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Comparing the Riskiness of Dependent Portfolios

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ABSTRACT

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A nonparametric test based on *nested L-statistics* and designed to compare the riskiness of portfolios was introduced by Brazauskas, Jones, Puri, and Zitikis (2007). Its asymptotic and small-sample properties were primarily explored for independent portfolios, though independence is not a required condition for the test to work. In this dissertation, we investigate how the performance of the test changes when insurance portfolios are dependent. To achieve that goal, we perform a simulation study where we consider three different risk measures: conditional tail expectation, proportional hazards transform, and mean. Further, three portfolios are generated from exponential, Pareto, and lognormal distributions, and their interdependence is modeled with the three-dimensional t and Gaussian copulas. It is found that the presence of comonotonicity makes the test very liberal for all the risk measures under consideration. For various other types of dependence, the results are mixed, i.e., they depend on the chosen risk measure, sample size, and even on the test's significance level. We illustrate how to incorporate such findings into sensitivity analysis of the decisions. The risks we analyze represent tornado damages in different regions of the United States from 1890 to 1999. In addition,

we provide a theoretical explanation to the behavior of the power function of the test by considering the usual stochastic orders of the Gini indexes of multivariate normal risks with the same marginals but different dependence structures. Finally, we generalize the comparison for the Gini indexes of multivariate elliptical risks.

Dedicated to my parents ...

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$\leq_c$	concordance order
EC	elliptical distribution
$\stackrel{d}{=}$	identity in distribution
$supp$	support of a random vector
$\leq_{st}$	usual stochastic order

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Chapter 1

Introduction

1.1 Motivation

Comparing the riskiness of portfolios is an area of practical importance that has received a fair share of attention from researchers in academia. Depending on the nature of the business, the purpose of comparing risks may vary. For example, a portfolio of automobile insurance policies may include policies from different geographic regions and comparing the riskiness of them may help to assign equal premiums for equally risky policies. In the finance industry, such tools may be used for solving portfolio selection problems or for comparing performances of investment funds. Hence, it is of interest to study the statistical methods of grouping equally risky portfolios by comparing their riskiness.

Brazauskas, Jones, Puri, and Zitikis (2007) introduced a nonparametric hypothesis test based on nested L-statistics to check the inequality of risk measures associated with the portfolios of insurance losses. The test statistic for this test is defined with the help of the Gini index (Gini, 1912) whose nonparametric estimator is an L-statistic (i.e., a linear combination of order statistics). The asymptotic

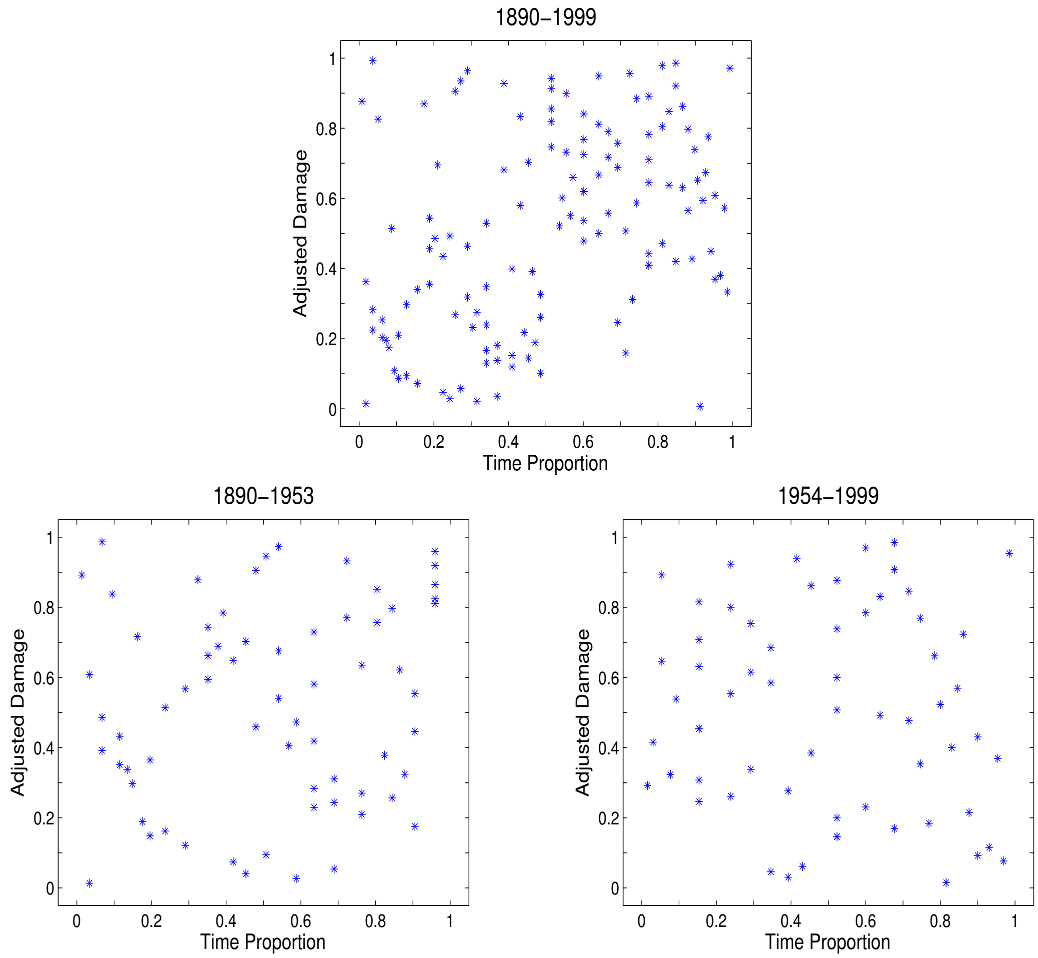

Figure 1.1: Inflation and wealth adjusted damages from major tornadoes in the U.S. for the periods 1890-1999 (top row), 1890-1953 (bottom row–left), and 1954-1999 (bottom row–right).

and small-sample properties of that test were primarily explored for independent portfolios though independence is not a required condition for the test to work. Independence is a very restrictive assumption to be satisfied by the data in use, and we find many examples of real data where the assumption of independence was violated. One such example can be found in the practical illustration of the nonparametric hypothesis test in Brazauskas *et al.* (2007). The practical performance of the test was illustrated using the tornado damage data taken from Brooks and Doswell (2001), who argued that, in order to compare tornado losses over time, it is appropriate to adjust for inflation and wealth. While this suggestion was followed, one important detail was overlooked.

As we can see in Figure 1.1 (top row), the adjusted data from 1890-1999 do not look independent (i.e., data points are not uniformly scattered on the square $[0,1]\times[0,1]$), but dependence patterns disappear when the data set is split into two at 1953 (bottom row-right and -left panels). Apparently, in 1953, the National Weather Service introduced its early warning program which shifted the subsequent losses. Therefore, we believe that it is important to examine the performance of those procedures when the independence assumption is violated. In this dissertation, we investigate−using theoretical analysis, Monte Carlo simulation, and real data examples−the performance of the test in Brazauskas *et al.* (2007) when insurance portfolios are dependent, with varying strengths of dependence.

1.2 Literature Review

Comparing the risks is an interesting and important actuarial problem. Many researchers have made contributions to this literature. Wang and Young (1998) and Wirch and Hardy (2000) used the notion of the stochastic dominance to compare the risk measures. Jones and Zitikis (2005) approached the same problem from a different angle and suggested parametric and nonparametric tests to examine the order of two risk measures. They constructed the asymptotic distribution of the difference between the empirical estimators of two risk measures. Further, they constructed confidence intervals for the difference of the risk measures at a prescribed confidence level.

Jones, Puri, and Zitikis (2006) extended the idea of statistical tests about the equality of risk measures to higher dimensions. They considered one and two sided alternatives and unordered alternatives based on the specification of the risk measure in the null hypothesis. Moreover, they discussed the asymptotic distributions of the test statistics and obtained the asymptotically most powerful tests. While this paper made a considerable contribution to move the literature forward, the practical applications of these tests has been very limited as these tests were developed based on the assumption of independence among the populations.

Brazauskas, Jones, Puri, and Zitikis (2007) considered testing hypothesis about the equality of risk measures of multivariate risks. This nonparametric proposal was based on the Gini index (see Gini, 1936, for English translation of the original article) which is an L-statistics. We note in passing that statistical inferential tools based on L-statistics play a leading role in the actuarial literature which is mostly due to their computational efficiency and straightforward risk measure formulations (see Necir, Meraghni, and Meddi, 2007, and Necir and Meraghni, 2009, 2010). Similar tools have also been proposed in the empirical finance literature (see Darolles, Gourieroux, and Jasiak, 2009), where performance of hedge funds is measured using a metric based on L-moments (see Hosking, 1990).

The test proposed in Brazauskas, Jones, Puri, and Zitikis (2007) is the subject of this dissertation. The nonparametric test introduced by these authors primarily explored the performance of the test when the portfolios of risks are independent. This test can be applied to dependent portfolios and we examine the performance of the test when the underlying risks are dependent. This study is carried out in two stages. In the first stage, we develop a Monte Carlo simulation study where we consider three portfolios with different dependence structures modeled using copula. In the second stage, we theoretically explain the numerical results related to the performance of test that we obtained through the simulation study. In order to complete the goal of the second stage, we explore the notion of stochastic ordering of Gini indexes of multivariate normal random variables. Further, we illustrate how to incorporate such findings into a sensitivity analysis of the decisions associated with tornado damages in different regions of the United States from 1890 to 1999.

Ordering of Gini indexes of multivariate normal risks that we discuss in this dissertation opens up a new venue of research due to the popularity of the Gini index in economics and insurance literature. In addition, all the results developed for the multivariate normal distribution can be easily extended to more general elliptical distributions using the relationship between the multivariate normal and elliptical distributions described by McNeil *et al.* (2005; Theorem 3.25 and Definition 3.26). This generalization will move the literature of the central concentration of elliptical distributions forward.

1.3 Plan of the Thesis

The rest of the dissertation is arranged in the following manner. In Chapter 2, we discuss some preliminary concepts such as dependent risks, risk measures, and ordering of risks under several stochastic orders.

In Chapter 3, we give an overview of the nonparametric hypothesis test in Brazauskas, Jones, Puri, and Zitikis (2007), and the asymptotic behavior of the test statistic under the null hypothesis and alternative hypothesis when the insurance portfolios are dependent. Further, we discuss the decision making process using the bootstrap estimator of the critical value.

In Chapter 4, we explain Monte Carlo simulation procedure used to investigate the effect of the dependence on the power of the hypothesis test. As a practical application, we propose a sensitivity analysis method based on the numerical findings.

In Chapter 5, we give a theoretical explanation for the results of the simulation study. In particular, we discuss the dependence effect on the power function of the hypothesis test by comparing the Gini indexes of multivariate normal risks with the same marginal distributions. Moreover, we expand the discussion to the Gini indexes of multivariate elliptical risks.

Finally, in Chapter 6, we summarize the results of this dissertation and briefly discuss our future research plans.

Chapter 2

Preliminaries

Throughout the dissertation, we use bold letters to denote vectors or matrices. For example, $\mathbf{X} = (X_1, \ldots, X_k)$ is a random vector, $\mathbf{x} = (x_1, \ldots, x_k)$ is a row vector, and $\boldsymbol{\Sigma}$ is a $k \times k$ matrix. In particular, the symbol $\mathbf{0}$ denotes the row vector with all entries equal to 0, and $\mathbf{1}_{k\times k}$ denotes the $k \times k$ matrix with all entries equal to 1. The inequality between vectors or matrices denotes componentwise inequalities. For example, $(x_1, \ldots, x_k) \leq (y_1, \ldots, y_k)$ implies that $x_i \leq y_i$ for all $i = 1, \ldots k$.

2.1 Dependent Risks

Studying the notion of risk is the central idea of actuarial science. Oxford dictionary describes the meaning of risk as a situation involving exposure to danger. We can broaden this idea and think of risk as an event that brings some adverse financial consequences with uncertainty. Uncertainty is merged with risks and hence they are random events. In insurance, one party (policy holder) transfers the economic impact of their risks to another party (insurer). Based on the nature of insurance and the size of the impact, this policy holder transfers the risk fully or partially to the insurer. The following definition taken from Denuit *et al.* (2005) gives a broader idea of risk in the actuarial science context.

Definition 2.1.1. A risk X is a non-negative random variable representing the random amount of money paid by an insurance company to indemnify a policy-holder, a beneficiary and/or a third-party in execution of an insurance contract.

In this dissertation, we consider the impact of risks simultaneously rather than separately. In other words, we focus on dependence among risks. There are many real life situations with dependent risks. In joint life insurance or annuity policy, it is important to study the joint mortality patterns of a group of insureds or annuitants. Sharing a similar lifestyle, facing an accident, or "broken heart" syndrome can be considered as factors that show dependence among the mortalities of a husband and wife, a family with children, or twins. Therefore, we cannot simply neglect the dependent risks in that situation. If we consider some catastrophic weather events such as tornadoes, hurricanes, earthquakes, and tsunamis, several lines of insurance business may be affected simultaneously, and we may want to consider the dependence among risks in those situations as well.

There are three fundamental dependence structures of random variables; namely, independence, comonotonicity, and countermonotonicity. Perhaps the most common dependence structure used in modeling is independence. When portfolios of risks arise, we say that they are independent if the behavior of one risk does not influence the behavior of the other risks. In probability theory, the notion of independence is defined as follows:

Definition 2.1.2. The random variables $X_1, \ldots, X_k$ are independent if, and only if,

$$F(\mathbf{x}) = \prod_{i=1}^{k} F_i(x_i) \text{ for all } \mathbf{x} \in \mathbb{R}^k,$$

where F is the joint distribution function of the random vector $\mathbf{X}$ and F_i are the marginal distribution functions of X_i for $i = 1, \ldots, k$.

Strong positive dependence or *comonotonicity* has received a fair share of attention as a fundamental dependence structure. The comonotonicity of risks has important applications in actuarial science and finance. Dhaene *et al.* (2002) conducted a comprehensive study on the concept of comonotonicity and its applications. Below we cite their definition and several equivalent characterizations of comonotonicity.

Definition 2.1.3. A set $A \subset \mathbb{R}^k$ is said to be *comonotonic*, if for any $\mathbf{x}, \mathbf{y} \in A$, either $\mathbf{x} \leq \mathbf{y}$ or $\mathbf{y} \leq \mathbf{x}$ holds. Intuitively, a set is comonotonic if and only if it is totally ordered.

Definition 2.1.4. For a random vector, its *support* is defined by

$$supp(\mathbf{X}) = \{\mathbf{x} \in \mathbb{R}^k : \mathbb{P}\{\mathbf{X} \in B(\mathbf{x}, r)\} > 0, \text{ for any } r > 0\},$$

where $B(\mathbf{x}, r)$ denotes the ball centered at $\mathbf{x}$ with radius r.

Definition 2.1.5. A random vector $\mathbf{X}$ is *comonotonic* if its support is comonotonic.

Dhaene *et al.* (2002) also developed several well-known characterizations of comonotonicity.

Proposition 2.1.6. A random vector $\mathbf{X}$ is comonotonic if and only if one of the following equivalent conditions holds:

(1) $\mathbf{X}$ has a comonotonic support,

(2) $\mathbb{P}\{X_1 \leq x_1, \ldots, X_k \leq x_k\} = \min\{\mathbb{P}\{X_1 \leq x_1\}, \ldots, \mathbb{P}\{X_k \leq x_k\}\}$ for all $(x_1, \ldots, x_k) \in \mathbb{R}^k$,

(3) There exists a random variable Z and increasing functions $f_1, \ldots, f_k$, such that $(X_1, \ldots, X_k) \stackrel{d}{=} (f_1(Z), \ldots, f_k(Z))$.

The other extreme dependence structure is *countermonotonicity* which is used only in the bivariate case. Bivariate risk is said to be countermonotonic if it is distributed as $(f_1(Z), f_2(Z))$ for some random variable Z, with an increasing function f_1 and a decreasing function f_2. This dependence structure can not be extended to higher dimensions.

Copulas are considered to be useful for understanding the relationship between risks. They are used for modeling dependence among random variables in a wide variety of disciplines. To understand their use in actuarial science, readers may be refered to Nelsen (2006), Chapter 5 of McNeil *et al.* (2005), Frees and Valdez (1998), or Chapter 4 of Denuit *et al.* (2005). The annotated bibliography in Frees and Valdez (1998) provides a collection of other references for researchers.

Definition 2.1.7 (copula). A k-dimensional copula is a joint distribution function of random variables whose marginal distributions are uniform on the interval $[0, 1]$.

We reserve the notation $C(\mathbf{u}) = C(u_1, \ldots, u_k)$ for a k-dimensional copula. The terminology copula is first used by Sklar, see Nelsen (2006), to link univariate distribution functions of random variables to their multivariate distribution function through the following theorem. It shows that all multivariate distribution functions contain copulas, and copulas may be used in conjunction with univariate distribution functions to construct multivariate distribution functions.

Theorem 2.1.8 (Sklar 1959). Let F be a joint distribution function with marginal distributions $F_1, \ldots, F_k$. Then there exists a copula $C : [0,1]^k \to [0,1]$ such that, for all $x_1, \ldots, x_k$ in $\overline{\mathbb{R}} = [-\infty, \infty]$,

$$F(x_1, \ldots, x_k) = C(F_1(x_1), \ldots, F_k(x_k)). \tag{2.1.1}$$

If the marginal distributions $F_1, \ldots, F_k$ are continuous, then C is uniquely determined. Otherwise, C is unique only on $\prod_{i=1}^{k} \mathrm{Ran}(F_i)$, where $\mathrm{Ran}(F_i)$ denotes the range of F_i. Conversely, if C is a copula and $F_1, \ldots, F_k$ are univariate distribution functions, then the function F defined in (2.1.1) is a joint distribution function with margins $F_1, \ldots, F_k$.

The *independence copula* (usually denoted as Π) characterizes the independent random variables and is given by

$$\Pi(u_1, \ldots, u_k) = \prod_{i=1}^{k} u_i \ \text{ for } \ 0 \leq u_i \leq 1.$$

Note that Theorem 2.1.8 and Definition 2.1.2 can be used to obtain the independence copula.

Comonotonicity of random variables is characterized by the *comonotonicity copula* (usually denoted as M), which can capture situations when the random variables are almost surely strictly increasing functions of each other, and countermonotonicity is characterized by the *countermonotonicity copula* (usually denoted as W), which applies to only two random variables where one is almost surely a decreasing function of the other. The comonotonicity copula is given by

$$M(u_1, \ldots, u_k) = \min(u_1, \ldots, u_k) \text{ for } 0 \leq u_i \leq 1.$$

The countermonotonicity copula is given by

$$W(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\} \text{ for } 0 \leq u_1, u_2 \leq 1.$$

The independence, comonotonicity, and countermonotonicity copulas are the fundamental copulas (see Chapter 5 of McNeil *et al.*, 2005), and they represent three important dependence structures. Likewise, many intermediate dependence structures can be described by identifying a relevant type of copula (see Frees and Valdez, 1998, Nelsen, 2006, or Joe, 2014).

In order to determine what effect, if any, the dependence structure between the portfolios has on the power function of the hypothesis test described in Chapter 3, we shall perform a simulation study. For the simulation study, we consider different types of dependent portfolios, which cover the full spectrum of dependence strength from negative dependence through the strong positive dependence . In particular, we select four types of dependent portfolios: negative dependence (for two portfolios, it corresponds to countermonotonicity), zero dependence, moderate positive dependence, and strong positive dependence (comonotonicity). These dependence structures can be captured using the well-known Gaussian and t copulas.

The Gaussian and t copulas belong to elliptical copula family. Simply, the copula of multivariate normal random variables is the Gaussian copula and that of multivariate t-distributed random variables is the t copula. Therefore, the Gaussian copula is completely determined by the correlation matrix (Σ) of the random variables. We use the notation C_{Σ}^{Ga} for a Gaussian copula with a correlation matrix Σ. For example, the identity matrix characterizes independence among the variables, while the correlation matrix with all entries equal to 1 characterizes the comonotonicity. In addition to the correlation matrix, we need to know the degrees of freedom (df) to determine the t copula. We use the notation $C_{\nu,\Sigma}^{t}$ for the t copula with the correlation matrix Σ. In fact, the Gaussian copula is the limiting case of t copula. The following are examples of the three-dimensional correlation matrix for the dependence structures mentioned above. Note that for the Gaussian copula zero dependence is equivalent to independence.

- *Negative* (Σ_1) and *Zero* (Σ_2) Dependence:

$$\Sigma_1 = \begin{pmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{pmatrix} \quad \text{and} \quad \Sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- *Moderate Positive* (Σ_3) and *Strong Positive* (Σ_4) Dependence:

$$\Sigma_3 = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix} \quad \text{and} \quad \Sigma_4 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

In addition, Figure 2.1 illustrates the difference between the two-dimensional t copula (with $\nu = 3$ degrees of freedom) and the Gaussian copula, i.e., t with

$\nu \to \infty$, for normal marginals and varying strengths of dependence. (In this particular instance, the three-dimensional plots provide no new insights.)

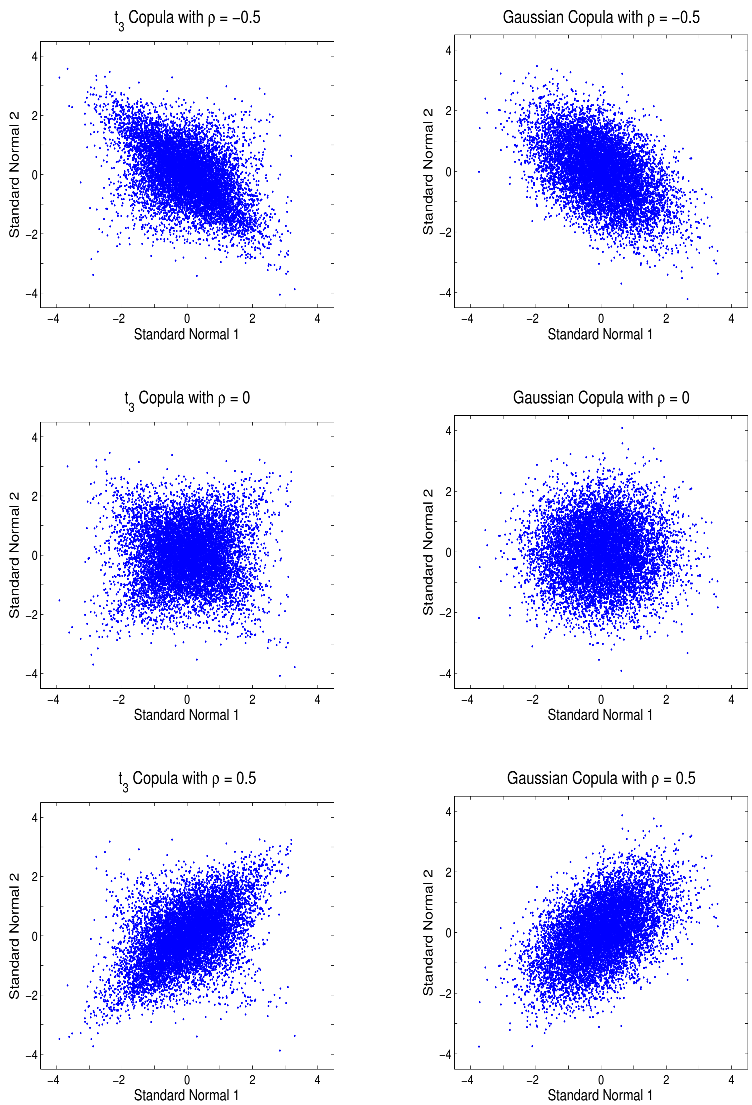

Figure 2.1: 2D copula realizations for negatively dependent, zero dependent, and moderately positively dependent normal marginals. Left column: t copulas. Right column: Gaussian copulas.

Notice how the tail dependence manifests itself for $\nu = 3$ and disappears as $\nu \to \infty$, i.e., in the latter case there are essentially no points in the corners of each plot.

