a dendrite $D(M)$, on which G acts as a convergence group, relative to $\mathbb{P}$, without global fixed point. Now we will use the action of G on $D(M)$ to obtain the action of G on a $\mathbb{R}$–tree. Toward that end, we prove a relative version of [Lev98, Theorem 1].

Suppose $T' = D(M) - \{\text{terminal points}\}$. We show that T' is a nontrivial real tree.

Lemma 4.0.1. *Suppose $D(M)$ is a dendron. S is the collection of terminal points of $D(M)$ and $T' = D(M) - S$. If $D(M)$ is not a point then T' is not a point.*

Proof. Assume, by way of contradiction, that $T' = \{x\}$, is a point. Since $D(M)$ is not a point, hence it must have at least another point $x_1 \neq x$. Clearly x_1 is terminal. But so are all points on the arc connecting x_1 and x (as all of them were removed while creating T').

Suppose $p \in [x_1, x], p \neq x, p \neq x_1$. Since p is terminal in $D(M)$ (as all points except x are terminal), x_1, x are in the connected component $D(M) - \{p\}$. As $D(M)$ is compact, connected, locally connected, hence by [HY88, Theorem 3.16] $D(M) - \{p\}$ is arc-wise connected. This implies there is an arc $[x_1, x]$ that does not contain p. But $D(M)$ is uniquely arc connected by definition of dendron. Hence we have a contradiction. $\square$

Next we show that the parabolic fixed points cannot be terminal.

Lemma 4.0.2. *Suppose G acts on a dendrite $D(M)$ as a convergence group. Then the parabolic fixed points are not terminal in $D(M)$.*

Proof. Assume by way of contradiction that p is a terminal point in $D(M)$ and p is a fixed point of the parabolic subgroup $P \in \mathbb{P}$.

Clearly $D(M)$ is a complete pretree as it is a dendrite. Notice that $T' = D(M) - \{\text{Terminal Points}\}$ is a non trivial full subset of $D(M)$ that is stabilized set-wise by P (as non terminal points map to non terminal points). Hence by Lemma 3.2.15, P fixes a point in T'. As parabolic fixed points must be unique, this leads to a contradiction. □

Hence G acts on T' relative to $\mathbb{P}$ by homeomorphism. We show that there exists an $\mathbb{R}$-tree T_0 on which G acts by isometries relative to $\mathbb{P}$. Additionally we show that the arc stabilizers of T_0 stabilize arcs in T'. Toward that end we will adapt the strategy used in [GL15, Theorem 9.9] and [Lev98].

Definition 4.0.3 (Nonnesting action)**.** Suppose $g \in G$ and I is a non-degenerate arc, then $gI \subseteq I$ implies $gI = I$.

Levitt proves the following in [Lev98]:

Theorem 4.0.4. *If a finitely presented group G admits a non-trivial non-nesting action by homeomorphisms on an $\mathbb{R}$–tree T, then it admits a non-trivial isometric action on some $\mathbb{R}$–tree T_0. A subgroup fixing an arc in T_0 fixes an arc in T. Moreover given a finite collection of finitely generated subgroups $G_j \subset G$, each fixing a point of T, one may require that each G_j fixes a point of T_0.*

The proof of Theorem 4.0.4 in [Lev98] has two components. In order to describe these two components, we need to make some definitions.

First, we define *topological resolution* of an action of G on a real tree T'. This is very similar to definition of (metric) resolution in [Gui98, Definition 2.2]. Guirardel defines resolutions for isometric action of G on a $\mathbb{R}$–tree T'. We do not assume any metric structure on the real tree T' and action of G on T' by homeomorphisms.

Definition 4.0.5 (Topological Resolution)**.** Suppose G is a group that acts by homeomorphisms on a real tree T'. A topological resolution of the action of G on T' includes

- a finite graph K whose components are 1-connected
- a system of homeomorphisms $\mathcal{K}$ with domain and codomain K providing a connected finite 2-complex Σ with a foliation $\mathcal{F}$, such that $K \subset \Sigma$. By *foliation* we mean an equivalence relation on Σ, where each equivalence class is a 1-complex. We say that each equivalence class is a *leaf.*
- a base point $* \in K \subset \Sigma$
- a morphism ρ from $\pi_1(\Sigma, *)$ onto G and a covering map $\pi\colon \bar{\Sigma}_p \to \Sigma$ such that $\pi_1(\bar{\Sigma}_p) = G$
- a set C of curves contained in leaves which are conjugate to loops based at $*$ that normally generate $ker\rho$ in $\pi_1(\Sigma)$
- a G-equivariant map $f_{\Sigma_p}\colon \bar{\Sigma}_p \to T'$, constant on every leaf, which homeomorphically embeds any connected component of $\pi^{-1}(K) \subset \bar{\Sigma}_p$ into T' .

The two components of the proof of Theorem 4.0.4 are as follows.

Theorem 4.0.6. *If a finitely presented group G admits a non-trivial non-nesting action by homeomorphisms on an $\mathbb{R}$–tree T, then the action admits a topological resolution.*

Theorem 4.0.7. *If the action by homeomorphisms of group G on a real tree T' relative to a finite collection of finitely generated subgroups $G_j \subset G$ has a resolution, then G admits a non-trivial isometric action on some $\mathbb{R}$–tree T_0. A subgroup fixing an arc in T_0 fixes an arc in T. Moreover each G_j fixes a point of T_0.*

We first prove a relative version of Theorem 4.0.6.

