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Optimal Trading Under the American Perpetual Put Option for Geometric Brownian Motion and Mean-reverting Processes

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Abstract

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This thesis is focused on the perpetual American put option under the geometric Brownian motion and mean-reverting models. Two approaches, which have been applied before to the call option of a mean-reverting process, will be studied in details for the two models. The first approach amounts to solving the associated quasi-variational inequality for the optimal stopping problem. A verification theorem is proved to demonstrate that the solution to the quasi-variational inequality agrees with the value function. The second approach is based on detailed analyses of an auxiliary two-point stopping problem, which leads to an explicit expression for the value function. Both approaches identify an optimal execution rule for the two models.

To my loving parents

who support me in my personal

and academic endeavors.

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List of Abbreviations

GBM	geometric Brownian motion
MRP	mean-reverting process
QVI	quasi-variational inequality
SDE	stochastic differential equation

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Chapter 1

Introduction

In different fields of life we want to find the optimal time to perform an action to gain a desired result. In the context of financial markets this is the time when the highest possible asset can be gained from selling a stock. Finding such an optimal point in time is a long discussed problem of theoretical and applied mathematics.

The behaviour of a stock is described using a mathematical process. The optimal stopping time depends on the behaviour of our stock and on the used trading rule.

In this thesis we want to verify two approaches to determine the optimal time for executing a perpetual put option. To do this we will describe in the first chapter the mathematical background. The perpetual put option and stochastic processes in general will be defined. This also includes a description of the different stochastic processes that will be used in this thesis to describe the behaviour of a traded stock. Furthermore, we briefly explain the ideas behind the different approaches that will be applied. The approaches to determine an optimal stopping time have previously been applied to the call option of a stock. The defined goal of this thesis is to show that these approaches can be applied to other trading rules.

The two approaches will be applied to stochastic processes in chapters 3 and 4. The trading rule that will be used is the perpetual put option. In both chapters a candidate solution with each approach will first be determined. After that we will try to verify that the calculated candidates are a valid solution.

The results will be summarized in the last chapter. We will also include a brief preview of how the results can be improved or which other stochastic processes can be analyzed by applying the approaches.

Chapter 2

Mathematical Background

In this chapter we want to clarify the different mathematical concepts to determine solutions for our given problem. We first explain the trading of stocks under the American perpetual put option. After that we briefly refer to the definition of stochastic processes and stochastic differential equations (SDEs) as a basis for the introduction of optimal stopping problems. We will close this chapter by presenting the two approaches we will use.

2.1 American Put Option

The American put option describes the situation of trading a stock. The buyer of a put option purchases the right to sell a stock for a defined value K. In order to gain a high asset the owner of the put option goes for a decreasing price x per stock because in this case he can gain a higher amount of money. If the price increases he is allowed not to make use of the put option. The profit for the asset of a put option is

$$h(x) = (K - x)^+ = \begin{cases} K - x & \text{for } x < K \\ 0 & \text{for } x \geqslant K \ . \end{cases} \tag{2.1}$$

Real-life put options expire after a defined time. The perpetual put option instead does not expire. It can be held until it is executed at some future time.

2.2 Stochastic Processes

Stochastic Processes are used in different contexts to define mathematical models for processes or applications with random behaviour. For example, some important fields of application are biology and financial markets. The different types of stochastic processes and important examples will be described before we give a short introduction to stochastic differential equations.

2.2.1 General Information and Important Types

Stochastic Processes can be classified into discrete-time and time-continuous (t-continuous) processes. Discrete-time processes are all processes which have a countable set of indices. In contrast t-continuous processes have uncountably many indices. In the case of stock trading we will use t-continuous processes, which are based on the process of Brownian motion. The Brownian motion W_t (also called Wiener Process) has three important properties:

1. The increments of disjoint time intervals are independent, which means that $W_{t_2} - W_{t_1}$ and $W_{t_3} - W_{t_4}$ are independent if $[t_1, t_2]$ and $[t_3, t_4]$ are disjoint.
2. For all $0 \leqslant s < t$, the increment $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$.
3. $W(0) = 0$ and the sample paths $t \to W_t$ is continuous for almost all $\omega \in \Omega$.

The Wiener Process is also part of SDEs which we will talk about in the following section.

2.2.2 Stochastic Differential Equation

Applications of differential equations assume that the described process does not include any randomness, which is also called "noise". stochastic differential equations instead not only consist of usual differential equations, but also include a term for the possible noise. We can compare normal differential equations and SDEs by examining the defined formulas

$$\frac{dX_t}{dt} = b(t, X_t) \qquad \text{for differential equations,} \tag{2.2}$$

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)W_t \qquad \text{for SDEs.} \tag{2.3}$$

The term W_t describes the noise or randomness in our process and is the derivative of a Brownian Motion. SDEs are solved by using Itô's formula so that (2.3) can be rewritten and solved as

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.4}$$

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s. \tag{2.5}$$

Just as for normal differential equations it is possible that a solution X_t does not exist or that several solutions exist instead of only one. In the case of solving SDEs an existence and uniqueness theorem exists which will briefly summarize the definition found in [3]:

Theorem 2.1. If for $T > 0$ the measurable functions $b(\cdot,\cdot) : [0,T] \times \mathcal{R}^n \to \mathcal{R}^n$ and $\sigma(\cdot,\cdot) : [0,T] \times \mathcal{R}^n \to \mathcal{R}^{n\times m}$ satisfy

$$|b(t,x)| + |\sigma(t,x)| \leqslant C(1 + |x|) \tag{2.6}$$

and

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leqslant D|x - y| \tag{2.7}$$

for some constants C, D and $x, y \in \mathcal{R}^n$, $t \in [0,T]$

and if there a random variable Z exists that is independent of the σ-Algebra generated by $W_s(\cdot)$ and for that

$$E[|Z|^2] < \infty \tag{2.8}$$

holds, then a unique solution X_t for SDE (2.4) exists and X_t has the property $E[\int_0^T |X_t|^2 dt] < \infty$ for any $T \geqslant 0$.

The inequality (2.6) guarantees that our solution is non-exploding almost surely. The Lipschitz condition (2.7) gives us the uniqueness of our solution.

The processes of geometric Brownian motion and the mean-reverting process, which we will use are defined by SDEs. For each of these SDEs an unique solution exists.

2.3 Optimal Stopping Problem

The optimal stopping problem describes the situation of maximizing the expected discount reward. For a reward function $h(X_t)$ and an underlying stochastic process X_t is the optimal stopping problem in an infinite time horizon defined as

$$V(x) = \sup_{\tau \geqslant 0} E_x[e^{-\rho\tau} h(X_\tau)], \tag{2.9}$$

where $\rho > 0$ is the discount factor and $\tau \geqslant 0$ is the stopping time. The goal is to find an optimal stopping rule τ_* such that

$$V(x) = \sup_{\tau \geqslant 0} E_x[e^{-\rho\tau} h(X_\tau)] = E_x[e^{-\rho\tau_*} h(X_{\tau_*})]. \tag{2.10}$$

By applying such an optimal stopping rule we can define stopping and continuation regions. In case of stock trading a stock will not be sold as long as the value of X_t is in a continuation region. If the value leaves this region the stock will be traded because it reaches a stopping region.

