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Determining the Alignment of Math 105 - Intermediate Algebra at the University of Wisconsin--Milwaukee to the Goals of the Common Core State Standards

Raymond Dempsey
University of Wisconsin-Milwaukee

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AT THE UNIVERSITY OF WISCONSIN - MILWAUKEE
TO THE GOALS OF THE COMMON CORE STATE STANDARDS

by

Raymond M Dempsey

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ABSTRACT

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In this analysis we examine the Common Core State Standards for Mathematics and compare them to content presented in Math 105 - Intermediate Algebra at the University of Wisconsin-Milwaukee, in order to determine how well the UW-Milwaukee course develops the skills described in the standards. This is done by examining the structure, textbook content, and assessments of the course. Examining relevant high school standards, we determine that many of the procedural elements of these standards are present in the course while many of the conceptual elements are absent or poorly developed. After, we discuss content that is present in the course but not detailed in the standards. We then determine the overlap of content from Math 105 with the content from Math 095. We finish by considering ways to better align Math 105 with the goals of the Common Core State Standards.

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CHAPTER 1: THE COMMON CORE STATE STANDARDS INITIATIVE

1.1 What is the Common Core State Standards Initiative?

The Common Core State Standards Initiative, or Common Core for short, is a state-led effort in standards-based K-12 education reform to bring the different mathematics and language arts curricula of the states into alignment with one another. It does this by providing a partially ordered list of skills, called "standards", that students should have. The standards are organized broadly by grade level for K-8 education and by topic for high school level mathematics. The mission statement reads [1], "The Common Core State Standards provide a consistent, clear understanding of what students are expected to learn, so teachers and parents know what they need to do to help them. The standards are designed to be robust and relevant to the real world, reflecting the knowledge and skills that our young people need for success in college and careers. With American students fully prepared for the future, our communities will be best positioned to compete successfully in the global economy." The Common Core is sponsored and led by both the National Governors Association and the Counsel of Chief State School Officers.

1.2 Brief History of the Common Core

In a report from 2004 titled "Ready or Not: Creating a High School Diploma That Counts", the American high school diploma is described as devalued because "what it takes to earn one is disconnected from what it takes for graduates to compete successfully beyond high school - either in the classroom or the workplace." [2] (p. 1). It also states, "the confidence that students and parents place in the [high school] diploma contrast sharply with the skepticism of employers and post-secondary institutions, who all but ignore the diploma, knowing that it often serves as little more than a certificate of attendance. In fact, in much of the United States, students can earn a high school diploma without having demonstrated the achievement of common academic standards or the ability to apply their knowledge in practical ways." [2] (p. 1). Also stated is that "college-bound students take national admissions exams that may not align with the high school curriculum the students have been taught." [2] (p. 2). Summarizing the problems, the report states that most high school graduates need remedial help in college, most college students never attain a degree, most employers say high school graduates lack basic skills., and most workers question the preparation that high schools provide [2] (p. 3). The proposed solution is that "state policy-makers need to anchor high school graduation requirements and assessments to the standards of the real world: to the knowledge and skills that colleges and employers actually expect if young people are to succeed in their institutions." [2] (p. 3).

In 2009, reacting to the need for education reform, the National Governors Association appointed teams to author curriculum standards for mathematics and language arts. The principal writers for the mathematics standards consisted of William McCallum Ph.D., head of the math department at the University of Arizona, Jason Zimba Ph.D., an astrophysicist/mathematician and previous Professor of Physics and Mathematics at

Bennington College, and Philip Daro, the Site Director of the Strategic Education Research Partnership (SERP) at the San Francisco Unified School District. Each has extensive experience in mathematics education. Though these were the principal writers, there were many other contributors. As the standards were being written and reviewed, many teachers, parents, school administrators, mathematical organizations, and experts from across the U.S. were providing feedback that was influential in shaping the standards. The standards were completed June 2010.