Theorem 4.0.8. *If a finitely generated, relatively finitely presented group* $(G, \mathbb{P})$ *admits a non-trivial non-nesting action by homeomorphisms on an real tree* T' *relative to* $\mathbb{P}$*, then the action admits a topological resolution.*

Proof. In the proof of [GL15, Theorem 9.9], Guirardel and Levitt explains how to create a resolution for a convergence action of a relatively finitely presented groups on a $\mathbb{R}$–tree (acting relative to the peripheral subgroups). In the construction of the resolution the metric structure of $\mathbb{R}$–tree is not used. Hence it is lends to the construction of a topological resolution as in Definition 4.0.5. $\square$

4.1 Proof of a relative version of Theorem 4.0.4

Theorem 4.1.1. *If a finitely generated, relatively finitely presented, group* $(G, \mathbb{P})$ *admits a non-trivial non-nesting action by homeomorphisms on a real tree* T' *relative to* $\mathbb{P}$*, then it admits a non-trivial isometric action on some* $\mathbb{R}$*–tree* T_0 *relative to* $\mathbb{P}$*. A subgroup fixing an arc in* T_0 *fixes an arc in* T'*. Moreover given a finite collection of finitely generated subgroups* $G_j \subset G$ *, each fixing a point of* T*, one may require that each* G_j *fixes a point of* T_0*.*

Proof. The action of G on T' satisfies the hypothesis of Theorem 4.0.4, except for the finite present-ability, possible lack of metric structure of T' and relative action to $\mathbb{P}$.

In particular, $(G, \mathbb{P})$ may not be finitely presented but relatively finitely presented. This issue can be remedied by the arguments presented by Guirardel and Levitt in [GL15, Theorem 9.9]. The only place finite presentation was used in the proof of Theorem 4.0.4 is in the construction of resolution. We obtain this by Theorem 4.0.8.

The existence of the metric tree T_0 and the remaining construction of [Lev98, Theorem 1] depended on the existence of the resolution of the action. Since [GL15, Theorem 9.9] provides for such a resolution, the arguments of [Lev98, Theorem 1] goes through. For a detailed account, please refer to [Lev98, Theorem 1] and [GL15, Theorem 9.9].

By [GL15, Theorem 9.9] and the above remark, the action G relative to $\mathbb{P}$ on T' admits a topological resolution. Hence by Theorem 4.0.7 it admits a non trivial isometric action on some $\mathbb{R}$–tree T_0 such that a subgroup fixing an arc in T_0 fixes an arc in T'.

Moreover, the action of $(G, \mathbb{P})$ on T_0 is relative to $\mathbb{P}$. Suppose $P_1 \in \mathbb{P}$. Let the fixed point of P_1 in T' be p_1. The construction of the resolution, in [GL15, Theorem 9.9], includes p_1 in K as in the Definition 4.0.5. By [Lev98, Corollary 6], this ensures that P_1 fixes a point in T_0. Hence the action of $(G, \mathbb{P})$ on T_0 is relative to $\mathbb{P}$. □

4.2 Isometric action of a relatively hyperbolic group

In thin section we will prove Theorem 4.2.2 which constructs a nontrivial isometric action of a relatively one-ended, finitely generate group $(G, \mathbb{P})$ on a $\mathbb{R}$–tree T_0 if $(G, \mathbb{P})$ acts on a compact metrizable space M as a convergence group and M has a global cut point that is not a parabolic fixed point.

In Chapter 3, we obtained in Theorem 3.0.18 a nontrivial dendrite $D(M)$ on which G acts as a convergence group relative to $\mathbb{P}$. Moreover in Lemma 4.0.1, the real tree $T' = D(M) -$ terminal points is not a point.

We begin by showing that the action of G on T' is non nesting. Bowditch indicated a different argument to reach similar conclusion in [Bow98a].

Theorem 4.2.1. *Suppose G acts as a convergence group on a compact, locally connected, real tree, $D(M)$. Then the restriction of the action to $T' = D(M) - \{\text{terminal points}\}$ is non nesting.*

Proof. Suppose otherwise. Then there is a group element g and a non generate arc $I = [a, b]$, such that $gI \subsetneq I$ (otherwise $gI \subseteq I$ would imply $gI = I$). Here a, b are the end points of the arc. Clearly g is not of finite order.

Consider the infinite sequence $\{g, g^2, g^3, ..\}$. Since G acts on $D(M)$ as a convergence group, there exists is an infinite subsequence $\{g^{k_1}, g^{k_2}, g^{k_3}, ..\}$, and points (possibly same), α

and β in $D(M)$, such that $\lim g^{k_n} \cdot x = \alpha$ for all $x \in D(M) - \{\beta\}$ and $\lim g^{-k_n} \cdot x = \beta$ for all $x \in D(M) - \{\alpha\}$. We claim $\alpha \in I$.

