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Anderson localization and saturable nonlinearity in one-dimensional disordered lattices

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ABSTRACT

We investigate numerically the propagation and the Anderson localization of plane waves in a one-dimensional lattice chain, where disorder and saturable nonlinearity are simultaneously present. Using a calculation scheme for solving the stationary discrete non-linear Schrödinger equation in the fixed input case, the disorder-averaged logarithmic transmittance and the localization length are calculated in a numerically precise manner. The localization length is found to be a non-monotonic function of the incident wave intensity, acquiring a minimum value at a certain finite intensity, due to saturation effects. For low incident intensities where the saturation effect is ineffective, the enhancement of localization due to Kerr-type nonlinearity occurs in a way similar to the case without saturation. For sufficiently high incident intensities, we find that the localization length is an increasing function of the incident wave intensity, which implies that localization is suppressed for stronger input intensities, and ultimately approaches a saturation value. This feature is associated with the fact that the non-linear system is reduced to an effectively linear one, when either the incident wave intensity or the saturation parameter is sufficiently large. The non-linear saturation effect is found to be stronger and more pronounced when the energy of the incident wave is larger. We also calculate the variance of the inverse localization length and find that it also shows a non-monotonic behaviour.

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Anderson localization; saturable nonlinearity; localization length

1. Introduction

Understanding the influence of nonlinearity on Anderson localization of waves in disordered systems has attracted much interest from researchers over the past years (*1*–*19*). In non-linear wave propagation problems, all aspects of the dynamics are not determined by the spectral properties unlike in linear problems. Due to the peculiarity of non-linear systems, different ways of posing the problem are not equivalent to each other (*20*). As a consequence, there exist many uncertainties and unanswered questions associated with the physical systems in which disorder and nonlinearity are simultaneously present (*21*, *22*). Within this context, a fundamental question is whether the presence of nonlinearity enhances or suppresses Anderson localization. The answer to this question can best be described as inconclusive at this point. The influence of nonlinearity on Anderson localization can be studied in three different contexts: (i) the transmission of plane waves through disordered non-linear media (*1*–*10*), (ii) the effect of non-linear perturbations on localized eigenstates in finite-size systems (*11*–*13*, *18*), (iii) the effect of nonlinearity on the spreading of initially localized wave packets (*5*, *14*–*17*, *19*). The discrete non-linear Schrödinger equation can be used in the study of all these cases (*23*).

Traditionally, much interest has been given to the standard cubic nonlinearity associated with the Kerr effect due to its relevance to a wide range of physical systems (*1*–*19*). In particular, it has been shown that the effect of nonlinearity on Anderson localization is qualitatively different for localized and extended excitations. It favours the propagation of localized excitations, while it inhibits that of extended excitations (*5*). By studying localized eigenstates and wave packet expansion in one-dimensional (1D) disordered lattices, Lahini et al. have observed that in the weakly non-linear regime, nonlinearity enhances localization for flat-phased states and induces delocalization for staggered states (*18*). Other papers have also reported that pure Anderson localization is destroyed and turns into a subdiffusive spreading of wave packets in the presence of nonlinearity (*16*, *17*, *19*).

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However, the cubic nonlinearity does not always reflect the physical reality and, in certain cases, other kinds of nonlinearity should be considered. For example, for short and high peak power pulses, the field-induced change in the refractive index cannot be described by a Kerr-type nonlinearity, since it is influenced by higher-order nonlinearities, such as saturable nonlinearity (*24*). The saturation of the non-linear response has been shown to have novel impacts on the dynamical properties of wave propagation in clean systems (*24*–*35*). For instance, it has been pointed out that the saturable nonlinearity allows the existence of stable two-state solitons with the same time duration (*24*) and of breathers with high power (*25*). In (*26*), genuinely localized travelling waves have been found in saturable non-linear Schrödinger lattices for the first time. Furthermore, in recent papers, the authors have shown that there exists asymmetric wave propagation through saturable non-linear oligomers and wave rectification devices can be produced based on such systems (*34*, *35*).