We will apply two approaches to solve the optimal stopping problem for the American perpetual put option. The two approaches are briefly explained in the following subsections.

2.3.1 Approach 1: Solving QVI

The first approach we use is based on [5], solving the quasi-variational inequality (QVI)

$$\min\{(\rho - L)V(x), V(x) - h(x)\} = 0 \tag{2.11}$$

where L is the infinitesimal generator for the process X. For example, when X is a geometric Brownian motion or X is a mean-reverting process

$$Lf(x) = \begin{cases} \theta(\mu - x)f'(x) + \frac{1}{2}\sigma^2 f''(x) & \textit{for the MRP} \\ \mu x f'(x) + \frac{1}{2}\sigma^2 x^2 f''(x) & \textit{for the GBM} \end{cases} \qquad f \in C^2. \tag{2.12}$$

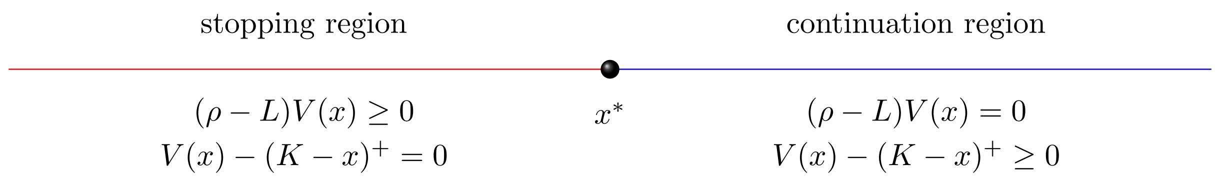

Figure 2.1: Numberline of stopping and continuation region

We apply this approach to the perpetual put option such that by the QVI (2.11) V should satisfy

$$(\rho - L)V(x) \geqslant 0 \ , \ V(x) \geqslant h(x) \qquad \forall x \tag{2.13}$$

$$\begin{cases} V(x) = h(x) & \textit{for } x < x_* \\ (\rho - L)V(x) = 0 & \textit{for } x > x_* \ . \end{cases} \tag{2.14}$$

This approach for the American perpetual put Option is applied to the process of GBM and the MRP. For each of these process types we first determine that a candidate solution $v(x)$ exists. Afterwards we verify that $v(x)$ is a solution by proving

$$\begin{cases} V(x) - h(x) \geqslant 0 & \textit{for } x > x_* \\ (\rho - L)V(x) \geqslant 0 & \textit{for } x < x_* \ . \end{cases} \tag{2.15}$$

If $v(x)$ satisifies (2.14) and (2.15) it is implied that $V(x) = v(x)$ is a solution to the QVI. The regions where the equalities and inequalities hold are shown at a numberline in Fig. 2.1.

2.3.2 Approach 2: Solving 2-Point Stopping Problem

The second approach we use is based on [2] in which $V(x)$ and the optimal stopping time for a mean-reverting process under the American call option are determined by solving a 2-Point stopping problem. For this approach the equation

$$(\rho - L)V(x) = 0 \tag{2.16}$$

with L from (2.12) has two fundamental solutions ϕ_1 and ϕ_2. It is well-known that ϕ_1 is increasing, ϕ_2 is decreasing and both are convex. ϕ_1 and ϕ_2 will be used to define

$$\Psi(x) = \frac{\phi_1}{\phi_2}(x). \tag{2.17}$$

It is evident that Ψ is strictly increasing and thus Ψ^{-1} exists, so

$$H(y) = \frac{h}{\phi_2}(\Psi^{-1}(y)) \tag{2.18}$$

can be defined. Evaluating the expected discounted reward we get according to [2]

$$\begin{aligned}
E_x[e^{-\rho(\tau_a \wedge \tau_b}h(X_{\tau_a \wedge \tau_b})] &= h(a)E_x[e^{-\rho\tau_a}\mathbb{1}_{\tau_a < \tau_b}] + h(b)E_x[e^{-\rho\tau_b}\mathbb{1}_{\tau_a > \tau_b}] \\
&= h(a)\frac{\phi_1(x)\phi_2(b) - \phi_1(b)\phi_2(x)}{\phi_1(a)\phi_2(b) - \phi_1(b)\phi_2(a)} + h(b)\frac{\phi_1(a)\phi_2(x) - \phi_1(x)\phi_2(a)}{\phi_1(a)\phi_2(b) - \phi_1(b)\phi_2(a)} \\
&= \phi_2(x)[\frac{h(a)}{\phi_2(a)}\frac{\Psi(b) - \Psi(x)}{\Psi(b) - \Psi(a)} + \frac{h(b)}{\phi_2(b)}\frac{\Psi(x) - \Psi(a)}{\Psi(b) - \Psi(a)}] \\
&= \phi_2(\Psi^{-1}(y))[H(y_a)\frac{y_b - y}{y_b - y_a} + H(y_b)\frac{y - y_a}{y_b - y_a}],
\end{aligned} \tag{2.19}$$

where $y = \Psi(x)$, $y_a = \Psi(a)$ and $y_b = \Psi(b)$.

Furthermore, we know by [2] that $V(x) = \sup_{a,b:a\leqslant x\leqslant b} E_x[e^{-\rho(\tau_a \wedge \tau_b}h(X_{\tau_a \wedge \tau_b})]$, so we define

$$W(y) = \sup_{y_a,y_b:y_a\leqslant y\leqslant y_b}[H(y_a)\frac{y_b - y}{y_b - y_a} + H(y_b)\frac{y - y_a}{y_b - y_a}]. \tag{2.20}$$

Then it follows that

$$V(x) = \phi_2(x)W(y) = \phi_2(x)W(\Psi(x)). \tag{2.21}$$

The idea behind $W(y)$ is to find the least concave majorant for $H(y)$ so that $V(x)$ is defined either by $\phi_2(x)H(\Psi(x))$ at the regions where $H(y)$ is concave or by $\phi_2(x)W(\Psi(x))$ where $H(y)$ is not concave at the complete region (a, b). When we apply the definitions given in this approach to the GBM and the MRP under the perpetual put option we have to determine the functions $\Psi(y)$, $H(y)$ and $W(y)$, where $W(y)$ is determined based on the behaviour of $H(y)$. Determining $W(y)$ includes the proof that $W(y)$ is the least concave majorant for $H(y)$. Only in this case does $V(x) = \phi_2(x)W(\Psi(x))$ hold.

Chapter 3

Geometric Brownian Motion

The process of geometric Brownian motion is defined by the SDE

$$dX_t = \mu X_t dt + \sigma X_t dW_t,\ X_0 = x > 0, \tag{3.1}$$

where μ and σ are constants and W_t is a Wiener process. Applying Ito's formula to $\ln(X_t)$ we get

$$X_t = xe^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{3.2}$$

We assume throughout the chapter that under some risk neutral probability measure $\mathbb{P}$, $e^{-\rho t}X_t$ is a martingale. In other words, we assume that the appreciation rate of the underlying risky asset under the measure $\mathbb{P}$ is $\mu = \rho$.