The Gaussian copulas do not effectively capture the tail dependence, while t copulas with small ν do. In risk management, focusing on these extreme events is required as they can have a significant impact on companies and the global economy. For example, after 2008 financial crisis, the Gaussian copulas models were partly blamed for not capturing the tail dependence of financial industry risks. In this dissertation, we introduce both tail independence and dependence through the Gaussian and t copulas, respectively.

As we mentioned, the Gaussian and t copulas belong to the elliptical copula family, and they are derived from the multivariate Gaussian and t distributions, respectively. It is important for us to discuss more about elliptical distributions, not only because it is important for simulating t copula, but also it is an important topic in our future research work. Therefore, we recall some basic concepts about elliptical distributions. The following definition and characterization of elliptical distribution are taken from McNeil *et al.* (2005).

Definition 2.1.9. A k-dimensional random vector $\mathbf{X}$ has an *elliptical distribution* if its characteristic function has the following form:

$$\mathbb{E}[e^{i\mathbf{t}\mathbf{X}^\intercal}] = e^{i\mathbf{t}\boldsymbol{\mu}^\intercal}\psi(\mathbf{t}\boldsymbol{\Sigma}\mathbf{t}^\intercal),$$

where $\boldsymbol{\mu}^\intercal \in \mathbb{R}^k$, $\boldsymbol{\Sigma} \in \mathbb{R}^{k\times k}$ is a positive semidefinite matrix, and ψ is a characteristic function. In this case we denote $\mathbf{X} \sim EC_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$. ψ is referred to as the characteristic generator of the elliptical distribution. $\boldsymbol{\mu}$ is referred to as location vector and is equal to the mean of $\mathbf{X}$ if it exists, and $\boldsymbol{\Sigma}$ is referred to as dispersion matrix.

McNeil *et al.* (2005) points out that, generally, characteristic generators may be used only in certain dimensions. In this dissertation, we shall focus on a special class of generators and the elliptical distributions induced by this class. Specifically, we consider all the generators that can be used in any arbitrary dimension and denote this class by Ψ_∞.

The elliptical distribution family induced by Ψ_∞ includes many important distributions, such as multivariate normal distribution and multivariate t distribution. For more discussion about this family, readers are referred to Chapter 3 of McNeil *et al.* (2005). Furthermore, a useful property about this family is that it has stochastic representation in terms of multivariate normal distribution, as shown by Proposition 2.1.10. It is essentially a combination of Theorem 3.25 and Definition 3.26 of McNeil *et al.* (2005), and the proof is thus omitted.

Proposition 2.1.10. Random vector $\mathbf{X} \sim EC_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ with $\psi \in \Psi_\infty$ if and only if there exist random vector $\mathbf{Z}$ and random variable R such that

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + R\,\mathbf{Z}, \tag{2.1.2}$$

where $\mathbf{Z} \sim MVN(\mathbf{0}, \boldsymbol{\Sigma})$ and $R \geq 0$ is a random variable independent of $\mathbf{Z}$.

Proposition 2.1.10 presents an important relationship between multivariate normal and elliptical distributions. With this representation, many properties of multivariate normal distributions can be easily generalized to elliptical distributions. In later chapters, we shall see some examples.

Theorem 5 of Dhaene *et al.* (2002) develops a characterization of the comonotonicity for multivariate normal distribution by its covariance matrix. Specifically, $\mathbf{X} = (X_1, \ldots, X_k) \sim MVN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is comonotonic if and only if $corr(X_i, X_j) = 1$ for all i, j (i.e., $rank(\boldsymbol{\Sigma}) = 1$). Furthermore, if all marginal distributions have the same variance 1, then the comonotonicity of $\mathbf{X}$ is equivalent to $\boldsymbol{\Sigma} = \mathbf{1}_{k\times k}$.

The characterization of comonotonicity of multivariate normal distributions can be generalized to elliptical distributions induced by Ψ_∞. Specifically, an elliptical distribution with $\psi \in \Psi_\infty$ is comonotonic if and only if its dispersion matrix has rank 1. Below, we formally state the characterization and prove it in general case.

Proposition 2.1.11. Let $\mathbf{X} \sim EC_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ with $\psi \in \Psi_\infty$. $\mathbf{X}$ is comonotonic if and only if $rank(\boldsymbol{\Sigma}) = 1$.

Proof. See Appendix A. □

2.2 Risk Measures

After the discussion of risks and their dependence structures, we consider the methods for measuring risks. Next section is devoted to the discussion of risk measures.

Risks are non-negative random variables (see Definition 2.1.1), and a risk measure assigns a single nonnegative value to a risk to reflect the riskiness associated with the distribution of the risk. More formally, a risk measure is a functional mapping from the set $\mathcal{F}$ of distribution functions to the extended real line. Risk measure is a useful tool for quantifying the riskiness. The interpretation of the risk measure value may vary based on the nature of the risk. If X is a loss of a financial portfolio, we interpret the risk measure as the risk capital of the portfolio and this can be used to determine the provision and capital requirement in order to avoid insolvency (see Panjer (1998)). If X is an amount of exposure to liability in an insurance company, a premium calculation principle gives the minimum amount the insurer must raise from the insured. This premium calculation principle is an example of risk measure in the insurance industry. Also, some risk measures can be used as a premium associated with an insurance contract. Risk measures have to satisfy some axioms (see Chapter 2 of Denuit *et al.* (2005)). There is a vast literature on risk measures and their application to contract pricing, capital allocation, and risk management. For a quick introduction into these topics, the reader may be referred to the review papers by Albrecht (2004), Tapiero (2004), and Young (2004).

In order to compare the riskiness of portfolios of risks, we will utilize a special class of risk measures, namely, spectral risk measures. Such measures were first introduced in the finance literature with the intention that the user may wish to re-weight the initial distribution of the portfolio in order to reflect his/her risk aversion. In mathematical terms, a spectral risk measure $R = R[F]$ of a random variable X, with a cumulative distribution function (cdf) F, is defined as

$$R[F] = \int_0^1 F^{-1}(u)J(u)\,du, \tag{2.2.1}$$

where J is the weight function which controls the risk aversion, and $F^{-1}(u) = \inf\{x : F(x) \geq u\}$ denotes the quantile function of X. It is not easy to find a descriptive guidance on selecting the risk aversion function. But readers may find some developments of spectral risk measures and their applications in Acerbi (2002). The following are a few typical examples of spectral risk measures.

Example 2.2.1. (MEAN). Choosing $J(u) = 1$ for $0 \leq u \leq 1$, in equation (2.2.1), gives the expected value of X, and we denote it using the notation MEAN$[F]$. □

Example 2.2.2. (PHT, proportional hazards transform). Let r $(0 < r \leq 1)$ be a real valued constant which can be chosen depending on the risk aversion. (In the actuarial literature, r is known as the distortion level.) Choosing $J(u) = r(1-u)^{r-1}$ for $0 \leq u \leq 1$, in equation (2.2.1), gives the Proportional Hazards Transform of F, and we denote this measure using the notation PHT$[F]$. □

Example 2.2.3. (CTE, conditional tail expectation). Conditional Tail Expectation can be defined as spectral risk measure by setting $J(u) = 0$ for $0 \leq u < t$ and $J(u) = 1/(1-t)$ for $t \leq u \leq 1$ in equation (2.2.1), where t $(0 \leq t < 1)$ is a real valued constant known as the threshold level. We denote this measure using the notation CTE$[F]$. □

In practice, the cdf F is usually unknown and has to be estimated from the observed data. As discussed in the introduction, one can do that parametrically, non-parametrically, or semi-parametrically and then insert estimated F in equation (2.2.1), which would produce an estimator of $R[F]$. In this research, we will focus on the empirical nonparametric estimation. That is, in equation (2.2.1) we replace

F by the empirical cdf so that $\widehat{R} = R[\widehat{F}]$ with

$$
\begin{aligned}
R[\widehat{F}] &= \int_0^1 \widehat{F}^{-1}(u)J(u)\,du \\
&= \sum_{m=1}^{n} \left(\int_{(m-1)/n}^{m/n} \hat{F}^{-1}(u)J(u)du \right).
\end{aligned}
$$

Considering the ordered sample of X, $X_{1:n} \leq \cdots \leq X_{n:n}$, we can obtain the empirical estimator $\hat{F}^{-1}$ of the quantile function of F. For $u \in \left(\frac{m-1}{n}, \frac{m}{n}\right)$, $\hat{F}^{-1}(u) = X_{m:n}$. We can express the above equation by

$$
R[\widehat{F}] = \sum_{m=1}^{n} X_{m:n} \left(\int_{(m-1)/n}^{m/n} J(u)\,du \right),
$$

where $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the ordered values of data $X_1, \ldots, X_n$. Hence, the empirical estimator of a risk measure $R[F]$ is given by

$$
R[\widehat{F}] = \sum_{m=1}^{n} c_{mn} X_{m:n} \tag{2.2.2}
$$

with $c_{mn} = \int_{(m-1)/n}^{m/n} J(u)\,du$. Note that $R[\widehat{F}]$, as defined in (2.2.2), belongs to a general class of L-statistics, theoretical properties of which are well understood and have been thoroughly studied by Jones and Zitikis (2003), Necir and Meraghni (2009, 2010), and other authors.

In Chapter 5 and Chapter 6 of the dissertation, we will present the theoretical investigation of the simulation study results. In order to do that, we recall some definitions and theorems associated with ordering of elliptical risks.

2.3 Ordering of Risks

Definition 2.3.1. Let X and Y be two random variables. X is said to be smaller than Y in *usual stochastic order*, denoted as $X \leq_{st} Y$, if

$$\mathbb{P}\{X > t\} \leq \mathbb{P}\{Y > t\} \text{ for all } t \in \mathbb{R}. \tag{2.3.1}$$

Roughly speaking, (2.3.1) says that X is less likely than Y to take on large values, where "large" means any value greater than t for all $t \in \mathbb{R}$.
It is easy to prove that $X \leq_{st} Y$ if and only if,

$$\mathbb{P}\{X \leq t\} \geq \mathbb{P}\{Y \leq t\} \text{ for all } t \in \mathbb{R}. \tag{2.3.2}$$

In this work, we will use both (2.3.1) and (2.3.2) in proofs.

The above definitions are taken from Shaked and Shanthikumar (2007), which also provide the following characterization for the usual stochastic order.

Proposition 2.3.2. Let X, Y be two random variables with the respective distribution functions F and G. $X \leq_{st} Y$ if and only if $F^{-1}(u) \leq G^{-1}(u)$ for all $u \in (0, 1)$.

Now, we pay our attention to an important dependence order relations for multivariate distributions. It is concordance order. The following is the definition of the bivariate concordance order.

Definition 2.3.3. Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{X}' = (X_1', X_2')$ be bivariate random vectors with the same marginals. Then $\mathbf{X}$ is said to be smaller than $\mathbf{X}'$ in concordance order (written as $\mathbf{X} \leq_c \mathbf{X}'$ or $F_{\mathbf{X}} \leq_c F_{\mathbf{X}'}$), if

$$\mathbb{P}(X_1 \leq s, X_2 \leq t) \leq \mathbb{P}(X_1' \leq s, X_2' \leq t) \quad \text{for all } s \text{ and } t.$$

The following theorem taken from Müller and Stoyan (2002) gives several equivalent characterizations of the concordance order.

Theorem 2.3.4. Let $\mathbf{X}$ and $\mathbf{X}'$ be bivariate random vectors with the same marginals. Then the following conditions are equivalent:

(1) $\mathbf{X} \leq_c \mathbf{X}'$.

(2) $\mathbb{P}(X_1 > s, X_2 > t) \leq \mathbb{P}(X_1' > s, X_2' > t)$ for all s and t.

(3) $\mathrm{Cov}(f_1(X_1), f_2(X_2)) \leq \mathrm{Cov}(f_1(X_1'), f_2(X_2'))$ for all increasing f_1 and f_2.

Joe (1990) suggested the following definition as a generalization of the bivariate concordance order in Definition 2.3.3.

Definition 2.3.5. Let $\mathbf{X}$ and $\mathbf{X}'$ be k-dimensional random vectors with the same marginals. Then $\mathbf{X}$ is said to be smaller than $\mathbf{X}'$ in concordance order (written as $\mathbf{X} \leq_c \mathbf{X}'$ or $F_{\mathbf{X}} \leq_c F_{\mathbf{X}'}$), if $F_{\mathbf{X}}(\mathbf{t}) \leq F_{\mathbf{X}'}(\mathbf{t})$ as well as $\bar{F}_{\mathbf{X}}(\mathbf{t}) \leq \bar{F}_{\mathbf{X}'}(\mathbf{t})$ hold for all $\mathbf{t}$.

The requirement of equal marginals is not included in the definition as the inequalities $F_{\mathbf{X}}(\mathbf{t}) \leq F_{\mathbf{X}'}(\mathbf{t})$ and $\bar{F}_{\mathbf{X}}(\mathbf{t}) \leq \bar{F}_{\mathbf{X}'}(\mathbf{t})$ automatically satisfy the requirement.

The following are important properties of the multivariate concordance order (see Müller and Stoyan, 2002).

$(P1)$ (bivariate concordance) $(X_1, \ldots, X_k) \leq_c (X'_1, \ldots, X'_k)$ implies $(X_i, X_j) \leq_c (X'_i, X'_j)$ for all $1 \leq i < j \leq k$.

$(P2)$ (invariance with respect to increasing transforms) $(X_1, \ldots, X_k) \leq_c (X'_1, \ldots, X'_k)$ implies $(g_1(X_1), \ldots, g_k(X_k)) \leq_c (g_1(X'_1), \ldots, g_k(X'_k))$ for all increasing functions $g_1, \ldots, g_k$.

Nelsen (2006) discusses the concordance orders of copulas. The copula of an elliptically distributed random vector is an elliptical copula, and they are characterized by the correlation matrix. Therefore, the elliptically distributed random vectors ordered with respect to $\leq_c$, if and only if they have the same marginals and all their covariances are ordered. At this point, we can revisit some examples in the discussion of dependent risks (see Section 2.1) and order them as follows. For a Gaussian copula,

$$C^{\text{Ga}}_{\Sigma_1} \leq_c C^{\text{Ga}}_{\Sigma_2} \leq_c C^{\text{Ga}}_{\Sigma_3} \leq_c C^{\text{Ga}}_{\Sigma_4}.$$

Similarly for t copula,

$$C^{t}_{\nu,\Sigma_1} \leq_c C^{t}_{\nu,\Sigma_2} \leq_c C^{t}_{\nu,\Sigma_3} \leq_c C^{t}_{\nu,\Sigma_4}.$$

In general, if a portfolio $\mathbf{X}'$ with copula C' is more positively dependent than a portfolio $\mathbf{X}$ with copula C, then $C \leq_c C'$.

Chapter 3

Hypothesis Test

3.1 Hypotheses of Interest

Let $X(1), \ldots, X(k)$ be k portfolios of risks with distribution functions $F_1, \ldots, F_k$, respectively. These portfolios can be independent or dependent. Suppose their riskiness is measured using the risk measures $R_1 = R[F_1], \ldots, R_k = R[F_k]$, as defined by (2.2.1). The hypothesis of interest is to check whether or not the k risk measures $R_1, \ldots, R_k$ are all equal. That is, we can define the hypothesis test as follows.

$$H_0\colon\ R_1 = \cdots = R_k \quad \text{versus} \quad H_{\text{A}}\colon\ \text{at least one pair } R_i \neq R_j.$$

To test the above hypothesis, Brazauskas, Jones, Puri, and Zitikis (2007) proposed a nonparametric test statistic that constructs the Gini index based on $R_1, \ldots, R_k$. Hence, all information about the differences of portfolio riskiness can be summarized by the inequality index

$$\gamma \;=\; \frac{1}{k^2} \sum_{1 \le i,j \le k} \Big| R_i - R_j \Big|, \tag{3.1.1}$$

which leads to a more compact formulation of the problem:

$$H_0: \;\gamma = 0 \quad \text{versus} \quad H_{\mathrm{A}}: \;\gamma > 0.$$

3.2 Test Statistic

Using standard techniques for order statistics (see, e.g., David and Nagaraja, 2003, Section 9.4), the Gini index in (3.1.1) can be reexpressed in the following manner:

$$\begin{aligned} \gamma &= \frac{1}{k^2} \sum_{i=1}^{k} (4i - 2(k+1)) R_{i:k} \\ &= \sum_{i=1}^{k} \left(\int_{(i-1)/k}^{i/k} K(u) du \right) R_{i:k}, \end{aligned} \tag{3.2.1}$$

where $K(u) := 4u - 2$ for all $0 \le u \le 1$, and $R_{1:k}, \ldots, R_{k:k}$ are the ordered values of $R_1, \ldots, R_k$. This equation helps us to define the Gini index as a nested L-statistic. From equation (3.2.1), we can define an estimator of the Gini index by replacing the $R_{i:k}$ with $\hat{R}_{i:k}$,

$$\hat{\gamma} = \sum_{i=1}^{k} \left(\int_{(i-1)/k}^{i/k} K(u) du \right) \hat{R}_{i:k}.$$

That is, $\hat{\gamma}$ can be defined as a linear combination of ordered values of $\hat{R}_i$, in other words an L-statistic based on $\hat{R}_1, \ldots \hat{R}_k$.

Now, lets consider a sample $X_1(i), \ldots, X_n(i)$ of size n drawn from portfolio $X(i)$ for any $1 \leq i \leq k$. Then we can derive an empirical distribution $\hat{F}_i$ of $X(i)$ based on the above sample. By equation (2.2.2), we have

$$\hat{R}_i = \sum_{m=1}^{n} c_{mn} X_{m:n}(i) \tag{3.2.2}$$

with $c_{mn} = \int_{(m-1)/n}^{m/n} J(u)du$. Since $\hat{R}_1, \ldots, \hat{R}_k$ are L-statistics, it is not hard to see that $\hat{\gamma}$ is an L-statistic of L-statistics, hence the name "nested L-statistics".

To test the hypothesis stated in the previous section, the following test statistic is used:

$$T := \sqrt{\frac{n}{k}}\ \hat{\gamma}.$$

Let $D_i = \hat{R}_i - R_i$; then under the null hypothesis H_0, $D_i - D_j = \hat{R}_i - \hat{R}_j$, and hence

$$\hat{\gamma} = \frac{1}{k^2} \sum_{1 \leq i,j \leq k} |D_i - D_j|$$

Then the test statistic can be defined as follows:

$$T = \frac{1}{k^2} \sum_{1 \leq i,j \leq k} |\Delta_i - \Delta_j|, \tag{3.2.3}$$

where $\Delta_i := \sqrt{\frac{n}{k}}\ D_i$, or

$$T = \frac{1}{k^2} \sum_{i=1}^{k} (4i - 2(k+1))\, \Delta_{i:k}, \tag{3.2.4}$$

where $\Delta_{1:k} \leq \cdots \leq \Delta_{k:k}$ are ordered values of Δ_i. With the function $\mathcal{T}_k : \mathbb{R}^k \mapsto \mathbb{R}$ defined by

$$\mathcal{T}_k(x_1, \ldots, x_k) := \frac{1}{k^2} \sum_{i=1}^{k} (4i - 2(k+1))\, x_{i:k},$$

we rewrite formula (3.2.3) as follows:

$$T = \mathcal{T}_k(\Delta_1, \ldots, \Delta_k), \tag{3.2.5}$$

3.3 Asymptotic Properties

Now, we want to obtain the asymptotic distribution of the test statistic T under H_0. Under the following assumptions (Brazauskas *et al.*, 2007; Jones *et al.*, 2006),

(A1) The weight function J is continuous on the interval $(0, 1)$, except possibly at a finite number of points at which F_i^{-1} is continuous,

(A2) There exist $a, b > 1/2$ and $c < \infty$ such that $|J(t)| \leq ct^{a-1}(1-t)^{b-1}$ on the interval $(0, 1)$,

(A3) The moment $\mathbb{E}[|X_1(i)|^p]$ is finite for some p such that $p > 1/(a - 1/2)$ and $p > 1/(b - 1/2)$, it can be shown that, when $n \longrightarrow \infty$,

$$\sqrt{n}(\hat{R}_i - R_i) \longrightarrow_d \sigma_i G_i, \quad 1 \le i \le k$$

where $\sigma_i^2 := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F_i(x \wedge y) - F_i(x)F_i(y)) \, J(F_i(x))J(F_i(y))dxdy$ and $G_i \sim \mathcal{N}(0,1)$. For $1 \le i, j \le k$,

$$\mathrm{Cov}(G_i, G_j) = \sigma_i^{-1}\sigma_j^{-1}\mathrm{Cov}(A_i(X_1(i)), A_j(X_1(j))),$$

where

$$A_i(y) := -\int_{-\infty}^{\infty} \Big(\mathbb{I}\{y \le x\} - F_i(x)\Big) J(F_i(x))dx. \tag{3.3.1}$$

Then we have $\Delta_i \longrightarrow_d \frac{1}{\sqrt{k}}\sigma_i G_i$, and hence by equation (3.2.5)

$$T \longrightarrow_d \mathcal{T}_k\left(\frac{1}{\sqrt{k}}\sigma_1 G_1, \dots, \frac{1}{\sqrt{k}}\sigma_k G_k\right).$$

Further, it is proved that the asymptotic power of T under the alternative hypothesis H_{A} is 1. That is under H_{A},

$$\begin{aligned} T = \;& \frac{1}{k^2}\sqrt{\frac{n}{k}} \sum_{1 \le i,j \le k} |(D_i - D_j) + (R_i - R_j)| \\ \ge \;& -\frac{1}{k^2}\sqrt{\frac{n}{k}} \sum_{1 \le i,j \le k} |D_i - D_j| + \frac{1}{k^2}\sqrt{\frac{n}{k}} \sum_{1 \le i,j \le k} |R_i - R_j| \end{aligned}$$

The first summand on the right-hand side has a non-degenerate distribution, while $\sum_{1 \le i,j \le k} |R_i - R_j| \ge 0$ under H_{A}. Therefore, the asymptotic power is 1.

3.4 Decision Making

Brazauskas *et al.* (2007) suggested a bootstrap approximation to the critical value of the test. For this study, we cannot use the exact same procedure, because we consider dependent random variables. We first introduce a simple bootstrap resampling technique to replicate the original sample. But as further research we can check some other resampling techniques, such as blocked bootstrap (overlapping and nonoverlapping) as explained in Lahiri (2003), to find the bootstrap estimate of the critical value. For $1 \leq j \leq n$, let $(X_j(1), \ldots, X_j(k))$ be the jth realization of the dependent random vector $(X(1), \ldots, X(k))$. Then we obtain the bootstrap samples (n samples), which we denote by $(X_l^*(1), \ldots, X_l^*(k))$, such that

$$(X_l^*(1), \ldots, X_l^*(k)) = (X_j(1), \ldots, X_j(k))$$

for each $1 \leq l \leq n$ and for some randomly selected $1 \leq j \leq n$. Then we obtain the bootstrap estimate, denoted by $\hat{R}_i^*$ of $\hat{R}_i$ for every $1 \leq i \leq k$ by replacing $X_{m:n}(i)$ with $X_{m:n}^*(i)$ in formula (3.2.2). After that, we estimate the Gini index using the following relationship:

$$\hat{\gamma}^* = \frac{1}{k^2} \sum_{1 \leq i,j \leq k} |D_i^* - D_j^*|,$$

where $D_i^* = \hat{R}_i^* - \hat{R}_i, i = 1, \ldots, k$. Then the bootstrap version of the test statistic can be defined as follows:

$$T^* := \sqrt{\frac{n}{k}}\ \hat{\gamma}^*.$$

The bootstrap estimate of the critical value of the test is the $(1-\alpha)$ quantile of the T^*, denoted by $x_\alpha[T^*]$. That quantile can be found by repeating the above process sufficient number of times, denoted by B, and by finding the $\lfloor B(1-\alpha) \rfloor$th order statistic of the bootstrap replicates $\left\{T^{*(1)}, \ldots, T^{*(B)}\right\}$ of the test statistic T. Once we find the approximated critical value $x_\alpha[T^*]$, we reject the null hypothesis H_0 in favor of the alternative hypothesis H_{A} if the actual value of the test statistic T (the value obtained from the original samples) exceeds the approximated critical value $x_\alpha[T^*]$. Otherwise, we do not reject H_0.