Until now, many researches have been devoted to soliton propagation in saturable non-linear systems. To the best of our knowledge, there has been very little research on the interplay between saturable nonlinearity and disorder (*36*–*38*). The authors of (*36*) have investigated light localization in disordered photo-refractive lattices with saturable self-defocusing nonlinearity. It has been demonstrated that by increasing the value of the nonlinearity strength continuously, a phase transition from a localized state to an extended state can be obtained. In a very recent paper, the authors have studied numerically the dynamics of an initially localized wave packet in 1D disordered chains with saturable nonlinearity (*38*). From a detailed numerical analysis, they have found that saturable nonlinearity can promote a sub-diffusive spreading of the wave packet even in the presence of diagonal disorder for a long time. In addition, they have also investigated the effect of saturated nonlinearity for initial times of the electronic evolution thus showing the possibility of mobile breather-like modes.

In this paper, we investigate the transmission and localization properties of plane waves in a 1D disordered lattice chain with saturable nonlinearity. Using a disordered version of the stationary discrete non-linear Schrödinger equation, the disorder-averaged logarithmic transmittance and the localization length are calculated in a numerically precise manner. Unlike in many previous works, we strictly fix the intensity of the incident wave and compute physical quantities as a function of other parameters. We find that the localization length and its fluctuation are a non-monotonic function of the incident wave intensity. For low incident intensities, the enhancement of localization due to nonlinearity occurs in a way similar to the case without saturation. For sufficiently high incident intensities, we find that the localization length is an increasing function of the incident wave intensity and ultimately approaches a saturation value. It is found that the non-linear saturation effect is stronger and more pronounced when the energy of the incident wave is larger. We also calculate the variance of the inverse localization length and find that it also shows a non-monotonic behaviour.

2. Theoretical model and numerical method

In order to study the effects of saturable nonlinearity on Anderson localization in a 1D disordered lattice, we use the discrete non-linear Schrödinger equation given by

$$i\hbar\frac{\mathrm{d}C_n(t)}{\mathrm{d}t} = \epsilon_n C_n(t) - V\left[C_{n+1}(t) + C_{n-1}(t)\right] + \frac{\alpha|C_n(t)|^2}{1+\beta|C_n(t)|^2}C_n(t), \tag{1}$$

where $C_n(t)$ is the wave function amplitude and V is the hopping integral between the nearest-neighbour sites. The on-site potential ϵ_n varies randomly as a function of the site index n. We assume that ϵ_n's are independent random variables distributed uniformly in the range $[-W/2, W/2]$. The parameter α is the strength of a third-order Kerr-type non-linear response in the regime of low-intensity waves, while β is the degree of saturation of the nonlinearity. We set $\hbar = V = 1$ for convenience. We note that when the intensity of the incident wave is sufficiently weak so that $|C_n(t)|^2 \ll 1$, the last term of Equation (1) can be approximated by the expression $[\alpha|C_n(t)|^2C_n(t) - \alpha\beta|C_n(t)|^4C_n(t)]$. With this approximation, we recover the standard cubic discrete non-linear Schrödinger equation with an additional quartic term $-\alpha\beta|C_n(t)|^4C_n(t)$.

In a recent paper, the authors have studied the influence of saturable nonlinearity on Anderson localization of an initially localized wave packet by integrating Equation (1) directly (*38*). The main focus in that paper was to study the time-dependent behaviour when a wave packet was locally excited in the middle of a 1D system. It has been widely known that in non-linear disordered systems described by discrete Schrödinger equations, there occurs a sub-diffusive spreading of an initially localized wave packet, without any indication of saturation for long-time runs (*17*). On the contrary, in the present paper, we focus on the transmission problem and study how an incident plane wave is transmitted through a finite segment of a saturable non-linear disordered medium. Such a set-up may be realized in experiments on layered photonic structures, where wave transmission through a non-linear disordered medium can be studied as afunction of

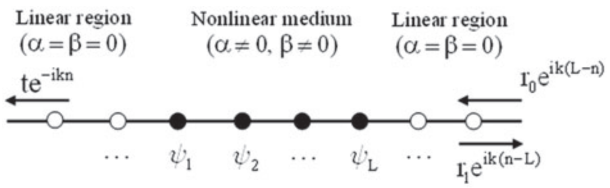

Figure 1. Schematic of the transmission problem. A plane wave is incident on a saturable non-linear disordered medium with $\alpha \neq 0$ and $\beta \neq 0$ from a uniform linear region on the right ($n > L$) and is transmitted to a uniform linear region on the left ($n < 1$). r_0, r_1 and t are the complex amplitudes of the incident, reflected and transmitted waves, respectively.