States choose to adopt the Common Core Standards, so adoption is not mandated by the federal government. To encourage education reform, the federal government gave states an incentive to adopt "internationally benchmarked standards and assessments that prepare students for success in college and the workplace" by awarding Race to the Top competitive grants to qualifying states [3]. Adoption of the Common Core Standards can serve as part of what qualifies a state for these grants. With this additional incentive, many states adopted the Common Core Standards shortly after their completion in 2010 and as of May 2013, forty-five states have adopted both the language arts and mathematics standards; Minnesota has adopted only the language arts standards.

1.3 Creation and Organization of the Standards

In a video from The Hunt Institute, Jason Zimba states [4], "When the states came together, the goal wasn't simply for them all to begin doing the same thing, the goal was for all of them to raise their game and do math education better. So our charge wasn't to combine, or average, or concatenate all of the standards together, or regress to the mean but to build on what the best states were doing." The goal was that [5] "the standards as a whole must be essential, rigorous, clear and specific, coherent, and internationally benchmarked." The authors designed the standards to focus on conceptual understanding, modeling, and procedural accuracy.

Each standard is stated as an individual statement that describes what students should be able to do. Almost all standards begin with a strong verb that describes the type of action that is taken to satisfy the standard. For example, many of these statements require students to perform something procedural (Rewrite, Factor, Graph, Rearrange, Solve, etc), to understand something (Interpret, Recognize, Explain, Identify, Relate, etc), or perform something that combines the two (Derive, Prove, Create, Use, etc). There are two main types of standards: Standards for Mathematical Practice and Standards for Mathematical Content.

Standards for Mathematical Practice describe overarching skills that instructors should look to instill in their students. They describe broad abilities that mathematically proficient students should possess. There are eight Standards for Mathematical Practice [1]:

- Make sense of problems and persevere in solving them
- Reason abstractly and quantitatively

- Construct viable arguments and critique the reasoning of others
- Model with mathematics
- Use appropriate tools strategically
- Attend to precision
- Look for and make use of structure
- Look for and express regularity in repeated reasoning

The Standards for Mathematical Content, or content standards, are designed to be specific to a skill learned in mathematics, for example, using polynomial identities to solve problems. Some standards have sub-standards that list specific tasks within a standard. Standards with connections to modeling are indicated with a star symbol. Optional advanced standards are indicated with a plus symbol. We will call standards *unplussed* if they are not optional advanced standards. Similar standards are grouped together into clusters. Similar clusters are grouped together into domains. Sometimes, different domains will have similar standards. For K-8 education, domains may appear in multiple grades, though the standards become more advanced in higher grades. In the high school standards, domains are grouped into conceptual categories, for example Geometry or Functions. Another organization of the standards includes listing all standards in each domain, organized by grade level. The organization we will be using in this thesis is the organization of the high school standards by conceptual category.

Since this analysis focuses on the high school standards, it is important to understand the motivation for these standards [6]:

- "The high school standards call on students to practice *applying mathematical ways of thinking to real world issues and challenges;* they prepare students to think and reason mathematically."
- "The high school standards set a *rigorous definition of college and career readiness,* by helping students to develop a depth of understanding and ability to apply mathematics to novel situations, as college students and employees regularly do."
- "The standards emphasize mathematical modeling, the use of mathematics and statistics to analyze empirical situations, understanding them better, and improve decisions. For example, the draft standards state: 'Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. It is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.'"

CHAPTER 2: FORMAT OF MATH 105 COURSE

2.1 Structure and Content

It is important to understand how the Math 105 course is organized in order to be able to analyze it. Math 105 is a three-credit undergraduate course that is commonly taken to satisfy a prerequisite for another course or satisfy a general education requirement. Students must earn a C or better in the course to satisfy the general education requirement. For most students, earning a grade below a C will accomplish very little. The format of the course is either lecture, or large lecture with a discussion section. Large lectures are typically taught by the course coordinator and paired with a discussion section taught by a graduate teaching assistant. This course coordinator also supervises graduate teaching assistants who, in turn, teach the majority of the lectures or discussion sections. Instructors teach from the same predetermined sections of the textbook. The topics covered are:

