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ABSTRACT
This paper presents a sound and complete five-sided sequent calculus for first-order weak Kleene valuations which permits not only elegant representations of four logics definable on first-order weak Kleene valuations, but also admissibility of five cut rules by proof analysis.

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1. Introduction

Recent literature on philosophical logic offers various sequent calculi for logics definable on weak Kleene valuations for propositional languages, most recently Coniglio and Corbalan (2012), Szmuc (2019), Da Ré et al. (2018) and Paoli and Baldi (in press). I have, however, not come across any sequent calculi that also accommodate the quantifiers. This is probably due to the somewhat awkward nature of the weak Kleene quantifiers as compared to for example strong Kleene valuations: for a universally quantified formula to be assigned 0, it is not sufficient that some instance is assigned 0, but every other instance must also be *crisp*, that is, assigned either 0 or 1. The aim of this paper is to present a sound and complete five-sided sequent calculus for first-order weak Kleene valuations which not only permits elegant representations of four logics definable on first-order weak Kleene valuations, but also admissibility of five cut rules by proof analysis using standard techniques from structural proof theory as presented by Negri and von Plato (2001).

Section 2 presents first-order weak Kleene valuations. Section 3 briefly discusses desirable features for sequent calculi and Section 4 articulates the problem with the weak Kleene quantifiers using as a starting point the four-sided sequent calculus for first-order strong Kleene valuations presented by Fjellstad (2017). Section 5 presents a five-sided sequent calculus that solves the problem and Section 6 shows that five cut rules are admissible by proof analysis and that the sequent calculus is complete with regard to first-order weak Kleene valuations.

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2. Weak Kleene valuations and their logics

We shall, in this paper, work with a simple first-order language without term constants, function symbols or designated predicates (i.e. predicates that are assigned some particular meaning such as equality or truth).

Definition 2.1 (Language): Let $\mathcal{L}$ be a first-order language based on a countable set of n-ary predicates, a countable set of variables, and the connectives $\wedge$, $\neg$ and $\forall$.

We shall use Latin letters A, B and C to refer to formulas of $\mathcal{L}$, and A_{at} to refer to atomic formulas of $\mathcal{L}$. x, y and z will be used to refer to variables of $\mathcal{L}$. The expression $A(z/y)$ represents the result of replacing every occurrence of y with z in A. $\vee$, $\supset$ and $\exists$ are treated as defined symbols in the usual way in order to reduce the number of cases in the definitions and proofs.

The crucial aspect of weak Kleene valuations is that a complex formula is crisp if and only if each immediate subformula is crisp. Correspondingly, a complex formula is assigned $\frac{1}{2}$ if and only if some immediate subformula is assigned $\frac{1}{2}$. We shall use the following definition based on the presentation by Malinowski (2001) but simplified in order to avoid a domain of quantification:

Definition 2.2 (First-order weak Kleene valuations): A function $\mathcal{V}$ from the $\mathcal{L}$-formulas to $\{1, \frac{1}{2}, 0\}$ is a QWK-valuation just in case $\mathcal{V}$ satisfies the following conditions:

$$\mathcal{V}(A \wedge B) = \begin{cases} 1 & \mathcal{V}(A) = 1 \text{ and } \mathcal{V}(B) = 1 \\ \dfrac{1}{2} & \mathcal{V}(A) = \dfrac{1}{2} \text{ or } \mathcal{V}(B) = \dfrac{1}{2} \\ 0 & \text{otherwise} \end{cases} \qquad \mathcal{V}(\neg A) = \begin{cases} 1 & \mathcal{V}(A) = 0 \\ \dfrac{1}{2} & \mathcal{V}(A) = \dfrac{1}{2} \\ 0 & \mathcal{V}(B) = 1 \end{cases}$$

$$\mathcal{V}(\forall x A) = \begin{cases} 1 & \text{for every } z, \mathcal{V}(A(z/x)) = 1 \\ \dfrac{1}{2} & \text{for some } y, \mathcal{V}(A(y/x)) = \dfrac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

In the same way as with strong Kleene valuations, we can use weak Kleene valuations to define various logics by tweaking the conditions for an inference to be satisfied by a valuation.

Definition 2.3 (Weak Kleene logics): Suppose that Γ and Δ are finite sets of $\mathcal{L}$-formulas, then

$\langle \Gamma, \Delta \rangle \in \text{QWK}_{SS}$ iff no QWK-valuation is s.t. $\forall A \in \Gamma, \mathcal{V}(A) = 1$ and $\forall B \in \Delta, \mathcal{V}(B) \neq 1$
$\langle \Gamma, \Delta \rangle \in \text{QWK}_{TT}$ iff no QWK-valuation is s.t. $\forall A \in \Gamma, \mathcal{V}(A) \neq 0$ and $\forall B \in \Delta, \mathcal{V}(B) = 0$
$\langle \Gamma, \Delta \rangle \in \text{QWK}_{ST}$ iff no QWK-valuation is s.t. $\forall A \in \Gamma, \mathcal{V}(A) = 1$ and $\forall B \in \Delta, \mathcal{V}(B) = 0$
$\langle \Gamma, \Delta \rangle \in \text{QWK}_{TS}$ iff no QWK-valuation is s.t. $\forall A \in \Gamma, \mathcal{V}(A) \neq 0$ and $\forall B \in \Delta, \mathcal{V}(B) \neq 1$

With this labelling, SS is paracomplete, TT is paraconsistent, ST is non-transitive and TS is non-reflexive.[1] The logics are presented as multiple-conclusion consequence relations, that is, as sets of pairs of sets of formulas. The sets are finite to avoid failure of ω-compactness for each logic except QWK_{TS} as a consequence of substitutional quantification.[2]

The recent literature on weak Kleene valuations offers some discussion on how to interpret weak Kleene valuations, examples include Beall (2016), Ciuni and Carrara (2019), Francez (2019) and Szmuc (2019). We shall not engage in that discussion but rather stay focused on our task at hand.

3. Desirable features of sequent calculi

A sequent calculus is a *tool* for establishing facts about something, for example a logic, and as a tool it can be better or worse for various purposes. While there is a sense in which a sequent calculus can be *correct* for a logic by telling us whether Γ entails Δ according to some logic just in case Γ actually entails Δ according to that logic, there can be further reasons for working with one sequent calculus for a logic as opposed to another sequent calculus for the same logic. This section presents one sequent calculus for classical logic and describe features that make it *desirable* not only from the perspective of structural proof theory, but also with regard to how it represents that an inference is valid. That these are desirable features from these perspectives does not imply that the calculus should be adopted for any purpose whatsoever. There could be other reasons for preferring another calculus for the same logic(s); the calculus is merely a tool for establishing facts about the logic(s).

The following sequent calculus where a sequent of the form $\Gamma \Rightarrow \Delta$ represents a pair of multisets of $\mathcal{L}$-formulas, which I label $\mathcal{C}^{CL}$, is a variant of the sequent calculus G3c presented by Negri and von Plato (2001) obtained by treating negation as primitive. It consists of *initial sequents* of the form

$$A_{at}, \Gamma \Rightarrow \Delta, A_{at}$$

and the following *primitive rules* where the label (y) for a rule means that the variable y is an *eigenvariable* in the sense that it does not occur free in Γ, $\forall xA$ and Δ:

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \quad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \quad \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\frac{\forall xA, A(z/x), \Gamma \Rightarrow \Delta}{\forall xA, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, A(y/x)}{\Gamma \Rightarrow \Delta, \forall xA}\,(y)$$

The displayed formulas in the conclusion of each rule are the principal formulas of that rule. The displayed formulas in the premise(s) of each rule are the active formulas of that rule. Γ and Δ are the contexts.