The values of a geometric Brownian motion are always greater than 0 and the expectation value is $E(X_t) = xe^{\mu t}$. In case of our optimal stopping problem under the perpetual put option we can use this information to determine a boundary condition for our value function in the following way:

$$V(x) = \sup_{\tau \geqslant 0} E[e^{-\rho\tau}(K - X_\tau)^+] \geqslant (K - x)^+ \ , \tag{3.3}$$

$$e^{-\rho\tau}(K - X_\tau)^+ \leqslant 1(K - 0)^+ = K. \tag{3.4}$$

In consideration of (2.15) we know that $0 \leqslant V(x) \leqslant K$ are our boundary conditions for $V(x)$. These conditions guarantee that $V(x)$ does not explode.

3.1 Approach 1

Applying the first approach to the process of geometric Brownian motion as defined in (3.2), $v(x)$ will be calculated in the following way. We first solve

$$(\rho - L)v(x) = \rho v - \rho x v' - \frac{1}{2}\sigma^2 x^2 v'' = 0. \tag{3.5}$$

Recall that we are considering the risk neutral pricing of the American perpetual put option we consequently assume $\mu = \rho$. Furthermore, we guess that $v(x)$ has the form $v(x) = x^p$. Applying this to equation (3.5) we solve the equation

$$\frac{1}{2}\sigma^2 p^2 + (\rho - \frac{1}{2}\sigma^2)p - \rho = 0, \tag{3.6}$$

and we get the results

$$p_1 = -\frac{2\rho}{\sigma^2},\ p_2 = 1. \tag{3.7}$$

So our general solution is $v(x) = A_1 x^{p_1} + A_2 x^{p_2} = A_1 x^{-\frac{2\rho}{\sigma^2}} + A_2 x$. For $x \to \infty$ we know that $V(x)$ should be bounded but $\lim_{x\to\infty} x = \infty$. According to this we conclude that $A_2 = 0$ and so

$$v(x) = \begin{cases} A_1 x^{-\frac{2\rho}{\sigma^2}} & \textit{for } x > x_* \\ (K-x)^+ & \textit{for } x < x_* \ . \end{cases} \tag{3.8}$$

Checking the two possible cases for $(K-x)^+$

$$(K-x)^+ = \begin{cases} 0 & \textit{for } x \geqslant K \\ (K-x) & \textit{for } x < K \end{cases}$$

under the condition that $v(x_*) > 0$ we can conclude that $x_* < K$. So we can rewrite (3.8) for $x < x_*$ as

$$v(x) = (K-x)\ .$$

Determination of constants We next use smooth pasting to determine the constants A_1 and x_*. By using the continuity in x_* of our function and its first derivative we get the following equations:

$$A_1 {x_*}^{-\frac{2\rho}{\sigma^2}} = (K - x_*), x = x_* \tag{3.9}$$

$$v'(x) = A_1(-\frac{2\rho}{\sigma^2}){x_*}^{-\frac{2\rho}{\sigma^2}-1} = -1 \tag{3.10}$$

$$A_1 = \frac{K - x_*}{{x_*}^{-\frac{2\rho}{\sigma^2}}} \tag{3.11}$$

(3.11) in (3.10)

$$\frac{K - x_*}{{x_*}^{-\frac{2\rho}{\sigma^2}}}(-\frac{2\rho}{\sigma^2}){x_*}^{-\frac{2\rho}{\sigma^2}-1} = -1$$

$$\frac{K}{x_*}(-\frac{2\rho}{\sigma^2}) + \frac{2\rho}{\sigma^2} = -1$$

$$x_* = \frac{K2\rho}{2\rho + \sigma^2} < K\ . \tag{3.12}$$

If we plug this back into (3.11) we get

$$A_1 = \frac{K - x_*}{{x_*}^{-\frac{2\rho}{\sigma^2}}} = \sigma^2(2\rho)^{\frac{2\rho}{\sigma^2}}(\frac{K}{\sigma^2 + 2\rho})^{\frac{\sigma^2+2\rho}{\sigma^2}} \tag{3.13}$$

Combining our calculated values for A_1 and x_* with the information that $x_* < K$, we rewrite (3.8) as

$$v(x) = \begin{cases} \sigma^2(2\rho)^{\frac{2\rho}{\sigma^2}}(\frac{K}{\sigma^2+2\rho})^{\frac{\sigma^2+2\rho}{\sigma^2}} x^{-\frac{2\rho}{\sigma^2}} & \textit{for } x > x_* \\ (K - x) & \textit{for } x < x_*\ . \end{cases} \tag{3.14}$$

So we have found a candidate solution for our QVI.

Check of Inequalities Next we verify that the function $v(x)$ given in (3.14) also satisfies (2.15). Consequently $v \in C^1(0,\infty) \cap C^2((0,\infty)\backslash\{x_*\})$ is a solution to the quasi-variational inequality.

First, we verify $v(x) \geqslant (K - x)$ for $x > x_*$:

To this end, consider the function

$$
\begin{aligned}
f(x) &= v(x) - (K - x) \\
&= \sigma^2 (2\rho)^{\frac{2\rho}{\sigma^2}} (\frac{K}{\sigma^2 + 2\rho})^{\frac{\sigma^2+2\rho}{\sigma^2}} x^{-\frac{2\rho}{\sigma^2}} - (K - x) \\
f'(x) &= -(2\rho)^{\frac{2\rho+\sigma^2}{\sigma^2}} (\frac{K}{\sigma^2 + 2\rho})^{\frac{\sigma^2+2\rho}{\sigma^2}} x^{-\frac{2\rho}{\sigma^2}-1} + 1 \\
&\geqslant -(2\rho)^{\frac{2\rho+\sigma^2}{\sigma^2}} (\frac{K}{\sigma^2 + 2\rho})^{\frac{\sigma^2+2\rho}{\sigma^2}} x_*^{-\frac{(2\rho+\sigma^2)}{\sigma^2}} + 1 \\
&= -(2\rho)^{\frac{2\rho+\sigma^2}{\sigma^2}} (\frac{K}{\sigma^2 + 2\rho})^{\frac{\sigma^2+2\rho}{\sigma^2}} \frac{K2\rho}{2\rho + \sigma^2}^{-\frac{(2\rho+\sigma^2)}{\sigma^2}} + 1 \\
&= 0 \ .
\end{aligned}
$$

Since $f(x_*) = 0$, we have $f(x) \geqslant 0$ for all $x \geqslant x_*$. Consequently, $v(x) \geqslant (K - x)$ for all $x \geqslant x_*$, which we wanted to show.

Secondly, we verify $(\rho - L)v(x) \geqslant 0$ for $x < x_*$:

For that we use

$$v(x) = (K - x),\ v'(x) = -1,\ v''(x) = 0$$

for all $x < x_*$ to determine

$$
\begin{aligned}
(\rho - L)v(x) =& \rho v(x) - \rho x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \\
=& \rho(K - x) + \rho x = \rho K \geqslant 0 \ .
\end{aligned}
$$

So we have shown that (3.14) is a solution for the QVI.