Chapter 4

Simulation Study

4.1 Study Objectives

Since the sampling distribution of the test statistic does not have a manageable closed form expression, we use Monte Carlo simulations to investigate how the performance of the test changes when insurance portfolios are dependent. More specifically, we are interested in quantifying the relationship between the power of the test and the strength of portfolio dependence, for selected types of alternatives.

We first generate three dependent portfolios of insurance losses such that they are either equally risky (H_0 setting) or unequally risky (H_{A} setting), according to a fixed risk measure. For this study, we choose MEAN, PHT, and CTE as the risk measures (see Examples 2.2.1, 2.2.2, and 2.2.3). We then perform the hypothesis test of Section 3.1 using the generated portfolios and compute its proportion of rejections. (Such a proportion estimates the nominal level of significance under H_0 and the power of the test under H_{A}.) By executing this process for the four types of dependence listed in Section 2.1 (negative dependence, zero dependence, moderate positive dependence, and strong positive dependence), we obtain the proportion of rejections corresponding to each of the dependence structures. Specific parameters and other details of the study design are described in Sections 4.2 and 4.3.

4.2 Marginal Distributions

For generation of insurance portfolios with specified riskiness, we follow the simulation studies of Brazauskas and Kaiser (2004), Kaiser and Brazauskas (2006), Brazauskas, Jones, Puri, and Zitikis (2007) and choose the following three parametric families:

- *Exponential* with the cdf

$$F_1(x) = 1 - e^{-(x-x_0)/\theta}, \qquad x > x_0,\ \theta > 0. \tag{4.2.1}$$

- *Pareto* with the cdf

$$F_2(x) = 1 - (x_0/x)^{\beta}, \qquad x > x_0,\ \beta > 0. \tag{4.2.2}$$

- *Lognormal* with the cdf

$$F_3(x) = \Phi\Big(\log(x - x_0) - \mu\Big), \qquad x > x_0,\ -\infty < \mu < \infty, \tag{4.2.3}$$

  where $\Phi(\cdot)$ denotes the standard normal cdf.

The parameter x_0 in the above distributions can be interpreted as a deductible or a retention level of an insurance policy. (Note that due to x_0, the distributions F_1, F_2, and F_3 have the same support.) Although in general x_0 could be any positive real number, for this study we set $x_0 = 1$. The other parameters θ, β, and μ are selected in such a way that the cdfs F_1, F_2, and F_3 follow the hypothesized portfolio riskiness with respect to a fixed risk measure. In particular, if they are equally risky (under H_0), then they must satisfy the equation

$$R[F_1] = R[F_2] = R[F_3], \tag{4.2.4}$$

where $R[\cdot]$ represents either MEAN, PHT, or CTE. Evaluation of these measures for the distributions F_1, F_2, F_3 yields the following expressions of (4.2.4).

- For the MEAN risk measure (when $R[F_i] = \text{MEAN}[F_i]$):

$$x_0 + \theta \;=\; \frac{x_0\beta}{\beta - 1} \;=\; x_0 + e^{\mu+0.5}. \tag{4.2.5}$$

- For the PHT risk measure (when $R[F_i] = \text{PHT}[F_i]$):

$$x_0 + \frac{\theta}{r} \;=\; x_0 + \frac{x_0}{r\beta - 1} \;=\; x_0 + C_r e^{\mu}, \tag{4.2.6}$$

  where for fixed r, the integral $C_r = \int_{-\infty}^{\infty} (1 - \Phi(z))^r \, e^z \, dz$ is found numerically. For example, as reported by Brazauskas and Kaiser (2004), $C_{0.55} = 3.896$, $C_{0.70} = 2.665$, $C_{0.85} = 2.030$, $C_{0.95} = 1.758$. Note that when $r = 1$, the PHT measure becomes the MEAN.

- For the CTE risk measure (when $R[F_i] = \text{CTE}[F_i]$):

$$\begin{aligned} x_0 - \theta(\log(1-t) - 1) &= \frac{x_0\beta}{\beta - 1}(1-t)^{-1/\beta} = \\ & x_0 + \frac{1}{1-t}e^{\mu+0.5}\Phi(1 - \Phi^{-1}(t)). \end{aligned} \quad (4.2.7)$$

Note that when $t = 0$, the CTE measure becomes the MEAN.

For the simulation study we fix $x_0 = 1$ and $\beta = 5.5$, and then compute the corresponding values of θ and μ for each risk measure. Table 4.1 provides all distribution related parameters under H_0, which are calculated using equations (4.2.5), (4.2.6), and (4.2.7).

Table 4.1: The distribution related parameters under H_0.

Risk Measure	*Parametric Distribution*	*Distribution-Related Parameters Under H_0*: $H_0 : R[F_1] = R[F_2] = R[F_3]$
MEAN	Exponential	$x_0 = 1, \theta = 0.222$
	Pareto	$x_0 = 1, \beta = 5.5$
	Lognormal	$x_0 = 1, \mu = -2.004, \sigma = 1$
PHT ($r = 0.85$)	Exponential	$x_0 = 1, \theta = 0.231$
	Pareto	$x_0 = 1, \beta = 5.5$
	Lognormal	$x_0 = 1, \mu = -2.010, \sigma = 1$
CTE ($t = 0.75$)	Exponential	$x_0 = 1, \theta = 0.240$
	Pareto	$x_0 = 1, \beta = 5.5$
	Lognormal	$x_0 = 1, \mu = -1.978, \sigma = 1$

Under H_A, the riskiness of portfolios can be unequal in numerous ways. In this study, we consider the following two types of alternatives:

- Two portfolios are equally risky but the third one differs; that is,

$$R[F_1^*] = c_* R[F_1], \quad R[F_2^*] = R[F_2], \quad R[F_3^*] = R[F_3], \tag{4.2.8}$$

  where F_1^*, F_2^*, and F_3^* are parametric distributions of portfolios under this alternative, $c_* \neq 1$, and $R[F_1] = R[F_2] = R[F_3]$.

- Relative riskiness of all three portfolios is equally-spaced; that is,

$$R[F_1^{**}] = c_{**} R[F_1], \quad R[F_2^{**}] = R[F_2], \quad R[F_3^{**}] = c_{**}^2 R[F_3], \tag{4.2.9}$$

  where F_1^{**}, F_2^{**}, and F_3^{**} are parametric distributions of portfolios under this alternative, $c_{**} > 1$, and $R[F_1] = R[F_2] = R[F_3]$.

To simulate these scenarios, we choose parameters θ and μ to be identical to their values under H_0. Also, constants c_* and c_{**} are such that $c_* = 0.85, 0.90, 0.95$, $1.05, 1.10, 1.15, 1.25$ and $c_{**} = 1.05, 1.10$,$1.15, 1.20, 1.25$. The remaining distribution related parameters are derived from equations (4.2.8) and (4.2.9), and their values or formulas are presented in Table 4.2 and 4.3.

Table 4.2: The distribution related parameters under H_{A} : First type of alternatives.

Risk Measure	*Parametric Distribution*	*Distribution-Related Parameters Under H_A* $R[F_1^*] = c_* R[F_2^*]$ and $R[F_2^*] = R[F_3^*]$
MEAN	Exponential	$x_0 = 1, \theta^* = x_0(c_* - 1) + c_*\theta$
	Pareto	$x_0 = 1, \beta = 5.5$
	Lognormal	$x_0 = 1, \mu = -2.004, \sigma = 1$
PHT $(r = 0.85)$	Exponential	$x_0 = 1, \theta^* = x_0 r(c_* - 1) + c_*\theta$
	Pareto	$x_0 = 1, \beta = 5.5$
	Lognormal	$x_0 = 1, \mu = -2.010, \sigma = 1$
CTE $(t = 0.75)$	Exponential	$x_0 = 1, \theta^* = \frac{x_0(c_*-1)}{1-\log(1-t)} + c_*\theta$
	Pareto	$x_0 = 1, \beta = 5.5$
	Lognormal	$x_0 = 1, \mu = -1.978, \sigma = 1$

Table 4.3: The distribution related parameters under H_{A}: Second type of alternatives.

Risk Measure	*Parametric Distribution*	*Distribution-Related Parameters Under H_A* $R[F_1^{**}] = c_{**} R[F_2^{**}]$ and $R[F_3^{**}] = c_{**}^2 R[F_2^{**}]$
MEAN	Exponential	$x_0 = 1, \theta^{**} = x_0(c_{**} - 1) + c_{**}\theta$
	Pareto	$x_0 = 1, \beta = 5.5$
	Lognormal	$x_0 = 1, \mu^{**} = \log(x_0(c_{**}^2 - 1) + c_{**}^2 e^{\mu+0.5}) - 0.5$
PHT $(r = 0.85)$	Exponential	$x_0 = 1, \theta^{**} = x_0 r(c_{**} - 1) + c_{**}\theta$
	Pareto	$x_0 = 1, \beta = 5.5$
	Lognormal	$x_0 = 1, \mu^{**} = \log(\frac{x_0(c_{**}^2-1)}{C_r} + c_{**}^2 e^{\mu})$
CTE $(t = 0.75)$	Exponential	$x_0 = 1, \theta^{**} = \frac{x_0(c_{**}-1)}{1-\log(1-t)} + c_{**}\theta$
	Pareto	$x_0 = 1, \beta = 5.5$
	Lognormal	$x_0 = 1, \mu^{**} = \log(\frac{x_0(1-t)(c_{**}^2-1)}{\Phi(1-\Phi^{-1}(t))} + c_{**}^2 e^{\mu+0.5}) - 0.5$

4.3 Dependence Structures

This section presents algorithms and major steps for generation of dependent portfolios with exponential, Pareto, and lognormal margins and the dependence structures specified by the correlation matrices of Section 2.1. Briefly, a key idea is to use the meta-t_ν distribution which is a multivariate distribution with arbitrary margins and the dependence structure governed by t copula. In our examples, the degrees of freedom parameter is either $\nu = 3$ or $\nu \to \infty$ (the latter case corresponds to the meta-Gaussian distribution). Specifically, we implement the following three-step procedure:

> **Step 1.** For a fixed risk measure and a fixed scenario of riskiness, we first generate a random realization of the trivariate variable t_ν, with the location vector $\mathbf{0}$ and the correlation matrix Σ (examples of which are specified in Section 2.1). The sample size of each margin is n, and we denote this variable as $\mathbf{Y} = (Y_1, Y_2, Y_3)$.
>
> **Step 2.** Next, we transform $\mathbf{Y}$ into $\mathbf{U}$ such that $U_i = G_\nu(Y_i)$ for $i = 1, 2, 3$, where G_ν is the cdf of the standard t_ν variable (i.e., with location 0 and scale 1). The distribution of $\mathbf{U}$ is the trivariate t copula with the correlation matrix Σ.
>
> **Step 3.** Finally, as Theorem 2.1.8 ensures, the quantile transformation of the uniform margins returns the output with the desired probabilistic features. That is, the trivariate vector $\mathbf{X} = (X_1, X_2, X_3) =$

$\left(F_1^{-1}(U_1),\ F_2^{-1}(U_2),\ F_3^{-1}(U_3)\right)$, where

$$\begin{aligned} F_1^{-1}(u) &= \ x_0 - \theta \log(1-u), \\ F_2^{-1}(u) &= \ x_0(1-u)^{-1/\beta}, \\ F_3^{-1}(u) &= \ x_0 + \exp\left(\Phi^{-1}(u) + \mu\right), \end{aligned}$$

represents portfolios X_1, X_2, X_3 with marginal cdfs F_1, F_2, F_3, defined by (4.2.1)–(4.2.3), and their interdependence governed by t copula with the correlation matrix Σ.

Further, since t copula is fully characterized by its correlation matrix Σ and df, one can easily see that setting Σ equal to $\Sigma_1, \Sigma_2, \Sigma_3$, or Σ_4 (see Section 2.1) in Step 1 produces portfolio realizations with negative dependence, zero dependence, moderate positive dependence, or strong positive dependence, respectively. Also, to generate equally and unequally risky portfolios, we change the parameters of the quantile functions according to the specifications of Tables 4.1, 4.2, and 4.3, respectively.

Finally, while Steps 2 and 3 are straightforward transformations of random variables, Step 1 requires a more careful explanation. For Σ's with non-diagonal elements strictly less than 1, we generate the trivariate variable t_ν (with the location vector $\mathbf{0}$) by implementing Algorithm 5.2 of Embrechts, Lindskog, and McNeil (2003):

(a) Find the Cholesky decomposition M of Σ.

(b) Simulate three independent standard normal random variables Z_1, Z_2, Z_3.

(c) Simulate a random variable V from χ^2_ν that is independent of $\mathbf{Z} = (Z_1, Z_2, Z_3)$.

(d) Then $\mathbf{Y} = \sqrt{\nu/V}\, M\mathbf{Z}$ is the trivariate t_ν variable with location $\mathbf{0}$ and correlation Σ.

In the case when $\nu \to \infty$, the (c) step can be skipped and the transformation of variables in (d) replaced with $\mathbf{Y} = M\mathbf{Z}$. This results in the trivariate Gaussian variable with location $\mathbf{0}$ and correlation Σ. In addition, for commonotonic cases (e.g., Σ_4 in Section 2.1), the tail-dependence differences between the t and Gaussian copulas vanish (see McNeil *et al.*, 2005, Section 5.3.1). Thus the strong positively dependent portfolios can be generated by ignoring Steps 1 and 2 and modifying Step 3 as follows: simulate a standard uniform random variable U and then compute $\mathbf{X} = \left(F_1^{-1}(U), F_2^{-1}(U), F_3^{-1}(U)\right)$, where $F_1^{-1}, F_2^{-1}, F_3^{-1}$ are defined as in Step 3 above (see McNeil *et al.*, 2005, Proposition 5.16).

4.4 Numerical Findings and Observations

Once a set of portfolios is generated then they are resampled according to the bootstrap procedure of Section 3.4, an α-level test is performed, and its decision—reject H_0 or not—is recorded. This procedure is repeated 5000 times, for each of the three risk measures, four dependence structures, and for each of the hypothesized scenarios. Using the recorded 5000 decisions for the tests based on the MEAN, PHT, and CTE measures, respectively, we estimate the proportion $\widehat{p}$ of test's rejections. Under H_0, if $\widehat{p}$ falls within the 99% confidence interval $\alpha \pm z_{0.005}\sqrt{\alpha(1-\alpha)/5000}$, where $z_{0.005}$ is a critical value of the standard normal variable, then the test performs as expected. If $\widehat{p}$ exceeds the upper bound of the interval, then the test is labeled as liberal. And if it is below the lower bound, then the test is called conservative. The study is performed for the following choices of simulation parameters:

- *Level of significance*: $\alpha = 0.01,\ 0.05,\ 0.10$.
- *Sample size*: $n = 50,\ 100,\ 200$.
- *Number of bootstrap samples*: $B = 1000$.

Our simulation results are summarized in Table 4.4, where probabilities of type I error are reported, as well as in Figures 4.1 and 4.2, where estimated power curves are plotted. Specifically, we notice from Table 4.4 that in the presence of strong positive dependence (comonotonicity), the probability of the type I error exceeds the nominal level several times, sometimes even more than four times (see, e.g., the entries for $\alpha = 0.01$), for all the risk measures under consideration. This means that the test is very liberal under this scenario of dependence, which is most extreme. For the less extreme strengths of dependence, however, the results are mixed. That is, they depend on the chosen risk measure (MEAN is never liberal, PHT almost always, and CTE sometimes), sample size (liberal performances are most common for $n = 50$, less for $n = 100$, and least for $n = 200$), and even on the test's significance level (for $\alpha = 0.10$, the bold entries are most frequent, but their frequency declines as α decreases). Further, outside of the comonotonic case, there is no statistical evidence to suggest that the strength of dependence monotonically affects the test's level. Finally, except for several borderline cases, the effect of tail dependence is also undetectable (compare the corresponding entries for $\nu = 3$ and $\nu \to \infty$).

Table 4.4: Estimated probabilities of the type I error of the tests based on the MEAN, PHT, CTE measures, for selected n, α, ν, and various dependence structures.

n	α	*Risk Measure*	*Dependence Structure (characterized by Σ_i's of Section 2.1)*							
			Negative		*Zero*		*Mod. Positive*		*Strong Positive*	
			$\nu = 3$	$\nu \to \infty$	$\nu = 3$	$\nu \to \infty$	$\nu = 3$	$\nu \to \infty$	$\nu = 3$	$\nu \to \infty$
50	0.01	MEAN	0.007	0.008	0.008	0.007	0.008	0.008	0.013	**0.014**
		PHT ($r = 0.85$)	0.013	**0.014**	**0.015**	0.013	**0.015**	0.013	**0.036**	**0.038**
		CTE ($t = 0.75$)	0.012	**0.014**	**0.016**	0.012	0.011	0.013	**0.019**	**0.026**
	0.05	MEAN	0.049	0.052	0.052	0.049	0.044	0.044	**0.063**	**0.059**
		PHT ($r = 0.85$)	**0.065**	**0.070**	**0.074**	**0.065**	**0.068**	**0.069**	**0.115**	**0.116**
		CTE ($t = 0.75$)	**0.063**	**0.065**	**0.063**	**0.063**	**0.060**	0.058	**0.080**	**0.085**
	0.10	MEAN	0.106	0.111	**0.115**	0.107	0.104	0.103	**0.124**	**0.125**
		PHT ($r = 0.85$)	**0.130**	**0.130**	**0.141**	**0.135**	**0.138**	**0.140**	**0.192**	**0.193**
		CTE ($t = 0.75$)	**0.124**	**0.120**	**0.129**	**0.126**	**0.118**	**0.121**	**0.140**	**0.154**
100	0.01	MEAN	0.009	0.008	0.009	0.011	0.008	0.010	**0.015**	0.012
		PHT ($r = 0.85$)	**0.015**	**0.014**	**0.016**	**0.018**	**0.014**	**0.017**	**0.035**	**0.040**
		CTE ($t = 0.75$)	0.011	0.011	0.013	0.013	0.009	0.012	**0.020**	**0.022**
	0.05	MEAN	0.049	0.050	0.050	0.050	0.048	0.051	**0.062**	**0.060**
		PHT ($r = 0.85$)	**0.064**	**0.069**	**0.065**	**0.069**	**0.070**	**0.070**	**0.110**	**0.110**
		CTE ($t = 0.75$)	0.056	0.057	0.054	**0.059**	0.055	0.057	**0.074**	**0.074**
	0.10	MEAN	0.100	0.104	0.098	0.106	0.104	0.105	**0.118**	**0.119**
		PHT ($r = 0.85$)	**0.128**	**0.126**	**0.123**	**0.128**	**0.141**	**0.131**	**0.182**	**0.189**
		CTE ($t = 0.75$)	**0.115**	**0.114**	**0.115**	**0.115**	**0.114**	0.111	**0.130**	**0.136**
200	0.01	MEAN	0.007	0.008	0.010	0.010	0.009	0.010	**0.014**	**0.015**
		PHT ($r = 0.85$)	0.011	**0.015**	**0.015**	**0.018**	**0.015**	**0.017**	**0.035**	**0.041**
		CTE ($t = 0.75$)	0.008	0.010	0.010	**0.014**	0.011	0.011	**0.016**	**0.021**
	0.05	MEAN	0.045	0.049	0.046	0.051	0.047	0.048	0.051	**0.057**
		PHT ($r = 0.85$)	0.058	**0.062**	**0.060**	**0.063**	**0.064**	**0.064**	**0.092**	**0.105**
		CTE ($t = 0.75$)	0.053	0.053	0.048	0.056	0.050	0.050	**0.061**	**0.074**
	0.10	MEAN	0.096	0.097	0.102	0.106	0.098	0.097	0.101	**0.112**
		PHT ($r = 0.85$)	**0.121**	**0.120**	**0.119**	**0.125**	**0.126**	**0.120**	**0.156**	**0.166**
		CTE ($t = 0.75$)	0.109	0.111	0.105	**0.115**	0.105	0.103	0.111	**0.132**

NOTE: The 99% margins of error are: ± 0.004 (for $\alpha = 0.01$), ± 0.008 (for $\alpha = 0.05$), ± 0.011 (for $\alpha = 0.10$). The bold entries correspond to the cases when the test performance is liberal.

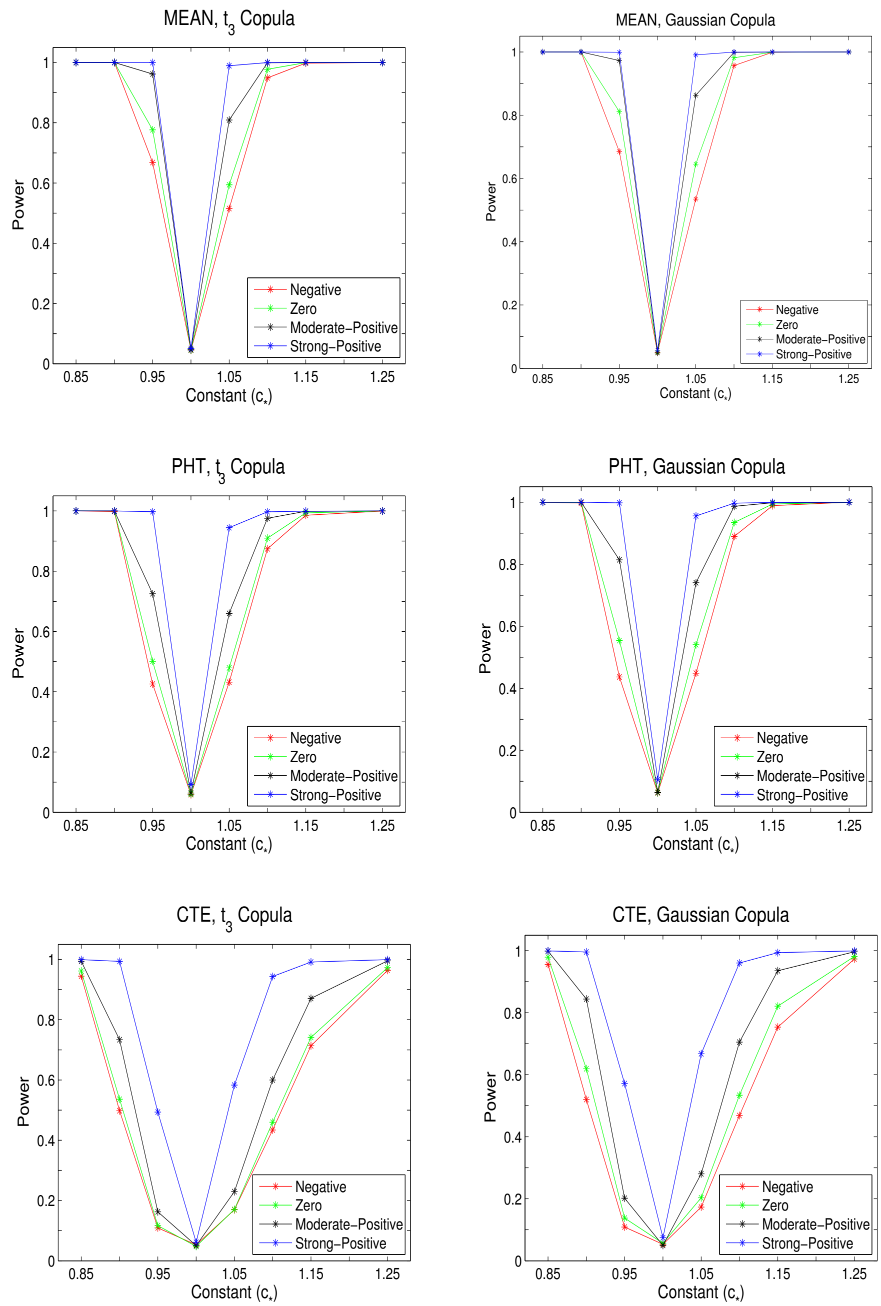

Figure 4.1: The first type of alternatives. Estimated power curves of the tests based on the MEAN, PHT, CTE measures, for various dependence structures, $n = 200$, and $\alpha = 0.05$. Left column: t_3 copulas. Right column: Gaussian copulas.