Derivations take the shape of trees where every leaf is an initial sequent and the root is obtained from the leaf(s) using the primitive rules. As opposed to for example natural deduction and natural deduction-inspired sequent calculi, derivations in $\mathcal{C}^{CL}$ do not proceed from assumptions. The height of a $\mathcal{C}^{CL}$-derivation $\mathcal{D}$ is defined inductively on its construction: a derivation $\mathcal{D}$ ending with an initial sequent has height 0, and if $\mathcal{D}$ is obtained with a n-premise rule from derivations $\mathcal{D}_0, \ldots, \mathcal{D}_{n-1}$, then the height of $\mathcal{D}$ is the supremum of the heights of $\mathcal{D}_i + 1$.

Following, for example, Negri and von Plato (2001), the following rules can be shown by proof analysis to be admissible; if there are derivations of the premise-sequents, then there is a derivation of the conclusion-sequent

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,\mathrm{cL} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\,\mathrm{cR} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,\mathrm{w}$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathrm{Cut}$$

In the case of the rules for weakening (w) and contraction (cL and cR), the proofs establish that the rules are *height-preservingly* admissible; whenever there is a derivation of the premise-sequent with height n, then there is a derivation of the conclusion-sequent with height $\leq n$. To show that the contraction rules are admissible, one first shows that every primitive rule is height-preservingly *invertible*: if there is a derivation of the conclusion-sequent of height n, then there is a derivation of each of the premise-sequents with height $\leq n$. In each case, the proof proceeds by induction on the height of a derivation.

The height-preserving admissibility of w, cL and cR is employed to show that the rule Cut is admissible by a double induction on the number of connectives in the active formula A in the Cut rule and the sum of the heights of the derivations of the premise-sequents of the Cut rule. A proof showing that Cut is admissible by proof analysis are typically referred to as a 'cut-elimination' proof. The strategy for cut-elimination applicable to $\mathcal{C}^{CL}$ presented by Negri and von Plato (2001) is significantly simpler than for example that presented by Gentzen (1935) for the calculus LK. This is partly due to the height-preserving admissibility of contraction. While Gentzen (1935) replaced Cut with a more complex rule typically referred to as 'multicut' in order to provide local transformations of derivations that push applications of multicut upwards in a derivation, the height-preserving admissibility of the contraction rules permits simpler local transformations involving merely Cut.[3]

In addition to possessing desirable features from the perspective of structural proof theory, the sequent calculus $\mathcal{C}^{CL}$ also offers a straightforward representation of a valid inference of classical logic. A completeness theorem will tell us that $\Gamma \Rightarrow \Delta$ is derivable if and only if Γ entails Δ according to first-order classical logic as defined for the language in question. This might seem like a triviality, but the literature is rife with sequent calculi in which the connection between a derivable sequent and a valid inference is more obscure.

Consider for example the three-sided 'negated-conjunctive' sequent calculus for strong Kleene valuations with transparent truth presented by Ripley (2012) where a sequent $\Gamma|\Theta|\Delta$ is interpreted as that there is no valuation at which every formula in Γ is assigned 1, every formula in Θ is assigned $\frac{1}{2}$ and every formula in Δ is assigned 0.[4] These sequents do not permit straightforward representation of the conditions on trivalent models utilised in Definition 2.3 with the exception of the condition for an ST-inference. While it is the case that $\langle \Gamma, \Delta \rangle$ is ST-valid if and only if $\Gamma \mid \quad \mid \Delta$ is derivable, it is, for example, the case $\langle \Gamma, \Delta \rangle$ is TT-valid if and only if every sequent of the form $\Gamma_0|\Gamma_1|\Delta$ is derivable where $\Gamma = \Gamma_0 \cup \Gamma_1$.

Consider instead the sequent calculus presented by Fjellstad (2017) for the same theories of transparent truth as that by Ripley (2012). With sequents of the form

$$\Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'$$

under the intended interpretation that there is no valuation at which everything in Γ is assigned 1, everything in Δ is assigned 0, everything in Γ' is assigned $\{1, \frac{1}{2}\}$ and everything in Δ' is assigned $\{\frac{1}{2}, 0\}$, it is straightforward to represent the various conditions for a valid inference on trivalent valuations such as those we utilised to define the four logics above in Definition 2.3. For example, $\langle \Gamma, \Delta \rangle$ is now TT-valid if and only if $\Rightarrow \Delta \mid \Gamma \Rightarrow$ is derivable.

Our aim is thus to develop a sequent calculus suitable for elegantly representing logics definable on QWK-valuations for which we can prove the analogous theorems as those for $\mathcal{C}^{CL}$ presented in this section using the techniques presented by Negri and von Plato (2001). The next section shows that four-sided sequents of the kind used in Fjellstad (2017) are unsuitable for our purposes because they cannot accommodate rules that capture the weak Kleene quantifiers.

4. Quantification and crispiness

With sequents of the form $\Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'$ together with suitable initial sequents and rules guaranteeing that a sequent is underivable if and only if that sequent has a QWK-valuation under the above interpretation, the following *would* hold:

$\langle \Gamma, \Delta \rangle \in \text{QWK}_{SS}$ iff $\Gamma \Rightarrow \mid \Rightarrow \Delta$ is derivable.
$\langle \Gamma, \Delta \rangle \in \text{QWK}_{TT}$ iff $\Rightarrow \Delta \mid \Gamma \Rightarrow$ is derivable.
$\langle \Gamma, \Delta \rangle \in \text{QWK}_{ST}$ iff $\Gamma \Rightarrow \Delta \mid \Rightarrow$ is derivable.
$\langle \Gamma, \Delta \rangle \in \text{QWK}_{TS}$ iff $\Rightarrow \mid \Gamma \Rightarrow \Delta$ is derivable.

Considering the difference between weak and strong Kleene connectives, we can expect that any shortcomings will concern rules for $\wedge$ or $\forall$.

With regard to rules for $\wedge$, we observe first that we can use the same rules as in Fjellstad (2017) for the left-most and the right-most position, i.e.

$$\frac{A, B, \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}{A \wedge B, \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'} \wedge LL \qquad \frac{\Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta', A \qquad \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta', B}{\Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta', A \wedge B} \wedge RR$$

Instead, the particularity of weak Kleene valuations shows up in the remaining two cases. To that purpose, we observe the following equivalences:

- $\mathcal{V}(A \wedge B) = 0$ iff $\mathcal{V}(A) = 0$ and $\mathcal{V}(B) \in \{1, 0\}$ *or* $\mathcal{V}(B) = 0$ and $\mathcal{V}(A) \in \{1, 0\}$.
- $\mathcal{V}(A \wedge B) \in \{1, \frac{1}{2}\}$ iff $\mathcal{V}(A) = 1$ and $\mathcal{V}(B) = 1$ *or* $\mathcal{V}(A) = \frac{1}{2}$ *or* $\mathcal{V}(B) = \frac{1}{2}$.

They result in the following rules (which we assume would be invertible):

$$\frac{A, \Gamma \Rightarrow \Delta, B \mid \Gamma' \Rightarrow \Delta' \qquad B, \Gamma \Rightarrow \Delta, A \mid \Gamma' \Rightarrow \Delta' \qquad \Gamma \Rightarrow \Delta, A, B \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta, A \wedge B \mid \Gamma' \Rightarrow \Delta'} \wedge LR$$

$$\dfrac{\Gamma \Rightarrow \Delta \mid A, \Gamma' \Rightarrow \Delta', A \qquad \Gamma \Rightarrow \Delta \mid B, \Gamma' \Rightarrow \Delta', B \qquad A, B, \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid A \wedge B, \Gamma' \Rightarrow \Delta'} \wedge\text{RL}$$

So far so good.