3.2 Verification Theorem

The next step is to verify that our calculated $v(x)$ is a value function $V(x)$ and that $\tau_* = \inf\{t \geqslant 0 : X_t \leqslant x_*\}$ is an optimal stopping rule, so that $v(x) = V(x) = E[e^{-\rho\tau_*}(K - X_{\tau_*}^+]$.

First, we show that $v(x) \geqslant E[e^{-\rho\tau}h(X_\tau)]$, $\forall\tau$ (stopping times):

Since $v(x)$ is smooth we can apply Itô's formula

$$e^{-\rho(t\wedge\tau)}v(X_{t\wedge\tau}) = v(x) + \int_0^{t\wedge\tau} e^{-\rho s}(L-\rho)v(X_s)ds + \int_0^{t\wedge\tau} e^{-\rho s}v'(X_s)\sigma X_s dW_s$$

using the inequalities

$$v(x) \geqslant h(x),\ (\rho - L)v(x) \geqslant 0 \ \text{ and so } (L-\rho)v(x) \leqslant 0$$

we can determine

$$\begin{aligned} E[e^{-\rho(t\wedge\tau)}v(X_{t\wedge\tau})] &= v(x) + E[\int_0^{t\wedge\tau} e^{-\rho s}(L-\rho)v(X_s)ds] \\ &\leqslant v(x) + E[\int_0^{t\wedge\tau} e^{-\rho s}0ds] \leqslant v(x) \end{aligned}$$

and

$$v(x) \geqslant E[e^{-\rho(t\wedge\tau)}v(X_{t\wedge\tau})] \geqslant E[e^{-\rho(t\wedge\tau)}h(X_{t\wedge\tau})]\ .$$

The above inequality is true for any stopping time τ. Note also $h(x) = (K-x)^+ \geqslant 0$. Hence we can apply Fatou's Lemma to obtain

$$E[e^{-\rho(\tau)}h(X_\tau)] \leqslant \liminf_{t\to\infty} E[e^{-\rho(t\wedge\tau)}h(X_{t\wedge\tau})] \leqslant v(x) \tag{3.15}$$

which is what we wanted to show. Now taking sup over all stopping times $\tau \geqslant 0$ yields $v(x) \geqslant \sup_{\tau\geqslant 0} E[e^{-\rho(\tau)}h(X_\tau)]$.

Next we have to show that $v(x) = V(x)$ for our stopping rule τ_*. For this we use the information that in our stopping region $v(x) = h(x)$, so that

$$v(x) = E[e^{-\rho(\tau_*)}h(X_{\tau_*})] \leqslant \sup_{\tau\geqslant 0} E[e^{-\rho(\tau)}h(X_\tau)] = V(x)\ . \tag{3.16}$$

The inequalities (3.15) and (3.16) together reveal that $v(x) = V(x)$.

So we have proven that our candidate solution $v(x)$ is really the value function $V(x)$ we wanted to determine and that τ_* is an optimal stopping rule.

3.3 Approach 2

Applying the second approach to the process of geometric Brownian motion as defined in (3.2) $V(x)$ will be calculated. First we solve the equation (3.5) just as in chapter (3.1) which gives us

$$\phi_1(x) = x \ , \ \phi_2(x) = x^{-\frac{2\rho}{\sigma^2}} \ .$$

Using the second approach now we get

$$\Psi(x) = \frac{\phi_1}{\phi_2}(x) = x^{\frac{\sigma^2+2r}{\sigma^2}} \ , \tag{3.17}$$

$$\Psi^{-1}(y) = y^{\frac{\sigma^2}{\sigma^2+2\rho}} \ , \tag{3.18}$$

$$h(x) = (K-x)^+ \ . \tag{3.19}$$

Referring to (2.18) and the definition of $h(x)$ we can determine an interval $(0, y_0)$ in which $H(y) > 0$, for $y \notin (0, y_0)$ is $H(y) = 0$. Solving the equation $H(y) = 0$ gives us

$$y_0 = K^{\frac{\sigma^2+2\rho}{\sigma^2}} \tag{3.20}$$

and in consequence

$$H(y) = \begin{cases} (K - y^{\frac{\sigma^2}{\sigma^2+2\rho}})y^{\frac{2\rho}{\sigma^2+2\rho}} & \textit{for } y \in (0, K^{\frac{\sigma^2+2\rho}{\sigma^2}}) \\ 0 & \textit{for } y \geqslant K^{\frac{\sigma^2+2\rho}{\sigma^2}} \ . \end{cases}$$

Analysis of H(y) We assume that an extremum for $H(y)$ on the interval $(0, y_0)$ exists. So solving

$$H'(y) = -\frac{\sigma^2}{\sigma^2+2\rho} + \frac{2\rho}{\sigma^2+2\rho}(K - y^{\frac{\sigma^2}{\sigma^2+2\rho}})y^{-\frac{\sigma^2}{\sigma^2+2\rho}} = 0 \tag{3.21}$$

we get

$$y_* = \frac{2\rho K}{\sigma^2 + 2\rho}^{\frac{2\rho+\sigma^2}{\sigma^2}} \tag{3.22}$$

as a possible extremum.

Next, we claim that $H(y)$ is concave for $y \in (0, y_0)$, which will be shown in two steps. First, we check the behaviour of $H''(y)$ for $y \in (0, y_0)$:

$$H'(y) = -\frac{\sigma^2}{\sigma^2 + 2\rho} + \frac{2\rho}{\sigma^2 + 2\rho}(K - y^{\frac{\sigma^2}{\sigma^2+2\rho}})y^{-\frac{\sigma^2}{\sigma^2+2\rho}} \ , \tag{3.23}$$

$$H''(y) = -\frac{2\rho\sigma^2}{\sigma^2 + 2\rho}\frac{1}{y}Ky^{-\frac{\sigma^2}{\sigma^2+2\rho}} \ . \tag{3.24}$$

Because the constants and y are positive we conclude

$$H''(y) < 0 \qquad \textit{for } y \in (0, y_0)$$

thus $H(y)$ is concave on the interval $(0, y_0)$. For $y > y_0$ we can neither testify that $H(y)$ is concave nor that the function is convex, because $H(y)$ is constant on this interval.

Based on the information that $H(y)$ is concave and $H(y) > 0$ on the interval $(0, y_0)$ we conclude that y_* is the maximum of $H(y)$. Combined with the concavity property we know that

$$H(y) = \begin{cases} \textit{increases} & \textit{for } y \in (0, y_*) \\ \textit{decreases} & \textit{for } y > y_* \ . \end{cases} \tag{3.25}$$

Definition of W(y) Similarly to (2.20) we determine $W(y)$ based on our $H(y)$.

Proposition 3.1. The function $W(y)$ defined by

$$W(y) = \begin{cases} H(y_*) & \textit{for } y \geqslant y_* \\ H(y) & \textit{for } y \in (0, y_*) \end{cases} \tag{3.26}$$

is the least concave majorant of $H(y)$.

