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Optimal Control of Energy Production in a Market with Emission Allowances

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ABSTRACT

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With a growing awareness for preserving the environment, governments started to regulate the greenhouse gas emissions of energy producers by implementing markets for CO2 allowances. Such markets can be found in the European Union with the Emission Trading Scheme (EU ETS). The CO2 emission permit trading is one approach to provide incentives to the power firms to reduce their CO2 emission.

This thesis proposes two models for an optimal control of the energy production rate depending on the energy unit price as well as on the trading of emission derivatives. One model aims to maximize the wealth of the power firm on the short term basis, whereas the other model focuses on the long term wealth maximization. This thesis examines different ways to solve these optimal control problems using techniques like the Hamilton-Jacobian-Bellman equation or convex optimization and compares the different results.

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List of Symbols

α	Markov chain modeling the electricity price changes due to external variations
β	Infinite time horizon discounting factor
γ	Shock intensity of the electricity spot price
$\hat{\sigma}$	Volatility of the electricity unit price
κ	Mean-reverting speed towards the mean electricity unit price
μ	Mean price of the deterministic emission permits process
μ_1	Mean value of the geometric Ornstein-Uhlenbeck process
ν	Mean electricity unit price
Φ	Legendre-Fenchel transformation
Ψ	Maximizer of the Legendre-Fenchel transformation
σ	Volatility function
σ_1	Volatility of the geometric Ornstein-Uhlenbeck process
θ	Amount of allowances

b	Drift function
c	Production cost function
E	Measure of the cumulative emission of greenhouse gas
F	Friction costs associated with trading one emission permit
k_1	Emission factor of the power production
k_2	Percentage of cost for holding one emission allowance
k_3	Percentage of cost for selling one emission allowance
M_0	Initial amount of allowances
P	Electricity unit price
q	Productionrate
r	Mean-reverting speed of the deterministic emission permit price process
r_1	Mean-reverting speed of the geometric Ornstein-Uhlenbeck process
s_2	Drift of the geometric Brownian motion
s_2	Volatility of the geometric Brownian motion
U	Utility function
u	Control
V	Value function
W_t	Brownian motion
X	Wealth of the power firm
Y	Price of one emission permit

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Introduction

Motivation

The development of new machines, the ongoing automation and the rapidly increasing number of electronic devices, creates a quickly growing demand for electric power. To satisfy the worldwide energy needs, more and more power plants are built. This increases not only the energy production but as well the emission of environmentally harmful greenhouse gases. In the climate change report from 2014 the Intergovernmental Panel on Climate Change [IPCC, 2015] encountered that 25% of the global greenhouse gas emission is produced by the electricity production sector.

A growing awareness for preserving the environment and the need to reduce the effects of a human caused climate change, sparked a strong interest in lowering the greenhouse gas emission. This process lead to the Kyoto Protocol in which many governments committed to reduce their emission. One part of the Kyoto protocol is the implementation of CO2 emission markets throughout the world. Such markets can for example be found in the European Union with the Emission Trading Scheme (EU ETS) or in the United States with the Regional Greenhouse Gas Initiative. These markets should provide incentives to the power firms to reduce their greenhouse gas emission in order the reach the goals from the Kyoto protocol.

This thesis proposes two models for maximizing the wealth of an energy producer in such a market with emission derivatives. It further solves the associated optimal control problems and provides an optimal scheme for buying emission certificates and for controlling the energy production rate, depending on the energy and allowance unit price. These solutions are not only intended to help power producing companies to control their production, but also to provide insight into market changes that accompany the implementation of the emission derivative markets.

The Problem

Suppose a power firm produces electricity whose unit price $P(t)$ evolves according to the following mean-reverting stochastic differential equation (SDE):

$$dP_t = \kappa(\alpha(t))[\nu(\alpha(t)) - P_t]dt + \hat{\sigma}(\alpha(t))dW_t + \int_{\mathbb{R}_0} \gamma(\alpha(t-), z)\tilde{N}(dt, dz) \tag{I.2.1}$$

where W is an one-dimensional Brownian motion, $\alpha \in M$ is a continuous-time Markov chain modeling the structural changes of the price due to external economic and/or seasonal variations and N is a Poisson random measure with the compensator $\tilde{N}(t, E) := N(t, E) - t\nu(E)$, in which ν is a Lévy measure satisfying $\int_{\mathbb{R}_0}(1 \wedge |z|^2)\nu(dz) < \infty$. The jump term in (I.2.1) depicts the spot price shocks due to unexpected events such as extreme weather. For each $i \in M$, the constants $\kappa(i)$ represents the mean-reverting speed towards the mean price $\nu(i)$.

It is a common approach to model the energy unit price by a mean-reverting differential equation with jumps to take the spot prices shocks into account. A model similar to (I.2.1) is proposed in [Gonzalez et al., 2017] and can as well be found in [Benth et al., 2008].

Figure I.1 shows a simulated path of the energy unit price process from (I.2.1) without regime switching, initial price $P_0 = 10$, mean value $\nu = 9.25$, mean-reverting speed $\kappa = 10$, volatility of the mean-reversion $\hat{\sigma} = 0.95$, shock intensity $\gamma = 0.9$ and $N \sim Pois(0.95)$. It is to note that an analytical solution to this process exist and it is examined in Appendix A. The following work only relies on the solution $P(t)$ to the process. No further knowledge about the process is needed, thus the energy price model could be changed without any complications.

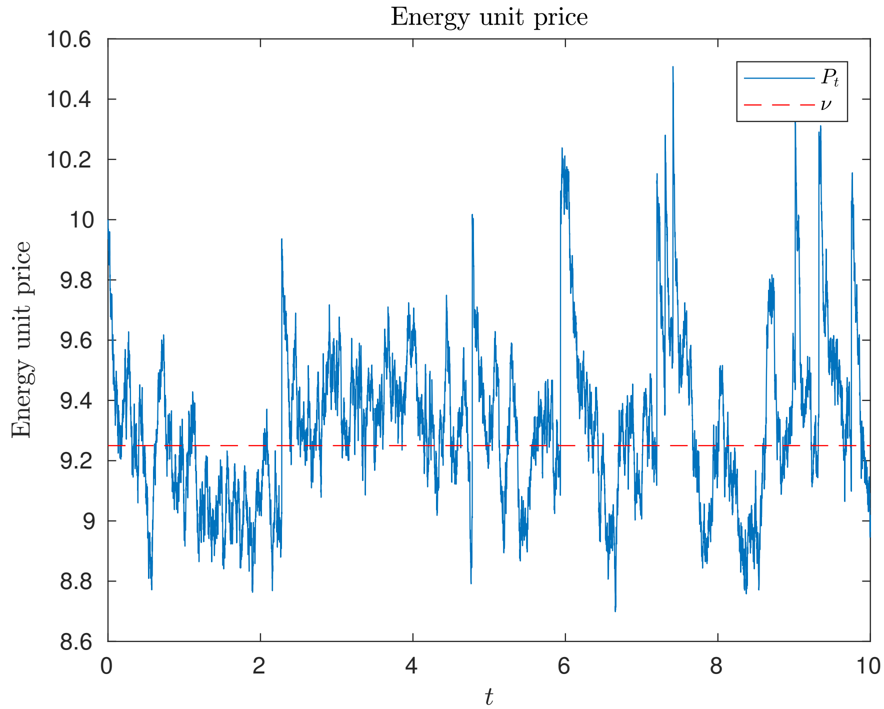


Figure I.1: Simulated path of the energy unit price without regime switching.

At the time t, the firm regulates the instantaneous rate of production $q(t)$ with the associated production cost $c(q(t))$, where $c\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ is assumed to be continuously differentiable (C^1 in notation), strictly convex and satisfying the Inada-like conditions:

$$\lim_{x\to\infty} c'(x) = \infty.$$

Consequently the profit accumulated during the time interval $[0,T]$ is $\int_0^T [P(t)q(t) - c(q(t))]dt$.

The production rate $q(t)$ is a non-negative and bounded function, such that

$$q(t) \in [q_{\min}, q_{\max}]$$

holds, with $q_{\min}, q_{\max} \in \mathbb{R}_+$. This can be interpreted in the sense that an energy plant has a maximum production rate and thus can only produce a limited amount of energy. On the other hand there might be some restrictions, not allowing a full shutdown. To shutdown and restart a coal-firing power station is a very expensive and time consuming process, which is why the production rate should not run below a minimum power output.

The emission regulation mandates that the cumulative emissions of each firm must be measured and that one emission permit has to be held per unit of emission. The price of one emission permit at time t is denoted by Y_t, with Y_t being non-negative.

At the beginning of the emission measurement every power firm receives a specific number of allowances denoted by M_0. The firm may trade the permits, depending on its needs. The amount of allowances held by the energy producer at time t is denoted by $\theta(t)$. For example if the firm decides to sell allowances, θ will be less than M_0, while buying more allowances yields $\theta > M_0$. In the case where the company does not participate in the market, the mount of allowances does not change and thus $\theta = M_0$.

It is assumed that the amount of allowances held by a firm is bounded and thus given by

$$\theta \in [\theta_{\min}, \theta_{\max}],$$

with $\theta_{\min}, \theta_{\max} \in \mathbb{R}_+$. The minimal amount of allowances that a company can hold is restricted by $\theta_{\min} \geq 0$. Otherwise the company has to consume greenhouse gas instead of emitting it. A natural upper bound to the amount of emission certificates a company can buy, is given by total amount of certificates in the market

Next we derive two models to maximize the wealth of the firm on different time scales.

Finite time horizon

Assume that the wealth of an energy producer is modeled on a finite time interval $[0, T]$ with $T > 0$. This could be interpreted in the way that the energy producer is interested in maximizing its wealth within the near future, e.g. the next two years. This model allows the company to plan for a short amount of time.

For this case it is assumed that the cumulative emission of greenhouse gas up to time t is given by $E_t \colon = \int_0^t k_1 q(t)dt$. This can be understood in the sense that the emission of greenhouse gas is proportional to the production rate q by some constant k_1, where k_1 reflects the emission efficiency of the company's power plants. At the end of the period $[0, T]$, the cost due to the emission regulation is $Y_T E_T$.

Under the above setup, the terminal wealth of a firm at time T is modeled by

$$X(T) \colon = X^{q,\theta}(T) = x + \int_0^T \theta(t)dY_t + \int_0^T [P(t)q(t) - c(q(t))]dt - Y_T E_T, \tag{I.3.1}$$

where x is the initial wealth of the firm. For a given concave utility function $U \colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying the Inada conditions, our task is to maximize the expected utility

$$V(x) \colon = \sup_{(q,\theta)\in\mathcal{A}} \mathbb{E}[U(X^{q,\theta}(T))] \colon X^{q,\theta}(t) \geq 0 \text{ for all } 0 \leq t \leq T\,, \tag{I.3.2}$$

where $\mathcal{A}$ is the set of admissible production and trading strategies (q, θ).

Infinite time horizon

The case for a power firm that is more interested in maximizing the long-term wealth is developed in this section.

Now the power producer wants to maximize its wealth for a long period of time. It can be assumed that an energy supplier has long-term contracts with the government that come with the obligation to secure the energy production for a certain amount of time, for example 20 years. In this case the company might be more interested in a strategy that maximizes its wealth in the long run and therefore the wealth is modeled on the infinite time interval $[0, \infty)$.

For this case assume that there are friction costs associated with holding or trading an amount of emission permits. This is expressed by

$$F(\theta_t) := k_2 I_{\{\theta_t \geq M_0\}} - k_3 I_{\{\theta_t < M_0\}}, \tag{I.4.1}$$

where $k_2 \geq 0$ is the percentage cost for holding one allowance and $k_3 \geq 0$ represents the percentage cost of selling one emission permit. Thus the friction cost F is the percentage cost for buying or selling one emission permit on the market.