- Graphing linear functions
- Understanding slope-intercept form
- Domain and range
- Solving systems of linear equations
- Solving inequalities
- Interval notation
- Factoring
- Solving equations with rational expressions
- Division of polynomials
- Synthetic division

- Simplifying radical expressions
- Complex numbers
- Completing the square
- Quadratic formula
- U-substitution
- Graphing quadratic functions
- Composite and inverse functions
- Exponential and logarithmic functions

The prerequisite for the course is a Math Placement Score of 20 or earning a C or better in Math 095 - the non-credit mathematics course that precedes Math 105. The course uses the text *Intermediate Algebra, Bittinger/Ellenbogen, 8th Edition* [7], which comes packaged with an activation code for MyMathLab [8] or MML for short - the online homework portion of the course. All instructors teaching Math 105 must use the same grading scale and grading template as follows:

Grading Scale (minimum cutoffs)

A	A-	B+	B	B-	C+	C	C-	D+	D	D-
93	90	87	83	80	77	73	70	67	63	60

Grading Template

Attendance 30 pts

Homework	50 pts
Quizzes	70 pts
Exams	300 pts
Final Exam	200 pts
Total	650 pts

Attendance must be recorded every class period and instructors are free to decide the number of points subtracted for absences. The homework grade is a combination of grades recorded automatically in MML and the homework assigned from the text. Instructors are free to decide how much homework to collect, what proportion of the homework grade comes from MML, and what proportion comes from the homework assigned from the text. Instructors are given the homework set to give the students, so each class assigns the same homework problems. There are ten quizzes given and spaced evenly throughout the course. Instructors drop the lowest three quiz scores of each student. There are three exams given and spaced evenly throughout the course. No exams are dropped. The instructors are free to write their own quizzes and exams but the course coordinator offers past quizzes to new instructors and if instructors decide to use them, they should make multiple copies with minor numeric alterations to reduce cheating during quizzes and exams. The course coordinator does not review the quizzes and exams written by the instructors. All sections take the same final exam with only minor numeric alterations between versions.

CHAPTER 3: ANALYSIS OF THE STANDARDS

3.1 Methods of Analysis

This thesis is an analysis of the content of Math 105 to see how well it aligns with the goals of the Common Core. We are specifically analyzing content which is presented in class, written in the text, assigned as homework, and assessed on quizzes, exams, and the final exam. Since instructors are given freedom on how this content is presented and assessed, the scope of this content analysis must be clarified. First, we are assuming that instructors are lecturing based on content in the text. The text is the main source of content for the course and instructors are using class time to explain the topics detailed in the text and prepare students to understand concepts needed to do well on the assessments.

Second, we are analyzing all unplussed high school standards that are not topics oriented to geometry, trigonometry, vectors, matrices, probability, and statistics. Math 105 is an intermediate algebra course and we are ignoring standards of the types listed because these topics typically appear only marginally in intermediate algebra courses and are not present in Math 105 at all.

There are six conceptual categories in the high school standards: Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics & Probability. We are not analyzing any standards in the Geometry or Statistics & Probability conceptual categories. The Modeling conceptual category is analyzed in relation to other standards that feature modeling. we are therefore analyzing twenty-three clusters from the three remaining conceptual categories. Many clusters have multiple standards and we will analyze the standards together as a group because each are closely related and are often studied and assessed concurrently. The analysis is broken into six main sections: *Prior Knowledge*, *Interpretation of the Standards*, *Place in the Course*, *Closest-Related Assessment Problems*, *Concepts Regularly Discussed*, and *Concepts Often Overlooked*.

Prior Knowledge is meant to give a brief history of how students typically gain the necessary prerequisite knowledge and skills to be able to begin working on the topics of each standard in the cluster. It is meant to show the evolution of the foundational concepts and helps to emphasize how the standards help further their development. Prior Knowledge is discussed at the cluster level.