For the quantifiers the corresponding equivalences are as follows:

- $\mathcal{V}(\forall xA) = 0$ iff for some y, $\mathcal{V}(A(y/x)) = 0$ and for every z, $\mathcal{V}(A(z/x)) \in \{1, 0\}$.
- $\mathcal{V}(\forall xA) \in \{1, \frac{1}{2}\}$ iff for every z, $\mathcal{V}(A(z/x)) = 1$ or for some y, $\mathcal{V}(A(y/x)) = \frac{1}{2}$.

The latter equivalence corresponds to the following rule:

$$\dfrac{\Gamma \Rightarrow \Delta \mid A(y/x), \Gamma' \Rightarrow \Delta', A(y/x) \qquad A(z/x), \Gamma \Rightarrow \Delta \mid \forall xA, \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid \forall xA, \Gamma' \Rightarrow \Delta'} (y)\forall\text{RL}$$

The former equivalence, on the other hand, is significantly more problematic. In the case of $\wedge$, we only had to consider two formulas, A and B, and we could thus transform the condition into the following:

Either $\mathcal{V}(A) = 1$ and $\mathcal{V}(B) = 0$ or $\mathcal{V}(B) = 1$ and $\mathcal{V}(A) = 0$ or $\mathcal{V}(A) = 0$ and $\mathcal{V}(B) = 0$.

This is not really an option when we have a countable number of instances where each can have any value in $\{1, 0\}$ in which case we obtain an uncountable number of alternatives.

5. Introducing the crisp side

One straightforward way to make room for representing the condition for $\mathcal{V}(\forall xA) = 0$ as a sequent calculus rule is to expand the sequents with a new position that express the desired *crispiness* of a formula.[5] This trick seems acceptable since we are not committed to a particular number of positions in the sequents as long as we obtain elegant representations of the logics and the sequent calculus is sound and complete with regard to QWK-valuations. One reason to be sceptical towards this trick, however, consists in that the new position will not actually be used to display that an inference is valid in any of the logics. One could thus argue that this should be considered a less elegant feature of the calculus. That would be a fair point which I am happy to acknowledge. On the other hand, it is important to keep in mind that a sequent calculus is a tool intended to be used to establish facts about the logics, and if the addition of the new position is useful to that purpose, then that should suffice as a defence of it.

Going forward, we shall thus work with sequents of the form

$$\Gamma \Rightarrow \Delta|\Theta|\Gamma' \Rightarrow \Delta'$$

with the intended interpretation that there is no QWK-valuation such that

for every formula $A \in \Gamma, \mathcal{V}(A) = 1$, for every formula $A \in \Delta, \mathcal{V}(A) = 0$, for every formula $A \in \Theta, \mathcal{V}(A) \in \{1, 0\}$, for every formula $A \in \Gamma', \mathcal{V}(A) \in \{1, \frac{1}{2}\}$ and for every formula $A \in \Delta', \mathcal{V}(A) \in \{\frac{1}{2}, 0\}$.

The new position in the middle represents crispiness. The other positions keep their interpretation from Fjellstad (2017).[6]

The new sequents with their intended interpretation suggest the following initial sequents:

$$\overline{A_{at}, \Gamma \Rightarrow \Delta, A_{at} \mid \Theta \mid \Gamma' \Rightarrow \Delta'} \qquad \overline{A_{at}, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A_{at}}$$

$$\overline{\Gamma \Rightarrow \Delta, A_{at} \mid \Theta \mid A_{at}, \Gamma' \Rightarrow \Delta'} \qquad \overline{\Gamma \Rightarrow \Delta \mid \Theta, A_{at} \mid A_{at}, \Gamma' \Rightarrow \Delta', A_{at}}$$

It is trivial to confirm that they are sound with regard to the models. Consider for example the last one which corresponds to the claim that there is no valuation at which $\mathcal{V}(A_{at}) \in \{1, 0\} \cap \{1, \frac{1}{2}\} \cap \{\frac{1}{2}, 0\}$.

For $\wedge$, we obtain the following rules:

$$\frac{A, B, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'}{A \wedge B, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'}\ \Gamma\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, A \mid \Theta, B \mid \Gamma' \Rightarrow \Delta' \qquad \Gamma \Rightarrow \Delta, B \mid \Theta, A \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta, A \wedge B \mid \Theta \mid \Gamma' \Rightarrow \Delta'}\ \Delta\wedge$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta, A, B \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid \Theta, A \wedge B \mid \Gamma' \Rightarrow \Delta'}\ \Theta\wedge$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta \mid A, \Gamma' \Rightarrow \Delta', A \qquad \Gamma \Rightarrow \Delta \mid \Theta \mid B, \Gamma' \Rightarrow \Delta', B \qquad A, B, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid \Theta \mid A \wedge B, \Gamma' \Rightarrow \Delta'}\ \Gamma'\wedge$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A \qquad \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', B}{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A \wedge B}\ \Delta'\wedge$$

The rule for introducing $A \wedge B$ into the Δ-position has fewer premises than the corresponding four-sided rule thanks to the crispy position (i.e. Θ). The rule for introducing $A \wedge B$ into the crispy position expresses that conjunction is crisp if and only if its conjuncts are crisp.

Moving on to negation, we obtain the following rules:

$$\frac{\Gamma \Rightarrow \Delta, A \mid \Theta \mid \Gamma' \Rightarrow \Delta'}{\neg A, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'}\ \Gamma\neg \qquad \frac{A, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta, \neg A \mid \Theta \mid \Gamma' \Rightarrow \Delta'}\ \Delta\neg$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A}{\Gamma \Rightarrow \Delta \mid \Theta \mid \neg A, \Gamma' \Rightarrow \Delta'}\ \Gamma'\neg \qquad \frac{\Gamma \Rightarrow \Delta \mid \Theta \mid A, \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', \neg A}\ \Delta'\neg$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta, A \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid \Theta, \neg A \mid \Gamma' \Rightarrow \Delta'}\ \Theta\neg$$

They are straightforwardly justified by considering the interpretation of $\neg$ in weak Kleene valuations. It is, for example, the case that $\mathcal{V}(\neg A) \in \{1, \frac{1}{2}\}$ if and only if $\mathcal{V}(A) \in \{0, \frac{1}{2}\}$, and assuming invertibility, this is precisely what is expressed by the rule $\Gamma'\neg$.

Finally, we have the following rules for the quantifiers:

$$\frac{\forall xA, A(z/x), \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'}{\forall xA, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'}\ \Gamma\forall$$

$$\frac{\Gamma \Rightarrow \Delta, A(y/x), \forall xA \mid \Theta, A(z/x) \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta, \forall xA \mid \Theta \mid \Gamma' \Rightarrow \Delta'}\ \Delta\forall \text{ (y)}$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta, \forall xA, A(z/x) \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid \Theta, \forall xA \mid \Gamma' \Rightarrow \Delta'}\ \Theta\forall$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta \mid A(y/x), \Gamma' \Rightarrow \Delta', A(y/x) \qquad A(z/x), \Gamma \Rightarrow \Delta \mid \Theta \mid \forall xA, \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid \Theta \mid \forall xA, \Gamma' \Rightarrow \Delta'}\ \Gamma'\forall \text{ (y)}$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A(y/x)}{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', \forall xA}\ \Delta'\forall \text{ (y)}$$

The rule $\Delta\forall$ is the reason for introducing the crisp side; what was previously unrepresentable is now a simple one-premise rule stating the obvious: some instance is assigned 0 and the rest are crisp. The additional copies of the principal formula as active formula are included to ensure admissibility of contraction.

Definition 5.1: Let $\mathcal{C}^{\mathrm{QWK}}$ be the sequent calculus for sequents of the form

$$\Gamma \Rightarrow \Delta|\Theta|\Gamma' \Rightarrow \Delta'$$

where $\Gamma, \Delta, \Theta, \Gamma'$ and Δ' represent finite multisets of $\mathcal{L}$-formulas obtained with the initial sequents and rules for five-sided sequents presented in this section prior to this definition.