Proof. This claim is proven first for $y \in (0, y_*)$. Therefore, the solution $W(y) = H(y)$ on

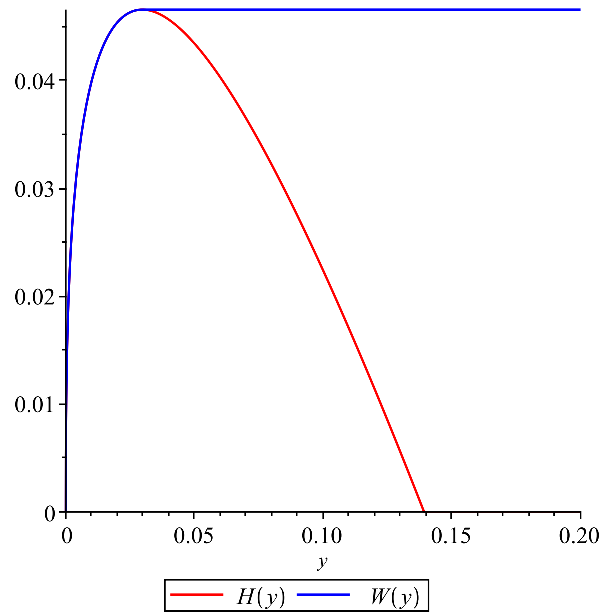

Figure 3.1: $H(y)$ and $W(y)$ for a GBM with $\mu = 0.10$, $\sigma = 0.56$, $\rho = 0.10$, $K = 0.3$

this interval is obvious by the argument that $H(y)$ is increasing and concave on $(0, y_*)$. So the tangent at every $y \in (0, y_*)$ will be the least concave majorant at this point.

For the region $y > y_*$ we proof by contradiction. We suppose $\exists \varphi$ concave such that

1. $\varphi(y) \geqslant H(y)\ \forall y \in (0, \infty)$

2. $\exists y_0 > y_*$ such that $W(y_0) > \varphi(y_0) \geqslant H(y_0)$.

Since φ is concave, we can find a dominating line, passing through $(y_0, \varphi(y_0))$ i.e. $\exists m \in (R)$ such that

$$H(y) \leqslant \varphi(y) \leqslant \varphi(y_0) + m(y - y_0) \qquad \forall y \in [0, \infty). \tag{3.27}$$

Note that $H(y_*) = W(y_0) > \varphi(y_0)$. Considering the case where $m \geqslant 0$ and plugging $y = y_*$ into (4.33) we get

$$\varphi(y_0) < H(y_*) \leqslant \varphi(y_*) \leqslant \varphi(y_0) + m(y_* - y_0) \leqslant \varphi(y_0)$$

which is a contradiction.

For $m < 0$ and if we plug in $y = y_*$, we get

$$\varphi(y_0) < H(y_*) \leqslant \varphi(y_*) \leqslant \varphi(y_0) + m(y_* - y_0).$$

When $y_0 \to \infty$ on the RHS $\varphi(y_0) + m(y_* - y_0) \to -\infty$ but on the LHS $H(y_0) \to 0$ which is also a contradiction. So we have proven that no smaller concave majorant φ of $H(y)$ exists. Consequently, $W(y)$ is the least concave majorant of $H(y)$. An example for $H(y)$ and $W(y)$ is shown in Fig. 3.1. □

After we have shown that (3.26) is the least concave majorant we then get the value function

$$V(x) = \phi_2(x)W(\Psi(x)) = \begin{cases} \sigma^2(2\rho)^{\frac{2\rho}{\sigma^2}}(\frac{K}{\sigma^2+2\rho})^{\frac{\sigma^2+2\rho}{\sigma^2}} x^{-\frac{2\rho}{\sigma^2}} & \textit{for } x > x_* \\ (K - x) & \textit{for } x < x_* \end{cases} \tag{3.28}$$

which is identical to our solution (3.14) we obtained using the first approach.

Chapter 4

Mean-reverting Process

The mean-reverting process is defined by the SDE

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t,\ X_0 = x \in \mathbb{R} \tag{4.1}$$

where $\theta > 0, \mu$ and $\sigma > 0$ are constants and W_t is a Wiener process. Applying Itô's formula to the SDE (4.1) we get

$$X_t = xe^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma e^{-\theta t}\int_0^t e^{\theta s} dW_s\ . \tag{4.2}$$

The MRP oscillates around its long-term mean value μ. Also observe that the values of X_t can be negative. Because we want to solve the optimal stopping problem for this process type under the perpetual put option we can determine a boundary condition for our value function in the following way:

$$V(x) = \sup_{\tau \geqslant 0} E[e^{-\rho\tau}(K - X_\tau)^+] = E[e^{-\rho\tau_*}(K - X_{\tau_*})^+]\ . \tag{4.3}$$

So we obviously have the lower bound $V(x) \geqslant (K - x)^+, \forall x \in \mathbb{R}$.

The upper bound is more difficult to compute: We have

$$E[X_t] = xe^{-\theta t} + \mu(1 - e^{-\theta t}) \tag{4.4}$$

and

$$\begin{aligned} Var(X_t) &= Var[e^{-\theta t}\int_0^t \sigma e^{\theta s} dW_s] \\ &= e^{-2\theta t} E[|\int_0^t \sigma e^{\theta s} dW_s|^2] \\ &= e^{-2\theta t} E[\int_0^t (\sigma e^{\theta s})^2 ds] = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\ . \end{aligned} \tag{4.5}$$

Then

$$
\begin{aligned}
E[X_t^2] &= (\mu + (x-\mu)e^{-2\theta t})^2 + \frac{\sigma^2}{2\theta}(1-e^{-2\theta t}) \\
&= (\mu(1-e^{-\theta t}) + xe^{-\theta t})^2 + \frac{\sigma^2}{2\theta}(1-e^{-2\theta t}) \\
&\leqslant (\mu(1-e^{-\theta t}) + xe^{-\theta t})^2 + \frac{\sigma^2}{2\theta} \ . \qquad (4.6)
\end{aligned}
$$

This in combination with the Cauchy-Schwarz Inequality gives

$$
\begin{aligned}
E[|X_t|] &\leqslant (E[|X_t|^2])^{\frac{1}{2}} \leqslant [(\mu(1-e^{-\theta t}) + xe^{-\theta t})^2 + \frac{\sigma^2}{2\theta}]^{\frac{1}{2}} \\
&\leqslant [(|\mu| + |x|e^{-\theta t})^2 + \frac{\sigma^2}{2\theta}]^{\frac{1}{2}} \leqslant e^{-\theta t}|x| + |\mu| + \frac{\sigma}{\sqrt{2\theta}} \\
&\leqslant |\mu| + \frac{\sigma}{\sqrt{2\theta}} + |x| \ . \qquad (4.7)
\end{aligned}
$$

So we know that $E[(K-X_t)^+] \leqslant |\mu| + \frac{\sigma}{\sqrt{2\theta}} + |x|$. Based on this we know that

$$
(K-x)^+ \leqslant V(x) \leqslant |\mu| + \frac{\sigma}{\sqrt{2\theta}} + |x| \qquad (4.8)
$$

are our boundary conditions for $V(x)$. These conditions guarantee that $V(x)$ does not explode.