Interpretation of the Standards gives my personal analysis of what the standards are asking students to do. Many standards are straightforward while others require some interpretation. When interpreting standards, I am using my best judgment to relate the concepts to the Math 105 course. I search for evidence for interpretations in assessments that are featured on Illustrative Mathematics [9]. Also, I often mention why mastering the standards is important and mention concepts that are developed as a result of mastering concepts of the cluster.

Place in the Course states where in the course the standards are studied. I state the sections in the text where the principles of the standards are developed and what topics the sections explore. Depending on the cluster, I may also list the sections that develop the necessary background information and/or list the sections that use concepts developed in the standards.

Closest-Related Assessment Problems gives examples of assessment problems from the five forms of assessment: written homework, abbreviated as HW, MyMathLab problems, abbreviated as MML, quizzes, exams, and the final exam. The written homework is the same for all sections of Math 105 as are MML assessment problems. As stated earlier, instructors are free to write their own quizzes and exams but most use the quizzes given by the course coordinator with minor alterations. In the analysis, we are assuming that instructors are using these quizzes and exams or creating ones very similar. Instructors that create their own often include problems that are very close to these quizzes and exams and rarely create ones that are drastically different in format or scope. Every section takes the same final exam with minor alterations. From this, instructors are implicitly encouraged to prepare their students for this final exam, so again instructors who create their own quizzes and exams often do so to mirror the format and scope of the final exam. When selecting my closest-related assessment problems, I am providing what I would consider a sample of problems that are the closest match to problems that assess the students' attainment of the standards. Sometimes no assessment problems resemble the standards.

Concepts Regularly Discussed and *Concepts Often Overlooked* examine how well the course achieves the goals of the standards. We review the approaches the course takes to discussing the topics, and typically comment regarding the strength of the assessments. These are based on my interpretation of the standards. How well principles of the standards are covered is weighted according to how prominently they appear in the course and how often they appear in the assessments. Sometimes the text has assessment problems that strongly align to the principles of the standards, but if those problems are not assigned or assessed, then I will not consider them part of the course. We weigh the prominence of a concept by its appearances in assessments that are weighted highly towards the course grade. The final exam is worth more than the attendance, written homework, MML homework, and quizzes combined, so principles that appear on the final exam are considered a more essential part of the course than those that appear in lesser-weighted assessments. Also considered is the amount of lecture time dedicated to exploring the concepts, and the explanations and examples provided in the text.

Illustrative Mathematics Sample Assessments are drawn from Illustrative Mathematics [9], and are assessments that accurately reflect the goals of the Common Core State Standards. Comparing these assessments with the assessments currently in Math 105 allows the reader to see the how well the current assessments align to the standards.

CHAPTER 4: NUMBER AND QUANTITY

4.1 The Real Number System

Prior Knowledge

Students are expected to understand the definition of a rational number and recognize the rational numbers as an extension of the integers. A student's progress in mathematics is often measured by an expanded sense of quantity. First with natural numbers, then integers, and later rational numbers, students are able to assign exact value to more things. These new values can be used arithmetically and algebraically in much the same way.

Students are expected to enter the course with experience working with exponents that are natural numbers. The domain of comprehension of the core concepts of exponents is restricted to natural numbers in developmental stages, because applying a natural number exponent can be presented computationally: an exponent greater than or equal to two represents how many times the base is multiplied by itself. Still only working with natural number exponents and investigating relations naively, we can establish the Laws of Exponents. This allows us to expand the scope of exponents by establishing relationships of numbers with the same bases as we multiply and divide them with one another. In this, we expand the scope of exponents from natural number exponents to integer exponents.

Students are also expected to enter the course with a basic understanding of a square root. In developmental courses, a square root is an expression that is presented as a question: what number multiplied by itself equals the radicand? Initially, examples are given where the radicand is a perfect square, and then a rational number. Examples are later given in which there is no rational number that satisfies the question, which introduces the idea of an irrational number. This allows us to expand our concept of quantity to include a notion of irrational numbers. Students learn to treat these irrational square roots as numbers and can approximate them with a decimal expansion using a calculator. Students should understand that including irrational numbers with the rational numbers represents an extension of a larger number system (the real numbers), that classification of a number as rational or irrational partitions the real number system, and recognize that the decimal representations of rational numbers are either terminating or repeating and the decimal representations of irrational numbers are both non-terminating and non-repeating.