the intensity of the incident wave. To this end, we will look for stationary solutions to Equation (1) of the form $C_n(t) = \psi_n e^{-iEt}$, where E is the energy of the incident wave. This leads to a set of coupled algebraic equations for ψ_n:

$$E\psi_n = \epsilon_n\psi_n - \psi_{n+1} - \psi_{n-1} + \frac{\alpha|\psi_n|^2}{1+\beta|\psi_n|^2}\psi_n. \quad (2)$$

In order to allow the free propagation of the wave outside the disordered non-linear region, we take ϵ_n, α and β to be non-zero only in the region $1 \leq n \leq L$. We note that the transformation $C_n \to (-1)^n C_n^*$, $\alpha \to -\alpha$ and $\epsilon_n \to -\epsilon_n$ leaves Equation (1) invariant (39). Therefore the sign of α can be fixed to be positive when we consider disorder-averaged quantities.

We assume that a plane wave is incident from the right side (see Figure 1) and define the amplitudes of the incident, reflected and transmitted waves, r_0, r_1 and t, by

$$\psi_n = \begin{cases} r_0 e^{ik(L-n)} + r_1 e^{ik(n-L)}, & n \geq L, \\ te^{-ikn}, & n \leq 0, \end{cases} \quad (3)$$

where the wave number k satisfies $E = -2\cos k$. When there is no dissipation, the conservation law $|r_1|^2 + |t|^2 = |r_0|^2$ is satisfied. In our previous works to which we refer the details, we have presented a method for solving Equations (2) and (3) numerically in the situation where the incident wave intensity, $|r_0|^2$, is fixed (*9*, *10*). We first choose a positive real number for t and solve Equation (2) iteratively until we obtain ψ_L and ψ_{L+1}. The quantities r_0 and r_1 are given by

$$r_0 = \frac{\psi_L e^{ik} - \psi_{L+1}}{e^{ik} - e^{-ik}}, \quad r_1 = \frac{\psi_{L+1} - \psi_L e^{-ik}}{e^{ik} - e^{-ik}}, \quad (4)$$

and the reflectance R and the transmittance T are expressed as

$$R = \left|\frac{r_1}{r_0}\right|^2 = \frac{|\psi_L e^{-ik} - \psi_{L+1}|^2}{|\psi_L e^{ik} - \psi_{L+1}|^2},$$

$$T = \left|\frac{t}{r_0}\right|^2 = |t|^2 \frac{4\sin^2 k}{|\psi_L e^{ik} - \psi_{L+1}|^2}. \quad (5)$$

This calculation is repeated for many different initial values of t ($t = \delta, 2\delta, 3\delta, \ldots$) until we obtain the value of $|r_0|^2$ sufficiently close to the pre-chosen value. The step size δ is chosen properly to achieve desired accuracy. The main quantity we calculate is the localization length ξ defined by

$$\frac{1}{\xi} = -\lim_{L\to\infty} \frac{\langle \ln T\rangle}{L}, \quad (6)$$

where $\langle\cdots\rangle$ represents averaging over a large number of random energy configurations. We will also present some results on the variance of the inverse localization length, σ^2, which is defined by

$$\sigma^2 = \lim_{L\to\infty} \frac{\langle(\ln T)^2\rangle - \langle \ln T\rangle^2}{L^2}. \quad (7)$$

3. Results

In order to calculate the localization length ξ, we need to obtain the disorder-averaged logarithm of the transmittance $\langle \ln T\rangle$. We have computed $\langle \ln T\rangle$ by averaging over 10,000 random configurations of ϵ_n. Calculations were performed for the system size L up to 60. In the linear case, it is necessary to do the calculation for substantially larger values of L in order to extract accurate results for the localization length. In the non-linear case, however, the exponential decay of $\langle T\rangle$ or the linear decay of $\langle \ln T\rangle$ with L is achieved for much smaller values of L. In the present study, we have verified numerically that calculations up to $L = 60$ are sufficient and $\langle \ln T\rangle$ approaches values smaller than -6 at this value of L. Our main aim is to investigate the effects of nonlinearity on localization properties and all of our results were obtained for a fixed disorder strength $W = 2$. The step size for t was $\delta = 10^{-7}$. The error in the calculated value of $|r_0|^2$ was always smaller than 10^{-5}.