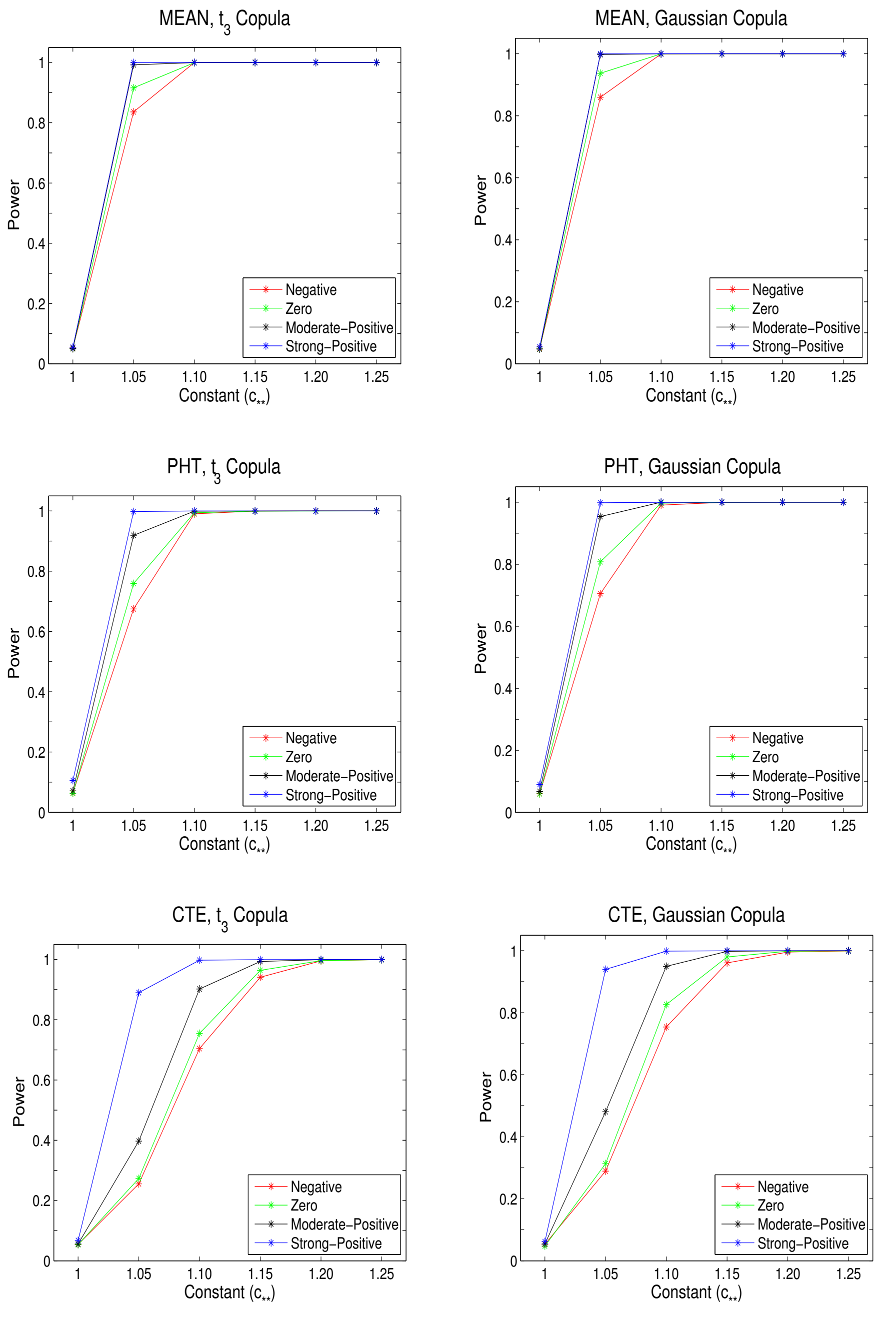

Figure 4.2: The second type of alternatives. Estimated power curves of the tests based on the MEAN, PHT, CTE measures, for various dependence structures, $n = 200$, and $\alpha = 0.05$. Left column: t_3 copulas. Right column: Gaussian copulas.

Figures 4.1 and 4.2 provide power estimates against the two types of alternatives described above, for $n = 200$ and $\alpha = 0.05$ (see the Appendix B figures for other sample sizes and significance levels). Similar to the type I error investigations, we notice that the power of the test is uniformly highest in the strong positive dependence case, for all risk measures and both types of copulas. Of course, this finding is not unexpected because the test exceeds the nominal level under H_0 and its power curve is simply shifted across all scenarios of riskiness. We also notice that the power of the test depends on the underlying risk measures. That is, all things being equal, the test is more powerful for the 'light' measure (such as the MEAN) than for the 'heavy' one (such as the PHT or CTE). There is no effect of tail dependence on the power curves, i.e., t_3 and Gaussian copulas produce similar power curves, but there is some effect of the strength of dependence. In particular, while negative dependence slightly decreases the power of the test when compared to the zero dependence case, the positive dependence improves the test's performance. Other features of the estimated power curves are typical: the test becomes more powerful as c_* (c_{**}) moves further away from $c_* = 1$ ($c_{**} = 1$), i.e., when data go deeper into the alternative. Further, comparison of the two types of alternatives reveals that the test is more powerful against the second type of alternatives, which can be anticipated because under the second scenario the differences in portfolio riskiness are more pronounced. Finally, we conclude that the test—which was designed for independent portfolios—performs adequately when portfolios are dependent, and it will successfully detect, with the probability substantially above 0.50, the differences in portfolio riskiness of at least 15% (corresponding to $c_* \leq 0.85$ or $c_* \geq 1.15$, and $c_{**} \geq 1.15$) for portfolios of

$n \geq 200$ losses. Of course, a caveat to this conclusion is the comonotonic case which requires a separate analysis.

Some natural questions arise based on the simulation study: in general, does the power of the hypothesis test increase as the dependence among the portfolios become positive? or does the power associated with strong positive dependence set an upper bound for the power of the test associated with the other dependences? Theoretical explanations for these problems have been achieved for some special cases (see Chapter 5). The remaining questions will be deferred to future research.

4.5 Practical Considerations

In this section, we illustrate how to apply the findings of Section 4.4 in practice. Using the tornado damage data of Brooks and Doswell (2001), normalized values of which (i.e., data adjusted for wealth and inflation) are available in Table A.3 of Brazauskas, Jones, Puri, and Zitikis (2007), we reanalyze the real-data example of the latter paper by investigating potential effects of portfolio dependence on the decision making procedure.

The given data set was sorted in two ways: (*i*) by time period, and (*ii*) by census region. The first way of sorting yields three time periods—from 1890 to 1929, 1930-1969, and 1970-1999—with the respective number of losses $n_{1890-1929} = 42$, $n_{1930-1969} = 57$, and $n_{1970-1999} = 38$. Thus three portfolios of tornado damages can be formed and their riskiness compared according to a selected risk measure $R[F]$. In the second case, the portfolios are formed for two regions—Midwest and South—with the respective sample sizes $n_{\text{midwest}} = 47$ and $n_{\text{south}} = 86$. (The data set also contains a third region, Northeast, but it has only four observations, which is way too small to assure valid statistical inference.) Both hypotheses (that the portfolios are equally risky) were tested by applying the procedure of Section 3.4. We used the same risk measures as in the simulation study: MEAN, PHT ($r = 0.85$), and CTE ($t = 0.75$). Also, $B = 1000$ bootstrap samples were generated to calculate the critical values at 1%, 5%, and 10% levels of significance. Tables 4.5 and 4.6 provide summary estimates and decisions of the tornado damage data by time period and region, respectively.

Table 4.5: Estimates and decisions for analysis of the tornado damage data by time period.

	MEAN	PHT ($r = 0.85$)	CTE ($t = 0.75$)
$\widehat{R}_{1890\text{-}1929}$	7,120	9,531	23,549
$\widehat{R}_{1930\text{-}1969}$	7,244	8,625	18,067
$\widehat{R}_{1970\text{-}1999}$	11,693	13,885	30,832
$\widehat{\gamma}$	2,032	2,342	5,673
$x_{0.10}[\widehat{\gamma}^*]$	2,497	3,016	8,281
$x_{0.05}[\widehat{\gamma}^*]$	2,828	3,407	9,507
$x_{0.01}[\widehat{\gamma}^*]$	3,757	4,482	13,006
Reject H_0 (at level α)?	No (at $\alpha \leq 0.10$)	No (at $\alpha \leq 0.10$)	No (at $\alpha \leq 0.10$)

NOTE: $n_{1890-1929} = 42$, $n_{1930-1969} = 57$, and $n_{1970-1999} = 38$.

According to the estimates of MEAN, PHT ($r = 0.85$), and CTE ($t = 0.75$) in Table 4.5, the third portfolio (period 1970–1999) seems riskier than the other two. (Notice that a similar observation can be gleaned from data in Figure 1.1.) Also, since sample sizes for the three portfolios are not too different from each other and the riskiness of the first two periods is similar, one may suspect that the underlying situation belongs to the first type of alternatives of unequal riskiness (see Section 4.2). This, however, is just a coincidence because all three risk measures failed to reject the null hypothesis at all typical levels of significance ($\alpha = 0.01, 0.05, 0.10$). Hence for this problem, there is no need to consider possible effects of dependence since the statistical decision is not to reject H_0.

Table 4.6: Estimates and decisions for analysis of the tornado damage data by region.

	MEAN	PHT ($r = 0.85$)	CTE ($t = 0.75$)
$\widehat{R}_{\text{midwest}}$	12,287	14,819	31,315
$\widehat{R}_{\text{south}}$	5,787	7,381	16,884
$\widehat{\gamma}$	3,250	3,719	7,215
$x_{0.10}[\widehat{\gamma}^*]$	1,940	2,421	6,580
$x_{0.05}[\widehat{\gamma}^*]$	2,332	2,918	7,671
$x_{0.01}[\widehat{\gamma}^*]$	3,122	3,788	10,106
Reject H_0 (at level α)?	Yes ($\alpha \geq 0.01$)	Yes ($\alpha \geq 0.05$)	Yes ($\alpha = 0.10$)

NOTE: $n_{\text{midwest}} = 47$ and $n_{\text{south}} = 86$.

The results in Table 4.6 are less clear-cut and thus more interesting. Indeed, as the point estimates of all three risk measures suggest, the Midwest region is roughly twice as risky as the South. More formally, according to the MEAN measure, the difference is statistically significant at all typical levels of significance. And the PHT ($r = 0.85$) and CTE ($t = 0.75$) measures reject H_0 at $\alpha = 0.05, 0.10$ and $\alpha = 0.10$, respectively. Further, we need to check how sensitive these decisions are due to (potentially) misspecified portfolio dependence. Aside from the comonotonic case, the results of Section 4.4 suggest that the decision to reject H_0 at the significance level α will remain at that level as long as portfolios are compared according to the MEAN measure. For the PHT ($r = 0.85$) and CTE ($t = 0.75$) measures, a premium of 20%–40% has to be added to α. That is, in many practical situations, the actual probability of type I error for PHT ($r = 0.85$) and CTE ($t = 0.75$) can reach 1.20α to 1.40α. Finally, the comonotonic case—no matter how rare it may be—represents a perfect-storm scenario that can break down the test and easily yield probabilities for the type I error as high as 0.30 or even higher. Thus the user of the test should keep such a possibility in mind.

Chapter 5

Theoretical Properties of Gini Indexes

From the simulation study we acquired some evidence of a relationship between the power of the test and the dependence structure. That is, in the presence of positive dependence among the portfolios the test is more conservative for the risk measures under consideration. In order to explain the monotonicity of the test power function, with respect to the strength of dependence, we propose the following conjecture.

Conjecture 5.0.1. Let $(Y_1, \ldots, Y_k)$ follow a multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}$, i.e., $(Y_1, \ldots, Y_k) \sim MVN(\mathbf{0}, \mathbf{\Sigma})$. Then its Gini index $\frac{1}{k^2}\sum_{1\leq i,j\leq k}|Y_i - Y_j|$ decreases in the sense of usual stochastic order (see definition 2.3.1) as the covariance matrix $\mathbf{\Sigma}$ increases component-wise with diagonal elements fixed.

For notation convenience, denote

$$G(\mathbf{Y}) = \sum_{1\leq i,j\leq k} |Y_i - Y_j| \,. \tag{5.0.1}$$

Basically, Conjecture 5.0.1 aims to order Gini indexes of multivariate normal risks with the same marginals but different strength of dependence. Proving Conjecture 5.0.1 is a challenging task but gives us a theoretical explanation for the questions that we raised in Section 4.4. If we denote the asymptotic distribution function of the test statistic T under H_0 with a certain dependence structure by F_{T_0} and denote that of more positive dependence by $F_{T_0'}$, then the power of the test increases as the strength of positive dependence increases if

$$F_{T_0'}^{-1}(1-\alpha) \leq F_{T_0}^{-1}(1-\alpha) \ \text{ for } \ \alpha \in (0,1),$$

where $F_{T_0}^{-1}$ and $F_{T_0'}^{-1}$ are the quantile functions of T_0 and T_0' respectively.

By Proposition 2.3.2, the above inequality is true if $T_0' \leq_{st} T_0$. Therefore, comparing the test statistics, T, of portfolios with different dependence can be achieved by comparing the Gini indexes of multivariate normal risks. In this dissertation, we partially complete the task of proving Conjecture 5.0.1 and generalizes the conclusion to elliptical distributions, yet still leaves some open problems.

Besides its actuarial application, the comparison of Gini indexes of multivariate elliptical risks shows its own independent interest. Intuitively, Conjecture 5.0.1 suggests that $\mathbb{P}\left\{\frac{1}{k^2}\sum_{1\leq i,j\leq k}|Y_i - Y_j| \leq t\right\}$ increases as $\mathbf{\Sigma}$ increases for any $t \geq 0$ or $\mathbb{P}\left\{G(\mathbf{Y}) \leq t\right\}$ increases as $\mathbf{\Sigma}$ increases for any $t \geq 0$. In this sense, the study of Conjecture 5.0.1 falls into the scope of the problem of central concentration of elliptical distributions, which is formulated as follows: how the probability

$$\mathbb{P}_{\Sigma}(D) = \mathbb{P}\{(Y_1, \ldots, Y_k) \in D\} \tag{5.0.2}$$

changes according to the change of $\mathbf{\Sigma}$? Here $(Y_1, \ldots, Y_k)$ follows an elliptical distribution with mean $\mathbf{0}$ and dispersion matrix $\mathbf{\Sigma}$.

This problem was first studied by Slepain (1962), which states that if $(Y_1, \ldots, Y_k)$ follows a multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}$, then $\mathbb{P}_{\Sigma}(D)$ increases as $\mathbf{\Sigma}$ increases component-wise with diagonal elements remaining unchanged for any lower orthant set D. Later literature has generalized the study to elliptical distributions while regions of different shapes have been considered, such as upper orthant sets, rectangles, and convex and centrally symmetric regions. Interested readers are referred to Das Gupta *et al* (1972), Joe (1990), Eaton and Perlman (1991), and Anderson (1996). These studied all imposed certain assumptions on the structure of the covariance matrix. The results derived in this paper enriches the studies on this problem in the sense that it broadens the choice of the set D.

We start proving the Conjecture 5.0.1 from two dimensions and implement the proof for higher dimensions by imposing some assumptions. We can prove this two dimensions case without much effort, but for higher dimensions it gets more complicated as it is hard to visualize the geometric shapes.

5.1 2-Dimensional Gaussian Risks

We first use a geometric argument to show a special two dimensional version of Conjecture 5.0.1 and then prove a general two dimensional version using much simpler technique. Consider the following proposition. The geometric argument used in this section is motivated by Theorem A2 of Joe (1990), which establishes the concordance order between elliptical distributions. We shall restate the Conjecture 5.0.1 for $k = 2$.

Proposition 5.1.1. Let $(Y_1, Y_2) \sim BVN(0, 0, 1, 1, \rho)$. Then $G(\mathbf{Y})$ decreases in the sense of usual stochastic order as ρ increases.

Proof. By using Cholesky decomposition of variance-covariance matrix of (Y_1, Y_2), we obtain, $Y_1 = Z_1$ and $Y_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2$, where Z_1 and Z_2 are independent standard normal random variables. Then,

$$Pr(|Y_1 - Y_2| \leq t) = Pr\left(\frac{(1-\rho)Z_1 - t}{\sqrt{1-\rho^2}} \leq Z_2 \leq \frac{(1-\rho)Z_1 + t}{\sqrt{1-\rho^2}}\right). \tag{5.1.1}$$

Now, we want to show that the probability on the right hand side of (5.1.1) is increasing in ρ. By changing (Z_1, Z_2) to the polar coordinates and using the transformation

$$Z_1 = R\cos(\Theta) \quad \text{and} \quad Z_2 = R\sin(\Theta)$$

we obtain the joint density function of (R, Θ) as follows:

$$g_{R,\Theta}(r, \theta) = r f_{(Z_1,Z_2)}(z_1, z_2),$$

where $r > 0$, $0 \leq \theta \leq 2\pi$, and $f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{2\pi}\exp\{-\frac{1}{2}(z_1^2 + z_2^2)\}$ is the joint density function of (Z_1, Z_2). That is,

$$g_{R,\Theta}(r, \theta) = \frac{1}{2\pi} r e^{-r^2/2},$$

which implies that R and Θ are independent and Θ is uniform on $[0, 2\pi]$ and R^2 is exponential with mean $\frac{1}{2}$. Therefore, to prove that the probability on the right hand side of (5.1.1) is increasing in ρ, it suffices to prove that, for a given r, the length of the shaded arc of the circle $x^2 + y^2 = r^2$ in Figure 5.1 is increasing in ρ.

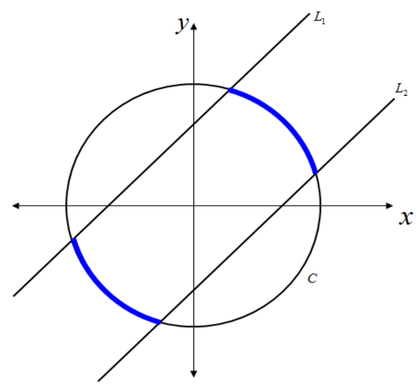

Figure 5.1: An illustrative diagram.

Let L_1 be the line with the equation $y = \frac{(1-\rho)x+t}{\sqrt{1-\rho^2}}$ and L_2 be the line with the equation $y = \frac{(1-\rho)x-t}{\sqrt{1-\rho^2}}$. Since the shaded arc length is directly proportional to the distance between the line L_1 and L_2, denoted by $d(L_1, L_2)$, we can easily obtain the desired result because $d(L_1, L_2) = \frac{\sqrt{2}t}{\sqrt{1-\rho}}$ is increasing in ρ. □

The reason for using this geometric proof is to put a foundation for the three dimensional proof. On the other hand, we can introduce a much simpler proof for more general version of Proposition 5.1.1 as follows.

Theorem 5.1.2. Let $(Y_1, Y_2) \sim BVN(0, 0, \sigma_1, \sigma_2, \rho)$. Then $G(\mathbf{Y})$ decreases in the sense of usual stochastic order as ρ increases.

Proof. We want to prove that $Pr\{|Y_1 - Y_2| \leq t\}$ increases in ρ for any $t \geq 0$.

$$\begin{aligned} E[Y_1 - Y_2] &= 0, \text{ and} \\ \text{Var}(Y_1 - Y_2) &= \text{Var}(Y_1) + \text{Var}(Y_2) - 2\text{Cov}(Y_1, Y_2), \\ &= \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho. \end{aligned}$$

Therefore, $|Y_1 - Y_2| \stackrel{d}{=} \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}\ |Z|$ where $Z \sim N(0, 1)$. The rest of the proof is obvious as $\sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}$ decreases in ρ.

□

Now, we move on to three dimensional version of Conjecture 5.0.1. As the matrix Σ gets complicated, checking the proof of the Conjecture 5.0.1 gets complicated. Therefore, we break down the problem and try to prove that Conjecture 5.0.1 is true for some special three dimensional Σ.

In the following proposition, we assume that only two random variables are independent of each other, but there is a dependency among the other random variables. Further, in the proof of the following theorem, we use the same initial procedure used in Proposition 5.1.1. In particular, we find Cholesky's decomposition and then transform the variables into the spherical coordinate system. As we used the length of arcs created by two parallel chords on a circle, in three dimensional case, we look at the surface area of a sphere created by a hexagon based cylinder.

5.2 3-Dimensional Gaussian Risks

Proposition 5.2.1. Let $(Y_1, Y_2, Y_3) \sim MVN(\mathbf{0}, \sigma^{\mathbf{2}}\mathbf{\Sigma})$. If,

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 & \rho_1 \\ 0 & 1 & \rho \\ \rho_1 & \rho & 1 \end{pmatrix} \text{ with } 0 \leq \rho_1 \leq \rho \leq 1 \text{ and } \rho + \rho_1 \leq 1,$$

then $G(\mathbf{Y})$ decreases in the sense of usual stochastic order as $\mathbf{\Sigma}$ increases in ρ.

Proof. See Appendix A. □

As another special 3-dimensional case of Conjecture 5.0.1, we shall prove that comonotonicity produces the stochastically smallest Gini index among all multivariate Gaussian risks with common marginals.

Proposition 5.2.2. Let $\mathbf{Y} = (Y_1, Y_2, Y_3)$ and $\mathbf{Y}' = (Y_1', Y_2', Y_3')$ follow multivariate normal distributions with mean $\mathbf{0}$ and common marginal distributions. If (Y_1', Y_2', Y_3') is comonotonic, then $G(\mathbf{Y}') \leq_{st} G(\mathbf{Y})$.

Proof. See Appendix A. □

5.3 More General Results

We say that a random vector $\mathbf{Y}$ is exchangeable if $\Big(Y_1, \ldots, Y_k\Big) \overset{d}{=} \Big(Y_{\Pi(1)}, \ldots, Y_{\Pi(k)}\Big)$ for any permutation $\Big(\Pi(1), \ldots, \Pi(k)\Big)$ of $(1, \ldots, k)$. For Gaussian random variable, we can explain the exchangeability using the correlation matrix. If the correlation matrix is equicorrelation matrix, we say that multivariate Gaussian random vector is exchangeable, i.e., $\Sigma = \rho \mathbf{1}_{k\times k} + (1-\rho)\mathbf{I}_{k\times k}$. Here we only talk about the correlation matrix, because the variances of all random variables should be the same in order to be exchangeable. We use this idea to prove the following proposition.

Proposition 5.3.1. Let $(Y_1, \ldots, Y_k) \sim MVN(\mathbf{0}, \mathbf{\Sigma})$. If $Y_1, \ldots, Y_k$ are exchangeable with common correlation ρ, then $G(\mathbf{Y})$ decreases in the sense of usual stochastic order as ρ increases.

Proof. If $(Y_1, \ldots, Y_k)$ is exchangeable with correlation coefficient ρ, then there exist $W \sim N(0,1)$ which is independent of $\mathbf{Z} \sim MVN(\mathbf{0}, \mathbf{I}_{k\times k})$, such that $Y_i = \sqrt{\rho}\, W + \sqrt{1-\rho}\, Z_i$ for all $1 \le i \le k$. Then $G(\mathbf{Y}) = \sqrt{1-\rho}\, G(\mathbf{Z})$, and the result follows as $(1-\rho)$ is decreasing in ρ. □

Proposition 5.3.2. Let $\mathbf{Y} = (Y_1, \ldots, Y_k)$ follow a multivariate normal distribution with mean $\mathbf{0}$ and positive definite covariance matrix $\mathbf{\Sigma} = (\sigma_{ij}) \in \mathbb{R}^{k\times k}$. If $\sigma_{1i} = \sigma_{2i}$ for all $i = 3, \ldots, k$ and $\sigma_{11} = \sigma_{22}$, then its Gini index $\frac{1}{k^2}\sum_{1\le i,j\le k} |Y_i - Y_j|$ decreases in the sense of usual stochastic order as σ_{12} increases.