4.1.1 Extend the Properties of Exponents to Rational Exponents

CCSS.Math.Content.HSN-RN.A.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we* *define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5*. [1]

CCSS.Math.Content.HSN-RN.A.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents. [1]

Interpretation of the Cluster

The first purpose of this cluster is to expand the scope of exponents to include rational exponents. The cluster expects students to understand that the basic axioms and properties of real-number arithmetic remain unchanged. We previously saw that the associative, commutative, and distributive properties still apply as we develop an intuition that a fraction is not merely a representation of a part of something, but a number itself. Students must develop the same intuition regarding rational exponents. They should comprehend that they are not learning an entirely new concept but rather they are expanding a concept to include rational numbers, with which they have experience in a different context.

The second purpose of this cluster is to relate roots to rational exponents. Using the Laws of Exponents naively, we can relate that the square root of a number k can be represented as $k^{\frac{1}{2}}$, because $k^{\frac{1}{2}} \cdot k^{\frac{1}{2}} = k^1 = k$, answering the question: What number times itself equals k? Using this as a conceptual foundation, students are expected to expand their comprehension of roots to include nth roots and to take a radical and represent it as a rational exponent and vice versa. We can then recognize the advantages of changing how these values are represented, that is b^k represents the positive solution to $x^k = b$, as does $\sqrt[k]{b}$.

Place in the Course

The course establishes the necessary background information in 7.1 by giving the definition of the nth root, introducing the graphs of root functions, and beginning to simplify nth roots by first observing whether the index of the root is even or odd. The topics of the cluster are covered directly in Section 7.2 [7]. The lesson reflects the principles of the cluster by establishing how to represent nth roots as rational exponents, stating the Laws of Exponents, and using rational exponents to simplify expressions. The topics that use principles learned in the cluster are covered in Sections 7.2 - 7.6, and 8.5 [7]. The principles of the cluster are used in these lessons to establish rules for roots like that of the Laws of Exponents, to simplify more complicated roots, to solve radical equations, and to recognize equations that are reducible to standard quadratic form.

Closest-Related Assessment Problems

HW 7.2 #13 - Write an equivalent expression using radical notation and, if possible, simplify. [7]

$32^{\frac{1}{5}}$

MML 7.3 #12 - Multiply and simplify. Assume that no radicands were formed by raising negative numbers to even powers. [8]

$\sqrt{3x^3}\sqrt{6x^6}$

Quiz 8 #3 - Write the answer in simplest radical notation. [10]

$\sqrt{\sqrt[7]{B^{11}}}$

Exam 3 #1 - Simplify (no radicals in the denominator). Write as a single radical. [10]

$\sqrt{\dfrac{81n^4}{A^{12}Z^4}}$

Final Exam G #21 - Rationalize the denominator. [10]

$\dfrac{\sqrt[4]{3}}{\sqrt[4]{L^2M}}$

Concepts Regularly Discussed

The course makes efficient use of examples and problem sets to illustrate that properties of rational exponents can be considered nearly identical to the properties of integer exponents. This reinforces the Laws of Exponents gently and reintroduces students to the idea of an expansion of a number system. The course also provides good examples in the book and homework set that illustrate the procedural advantages of converting an expression from root form to rational exponents and vice versa.

Concepts Often Overlooked

The course has a way of oversimplifying the use of rational exponents. In expanding our use of exponents to include rational exponents, students are encouraged to treat rational exponents as they would integer exponents, with the exception that they have the ability to be converted into root form. The aspect that is often overlooked in the course is the importance that equivalent fractions are not necessarily equivalent exponents. The only direct reference of this in the text is a group exercise after the problem set that suggests comparing the graphs of $f(x) = x^{1/3}$, $g(x) = (x^{1/6})^2$, $h(x) = (x^2)^{1/6}$, and $k(x) = x^{2/6}$. No allusion as to why this occurs is given, only that $g(x) = h(x) = k(x)$ when $\sqrt[6]{x}$ exists [7].