Before we engage in the project of establishing that the sequent calculus has the features described as desirable above in Section 3 it is appropriate to clarify what the relevant structural rules are. While it is relatively easy to identify both weakening and contraction rules since weakening rules add formulas into a position and contraction rules contract two or more copies of a formula occurring in a position, the cut rule deserves a brief discussion.

One might at first think that the obvious candidate for the rule corresponding to the rule

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{Cut}$$

for five-sided sequents is the analogous five-premise rule, one for each position of a sequent for $\mathcal{C}^{\mathrm{QWK}}$.

However, an alternative way of approaching the issue consists in observing that a cut-rule is a rule that permits certain 'derivational shortcuts' as compared to the introduction rules for the connectives. The relevant question is therefore which 'shortcuts' we can permit that would not lead us astray to sequents that are not sound with regard to QWK-valuations under the intended interpretation. One way to uncover these shortcuts is by relying on the semantic role of a cut rule as articulated by Restall (2005) and

Ripley (2012, 2013): they are supposed to exhaust the options for a formula on a valuation when sequents are interpreted 'negated-conjunctively'. In the case of $\mathcal{C}^{CL}$, Cut tells us with a negated-conjunctive reading of sequents that every formula is assigned either 1 or 0. This exhausts the options for a formula on bivalent valuations. With trivalent valuations and five-sided sequents under the intended interpretation, we obtain the following five possibilities:

- O(uter)Cut: every formula is either assigned 1 or assigned a value in $\{\frac{1}{2}, 0\}$.
- I(nner)Cut: every formula is either assigned 0 or assigned a value in $\{1, \frac{1}{2}\}$.
- L(eft)Cut: every formula is either assigned a value in $\{1, 0\}$ or in $\{1, \frac{1}{2}\} \cap \{\frac{1}{2}, 0\}$.
- T(olerant-)S(trict)Cut: every formula has a value either in $\{1, \frac{1}{2}\}$ or in $\{\frac{1}{2}, 0\}$.
- 3(-premise)Cut: every formula is assigned either 1, 0 or $\frac{1}{2}$:

This would give us the following five rules:

$$\frac{A, \Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0 \qquad \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \Gamma'_1 \Rightarrow \Delta'_1, A}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_1, \Delta_0 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1} \text{ OCut}$$

$$\frac{\Gamma_0 \Rightarrow \Delta_0, A \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0 \qquad \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid A, \Gamma'_1 \Rightarrow \Delta'_1}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_1, \Delta_0 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1} \text{ ICut}$$

$$\frac{\Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0, A \mid \Gamma'_0 \Rightarrow \Delta'_0 \qquad \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid A, \Gamma'_1 \Rightarrow \Delta'_1, A}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_1, \Delta_0 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1} \text{ LCut}$$

$$\frac{\Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid A, \Gamma'_0 \Rightarrow \Delta'_0 \qquad \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \Gamma'_1 \Rightarrow \Delta'_1, A}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_1, \Delta_0 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1} \text{ TSCut}$$

$$\frac{A, \Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0 \qquad \Gamma_1 \Rightarrow \Delta_1, A \mid \Theta_1 \mid \Gamma'_0 \Rightarrow \Delta'_0 \qquad \Gamma_2 \Rightarrow \Delta_2 \mid \Theta_2 \mid A, \Gamma'_2 \Rightarrow \Delta'_2, A}{\Gamma_0, \Gamma_1, \Gamma_2 \Rightarrow \Delta_0, \Delta_1, \Delta_2 \mid \Theta_0, \Theta_1, \Theta_2 \mid \Gamma'^0, \Gamma'^1, \Gamma'^2 \Rightarrow \Delta'_0, \Delta'_1, \Delta'_2} \text{ 3cut}$$

Each of 3Cut, LCut, TSCut, OCut and ICut exhaust the options for a formula. This makes them 'safe' shortcuts. Moreover, it is also clear that the admissibility of either OCut, ICut or TSCut implies the admissibility of the five-premise rule.

6. Results by proof analysis and completeness

The previous section introduced a sequent calculus $\mathcal{C}^{\mathrm{QWK}}$ intended to be sound and complete with regard to quantified weak Kleene valuations and thus to represent the four logis definable on quantified weak Kleene valuations presented in Section 2. The aim of this section is to show that $\mathcal{C}^{\mathrm{QWK}}$ has the features described as desirable in Section 3. We thus have two items on the agenda. The first is to present results obtainable by proof analysis with the aim being, of course, admissibility of the cut rules. The second is to show that a sequent is underivable in $\mathcal{C}^{\mathrm{QWK}}$ if and only if that sequent has a QWK-valuation under the intended interpretation.

Definition 6.1: The height of a $\mathcal{C}^{\mathrm{QWK}}$-derivation $\mathcal{D}$, $\mathcal{H}(\mathcal{D})$, is defined inductively as follows:

- If $\mathcal{D}$ consists of an initial sequent or a zero-premise rule, then $\mathcal{H}(\mathcal{D}) = 0$.
- If $\mathcal{D}$ is obtained with a κ-premise rule $\mathcal{R}$ where $0 < \kappa < \omega$ from derivations $\mathcal{D}_0, \dots, \mathcal{D}_{\kappa-1}$, then $\mathcal{H}(\mathcal{D}) = \sup_{i<\kappa}(\mathcal{H}(\mathcal{D}_i) + 1)$.

The notation $n \vdash \Gamma \Rightarrow \Delta|\Theta|\Gamma' \Rightarrow \Delta'$ means that there is a derivation of $\Gamma \Rightarrow \Delta|\Theta|\Gamma' \Rightarrow \Delta'$ in $\mathcal{C}^{\mathrm{QWK}}$ with height $\leq n$.

Definition 6.2: A single-premise rule $\mathcal{R}$ is height-preservingly admissible in $\mathcal{C}^{\mathrm{QWK}}$ just in case if there is a $\mathcal{C}^{\mathrm{QWK}}$-derivation of the premise of $\mathcal{R}$ with height n then there is a $\mathcal{C}^{\mathrm{QWK}}$-derivation of the conclusion of $\mathcal{R}$ with height $\leq n$.

The proofs of the next four lemmas proceed in each case by induction on the height of a derivation. They are straightforward generalisations of the proofs for the corresponding lemmas for a two-sided sequent calculus for classical logic presented by Negri and von Plato (2001) and are thus too obvious to warrant a presentation.

Lemma 6.3 (Weakening): *The following rule is height-preservingly admissible in* $\mathcal{C}^{\mathrm{QWK}}$:

$$\frac{\Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1}$$

Lemma 6.4 (Inversion): *For every primitive rule $\mathcal{R}$ of $\mathcal{C}^{\mathrm{QWK}}$, if there is a derivation of the conclusion with height n, then there are derivations of each premise with height $\leq n$.*

Lemma 6.5 (Contraction): *The following rules are height-preservingly admissible in* $\mathcal{C}^{\mathrm{QWK}}$:

$$\frac{A, A, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'}{A, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'} \qquad \frac{\Gamma \Rightarrow \Delta, A, A \mid \Theta \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta, A \mid \Theta \mid \Gamma' \Rightarrow \Delta'}$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta, A, A \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid \Theta, A \mid \Gamma' \Rightarrow \Delta'}$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta \mid A, A, \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid \Theta \mid A, \Gamma' \Rightarrow \Delta'} \qquad \frac{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A, A}{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A}$$