Taking the friction cost into account, the wealth of the power firm is given by

$$X(T) = x + \int_0^T \theta(t) dY_t + \int_0^T [P(t)q(t) - c(q(t))]dt - \int_0^T \theta(t) Y_t F(\theta_t) dt \tag{I.4.2}$$

Note that the terminal condition in (I.3.1) is now expressed by the friction cost, which leads to the following definition of the discounted optimization problem for the total expected wealth

$$V(x) := \sup_{(q,\theta)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(X^{q,\theta}(t))dt\right], \tag{I.4.3}$$

where $\beta \geq 0$ is the discounting factor while the utility function U satisfies the same assumptions stated in (I.3.1). Again $\mathcal{A}$ is the set of admissible production and trading strategies (q, θ).

Structure of the thesis

This thesis is structured into 5 Chapter. The first chapter provides general background information and motivates the thesis in Section I.1. It introduces the reader to the central optimization problem in Section I.2 and explains the energy spot price model that will be used throughout the thesis. Section I.3 states the model and formulates the problem for the finite time horizon wealth optimization and Section I.4 does the same for the infinite time horizon optimization.

The intention of Chapter II is to provide the mathematical background for solving the stochastic optimal control problems. Section II.1 introduces the theory for stochastic control with a focus on infinite time horizon optimal control. In addition to the theory it states the Hamilton-Jacobi-Bellman equation as a central tool for approaching such control problems. Section II.2 provides some results from the theory of convex optimization, in particular for minimizing the cost function that is associated to the production rate.

Chapter III deals with the finite time horizon optimal control problem. It firstly examines an optimal production rate for an energy producer that is not participating in a greenhouse gas emission allowance market in Section III.1. Section III.2 derives two different models for the case of a deterministic emission permits price and compares them. One of the models follows the idea presented in [Carmona et al., 2012]. The last section in Chapter III examines a solution to the finite time horizon optimal control problem where the price dynamic of the CO2 allowances is given by an Itó diffusion processes. The Chapter is concluded by two examples.

Finally Chapter IV examines the infinite time horizon optimal control problem. Section IV.1 utilizes the Hamilton-Jacobi-Bellman equation to determine the optimal controls. Finally Section IV.2 solves the control problem for arbitrary Itó diffusion processes in the spirit of Section III.3. The two examples from the finite time horizon are revisited to apply the theoretic results. The thesis closes with a summary of the results.

Mathematical Background and Stochastic Control Theory

In this Chapter we recall the theoretical background for solving stochastic control problems. Therefore Section II.1 starts with a short motivation for using stochastic models in control theory. It further introduces common nomenclature and notation for a general description of a stochastic control problem in Section II.1.1. Finally we motivate the HJB equation by the Dynamic Programming Principle in Section II.1.2 as a standard methodology for solving stochastic control problems.

Section II.2 states the general form of the Legendre-Fenchel transformation from the theory of convex optimization. It further examines some properties of the Legendre-Fenchel transformation, when it is applied to the cost function introduced in section I.2. These results will be essential for the analysis of the wealth models in Chapter III and Chapter IV.

Stochastic Control Theory

In a world driven by uncertainty, science aims to describe the randomness in every event to control the uncertainty in the environment. The highly developed theory of stochastics is an effort made by mathematicians to describe randomness and to provide tools to solve problems which have random influence. One part of this theory is to describe controllable dynamical systems, with the aim to optimize their output. This is called stochastic control theory. The following gives a short introduction into this extensive theory.

Stochastic Optimal Control on an infinite time horizon

The state of a stochastic system at time t, with the known *initial state* $x_0 \in \mathbb{R}^n$ at time 0, can be described by the stochastic differential equation

$$\begin{cases} dX_t = dX_t^{x_0} = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t, \\ X_0 = x_0 \end{cases} \tag{II.1.1}$$

where $X_0, X_t \in \mathbb{R}^n$, $b : \mathbb{R} \times \mathbb{R}^n \times U \mapsto \mathbb{R}^n$, $\sigma : \mathbb{R} \times \mathbb{R}^n \times U \mapsto \mathbb{R}^{n \times m}$ and W_t is a m-dimensional Brownian motion. We call $u_t \in U \subset \mathbb{R}^k$ the *control* of the system and a solution X_t of the differential equation is the *state trajectory*.

The functions b and σ are required to satisfy the Lipschitz condition, which means that there exists a constant K, such that

$$|b(t, x, u) - b(t, y, u)| + |\sigma(t, x, u) - \sigma(t, y, u)| \leq K|x - y| \quad \forall x, y \in \mathbb{R}^n, u \in U$$

is satisfied. We further denote by $\mathcal{A}_0$ the set of control processes and $u_t \in \mathcal{A}_0$ if

$$\mathbb{E}\left[\int_0^T |b(0, u_t)|^2 + |\sigma(0, u_t)|^2 dt\right] < \infty, \quad \forall\, T > 0.$$

Thus u_t needs to be $\mathcal{F}_t$-adapted, where $\mathcal{F}_t$ is the filtration generated by the Brownian motion W_t. This necessity makes sense in the way that when a new control needs to be chosen at time t, all the past information about the process up-to this time point is known. Thus the decision can be based on the previous behavior of the system. The control process $u_t : [0, T] \times \mathbb{R}^n \mapsto U$ is a *Markov Control*, if the decision at time t only depends on the state X_t of the system at that time. It is called a Markov control, because with u_t chosen this way, X_t becomes a Markov process.

Under the assumption that a solution X_t to (II.1.1) exists, the control can be interpreted in the way that at the time t, the *input* u_t to the system will result in the output X_t. Therefore we call such a system a *controlled system*. The solution $X_t^{x_0}$ denotes, the state of the system at time t for the initial value x_0 at starting time 0. This can be also expressed by

$$X_t^{x_0} = x_0 + \int_0^t b(s, X_s^{x_0}, u_s)ds + \int_0^t \sigma(s, X_s^{x_0}, u_s)dW_s.$$

To compare different controls and to measure their performance, a *cost functional* or *performance functional* is defined by

$$J(x, u(\cdot)) = \mathbb{E}\left[\int_0^\infty e^{-\beta s} f(X_s^x, u_s)ds\right], \tag{II.1.2}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}$ represents the *profit rate* or *utility rate*, $\beta > 0$ the discount factor and $u_t \in \mathcal{A}(x)$, where $\mathcal{A}(x)$ denotes the set of *admissible controls* given the initial condition x. A control is admissible, if it satisfies

$$\mathbb{E}\left[\int_0^\infty e^{-\beta s}|f(X_s^x, u_s)|ds\right] < \infty.$$

With this notation we can define the corresponding value function

$$v(x) = \sup_{u(\cdot)\in\mathcal{A}(x)} J(x, u(\cdot)). \tag{II.1.3}$$

We say that a control $u^\star(\cdot) \in \mathcal{A}(x)$ is *optimal* for the initial condition x, if $v(x) = J(x, u^\star(\cdot))$. Thus the goal of the optimal control problem is to find such an optimal $u^\star(\cdot)$.

This introduction follows the ideas presented in [Øksendal, 2003], [Pham, 2009] and [Yong and Zhou, 1999]. The interested reader is referred to these references for a deeper and more detailed introduction to stochastic control theory.

Hamilton-Jacobi-Bellman equation

After stating the optimal control problem in (II.1.3), we will now introduce one method to solve this problem. This method relies on the dynamic programming principle (DPP).

Theorem II.1.4 (Dynamic programming principle on an infinite horizon)**.** *Let $x \in \mathbb{R}^n$ and $\mathcal{T}$ be a collection of stopping times, then we have*

$$\begin{aligned} v(x) &= \sup_{u \in \mathcal{A}(x)} \sup_{\theta \in \mathcal{T}} \mathbb{E}\left[\int_0^\theta e^{-\beta s} f(X_s^x, u_s) ds + e^{-\beta\theta} v(X_\theta^x)\right] \\ &= \sup_{u \in \mathcal{A}(x)} \inf_{\theta \in \mathcal{T}} \mathbb{E}\left[\int_0^\theta e^{-\beta s} f(X_s^x, u_s) ds + e^{-\beta\theta} v(X_\theta^x)\right], \end{aligned}$$

with the convention that $e^{-\beta\theta} = 0$ when $\theta(\omega) = \infty$.

Proof. The proof can be found in [Pham, 2009]. □

The DPP basically says that the value function in (II.1.3) can be obtained by splitting the problem into two parts. Thus an optimal control on the interval $[0, T]$ can be found by first searching for an optimal control starting from $\theta \in [0, T]$ with the state $X_\theta^{0,x}$ and then maximizing over controls from $[0, \theta]$ with the performance function v. Thus we optimize over

$$\mathbb{E}\left[\int_0^\theta e^{-\beta s} f(X_s^x, u_s) ds + e^{-\beta\theta} v(X_\theta^x)\right]$$

Using the DPP we can now derive the Hamilton-Jacobi-Bellman equation, one of the most important theorems in stochastic control theory.

Theorem II.1.5 (Hamilton-Jacobi-Bellman (HJB) equation and verification theorem)**.**
i) Assume that $w \in C^2(\mathbb{R}^n)$ is a smooth function fulfilling a quadratic growth condition and satisfying the HJB equation given by

$$\beta w(x) - \sup_{u \in U} [\mathcal{L}^u w(x) + f(x, u)] = 0 \quad \forall x \in \mathbb{R}^n$$

with

$$\mathcal{L}^u w = b(x,u)D_x(w) + \frac{1}{2}tr(\sigma(x,u)\sigma^T(x,u)D_x^2 w).$$

ii) Further assume that for all $x \in \mathbb{R}^n$, there exists a measurable function $u^\star(x), x \in \mathbb{R}^n$, valued in U such that

$$\beta w(x) - \sup_{u\in U}[\mathcal{L}^u w(x) + f(x,u)] = \beta w(x) - \mathcal{L}^{u^\star(x)} w(x) - f(x,u^\star(x)) = 0,$$

the SDE

$$dX_s = b(X_s, u^\star(X_s))ds + \sigma(X_s, u^\star(X_s))dW_s$$

admits a unique solution, denoted by $X_s^{\star x}$, given an initial condition $X_0 = x$, satisfying

$$\liminf_{T\to\infty} e^{-\beta T}\mathbb{E}[w(X_T^{\star x})] \leq 0$$

and the process $\{u^\star(X_s^{\star,x}), s \geq 0\}$, lies in $\mathcal{A}(x)$. Then

$$w(x) = v(x), \forall x \in \mathbb{R}^n,$$

where v(x) is the value function defined in (II.1.3) *and $u^\star$ is an optimal Markovian control.*

Proof. A derivation of the HJB equation and the verification theorem can be found in Chapter 3 of [Pham, 2009]. □

Solving the HJB equation not only yields a candidate for the value function defined in (II.1.3), it also reveals the optimal control $u^\star$. The verification theorem from *ii*) of Theorem II.1.5 proves that the obtained candidate and the value function coincide.

The Hamilton-Jacobi-Bellman equation can be interpreted in the way that the stochastic control problem can be transformed into a deterministic ordinary differential equation and then solved in a deterministic setting. It is important to note that the HJB equation can not always be solved and yields only in some cases a closed-form solution. But even if the analytic solution can not be found it is normally sufficient to numerically approximate a solution.

This introduction represents the result stated in [Pham, 2009].

Legendre-Fenchel transform

In this Section we recall convex optimization, in particular the properties of the Legendre-Fenchel transformation. One of the early papers published in this area is [Fenchel, 1949] by Werner Fenchel. It provides a method to solve a convex primal problem by its associated dual problem.

Definition II.2.1 (Legendre-Fenchel transformation)**.** *Let $c : \mathbb{R} \mapsto \mathbb{R} \cup \{-\infty, \infty\}$ be a lower semicontinous, convex function and $A = dom(c)$. The Legendre-Fenchel transformation of the function $c(\cdot)$ is defined by*

$$\Phi(y) = \sup_{x \in A} \{yx - c(x)\} \tag{II.2.2}$$

For our purpose we chose $c(\cdot)$ to be the production cost function as defined in I.4. Thus $c(\cdot)$ satisfies the conditions of the Definition II.2.1.