Proposition 5.3.2 suggests that if we impose a conditional exchangeable structure on the multivariate normal random vector $\mathbf{Y}$, i.e., (Y_1, Y_2) is exchangeable conditioning on the remaining components, then $G(\mathbf{Y})$ is stochastically decreasing in $\mathrm{Cov}(Y_1, Y_2)$, which is one component of the covariance matrix.

Now, we look at a new structure of the covariance matrix. We first cite Theorem 4.1 of Eaton and Perlman (1991) below.

Lemma 5.3.3. Let $\mathbf{Y}_i \sim MVN(\mathbf{0}, \mathbf{\Sigma}_i)$ for $i = 1, 2$. If $\mathbf{\Sigma}_2 - \mathbf{\Sigma}_1$ is positive semidefinite, then $\mathbb{P}\{\mathbf{Y_2} \in D\} \le \mathbb{P}\{\mathbf{Y_1} \in D\}$ for any convex and centrally symmetric set D (i.e., $D = -D$).

Proposition 5.3.4. Let $\mathbf{Y} \sim MVN(\mathbf{0}, \mathbf{\Sigma}_Y)$ and $\mathbf{Y}' \sim MVN(\mathbf{0}, \mathbf{\Sigma}_{Y'})$. If there exists $a \in \mathbb{R}$ such that

$$a\mathbf{1}_{k\times k} + \mathbf{\Sigma}_Y - \mathbf{\Sigma}_{Y'} \qquad \text{is positive semidefinite,} \tag{5.3.1}$$

where $\mathbf{1}_{k\times k}$ denotes the $k \times k$ matrix with all entries equal to 1, then $G(\mathbf{Y}) \geq_{st} G(\mathbf{Y}')$.

Proof. See Appendix A. □

A matrix P is said to dominate another matrix Q if $P - Q$ is positive semidefinite. In this sense, condition (5.3.1) in Proposition 5.3.4 is referred to as "quasi" dominance. Intuitively, Lemma 5.3.3 indicates that the covariance matrix determines the degree of central concentration of a multivariate normal distribution. Specifically, the "smaller" the covariance matrix is, the more concentrated the normal random vector is on a convex and centrally symmetric region.

In this work, we focus on the comparison of dependence structure without changing marginals. That means the dispersion matrices we compare have the same diagonal elements. When taking difference, the diagonal elements become 0. In this sense, we do not expect one dispersion matrix to dominate another since the difference matrix is not positive semidefinite. Therefore, the dominance condition in Lemma 5.3.3 is relaxed to "quasi" dominance condition in Proposition 5.3.4 to deal with this situation.

Example 5.3.5. Examples satisfying condition (5.3.1).

(i) All the off-diagonal elements of the covariance matrix increase by same amount, i.e., $\mathbf{\Sigma}_{Y'} = \mathbf{\Sigma}_Y + \sigma(\mathbf{1}_{k\times k} - \mathbf{I}_k)$ with $\sigma > 0$. This includes the case that $\mathbf{Y}$ and $\mathbf{Y}'$ are both exchangeable. The conclusion of Proposition 5.3.4 for exchangeable $\mathbf{Y}$ and $\mathbf{Y}'$ has been verified from other approaches, see for example, Theorem 6.25 of Tong (1990).

(ii) The off-diagonal elements of the covariance on the i^{th} row and column increase by same amount, i.e., $\mathbf{\Sigma}_{Y'} = \mathbf{\Sigma}_{Y} + \sigma \sum_{j \neq i}(\mathbf{\Delta}_{ji} + \mathbf{\Delta}_{ij})$ with $\sigma > 0$, where $\mathbf{\Delta}_{ij}$ denotes the matrix with 1 on the (i, j) position and 0 on others.

Now, we generalize Propositions 5.2.1, 5.2.2, and 5.3.1–5.3.4 for elliptical risks using Proposition 2.1.10. We restate those propositions for elliptical distributions as follows. We show the proofs of Proposition 5.3.6 and omit the other proofs as they are similar to the proof of Proposition 5.3.6.

Proposition 5.3.6. Let $(Y_1, Y_2, Y_3) \sim EC_3(\mathbf{0}, \sigma^{\mathbf{2}}\mathbf{\Sigma}, \psi)$ with $\psi \in \Psi_\infty$. If,

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 & \rho_1 \\ 0 & 1 & \rho \\ \rho_1 & \rho & 1 \end{pmatrix} \text{ with } 0 \leq \rho_1 \leq \rho \leq 1 \text{ and } \rho + \rho_1 \leq 1,$$

then $G(\mathbf{Y})$ decreases in the sense of usual stochastic order as $\mathbf{\Sigma}$ increases in ρ.

Proof. According to Proposition 2.1.10, there exist $\mathbf{Z} \sim MVN(\mathbf{0}, \mathbf{\Sigma})$ and a random variable $R \geq 0$ independent of $\mathbf{Z}$ such that $\mathbf{Y} \overset{d}{=} R\,\mathbf{Z}$.

Note that for any given $r > 0$, $r\mathbf{Z}$ follows multivariate normal distributions with covariance matrices satisfying the condition in Proposition 5.2.1. Therefore, $[G(\mathbf{Y}) \mid R = r]$ decreases in the sense of usual stochastic order as ρ increases for all $r \geq 0$.

Since the usual stochastic order is closed under mixture (see Theorem 1.2.15 of Müller and Stoyan, 2002), $G(\mathbf{Y})$ decreases in the sense of usual stochastic order as ρ increases. $\square$

The proofs of the following propositions are similar to the proof of Proposition 5.3.6 and thus are omitted.

Proposition 5.3.7. Let $\mathbf{Y} = (Y_1, Y_2, Y_3)$ and $\mathbf{Y}' = ({Y'}_1, {Y'}_2, {Y'}_3)$ follow elliptical distributions with mean vector $\mathbf{0}$, common marginal distributions and a common generator $\psi \in \Psi_\infty$. If $\mathbf{Y}'$ is comonotonic, then $G(\mathbf{Y}') \leq_{st} G(\mathbf{Y})$.

Proposition 5.3.8. Let $(Y_1, \ldots, Y_k) \sim EC_k(\mathbf{0}, \mathbf{\Sigma}, \psi)$ with $\psi \in \Psi_\infty$. If $Y_1, \ldots, Y_k$ are exchangeable, then $G(\mathbf{Y})$ decreases in the sense of usual stochastic order as ρ increases.

Proposition 5.3.9. Let $\mathbf{Y} \sim EC_k(\mathbf{0}, \mathbf{\Sigma}, \psi)$ with $\mathbf{\Sigma} = (\sigma_{ij}) \in \mathbb{R}^{k \times k}$ and $\psi \in \Psi_\infty$. If $\sigma_{1i} = \sigma_{2i}$ for all $i = 3, \ldots, k$ and $\sigma_{11} = \sigma_{22}$, $\mathbb{P}\{G(\mathbf{Y}) \leq t\}$ is increasing in σ_{12} for any $t \geq 0$.

Proposition 5.3.10. Let $\mathbf{Y} \sim EC_k(\mathbf{0}, \mathbf{\Sigma}_Y, \psi)$ and $\mathbf{Y}' \sim EC_k(\mathbf{0}, \mathbf{\Sigma}_{Y'}, \psi)$ with $\psi \in \Psi_\infty$. If there exists $a \in \mathbb{R}$ such that

$$a\mathbf{1}_{k \times k} + \mathbf{\Sigma}_Y - \mathbf{\Sigma}_{Y'} \qquad \text{is positive semidefinite,}$$

where $\mathbf{1}_{k \times k}$ denotes the $k \times k$ matrix with all entries equal to 1, then $G(\mathbf{Y}) \geq_{st} G(\mathbf{Y}')$.

Chapter 6

Conclusions and Future Research

6.1 Concluding Remarks

In this dissertation, we have considered a hypothesis testing problem about the equality of risk measures using a nested L-statistic. Asymptotic and small-sample properties of the test have been studied by Brazauskas, Jones, Puri, and Zitikis (2007) under the assumption of independent insurance portfolios. Here, using Monte Carlo simulations, we have investigated the performance of the test when portfolios are dependent. We have concluded that the presence of strong positive dependence (comonotonicity) makes the test very liberal for the PHT, CTE, and MEAN risk measures, when marginal portfolios follow exponential, Pareto, and lognormal distributions and their interdependence is governed by the three-dimensional t and Gaussian copulas. For non-comonotonic scenarios of dependence, the test performs adequately, with its probabilities of type I error being on target for the MEAN measure and getting inflated by about 20% to 40% for the PHT and CTE measures. In addition, for the alternative hypotheses considered in this dissertation, we have not observed any significant effects of tail dependence, but detected some effect of the strength of dependence. In particular, while negative dependence slightly decreases the power of the test when compared to the zero dependence case, the positive dependence improves the test's performance. We have also demonstrated how to incorporate such findings into sensitivity analysis of the decisions by providing a real-data example.

Further, it is of interest to understand the mathematical phenomenon of how the power function of the test behaves due to changes in the correlation matrix that controls the interdependence of portfolios. This problem is related to the usual stochastic ordering of multivariate elliptical risks, and we have managed to order Gini indexes when the dispersion matrices follow special structures. Furthermore, we have demonstrated that among all dependence structures, comonotonicity produces the smallest Gini index in the sense of usual stochastic order. This explains the devastating effect of comonotonic case on the hypothesis test of Chapter 3. Apart from its usefulness in actuarial applications, the comparison of Gini indexes presented in this paper enriches the studies of the concentration of elliptical random vectors on convex centrally symmetric regions.

Finally, the results of this dissertation motivate open problems and generate several ideas for further research. First, to what extent can Gini indexes of multivariate elliptical risks be ordered in the sense of usual stochastic order? and does the conclusion still hold for higher dimensional risks with general elliptical distributions? Second, what is the relationship between the covariance of risks and the covariance of the empirical estimates of risk measures? Finally, one may abandon the idea of using the Gini index on risk measures and construct a completely different test. Some of these problems are discussed in more detail in Section 6.2 and 6.3.

6.2 Ordering of Multivariate Elliptical Risks

Over a hundred years ago, Corrado Gini introduced an index to measure concentration or inequality of incomes (see Gini, 1936, for English translation of the original article). It later became known as the Gini index and has been extensively studied in many field such as economics, insurance, finance, and statistics. This is a well-known tool in economics that is often used for measuring income inequalities. In insurance, the index and its modifications have been used to compare the riskiness of portfolios and to summarize insurance scores. At the intersection of insurance and statistics, for example, the index has been used for comparing distributions of risks and prices (see Frees *et al.*, 2011). The comparisons are usually based on insurance scores relative to price, also known as "relativities," that point to areas of potential discrepancies between risk and price distributions. After ordering both risks and prices based on relativities, one arrives at an ordered Lorenz curve that can be summarized using a Gini index. Interestingly, the Lorenz curve and Gini index defined via relativities can cope with adverse selection, help measure potential profit, and serve as useful tools in predictive modeling (for more information, see Frees *et al.*, 2014). Moreover, Lorenz curve and Lorenz order, the concepts closely related to Gini index, have been employed by Denuit and Vermandele (1999) to order reinsurance contracts. Therefore, we believe that devoting our time to study the behavior of the Gini index of dependent random variables is beneficial to those who are interested in studying the Gini index. Thus, we intend to continue the study of stochastic ordering of the Gini index on multivariate random variables in general.

6.3 Dependence of Risk Measures

In practice, we can obtain the covariance of the risks. Therefore, if we can develop the relationship between the covariance of risks and the covariance of the nonparametric estimators of risk measures it will be much easier to use the above mentioned ordering of the Gini indexes of multivariate normal risks. However, it is not easy to obtain the covariance matrix of the empirical estimators of the risk measures even by facilitating the asymptotic distributions of those estimators. So far, we can show that more positively dependent risks have more positively dependent risk measures, that is, they have a one-to-one correspondence. When the underlying risk measure is the mean, we can easily obtain that the covariance matrix of the losses is equal to the covariance matrix of the sample means (an estimator of the population mean). Moreover, when the risks are independent so are their risk measures. For other spectral risk measures, we have established some results and hope to study it further in our future research work. Studying these types of asymptotic behavior of estimators of risk measures is of interest to actuaries.

As discussed in Section 2.3, if $\mathbf{X}'$ with copula C' is more positive than $\mathbf{X}$ with copula C, then $C \leq_c C'$. By using Theorem 2.3.3 and properties $P1$ and $P2$ (see

Section 2.3), it is clear that $\text{Cov}\Big(X_i, X_j\Big) \le \text{Cov}\Big(X_i', X_j'\Big)$ implies

$$\text{Cov}\Big(A_i(X_i), A_j(X_j)\Big) \le \text{Cov}\Big(A_i(X_i'), A_j(X_j')\Big).$$

since function $A_i(y)$ is an increasing function in y (see equation (3.3.1)). Now, we want to check the possibility of obtaining the $\text{Cov}\Big(A_i(X_i), A_j(X_j)\Big)$ using the $\text{Cov}\Big(X_i, X_j\Big)$. It is easy to see that $\mathbb{E}\Big[A_i(X_i)\Big] = 0$. Then, $\text{Cov}\Big(A_i(X_i), A_j(X_j)\Big) = \mathbb{E}\Big[A_i(X_i)A_j(X_j)\Big]$

$$\begin{aligned} &= \mathbb{E}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\mathbb{I}_{\{X_i \le x\}} - F_i(x)\Big)\Big(\mathbb{I}_{\{X_j \le y\}} - F_j(y)\Big)J(F_i(x))J(F_j(y))dxdy\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(F(x,y) - F_i(x)F_j(y)\Big)J(F_i(x))J(F_j(y))dxdy, \end{aligned}$$

where the last result is obtained by taking the expected value of

$$\mathbb{I}_{\{X_i \le x\}}\mathbb{I}_{\{X_j \le y\}} - F_i(x)\mathbb{I}_{\{X_j \le y\}} - F_j(y)\mathbb{I}_{\{X_i \le x\}} + F_i(x)F_j(y),$$

where F is the joint distribution function of X_i and X_j. With the risk measure MEAN, $J(F_i(x)) = J(F_j(y)) = 1$ for all $x, y \in \mathbb{R}$. Hence,

$$\text{Cov}\Big(A_i(X_i), A_j(X_j)\Big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(F(x,y) - F_i(x)F_j(y)\Big)dxdy \overset{(*)}{=} \text{Cov}(X_i, X_j).$$

Note that equality (*) is nothing else but Hoeffding's Lemma (see Nelsen, 2006 and McNeil *et al.*, 2005). Therefore, if the risk measure is MEAN, the above equality holds regardless of the distribution.

If X_i and X_j are independent, then $F(x,y) = F_i(x)F_j(y)$ and

$$\mathrm{Cov}\Big(A_i(X_i), A_j(X_j)\Big) = 0 = \mathrm{Cov}(X_i, X_j).$$

Therefore, regardless of the risk measure, when the dependence structure is independence, we have

$$\mathrm{Cov}\Big(X_i, X_j\Big) = \mathrm{Cov}\Big(A_i(X_i), A_j(X_j)\Big) = 0.$$

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Appendix A: Proofs

Proof of Proposition 2.1.11.

Recalling the stochastic representation (2.1.2), there exists $\mathbf{Z} \sim MVN(\mathbf{0}, \mathbf{\Sigma})$ and $R \geq 0$ independent of $\mathbf{Z}$ such that $\mathbf{X} \overset{d}{=} \boldsymbol{\mu} + R\,\mathbf{Z}$.

The "if" part. Assume $rank(\mathbf{\Sigma}) = 1$, then $corr(Z_i, Z_j) = 1$ for all i, j. Therefore, there exists $Z \sim N(0, 1)$ such that $Z_i = a_i Z$ with $a_i \geq 0$ for all $i = 1, \ldots, k$. It immediately follows that $\mathbf{X}$ is comonotonic from the stochastic representation (2.1.2) and the functional characterization of comonotonicity.

The "only if" part. Assume that $\mathbf{X}$ is comonotonic. Consider any $\mathbf{y}, \mathbf{z} \in$ *supp*$(\mathbf{Z})$. Since R is independent of $\mathbf{Z}$, then $\boldsymbol{\mu} + r\mathbf{y}, \boldsymbol{\mu} + r\mathbf{z} \in$ *supp*$(\mathbf{X})$ for any $0 < r \in$ *supp*$(\mathbf{Z})$. From the comonotonicity of $\mathbf{X}$, it holds that $\boldsymbol{\mu} + r\mathbf{y} \leq \boldsymbol{\mu} + r\mathbf{z}$ or $\boldsymbol{\mu} + r\mathbf{y} \geq \boldsymbol{\mu} + r\mathbf{z}$, which implies that $\mathbf{y} \leq \mathbf{z}$ or $\mathbf{y} \geq \mathbf{z}$. Therefore, we conclude that $\mathbf{Z}$ is comonotonic and thus $rank(\mathbf{\Sigma}) = 1$.

Proof of Proposition 5.2.1.

By considering Cholesky decomposition of $\mathbf{\Sigma}$ we obtain,

$$\mathbf{Y} = \mathbf{A}\mathbf{Z},$$

where $\mathbf{Z} = (Z_1, Z_2, Z_3)$ is multivariate standard normal vector, and

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \rho_1 & \rho & \sqrt{1-\rho^2-\rho_1^2} \end{pmatrix}.$$

By transforming (Z_1, Z_2, Z_3) to the spherical coordinates system

$$Z_1 = R\sin(\Theta)\cos(\Phi)\,, Z_2 = R\sin(\Theta)\sin(\Phi) \text{ and } Z_3 = R\cos(\Theta),$$

we obtain the joint distribution function of (R, Θ, Φ) as follows:

$$g_{(R,\Theta,\Phi)}(r,\theta,\phi) = r^2\sin(\theta) f_{\mathbf{Z}}(\mathbf{z}),$$

where $r \in (0,\infty), \theta \in [0, 2\pi], \phi \in [0,\pi),$ and $f_{\mathbf{Z}}(\mathbf{z}) = (2\pi)^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}\mathbf{z}\mathbf{z}^{\mathsf{T}}\right\}$. That is,

$$g_{(R,\Theta,\Phi)}(r,\theta,\phi) = (2\pi)^{-\frac{3}{2}} r^2 \sin(\theta) e^{-\frac{r^2}{2}} = \frac{1}{2\pi} \times \frac{1}{2}\sin(\theta) \times \sqrt{\frac{2}{\pi}} r^2 e^{-\frac{r^2}{2}},$$

which implies that $R, \Theta,$ and Φ are independent. Therefore, to prove Proposition 5.2.1, we study the surface area on the sphere created by the set, E_ρ, given by the following inequality

$$|z_1 - z_2| + \left|(1-\rho_1)z_1 - \rho z_2 - \sqrt{1-\rho^2-\rho_1^2}z_3\right| + \left|-\rho_1 z_1 + (1-\rho)z_2 - \sqrt{1-\rho^2-\rho_1^2}z_3\right| < t,$$

for $t > 0$.

The set E_ρ is a hexagon base cylinder and its axis of symmetry changes as we change ρ. We want to show that, for fixed $r > 0$, the surface area of the intersection of E_ρ with the sphere is an increasing function of ρ. To simplify this problem we transform the above set so that the axis of symmetry of the hexagon base cylinder remains the same as we change ρ. We use the orthogonal matrix

$$Q = \begin{pmatrix} 0 & \frac{1-\rho-\rho_1}{V} & -\frac{\sqrt{1-\rho^2-\rho_1^2}}{V} \\ -\frac{V^2}{VW} & \frac{1-\rho^2-\rho_1^2}{VW} & \frac{(1-\rho-\rho_1)\sqrt{1-\rho^2-\rho_1^2}}{VW} \\ \frac{\sqrt{1-\rho^2-\rho_1^2}}{W} & \frac{\sqrt{1-\rho^2-\rho_1^2}}{W} & \frac{1-\rho-\rho_1}{W} \end{pmatrix},$$

where $V = \sqrt{2(1-\rho)(1-\rho_1)}$ and $W = \sqrt{(1-\rho)(3+\rho-2\rho_1)-\rho_1^2}$.

We transform $\mathbf{u} = Q\mathbf{z}$, where $\mathbf{u} = [u_1, u_2, u_3]^\mathsf{T}, \mathbf{z} = [z_1, z_2, z_3]^\mathsf{T}$. Then

$$z_1 - z_2 = \frac{(1-\rho-\rho_1)}{V}u_1 + \frac{W}{V}u_2,$$

$$(1-\rho_1)z_1 - \rho z_2 - \sqrt{1-\rho^2-\rho_1^2}z_3 = \frac{(1-\rho_1)(1-\rho+\rho_1)}{V}u_1 + \frac{(1-\rho_1)W}{V}u_2,$$

and $$-\rho_1 z_1 + (1-\rho)z_2 - \sqrt{1-\rho^2-\rho_1^2}z_3 = \frac{((1-\rho)(2-\rho_1)-\rho_1^2)}{V}u_1 + \frac{\rho_1 W}{V}u_2.$$

The set given by the inequality

$$\left|\tfrac{(1-\rho-\rho_1)}{V}u_1 + \tfrac{W}{V}u_2\right| + \left|\tfrac{(1-\rho_1)(1-\rho+\rho_1)}{V}u_1 + \tfrac{(1-\rho_1)W}{V}u_2\right| + \left|\tfrac{\left((1-\rho)(2-\rho_1)-\rho_1^2\right)}{V}u_1 + \tfrac{\rho_1 W}{V}u_2\right| < t$$

is a hexagon in the $x, y-$plane, i.e, the axis of symmetry of the cylinder is $z-$axis.

Figure A.1 shows the above mentioned hexagon when $\rho = 0.5$ and $\rho_1 = 0.3$.

In that figure, d_i = perpendicular distance from the origin to each side and v_i = distance between the origin and the vertexes as indicated in the figure.

$$d_1 = \frac{1}{\sqrt{1-\rho}}, d_2 = 1, d_3 = \frac{1}{\sqrt{1-\rho_1}}, v_1 = \frac{2\sqrt{1-\rho_1}}{W}, v_2 = \frac{2\sqrt{1-\rho}}{W}, \text{ and } v_3 = \frac{2}{W}.$$

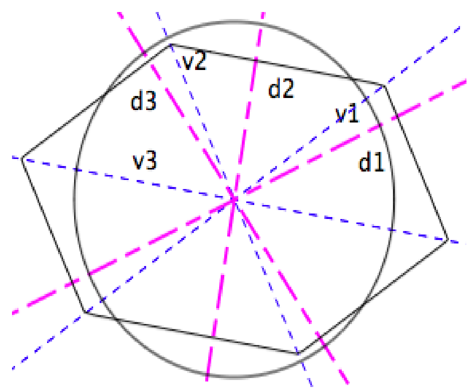


Figure A.1: The cross section through the origin perpendicular to the axis of symmetry.

We consider the area $S(\rho, r)$ of the sphere $x^2 + y^2 + z^2 = r^2$ which lies above the hexagon region. Let $L(\rho, s)$ be the arc length of the part of the of the circle $x^2 + y^2 = s^2$ which lies within hexagon in the x, y-plane.

$$S(\rho, r) = 2\int_0^r L(\rho, s)\frac{k}{\sqrt{k^2 - s^2}}ds.$$

In order to show that $S(\rho, r)$ is an increasing function of ρ it is enough to show that $L(\rho, s)$ is an increasing function of ρ for every $s > 0$. Let θ be the total central angle such that the circle lies outside the hexagon. Then, $L(\rho, s) = s(2\pi - \theta)$ and we want to show θ is a decreasing function in ρ. To show this we consider two cases.

(1) $d_2 < d_3 < d_1 < v_2 < v_1 < v_3$.

(2) $d_2 < d_3 < v_2 < d_1 < v_1 < v_3$.