Lemma 6.6 (Substitution): *The following rules are height-preservingly admissible in* $\mathcal{C}^{\mathrm{QWK}}$ *where z is any variable, y is a eigenvariable and $A(z/y)$ is the result of replacing every occurrence of y with z in $A(y)$:*

$$\frac{\Gamma \Rightarrow \Delta, A(y) \mid \Theta \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta, A(z/y) \mid \Theta \mid \Gamma' \Rightarrow \Delta'} \qquad \frac{\Gamma \Rightarrow \Delta \mid \Theta \mid A(y), \Gamma' \Rightarrow \Delta', A(y)}{\Gamma \Rightarrow \Delta \mid \Theta \mid A(z/y), \Gamma' \Rightarrow \Delta', A(z/y)}$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A(y)}{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A(z/y)}$$

With the standard lemmas out of the way, we can move on to a more interesting feature of the calculus. As the interpretation of the calculus suggest, it should be possible to move formulas around; we should, for example, be in a position to show that if a formula is not assigned a value in $\{1, 0\}$ then it is not assigned 1. The following lemma confirms this:

Lemma 6.7 (Transfer): *The following rules are height-preservingly admissible in* $\mathcal{C}^{\mathrm{QWK}}$:

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta, A \mid \Gamma' \Rightarrow \Delta'}{A, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'}\ \Theta/\Gamma \qquad \frac{\Gamma \Rightarrow \Delta \mid \Theta, A \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta, A \mid \Theta \mid \Gamma' \Rightarrow \Delta'}\ \Theta/\Delta$$

$$\frac{\Gamma \Rightarrow \Delta \mid \Theta \mid A, \Gamma' \Rightarrow \Delta'}{A, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta'}\ \Delta'/\Delta \qquad \frac{\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A}{\Gamma \Rightarrow \Delta, A \mid \Theta \mid \Gamma' \Rightarrow \Delta'}\ \Gamma'/\Gamma$$

Proof: The proofs proceed by induction on the height of a derivation. Due to features the rules for $\neg$ and $\forall$, we must show Θ/Γ and Θ/Δ simultaneously. Correspondingly with Δ'/Δ and Γ'/Γ. We present some details in the proof for Θ/Γ and Θ/Δ.

Base case: $\Gamma \Rightarrow \Delta \mid \Theta, A \mid \Gamma' \Rightarrow \Delta'$ is an initial sequent. Then either A is atomic and in $\Gamma' \cap \Delta'$ or A is any formula. In the latter case, A is weakened in and we can thus move it wherever we want. In the former case, we observe that both $A, \Gamma \Rightarrow \Delta|\Theta|\Gamma' \Rightarrow \Delta'$ and $\Gamma \Rightarrow \Delta, A|\Theta|\Gamma' \Rightarrow \Delta'$ will also be initial sequents.

Inductive step: We distinguish between whether A is principal or not. If A is not principal, we simply 'backtrack', apply the inductive hypothesis and reapply the rule used to obtain $\Gamma \Rightarrow \Delta \mid \Theta, A \mid \Gamma' \Rightarrow \Delta'$ to thereby obtain the existence of a derivation of the desired sequent with the desired height. If A is principal, we separate cases based on the rule in question and present two cases. The others are similar.

$\Theta\neg$: A is $\neg B$ and $n{+}1 \vdash \Gamma \Rightarrow \Delta \mid \Theta, \neg B \mid \Gamma' \Rightarrow \Delta'$ is obtained from $n \vdash \Gamma \Rightarrow \Delta \mid \Theta, B \mid \Gamma' \Rightarrow \Delta'$. We apply the inductive hypothesis to obtain $n \vdash B, \Gamma \Rightarrow \Delta|\Theta|\Gamma' \Rightarrow \Delta'$ and $n \vdash \Gamma \Rightarrow \Delta, B|\Theta|\Gamma' \Rightarrow \Delta'$ before we apply $\Delta\neg$ and $\Gamma\neg$ to obtain $n{+}1 \vdash \neg B, \Gamma \Rightarrow \Delta|\Theta|\Gamma' \Rightarrow \Delta'$ and $n{+}1 \vdash \Gamma \Rightarrow \Delta, \neg B|\Theta|\Gamma' \Rightarrow \Delta'$.

$\Theta\forall$: A is $\forall xB$ and $n{+}1 \vdash \Gamma \Rightarrow \Delta \mid \Theta, \forall xB \mid \Gamma' \Rightarrow \Delta'$ is obtained from $n \vdash \Gamma \Rightarrow \Delta \mid \Theta, \forall xB, B(y) \mid \Gamma' \Rightarrow \Delta'$. For Θ/Δ, We apply the inductive hypothesis to obtain $n \vdash \Gamma \Rightarrow \Delta, \forall xB \mid \Theta, B(y) \mid \Gamma' \Rightarrow \Delta'$ and weakening to obtain $n \vdash \Gamma \Rightarrow \Delta, \forall xB, B(z) \mid \Theta, B(y) \mid \Gamma' \Rightarrow \Delta'$ where z is an eigenvariable. We now apply $\Delta\forall$ to obtain $n{+}1 \vdash \Gamma \Rightarrow \Delta, \forall xB|\Theta|\Gamma' \Rightarrow \Delta'$. For Θ/Γ, we use the inductive hypothesis to obtain $n \vdash \forall xB, B(y), \Gamma \Rightarrow \Delta, |\Theta|\Gamma' \Rightarrow \Delta'$ and then $\Gamma\forall$ to obtain $n{+}1 \vdash \forall xB, \Gamma \Rightarrow \Delta|\Theta|\Gamma' \Rightarrow \Delta'$. ■

Definition 6.8 (Formula complexity): The complexity of a $\mathcal{L}$-formula A, $|A|$, is defined inductively as follows: If A is an atomic formula, then $|A| = 0$, if A is of the form $\neg B$ or $\forall xB$, then $|A| = |B| + 1$, and if A is of the form $B \wedge C$, then $|A| = |B| + |C| + 1$.

Lemma 6.9 (Identity): *The following sequents are derivable in* $\mathcal{C}^{\mathrm{QWK}}$ *for every formula A:*

(i) $A, \Gamma \Rightarrow \Delta, A \mid \Theta \mid \Gamma' \Rightarrow \Delta'$

(ii) $A, \Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta', A$
(iii) $\Gamma \Rightarrow \Delta, A \mid \Theta \mid A, \Gamma' \Rightarrow \Delta'$
(iv) $\Gamma \Rightarrow \Delta \mid \Theta, A \mid A, \Gamma' \Rightarrow \Delta', A$

Proof: By induction on the complexity of a formula. With the base case being the initial sequents themselves, it suffices to show that if each of (i) –(iv) hold for formulas of complexity n then each of (i) –(iv) hold for formulas of complexity $n+1$. We illustrate the cases for $\forall$ where all contexts are omitted for readability:

$$\dfrac{\dfrac{\dfrac{A(y/x) \Rightarrow A(y/x) \mid \; \mid \Rightarrow}{\forall xA, A(y/x) \Rightarrow A(y/x), \forall xA \mid A(z/x) \mid \Rightarrow}\,\text{Weak}}{\forall xA \Rightarrow A(y/x), \forall xA \mid A(z/x) \mid \Rightarrow}\,\Gamma\forall}{\forall xA \Rightarrow \forall xA \mid \; \mid \Rightarrow}\,\Delta\forall(y)$$