In the context of the problems in (I.3.2) and (I.4.3), the Legendre-Fenchel transformation can be interpreted in the following manner. The primal problem is to minimize the cost function $c(\cdot)$ associated to the production rate. But instead of solving the primal problem, we solve the dual problem defined by the Legendre-Fenchel transformation.

Next we define the unique maximizer of the Legendre-Fenchel transformation.

Definition II.2.3. *Let Φ be given by Definition II.2.1. Then define Ψ to be the smallest*

$x^\star \in A$ such that Φ attains its maximum.

In addition to the conditions stated in section I.4, we further require that $c(0) = 0$ and

$$\lim_{x \to \infty} \frac{c(x)}{x} = \infty.$$

Since the cost function is defined on the production rate q, the set A of possible maximizer is given by $A = [q_{\min}, q_{\max}]$.

With the unique maximizer Ψ and the conditions on $c(\cdot)$, we can examine the convexity of the Legendre-Fenchel transformation in the spirit of [George and Harrison, 2001].

Theorem II.2.4. *Φ is convex on $[0, \infty)$*

Proof. Fix $y_0 \geq 0$ and define $x_0 = \Psi(y_0)$, thus $\Phi(y_0)$ becomes

$$\Phi(y_0) = y_0 x_0 - c(x_0). \tag{II.2.5}$$

Then for any $y \geq 0$

$$y x_0 - c(x_0) \leq \sup_{x \in A} \{yx - c(x)\} = \Phi(y). \tag{II.2.6}$$

By solving (II.2.5) for $c(x_0)$ and substitution the solution into (II.2.6) finally gives

$$\Phi(y) \geq \Phi(y_0) + x_0(y - y_0). \tag{II.2.7}$$

Thus Φ is convex on $[0, \infty)$. □

The convexity of Φ is helpful for comparing different values of Φ, but strict convexity of the Legendre-Fenchel transformation would be even more appealing. In the following, it will be shown that Φ is strictly convex assuming some properties of the cost function.

Theorem II.2.8. *Let* Φ *be given as defined in Definition II.2.1. Then* Φ *is strictly convex, if the cost function* $c(\cdot)$ *is strictly convex.*

Proof. To prove that Φ is strictly convex, it needs to be shown that Φ' is strictly increasing. To simplify the derivative of Legendre-Fenchel transformation, it is helpful to first express Φ by its maximal value. This can be done by replacing x with the maximizer Ψ

$$\Phi(y) = \Psi(y)y - c(\Psi(y)).$$

Hence the derivative of Φ becomes

$$\Phi'(y) = \Psi'(y)(y - c'(\Psi(y)) + \Psi(y) = \Psi(y).$$

Note that the difference $y - c'(\Psi(y)) = 0$ by the necessary condition of Ψ.

Thus Φ is strictly convex if Ψ is strictly increasing. Recall that Ψ is defined by

$$\Psi(y) = (c')^{-1}(y)$$

By assumption the cost function $c(\cdot)$ is strictly convex, such that its derivative is strictly increasing as well as the inverse $(c')^{-1}$. □

With this we have all properties that will be needed in the following for Φ and Ψ. [George and Harrison, 2001], [Pham, 2009] and [Karatzas and Shreve, 1998] provide a deeper analysis and more properties of the Legendre-Fenchel transform and its maximizer.

Optimal control for the finite time horizon problem

In this Chapter we examine multiple solutions for the optimal control problem for the finite time horizon, as stated in Section I.3. To introduce the general idea of finding the optimal control, we will firstly examine in Section III.1 the case that no emission derivatives exist. The next Section introduces a market for CO2 emission allowances, where the allowance prices will be determined by a deterministic mean-reverting process. Therefore we introduce the new model with an example and derive two solutions. One of the solutions follows the idea provided by [Carmona et al., 2012]. Finally we show that only one of the two solutions is optimal.

In the Section III.3 we derive a solution to the general case, where the price dynamic of the emission derivatives is given by an Itó diffusion process. To conclude the chapter we present two examples for the optimal control problem on the finite time horizon.

Green energy

Before we start to examine some more general approaches on how to maximize the wealth of an energy producer on a finite time horizon, we introduce the problem in the case of a market without emission permits. For the positive interpretation of the approach one can simply assume that the companies found a way to produce energy without emitting any greenhouse gas to the environment. A different interpretation for the same result would be that the government is not interested in the reduction of greenhouse gas emission and therefore there is no need for emission permits. This interpretation could be called the business as usual.

Assuming that there is no need for holding emission derivatives the initial wealth process from (I.3.1) simplifies into

$$X^q(T) = x + \int_0^T [P(t)q(t) - c(q(t))]dt. \tag{III.1.1}$$

Maximizing the expected utility of this wealth process leads to the following result.

Theorem III.1.2. *The expected utility of the wealth of a firm at the end of a time period* $[0, T]$*, described by* (III.1.1) *is maximized by the controlled wealth process with optimal production rate* $q^\star(t)$ *at time* $t \in [0, T]$

$$q^\star(t) = \Psi(P(t))$$

Proof. Applying the Legendre-Fenchel transformation from Definition II.2.1 to (III.1.1), yields the following

$$x + \int_0^T \Phi(P(t))dt.$$

Therefore the optimal production rate is given by the maximizer $\Psi(P(t))$ which is related to $\Phi(P(t))$. □

This simple example provides some intuition on how to obtain the optimal production rate.

Deterministic Case

In this Section emission allowances will be introduced into the wealth process and the result will be compared with one found in the literature.

Let us therefore first examine the control problem of maximizing the terminal wealth as formulated in (I.3.2) with a simple example and compare it to a solution found by [Carmona et al., 2012]. For the simple case we assume that the dynamic of the emission derivative is given by a deterministic process. Therefor let the price of the emission permit Y_t be a deterministic mean reverting process

$$dY_t = r(\mu - Y_t)dt \tag{III.2.1}$$

with the solution

$$Y_t = \mu + (Y_0 - \mu)e^{-rt}.$$

With these assumptions the terminal wealth from (I.3.1) simplifies into the following form

$$X(T) = x + \int_0^T \theta(t)r(\mu - Y_t)dt + \int_0^T [P(t)q(t) - c(q(t))]dt - Y_T E_T. \tag{III.2.2}$$

Figure III.2 compares the deterministic mean-reverting process defined by (III.2.1) with initial price $Y_0 = 6$, mean value $\mu = 6.5$ and the mean-reverting speed r for the values $[0.5, 1, 2]$. The plot shows that the process reverts faster to its mean value for a larger reversion rate.

Starting from (III.2.2) we will firstly follow the idea proposed in [Carmona et al., 2012] to find an optimal control. For simplicity we will denote the first approach with $X_1(T)$. Through integration by parts, $Y_T E_T$ can be written as

$$\begin{aligned} Y_T E_T &= \int_0^T Y_t dE_t + \int_0^T E_t dY_t \\ &= \int_0^T k_1 Y_t q(t)dt + \int_0^T k_1 \int_0^t q(s)ds dY_t. \end{aligned}$$

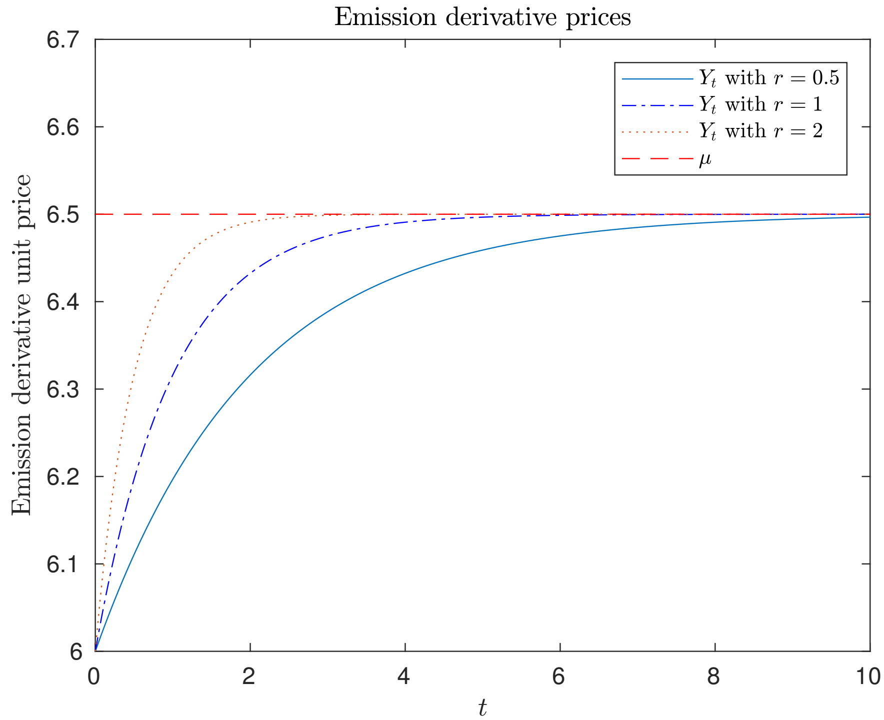


Figure III.2: The emission derivative unit price with different values for the mean-reverting speed.

Next substitute this expression for $Y_T E_T$ into (III.2.2), to obtain the process

$$\begin{aligned}X_1(T) &= x + \int_0^T \theta(t)dY_t + \int_0^T [P(t)q(t) - c(q(t))]dt \\ &\quad - \int_0^T k_1 Y_t q(t)dt - \int_0^T k_1 \int_0^t q(s)dsdY_t \\ &= x + \int_0^T (\theta(t) - k_1 \int_0^t q(s)ds)dY_t \\ &\quad + \int_0^T [P(t) - k_1 Y_t]q(t) - c(q(t))dt \\ &= A^{\tilde{\theta}}(T) + B^q(T),\end{aligned} \tag{III.2.3}$$

with

$$A^{\tilde{\theta}}(T) = \int_0^T \tilde{\theta}(t)dY_t \text{ in which } \tilde{\theta}(t) = \theta(t) - k_1 \int_0^t q(s)ds$$

and

$$B^q(T) = x + \int_0^T [P(t) - k_1 Y_t]q(t) - c(q(t))dt$$

independent of $\theta(t)$. Which means that the optimal solution $(q_1^\star, \tilde{\theta}_1^\star)$ can be found by firstly maximizing $B^q(T)$ to obtain $q^\star$ and then optimizing $A^{\tilde{\theta}}(T)$ using $q^\star$. The following lemma summarizes the result.

Lemma III.2.4. *The optimization of the wealth of a firm can be be solved by performing two optimizations*

$$\sup_{(q,\theta)\in\mathcal{A}} \mathbb{E}[U(X_1^{q,\theta}(T))] = \sup_{\tilde{\theta}\in\mathcal{A}} \sup_{q\in\mathcal{A}} \mathbb{E}[U(A^{\tilde{\theta}}(T) + B^q(T))] \tag{III.2.5}$$

or in other words, the maximum value of $q^\star$ does not depend on the value of $\tilde{\theta}$.

Proof. The proof can be found in [Carmona et al., 2012]. □

In the next step we will determine the optimal control $(q_1^\star, \tilde{\theta}_1^\star)$ for $X_1(T)$.