Since I is non degenerate, choose $x_0 \in I$ that is not equal to α, β. Hence, $\lim g^{k_n} \cdot x_0 = \alpha$. But $g^{k_n} \cdot x_0 \in I$ for each n as $g \cdot I \subsetneq I$. Therefore the limit point of the sequence is also in I. Therefore $\alpha \in I$.

Note that the action of the cyclic group generated by g on I is 'stretch-shrink' in the following sense: $...g^2 \cdot I \subsetneq g \cdot I \subsetneq I \subsetneq g^{-1} \cdot I \subsetneq g^{-2} \cdot I....$

This can be proven in the following manner. $g \cdot I \subsetneq I$ by given hypothesis. Chose $x \in I - g \cdot I$. Clearly $g^{-1} \cdot x \notin I$ because otherwise $g(g^{-1} \cdot x) \in gI \subsetneq I$ or $x \in I$. Hence there are points in $g^{-1} \cdot I$ which are not in I. Now for the other part, we use the following argument. Suppose $x \notin g^{-1} \cdot I$. This implies $gx \notin I$ This implies $x \notin I$ as whenever $x \in I$ we have $gx \in I$. Therefore we have the following implication $x \notin g^{-1} \cdot I \implies x \notin I$. Thus $x \in I \implies x \in g^{-1} \cdot I$. The 'stretch-shrink' property follows by induction.

Next we show that $\beta \notin I$. Recall the notation $I = [a, b]$ where a, b are the endpoints. Notice that we cannot have $g^{-n} \cdot a \in I$ and $g^{-n} \cdot b \in I$ as, $g^{-n} \cdot I$ is strictly larger than I.

Without loss of generality, suppose $b_{-k_n} = g^{-k_n} \cdot b \notin I$.

Clearly $\lim g^{-k_n} \cdot b \to \beta$. Since I is homeomorphic to the closed interval, its images $g^{-k_n} \cdot I = I_{-k_n}$ are also (nested) closed intervals (strictly expanding). $\lim b_{-k_n} = \beta$ is an endpoint of the closure of $\cup I_{-k_n}$.

Also $\beta \neq a$ otherwise we have an embedded circle in $D(M)$. This is not possible as by hypothesis, as $D(M)$ is a real tree.

Hence $\beta \notin I$. Therefore $\beta \neq \alpha$ as earlier we showed $\alpha \in I$ and then we proved $\beta \notin I$. Hence they cannot be the same point.

In fact $\alpha \notin (a, b]$. Consider the following subspace of T: $I^* = \cup I_{-k_n}$. This is homeomorphic to a line or a ray. If $\alpha \in (a, b]$, we have $\lim g^{-k_n} a = \beta$.

But earlier we found $\lim g^{-k_n} b = \beta$. This implies I is eventually sent inside a neighborhood U of β. As β is outside I, hence we can arrange this neighborhood U of β to be disjoint from I. But, $g^{-n} \cdot I$ cannot be disjoint from I (infact it contains I).

Finally we show $\alpha \neq a$ thus achieving contradiction. This is as follows: We show that it is a terminal point.

If a is not a terminal point of $D(M)$, then choose $a' \in D(M) - I$ such that a is between a' and b.

Therefore $\lim g^{-k_n} \cdot a' = \beta$. This implies $g^{-k_n} a'$ eventually goes inside a neighborhood V that can be arranged to be disjoint from I . As g^{-k_n} is a homeomorphism, it must preserve 'betweeness' and hence a goes inside that neighborhood, thus carrying I inside that neighborhood of β. This implies $g^{-k_n} I$ is disjoint from I. This is impossible.

Hence we showed $\alpha \notin I$. Hence we find a contradiction as earlier we found $\alpha \in I$. Therefore the $gI \not\subset I$. $\square$

Theorem 4.2.2. *Suppose a relatively one-ended finitely generated group* $(G, \mathbb{P})$*, that acts on a compact metrizable space* M *relative to* $\mathbb{P}$ *also acts on a nontrivial dendrite* $D(M)$ *as a convergence group relative to* $\mathbb{P}$*. Then there exists an* $\mathbb{R}$*–tree* T_0 *that admits a non-trivial isometric action by* G*. A subgroup fixing an arc in* T_0 *fixes an arc in* $D(M)$*. Moreover given a finite collection of finitely generated subgroups* $G_j \subset G$ *, each fixing a point of* $D(M)$*, one may require that each* G_j *fixes a point of* T_0*.*

Proof. By Lemma 4.0.2, Theorem 3.0.18 and Theorem 4.2.1 there exists a real tree T', which admits a non-trivial non-nesting action by homeomorphisms by G relative to $\mathbb{P}$. Therefore by Theorem 4.1.1 we have the group $(G, \mathbb{P})$ acts on some $\mathbb{R}$–tree T_0 by isometries relative to $\mathbb{P}$ and given a finite collection of finitely generated subgroups $G_j \subset G$, each fixing a point of $D(M)$, one may require that each G_j fixes a point of T_0. $\square$

5. Relative Accessibility

In this chapter we consider a JSJ decomposition of a relatively hyperbolic group $(G, \mathbb{P})$. We show in Lemma 5.0.21 and Lemma 5.0.18 that some of the vertex groups of this decomposition fixes some point in the dendrite obtained in Chapter 3. We finally show in Lemma 5.0.23 and Lemma 5.0.14 $(G, \mathbb{P})$ fixes some point in the $\mathbb{R}$–tree T_0 obtained in Section 4.2.