$$\dfrac{\dfrac{\dfrac{A(y/x) \Rightarrow \mid \; \mid \Rightarrow A(y/x)}{\forall xA, A(y/x) \Rightarrow \mid \; \mid \Rightarrow A(y/x)}\,\text{Weak}}{\forall xA \Rightarrow \mid \; \mid \Rightarrow A(y/x)}\,\Gamma\forall}{\forall xA \Rightarrow \mid \; \mid \Rightarrow \forall xA}\,\Delta'\forall(y)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \mid A(y/x) \mid A(y/x) \Rightarrow A(y/x)}{\Rightarrow \forall xA, A(z/x), A(z'/x) \mid A(y/x) \mid A(y/x) \Rightarrow A(y/x)}\,\text{Weak}}{\Rightarrow \forall xA, A(z'/x) \mid \; \mid A(y/x) \Rightarrow A(y/x)}\,\Delta\forall(z) \quad \dfrac{A(z'/x) \Rightarrow A(z'/x) \mid \; \mid \Rightarrow}{A(z'/x) \Rightarrow \forall xA, A(z'/x) \mid \; \mid \forall xA \Rightarrow}\,\text{Weak}}{\Rightarrow \forall xA, A(z'/x) \mid \; \mid \forall xA \Rightarrow}\,\Gamma'\forall(y)}{\Rightarrow \forall xA \mid \; \mid \forall xA \Rightarrow}\,\Delta\forall(z')$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \mid A(y/x) \mid A(y/x) \Rightarrow A(y/x)}{\Rightarrow \mid \forall xA, A(y/x) \mid A(y/x) \Rightarrow A(y/x), A(z/x)}\,\text{Weak}}{\Rightarrow \mid \forall xA \mid A(y/x) \Rightarrow A(y/x), A(z/x)}\,\Theta\forall \quad \dfrac{A(z/x) \Rightarrow \mid \; \mid \Rightarrow A(z/x)}{A(z/x) \Rightarrow \mid \forall xA \mid \forall xA \Rightarrow A(z/x)}\,\text{Weak}}{\Rightarrow \mid \forall xA \mid \forall xA \Rightarrow A(z/x)}\,\Gamma'\forall(y)}{\Rightarrow \mid \forall xA \mid \forall xA \Rightarrow \forall xA}\,\Delta'\forall(z)$$

■

We can now turn our attention to proving that the cut rules are admissible. As it turns out, 3Cut, LCut and TSCut are all implied by ICut, OCut and the transfer rules. Here is the derivation of LCut (which is also a derivation of 3Cut):

$$\dfrac{\dfrac{\Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0, A \mid \Gamma_0' \Rightarrow \Delta_0'}{\Gamma_0 \Rightarrow \Delta_0, A \mid \Theta_0 \mid \Gamma_0' \Rightarrow \Delta_0'}\,\text{Tr.} \quad \dfrac{\dfrac{\Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0, A \mid \Gamma_0' \Rightarrow \Delta_0'}{A, \Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma_0' \Rightarrow \Delta_0'}\,\text{Tr.} \quad \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid A, \Gamma_1' \Rightarrow \Delta_1', A}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_1, \Delta_0 \mid \Theta_0, \Theta_1 \mid A, \Gamma_0', \Gamma_1' \Rightarrow \Delta_0', \Delta_1'}\,\text{OCut}}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_1, \Delta_0 \mid \Theta_0, \Theta_1 \mid \Gamma_0', \Gamma_1' \Rightarrow \Delta_0', \Delta_1'}\,\text{ICut/Contr.}$$

TSCut follows from either OCut or ICut using transfer.

We proceed thus to show that ICut and OCut are admissible. Due to the rules for $\neg$ and $\forall$, we prove the following claim:

Theorem 6.10 (Cut-elimination): *$\mathcal{C}^{\mathrm{QWK}}$ is such that if either $n \vdash A, \Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma_0' \Rightarrow \Delta_0'$ and $m \vdash \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \Gamma_1' \Rightarrow \Delta_1', A$ or $n \vdash \Gamma_0 \Rightarrow \Delta_0, A \mid \Theta_0 \mid \Gamma_0' \Rightarrow \Delta_0'$*

and $m \vdash \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid A, \Gamma'_1 \Rightarrow \Delta'_1$, *then there is a natural number* i *such that* $i \vdash \Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1$

Proof: This is established by strong double induction on $|A|$ and $n+m$ (cut-height). With a strong double induction, we obtain two inductive hypotheses; the first is applicable with reduced complexity but any cut-height and the second is applicable with the same complexity but reduced cut-height. We consider some pertinent cases.

The premise $A, \Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0$ *is an initial sequent:* Assume that A is principal. If $A \in \Delta_0$, then we obtain $\Gamma_1 \Rightarrow \Delta_1, A \mid \Theta_1 \mid \Gamma'_1 \Rightarrow \Delta'_1$ by transfer from the other premise and thus $\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1$ by weakening. If $A \in \Delta'_0$, then we obtain $\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1$ directly by weakening from $\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \Gamma'_1 \Rightarrow \Delta'_1, A$. If A is not principal, then $\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1$ is also an initial sequent.

The premise $\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \Gamma'_1 \Rightarrow \Delta'_1, A$ *is an initial sequent:* If A is not principal, then $\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1$ is also an initial sequent. Assume that A is principal. If $A \in \Gamma_1$, then $\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1$ follows by weakening. If $A \in \Theta_1 \cap \Gamma'_1$, then we only need to consider the case where $A, \Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0$ is not an initial sequent. With A being atomic since A is principal in an initial sequent, A is not principal in $A, \Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0$. Instead, the latter sequent is obtained with some rule $\mathcal{R}$ from a set of sequents. We apply the second induction hypothesis on each of the sequents in that set and $A, \Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0$, and then $\mathcal{R}$ on the results thereof to obtain $\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1$.

A is of the form $\neg A$ and is principal in $\Gamma_0 \Rightarrow \Delta_0, \neg A \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0$ *and* $\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \neg A, \Gamma'_1 \Rightarrow \Delta'_1$*:* We thus have the following two subderivations:

$$\dfrac{A, \Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0}{\Gamma_0 \Rightarrow \Delta_0, \neg A \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0}\ \Delta\neg \qquad \dfrac{\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \Gamma'_1 \Rightarrow \Delta'_1, A}{\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \neg A, \Gamma'_1 \Rightarrow \Delta'_1}\ \Gamma'\neg$$

While the pair of conclusions call for an application of ICut, the pair of premises call for an application of OCut:

$$\dfrac{A, \Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0 \qquad \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \Gamma'_1 \Rightarrow \Delta'_1, A}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1}\ \text{Ind1}$$

A is of the form $\forall xA$ and is principal in $\Gamma_0 \Rightarrow \Delta_0, \forall xA \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0$ *and* $\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \forall xA, \Gamma'_1 \Rightarrow \Delta'_1$*:* We thus have the following two subderivations:

$$\dfrac{\Gamma_0 \Rightarrow \Delta_0, A(y), \forall xA \mid \Theta_0, A(z) \mid \Gamma'_0 \Rightarrow \Delta'_0}{\Gamma_0 \Rightarrow \Delta_0, \forall xA \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0}$$

$$\dfrac{\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid A(y), \Gamma'_1 \Rightarrow \Delta'_1, A(y) \qquad A(z), \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \forall xA, \Gamma'_1 \Rightarrow \Delta'_1}{\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \forall xA, \Gamma'_1 \Rightarrow \Delta'_1}$$

We proceed as follows, using both the second and the first inductive hypothesis:

$$\dfrac{\Gamma_0 \Rightarrow \Delta_0, \forall xA \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0 \qquad A(z), \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \forall xA, \Gamma'_1 \Rightarrow \Delta'_1}{A(z), \Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \mid \Theta_0, \Theta_1 \mid \Gamma'_0, \Gamma'_1 \Rightarrow \Delta'_0, \Delta'_1}\ \text{Ind2}$$

$$\dfrac{\dfrac{\vdots}{A(z),\Gamma_0,\Gamma_1 \Rightarrow \Delta_0,\Delta_1 \mid \Theta_0,\Theta_1 \mid \Gamma_0',\Gamma_1' \Rightarrow \Delta_0',\Delta_1'} \quad \dfrac{\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid A(y),\Gamma_1' \Rightarrow \Delta_1',A(y)}{\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid A(z),\Gamma_1' \Rightarrow \Delta_1',A(z)}\,\text{Sub}}{\Gamma_0,\Gamma_1,\Gamma_1 \Rightarrow \Delta_0,\Delta_1,\Delta_1 \mid \Theta_0,\Theta_1,\Theta_1 \mid A(z),\Gamma_0',\Gamma_1',\Gamma_1' \Rightarrow \Delta_0',\Delta_1',\Delta_1'}\,\text{Ind1}$$