Theorem III.2.6. *The expected utility of the wealth of a power firm at the end of a time period $[0,T]$, described by* (III.2.3) *is maximized by the controlled wealth process with optimal production rate*

$$q_1^\star(t) = \Psi(P(t) - k_1 Y_t)$$

at time $t \in [0,T]$ and the optimal amount of greenhouse gas allowances

$$\tilde{\theta}^\star(t) = \sup_{\tilde{\theta}\in\mathcal{A}} \int_0^T \tilde{\theta}(t) dY_t = \int_0^T (\theta_1^\star(t) - k_1 \int_0^t q^\star(s) ds) dY_t$$

with

$$\theta_1^\star(t) = \begin{cases} \theta_{\max} & \text{if } (\mu - Y_0) \geq 0. \\ \theta_{\min} & \text{otherwise} \end{cases}$$

Proof. First note that instead of maximizing $X_1(T)$ we can firstly maximize $A^{\tilde{\theta}}(T)$ and then $B^q(T)$ by Lemma III.2.4, since the utility function $U(x)$ satisfies the Inada conditions.

The optimal production rate $q_1^\star(t)$ is determined by applying the Legendre-Fenchel transformation to $B^q(T)$. This gives the following

$$\int_0^T \Phi(P(t) - k_1 Y_t)dt$$

and thus the optimal control is then given by $\Psi(P(t) - k_1 Y_t)$.

Finally we have to find the optimal $\theta_1^\star$ that maximizes $A^{\tilde{\theta}}(T)$. With dY_t being the deterministic mean reverting process as defined in (III.2.1), leads to

$$A^{\tilde{\theta}}(T) = \int_0^T \tilde{\theta}(t) r(\mu - Y_0) e^{-rt} dt$$

Without loss of generality we can assume that $(\mu - Y_0) \geq 0$. Now the maximum of $A^{\tilde{\theta}}(T)$ only depends on $\tilde{\theta}$ since r and e^{-rt} are positive. Using Lemma III.2.4 and $q^\star(t)$, we know that $\tilde{\theta}(t)$ is maximized if $\theta(t)$ is maximized.

To end the proof we only miss the case if $(\mu - Y_0) < 0$. In this case $\theta(t)$ will be chosen to be minimal in order to maximize $A^{\tilde{\theta}}(T)$ and finally the optimal amount of emission permits is given by

$$\theta_1^\star(t) = \begin{cases} \theta_{\max} & \text{if } (\mu - Y_0) \geq 0, \\ \theta_{\min} & \text{otherwise.} \end{cases}$$

□

In the next step we derive a different approach to solve the optimal control problem in (I.3.1). We denote this approach with $X_2(T)$. Starting from (III.2.3), we apply Fubini's theorem to the double integral and solving the inner integral gives

$$
\begin{aligned}
X_2(T) =& x + \int_0^T \theta(t) r(\mu - Y_0) e^{-rt} dt \\
&+ \int_0^T [P(t) - k_1 Y_T] q(t) - c(q(t)) dt \\
=& A^\theta(T) + B^q(T)
\end{aligned}
\tag{III.2.7}
$$

with

$$A^\theta(T) = x + \int_0^T \theta(t) r(\mu - Y_0) e^{-rt} dt$$

independent of $q(t)$ and

$$B^q(T) = \int_0^T [P(t) - k_1 Y_T] q(t) - c(q(t)) dt$$

only depending on $q(t)$.

To find the optimal controls to the process $X_2(T)$, we simply need to maximize $A^\theta(T)$ and $B^q(T)$ independently of each other. The solution to this is given by the next theorem.

Theorem III.2.8. *The expected utility of the wealth of a power firm at the end of a time period* $[0, T]$ *described by* (III.2.7) *is maximized by the controlled wealth process with optimal production rate*

$$q_2^\star(t) = \Psi(P(t) - k_1 Y_T)$$

at time $t \in [0, T]$ *with and the optimal amount of greenhouse gas allowances*

$$\theta_2^\star = \sup_{\theta \in \mathcal{A}} \int_0^T \theta(t) r(\mu - Y_0) e^{-rt} dt$$

with the solution

$$\theta_2^\star = \begin{cases} \theta_{\max} & \text{if } (\mu - Y_0) \geq 0, \\ \theta_{\min} & \text{otherwise.} \end{cases}$$

Proof. As already mentioned the process $X_2(T)$ defined in (III.2.7) attains its maximum, if both $A^\theta(T)$ and $B^q(T)$ are maximized.

To obtain the optimal production rate $q_2^\star(t)$, we again apply the Legendre-Fenchel transformation to $B^q(T)$ from (III.2.7), to get

$$\int_0^T \Phi(P(t) - k_1 Y_T)dt. \tag{III.2.9}$$

With the optimal production rate given by the maximizer $\Psi(P(t)-k_1Y_T)$ from the Legendre-Fenchel transformation.

Now it remains to show that

$$\theta_2^\star = \begin{cases} \theta_{\max} & (\mu - Y_0) \geq 0, \\ \theta_{\min} & \text{otherwise}, \end{cases}$$

maximizes $A^\theta(T)$. But this is true, since it is given by

$$\sup_{\theta \in \mathcal{A}} \int_0^T \theta(t) r(\mu - Y_0) e^{-rt} dt$$

and $r > 0$ and e^{-rt} are positive. □

With the two optimal controls $(q_1^\star, \tilde{\theta}^\star)$ and $(q_2^\star, \theta^\star)$ for $X_1(T)$ and $X_2(T)$ the question remains, whether the controls result in the same value for $X_1^{q_1^\star, \tilde{\theta}^\star}(T)$ and $X_2^{q_2^\star, \theta^\star}(T)$ or not?

Before we are going to answer this question, we first compare $\theta_1^\star$ and $\theta_2^\star$ in the next lemma.

Lemma III.2.10. *The optimal amount of allowances $\theta^\star(t)$ held at time $t \in [0,T]$ is the same for the two different models $X_1(T)$ and $X_2(T)$.*

Proof. This result follows immediately from the definition of $\theta_1^\star(t)$ and $\theta_2^\star(t)$ in (III.2.6) and (III.2.8). □

Lemma III.2.10 implies that only $q_1^\star$ and $q_2^\star$ have to be compared to analyze the differences in the processes $X_1^{q_1^\star,\tilde{\theta}^\star}(T)$ and $X_2^{q_2^\star,\theta^\star}(T)$.

Using the properties of the Legendre-Fenchel transformation we can show that only $X_2^{q_2^\star,\theta^\star}(T)$ is optimal. This result is summarized in the following theorem.

Theorem III.2.11. *Under the assumption that the emission allowance price is determined by the deterministic mean-reverting process given by* (III.2.1) *and the condition that the cost function c is strictly convex, the following relation holds:*

$$X_1^{q_1^\star,\tilde{\theta}^\star}(T) < X_2^{q_2^\star,\theta^\star}(T)$$

Proof. To begin the proof recall that by Lemma III.2.10 the optimal amount of allowances $\theta^\star$ is the same for the two processes and therefore can be omitted in the following. The maximization of $B^q(T)$ for the $X_1(T)$ process given by (III.2.3), can be stated in terms of Legendre-Fenchel transformation and its unique maximizer Ψ and thus

$$X_1^{q_1^\star,\theta^\star}(T) = x + \int_0^T \theta^\star(t)dY_t - \int_0^T k_1 \left(\int_0^t \Psi(P(s) - k_1 Y_s)ds \right) dY_t + \int_0^T \Phi(P(t) - k_1 Y_t)dt. \tag{III.2.12}$$

Applying Fubini's theorem to the second integral in (III.2.12) to change the order of integration and evaluating the inner integral gives the following

$$X_1^{q_1^\star,\theta^\star}(T) = x + \int_0^T \theta^\star(t)dY_t + \int_0^T \Phi(P(t) - k_1 Y_t) - \Psi(P(t) - k_1 Y_t)k_1(Y_T - Y_t)dt$$

Now using (II.2.7) to conclude

$$
\begin{aligned}
X_1^{q_1^\star,\tilde{\theta}^\star}(T) &< x + \int_0^T \theta^\star(t) dY_t + \int_0^T \Phi(P(t) - k_1 Y_T) dt \\
&< X_2^{q_2^\star,\theta^\star}(T).
\end{aligned}
$$

The strict inequality holds since the cost function is strictly convex by assumption and thus by Theorem II.2.8 the Legendre-Fenchel transformation is strictly convex. □

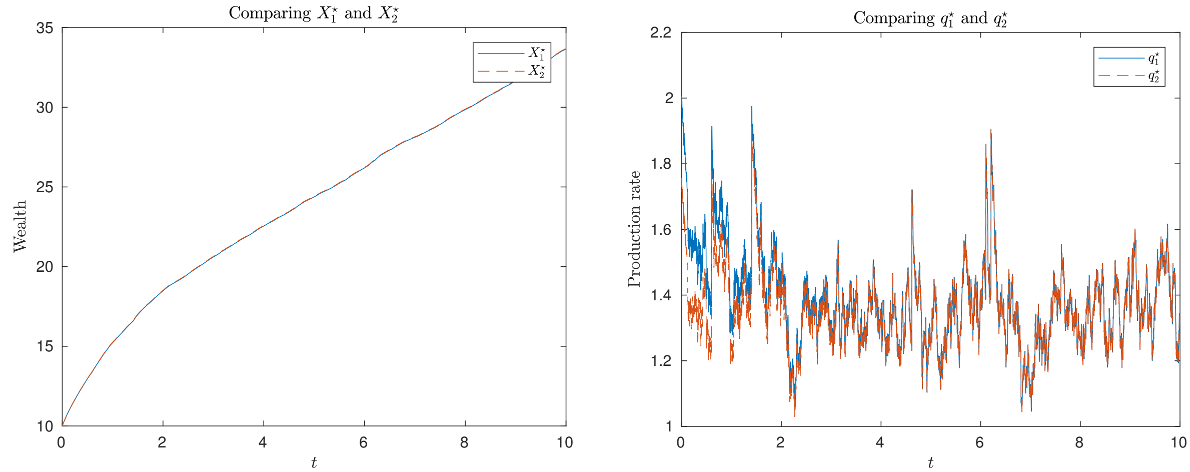
This result shows that if the prices of the emission permits is given by a deterministic mean reverting process, the maximal wealth of a power producer at time T is given by $X_2^{q_2^\star,\theta^\star}(T)$.

Remark III.2.13. *For both processes, we are actually interested in maximizing the expected utility U of the wealth process $X(T)$. But under the given assumptions and the absence of any randomness the utility is maximized, if the wealth process is maximized.*

Remark III.2.14. *It is further to note that this result does not contradict the work of [Carmona et al., 2012]. Under the given assumptions it is possible to find a pair of controls such that the solution found by [Carmona et al., 2012] is not optimal. But [Carmona et al., 2012] requires that there exists a measure $\mathbb{Q}$ under which the Y_t process is martingale. Since we chose Y_t to be deterministic, this requirement is not satisfied.*

Numerical solutions

In the following we examine numerical results to the solutions given by Theorem III.2.6 and Theorem III.2.8. To compare the solutions for the two processes the emission permit price is generated with initial price $Y_0 = 6$, mean value $\mu = 6.5$ and mean-reverting rate $r = 1$. For computing the wealth processes, the process parameters are chosen in the following way: the initial wealth $X_0 = 10$, the greenhouse gas emission rate $k_1 = 1$, the cost function $c(x) = x^2$ and the controls are defined on $\theta \in [0, 10]$ and $q \in [0.5, 10]$.



(a) Comparing the wealth processes $X_1^{q_1^\star,\tilde{\theta}_1^\star}(t)$ and $X_2^{q_2^\star,\theta_2^\star}(t)$.

(b) Comparing the production rates $q_1^\star$ and $q_2^\star$.

Figure III.3: Comparing the wealth process $X_1^{q_1^\star,\tilde{\theta}_1^\star}(t)$ and $X_2^{q_2^\star,\theta_2^\star}(t)$ and their optimal production rates $q_1^\star$ and $q_2^\star$.

Figure III.3a compares the wealth processes for the optimal controls. The difference between the two processes is very small and can not be determined by Figure III.3a. Therefore the difference $X_2 - X_1$ is shown in Figure III.4. The X_2 process is in total 0.03 larger than X_1, which agrees with the result from Theorem III.2.11.