A practical explanation of why this is ignored is because negative bases with rational exponents cannot be fully understood without an advanced discussion of the roots of unity, which delves into trigonometry and complex numbers prematurely. Also, it is highly unlikely that negative bases are used in a real-world application relevant to a learner of that level.

Illustrative Mathematics Sample Assessment

Alignment: N-RN.A.2

Checking a Calculation of a Decimal Exponent [9]

Alicia and Zara are scientists working together. Alicia uses a calculator to evaluate $3^{1.4}$ and gets an answer 6.473. Zara thinks for a moment, makes some calculations on paper, and says "That cannot be right, because $3^{1.4}$ must be less than 6. Find some hand calculations which show that, as Zara says, $3^{1.4}$ must be less than 6.

4.1.2 Use Properties of Rational or Irrational Numbers

CCSS.Math.Content.HSN-RN.B.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to reason and draw conclusions about the results of how rational and irrational numbers interact with one another through addition and multiplication. By investigating the results of adding and multiplying combinations of these numbers, this cursorily introduces students to the concept of closure and gives them practice using various methods of proof.

Place in the Course

The course establishes the necessary background information in Section 7.1 [7] by giving a formal definition of the nth root, beginning to simplify nth roots by first observing whether the index of the root is even or odd, and giving a definition of an irrational number. The course never directly reflects the principles of the cluster but rather only alludes to them obliquely in the homework sets of Sections 7.1 and 8.3 [7] by asking questions about the classification of a number as rational or irrational.

Closest-Related Assessment Problems

(none)

Concepts Regularly Discussed

The course focuses on the classification of a number as rational or irrational while studying two concepts. First, the course briefly asserts that the expression $\sqrt{a}$ is irrational if a is not a perfect square. Also, the course puts some attention on the idea that when we solve a quadratic equation using the quadratic formula, we can expect certain types of results. In this, students can make a connection between quadratic equations and functions and the rationality of its roots.

Concepts Often Overlooked

The course almost entirely neglects the principles of the cluster. First, the course neglects that the classification of a number as rational or irrational partitions the real numbers. To begin this discussion, an unconventional definition of an irrational number is given: $\sqrt{a}$ is irrational if its decimal representation is non-terminating and non-repeating. The only observation that alludes to the partition is the term irrational itself, which literally means not rational but still does not directly state that a real number must be either rational or irrational. If students equate rationality of a number as the negation of the given definition of an irrational number, students may incorrectly interpret the negation and may come to a number of incorrect conclusions. First, they may believe that any expression that does not feature a radical, like π, will be considered rational. Likewise, they may believe an expression like $\sqrt{x}$ is irrational because it belongs in a radical, which of course the correct answer is that it depends whether x is a perfect square or not. Students may focus on the decimal presentation of an expression and make conclusions based on the reading on their calculators. They may assume a number pattern that appears to be repeating will classify the quantity as rational. Consider $e \approx 2.718281828$. Using a typical 10-digit calculator display, a student may incorrectly conclude that e is rational. Likewise, a non-repeating decimal representation may terminate beyond the capacity of the calculator display, in which a student may incorrectly conclude a quantity like this to be irrational. If a student understands that the capacity of their calculator display could hinder them from ever really knowing whether any irrational numbers exist, then a student will not understand how or why irrational numbers exist but rather take the instructor's word for it.

The concept that is not conveyed with this definition of irrationality is the notion of an extension of the rational number system. There is no explanation as to why we care if the decimal representation is non-terminating and non-repeating. There would be advantages and disadvantages if the course focused on establishing the definition of an irrational number as a number that cannot be represented as $\frac{p}{q}$, with p and q integers and q non-zero. An advantage would be that the course could demonstrate the proof of the existence of an irrational number, namely $\sqrt{2}$. This also gives students a manageable proof to follow and gives them much needed practice with proofs. A disadvantage would be that the classification of numbers as irrational is often not a trivial task and proving irrationality of quantities like π this way is typically too complicated to be included with the introduction of irrational numbers.