We begin by considering the elementary splittings of G relative to $\mathbb{P}$. First, we give the definition of an elementary subgroup.

Definition 5.0.3. A subgroup of a finitely generated relatively hyperbolic group $(G, \mathbb{P})$ is *elementary* if it is virtually cyclic (possibly finite) or parabolic (as in Definition 2.0.9).

Lemma 5.0.4. *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group. Suppose G acts as a convergence group on a dendrite $D(M)$ (not necessarily relative to $\mathbb{P}$). Then the arc stabilizers of this action restricted to $T' = D(M) - \{Terminal\ points\}$ are finite.*

Proof. Suppose $[a, b]$ is a non trivial arc (homeomorphic to closed interval $[0, 1]$) in T'. Assume, by way of contradiction, there is an infinite sequence of group elements $(g_1, g_2, ...)$ that stabilizes the arc $[a, b]$.

Since the action of $(G, \mathbb{P})$ on $D(M)$ is a convergence action, hence there exists an infinite subsequence $\langle g_{k_1}, g_{k_2}, ...\rangle$ and points $\alpha, \beta \in D(M)$ (possibly equal), such that $g_{k_n}(x) \to \alpha$ for all $x \in D(M) - \{\beta\}$ locally uniformly. By locally uniformly we mean, if C is a compact subset of $D(M) - \{\beta\}$ and U is any open neighborhood of α, then there is an $N \in \mathbb{N}$ such that $g_{k_i}C \subset U$ for all $i > N$.

If $\beta \notin [a, b]$ then we have two cases:

1. $\alpha \notin [a,b]$: In this case, choose an open neighborhood U of α disjoint from $[a,b]$. By the property of locally uniform action, $g_{k_n}([a,b]) \subset U$ for all $n > N_0$ for some $N_0 \in \mathbb{N}$, hence contradiction.

2. $\alpha \in [a,b]$: In this case, choose an open neighborhood U of α strictly smaller than $[a,b]$. By the property of locally uniform action, $g_{k_n}([a,b]) \subset U$ for all $n > N_0$ for some $N_0 \in \mathbb{N}$, hence contradiction (as the homeomorphisms $\langle g_{k_n} \rangle$ map $[a,b]$ strictly inside $[a,b]$).

If $\beta \in [a,b]$ then the same argument applies with the sequence $\langle g_{k_1}^{-1}, g_{k_2}^{-1}, ... \rangle$ □

We will need the following definitions next. In the following definitions T_S is any simplicial tree.

Let G be a finitely generated group acting on a simplicial tree T_S without inversions. Suppose there is no proper G invariant subtree.

Definition 5.0.5. A subgroup $H < G$ acts *elliptically* on T_S if it fixes a point of T_S.

We can choose a preferred collection of subgroups and study splittings of G over this collection of subgroups. Toward that end, we have the following definition.

Definition 5.0.6. Suppose $\mathbb{A}$ is a collection of subgroups of G that is closed under conjugation and passing to subgroups. We say T_S is an $\mathbb{A}$–*tree* if every edge stabilizer is a member of the collection $\mathbb{A}$.

Similarly, we often specify a collection of subgroups which are elliptic in T_S.

Definition 5.0.7. Suppose $\mathbb{P}$ is an arbitrary family of subgroups of G, an $(\mathbb{A}, \mathbb{P})$–*tree* is an $\mathbb{A}$–tree T_S such that every $P \in \mathbb{P}$ acts elliptically on T_S.

Next we define universally elliptic trees.

Definition 5.0.8 (universally elliptic)**.** An $(\mathbb{A}, \mathbb{P})$–tree is *universally elliptic* if its edge stabilizers act elliptically on every $(\mathbb{A}, \mathbb{P})$–tree.

Finally we need the notion of domination.

Definition 5.0.9. If G acts on trees T_S and T'_S, we say T_S *dominates* T'_S if there is a G-equivariant map $T_S \to T'_S$. This is equivalent to saying that each vertex stabilizer of T_S also stabilizes a vertex of T'_S. Two $(\mathbb{A}, \mathbb{P})$–trees T_S and T'_S are *equivalent* if T_S dominates T'_S and T'_S dominates T_S.

Finally, we define JSJ tree.

Definition 5.0.10 (JSJ Tree)**.** An $(\mathbb{A}, \mathbb{P})$–tree T_S is a *JSJ tree for splittings of* G *over* $\mathbb{A}$ *relative to* $\mathbb{P}$ if it satisfies the following:

1. T_S is universally elliptic among all $(\mathbb{A}, \mathbb{P})$–trees.

2. T_S dominates any other universally elliptic $(\mathbb{A}, \mathbb{P})$–tree.

We will be interested in a particular JSJ tree. We first define QH subgroup and flexible subgroup.

Definition 5.0.11 (flexible)**.** A vertex stabilizer G_v of a JSJ tree over $\mathbb{A}$ relative to $\mathbb{P}$ is *flexible* if there is another $(\mathbb{A}, \mathbb{P})$–tree on which G_v does not act elliptically.