The convergence of the processes to the constant difference is based on the convergence of the emission permits price to its mean value μ. The controls q_1 and q_2, as shown in Figure III.3b, only differ in the time of the emission price process. The optimal production rate q_2 only depends on Y_T, while q_1 takes all emission prices into account. This is the reason why the controls converge to the same value, as well as the difference between X_1 and X_2.

Figure III.5 shows the value functions for: $T = 100$, $X_0 = [0, 100]$, $k_1 = 1$, $c(x) = x^2$, $U(x) = 4x^{\frac{1}{4}}$, $\theta \in [0, 10]$, $q \in [0.5, 10]$. The difference between the value functions in Figure III.5a is very small and therefore Figure III.5b depicts the difference between the two functions.

From Figure III.5b it can be seen that the value function of the X_2 process is larger than the one for the X_1 process, since it is the larger process. Finally it is to note that for increasing initial wealth, the difference between the value functions decreases.

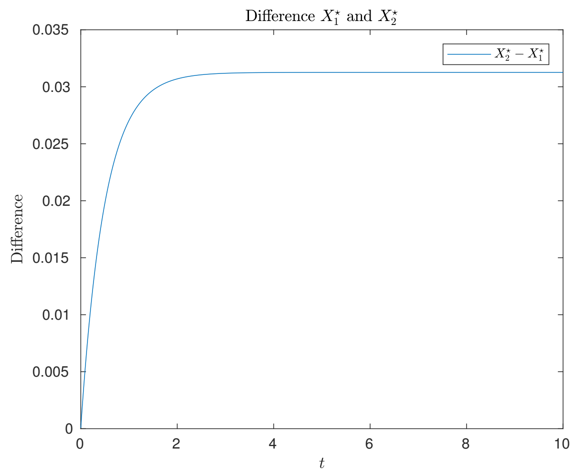
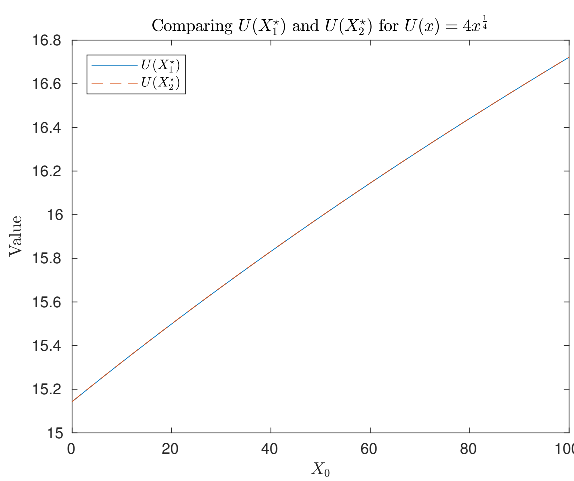
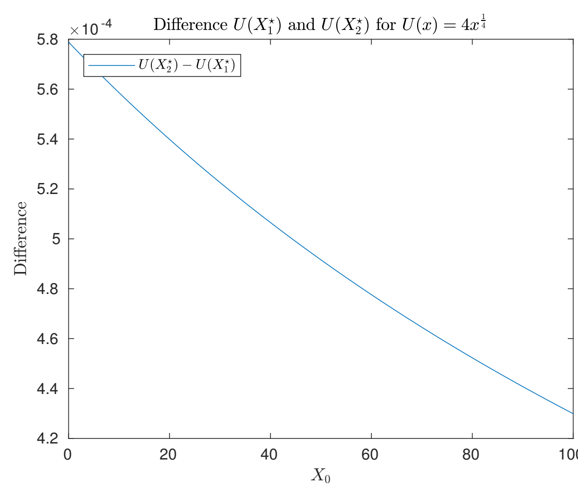


Figure III.4: The difference between the wealth processes $X_1^{q_1^\star,\tilde{\theta}_1^\star}(t)$ and $X_2^{q_2^\star,\theta_2^\star}(t)$

.



(a) Comparing $U(X_1^{q_1^\star,\tilde{\theta}_1^\star}(T))$ and $U(X_2^{q_2^\star,\theta_2^\star}(T))$.



(b) Difference of $U(X_1^{q_1^\star,\tilde{\theta}_1^\star}(T))$ and $U(X_2^{q_2^\star,\theta_2^\star}(T))$.

Figure III.5: Comparing the value functions for the optimal wealth and their difference.

Solution for general emission dynamics

In this Section we will solve, the control problem given by (I.3.2) for a family of stochastic processes modeling the emission permit price.

Assume that the price dynamics for the emission derivatives is given by

$$\begin{cases} dY_t = b(Y_t)dt + \sigma(Y_t)dW_t, \\ Y_0 > 0 \end{cases} \tag{III.3.1}$$

where $b : \mathbb{R} \mapsto \mathbb{R}$, $\sigma : \mathbb{R} \mapsto \mathbb{R}$, W_t is a one-dimensional standard Brownian motion and Y_0 is the initial price of the emission permits. Further assume that a solution to this SDE exists and its integral equation given by

$$Y_t = Y_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dW_s. \tag{III.3.2}$$

Under these assumptions, the wealth process of an energy producer as described in (I.3.1) can be written as

$$X(T) = x + \int_0^T \theta(t)[b(Y_t)dt + \sigma(Y_t)dW_t] \quad + \int_0^T [P(t) - k_1 Y_T]q(t) - c(q(t))dt. \tag{III.3.3}$$

With the explicit formulation of the wealth process in (III.3.3), we can solve the optimization problem stated in (I.3.2). The next theorem provides an upper bound to the maximal wealth, given that the price of the greenhouse gas permits evolves according to the process Y_t as defined in (III.3.2). The upper bound is defined by the optimal solution for the production rate $q^\star$ and the optimal amount of emission derivatives $\theta^\star$.

Theorem III.3.4. *The expected utility of the wealth of an energy producer is bounded above by the controlled wealth process with production rate*

$$q^\star(t) = \Psi(P(t) - k_1 \mathbb{E}_t[Y_T])$$

at time $t \in [0, T]$, where $\mathbb{E}_t[Y_T]$ is the conditional expectation with respect to $\mathcal{F}_t$ and the *optimal amount of emission derivatives*

$$\theta^\star(t) = \begin{cases} \theta_{\max} & \textit{if } b(Y_t) \geq 0, \\ \theta_{\min} & \textit{otherwise.} \end{cases}$$

Proof. The maximization problem that we have to solve is the following

$$V(x) = \sup_{(q,\theta)\in\mathcal{A}} \mathbb{E}[U(X^{q,\theta}(T))] \colon X^{q,\theta}(t) \geq 0 \quad \forall 0 \leq t \leq T\,,$$

where $X(T)$ is given by (III.3.3). Using Jensen's inequality to pull the expectation into the utility function gives

$$V(x) \leq U(\mathbb{E}[X^{q,\theta}(T)]), \tag{III.3.5}$$

since it is concave. With the assumptions on the utility function it is maximized, if its argument is maximized and thus it is enough to solve

$$\sup_{(q,\theta)\in\mathcal{A}} \mathbb{E}\left[x + \int_0^T \theta(t)[b(Y_t)dt + \sigma(t,Y_t)dW_t] + \int_0^T [P(t) - k_1 Y_T]q(t) - c(q(t))dt\right].$$

Note that $\theta(t)$ and $q(t)$ are independent of each other and thus can be maximized independently. By the linearity of the expectation, $\theta^\star$ the optimal amount of allowances at time t is given by

$$\theta^\star(t) = \begin{cases} \theta_{\max} & \text{if } b(Y_t) \geq 0 \\ \theta_{\min} & \textit{otherwise,} \end{cases} \tag{III.3.6}$$

since under the given assumption the integral with respect to the Brownian motion is a martingale and $b(Y_t)$ and $\theta(t)$ are $\mathcal{F}_t$-adapted.

To determine the optimal production rate, we take the conditional expectation with respect to t. Next, using Fubini's theorem allows to interchange the order of integration and yields

$$\sup_{q\in\mathcal{A}} \mathbb{E}\left[\int_0^T \mathbb{E}\left[(P(t)-k_1 Y_T)q(t)-c(q(t))dt|\mathcal{F}_t\right]\right]. \qquad \text{(III.3.7)}$$

Since only Y_T is not $\mathcal{F}_t$-adapted (III.3.7) can be written as

$$\sup_{q\in\mathcal{A}} \mathbb{E}\left[\int_0^T (P(t)-k_1\mathbb{E}\left[Y_T|\mathcal{F}_t\right])q(t)-c(q(t))dt\right].$$

Due to the monotone convergence theorem and through applying the Legendre-Fenchel transformation, the optimal production rate is

$$q^\star(t)=\Psi(P(t)-k_1\mathbb{E}_t[Y_T]),$$

with the conditional expectation expressed by $\mathbb{E}_t[Y_T]$. □

Remark III.3.8. *We are only able to specify an upper bound that maximizes the wealth process. By using Jensen's inequality in* (III.3.5) *we can show that the process is bounded above, but we can not find the actual maximizer.*

The approach proposed by [Carmona et al., 2012] using the theory of portfolio optimization as for example introduced in [Karatzas and Shreve, 1998], overcomes this problem and obtains an optimal production rate. But the approach fails to provide an explicit and easy to compute formula for the optimal amount of CO2 certificates.

With this general solution for the maximal terminal wealth, we are able to examine some examples for different emission derivative price processes.

Geometric Brownian motion

For the first example we let the price of an emission permit evolve according to a geometric Brownian motion. The geometric Brownian motion is a non-negative process and therefore satisfies the conditions on the price process for the greenhouse gas allowances. It has the disadvantage that it is always increasing or decreasing. Thus the emission permits will get more expansive or worthless as the time advances.

The geometric Brownian motion is defined by the following SDE

$$dY_t = s_1 Y_t dt + s_2 Y_t dWt, \tag{III.3.9}$$

where again W_t is a Brownian motion with the constant parameters $s_1 \in \mathbb{R}_+$ the percentage of drift and $s_2 \in \mathbb{R}_+$ the percentage of volatility in the process. The SDE in (III.3.9) has the well known analytical solution

$$Y_t = Y_0 e^{(s_1 - \frac{s_2^2}{2})t + s_2 W_t}. \tag{III.3.10}$$

The following theorem gives the solution to the optimal control problem defined in (I.3.2), in virtue of Theorem III.3.4 and Y_t given by (III.3.10).

Theorem III.3.11. *The expected wealth of an electricity producer modeled by* (I.3.1) *under the assumption that the unite price of an emission permit is determined by a geometric Brownian motion is maximized by the optimal amount of emission permits*

$$\theta^\star(t) = \theta_{\max}$$

at time $t \in [0, T]$ *and the production*

$$q^\star(t) = \Psi(P(t) - k_1 \mathbb{E}_t[Y_T]),$$

where $\mathbb{E}_t[Y_T]$ *is given by*

$$\mathbb{E}_t[Y_T] = e^{s_1(T-t)} Y_t.$$

Proof. First of all, it is to note that for the geometric Brownian motion the parameters of (III.3.3) are chosen such that

$$b(Y_t) = s_1 Y_t \quad \text{and} \quad \sigma(Y_t) = s_2 Y_t.$$

Since Y_t and s_1 are non-negative, (III.3.6) has only one case and the optimal amount of greenhouse gas allowances is given by

$$\theta^\star(t) = \theta_{\max}.$$

To determine the optimal production rate $q^\star$, we need to find the conditional expectation of the geometric Brownian motion. Therefore we need to solve

$$\begin{aligned} \mathbb{E}_t[Y_T] &= \mathbb{E}[Y_0 e^{(s_1 - \frac{s_2^2}{2})T + s_2 W_T} | \mathcal{F}_t] \\ &= Y_0 e^{s_1 T} \mathbb{E}[e^{-\frac{s_2^2 T}{2} + s_2 W_T} | \mathcal{F}_t]. \end{aligned}$$

Since $\{e^{-\frac{s_2 T}{2} + s_2 W_T}, t \in [0, T]\}$ is a martingale, it follows directly from the martingale property that

$$\mathbb{E}_t[Y_T] = e^{s_1(T-t)} Y_t$$

and thus the proof is concluded. $\square$

As mentioned before, the geometric Brownian motion might not be the best choice to model the price of the emission allowances, since it either increases or decreases to 0 over time. This can be seen in Figure III.6. The figure shows a path obtained from the geometric Brownian motion for the initial derivative price $Y_0 = 6$, 10% drift and 12% volatility.