In Figure 2, we plot the inverse localization length $1/\xi$ as a function of the incident wave intensity $|r_0|^2$ in the absence of saturation effects ($\beta = 0$). The energy of the incident wave, E, is equal to 0, 0.5 and 1 and the nonlinearity parameter α is fixed to 0.5. We find that the enhancement of localization occurs in the presence of weak nonlinearity for all cases. When $|r_0|^2$ is small, the localization length is a decreasing function of it. For sufficiently large values of $|r_0|^2$, the localization length is found to approach a saturation value. The specific saturation value of ξ depends on the energy E and the disorder strength W. A similar behaviour has also been demonstrated in our previous work, though there is a

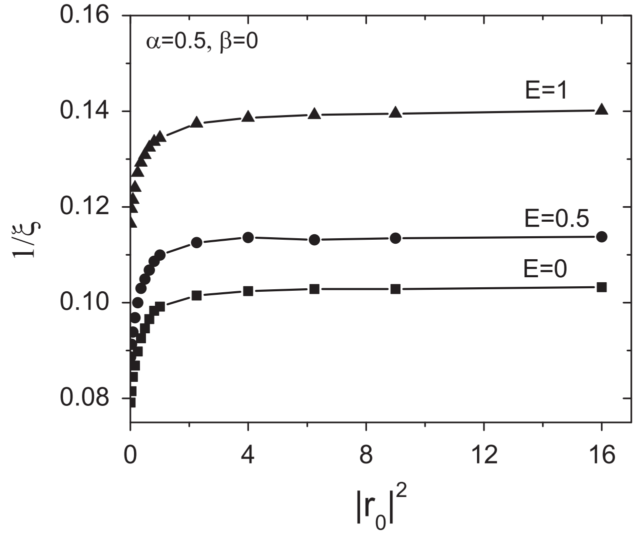

Figure 2. Inverse localization length $1/\xi$ plotted as a function of the incident wave intensity $|r_0|^2$ for different values of the incident wave energy, $E = 0$, 0.5 and 1, in the absence of saturation effects ($\beta = 0$). The value of the nonlinearity parameter α is equal to 0.5 and the disorder strength W is equal to 2. For small values of the incident wave intensity, the localization length is a decreasing function of it. For a sufficiently large $|r_0|^2$, however, the localization length is found to approach a saturation value.

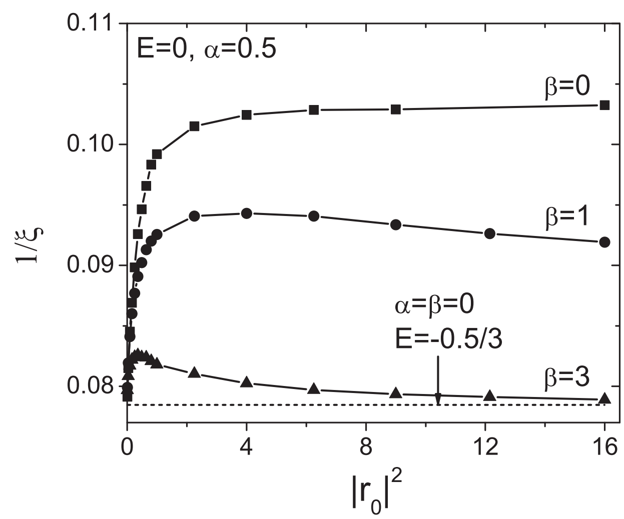

Figure 3. Inverse localization length $1/\xi$ plotted as a function of the incident wave intensity, $|r_0|^2$, when $E = 0$, $\alpha = 0.5$ and $\beta = 0, 1, 3$. When $|r_0|^2$ or β is sufficiently large, an effective linear behaviour is achieved. The curve for $\beta = 3$ is compared with the dashed straight line corresponding to the linear result obtained when $E = -0.5/3$.

subtle difference in the way in which the results are obtained (*9*). In contrast to the present study, in (*9*), the incident wave intensity $|r_0|^2$ was fixed and the localization length was calculated as a function of the nonlinearity parameter α. When β is zero, varying $|r_0|^2$ is theoretically equivalent to varying α. When β is non-zero, however, they are not equivalent.