Definition 5.0.12 (Quadratically hanging)**.** A vertex stabilizer G_v of an $(\mathbb{E}, \mathbb{P})$–tree is *quadratically hanging* if it is an extension

$$1 \to F \to G_v \to \pi_1(\Sigma) \to 1,$$

where Σ is a compact hyperbolic two-orbifold and F is an arbitrary group called the *fiber*. Additionally, it is required that each incident edge stabilizer and each group $G_v \cap gPg^{-1}$ for $P \in \mathbb{P}$ has image in $\pi_1(\Sigma)$ that is either finite or contained in a boundary subgroup of $\pi_1(\Sigma)$.

We examine a JSJ splitting of $(G, \mathbb{P})$ over elementary arc stabilizers relative to $\mathbb{P}$. Such a splitting exists. In [GL17, Corollary 9.20], Guirardel and Levitt proves the following:

Theorem 5.0.13. *[GL17, Corollary 9.20] Let G be hyperbolic relative to a finite family of finitely generated subgroups $\mathbb{P} = \{P_1, ..., P_p\}$. Let $\mathcal{A}$ be the family of elementary subgroups of G. If G is one-ended relative to P, there is a JSJ tree T_{CAN} over A relative to P which is equal to its tree of cylinders, invariant under automorphisms of G preserving P, and compatible with every $(\mathbb{A}, \mathbb{P})$-tree. Its non-elementary flexible vertex stabilizers are QH with finite fiber.*

From this point onward, T_{CAN} refers to the special JSJ tree in the Theorem 5.0.13. The edge stabilizers of T_{CAN} are not finite as G is one ended relative to $\mathbb{P}$. In [GL15, Section 3.3], Guirardel and Levitt provides the following classification of the vertex groups $G(v)$:

- rigid: Let $\mathcal{H}$ be the collection of stabilizers of edges incident on a vertex v. We say a vertex group $G(v)$ is *rigid* if it is non-elementary and is elliptic in every $(A, \mathbb{P} \cup \mathcal{H})$-tree.
- (flexible) QH : $G(v)$ is non-elementary and not universally elliptic. Then v is a flexible QH vertex with finite fiber
- maximal parabolic: $G(v)$ is conjugate to a P_i.
- maximal loxodromic: $G(v)$ is a maximal virtually cyclic subgroup of G, and $G(v)$ is not parabolic.

Notice that we have two actions of G relative to $\mathbb{P}$: on T_{CAN} and on $\mathbb{R}$–tree T_0. Moreover the action on T_0 is without global fixed point (in particular T_0 itself is not a point).

Lemma 5.0.14. *Suppose a finitely generated group G has a graph of group decomposition T_S over infinite edge stabilizers. Let G act minimally by isometries on a $\mathbb{R}$–tree T_0 such that all arc stabilizers are finite. If all vertex groups of T_S fix points in T_0, then T_0 is a point.*

Proof. Assume by way of contradiction that u, v are adjacent vertices in T_S with $G(u), G(v)$ (respective vertex stabilizers) fixing distinct points $u', v' \in T_0$. Let e be the edge connecting $u, v \in T_S$. Let $G(e)$ be the infinite edge stabilizer.

Consider the arc $[u', v'] \subset T_0$. Both end points of the arc are fixed by the infinite group $G(e)$. Since the action is by isometry, if two end points are fixed by $G(e)$ then the entire arc is fixed by $G(e)$. But that is a contradiction as all arc stabilizers of T_0 are finite by Lemma 5.0.4.

Therefore the entire group G fixes a single point. By minimality, T_0 is a point. $\square$

Hence, to achieve a contradiction we will show that each of the vertex groups of T_{CAN} fixes some point on T_0. This would imply that G fixes a point of T_0, but that is a contradiction.

The maximal parabolic subgroups are elliptic in T_0 by Theorem 4.1.1. We will show that maximal loxodromic, rigid and QH vertices are elliptic on T_0 as well.

We need the following definition to understand the peripheral structures. More details can be found in [GL15, Section 4.2.1].

Let T be a tree (minimal, relative to $\mathbb{P}$, with edge stabilizers in $\mathcal{A}$). Let v be a vertex, with stabilizer G_v.

Definition 5.0.15 (Incident edge groups Inc_v)**.** Given a vertex v of a tree T, there are finitely many G_v-orbits of edges with origin v. We choose representatives e_i and we define Inc_v (or Inc_{G_v}) as the family of stabilizers G_{e_i} . We call Inc_v the set of incident edge groups. It is a finite family of subgroups of G_v, each well-defined up to conjugacy.

Definition 5.0.16 (Restriction $\mathbb{P}|_{G_v}$)**.** Given v, consider the family of conjugates of groups in $\mathbb{P}$ that fix v and no other vertex of T. We define the restriction $\mathbb{P}|_{G_v}$ by choosing a representative for each G_v-conjugacy class in this family.

Definition 5.0.17. The collection of subgroups $Q_v = Inc_v^{\mathbb{P}}$ is defined as the union of Inc_v and $\mathbb{P}|_{G_v}$.