Lastly, there is no mention of the results we expect when we add or multiply rational and irrational numbers. In this, a discussion involving the principles of closure never occurs. Working from the classification of numbers as rational or irrational based on their decimal representations, this makes drawing conclusions about their combinations by addition or subtraction a complicated if not impossible task that would be handled better by considering the definition of a rational number as a ratio of integers and an irrational number as a number that cannot be expressed that way.

Illustrative Mathematics Sample Assessment

Alignment: N-RN.A.2, N-RN.B

Rational or Irrational? [9]

In each of the following problems, a number is given. If possible, determine whether the given number is rational or irrational. In some cases, it may be impossible to determine whether the given number is rational or irrational. Justify your answers.

a. $4 + \sqrt{7}$

b. $\frac{\sqrt{45}}{\sqrt{5}}$

c. $\frac{6}{\pi}$

d. $\sqrt{2} + \sqrt{3}$

e. $\frac{2+\sqrt{7}}{2a+\sqrt{7a^2}}$, where a is a positive integer

f. $x + y$, where x and y are irrational numbers

4.2 Quantities

Prior Knowledge

Students are expected to enter the course with experience solving problems that involve the relationship between quantities and units. From an early age, students have a basic intuition of a unit: three apples; three dogs. We see that these are different things because apples are different from dogs, but have an identical quantity with similar properties. Starting with this basic relationship between quantity and unit, students advance their understanding of units by identifying types of units, comparing and converting different units, working with rates, and modeling by selecting appropriate units. Students should feel comfortable solving simpler problems involving perimeter, area, time, relating distance and velocity, and converting similar units like miles and kilometers. Students are expected to have the understanding that quantities in the real world are frequently represented as a number paired with a unit and that we must use this relationship consistently to come to accurate conclusions.

4.2.1 Reason Quantitatively and Use Units to Solve Problems

CCSS.Math.Content.HSN-Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. [1]

CCSS.Math.Content.HSN-Q.A.2 Define appropriate quantities for the purpose of descriptive modeling. [1]

CCSS.Math.Content.HSN-Q.A.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to better establish a connection between quantity and unit and use them in descriptive modeling. They should use units in modeling to relate results to real-world problems and recognize that we measure things first by establishing a benchmark for the magnitude 1. Students should identify quantity as the number of these units present. Also, students should identify units by type, for example distance-related units like miles, meters, or feet. They should see that any of these units could be used in distance-related problems but some are more appropriate than others due to the context of the problem. They should understand that observations are limited to the precision of the equipment used, so readings on these instruments often represent an estimation of the exact value. Hence, measuring the current distance from the earth to the moon is better represented in miles than millimeters because miles more accurately reflects the estimation, the quantity for millimeters would be extremely large, and it's almost certainly easier for someone to intuitively picture the distance in miles.

Place in the Course

There is no section that is devoted directly to expanding the students' concept of the relationship between quantity and unit. Quantity and unit appear moderately in the course in word problems in the homework, quiz, and exam problem set. Problems that involve this relationship are featured in most sections of the course in at least some loose sense.

Closest-Related Assessment Problems

HW 2.3 #77 - Model in the form $y = mx + b$. Determine what m and b signify.

After Lauren donated her hair to Locks of Love, the length $L(t)$ of her hair, in inches, was given by $L(t) = \frac{1}{2}t + 5$, where t is the number of months after she had the haircut. [7]

MML 5.8 #15 - The height $h(t)$, in feet, of an airborne tee-shirt t seconds after being launched can be approximated by $h(t) = -15t^2 + 75t + 10$. How long would a tee-shirt be airborne if it is caught on the way down by a fan 100 feet above ground level? Fill in the blank: The tee-shirt would be airborne for ___ second(s). [8]