In Figure 3, we plot the inverse localization length $1/\xi$ as a function of the incident wave intensity $|r_0|^2$, when $E = 0$, $\alpha = 0.5$ and $\beta = 0, 1, 3$. When the saturation of the non-linear response is taken into account ($\beta \neq 0$), the localization behaviour of waves is substantially changed. A *non-monotonic* dependence of the localization length on the incident wave intensity occurs for finite saturation strengths. The inverse localization length takes a maximum value, which is a decreasing function of β, at a certain value of $|r_0|^2$. The reason for this behaviour is fairly simple. For sufficiently low incident intensities where the saturation effect is ineffective, the enhancement of localization due to nonlinearity still occurs in a way similar to the case with $\beta = 0$, and therefore the inverse localization length $1/\xi$ grows rapidly as $|r_0|^2$ increases. For larger values of $|r_0|^2$, however, the saturation effect begins to affect the localization behaviour and the growth of $1/\xi$ is suppressed (more strongly when β is larger). For sufficiently high incident intensities, the inverse localization length is a decreasing function of the incident wave intensity, which implies that localization is suppressed for stronger input intensities, and ultimately approaches a saturation value. This feature is associated with the fact that the non-linear system is reduced to an effectively linear one with $\alpha = \beta = 0$ and the *renormalized energy* $\tilde{E} = E - \alpha/\beta$ due to the saturation of the non-linear response, when either $|r_0|^2$ or β is sufficiently large. We illustrate this by comparing the curve for $\beta = 3$ with the dashed straight line corresponding to the linear result obtained when $E = -0.5/3$ in Figure 3. When the linear value of $1/\xi$ at $\tilde{E}$ is substantially smaller than the non-linear saturation value of $1/\xi$ at E for the Kerr-type non-linear model, the non-monotonic dependence of $1/\xi$ is bound to occur.

In a recent paper, where the wave propagation through a saturable non-linear asymmetric dimer has been studied, it has been shown that the saturation of the non-linear response has distinct impacts on the wave transmission properties for short-and long-wavelength input signals (*34*). In other words, the saturation effect depends strongly on the energy of incident waves. A similar behaviour is also observed in the present study. In Figure 4, we plot the inverse localization length as a function of the incident wave intensity for three values of the energy, $E = 0$, 0.5, 1, and $\alpha = 0.5$, when β is equal to 1 and 3. From both figures, we observe that the non-linear saturation effect is stronger and more pronounced when the energy of the incident wave E is larger.

In Figure 5, we show some examples of the spatial distribution of the intensity of the wave function calculated for a particular realization of disorder, when the system size N is 60 and a plane wave is incident from the region where $n > 60$. The wave function intensity is normalized