We begin by analyzing the maximal loxodromic vertex groups. The proof of the following lemma is largely inspired by [Bow99b, Theorem 0.1]. It differs from [Bow99b, Theorem 0.1] in that we assume splitting of G relative to $\mathbb{P}$.

Lemma 5.0.18. *Let $G(v)'$ be a maximal loxodromic vertex group in T_{CAN}. Then $G(v)'$ fixes some point in T'.*

Proof. Let $G(v)$ be an edge group that is subgroup of $G(v)'$. Note that $G(v)$ cannot be finite as G is relatively one ended. Hence $G(v)$ is an infinite subgroup of $G(v)'$. Therefore $G(v)$ must be a two-ended edge group. Moreover as $G(v)$ is a finite index two ended subgroup of the two ended group $G(v)'$, hence the limit set of $G(v)$ is same as the limit set of $G(v)'$ in M. We show that limit set of $G(v)$ collapses to a point in $D(M)$ implying that the limit set of $G(v)'$ collapses to a point in $D(M)$

In fact, we claim that $G(v)$ is parabolic on T' $(= D(M) - \{\text{terminal points}\})$ and hence on T_0. If $G(v)$ is parabolic on M, then it is certainly parabolic on T' as by Lemma 4.0.2. So we can assume that it is loxodromic on M.

Thus the limit set of $\Lambda G(v)$ consists of precisely two points, say a and b. By [HH19, Proposition 5.6] we have a separation of $M - \{a, b\}$ into two disjoint, nonempty open sets of M and this partition is $G(v)$-equivariant. Moreover $(M - \Lambda G(v))/G$ is compact Hausdorff. By hypothesis $(M - \Lambda G(v))/G$ is disconnected, so we can write it as a disjoint union, $A_1 \sqcup A_2$, of nonempty closed subsets. Now, the preimage, U_i, of A_i in $M - \Lambda G(v)$ is open in $M - \Lambda G(v)$ and hence in M. Thus $B_i = U_i \cup \Lambda G \subseteq M$ is closed and $G(v)$-invariant. Moreover $M = B_1 \cup B_2$ and $\Lambda G(v) = B_1 \cap B_2$.

We claim that B_i is connected. To see that, let K be a connected component of B_i. If $K \cap \Lambda G$ were empty, then we could find a closed and open subset, L, of B_i containing K, and which does not meet $\Lambda G(v)$. We see that L must be closed and open in M, contradicting the fact M is connected. This shows that $K \cap \Lambda G(v) \neq \varnothing$. Suppose that $a \in K \cap \Lambda G(v)$. Let $H \neq G(v)$ be the subgroup (of index at most 2) of $G(v)$ which fixes a. Now K is H-invariant, so either $\Lambda G(v) \subseteq K$ or $K = \{a\}$. In the former case, we see that $B_i = K$ is connected as required. In the latter case, we deduce similarly, that $\{b\}$ is a component of B_i, giving the contradiction that $B_i = \Lambda G(v)$.

It now follows that no point of M separates the two points of $\Lambda G(v)$ collapses to a point in $D(M)$ implying that two points of $\Lambda G(v)$ collapses to a point in $D(M)$. So $G(v)'$ is parabolic in $D(M)$ therefore by Lemma 4.0.2, $G(v)'$ fixes some point in T'. $\square$

Next we show that each flexible vertex group also fixes some point in $D(M)$. We need the following lemmas toward that end.

Lemma 5.0.19. *Suppose $v_1, v_2 \in M$ are fixed points of some loxodromic element g (not conjugated in any boundary subgroup) and there is a separation of $M - \{v_1, v_2\}$ into two disjoint nonempty open sets of M. Similarly $u_1, u_2 \in M$ are fixed points of another loxodromic element h and there is a separation of $M - \{u_1, u_2\}$ into two disjoint nonempty open sets of M. Then v_1, v_2, u_1, u_2 are identified in the dendrite $D(M)$.*

Proof. Clearly $v_1 \sim v_2$ and $u_1 \sim u_2$ by Lemma 5.0.18. Suppose $v_1 \nsim u_1$. Therefore they are separated by a set of cut points of M order-isomorphic to the rationals (by definition). Suppose c is one such cut point. Hence $M = UcV$. Without loss of generality, assume $u_1, u_2 \in U$ and $v_1, v_2 \in V$. In the boundary the pair (v_1, v_2) separates (u_1, u_2). Suppose W_1, W_2 be the two connected components of the separation. Without loss of generality we can assume $u_1 \in W_1, u_2 \in W_2$. But that is in contradiction with our previous finding that u_1, u_2 are in the same connected open set U. $\square$

Definition 5.0.20. Let Σ be a compact hyperbolic 2-orbifold, and let $\mathcal{C}$ be a non-empty collection of (non-disjoint) essential simple closed geodesics in Σ. We say that $\mathcal{C}$ fills Σ if the following equivalent conditions hold:

- For every essential simple closed geodesic α in Σ, there exists $\gamma \in \mathcal{C}$ that intersects α non-trivially (with $\alpha \neq \gamma$).