In case (1),

$$\theta = \begin{cases} 0 & ; s < d_2, \\ 4\cos^{-1}\left(\frac{d_2}{s}\right) & ; d_2 < s < d_3, \\ 4\sum_{i=2}^{3}\cos^{-1}\left(\frac{d_i}{s}\right) & ; d_3 < s < d_1, \\ 4\sum_{i=1}^{3}\cos^{-1}\left(\frac{d_i}{s}\right) & ; d_1 < s < v_2, \\ 4\cos^{-1}\left(\frac{d_1}{s}\right) + 2\sum_{i=2}^{3}\left(\cos^{-1}\left(\frac{d_i}{s}\right) + \cos^{-1}\left(\frac{d_i}{v_2}\right)\right) & ; v_2 < s < v_1, \\ \pi + 2\cos^{-1}\left(\frac{d_1}{s}\right) + 2\cos^{-1}\left(\frac{d_3}{s}\right) + 2\cos^{-1}\left(\frac{d_3}{v_2}\right) & ; v_1 < s < v_3, \\ 2\pi & ; s > v_3. \end{cases}$$

Notice that, $d_1 = \frac{1}{\sqrt{1-\rho}}$ is an increasing function of ρ and d_2, d_3 are independent of ρ.

$$\frac{d_2}{v_2} = \frac{W}{2\sqrt{(1-\rho)}} = \frac{\sqrt{(1-\rho)(3+\rho-2\rho_1)-\rho_1^2}}{2\sqrt{(1-\rho)}},$$

$$\frac{d_3}{v_2} = \frac{W}{2\sqrt{(1-\rho_1)(1-\rho)}} = \frac{\sqrt{(1-\rho)(3+\rho-2\rho_1)-\rho_1^2}}{2\sqrt{(1-\rho_1)(1-\rho)}},$$

$$\frac{d_1}{v_1} = \frac{W}{2\sqrt{(1-\rho_1)(1-\rho)}} = \frac{\sqrt{(1-\rho)(3+\rho-2\rho_1)-\rho_1^2}}{2\sqrt{(1-\rho_1)(1-\rho)}},$$

are increasing functions of ρ, if $\rho + \rho_1 \leq 1$.

Hence, θ is a decreasing function of ρ for $0 < \rho_1 < \rho$, and $d_2 < d_3 < d_1 < v_2 < v_1 < v_3$.

In case (2),

$$\theta = \begin{cases} 0 & ; s < d_2, \\ 4\cos^{-1}\left(\frac{d_2}{s}\right) & ; d_2 < s < d_3, \\ 4\sum_{i=2}^{3}\cos^{-1}\left(\frac{d_i}{s}\right) & ; d_3 < s < v_2, \\ 2\sum_{i=2}^{3}\left(\cos^{-1}\left(\frac{d_i}{s}\right) + \cos^{-1}\left(\frac{d_i}{v_2}\right)\right) & ; v_2 < s < d_1, \\ 2\sum_{i=1}^{3}\cos^{-1}\left(\frac{d_i}{s}\right) + 2\sum_{i=2}^{3}\cos^{-1}\left(\frac{d_i}{v_2}\right) & ; d_1 < s < v_1, \\ \pi + 2\cos^{-1}\left(\frac{d_1}{s}\right) + 2\cos^{-1}\left(\frac{d_3}{s}\right) + 2\cos^{-1}\left(\frac{d_3}{v_2}\right) & ; v_1 < s < v_3, \\ 2\pi & ; s > v_3. \end{cases}$$

As mentioned above, d_1, $\frac{d_2}{v_2}$, $\frac{d_3}{v_2}$, and $\frac{d_1}{v_1}$ are increasing functions of ρ, if $\rho+\rho_1 \leq 1$.

Hence, θ is a decreasing function of ρ for $0 < \rho_1 < \rho$, and $d_2 < d_3 < v_2 < d_1 < v_1 < v_3$. □

Proof of Proposition 5.2.2. Denote the variances of the marginal distributions by $\sigma_1^2 \leq \sigma_2^2 \leq \sigma_3^2$. There exists $\{Z_1, Z_2, Z_3\} \overset{i.i.d.}{\sim} N(0,1)$, such that $(Y'_1, Y'_2, Y'_3) \overset{d}{=} (\sigma_1 Z_1, \sigma_2 Z_1, \sigma_3 Z_1)$ and

$$(Y_1, Y_2, Y_3) \overset{d}{=} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

where $\mathbf{L}_Y = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$ is the Cholesky decomposition of the correlation matrix of (Y_1, Y_2, Y_3), which means that $l_{11} = 1, l_{21}^2 + l_{22}^2 = 1$ and $l_{31}^2 + l_{32}^2 + l_{33}^2 = 1$.

Therefore, $\mathbb{P}\{G(\mathbf{Y}) \le t\} = \mathbb{P}\{(Z_1, Z_2, Z_3) \in R_Y(t)\}$ and $\mathbb{P}\{G(\mathbf{Y}') \le t\} = \mathbb{P}\{(Z_1, Z_2, Z_3) \in R_{Y'}(t)\}$, where

$$\begin{aligned} R_{Y'}(t) &= \{(z_1, z_2, z_3) : 4(\sigma_3 - \sigma_1)|z_1| \le t\} \\ R_Y(t) &= \{(z_1, z_2, z_3) : 2|\sigma_1 z_1' - \sigma_2 z_2'| + 2|\sigma_2 z_2' - \sigma_3 z_3'| + 2|\sigma_3 z_3' - \sigma_1 z_1'| \le t\} \\ &= \{(z_1, z_2, z_3) : 4|\sigma_1 z_1' - \sigma_2 z_2'| \le t\} \cap \{(z_1, z_2, z_3) : 4|\sigma_2 z_2' - \sigma_3 z_3'| \le t\} \\ &\qquad \cap \{(z_1, z_2, z_3) : 4|\sigma_3 z_3' - \sigma_1 z_1'| \le t\}, \end{aligned}$$

with $z_1' = z_1, z_2' = l_{21} z_1 + l_{22} z_2$ and $z_3' = l_{31} z_1 + l_{32} z_2 + l_{33} z_3$.

Now we compare the two regions $R_{Y'}(t)$ and $R_Y^1(t) = \{(z_1, z_2, z_3) : 4|\sigma_3 z_3' - \sigma_1 z_1'| \le t\}$. Note that both of them are regions between a pair of parallel planes. For $R_{Y'}(t)$, the distance between the boundary planes is $\frac{t}{2(\sigma_3 - \sigma_1)}$. For $R_Y^1(t)$, the distance between the boundary planes is $\frac{t}{2\sqrt{\sigma_1^2 + \sigma_3^2 - 2 l_{31} \sigma_1 \sigma_3}} \le \frac{t}{2(\sigma_3 - \sigma_1)}$. Since both $R_{Y'}(t)$ and $R_Y^1(t)$ are centered at the origin, we conclude that $R_Y^1(t)$, and thus $R_Y(t)$ as a subset of $R_Y^1(t)$, can be moved inside $R_{Y'}(t)$ through certain rational transformations. Since the distribution of (Z_1, Z_2, Z_3) is rational invariant, it immediately follows that $\mathbb{P}\{(Z_1, Z_2, Z_3) \in R_Y(t)\} \le \mathbb{P}\{(Z_1, Z_2, Z_3) \in R_{Y'}(t)\}$, i.e., $\mathbb{P}\{G(\mathbf{Y}) \le t\} \le \mathbb{P}\{G(\mathbf{Y}') \le t\}$ for any $t \ge 0$, which implies that $G(\mathbf{Y}) \ge_{st} G(\mathbf{Y}')$. □

In order to prove Proposition 5.3.2, we propose the following lemma.

Lemma A.1. Assume (X, Y) have an exchangeable bivariate normal random vector with correlation coefficient ρ. Let $D \subset \mathbb{R}^2$ be any convex set such that $\{(x, y) \mid (y, x) \in D\} = D$. Then $\mathbb{P}\{(X, Y) \in D\}$ is increasing in ρ.

Proof. For simplicity, assume D is bounded. The unbounded case can be approached by limiting argument. Denote by ∂D the boundary of D with positive (counterclockwise) orientation. The ∂D is piecewise smooth due to the convexity of D. Furthermore, denote $\partial D_+ = \{(x, y) \in \partial D \mid y \geq x\}$ and $\partial D_- = \{(x, y) \in \partial D \mid y \leq x\}$, then $\partial D = \partial D_+ \cup \partial D_-$. Let $\partial D'_+$ be same as ∂D_+ but with the opposite (clockwise) orientation. Then $\partial D'_-$ be same as ∂D_- are reflection to each other with respect to the line $y = x$. Figure A.2 provides an illustration (not an accurate representation) of these orientation curves.

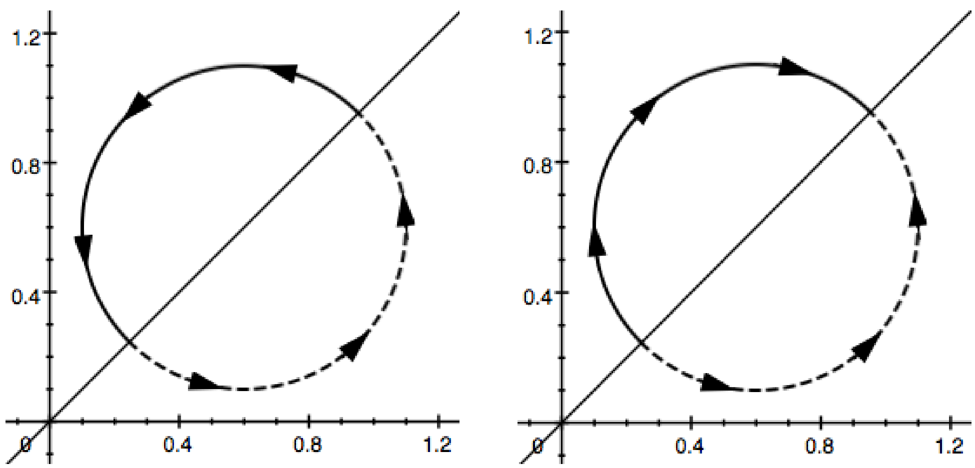


Figure A.2: Orientation Curves. Left: ∂D_+ (solid) and ∂D_- (dashed). Right: $\partial D'_+$ (solid) and ∂D_- (dashed).

Without loss of generality, assume $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ and $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$. Then the density function of (X, Y) by $f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\Big\{ - \frac{x^2+y^2-2\rho xy}{2(1-\rho^2)} \Big\}$. Plackett's identity (Plackett, 1954) states that $\frac{\partial}{\partial \rho} f(x, y) = \frac{\partial^2}{\partial x \partial y} f(x, y)$. According to Fubini's theorem, we have

$$
\begin{aligned}
\frac{\partial}{\partial\rho}\mathbb{P}\{(X,Y)\in D\} &= \frac{\partial}{\partial\rho}\int_D f(x,y)dxdy = \int_D \frac{\partial}{\partial\rho} f(x,y)dxdy \\
&= \int_D \frac{\partial^2}{\partial x\partial y} f(x,y)dxdy \overset{(*)}{=} \oint_{\partial D} \frac{\partial}{\partial y} f(x,y)dy \\
&= \Big(\int_{\partial D_+} + \int_{\partial D_-}\Big)\frac{\partial}{\partial y} f(x,y)dy \\
&= \Big(-\int_{\partial D'_+} + \int_{\partial D_-}\Big)\frac{\partial}{\partial y} f(x,y)dy, \qquad \text{(A.1)}
\end{aligned}
$$

where Equality $(*)$ follows from Green's theorem.

Note that $\frac{\partial}{\partial y} f(x,y) = \frac{\rho x - y}{1-\rho^2} f(x,y)$. Following equation (A.1), we have

$$
\begin{aligned}
\frac{\partial}{\partial\rho}\mathbb{P}\{(X,Y)\in D\} &= -\int_{\partial D'_+} \frac{\rho x - y}{1-\rho^2} f(x,y)dy + \int_{\partial D_-} \frac{\rho x - y}{1-\rho^2} f(x,y)dy \\
&= -\int_{\partial D'_+} f(x,y)dy \times \int_{\partial D'_+} \frac{\rho x - y}{1-\rho^2}\frac{f(x,y)}{\int_{\partial D'_+} f(x,y)dy}dy \\
&\quad + \int_{\partial D_-} f(x,y)dy \times \int_{\partial D_-} \frac{\rho x - y}{1-\rho^2}\frac{f(x,y)}{\int_{\partial D_-} f(x,y)dy}dy \\
&= -\int_{\partial D'_+} f(x,y)dy \times \mathbb{E}\Big[\frac{\rho X - Y}{1-\rho^2}\;\Big|\;(X,Y)\in\partial D'_+\Big] + \\
&\quad \int_{\partial D_-} f(x,y)dy \times \mathbb{E}\Big[\frac{\rho X - Y}{1-\rho^2}\;\Big|\;(X,Y)\in\partial D_-\Big].
\end{aligned}
$$

Due to the symmetry between $\partial D'_+$ and ∂D_- and the exchangeability of (X,Y)

(or $f(x,y)$), we know that $\int_{\partial D'_+} f(x,y)dy = \int_{\partial D_-} f(x,y)dy$ and

$$\begin{aligned}\mathbb{E}\Big[\frac{\rho X - Y}{1-\rho^2}\ \Big|\ (X,Y) \in \partial D'_+\Big] &= \mathbb{E}\Big[\frac{\rho Y - X}{1-\rho^2}\ \Big|\ (Y,X) \in \partial D'_+\Big] \\ &= \mathbb{E}\Big[\frac{\rho Y - X}{1-\rho^2}\ \Big|\ (X,Y) \in \partial D_-\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial \rho}\mathbb{P}\{(X,Y) \in D\} &= \int_{\partial D_-} f(x,y)dy \times -\mathbb{E}\Big[\frac{\rho X - Y}{1-\rho^2}\ \Big|\ (X,Y) \in \partial D'_+\Big] + \\ &\quad \int_{\partial D_-} f(x,y)dy \times \mathbb{E}\Big[\frac{\rho X - Y}{1-\rho^2}\ \Big|\ (X,Y) \in \partial D_-\Big] \\ &= \int_{\partial D_-} f(x,y)dy \times -\mathbb{E}\Big[\frac{\rho Y - X}{1-\rho^2}\ \Big|\ (X,Y) \in \partial D_-\Big] + \\ &\quad \int_{\partial D_-} f(x,y)dy \times \mathbb{E}\Big[\frac{\rho X - Y}{1-\rho^2}\ \Big|\ (X,Y) \in \partial D_-\Big] \\ &= \int_{\partial D_-} f(x,y)dy \times \mathbb{E}\Big[\frac{(1+\rho)(X-Y)}{1-\rho^2}\ \Big|\ (X,Y) \in \partial D_-\Big] \geq 0.\end{aligned}$$

The last inequality hods because $(X,Y) \in \partial D_-$ implies $X \geq Y$. It immediately follows that $\mathbb{P}\{(X,Y) \in D\}$ is increasing in ρ.

Proof of Proposition 5.3.2. We shall use conditioning argument. According to the property of multivariate normal distribution, we know that conditioning on $\{Y_3 = y_3, \ldots, Y_k = y_k\}, (Y_1, Y_2)$ has an exchangeable bivariate normal distribution with covariance $\sigma^*_{12} = \sigma_{12} - s$, where s is determined by other components of the covariance matrix.

For any fixed $y_3, \ldots, y_k$ and $t \geq 0$, denote $D = \{(y_1, y_2) \mid G(y_1, \ldots, y_k) \leq t\}$. Note that $D \subset \mathbb{R}^2$ is a convex polygon and symmetric with respect to the line $y_1 = y_2$. According to Lemma A.1, $\mathbb{P}\{G(\mathbf{Y}) \leq t \mid Y_3 = y_3, \ldots, Y_k = y_k\} = \mathbb{P}\{(Y_1, Y_2) \in D \mid Y_3 = y_3, \ldots, Y_k = y_k\}$ is increasing in σ_{12}^* and thus in σ_{12}. Therefore, $\mathbb{P}\{G(\mathbf{Y}) \leq t\} = \mathbb{E}[\mathbb{P}\{G(\mathbf{Y}) \leq t \mid Y_3, \ldots, Y_k\}]$ is also increasing in σ_{12}. □

Proof of Proposition 5.3.4. Recall expression $G(\mathbf{Y})$ can be expressed as follows,

$$G(\mathbf{Y}) = \sum_{i=1}^{k} (4i - 2k - 2) Y_{i:k} := \sum_{i=1}^{k} c_i Y_{i:k},$$

where $c_i = 4i - 2k - 2$. Noting that $\{c_i, i = 1, 2, \ldots, k\}$ is an increasing sequence, according to arrangement inequality, we have

$$G(\mathbf{Y}) = \max_{\pi \in \mathcal{P}} \left\{ \sum_{i=1}^{k} c_i Y_{\pi(i)} \right\} = \max_{\pi \in \mathcal{P}} \left\{ \sum_{i=1}^{k} c_{\pi(i)} Y_i \right\},$$

where $\mathcal{P}$ denotes the collection of all permutations of $(1, 2, \ldots, k)$. Let $\mathbf{C} \in \mathbb{R}^{k! \times k}$ be the matrix generated by all different permutations of $(c_1, \ldots, c_k)$. Then $G(\mathbf{Y})$ and $G(\mathbf{Y}')$ are the largest order statistics of random vectors $\mathbf{C}\mathbf{Y}^{\mathsf{T}}$ and $\mathbf{C}\mathbf{Y}'^{\mathsf{T}}$ respectively. On the other hand, since $\mathbf{Y}$ and $\mathbf{Y}'$ follow multivariate normal distributions, so do $\mathbf{C}\mathbf{Y}^{\mathsf{T}}$ and $\mathbf{C}\mathbf{Y}'^{\mathsf{T}}$. Specifically, $\mathbf{C}\mathbf{Y}^{\mathsf{T}} \sim MVN(\mathbf{0}, \mathbf{C}\mathbf{\Sigma}_Y \mathbf{C}^{\mathsf{T}})$ and $\mathbf{C}\mathbf{Y}'^{\mathsf{T}} \sim MVN(\mathbf{0}, \mathbf{C}\mathbf{\Sigma}_{Y'} \mathbf{C}^{\mathsf{T}})$.

Noting that $\mathbf{C}\mathbf{1}_{k \times k}\mathbf{C}^{\mathsf{T}} = \mathbf{0}$ since $\sum_{i=1}^{k} c_i = 0$, comparing covariance matrices of

$\mathbf{CY}^\mathsf{T}$ and $\mathbf{CY'}^\mathsf{T}$ yields that

$$\begin{aligned}\mathbf{C\Sigma}_Y\mathbf{C}^\mathsf{T} - \mathbf{C\Sigma}_{Y'}\mathbf{C}^\mathsf{T} &= \mathbf{C\Sigma}_Y\mathbf{C}^\mathsf{T} - \mathbf{C\Sigma}_{Y'}\mathbf{C}^\mathsf{T} + a\mathbf{C}\mathbf{1}_{k\times k}\mathbf{C}^\mathsf{T} \\ &= \mathbf{C}(\mathbf{\Sigma}_Y - \mathbf{\Sigma}_{Y'} + a\mathbf{1}_{k\times k})\mathbf{C}^\mathsf{T},\end{aligned}$$

which is positive semidefinite since $\mathbf{\Sigma}_Y - \mathbf{\Sigma}_{Y'} + a\mathbf{1}_{k\times k}$ is positive semidefinite.

Recall that $G(\mathbf{Y}) = \max\{row_i\mathbf{C}.\mathbf{Y}^\mathsf{T}, i = 1,\ldots,k!\}$. Since $\{c_1,\ldots,c_k\} = \{-c_1,\ldots,-c_k\}$, then $\{row_i\mathbf{C}.\mathbf{Y}^\mathsf{T}, i = 1,\ldots,k!\} = \{-row_i\mathbf{C}.\mathbf{Y}^\mathsf{T}, i = 1,\ldots,k!\}$. Therefore

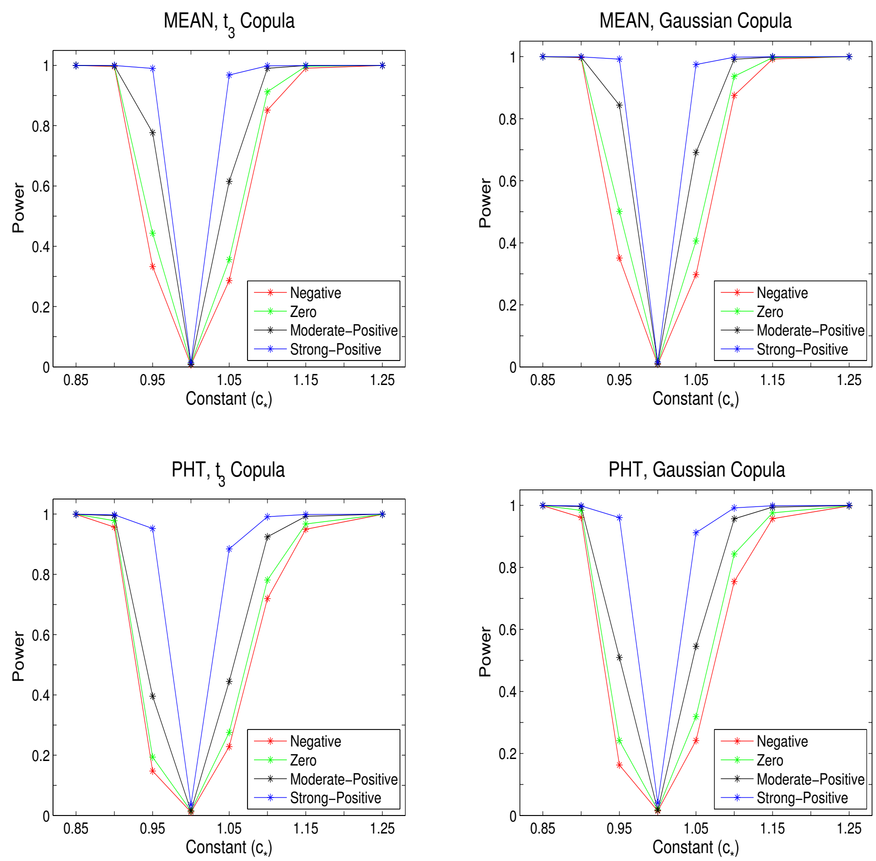
$$\begin{aligned}\mathbb{P}\{G(\mathbf{Y}) \le t\} &= \mathbb{P}\{\mathbf{CY}^\mathsf{T} \le (t,..,t)^\mathsf{T}\} = \mathbb{P}\{-\mathbf{CY}^\mathsf{T} \le (t,..,t)^\mathsf{T}\} \\ &= \mathbb{P}\{(-t,\ldots,-t)^\mathsf{T} \le \mathbf{CY}^\mathsf{T} \le (t,..,t)^\mathsf{T}\} := \mathbb{P}\{\mathbf{CY}^\mathsf{T} \in Q_t\},\end{aligned}$$

where Q_t is a super cube centered at origin with side length of $2t$. It is clear that Q_t is convex and centrally symmetric. According to Lemma 5.3.3,

$$\mathbb{P}\{G(\mathbf{Y}) \le t\} = \mathbb{P}\{\mathbf{CY}^\mathsf{T} \in Q_t\} \le \mathbb{P}\{\mathbf{CY'}^\mathsf{T} \in Q_t\} = \mathbb{P}\{G(\mathbf{Y'}) \le t\},$$

for any t, which implies that $G(\mathbf{Y}) \ge_{st} G(\mathbf{Y'})$ from Definition 2.3.1. □

Appendix B: Figures



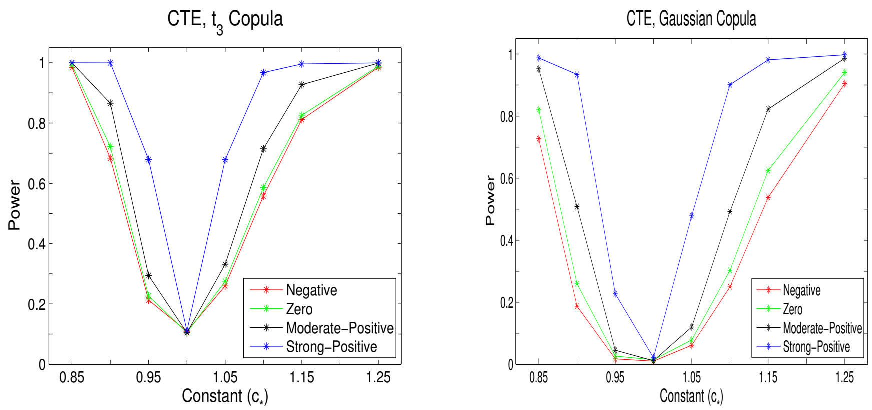


Figure B.1: The first type of alternatives. Estimated power curves of the tests based on the MEAN, PHT, CTE measures, for various dependence structures, $n = 200$, and $\alpha = 0.01$. Left column: t_3 copulas. Right column: Gaussian copulas.