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \forall xA,\Gamma_1' \Rightarrow \Delta_1' \qquad \Gamma_0 \Rightarrow \Delta_0,A(y),\forall xA \mid \Theta_0,A(z) \mid \Gamma_0' \Rightarrow \Delta_0'}{\Gamma_0,\Gamma_1 \Rightarrow \Delta_0,\Delta_1,A(y) \mid \Theta_0,\Theta_1,A(z) \mid \Gamma_0',\Gamma_1' \Rightarrow \Delta_0',\Delta_1'}\,\text{Ind2}}{\Gamma_0,\Gamma_1 \Rightarrow \Delta_0,\Delta_1,A(z) \mid \Theta_0,\Theta_1,A(z) \mid \Gamma_0',\Gamma_1' \Rightarrow \Delta_0',\Delta_1'}\,\text{Sub}}{\Gamma_0,\Gamma_1 \Rightarrow \Delta_0,\Delta_1,A(z) \mid \Theta_0,\Theta_1 \mid \Gamma_0',\Gamma_1' \Rightarrow \Delta_0',\Delta_1'}\,\text{Trans}$$

The desired conclusion is now obtained with one application of the first inductive hypothesis and some applications of contraction:

$$\dfrac{\dfrac{\vdots}{\Gamma_0,\Gamma_1 \Rightarrow \Delta_0,\Delta_1 \mid \Theta_0,\Theta_1 \mid A(z),\Gamma_0',\Gamma_1' \Rightarrow \Delta_0',\Delta_1'} \quad \dfrac{\vdots}{\Gamma_0,\Gamma_1 \Rightarrow \Delta_0,\Delta_1,A(z) \mid \Theta_0,\Theta_1 \mid \Gamma_0',\Gamma_1' \Rightarrow \Delta_0',\Delta_1'}}{\Gamma_0,\Gamma_1 \Rightarrow \Delta_0 \mid \Theta_0,\Theta_1 \mid \Gamma_0',\Gamma_1' \Rightarrow \Delta_0',\Delta_1'}$$

■

Corollary 6.11: *The rules TSCut, LCut and 3Cut are admissible in $\mathcal{C}^{\text{QWK}}$.*

With ICut and OCut admissible, we also obtain the admissibility of the following rule which will be utilised in the completeness proof below:

Corollary 6.12 (Split-transfer): *The following rule is admissible:*

$$\dfrac{A,\Gamma \Rightarrow \Delta \mid \Theta \mid \Gamma' \Rightarrow \Delta' \qquad \Gamma \Rightarrow \Delta,A \mid \Theta \mid \Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta \mid \Theta,A \mid \Gamma' \Rightarrow \Delta'}$$

Proof: ICut, OCut and contraction on the premises and $\Rightarrow \mid A \mid A \Rightarrow A$. ■

This completes the applications of techniques from structural proof theory as presented by Negri and von Plato (2001), and we have thus illustrated that the sequent calculus has the features desirable from the perspective of structural proof theory.

We can thus turn our attention to showing that that the sequent calculus is sound and complete with regard to QWK-valuations in order to establish that the sequent calculus also permit elegant representations of each of the four logics.

Theorem 6.13 (Completeness): *A sequent $\Gamma \Rightarrow \Delta|\Theta|\Gamma' \Rightarrow \Delta$ is derivable in $\mathcal{C}^{\text{QWK}}$ if and only if there is no QWK-valuation such that for every formula $A \in \Gamma, \mathcal{V}(A) = 1$, for every formula $A \in \Delta, \mathcal{V}(A) = 0$, for every formula $A \in \Theta, \mathcal{V}(A) \in \{1, 0\}$, for every formula $A \in \Gamma', \mathcal{V}(A) \in \{1, \frac{1}{2}\}$ and for every formula $A \in \Delta', \mathcal{V}(A) \in \{\frac{1}{2}, 0\}$.*

Proof: The left-to-right direction proceeds by induction on the height of a derivation. For the contrapositive of the right-to-left direction, we proceed in the standard way through the construction of a reduction tree for an underivable sequent which is used to define a QWK-valuation for that sequent. We will here sketch the pertinent details of the proof.

Assume that the sequent $\Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta_0$ is underivable. We can then construct a tree with that sequent as root by repeated 'upwards' applications of the primitive rules and split-transfer (Corollary 6.12) on the formulas in the underivable sequent and their subformulas (as they are unpacked) with the twist that the principal formula is always included as the active formula in the same position. This proceeds in stages, and at each stage every branch with an initial sequent as leaf is closed. The exact procedure for constructing the tree is a straightforward modification to the procedures presented by, for example, Takeuti (1987) and Negri and von Plato (2001) and we thus skip the finer details. This tree is guaranteed to have at least one open branch $\mathcal{B} = \{\Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta'_0;\ \Gamma_1 \Rightarrow \Delta_1 \mid \Theta_1 \mid \Gamma'_1 \Rightarrow \Delta'_1;\ \ldots\}$ of length $\kappa \leq \omega$. That the tree contains an open branch follows from the assumption that $\Gamma_0 \Rightarrow \Delta_0 \mid \Theta_0 \mid \Gamma'_0 \Rightarrow \Delta_0$ is underivable; had the sequent been derivable, then each branch would close.[7] The 'upwards' applications of split-transfer ensure that for every formula occurring in Θ-position of a sequent in an open branch, there is a sequent 'higher up' in that branch at which that formula also occurs in either Γ-position or Δ-position.

Using $\mathcal{B}$ we define five sets as follows: $\Gamma^{\mathcal{B}} = \bigcup_{i<\kappa}(\Gamma_i)$; $\Gamma'^{\mathcal{B}} = \bigcup_{i<\kappa}(\Gamma'_i)$; $\Theta^{\mathcal{B}} = \bigcup_{i<\kappa}(\Theta_i)$; $\Delta^{\mathcal{B}} = \bigcup_{i<\kappa}(\Delta_i)$; and $\Delta'^{\mathcal{B}} = \bigcup_{i<\kappa}(\Delta'_i)$. $\mathcal{V}$ is now defined as follows:

- If $A_{at} \in \Gamma^{\mathcal{B}}$, then $\mathcal{V}(A_{at}) = 1$.
- If $A_{at} \in \Delta^{\mathcal{B}}$, then $\mathcal{V}(A_{at}) = 0$.
- For every other atomic formula of $\mathcal{L}$, $\mathcal{V}(A_{at}) = \frac{1}{2}$.
- Weak Kleene clauses for $\neg$, $\wedge$ and $\forall$.

It is left to show by induction on the complexity of a formula that the following holds for every formula A:

- if $A \in \Gamma^{\mathcal{B}}$, then $\mathcal{V}(A) = 1$,
- if $A \in \Delta^{\mathcal{B}}$, then $\mathcal{V}(A) = 0$,
- if $A \in \Theta^{\mathcal{B}}$, then $\mathcal{V}(A) \in \{1, 0\}$,
- if $A \in \Gamma'^{\mathcal{B}}$, then $\mathcal{V}(A) \in \{1, \frac{1}{2}\}$,
- if $A \in \Delta'^{\mathcal{B}}$, then $\mathcal{V}(A) \in \{\frac{1}{2}, 0\}$.

We illustrate some cases regarding atomic formulas and universally quantified formulas.