- For every element $g \in \pi_1(\Sigma)$ of infinite order that is not conjugate into a boundary subgroup, there exists $\gamma \in \mathcal{C}$ such that g acts hyperbolically in the splitting of $\pi_1(\Sigma)$ dual to γ.

- The full preimage $\tilde{\mathcal{C}}$ of $\mathcal{C}$ in the universal covering $\tilde{\Sigma}$ is connected.

Lemma 5.0.21. *QH vertex group $G(v)$ fixes a point in T'.*

Proof. $G(v)$ is hyperbolic relative to the finite family of finitely genrated subgroups Q_v by [GL15, Lemma 3.8]. By definition $G(v)$ is a finite extension of a fundamental group of a two dimensional hyperbolic orbifold Σ.

$$1 \to F \to G(v) \to \pi_1(\Sigma) \to 1$$

Since F is a finite normal subgroup and $G(v)/F = \pi_1(\Sigma)$, hence $G(v)$ is quasi isometric to $\pi_1(\Sigma)$ and their boundaries are homeomorphic.

As $G(v)$ is flexible, (and F being finite, fixes some point in every tree), by [GL17, Proposition 5.20], Σ contains an essential simple closed geodesic. Hence by [GL17, Corollary 5.10] Σ has a filling set of geodesics $\mathcal{C}$. Let $\tilde{\mathcal{C}}$ be the full preimage of $\mathcal{C}$ in the universal covering $\tilde{\Sigma}$. As $\mathcal{C}$ is filling, $\tilde{\mathcal{C}}$ is connected.

Lifts of each essential simple closed geodesic in Σ are, by definition, a disjoint collection of bi-infinite geodesics in $\tilde{\Sigma}$ (not contained in $\partial\tilde{\Sigma}$). Whenever two such bi-infinite geodesics in $\mathbb{H}^2$ intersect each other non-trivially, their endpoints are identified in $D(M)$ by Lemma 5.0.19. Hence all the endpoints of the filling set of geodesics are identified in $D(M)$ by the remark after [GL17, Lemma 5.28]. Note that $G(v)$ is hyperbolic relative to Q_v, hence action of $G(v)$ on its boundary is minimal. Therefore the closure of collection of endpoints of the filling set of geodesics is the entire boundary. Since the quotient map from $M \to D(M)$ is upper semicontinuous by [Bow99c, Lemma 6.5], hence the equivalence classes are closed. Therefore the entire boundary of $G(v)$ is mapped to a point in $D(M)$. Since the infinite subgroup $G(v)$ fixes a unique point in $D(M)$, this fixed point is not terminal by Lemma 4.0.2. Hence $G(v)$ fixes a point in T' $\square$

Next we analyze the rigid vertex groups. First we need a lemma.

Lemma 5.0.22. *Suppose $G(v)$ is a vertex group in T_{CAN}. If it acts on T_0 without a global fixed point such that each loxodromic (maximal, virtually cyclic) subgroup fixes some point* *in T_0. Then $G(v)$ splits over an elementary subgroup relative to Q_v.*

Proof. By [GL15, Lemma 3.8] $G(v)$ is hyperbolic relative to the finite family of finitely genrated subgroups Q_v. We show that $G(v)$ acts on the $\mathbb{R}$–tree T_0 relative to Q_v. By [GL15, Lemma 3.8]) each member of $Q_v = Inc_v \cup \mathbb{P}|_{G_v}$ is either $G(v) \cap gP_ig^{-1}$, loxodromic (maximal virtually cyclic) or finite. Observe that,

1. $G(v) \cap gP_ig^{-1}$ fixes some point in T_0 as P_i's fix points in T_0
2. Loxodromic maximal virtually cyclic subgroups of $G(v)$ fixes some point in T_0 by hypothesis
3. Finite subgroups fixes some point in T_0 as finite subgroups have global fixed points

Note that $G(v)$ acts as a convergence group on $D(M)$, hence all arc stabilizers of the action of $G(v)$ on T' are elementary (in fact finite) by Lemma 5.0.4. By Theorem 4.1.1, each arc stabilizer of $G(v)$ in T_0, stabilizes an arc in T'. Therefore all arc stabilizers of $G(v)$ on T_0 are elementary (in fact finite).

We apply Theorem 1.3.3 with $G = G(v)$, $\mathbb{P} = Q_v$ and $\mathcal{H} = \Phi$. If $G(v)$ does not fix a point in T_0, then it must split over an elementary subgroup relative to Q_v. □

Lemma 5.0.23. *Rigid vertex group $G(v)$ fixes a point in T_0.*

Proof. Assume by way of contradiction that $G(v)$ does not fix a point in T_0. Then by Lemma 5.0.22, G(v) splits over an elementary subgroup relative to Q_v. But that is not possible as by definition, rigid vertex groups have no such splitting.

Therefore rigid vertex groups fix some point in T_0. □

6. Proof of Theorem 1.2.2

We now give a proof of Theorem 1.2.2 using results obtained in the previous chapters.

Proof. Assume by way of contradiction that there is a global cut point p that is not a parabolic fixed point. By Theorem 3.0.18, M has an equivariant quotient $D(M)$ that is a nontrivial dendrite such that the induced action of $(G, \mathbb{P})$ on $D(M)$ is a minimal convergence action, relative to $\mathbb{P}$.