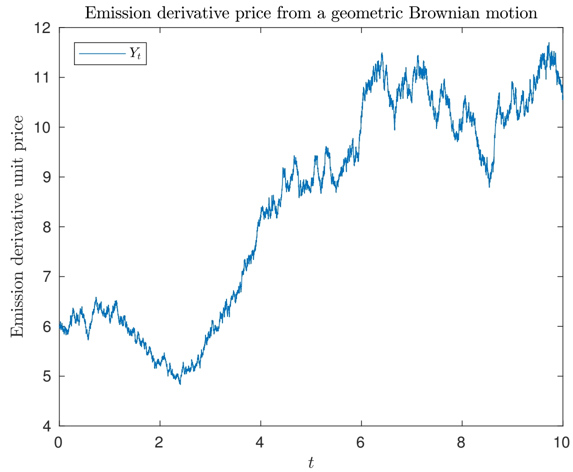
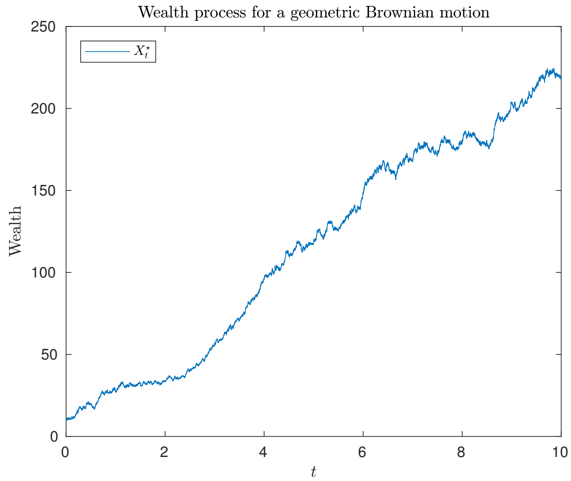


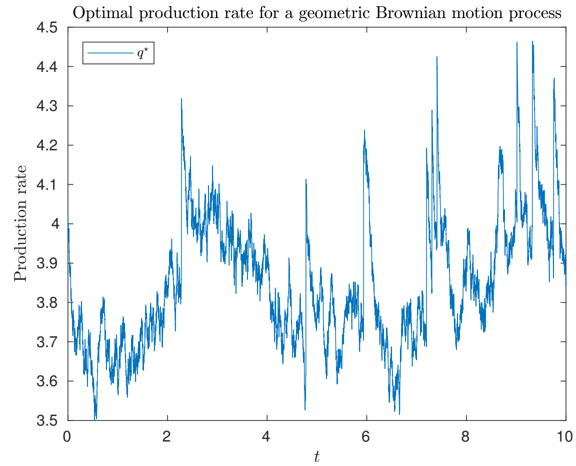
Figure III.6: Simulated path of the geometric Brownian motion.

With the simulated emission price, the wealth process and its optimal production rate can be computed. Figure III.7a shows the wealth process for the emission permit price given by Figure III.6 and the following parameter choice: initial wealth $X_0 = 10$, CO2 emission rate $k_1 = 0.12$, the controls defined by $\theta = 10$ and $q \in [0.5, 10]$ and utility function $U(x) = \frac{x^\gamma}{\gamma}$.

The wealth of the company, as pictured in Figure III.7a, increases sharply over time due to the increasing emission permit price. The optimal production rate shown in Figure III.7b increases slightly over time, but it is not heavily influenced by the rising values of Y_t. It only fluctuates due to the spot price shocks of the energy unit price $P(t)$.



(a) Wealth processes for a geometric Brownian motion process



(b) Optimal production rate for a geometric Brownian motion process

Figure III.7: The wealth and its associated optimal production rate for the geometric Brownian motion.

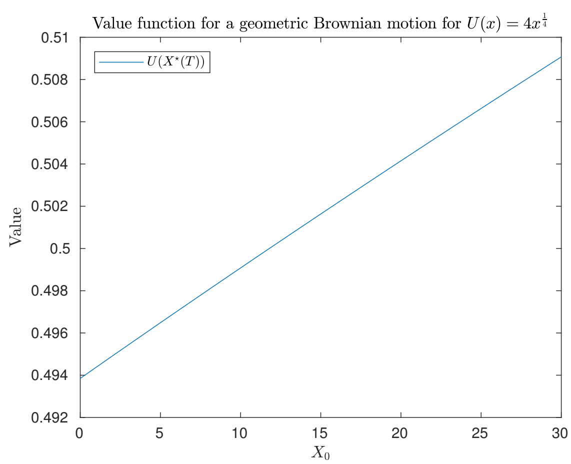


Figure III.8: Simulated value function for the geometric Brownian motion.

Figure III.8 shows the value function (I.3.2) for the initial wealth $X_0 \in [0, 30]$ and power parameter $\gamma = 0.25$ for the power utility function. From Figure III.8 it seems that the value function is linear and not concave as expected. But this based on the fact, that it increases very slowly.

Mean-reverting Process

In this section we are going to examine a solution to the finite time horizon wealth optimization of an energy producer, when the emission permit price is determined by a mean-reverting process. The most popular mean reverting process is the Ornstein-Uhlenbeck process (OU-process), defined by the SDE

$$dY_t = r_1(\mu_1 - Y_t)dt + \sigma_1 dW_t, \tag{III.3.12}$$

where W_t is an one-dimensional Brownian motion with the parameters $r_1 > 0$ determining the speed with which the process reverses to the mean price μ_1 of the emission permits and σ_1 the volatility of the process. It is assumed that $\mu_1 > 0$ since the mean price of an emission permit should not be negative. Note that the volatility controls the disturbance of the mean reversion.

One example of this type of processes, can be found in Appendix A, where it is extended by an additional jump term. The SDE in (III.3.12) can be solved explicitly and has the well known solution

$$Y_t = Y_0 e^{-r_1 t} + \mu_1(1 - e^{-r_1 t}) + \sigma_1 \int_0^t e^{r_1(s-t)} dW_s. \tag{III.3.13}$$

Since the diffusion is driven by a constant and does not depend on the position of Y_t, the process can take values below zero and therefore does not satisfy the conditions that we demand on the emission price.

This flaw can be resolved by taking the position of the process into account in the diffusion part of (III.3.12). This results in the following SDE

$$dY_t = r_1(\mu_1 - Y_t)dt + \sigma_1 Y_t dW_t, \tag{III.3.14}$$

with the same assumptions as made for the Ornstein-Uhlenbeck process.

Comparing (III.3.14) to (III.3.9) reveals that the only difference is the mean reverting part μ_1. Thus the process defined by (III.3.14) is a combination of an Ornstein-Uhlenbeck process and a geometric Brownian motion and therefore will be called a geometric Ornstein-Uhlenbeck process.

The detailed derivation of the solution to (III.3.14) can be found in Appendix B. In the following we only use the solution given by

$$Y_t = Y_0 e^{-(r_1+\frac{1}{2}\sigma_1^2)t+\sigma_1 W_t} + r_1\mu_1 \int_0^t e^{-(r_1+\frac{1}{2}\sigma_1^2)(t-s)+\sigma_1(W_t-W_s)} ds. \tag{III.3.15}$$

With the explicit formula for the price of the greenhouse gas allowances, the optimal controls that maximize the expected utility of the terminal wealth of the energy producer, can again be solved in virtue of Theorem III.3.4. The solution is summarized in the following theorem.

Theorem III.3.16. *The expected wealth of an electricity producer modeled by* (I.3.1) *under the assumption that the unite price of an emission permit is determined by a geometric Ornstein-Uhlenbeck process is maximized by the optimal amount of emission permits*

$$\theta^\star(t) = \begin{cases} \theta_{\max} & \text{if } (\mu_1 - Y_t) \geq 0 \\ \theta_{\min} & \text{otherwise} \end{cases}$$

at time $t \in [0, T]$ and the production rate

$$q^\star(t) = \Psi(P(t) - k_1 \mathbb{E}_t[Y_T]).$$

Proof. In the case of the geometric Ornstein-Uhlenbeck process as defined in (III.3.14), the parameters of (III.3.3) are chosen to be

$$b(Y_t) = r_1(\mu_1 - Y_t) \quad \text{and} \quad \sigma(Y_t) = \sigma_1 Y_t.$$

Thus, it immediately follows that the optimal amount of greenhouse gas allowances is determined by

$$\theta^{\star}(t) = \begin{cases} \theta_{\max} & \text{if } (\mu_1 - Y_t) \geq 0, \\ \theta_{\min} & \textit{otherwise}. \end{cases}$$

□

Figure III.9 shows a simulated path from a geometric Ornstein-Uhlenbeck process, with initial permit price $Y_0 = 6$, mean price $\mu_1 = 6.5$ of the allowance, mean-reverting rate $r_1 = 9$, $\sigma_1 = 0.08$ volatility. The figure clearly depicts the mean-reverting property of the geometric Ornstein-Uhlenbeck process and that it is positive. Thus it satisfies the desired properties.

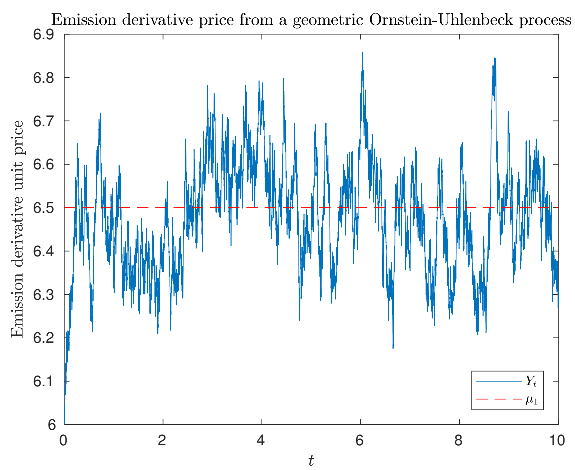


Figure III.9: Simulated path of the geometric Ornstein-Uhlenbeck process.

Optimal control for the infinite time horizon problem

In this chapter we examine two approaches to solve the optimal control problem for the infinite time horizon. We state the HJB equation in Section IV.1 and obtain its associated optimal controls depending on the value function. But the HJB equation constrains the possible processes for the energy price dynamics and does not yield a closed-form solution for the value function. Therefore we obtain a more general solution in Section IV.2 similarly to the finite time horizon problem.

Solution using the Hamilton-Jacobi-Bellman equation

Assume that the price of the emission permits evolves according to the following stochastic differential equation

$$dY_t = b(Y_t)dt + \sigma(Y_t)dW_t, \tag{IV.1.1}$$

where W_t is an one-dimensional Brownian Motion, $b : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$. Further assume that the solution to (IV.1.1) exists and its integral equation is given by

$$Y_t = Y_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dW_s,$$

where $Y_0 \in \mathbb{R}$ is the initial price at time 0. It is also assumed that the energy unit price is non-negative, constant and given by $P \in \mathbb{R}_+$. This could be a long-term contract with a fixed price per unit of produced energy.

In the following we will derive the HJB equation to solve the infinite time horizon optimal control problem

$$V(x,y) := \sup_{(q,\theta)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(X^{q,\theta}(t))dt\right] \tag{IV.1.2}$$

with the assumptions made in Section I.4. The wealth process X_t does not only depend on the initial wealth x, but also on Y_t, the price process of the CO2 emission permits. Therefore we need to consider the initial allowance price y in the value function as well.