4.1 Approach 1

Applying the first approach to the mean-reverting process as defined in (4.2) our $V(x)$ will be calculated based on

$$
(\rho - L)v(x) = \rho v - \theta(\mu - x)v' + \frac{1}{2}\sigma^2 x^2 v'' = 0 \ . \qquad (4.9)
$$

In view of [5], a general solution to (4.9) is given by $A_1\phi_1(x) + A_2\phi_2(x)$, where ϕ_1 and ϕ_2 are given by

$$
\phi_1(x) = \int_0^\infty (t^{\frac{\rho}{\theta}-1} e^{\frac{-t^2}{2}} e^{-\kappa(\mu-x)t})dt \ , \qquad (4.10)
$$

$$
\phi_2(x) = \int_0^\infty (t^{\frac{\rho}{\theta}-1} e^{\frac{-t^2}{2}} e^{\kappa(\mu-x)t})dt \ , \qquad (4.11)
$$

and

$$\kappa = \frac{\sqrt{\theta}}{\sigma},\ \rho > 0,\ \theta > 0,\ \sigma > 0\ . \tag{4.12}$$

It is well-known that ϕ_1 is strictly increasing, ϕ_2 is strictly decreasing, both are positive and convex. Therefore, a candidate solution to the QVI can be written as:

$$v(x) = \begin{cases} (K-x)^+ & \textit{for } x < x_* \\ \\ A_1\phi_1(x) + A_2\phi_2(x) & \textit{for } x > x_*\ . \end{cases}$$

Behaviour of ϕ_1 and ϕ_2 Because $V(x)$ should be bounded for $x \to \infty$ we have to examine the behaviour of ϕ_1 and ϕ_2. In the case of ϕ_1 when $x > 0$ we compute

$$\begin{aligned} \phi_1(x) &= \int_0^\infty (t^{\frac{\rho}{\theta}-1} e^{\frac{-t^2}{2}} e^{-\kappa(\mu-x)t}) dt \\ &\geqslant \int_1^\infty (t^{\frac{\rho}{\theta}-1} e^{\frac{-t^2}{2}-\kappa\mu t} e^{\kappa x t}) dt \\ &\geqslant \int_1^\infty (t^{\frac{\rho}{\theta}-1} e^{\frac{-t^2}{2}-\kappa\mu t} e^{\kappa x}) dt \\ &= e^{\kappa x} \int_1^\infty (t^{\frac{\rho}{\theta}-1} e^{\frac{-t^2}{2}-\kappa\mu t}) dt > 0\ . \end{aligned} \tag{4.13}$$

Then we observe the behaviour of ϕ_1 in relation to x by using the minorant (4.13)

$$\lim_{x\to\infty} \frac{\phi_1(x)}{x} \geqslant \lim_{x\to\infty} \frac{e^{\kappa x}}{x} = +\infty\ . \tag{4.14}$$

For the analysis of ϕ_2 we compute

$$\phi_2'(x) = -\kappa \int_0^\infty (t^{\frac{\rho}{\theta}} e^{\frac{-t^2}{2}} e^{\kappa(\mu-x)t}) dt < 0\ , \tag{4.15}$$

so that

$$\phi_2(x) < \phi_2(1) \qquad \text{if } x \geqslant 1 \tag{4.16}$$

then

$$\lim_{x\to\infty} \frac{\phi_2(x)}{x} = 0\ . \tag{4.17}$$

Similar analysis reveals that $\lim_{x\to-\infty} \frac{\phi_2(x)}{x} = -\infty$ and that $\phi_2(x)$ is bounded when $x > K$ for $K > -\infty$. Regarding this we know that ϕ_2 is decreasing and bounded whereas ϕ_1 is increasing and unbounded for $x \to \infty$. With respect to this we conclude that $A_1 = 0$, otherwise $v(x)$ isn't bounded. Based on that

$$v(x) = \begin{cases} (K-x)^+ & \textit{for } x < x_* \\ A_2\phi_2(x) & \textit{for } x > x_* \ . \end{cases} \tag{4.18}$$

Now we have to find A_2 and x_* by using the continuity of the function and its first derivative at the point x_* like in Chapter(3.1). So,

$$A_2\phi_2(x_*) = K - x_* \tag{4.19}$$

$$A_2\phi_2'(x_*) = -1$$

$$A_2 = \frac{-1}{\phi_2'(x_*)} > 0 \tag{4.20}$$

(4.20) in (4.19)

$$\frac{\phi_2(x_*)}{\phi_2'(x_*)} = x_* - K \qquad\qquad \phi_2(x_*) - (x_* - K)\phi_2'(x_*) = 0 \ . \tag{4.21}$$

Because of the complexity of solving the equation (4.21) we will only show that a $v(x)$ with constants x_* and A_2 exists.

Existence of v(x) Now we show that (4.21) has a solution. To do this we consider the function

$$G(x) = \phi_2(x) - (x - K)\phi_2'(x) \ . \tag{4.22}$$

That (4.21) has a solution is equivalent to $G(x_*) = 0$. $G(x)$ is continuous and furthermore on $[-\infty, K]$ is $G(K) = \phi_2(K) > 0$ and $\lim_{x\to-\infty} G(x) = -\infty$. Based on the continuity of $G(x)$ and the behaviour of the function at the boundaries of the interval we know by the intermediate value theorem that at least one $x_* < K$ exists such that $G(x_*) = 0$. From this we obtain that (4.21) has a solution x_* and consequently (4.20) determines the value of A_2.

So we have shown that a solution of the QVI exists and the solution has the following form

$$v(x) = \begin{cases} (K-x)^+ & \textit{for } x < x_* \\ \frac{-1}{\phi_2'(x_*)}\phi_2(x) & \textit{for } x > x_* \ . \end{cases} \tag{4.23}$$

Check of Inequalities Next we verify that the function $v(x)$ given in (4.23) also satisfies the inequalities (2.15). Consequently, $v \in C^1(0,\infty) \cap C^2((0,\infty)\backslash\{x_*\})$ is a solution to the quasi-variational inequality.

First, let's verify $V(x) \geqslant (K-x)$ for $x > x_*$:

To this end, consider the function

$$\begin{aligned} f(x) &= v(x) - (K-x) \\ &= A_2\phi_2(x) - (K-x) \ , \\ f'(x) &= A_2\phi_2'(x) + 1 \ , \end{aligned}$$

using (4.20) we get:

$$f'(x) = \frac{-1}{\phi_2'(x_*)}\phi_2'(x) + 1 > \frac{-1}{\phi_2'(x_*)}\phi_2'(x_*) + 1 = -1 + 1 = 0 \ .$$

This inequality holds, because with respect to $\phi_2'' > 0$ we know that ϕ_2 is convex but $\phi_2(x)$ is also bounded for $\lim_{x\to\infty}$. Therefore ϕ_2 and similar to that ϕ_2' decrease for $\lim_{x\to\infty}$. Consequently, is $\phi_2'(x_*) > \phi_2'(x)$ and therefore $\frac{\phi_2'(x)}{\phi_2'(x_*)} < 1$.