Remove the terminal points of $D(M)$ to produce a separable real tree T'. By Lemma 4.0.1 T' is not a point as $D(M)$ is not a point. Moreover none of the parabolic points of $D(M)$ are terminal by Lemma 4.0.2. So restriction of the action of G to T' is relative to $\mathbb{P}$.

Let T_{CAN} be a JSJ decomposition of $(G, \mathbb{P})$ over elementary arc stabilizers relative to $\mathbb{P}$. This JSJ tree exists by [GL17, Corollary 9.20]. The vertex stabilizers $G(v)$ of the JSJ tree are of four types by [GL15, Section 3.3]: rigid, flexible, maximal parabolic and maximal loxodromic. Each maximal parabolic subgroup fixes some point in T' as the action on T' is relative to $\mathbb{P}$. By Lemma 5.0.18, Lemma 5.0.21, each maximal loxodromic subgroup and each QH subgroup fixes some point in T'. Suppose $\mathcal{F} = \{G_1, ..., G_j\}$ be the collection of maximal parabolic, maximal loxodromic and QH vertex groups in T_{CAN}.

By Theorem 4.2.1 the restriction of the action of G to T' is non-nesting and without global fixed point. Therefore the action of $(G, \mathbb{P})$ on T' satisfies the hypothesis of Theorem 4.1.1. By Theorem 4.1.1, G acts on a non-trivial $\mathbb{R}$–tree T_0 by isometries such that each member of $\mathcal{F}$ fixes a point in T_0. Moreover this action is non-trivial.

By Lemma 5.0.23, each rigid vertex group of T_{CAN} fixes some point in T_0. Hence all four types of vertex groups fix some point in T_0. By Lemma 5.0.14, this implies that G fixes a point in T_0. This is a contradiction as earlier we found that G acts on T_0 without global fixed point.

Hence there is no cut point in M that is not a parabolic fixed point. $\square$

Bibliography

[BF91] Mladen Bestvina and Mark Feighn. Bounding the complexity of simplicial group actions on trees. *Invent. Math.*, 103(3):449–469, 1991.

[BM91] Mladen Bestvina and Geoffrey Mess. The boundary of negatively curved groups. *J. Amer. Math. Soc.*, 4(3):469–481, 1991.

[Bow98a] B. H. Bowditch. Group actions on trees and dendrons. *Topology*, 37(6):1275–1298, 1998.

[Bow98b] Brian H. Bowditch. Cut points and canonical splittings of hyperbolic groups. *Acta Math.*, 180(2):145–186, 1998.

[Bow99a] B. H. Bowditch. Boundaries of geometrically finite groups. *Math. Z.*, 230(3):509–527, 1999.

[Bow99b] B. H. Bowditch. Connectedness properties of limit sets. *Trans. Amer. Math. Soc.*, 351(9):3673–3686, 1999.

[Bow99c] B.H. Bowditch. Treelike structures arising from continua and convergence groups. *Mem. Amer. Math. Soc.*, 139(662):1–86, 1999.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012.

[GL15] Vincent Guirardel and Gilbert Levitt. Splittings and automorphisms of relatively hyperbolic groups. *Groups Geom. Dyn.*, 9(2):599–663, 2015.

[GL17] Vincent Guirardel and Gilbert Levitt. *JSJ decompositions of groups*, volume 395 of *Astérisque*. Société Mathématique de France, Paris, 2017.

[GM87] F. W. Gehring and G. J. Martin. Discrete quasiconformal groups. I. *Proc. London Math. Soc. (3)*, 55(2):331–358, 1987.

[Gro87] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[Gui98] Vincent Guirardel. Approximations of stable actions on **R**-trees. *Comment. Math. Helv.*, 73(1):89–121, 1998.

[HH19] Matthew Haulmark and G. Christopher Hruska. On canonical splittings of relatively hyperbolic groups. arXiv:1912.00886v2, 2019.

[Hru10] G. Christopher Hruska. Relative hyperbolicity and relative quasiconvexity for countable groups. *Algebr. Geom. Topol.*, 10(3):1807–1856, 2010.

[HY88] John G. Hocking and Gail S. Young. *Topology*. Dover Publications, Inc., New York, second edition, 1988.

[Lev98] Gilbert Levitt. Non-nesting actions on real trees. *Bull. London Math. Soc.*, 30(1):46–54, 1998.

[Osi06] Denis V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.

[Rol98] Martin Roller. *Poc sets, median algebras, and group actions: An extended study of Dunwoody's construction and Sageev's theorem.* Habilitation, University of Regensburg, 1998. arXiv:1607.07747.

[Ser77] Jean-Pierre Serre. *Arbres, amalgames,* SL_2, volume 46 of *Astérisque.* Société Mathématique de France, Paris, 1977. Written in collaboration with Hyman Bass.

[Swa96] G. A. Swarup. On the cut point conjecture. *Electron. Res. Announc. Amer. Math. Soc.*, 2(2):98–100, 1996.

[Tuk94] Pekka Tukia. Convergence groups and Gromov's metric hyperbolic spaces. *New Zealand J. Math.*, 23(2):157–187, 1994.

[Yam04] Aslı Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.

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