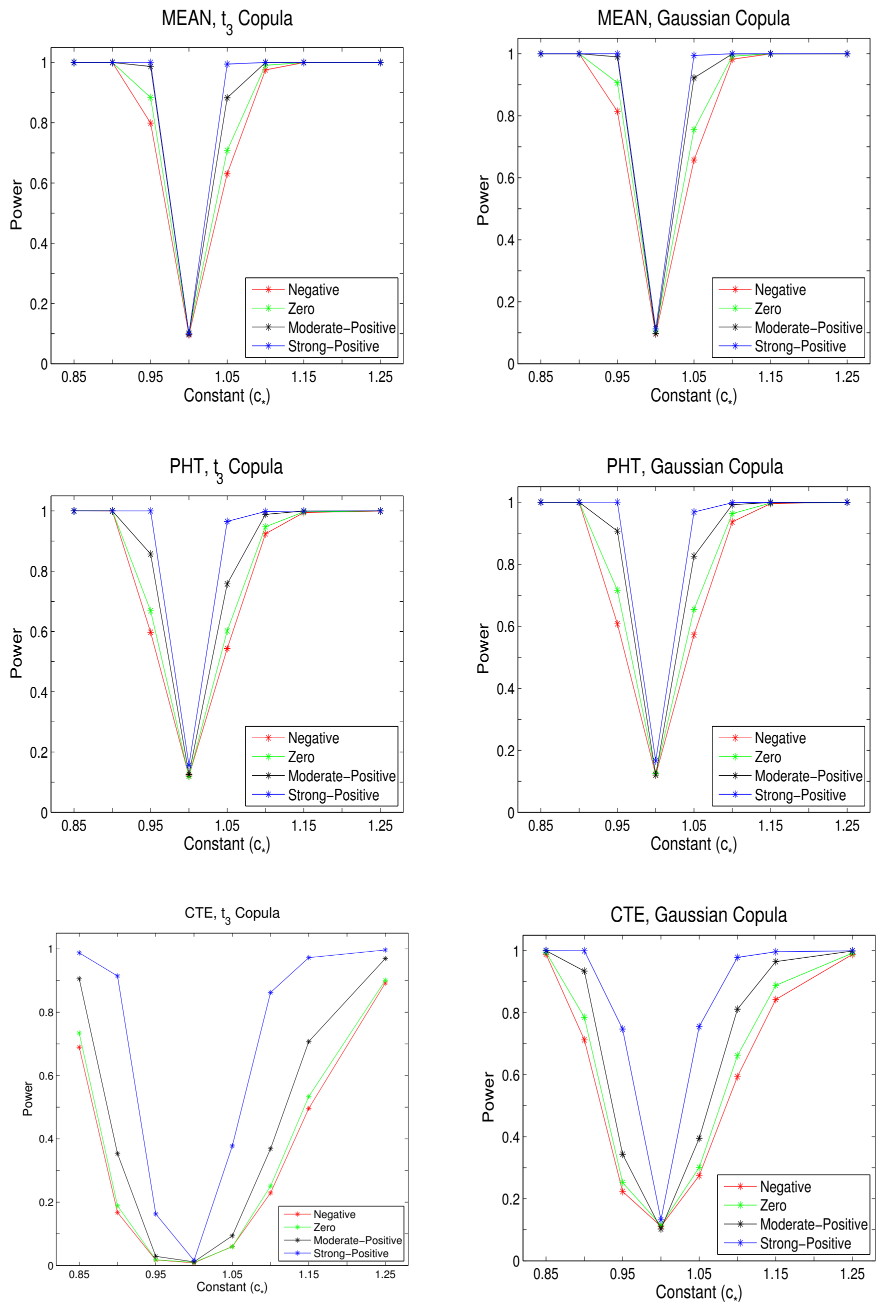


Figure B.2: The first type of alternatives. Estimated power curves of the tests based on the MEAN, PHT, CTE measures, for various dependence structures, $n = 200$, and $\alpha = 0.10$. Left column: t_3 copulas. Right column: Gaussian copulas.

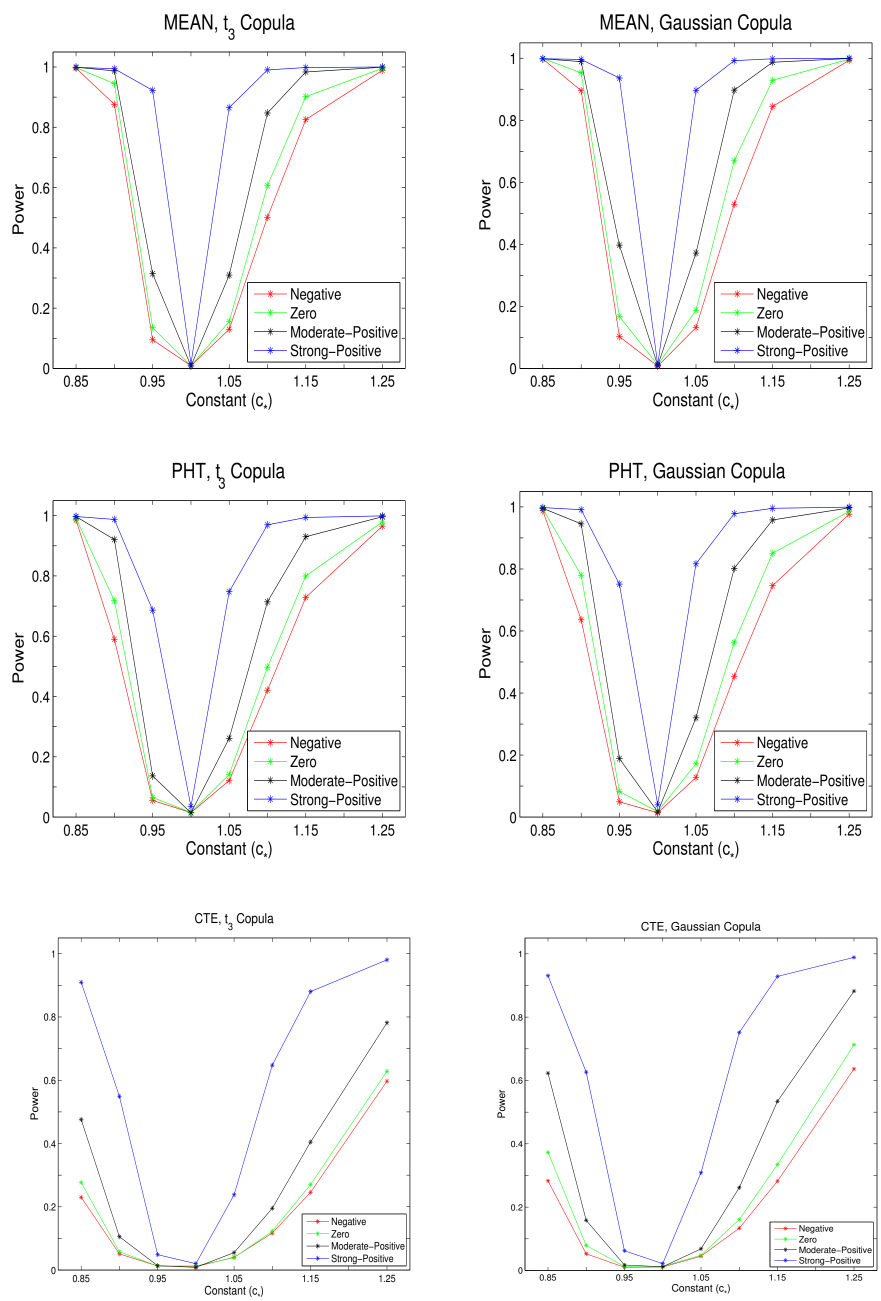


Figure B.3: The first type of alternatives. Estimated power curves of the tests based on the MEAN, PHT, CTE measures, for various dependence structures, $n = 100$, and $\alpha = 0.01$. Left column: t_3 copulas. Right column: Gaussian copulas.

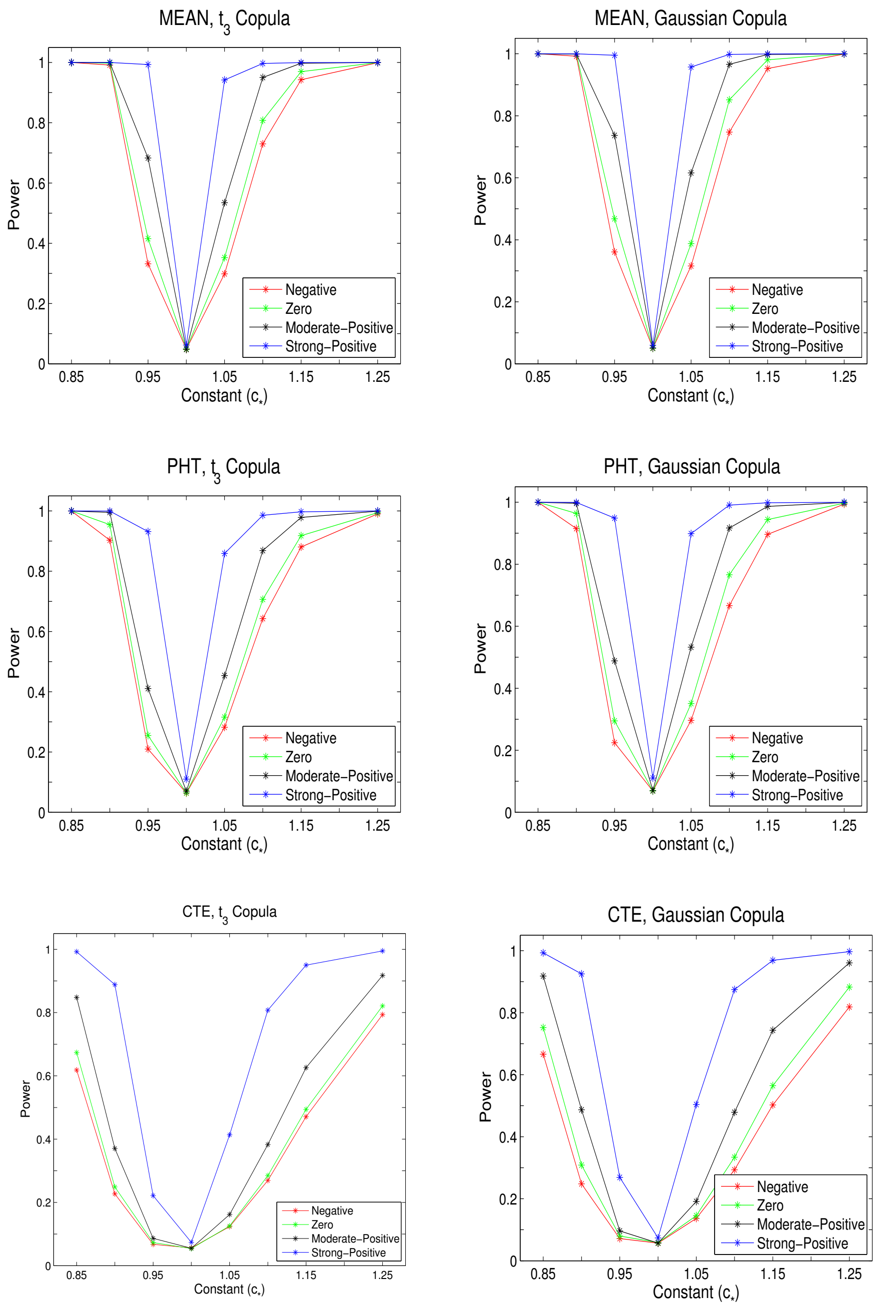


Figure B.4: The first type of alternatives. Estimated power curves of the tests based on the MEAN, PHT, CTE measures, for various dependence structures, $n = 100$, and $\alpha = 0.05$. Left column: t_3 copulas. Right column: Gaussian copulas.

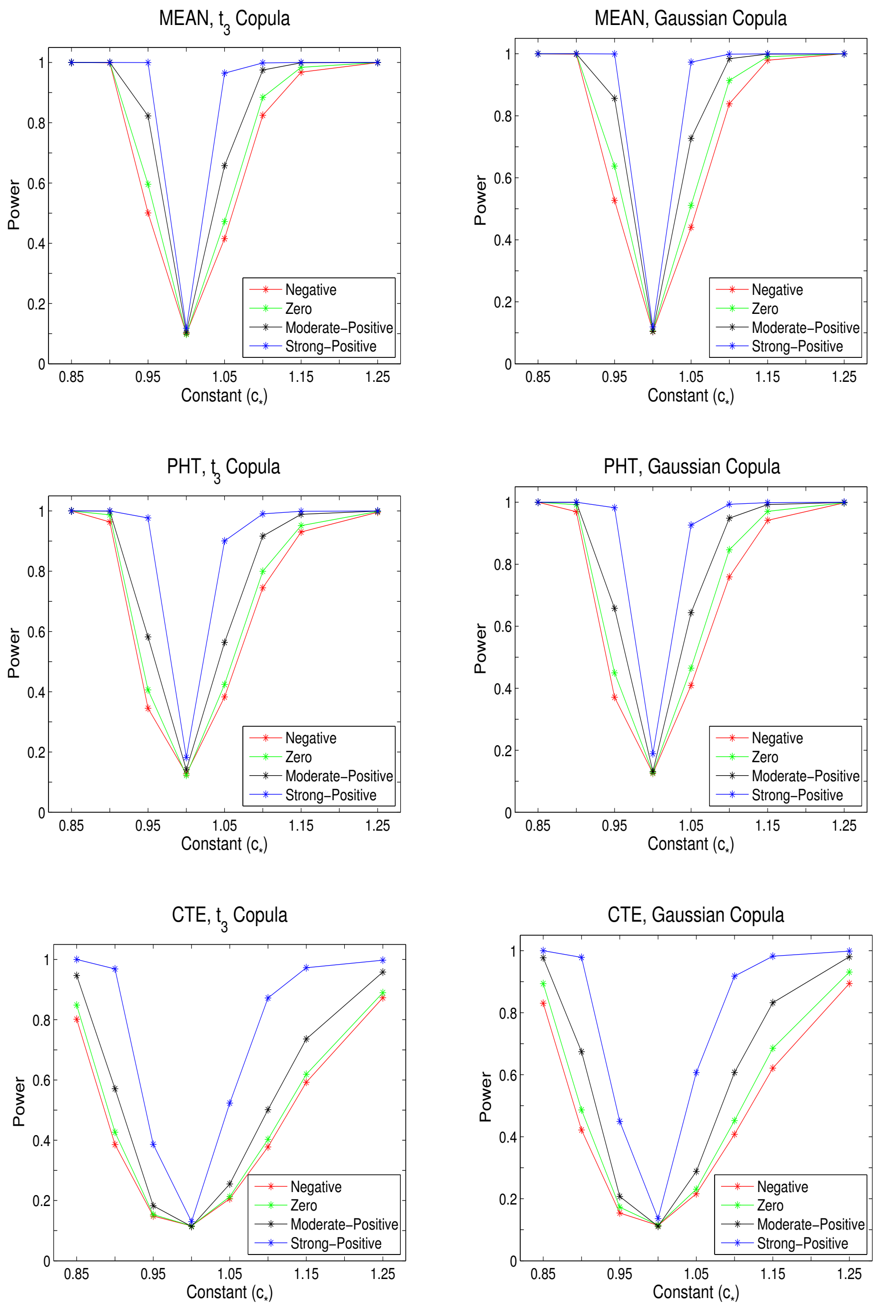


Figure B.5: The first type of alternatives. Estimated power curves of the tests based on the MEAN, PHT, CTE measures, for various dependence structures, $n = 100$, and $\alpha = 0.10$. Left column: t_3 copulas. Right column: Gaussian copulas.