Furthermore, since $f(x_*) = 0$, we have $f(x) \geqslant 0$ for all $x \geqslant x_*$. Consequently is $v(x) \geqslant (K-x)$ for all $x \geqslant x_*$, which we wanted to show.

Secondly, we verify $(\rho - L)v(x) \geqslant 0$ for $x < x_*$:

Therefore we will use the condition that at the point x_* our $v(x)$ from the region $x > x_*$ and our $v(x)$ for $x < x_*$ are equal. We use

$$v(x) = (K-x) \ , \ v'(x) = -1 \ , \ v''(x) = 0$$

for all $x < x_*$ to determine

$$\begin{aligned}(\rho - L)v(x) =&\rho v - \theta(\mu - x)v' + \frac{1}{2}\sigma^2 x^2 v'' \\ =&\rho(K - x) + \theta(\mu - x) = \rho K + \theta\mu - x(\theta + \rho) \\ \geqslant&\rho K + \theta\mu - x_*(\theta + \rho) = (\rho - L)v(x_*) + \frac{1}{2}\sigma^2 v''(x_*) \\ =&\frac{1}{2}\sigma^2 v''(x_*) \geqslant 0\ .\end{aligned}$$

So we have shown that (4.23) is a solution for the QVI.

4.2 Verification Theorem

The next step is to verify that our calculated $v(x)$ is a value function $V(x)$ and that $\tau_* = \inf\{t \geqslant 0 : X_t \leqslant x_*\}$ is an optimal stopping rule, so that $v(x) = V(x) = E[e^{-\rho\tau_*}(K - X^+_{\tau_*}]$.

First, we show that $v(x) \geqslant E[e^{-\rho\tau}h(X_\tau)]$, $\forall\tau$ (stopping times):

Since $v(x)$ is smooth we can apply Itô's formula

$$e^{-\rho(t\wedge\tau)}v(X_{t\wedge\tau}) = v(x) + \int_0^{t\wedge\tau} e^{-\rho s}(L - \rho)v(X_s)ds + \int_0^{t\wedge\tau} e^{-\rho s}v'(X_s)\sigma X_s dW_s$$

using the inequalities

$$v(x) \geqslant h(x),\ (\rho - L)v(x) \geqslant 0 \ \text{ and so } (L - \rho)v(x) \leqslant 0$$

we can determine

$$\begin{aligned}E[e^{-\rho(t\wedge\tau)}v(X_{t\wedge\tau})] = v(x) + E[\int_0^{t\wedge\tau} e^{-\rho s}(L - \rho)v(X_s)ds] \\ \leqslant v(x) + E[\int_0^{t\wedge\tau} e^{-\rho s}0ds] \leqslant v(x)\end{aligned}$$

and

$$v(x) \geqslant E[e^{-\rho(t\wedge\tau)}v(X_{t\wedge\tau})] \geqslant E[e^{-\rho(t\wedge\tau)}h(X_{t\wedge\tau})]$$

The above inequality is true for any stopping time τ. Note also $h(x) = (K - x)^+ \geqslant 0$. Hence we can apply Fatou's Lemma to obtain

$$E[e^{-\rho(\tau)}h(X_\tau)] \leqslant \liminf_{t\to\infty} E[e^{-\rho(t\wedge\tau)}h(X_{t\wedge\tau})] \leqslant v(x) \tag{4.24}$$

which is what we wanted to show. Now taking sup over all stopping times $\tau \geqslant 0$ yields $v(x) \geqslant \sup_{\tau\geqslant 0} E[e^{-\rho(\tau)}h(X_\tau)]$.

Next we have to show that $v(x) = V(x)$ for our stopping rule τ_*. For this we use the information that in our stopping region $v(x) = h(x)$, so that

$$v(x) = E[e^{-\rho(\tau_*)}h(X_{\tau_*})] \leqslant \sup_{\tau\geqslant 0} E[e^{-\rho(\tau)}h(X_\tau)] = V(x)\ . \tag{4.25}$$

The inequalities (4.24) and (4.25) together reveal that $v(x) = V(x)$.

So we have proven that our candidate solution $v(x)$ is really the value function $V(x)$ we wanted to determine and that τ_* is an optimal stopping rule.

4.3 Approach 2

Applying the second Approach to the mean-reverting process as defined in (4.2) $V(x)$ will be calculated. First we solve the equation (4.9) just as in chapter (4.1) which gives us

$$\phi_1(x) = \int_0^\infty (t^{\frac{\rho}{\theta}-1}e^{\frac{-t^2}{2}}e^{-\kappa(\mu-x)t})dt\ ,$$
$$\phi_2(x) = \int_0^\infty (t^{\frac{\rho}{\theta}-1}e^{\frac{-t^2}{2}}e^{\kappa(\mu-x)t})dt\ .$$

Using the second approach now we get

$$\Psi(x) = \frac{\phi_1}{\phi_2}(x) = \frac{\int_0^\infty (t^{\frac{\rho}{\theta}-1}e^{\frac{-t^2}{2}}e^{-\kappa(\mu-x)t})dt}{\int_0^\infty (t^{\frac{\rho}{\theta}-1}e^{\frac{-t^2}{2}}e^{\kappa(\mu-x)t})dt}\ . \tag{4.26}$$

According to [2] Ψ is strictly increasing and $\Psi(x) > 0, \forall x \in (R)$. In view of (2.18) and the definition of $h(x)$ we define

$$H(y) = \frac{h}{\phi_2}(\phi_2^{-1}(y)) = \frac{(K - \Psi^{-1}(y))^+}{\phi(\Psi^{-1}(y))} \qquad \textit{for } y > 0\ .$$

From [2] and the behaviour of ϕ_2 we know that

$$H(0) = \lim_{x \to -\infty} \frac{h(x)^+}{\phi_2(x)} = \lim_{x \to -\infty} \frac{K - x}{\phi_2(x)} = 0 \ .$$

So $H(y)$ can be written as

$$H(y) = \begin{cases} \frac{h}{\phi_2}(\Psi^{-1}(y)) & \textit{for } y > 0 \\ 0 & \textit{for } y = 0 \end{cases} \tag{4.27}$$

and is continuous on $[0, \infty)$.