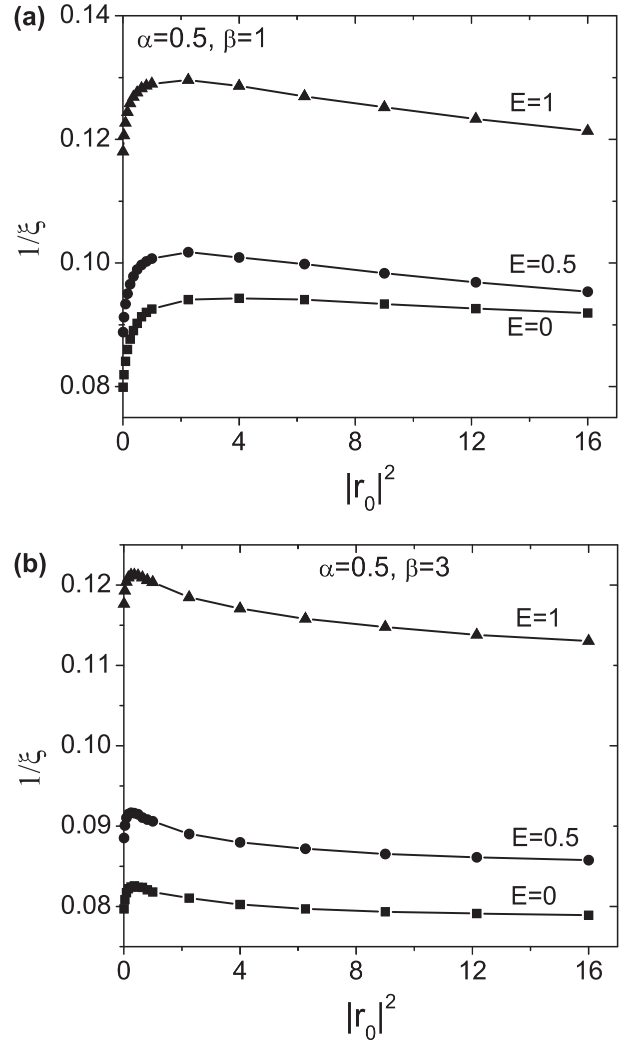

Figure 4. Inverse localization length $1/\xi$ plotted as a function of the incident wave intensity $|r_0|^2$ for three values of the energy, $E = 0, 0.5, 1$, and $\alpha = 0.5$, when (a) $\beta = 1$ and (b) $\beta = 3$. The non-linear saturation effect depends strongly on the energy of the incident wave and is more pronounced when E is large.

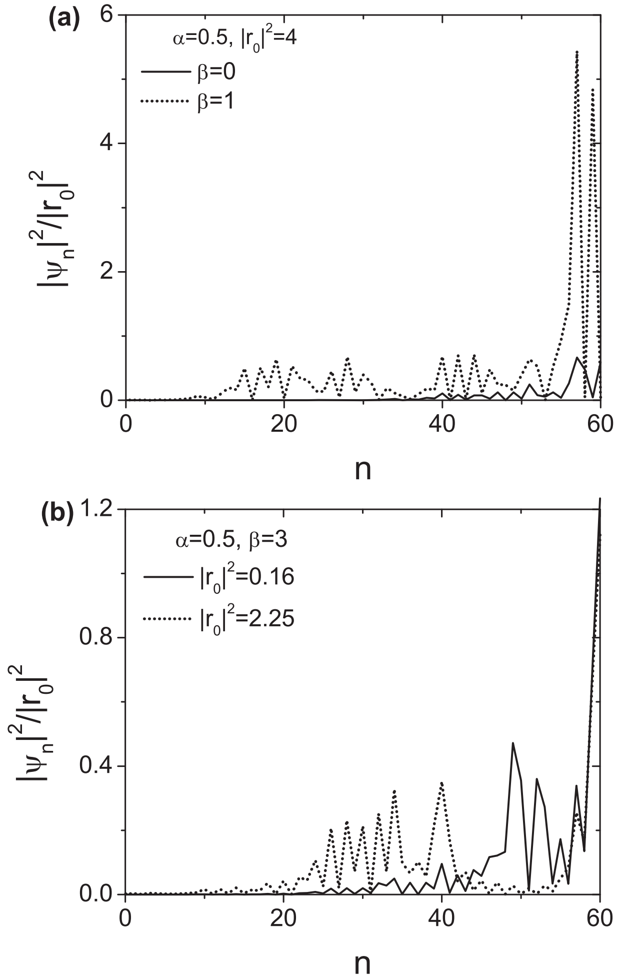

Figure 5. Spatial distribution of the intensity of the normalized wave function for one particular realization of the random potential, when (a) $\alpha = 0.5, \beta = 0, 1, |r_0|^2 = 4$ and (b) $\alpha = 0.5$, $\beta = 3$, $|r_0|^2 = 0.16, 2.25$. The values of N and E are fixed to 60 and 0, respectively. A plane wave is assumed to be incident from the right side.