Under the given assumption the wealth process is given by

$$dX_t = (Pq(t) - c(q(t)) + \theta(t)[b(Y_t) - Y_t F(\theta_t)])dt + \theta(t)\sigma(Y_t)dW_t. \tag{IV.1.3}$$

Note that the production rate q is independent of θ and does not depend on the Brownian motion. Therefore it can be be solved independently by determining the maximizer of $\Phi(P)$, which is given by

$$q^\star(t) = \Psi(P).$$

By substituting $q^\star$ into (IV.1.3), the wealth process can be written as

$$dX_t = (\Phi(P) + \theta(t)[b(Y_t) - Y_t F(\theta_t)])dt + \theta(t)\sigma(Y_t)dW_t. \tag{IV.1.4}$$

From (IV.1.4) we derive the HJB equation

$$0 = \beta V(x,y) - \sup_{\theta \in [\theta_{\min}, \theta_{\max}]} \Big\{ \left[\Phi(P) + \theta(b(y) - yF(\theta))\right] V_x(x,y) + \frac{\theta^2 \sigma^2(y)}{2} V_{xx}(x,y) + b(y) V_y(x,y) + \frac{\sigma^2(y)}{2} V_{yy}(x,y) + \sigma^2(y)\theta V_{xy}(x,y) + U(x) \Big\}. \tag{IV.1.5}$$

With the HJB equation we can obtain the optimal amount of emission permits $\theta^\star(t)$ held at time t, depending on the value function $V(x,y)$ and its derivatives, by taking the derivative with respect to θ. Thus we solve

$$0 = (b(y) - yF(\theta))V_x(x,y) + \theta\sigma^2(y)V_{xx} + \sigma^2(y)V_{xy},$$

for θ to obtain

$$\theta^\star = -\frac{(b(y) - yF(\theta))V_x(x) + \sigma^2(y)V_{xy}(x,y)}{\sigma^2(y)V_{xx}(x,y)}. \tag{IV.1.6}$$

To satisfy the sufficient condition for $\theta^\star$ to be a maximizer, we assume the condition on the value function that $V_{xx}(x,y) < 0$.

Substituting $\theta^\star$ into (IV.1.5), the HJB equation simplifies to

$$0 = \beta V(x,y) - \Big\{ -\frac{1}{2}\frac{((b(y) - F(\theta))V_x(x,y) + \sigma^2(y)V_{xy}(x,y))^2}{\sigma^2(y)V_{xx}(x,y)}) + b(y)V_y(x,y) + \frac{1}{2}\sigma^2(y)V_{yy}(x,y) + \Phi(P)V_x(x,y) + U(x) \Big\}. \tag{IV.1.7}$$

If there exist a smooth solution $V(x,y)$ to the HJB equation, such that

$$\liminf_{T\to\infty} e^{-\beta T}\mathbb{E}[V(X_T^{\star x}, Y_T)] \le 0$$

is satisfied, then $V(x,y)$ solves (IV.1.2). With the value function the solution the optimal amount of emission derivatives $\theta^\star$ can be examined by solving (IV.1.6) and the optimal production rate $q^\star$ is given by $\Psi(P)$.

Remark IV.1.8. *The HJB approach is only able to derive a solution for a constant electricity unit price P and when the dynamics of the greenhouse gas allowances only depends on the price of the allowances Y_t. This is based on the fact that the solution to the HJB equation in* (IV.1.7) *only yields the value function if P and dY_t do not depend on time.*

The last remark points out the weaknesses of the HJB approach.

General solution

From the HJB approach we already derived a solution to the control problem in (I.4.3). But as mentioned in Remark IV.1.8, this solution only works under some special conditions. In this Section we try to solve the control problem in a general setup in the spirit of Section III.3.

Assume again that the dynamics of the price of the emission derivatives is given by (III.3.1), with the integral equation (III.3.2). Under the given assumptions, the wealth process defined by (I.4.2) can be written as

$$X_t = x + \int_0^t [P(s)q(s) - c(q(s))]ds + \int_0^t \theta(s)[(b(Y_s) - Y_s F(\theta_s))ds + \sigma(s, Y_s)dW_s]. \quad \text{(IV.2.1)}$$

The next theorem gives an upper bound to the value function on an infinite time horizon.

Theorem IV.2.2. *The expected discounted utility of the wealth of an energy producer is bounded above by the controlled wealth process with the production rate*

$$q^\star(t) = \Psi(P(t))$$

at time $t \in [0, \infty)$ *and the optimal amount of emission derivatives*

$$\theta^\star(t) = \begin{cases} \theta_{\max} & b(Y_t) > Y_t F(\theta_t), \\ \theta_{\min} & \textit{otherwise.} \end{cases}$$

Proof. The goal is to maximize the expected discounted wealth, thus to find the value function defined by

$$V(x) = \sup_{(q,\theta)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(X^{q,\theta}(t))dt\right].$$

Under the given assumptions, Fubini's theorem holds and we can interchange the order of integration. With Jensen's inequality the value function is bounded by

$$V(x) \leq \sup_{(q,\theta)\in\mathcal{A}} \int_0^\infty e^{-\beta t} U(\mathbb{E}\left[X^{q,\theta}(t)\right])dt$$

and thus we only need to maximize the expectation of the wealth process defined by (IV.2.1). The expected wealth is given by

$$\mathbb{E}\left[X^{q,\theta}(t)\right] = x + \int_0^t [P(s)q(s) - c(q(s))]ds + \int_0^t \theta(s)(b(Y_s) - Y_s F(\theta_s))ds, \qquad \text{(IV.2.3)}$$

since under the given assumptions the Itô integral is a mean zero martingale.

It is to note that the utility function U is maximized, if X attains its maximal value, since U is concave and non-decreasing. Thus we can determine the optimal values for q and θ independently of each other, to maximize the expected wealth.

To derive the optimal production rate $q^\star$ we apply the Legendre-Fenchel transformation to the first integral in (IV.2.3) and directly obtain the optimal production rate

$$q^\star(t) = \Psi(P(t)).$$

For the optimal amount of greenhouse gas derivatives we find that

$$\theta^\star(t) = \begin{cases} \theta_{\max} & b(Y_t) > Y_t F(\theta_t) \\ \theta_{\min} & otherwise \end{cases}$$

maximizes the second integral in (IV.2.3). Thus we have determined an upper bound to the value function. □

Remark IV.2.4. *Comparing Theorem IV.2.2 and Theorem III.3.4 shows that the optimal amount of greenhouse gas allowances is the same. This result is surprising, since the two models use completely different approaches to take the cost due to emission regulation into account. Thus it follows that the optimal controls for the finite and the infinite time horizon only differ in terms of the production rate.*

With this result we can now examine some examples. In the following we state the results to the processes, we already have analyzed in Section III.3. We will only state the solutions without proof, since they are similar to the finite horizon case.

Theorem IV.2.5 (Geometric Brownian motion). *Assume the unit price of an emission permit is given by a geometric Brownian motion. Then the expected discounted wealth of an electricity producer modeled by* (I.4.3) *is maximized by the optimal amount of emission permits*

$$\theta^\star(t) = \theta_{\max}$$

at time $t \in [0, \infty]$ *and the production rate*

$$q^\star(t) = \Psi(P(t)).$$

Theorem IV.2.6 (Geometric Ornstein-Uhlenbeck process). *Assume the unit price of an emission permit is given by a geometric Ornstein-Uhlenbeck process. Then the expected* *discounted wealth of an electricity producer modeled by* (I.4.3) *is maximized by the optimal amount of emission permits*

$$\theta^\star(t) = \begin{cases} \theta_{\max} & \textit{if } (\mu_1 - Y_t) \geq 0, \\ \theta_{\min} & \textit{otherwise} \end{cases}$$

at time $t \in [0, \infty]$ *and the production rate*

$$q^\star(t) = \Psi(P(t)).$$

In the following we present numerical results for the geometric Brownian motion and the geometric Ornstein-Uhlenbeck process for the infinite time horizon wealth optimization. We further compare the result for the finite and the infinite time horizon for the geometric Brownian motion. The results are based on the same processes as already shown in Figure III.6 and Figure III.9. To compute the wealth processes, the same parameters as in the finite time horizon optimization are used. In addition the friction costs are chosen to be 2% for holding and selling allowances.

Figure IV.10 depicts the optimal production rate for both processes since the production only depends on the electricity unit price $P(t)$.

The infinite horizon wealth process shows the same result as the finite time horizon process. The geometric Brownian motion process in Figure IV.11a gives a larger terminal wealth compared to the geometric Ornstein-Uhlenbeck process in Figure IV.11b. But the terminal wealth is much less, compared to the finite time process in Figure III.7a. This is based on the friction cost that impacts the obtained wealth strongly. The result seems to be reasonable, since the long term maximum should be a low-key forecast and therefore less then the short term optimum.

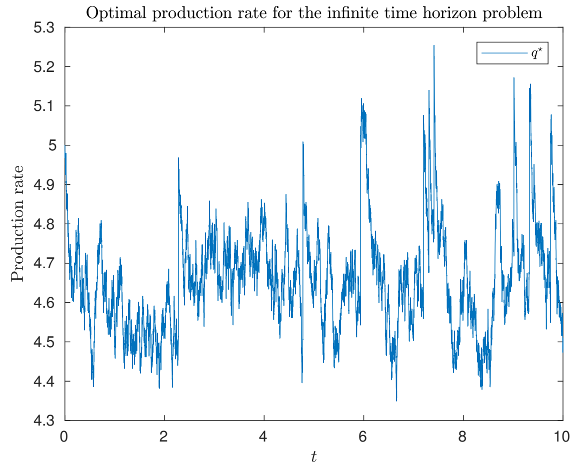
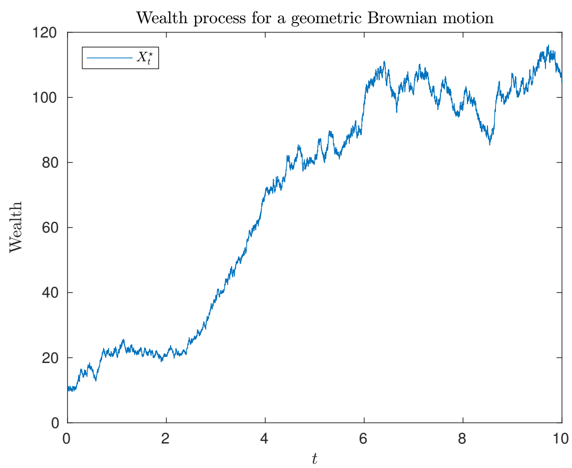
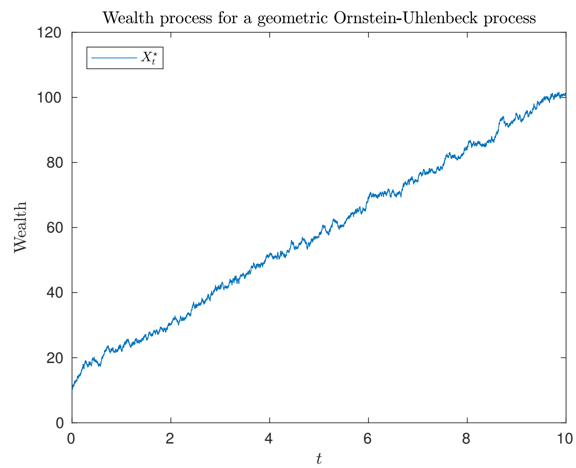


Figure IV.10: Optimal production rate on the infinite time horizon.



(a) Wealth process $X^\star(t)$ for a geometric Brownian motion process.



(b) Wealth process $X^\star(t)$ for a geometric Ornstein-Uhlenbeck process.

Figure IV.11: Comparing the optimal wealth processes on the infinite time horizon.