Analysis of H(y) $H(y)$ is concave for $y \in (0, \Psi(K))$, which will be shown by the behaviour of $H''(y)$. For that we determine the derivatives of $H(y)$, using the notation $\Psi^{-1}(y) = x$, so that

$$H'(y) = \frac{h'\phi_2 - \phi_2' h}{\phi_1'\phi_2 - \phi_1\phi_2'} = \frac{-\phi_2(x) - (K - x)\phi_2'(x)}{\phi_1'(x)\phi_2(x) - \phi_1(x)\phi_2'(x)} \ , \tag{4.28}$$

$$H''(y) = \frac{\phi_2^2[(\phi_1'\phi_2 - \phi_1\phi_2')h''\phi_2 + \phi_1''\phi_2(\phi_2' h - h'\phi_2)]}{(\phi_1'\phi_2 - \phi_1\phi_2')^3}(x) \tag{4.29}$$

$$= \frac{\phi_2^3(x)[\phi_1''(x)(-\phi_2(x) - (K - x)^+\phi_2'(x)) + \phi_2''(x)(-\phi_1(x) - (K - x)^+\phi_1'(x))]}{(\phi_1'(x)\phi_2(x) - \phi_1(x)\phi_2'(x))^3} \tag{4.30}$$

From the behaviour of ϕ_1 and ϕ_2 and their derivatives we conclude

$$H''(y) < 0 \qquad \textit{for } y \in (0, \Psi(K)) \ .$$

Thus $H(y)$ is concave on the interval $(0, \Psi(K))$.

For $y > \Psi(K)$ we can neither testify that $H(y)$ is concave nor that the function is convex, because $H(y)$ is constant on this interval.

Based on the information that $H(y)$ is concave and $H(y) > 0$ on the interval $(0, y_0)$ we assume that at y_* is the maximum of $H(y)$. So solving $H'(y) = 0$ we get the equation $-\phi_2(x_*) - (K - x_*)\phi_2'(x_*) = 0$; this is the same equation as (4.21). So we conclude that

$y_* = \Psi(x_*)$ exists. Combined with the concavity property we know that

$$H(y) = \begin{cases} \textit{increases} & \textit{for } y \in (0, y_*) \\ \textit{decreases} & \textit{for } y > y_* \ . \end{cases} \tag{4.31}$$

Definition of W(y) Similarly to (2.20) we determine $W(y)$ based on our $H(y)$.

Proposition 4.1. The function $W(y)$ defined by

$$W(y) = \begin{cases} H(y_*) & \textit{for } y \geqslant y_* \\ H(y) & \textit{for } y \in (0, y_*) \end{cases} \tag{4.32}$$

is the least concave majorant of $H(y)$.

Proof. This claim is proven first for $y \in (0, y_*)$. Therefore, the solution $W(y) = H(y)$ on this interval is obvious by the argument that $H(y)$ is increasing and concave on $(0, y_*)$. So the tangent at every $y \in (0, y_*)$ will be the least concave majorant at this point.

For the region $y > y_*$ we proof by contradiction.

We suppose $\exists \varphi$ concave such that

1. $\varphi(y) \geqslant H(y)\ \forall y \in (0, \infty)$

2. $\exists y_0 > y_*$ such that $W(y_0) > \varphi(y_0) \geqslant H(y_0)$

Since φ is concave, we can find a dominating line, passing through $(y_0, \varphi(y_0))$ i.e. $\exists m \in (R)$ such that

$$H(y) \leqslant \varphi(y) \leqslant \varphi(y_0) + m(y - y_0) \qquad \forall y \in [0, \infty) \ . \tag{4.33}$$

Note that $H(y_*) = W(y_0) > \varphi(y_0)$. Considering the case where $m \geqslant 0$ and plugging $y = y_*$ into (4.33), we get

$$\varphi(y_0) < H(y_*) \leqslant \varphi(y_*) \leqslant \varphi(y_0) + m(y_* - y_0) \leqslant \varphi(y_0)$$

which is a contradiction.

For $m < 0$ and if we plug in $y = y_*$, we get

$$\varphi(y_0) < H(y_*) \leqslant \varphi(y_*) \leqslant \varphi(y_0) + m(y_* - y_0) \ .$$

When $y_0 \to \infty$ on the RHS $\varphi(y_0) + m(y_* - y_0) \to -\infty$ but on the LHS $H(y_0) \to 0$ which is also a contradiction. So we have proven that no smaller concave majorant φ of $H(y)$ exists. Consequently, $W(y)$ is the least concave majorant of $H(y)$. $\square$

After we have shown that (4.32) is the least concave majorant we compute

$$\begin{aligned} \phi_2(\Psi^{-1}(y_*)) + (K - \Psi^{-1}(y_*))\phi_2'(\Psi^{-1}(y_*)) &= 0 \\ \frac{-1}{\phi_2(\Psi^{-1}(y_*))} = \frac{K - \Psi^{-1}(y_*)}{\phi_2(\Psi^{-1}(y_*))} &= H(y_*) \end{aligned}$$

so that we get then the value function

$$V(x) = \phi_2(x)W(\Psi(x)) = \begin{cases} (K-x)^+ & \textit{for } x < x_* \\ \frac{-1}{\psi'(x_*)}\phi_2(x) & \textit{for } x > x_* \end{cases} \tag{4.34}$$

which is identical to our candidate solution (4.23).

Chapter 5

Conclusion

Referring to the results of chapter 3 and chapter 4 we have shown that both approaches solve the optimal stopping problem for the GBM and the MRP under the perpetual put option. For the approach of solving the QVI we were able to verify that our determined solution $v(x)$ is our desired value function $V(x)$ for each process.

Additionally, we were able verify that $\tau_* = \inf\{t \geqslant 0 : X_t \leqslant x_*\}$ is the optimal stopping rule. Based on this we identified the optimal stopping time and the optimal stopping region. Furthermore, we found out that it does not matter if we solve the QVI or the 2-point stopping problem for each of the processes. For both the GBM and the MRP there were no differences between the solutions of the two approaches.

We also gained from our results other potential research ideas and points of interest. One point of interest is to find a more convenient formula to determine the optimal stopping time for the mean-reverting process. We were only able to show that a stopping time exists, but it was not possible to calculate this stopping time without support of computer-based simulations.

A second point of interest is to apply the two approaches to other stochastic processes. Traded stocks can not only be described by the GBM and MRP. It would be interesting to determine the optimal stopping times for other processes that describe a traded stock.

Furthermore, the one-point stopping problem, which we discussed in this thesis, could be changed to a multiple stopping problem. Both approaches were used to determine the multiple stopping times for a MRP under the call option. In case of the perpetual put option we would have to verify if the approaches also hold.

Another idea is to develop approaches for finding the value function of expected discounted reward that is not exponential. For example, in the case of hyperbolic discount ($E[\frac{1}{1+\tau}h(X_\tau)]$) could it be possible that the given two approaches would no longer be applicable.

Those are four possible routes of further research based on the verification that the two approaches can be applied to the geometric Brownian motion and the mean-reverting process under the American perpetual put option.

Bibliography

[1] S. Karlin and H. M. Taylor. *A first course in stochastic processes.* 2nd Edition. Academic Press, 1975.

[2] T. Leung and X. Li. “Optimal Mean Reversion Trading with Transaction Costs and Stop-Loss Exit”. In: *International Journal of Theoretical and Applied Finance* 18, Number 3 (May 2015).

[3] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications.* 5th Edition. Universitext. Springer, 1998.

[4] Z. Schuss. *Theory and Applications of Stochastic Differtential Equations.* 1st Edition. Wiley series in probability and mathematical statistics. John Wiley & Sons, 1980.

[5] Q. Song, Q. Zhang. “An optimal pairs-trading rule”. In: *Automatica* 49 (2013), pp. 3007–3014.

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