by the intensity of the incident wave, $|r_0|^2$. The parameters α and E are fixed to 0.5 and 0, respectively. In Figure 5(a), we fix $|r_0|^2 = 4$ and compare the results for $\beta = 0$ and $\beta = 1$. We find that the wave penetrates more deeply into the medium and its intensity is significantly larger in a wider region when $\beta = 1$. This is consistent with the result that the localization length for $\beta = 1$ is larger than that for $\beta = 0$ (see Figure 3). In Figure 5(b), we fix $\beta = 3$ and compare the results for $|r_0|^2 = 0.16$ and $|r_0|^2 = 2.25$. Again, we find that the wave penetrates more deeply into the medium when $|r_0|^2 = 2.25$, for which the localization length is larger as can be seen in Figure 4(b).

We have also considered the influence of the saturable nonlinearity on the fluctuation of the localization length. In Figure 6, we plot the variance of the inverse localization length, σ^2, as a function of the incident wave intensity $|r_0|^2$, for three values of the energy, $E = 0$, 0.5 and 1, when $\alpha = 0.5$ and $\beta = 3$ (cf. Figure 4(b)). We find that this quantity also shows a non-monotonic behaviour. The variance takes a minimum value at the same value of $|r_0|^2$, at which $1/\xi$ takes a maximum value.

The effects studied in this paper may be realized in experiments on layered photonic structures. In such systems, it has been demonstrated that the evolution of longitudinal Bloch waves can be approximated sensibly by the discrete non-linear Schrödinger equation (*7*, *40*). For instance, we can employ a layered photonic system that is fabricated from lithium niobate ($LiNbO_3$) crystals. This type of material exhibits a saturable self-defocusing nonlinearity corresponding to our consideration, via photorefractive effects (*41*, *42*). Spatial disorder is introduced into the system through a random variation of the layer width in the process of fabrication.

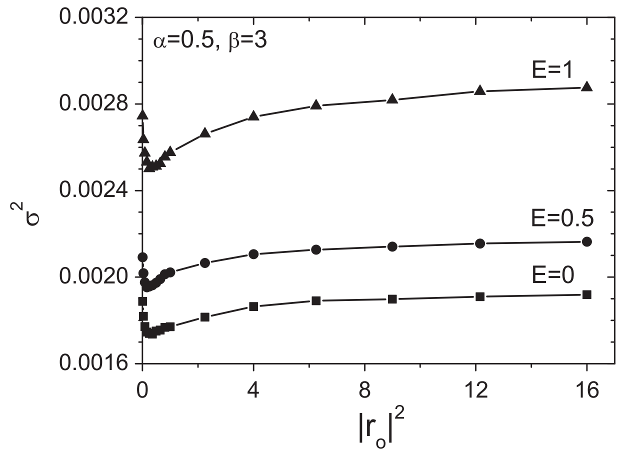

Figure 6. Variance of the inverse localization length, σ^2, plotted as a function of the incident wave intensity $|r_0|^2$ for three values of the energy, $E = 0, 0.5$ and 1, when $\alpha = 0.5$ and $\beta = 3$.

4. Conclusion

In this paper, we have presented a numerical study of the propagation and the Anderson localization of plane waves in a 1D lattice chain, where disorder and saturable nonlinearity are simultaneously present. Using a calculation scheme for solving a disordered version of the stationary discrete non-linear Schrödinger equation in the fixed input case, the disorder-averaged logarithmic transmittance and the localization length have been calculated in a numerically precise manner. For low incident intensities where the saturation effect is ineffective, the enhancement of localization due to nonlinearity has been found to occur in a way similar to the case without saturation. For sufficiently high incident intensities, we have found that the localization length is an increasing function of the incident wave intensity, which implies that disorder-induced localization is suppressed for stronger input intensities. This feature is associated with the fact that the non-linear system is reduced to an effectively linear one, when either the incident wave intensity or the saturation parameter is sufficiently large. It is found that the non-linear saturation effect is stronger and more pronounced when the energy of the incident wave is larger. We have also calculated the variance of the inverse localization length and found that it also shows a non-monotonic behaviour, taking a minimum value at the same value of the incident wave intensity, at which the localization length takes a minimum.

Disclosure statement

No potential conflict of interest was reported by the authors.

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