Case A_{at}: Assume $A_{at} \in \Delta'^{\mathcal{B}}$. By the construction of the reduction tree, there are three possibilities. A_{at} is only there, or also in $\Delta^{\mathcal{B}}$ or $\Gamma'^{\mathcal{B}}$. If also $A_{at} \in \Delta^{\mathcal{B}}$, then, $\mathcal{V}(A_{at}) = 0$. If not, then $\mathcal{V}(A_{at}) = \frac{1}{2}$. A similar reasoning will show that if $A_{at} \in \Gamma'^{\mathcal{B}}$, then $I(A_{at}) \in \{\frac{1}{2}, 0\}$. Assume $A_{at} \in \Theta^{\mathcal{B}}$. Then $A_{at} \in \Delta$ or $A_{at} \in \Gamma$ by the reduction according to split-transfer and thus $\mathcal{V}(A_{at}) \in \{1, 0\}$.

Case $\forall xA$: Assume $\forall xA \in \Delta^{\mathcal{B}}$. Then $A(z) \in \Theta^{\mathcal{B}}$ for every variable z and for some y, $A(y) \in \Delta^{\mathcal{B}}$. From the induction hypothesis and the clause for $\forall$ it follows that $\mathcal{V}(\forall xA) = 0$. Assume $\forall xA \in \Theta^{\mathcal{B}}$. Then $A(z) \in \Theta^{\mathcal{B}}$ for every variable z. It follows by the induction hypothesis that every instance is crisp, so $\mathcal{V}(\forall xA) \in \{1, 0\}$ follows. Assume $\forall xA \in \Gamma'^{\mathcal{B}}$. Then either for some y, $A(y) \in \Gamma'^{\mathcal{B}} \cap \Delta'^{\mathcal{B}}$ and thus $\mathcal{V}(A(y)) = \frac{1}{2}$ from which $\mathcal{V}(\forall xA) = \frac{1}{2}$ follows, or for every z, $A(z) \in \Gamma^{\mathcal{B}}$, and thus $\mathcal{V}(A(z)) = 1$, so $\mathcal{V}(\forall xA) = 1$.

Since every formula in Γ is also in $\Gamma^{\mathcal{B}}$ and so forth, we conclude that we have defined a QWK-valuation for the underivable sequent. With the sequent in question being arbitrary, it follows that for every underivable sequent there is a QWK-valuation under the intended interpretation. ■

Corollary 6.14:
$\langle \Gamma, \Delta \rangle \in \text{QWK}_{SS}$ *iff* $\Gamma \Rightarrow \mid\mid \Rightarrow \Delta$ *is derivable.*
$\langle \Gamma, \Delta \rangle \in \text{QWK}_{TT}$ *iff* $\Rightarrow \Delta \mid\mid \Gamma \Rightarrow$ *is derivable.*
$\langle \Gamma, \Delta \rangle \in \text{QWK}_{ST}$ *iff* $\Gamma \Rightarrow \Delta \mid\mid \Rightarrow$ *is derivable.*
$\langle \Gamma, \Delta \rangle \in \text{QWK}_{TS}$ *iff* $\Rightarrow \mid\mid \Gamma \Rightarrow \Delta$ *is derivable.*

The proof of Theorem 6.13 used the rule split-transfer which was established as admissible in Corollary 6.12 using the rules ICut and OCut which in turn were established as admissible in Theorem 6.10. Now, if we had established the admissibility of split-transfer without invoking the rules ICut and OCut, then the admissibility of ICut and OCut would have been a further corollary of Theorem 6.13. However, I have not found a proof of split-transfer without invoking ICut and OCut. The main issue is, perhaps unsurprisingly, the rules for the universal quantifier and the fact that the principal formula occurs as active formula.

This completes our agenda in this paper, namely to present *a* sequent calculus for four logics definable on quantified weak Kleene valuations that display the desirable features discussed in Section 3 with regard to structural proof theory and representation of valid inferences. In particular, it has been shown that techniques from structural proof theory presented by Negri and von Plato (2001) apply more or less directly to the sequent calculus $\mathcal{C}^{\text{QWK}}$, the main result being the cut-elimination theorem 6.10. Moreover, it has been shown with Corollary 6.14 as an immediate consequence of the completeness Theorem 6.13 that valid inferences can be read straight off sequents.

Notes

1. Regarding their labelling, it seems that *paraconsistent weak Kleene* is typically referred to as PWK; see, for example, Bonzio et al. (2017), Ciuni and Carrara (2019) and Paoli and Baldi (in press). That acronym is, however, not particularly enlightening if we also take into consideration *paracomplete weak Kleene*. Others, e.g. Coniglio and Corbalan (2012), refer to the paracomplete and the paraconsistent logics as the nonsense-operator free fragments of *Bochvar's logic* and *Halldén's logic*. That would still leave us without labels for the non-reflexive and the non-transitive logics, two logics that have not actually been 'claimed' by anyone in the case of weak Kleene. For the purposes of simplicity and uniformity, we will instead use the distinction between strict and tolerant satisfaction of a formula from Cobreros et al. (2012) to label the logics.
2. QWK_{TS} is actually empty because there is currently no formula which is crisp in every QWK-valuation. It is still included in the list because the valuations are easily augmented with formulas that are crisp in every valuation (e.g. $x = y$, $\perp$ or $\top$), and it is thus of interest to see that also that logic is representable in the sequent calculus.
3. For a discussion of this aspect with Gentzen (1935)'s original proof, see von Plato (2001).
4. The notation for three-sided sequents used by Ripley (2012) is significantly fancier than our three simple lines, but we can safely ignore such typographic differences.
5. Such a position in a sequent corresponds to Paoli and Baldi (in press)' label for crispiness in their semantic tableaux for propositional paraconsistent weak Kleene, and the

sequent calculus presented in this article is thus based on the same trick as the semantic tableaux presented by Paoli and Baldi (in press). There are nonetheless notable differences between the systems over and above the fact that their semantic tableaux are restricted to a propositional language. For example, the ambition to capture not only paraconsistent but also other weak Kleene logics allows us to obtain 'pure' rules for negation as opposed to the De Morgan-ish treatment of negation in the semantic tableaux by Paoli and Baldi (in press). However, it should be clear from the results, in this paper, that their semantic tableaux can be expanded to represent the four first-order logics definable on quantified weak Kleene models. The extended version of their semantic tableaux and the sequent calculus presented in this paper would thus be alternative proof theories for the same logics. Now, this does not make them competitors as if they present alternative metaphysical pictures. Instead, they would supplement each other. For example, the completeness proof for Theorem 6.13 implicitly treats the sequent calculus as a semantic tableaux to extract a model.

6. If the goal of the paper was to define only either QWK_{ST} or QWK_{TT}, then one could restrict the attention to three sides, i.e. either $\Gamma \Rightarrow \Delta \mid \Theta$ or $\Delta|\Theta|\Gamma'$, respectively, keeping the proposed interpretation of each side from the five-sided case. However, then negation would require a De Morgan-ish treatment along the lines of Paoli and Baldi (in press) or possibly a variable-restriction along the lines of Coniglio and Corbalan (2012). Another approach would be to use a Tait (1968)-style language (and thus implicitly giving negation a De Morgan-ish treatment) in which case we would obtain a sequent calculus for all four logics through three-sided sequents of the form $\Delta|\Theta|\Delta'$, keeping the proposed interpretation of each side from the five-sided case.
7. If the reduction tree had been constructed using only the inverses of primitive rules as opposed to also the inverse of split-transfer, then a tree in which each branch is closed would itself be a derivation of the sequent. In our case, however, a tree in which each branch is closed will not itself be a derivation of the root-sequent since split-transfer is merely admissible. Instead, such a tree only guarantees that the root-sequent is derivable.

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