Appendix C: Hypothesis Testing in R

```
#================================================================
    The code is created to carry out the simulation study explained in chapter 5. This code includes
    functions: (1) To simulate exponential with parameter theta, Pareto with parameters x0 and beta,
    lognormal with parameters mu and sigma. (2) To simulate Gaussian and t copulas with four different
    dependence structures, namely strong positive, moderate positive, zero, and negative dependence.(3) To
    find the bootstrap estimate of the critical value for the decision making process. (4) To check the decision
    making requirements.
#================================================================
# R− libraries
library(VGAM)
library(abind)
library(Matrix)
# Remove all the variables from the workspace
rm(list=ls())
#================================================================
    Defined Parameters as in Chapter 4
nk=200 # sample size − e.g. 50,100,200
alpha=c(0.01,0.05,0.1) # desired significance levels
B = 1000 # no of iterations for bootstrap sampling
M = 5000 # no of iterations for Monte Carlo
riskM = 3 # no of risk measures used in the study− e.g. MEAN, PHT, and CTE
t<− 0.25 # threshold level of PHT (see Example 2.2.2)
r<− 0.85 # distortion level of CTE (see Example 2.2.3)
x0<−1.0 # Pareto parameters
beta<− 5.5 # Pareto parameters
gamma=0 # user define:= Gini index when H0 is true
# define constants c∗ and c∗∗ in equations 4.2.8 and 4.2.9
alter1<−c(seq(0.85,1.15, by=0.05), 1.25, 1.50, 2.00) # first type of alternatives
alter2<−c(seq(1.00,1.25, by=0.05), seq(1.50,3.00, by=0.5)) # second type of alternatives
a0=1 # a0=1 if first type; a0=2 if second type of alternatives
EE=10^(−15) # a small number to make the following matrices positive definite (useful in Cholesky
    decomposition)
# copula correlation matrices to generate dependence structures: here we define four, namely
# Sig1 = negative dependence, Sig2 = independence,
```

```
# Sig3 = moderate positive dependence, Sig4 = strong positive dependence
Sig1=matrix(c(1,-0.5,-0.5,-0.5,1,-0.5+EE,-0.5,-0.5+EE,1),nrow=3)
Sig2=matrix(c (1,0,0,0,1,0,0,0,1) ,nrow=3)
Sig3=matrix(c  (1,0.5,0.5,0.5,1,0.5,0.5,0.5,1)   ,nrow=3)
Sig4=matrix(c(1,1-EE,1-EE,1-EE,1,1-EE,1-EE,1-EE,1),nrow=3)
#==============================================================
SigAll=list(Sig1,Sig2,Sig3,Sig4) # make a list of all correlation matrices
N=length(SigAll) # length of the above list

alter=if(a0==1) alter1 else alter2 # select the alternative according to the users' choice
a=length(alter)
indexNull= if(a0==1) 4 else 1

al=length(alpha) # no: of significance levels
#==============================================================
# True Parameters & Risk Measures
#==============================================================
# Distribution related parameters under H0 (see equation 4.2.5--4.2.7)
gb1<-function(x0,beta,t){ x0*((beta/(beta-1))*(t^(-1/beta))-1)}
gb2 <-function(t){pnorm(1-qnorm(1-t,0,1),0,1)}
thetaM<- x0/(beta-1)
thetaP<-(x0*r)/(beta*r-1)
thetaC<-(-(gb1(x0,beta,t))/(log(t)-1))

Cr<-2.03 # Used in calculation of the CTE of the lognormal
muM<-log(x0/(beta-1)) - 0.5
muP <-log(x0/(Cr*(beta*r-1)))
muC<-log(t*gb1(x0,beta,t)/gb2(t)) - 0.5

# Risk Measures under H0 (e - exponential, p-Pareto, l-lognormal)
MEANe<- x0 + thetaM
MEANp<- x0 + x0/(beta-1)
MEANl<- x0 + exp(1/2)*exp(muM)

PHTe<-x0 + thetaP/r
PHTp<-x0 + x0/(beta*r-1)
PHTl<-x0 + Cr*exp(muP)

CTEe<-x0 - thetaC*(log(t)-1)
CTEp<-x0 + gb1(x0,beta,t)
CTEl<-x0 + gb2(t)*exp(muC+0.5)/t

# Distribution related parameters under Ha (see Table 4.2-4.3)
thetaM_1 <-function(thetaM,x0,alter){ thetaM*alter + x0*(alter-1)}
thetaP_1<-function(thetaP,x0,alter){thetaP*alter + x0*r*(alter-1)}
thetaC_1 <-function(thetaC,x0,alter,t){
thetaC*alter+ x0*(alter-1)/(1-log(t))}

muM_1<-function(muM,x0,alter) {log(x0*(alter^2-1) + alter^2*exp(muM+0.5)) - 0.5}
muP_1<-function(muP,x0,alter,Cr) {log(x0*(alter^2-1)/Cr+alter^2*exp(muP))}
```

```
muC_1<-function(muC,x0,alter,t){ log(x0*(alter^2-1)*t/gb2(t)+alter^2*exp(muC+0.5))-0.5}

theta1=rbind(thetaM_1(thetaM,x0,alter1),thetaP_1(thetaP,x0,alter1),
thetaC_1(thetaC,x0,alter1,t)) # theta for the first type of alternatives
theta2=rbind(thetaM_1(thetaM,x0,alter2),thetaP_1(thetaP,x0,alter2),
thetaC_1(thetaC,x0,alter2,t)) # theta for the second type of alternatives

theta=if(a0==1) theta1 else theta2 # theta for the selected type alternatives

Mu1=rbind(rep(muM,length(alter1)),rep(muP,length(alter1)),rep(muC,length(alter1))) # Mu for the first type
    of alternatives
Mu2=rbind(muM_1(muM,x0,alter2),muP_1(muP,x0,alter2,Cr), muC_1(muC,x0,alter2,t)) # Mu for the second
    type of alternatives

Mu=if(a0==1) Mu1 else Mu2 # Mu for the selected type alternatives

#=============================================================
# Functions for Dependent Data Generation (see Algorithms in Section 4.3)
#=============================================================
# Name of the function: tCopula
# Purpose: generate uniformly distributed dependent data
# Input: nk= sample size; dim=dimention; Q=chol(Sig) Cholesky decomposition of correlation matrix
    (Sig1,...,Sig4); df=degrees of freedom
# Output: Ut= t copula; Ug = Gaussian copula
tCopula<-function(nk,dim,Q,df){
x<-array(NA, dim=c(nk,dim))
y<-array(NA, dim=c(nk,dim))
z<-array(NA, dim=c(nk,dim))
s<-array(NA, dim=c(nk,dim))
Ug<-array(NA, dim=c(nk,dim)) # Gaussian copula
Ut<-array(NA, dim=c(nk,dim)) # t copula
for(j in 1:dim){z[,j]=rnorm(nk,0,1)}
s=rchisq(nk, df, ncp = 0)
y = z%*%Q
Ug=pnorm(y)

for(i in 1:nk){
for(j in 1:dim){
   x[i,j]=sqrt(df)*y[i,j]/sqrt(s[i])
   Ut[i,j]=pt(x[i,j], df, lower.tail = TRUE, log.p = FALSE)}}
return( list (Ug,Ut))}

#generate exp in the form f(x)=(1/lambda)*e^(-x/lamda)
expFROMunif<-function(u2,lambda){ExP<-(-log(1-u2)*lambda)
return(ExP)}

#generate lognormal with mu and sigma
lognFROMunif<-function(u3,Muu,Sigma){logn1<-array(NA,dim=c(length(u3)))
for( i in 1:length(u3)){logn1[i]=exp(Sigma*qnorm(u3[i])+Muu)}
return(logn1)}
```

```
#=============================================================
# Data Generation for the Analysis
#=============================================================
# Name of the function: fnUData
# Purpose: generate uniformly distributed dependent data
# Input: nk = sample size
# Output: Ut= t copula; Ug = Gaussian copula
fnUData<-function(nk){
Ut<- array(NA,dim=c(N,nk,3))
Ug<- array(NA,dim=c(N,nk,3))
for(k1 in 1:N){
Q = chol(SigAll[[k1]])
Ug[k1,,]=tCopula(nk,riskM,Q,3)[[1]]
Ut[k1,,]=tCopula(nk,riskM,Q,3)[[2]]}
return( list (Ut,Ug))}

# Name of the function: fnDataANDsort
# Purpose: generating and sorting data for the analysis
# Input: nk = sample size ; U = output from fnUdata
# Output: xe = exponential; xp = Pareto; xl = lognormal; Xe,Xp,Xl = ordered xe,xp, and xpl respectively
fnDataANDsort <-function(nk,U){
xe<-array(NA,dim=c(N,nk,a,riskM))
xp<-array(NA, dim=c(N,nk,a,riskM))
xlp<-array(NA,dim=c(N,nk,a,riskM))

Xe<-array(NA,dim=c(N,nk,a,riskM))
Xp<-array(NA, dim=c(N,nk,a,riskM))
Xl<-array(NA,dim=c(N,nk,a,riskM))

for(k1 in 1:N){
for(q in 1: riskM){
for(i in 1:nk){
for(j in 1:a){
  xe[k1,i,j,q]=expFROMunif(U[k1,i,1],theta[q,j])
  xp[k1,i,j,q]=exp(expFROMunif(U[k1,i,2],1/beta))
  xlp[k1,i,j,q]=lognFROMunif(U[k1,i,3],Mu[q,j],1)}}
for(j in 1:a){
  Xe[k1,,j,q]=x0+xe[k1,order(xe[k1,,j,q]),j,q]
  Xp[k1,,j,q]=x0*xp[k1,order(xp[k1,,j,q]),j,q]
  Xl[k1,,j,q]=x0+xlp[k1,order(xlp[k1,,j,q]),j,q]}}}
return( list (xe,xp,xlp,Xe,Xp,Xl))}

#=============================================================
# Computing Risk Measures
#=============================================================
# Name of the function: fnRiskMeasure
# Purpose: computing the risk measures of samples
# Input: nk = sample size; DataANDsort = output of fnDataANDsort
```

```
# Output: RMe, RMp, RMl = risk measures of exponential, Pareto, and lognormal; RMAll = sorted out all
    the risk measure values.
fnRiskMeasure<-function(nk,DataANDsort){
meanE<-array(NA,dim=c(N,a))
meanP<-array(NA,dim=c(N,a))
meanL<-array(NA,dim=c(N,a))
MEAN<-array(NA,dim=c(N,a,riskM))

phtE<-array(NA,dim=c(N,a))
phtP<-array(NA,dim=c(N,a))
phtL<-array(NA,dim=c(N,a))
PHT<-array(NA,dim=c(N,a,riskM))
ttE<-array(NA,dim=c(N,nk,a))
ttP<-array(NA,dim=c(N,nk,a))
ttL<-array(NA,dim=c(N,nk,a))

cteE<-array(NA,dim=c(N,a))
cteP<-array(NA,dim=c(N,a))
cteL<-array(NA,dim=c(N,a))
CTE<-array(NA,dim=c(N,a,riskM))
w=floor(nk*t)
cteLevel=nk-w+1

for(k2 in 1:nk){
cons=(1-(k2-1)/nk)^r-(1-k2/nk)^r
for(k1 in 1:N){
  for(j in 1:a){
  ttE[k1,k2,j]=cons*DataANDsort[[4]][k1,k2,j,2]
  ttP[k1,k2,j]=cons*DataANDsort[[5]][k1,k2,j,2]
  ttL[k1,k2,j]=cons*DataANDsort[[6]][k1,k2,j,2]}}}
for(k1 in 1:N){
  for(j in 1:a){
  meanE[k1,j]=mean(DataANDsort[[4]][k1,,j,1])
  meanP[k1,j]=mean(DataANDsort[[5]][k1,,j,1])
  meanL[k1,j]=mean(DataANDsort[[6]][k1,,j,1])
  phtE[k1,j]=sum(ttE[k1,,j])
  phtP[k1,j]=sum(ttP[k1,,j])
  phtL[k1,j]=sum(ttL[k1,,j])
  cteE[k1,j]=sum(DataANDsort[[4]][k1,cteLevel:nk,j,3])/w
  cteP[k1,j]=sum(DataANDsort[[5]][k1,cteLevel:nk,j,3])/w
  cteL[k1,j]=sum(DataANDsort[[6]][k1,cteLevel:nk,j,3])/w}}
for(k1 in 1:N){
for(j in 1:a){
   MEAN[k1,j,]=sort(c(meanE[k1,j],meanP[k1,j],meanL[k1,j]))
   PHT[k1,j,]=sort(c(phtE[k1,j],phtP[k1,j],phtL[k1,j]))
   CTE[k1,j,]=sort(c(cteE[k1,j],cteP[k1,j],cteL[k1,j]))}}
RMe = list(meanE,phtE,cteE)
RMp = list(meanP,phtP,cteP)
RMl = list(meanL,phtL,cteL)
RMAll =list(MEAN,PHT,CTE)
```

```
return( list (RMe,RMp,RMl,RMAll))}

#=============================================================
# Computing Gamma
#=============================================================
# Coefficient of the order statistics in equation 3.2.1
K<-function(k){
   kk<-c(rep(NA, k))
   for(i in 1:k){kk[i]<-4*i-2*(k+1)}
   return(kk)}
KU=K(riskM)

# Name of the function: fnGammaT
# Purpose: computing Gamma using the equation 3.2.1
# Input: nk = sample size; RiskMeasure = output of fnRiskMeasure
# Output: gammaMEAN, gammaPHT, gammaPHT = calculated gamma for the risk measures MEAN, PHT,
    and CTE
fnGammaT<-function(nk,RiskMeasure){
gamMEAN<-array(NA,dim=c(N,a,riskM))
gammaMEAN<-array(NA,dim=c(N,a))
gamPHT<-array(NA,dim=c(N,a,riskM))
gammaPHT<-array(NA,dim=c(N,a))
gamCTE<-array(NA,dim=c(N,a,riskM))
gammaCTE<-array(NA,dim=c(N,a))
RMAll=RiskMeasure[[4]]

for(k1 in 1:N){
for(j in 1:a){
  for(j1 in 1:riskM){
  gamMEAN[k1,j,j1]=(KU[j1])*RMAll[[1]][k1,j,j1]
  gamPHT[k1,j,j1]=(KU[j1])*RMAll[[2]][k1,j,j1]
  gamCTE[k1,j,j1]=(KU[j1])*RMAll[[3]][k1,j,j1]}
  gammaMEAN[k1,j]=sum(gamMEAN[k1,j,])/(riskM^2)
  gammaPHT[k1,j]=sum(gamPHT[k1,j,])/(riskM^2)
  gammaCTE[k1,j]=sum(gamCTE[k1,j,])/(riskM^2)}}
return( list (gammaMEAN,gammaPHT,gammaCTE))}

#=============================================================
# Test Statistic
#=============================================================
# Name of the function: fnTstat
# Purpose: computing the test statistic, T
# Input: nk = sample size; GammaT = output of fnGammaT
# Output: Tmean, Tpht, Tcte = calculated T for the risk measures MEAN, PHT, and CTE
fnTstat<-function(nk,GammaT){
Tmean<-array(NA,dim=c(N,a))
Tpht<-array(NA,dim=c(N,a))
Tcte<-array(NA,dim=c(N,a))

for(k1 in 1:N){
```

```
for(j in 1:a){
   Tmean[k1,j]=(GammaT[[1]][k1,j]-gamma)/sqrt(riskM/nk)
   Tpht[k1,j]=(GammaT[[2]][k1,j]-gamma)/sqrt(riskM/nk)
   Tcte[k1,j]=(GammaT[[3]][k1,j]-gamma)/sqrt(riskM/nk)}}
return( list (Tmean,Tpht,Tcte))}

#==========================================================
# Generating Bootstrap Data
#==========================================================
# Name of the function: fnBootData
# Purpose: generating bootstrap data
# Input: nk = sample size; GammaT = output of fnDataANDsort
# Output: "B" represents bootstrap; xeB = exponential; xpB = Pareto; xlB = lognormal; XeB,XpB,XlB =
     ordered xeB, xpB, and xplB respectively
fnBootData<-function(nk,DataANDsort){
xeB<-array(NA,dim=c(N,nk,a,riskM))
xpB<-array(NA,dim=c(N,nk,a,riskM))
xlB<-array(NA,dim=c(N,nk,a,riskM))

XeB<-array(NA,dim=c(N,nk,a,riskM))
XpB<-array(NA,dim=c(N,nk,a,riskM))
XlB<-array(NA,dim=c(N,nk,a,riskM))
sam<-array(NA,dim=c(nk))
eplDataANDsort=DataANDsort[1:3]
sam=sample(1:nk,nk,replace = TRUE)

for( q in 1: riskM){
for(k3 in 1:N){
for(i1 in 1:nk){
for(j in 1:a){
    xeB[k3,i1,j,q]=eplDataANDsort[[1]][k3,sam[i1],j,q]
    xpB[k3,i1,j,q]=eplDataANDsort[[2]][k3,sam[i1],j,q]
    xlB[k3,i1,j,q]=eplDataANDsort[[3]][k3,sam[i1],j,q]}}
for(j in 1:a){
   XeB[k3,,j,q]=x0+xeB[k3,order(xeB[k3,,j,q]),j,q]
   XpB[k3,,j,q]=x0*xpB[k3,order(xpB[k3,,j,q]),j,q]
   XlB[k3,,j,q]=x0+xlB[k3,order(xlB[k3,,j,q]),j,q]}}}
return( list (xeB,xpB,xlB,XeB,XpB,XlB))}

#==========================================================
# Computing Risk Measure of the Bootstrap Data
#==========================================================
# Name of the function: fnBootRiskM
# Purpose: computing the risk measures of samples
# Input: nk = sample size; BootData = output of fnBootData; RiskMeasure = output of fnRiskMeasure
# Output: DmeanB, DphtB, DcteB = sorted risk measures, MEAN, PHT, and CTE, of the bootstrap data
fnBootRiskM<-function(nk,BootData,RiskMeasure){
meanEb<-array(NA,dim=c(N,a))
meanPb<-array(NA,dim=c(N,a))
meanLb<-array(NA,dim=c(N,a))
```

```
DmeanB<-array(NA,dim=c(N,a,riskM))

phtEb<-array(NA,dim=c(N,a))
phtPb<-array(NA,dim=c(N,a))
phtLb<-array(NA,dim=c(N,a))
DphtB<-array(NA,dim=c(N,a,riskM))
ttEb<-array(NA,dim=c(N,nk,a))
ttPb<-array(NA,dim=c(N,nk,a))
ttLb<-array(NA,dim=c(N,nk,a))

cteEb<-array(NA,dim=c(N,a))
ctePb<-array(NA,dim=c(N,a))
cteLb<-array(NA,dim=c(N,a))
cteB1<-array(NA,dim=c(N,a,riskM))
DcteB<-array(NA,dim=c(N,a,riskM))
w=floor(nk*t)
cteLevel=nk-w+1
eplRM=RiskMeasure[1:3]

for(k2 in 1:nk){
cons=(1-(k2-1)/nk)^r-(1-k2/nk)^r
   for(k1 in 1:N){
   for(j in 1:a){
   ttEb[k1,k2,j]=cons*BootData[[4]][k1,k2,j,2]
   ttPb[k1,k2,j]=cons*BootData[[5]][k1,k2,j,2]
   ttLb[k1,k2,j]=cons*BootData[[6]][k1,k2,j,2]}}}
for(k1 in 1:N){
   for(j in 1:a){
   meanEb[k1,j]=mean(BootData[[4]][k1,,j,1])
   meanPb[k1,j]=mean(BootData[[5]][k1,,j,1])
   meanLb[k1,j]=mean(BootData[[6]][k1,,j,1])
   phtEb[k1,j]=sum(ttEb[k1,,j])
   phtPb[k1,j]=sum(ttPb[k1,,j])
   phtLb[k1,j]=sum(ttLb[k1,,j])
   cteEb[k1,j]=sum(BootData[[4]][k1,cteLevel:nk,j,3])/w
   ctePb[k1,j]=sum(BootData[[5]][k1,cteLevel:nk,j,3])/w
   cteLb[k1,j]=sum(BootData[[6]][k1,cteLevel:nk,j,3])/w}}
for(k1 in 1:N){
for(j in 1:a){
   DmeanB[k1,j,]= sort(c((meanEb[k1,j]-eplRM[[1]][[1]][k1,j]),(meanPb[k1,j]-eplRM[[2]][[1]][k1,j]),
   (meanLb[k1,j]-eplRM[[3]][[1]][k1,j])))
   DphtB[k1,j,]= sort(c((phtEb[k1,j]-eplRM[[1]][[2]][k1,j]),(phtPb[k1,j]-eplRM[[2]][[2]][k1,j]),
   (phtLb[k1,j]-eplRM[[3]][[2]][k1,j])))
   DcteB[k1,j,]= sort(c((cteEb[k1,j]-eplRM[[1]][[3]][k1,j]),(ctePb[k1,j]-eplRM[[2]][[3]][k1,j]),
   (cteLb[k1,j]-eplRM[[3]][[3]][k1,j])))}}
return(list(DmeanB,DphtB,DcteB))}
```

```
#==========================================================
#    Computing Bootstrap Gamma
#==========================================================
# Name of the function: fnBootGammaT
# Purpose: computing Gamma of the bootstrap risk measures (see Section 3.4)
# Input: nk = sample size; BootRiskM = output of fnBootRiskM
# Output: gammaMEANb, gammaPHTb, gammaPHTb = calculated gamma for the bootstrap risk measures
      MEAN, PHT, and CTE
fnBootGammaT<-function(nk,BootRiskM){
gamMEANb<-array(NA,dim=c(N,a,riskM))
gammaMEANb<-array(NA,dim=c(N,a))
gamPHTb<-array(NA,dim=c(N,a,riskM))
gammaPHTb<-array(NA,dim=c(N,a))
gamCTEb<-array(NA,dim=c(N,a,riskM))
gammaCTEb<-array(NA,dim=c(N,a))

for(k1 in 1:N){
for(j in 1:a){
for(j1 in 1:riskM){
   gamMEANb[k1,j,j1]=(KU[j1])*BootRiskM[[1]][k1,j,j1]
   gamPHTb[k1,j,j1]=(KU[j1])*BootRiskM[[2]][k1,j,j1]
   gamCTEb[k1,j,j1]=(KU[j1])*BootRiskM[[3]][k1,j,j1]}
   gammaMEANb[k1,j]=sum(gamMEANb[k1,j,])/(riskM^2)
   gammaPHTb[k1,j]=sum(gamPHTb[k1,j,])/(riskM^2)
   gammaCTEb[k1,j]=sum(gamCTEb[k1,j,])/(riskM^2)}}
return( list (gammaMEANb,gammaPHTb,gammaCTEb))}

#==========================================================
# Decision Making
#==========================================================
# Name of the function: fnBootTstat
# Purpose: decision making based on sample T statistics and bootstrap risk measures
# Input: nk = sample size; DataANDsort, RiskMeasure, Tstat = output of fnDataANDsort, fnRiskMeasure,
      fnTstat; B = no of iterations in bootstrap; alpha = significance level
# Output: Tmean, Tpht, Tcte = calculated T for the risk measures MEAN, PHT, and CTE
fnBootTstat<-function(nk,DataANDsort,RiskMeasure,Tstat,B,alpha){
TmeanB<-array(NA,dim=c(B,N,a))
TmeanBB<-array(NA,dim=c(B,N,a))

TphtB<-array(NA,dim=c(B,N,a))
TphtBB<-array(NA,dim=c(B,N,a))

TcteB<-array(NA,dim=c(B,N,a))
TcteBB<-array(NA,dim=c(B,N,a))

decMEAN<-array(NA, dim=c(N,a,al))
decPHT<-array(NA, dim=c(N,a,al))
decCTE<-array(NA, dim=c(N,a,al))
al=length(alpha)
```

```
for(b in 1:B){
BootData=fnBootData(nk,DataANDsort)
BootRiskM =fnBootRiskM(nk,BootData,RiskMeasure)
BootGammaT=fnBootGammaT(nk,BootRiskM)

for(k1 in 1:N){
for(j in 1:a){
   TmeanB[b,k1,j]=(BootGammaT[[1]][k1,j]−gamma)/sqrt(riskM/nk)
   TphtB[b,k1,j]=(BootGammaT[[2]][k1,j]−gamma)/sqrt(riskM/nk)
   TcteB[b,k1,j]=(BootGammaT[[3]][k1,j]−gamma)/sqrt(riskM/nk)}}}
#sorting bootsrap data
for(k1 in 1:N){
for(j in 1:a){
   TmeanBB[,k1,j]=TmeanB[order(TmeanB[,k1,j]),k1,j]
   TphtBB[,k1,j]=TphtB[order(TphtB[,k1,j]),k1,j]
   TcteBB[,k1,j]=TcteB[order(TcteB[,k1,j]),k1,j]}}

for(i in 1:al){
cc=floor(B∗(1−alpha[i]))
for(k1 in 1:N){
for(j in 1:a){
   if( (Tstat [[1]][ k1,j ]) > (TmeanBB[cc,k1,j])){
   decMEAN[k1,j,i]=1}
   else decMEAN[k1,j,i]=0

   if( (Tstat [[2]][ k1,j ]) > (TphtBB[cc,k1,j])){
   decPHT[k1,j,i]=1}
   else decPHT[k1,j,i]=0

   if( (Tstat [[3]][ k1,j ]) > (TcteBB[cc,k1,j])){
   decCTE[k1,j,i]=1}
   else decCTE[k1,j,i]=0}}}
return( list (decMEAN,decPHT,decCTE))}

#=====================================================
# Monte Carlo Simulation
#=====================================================
# Name of the function: fnMonteC
# Purpose: Monte Carlo simlation with M no: of iterations using all the above functions
# Input: nk = sample size; alpha = significance level ; B = no of iterations in bootstrap; M = no of Monte
     Carlo iterations
# Output: MonteC.G, MonteC.T = collection of true or false decision for each iteration
fnMonteC <−function(nk,alpha,B,M){
MonteC.G<−array(NA, dim=c(N,a,al,riskM,M))
MonteC.t<−array(NA, dim=c(N,a,al,riskM,M))

for(i in 1:M){
   U = fnUData(nk)

    DataANDsort.G=fnDataANDsort(nk,U[[1]])
```

```
    DataANDsort.t=fnDataANDsort(nk,U[[2]])

    RiskMeasure.G=fnRiskMeasure(nk,DataANDsort.G)
    RiskMeasure.t=fnRiskMeasure(nk,DataANDsort.t)

    GammaT.G=fnGammaT(nk,RiskMeasure.G)
    GammaT.t=fnGammaT(nk,RiskMeasure.t)

    Tstat.G=fnTstat(nk,GammaT.G)
    Tstat.t=fnTstat(nk,GammaT.t)

   BootTstat.G=fnBootTstat(nk,DataANDsort.G,RiskMeasure.G,Tstat.G,B,alpha)
   BootTstat.t=fnBootTstat(nk,DataANDsort.t,RiskMeasure.t,Tstat.t,B,alpha)
for(j in 1:riskM){
   MonteC.G[,,j,i]=BootTstat.G[[j]]
   MonteC.t[,,,j,i]=BootTstat.t[[j]]}}
return( list (MonteC.G,MonteC.t))}

#=============================================================
# Power function for the Selected Alternatives under Different Dependence
#=============================================================
# Name of the function: fnFinalOUT
# Purpose: evaluating the power of the hypothesis test for each alternative based on four dependence
     structures and selected copula
# Input: MonteC = MonteC.G or MonteC.t output of fnMonte
# Output: MonteC.G, MonteC.T = collection of true or false decision for each iteration
fnFinalOUT<-function(MonteC){
    alMean<-array(NA, dim=c(N,a,al))
    alPht<-array(NA, dim=c(N,a,al))
    alCte<-array(NA, dim=c(N,a,al))

for(k1 in 1:N){
for(j in 1:a){
for(i in 1:al){
    alMean[k1,j,i]=mean(MonteC[k1,j,i,1,])
    alPht[k1,j, i]=mean(MonteC[k1,j,i,2,])
    alCte[k1,j, i]=mean(MonteC[k1,j,i,3,])}}}
return( list (alMean,alPht,alCte))}
```

CURRICULUM VITAE

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EDUCATION

2016 **Ph.D., Mathematics (concentration in Statistics & Actuarial Science)**,
University of Wisconsin - Milwaukee.
Thesis Title : "Comparing the Riskiness of Dependent Porfolios."
Advisors: Prof. Vytaras Brazauskas and Prof. Wei Wei.

2010 **M.S., Mathematical Sciences**,
University of Arkansas at Little Rock, AR, (GPA – 4.00/4.00).

2006 **B.S. (honors), Finance, Business, and Computational Mathematics**,
University of Colombo, Colombo, Sri Lanka.
Thesis Title: "Risk Analysis Involved in a Financial Investment."
Advisors: Dr. R.T. Samaratunga and Dr. Chula Jayawardene.

ACTUARIAL EDUCATION

Society of Actuaries, Schaumburg, IL, USA.

- Passed all preliminary exams: P/1, FM/2, MFE/3F, MLC, C/4.
- Obtained all VEE credits: VEE-Applied Statistics, VEE-Economics, VEE-Corporate Finance.

PROFESSIONAL SOCIETY MEMBERSHIPS

2015–Present American Statistical Association (ASA).

2010–Present American Mathematical Society (AMS).

PUBLICATIONS

2015 Samanthi, R.G.M., Wei, W., Brazauskas, V., "Ordering Gini Indexes of Multivariate Elliptical Risks". *Insurance: Mathematics and Economics* **68**, 84-91.

2015 Samanthi, R.G.M., Wei, W., Brazauskas, V., "Comparing the Riskiness of Dependent Portfolios via Nested L-Statistics". Submitted to *Insurance: Mathematics and Economics.*

CONFERENCE PRESENTATIONS

2015 "Comparing the Riskiness of Dependent Portfolios via Nested L-Statistics". 50th Actuarial Research Conference, University of Toronto, Toronto, Canada.

2014 "Comparing the Riskiness of Dependent Portfolios". 49th Actuarial Research Conference, University of California-Santa Barbara, Santa Barbara, CA.

HONORS AND AWARDS

2016 **Morris and Miriam Marden Award in Mathematics**,
Department of Mathematical Sciences, University of Wisconsin-Milwaukee, WI.

2015–Present **Research Excellence Award**,
Department of Mathematical Sciences, University of Wisconsin-Milwaukee, WI.

2014 & 2015 **SOA Travel Award**,
Society of Actuaries, Schaumburg, IL.

2014 & 2015 **Graduate School Travel Award**,
University of Wisconsin-Milwaukee, WI.

2014 & 2015 **Travel Award**,
Department of Mathematical Sciences, University of Wisconsin-Milwaukee, WI.

2010–2015 **Chancellor's Graduate Student Award**,
University of Wisconsin-Milwaukee, WI.

2010 **Award for Outstanding Teaching by a Graduate Student**,
Department of Mathematics and Statistics, University of Arkansas at Little Rock, AR.

2010 **Award for Outstanding Achievement by a Graduate Student**,
Department of Mathematics and Statistics, University of Arkansas at Little Rock, AR.

2009 & 2010 **Harambee Award for Outstanding Graduate GPA**,
University of Arkansas at Little Rock, AR.