Figure IV.12 compares the value functions defined in (I.4.3) for the geometric Brownian motion process from Figure IV.11a and the geometric Ornstein-Uhlenbeck process from Figure IV.11b for the initial wealth $X_0 = [0, 10]$ and the discount factor $\beta = 0.75$.

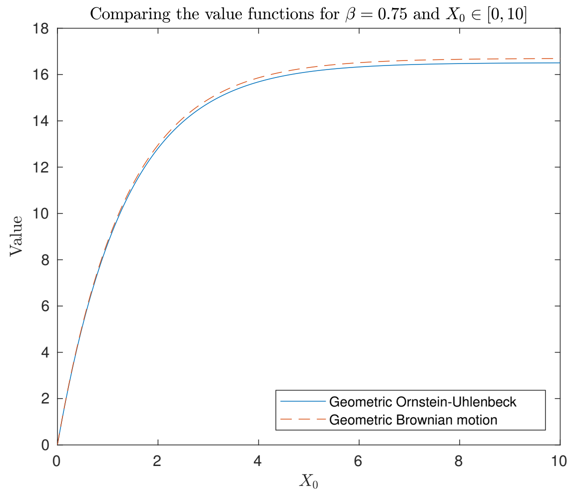


Figure IV.12: Comparing the value function for the geometric Brownian motion and the geometric Ornstein-Uhlenbeck process.

As expected from Figure IV.11 the geometric Brownian motion results in a higher value function, since its wealth is larger than for the geometric Ornstein-Uhlenbeck process. Figure IV.12 shows that the value function of the geometric Brownian motion converges to a slightly higher value.

Summary

The initial green energy production example from Section III.1 provided a first intuition on how to apply the Legendre-Fenchel transformation to solve the optimal control problem and what to expect for the solution. With this first result the comparison of the optimal control proposed by [Carmona et al., 2012] and our result revealed that our approach is optimal in the deterministic case. The strict convexity of the Legendre-Fenchel transformation played the key-role for this proof. But finally this result is not unexpected, since the solution from [Carmona et al., 2012] requires that a $\mathbb{Q}$ measure exists under which the emission price process is a martingale. In the deterministic case, this condition is not satisfied and therefore the solution is not optimal.

The solution approach for a general stochastic process for the CO2 emission allowances on the finite time horizon only yield an upper bound to the value function $V(x)$. The convexity of the utility function and Jensen's inequality only allows to obtain an upper bound. We faced the same problem for the infinite time horizon control problem and thus could again only provide an upper bound. Comparing the controls showed, that the optimal amount of emission allowances is the same for both models. The only difference between the two approaches is the optimal production rate. The finite time production rate depends on the energy unit price as well as on the emission permit price, whereas the infinite time production rate purely depends on the energy price.

Comparing the numerical examples for the finite and infinite horizon controls for the geometric Brownian motion, revealed that even small friction costs have a large impact on the wealth process. After the same amount of time the obtained wealth in the finite time model was almost twice as much as in the infinite. The other difference between the models was discovered by comparing the value functions. With the same choice of the parameters, the infinite time horizon value function converged to a final value while the finite time value function did not converged. This behavior can be interpreted in the way that the infinite horizon optimization is a low-key forecast, since it optimizes over a longer time. The finite time optimization on the other hand exploits every possibility to maximize the short term wealth. Therefore this model should only be reasonable for a specific amount of time.

Comparing the geometric Brownian motion and the geometric Ornstein-Uhlenbeck process on the infinite time horizon also provided interesting results. Although the wealth process of the geometric Brownian motion attained larger values, the difference between the value functions is not significantly. The value function of the geometric Brownian motion is just slightly larger. One explanation for this result could be that for the chosen discounting factor, the time where the wealth of the geometric Brownian motion is larger is not weighted that much.

Finally examining the HJB equation yielded a solution for the optimal controls. But the optimal amount of emission allowances depends on the value function and its derivatives and we were not able to derive a solution to the HJB equation. The value function could still be numerically approximated, to compute the amount of emission permits. The HJB equation showed one more weakness. We were only able to derive the equation for a fixed energy price and a restricted emission permit price process. Thus the optimization using the Legendre-Fenchel transformation provided a better method, even though it only yields an upper bound to the value function.

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Appendix

Solution to the electricity unit price

The electricity unit price $P(t)$ is given by the following stochastic differential equation

$$dP(t) = \kappa(\alpha(t))[\nu(\alpha(t)) - P(t)]dt + \hat{\sigma}(\alpha(t))dW(t) + \int_{\mathbb{R}_0} \gamma(\alpha(t-), z)\tilde{N}(dt, dz) \qquad \text{(A.1)}$$

where W is an one-dimensional Brownian motion and N is a Poisson random measure with the compensator $\tilde{N}(t, E) := N(t, E) - t\nu(E)$, in which ν is a Lévy measure satisfying $\int_{\mathbb{R}_0}(1 \wedge |z|^2)\nu(dz) < \infty$. A more detailed description can be found in Section I.2.

Without the last term in (A.1) the electricity price would be given by an Ornstein-Uhlenbeck process, which is a an Itô process and thus easy to solve with Itô's formula. But with the jump term, the process becomes a Lévy process. In literature it is often referred as Ornstein-Uhlenbeck-Lévy process.

Because (A.1) is an Lévy process, it can not be solved by the standard procedure with Itô's formula. But an extension of Itô's formula exists for Lévy processes that can be applied instead.

Theorem A.2 (Iô's formula for Lévy processes)**.** *Suppose $X_t \in \mathbb{R}$ is an Lévy process of the form*

$$dX_t = \alpha(t, \omega)dt + \beta(t, \omega)dW_t + \int_{\mathbb{R}} \gamma(t, z, \omega)\tilde{N}(dt, dz),$$

where

$$\tilde{N}(dt,dz) = \begin{cases} N(dt,dz) - \nu(dz)dt & if|z| < R \\ N(dt,dz) & if|z| \geq R \end{cases}$$

for some $R \in [0,\infty]$.

Let $f \in C^2(\mathbb{R}^2)$ *and define* $Y_t = f(t, X_t)$. *Then* Y_t *is again an Itô-Lévy process and*

$$\begin{aligned} dY_t = &\frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)[\alpha(t,\omega)dt + \beta(t,\omega)dW_t] + \frac{1}{2}\beta^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t,X_t)dt \\ &+ \int_{|z|<R} \{f(t, X(t^-) + \gamma(t,z)) - f(t, X_{t^-} - \frac{\partial f}{\partial x}(t, X_{t^-})\gamma(t,z)\}\nu(dz)dt \\ &+ \int_{\mathbb{R}} \{f(t, X(t^-)) + \gamma(t, X_t) - f(t, X_{t^-})\}\tilde{N}(dt,dz). \end{aligned}$$

It is to note that Itô's formula for Lévy process is almost the same as for Itô processes. The only difference is that an additional increment is added, to take the jumps of the process into account. This theorem and further theory for Lêvy processes can be found in [Øksendal and Sulem, 2005].

With the extension of Itô's formula, (A.1) can be solved similarly to the solution of the normal Ornstein-Uhlenbeck process. Therefore we collect all terms including $P(t)$ on the left hand side of the equation

$$dP(t) - \kappa(\alpha(t))P_t dt = \kappa(\alpha(t))\nu(\alpha(t))dt + \hat{\sigma}(\alpha(t))dW_t + \int_{\mathbb{R}_0} \gamma(\alpha(t-),z)\tilde{N}(dt,dz).$$

Next we choose $f(t,X_t) = e^{\kappa(\alpha(t))t}$ to obtain the new Lévy process Y_t associated to the SDE

$$d(e^{\kappa(\alpha(t)t}P_t) = \kappa(\alpha(t))\nu(\alpha(t))e^{\kappa(\alpha(t))t}dt + \hat{\sigma}(\alpha(t))e^{\kappa(\alpha(t))t}dW_t + \int_{\mathbb{R}_0} e^{\kappa(\alpha(t))t}\gamma(\alpha(t-),z)\tilde{N}(dt,dz).$$

Integrating both sides yields the solution for the energy unit price

$$\begin{aligned} P(t) =& \nu(t) + e^{-\kappa(\alpha(t))t}(P_0 - \nu(t)) + \int_0^t \hat{\sigma}(\alpha(s)) e^{\kappa(\alpha(t))(s-t)} dW_s \\ &+ \int_0^t \int_{\mathbb{R}_0} e^{\kappa(\alpha(t))(s-t)} \gamma(\alpha(s-), z) \tilde{N}(ds, dz). \end{aligned}$$

By comparing the solution of the Ornstein-Uhlenbeck-Lévy process and the Ornstein-Uhlenbeck process we see that the Ornstein-Uhlenbeck-Lévy process only has the additional term to take the jumps of the process into account.

Solution to the geometric Ornstein-Uhlenbeck process

Define the following SDE:

$$dY_t = r_1(\mu_1 - Y_t)dt + \sigma_1 Y_t dW_t, \tag{B.1}$$

where W_t is an one-dimensional Brownian motion with the parameters $r_1 > 0$ determining the speed with that the process reverses to its mean value μ_1 and σ_1 the volatility of the process controlling the disturbance of the mean reversion.

The process of solving (B.1) is similar to finding a solution to the geometric Brownian motion or the Ornstein-Uhlenbeck process. The major difference is guessing a solution for the variation of constants approach. The following gives a detailed solution for the SDE.

Starting from (B.1) we firstly collect all terms containing Y_t on the left hand side and multiply both sides with $M_t = e^{at+bW_t}$, in which a and b are constants to be determined, to derive

$$M_t dY_t + r_1 Y_t M_t dt - \sigma_1 Y_t M_t dW_t = r_1 \mu_1 M_t dt. \tag{B.2}$$

Next we use integration by parts to solve

$$d(M_t Y_t) = r_1 \mu_1 M_t dt \tag{B.3}$$

and finally determine the parameters a and b of M_t. But before we solve (B.3), we have to find the derivative of M_t using Itô's formula. Therefore we need to find a process X_t and a function $f(X_t) \in C^2$ such that

$$dM_t = df(X_t).$$

Choosing $f(x) = e^x$ and $X_t = at + bW_t$ with the derivative $dX_t = adt + bdW_t$, we can apply Itô's formula to determine

$$\begin{aligned} dM_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)dXt^2 \\ &= f(X_t)(adt + bdW_t) + \frac{1}{2}b^2 f(X_t)dt \\ &= M_t[(a + \frac{1}{2}b^2)dt + bdW_t]. \end{aligned}$$

Next we use the integration by parts formula to determine the left hand side of (B.3), to derive

$$d(M_tY_t) = M_tdY_t + Y_tdM_t + d[MY]_t, \tag{B.4}$$

where $d[MY]_t$ is the cross variation of Y_t and M_t. The cross variation is given by

$$d[MY]_t = \sigma_1 Y_t dW_t \cdot bM_t dW_t = \sigma_1 b Y_t M_t dt,$$

since the product $dW_t \cdot dW_t$ is just dt. Substituting all values in (B.4) gives

$$d(M_tY_t) = M_tdY_t + Y_tM_t(a + \frac{1}{2}b^2 + \sigma_1 b)dt + Y_tM_tbdW_t$$

and immediately from the last summand in (B.2) follows that $b = -\sigma_2$. To match the middle term of (B.2) we have to solve

$$a + \frac{1}{2}\sigma_1^2 - \sigma_1^2 = r_1$$

for a to obtain

$$a = r_1 + \frac{1}{2}\sigma_1^2$$

and so

$$M_t = e^{(r_1+\frac{1}{2}\sigma_1^2)t-\sigma_1 W_t}. \tag{B.5}$$

The final step is to integrate (B.2) on both sides using (B.5), to derive the solution

$$Y_t = Y_0 e^{-(r_1+\frac{1}{2}\sigma_1^2)t+\sigma_1 W_t} + r_1\mu_1 \int_0^t e^{-(r_1+\frac{1}{2}\sigma_1^2)(t-s)+\sigma_1(W_t-W_s)} ds.$$