Quiz 3 #3 - Mary paddled for 12 hours against a 4 km/h current to reach a campsite. The return trip with the same current took 4 hours. How fast can Mary paddle in still water? [10]

Exam 1 #8 - Bill can rent a van for either $65 per day with unlimited mileage or $40 per day with 120 free miles and an extra charge of $.20 for each mile over 120. For what numbers of miles traveled would the unlimited mileage plan save Bill money? [10]

Final Exam G #7 - Jim has a part time job selling newspaper and magazine subscriptions. One week he earned $54 by selling 12 newspaper and 6 magazine subscriptions. The following week he earned $73 by selling 10 newspaper and 12 magazine subscriptions. How much does he earn for each newspaper and each magazine subscriptions he sells? [10]

Concepts Regularly Discussed

The principles of quantity and unit are touched upon only when working with real-world problems. The word problems in the course partially illustrate the relationship between quantity and unit but only occasional focus on it. Typically, the main goal of these problems though is to provide real-world examples that make use of other concepts in the class like evaluating functions, solving simultaneous equations, or solving quadratic functions to name a few. Many of the foundational concepts of the quantity-unit relationship are evident in some form in the homework set but are rarely the focus of the quiz and exam problem set.

Concepts Often Overlooked

The relationship between quantity and unit is often no more than an afterthought in solving word problems. Units are usually given to the student in these word problems and they typically involve expressing the solution in the same unit as the given unit. Hence, students view units as a formality of expressing the correct answer rather than a way of understanding it. In exercises that require students to plot graphs, the units are given so students often plot graphs focusing entirely on quantity rather than the quantity-unit relationship. Students are rarely challenged to consider how the plot would change if the units were changed.

The most important concept overlooked in the course regarding the relationship between quantity and unit is the concept of modeling. Modeling is of utmost importance in the standards because it allows students to connect abstract concepts to the real world and vice versa in a more comprehensive way. It also reinforces the understanding of the quantity-unit relationship because it requires students to make informed decisions based on the foundations of the quantity-unit relationship. Problems that feature modeling, or interpreting models, typically appear at the end of the homework set of a section to reinforce the concepts of the section and to apply the concepts to real-world problems. In many cases, on homework problems that feature both quantity and unit, a student can still produce the correct answer without regarding the unit as the fundamental concept that gives meaning to the quantity. Students are rarely challenged to choose a unit appropriate to the context of the problem. Also, the course often regards the unit as a label for a quantity rather than a benchmark for the magnitude of 1. Students are not assessed on determining a level of accuracy of a measurement according to the context of the problem or the limitation of the instruments of observation. Students are also not often required to build models but rather work with models given to them.

Illustrative Mathematics Sample Assessment

Alignment: N-Q.A.1, N-Q.A.3

Felicia's Drive [9]

As Felicia gets on the freeway to drive to her cousin's house, she notices that she is a little low on gas. There is a gas station at the exit she normally takes, and she wonders if she will have to get gas before then. She normally sets her cruise control at the speed limit of 70 mph and the freeway portion of the drive takes about an hour and 15 minutes. Her car gets about 30 miles per gallon on the freeway, and gas costs $3.50 per gallon.

a. Describe an estimate that Felicia might do in her head while driving to decide how many gallons of gas she needs to make it to the gas station at the other end.

b. Assuming she makes it, how much does Felicia spend per mile on the freeway?

4.3 The Complex Number System

Prior Knowledge

Students are not expected to enter the course with knowledge or direct intuition of the complex number system but should have familiarity with the idea that number systems can be extended. As mentioned before, students should be familiar with the real number system, and be aware that real numbers obey the associative, commutative, and distributive properties just as did the rational numbers. Students should also be aware that one cannot take the square root of a negative number in the real numbers system, because there is no real number whose square is negative. Students learn earlier in the course how to solve simple quadratic equations that have rational roots.

4.3.1 Perform Arithmetic Operations with Complex Numbers

<u>CCSS.Math.Content.HSN-CN.A.1</u> Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real. [1]

CCSS.Math.Content.HSN-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers. [1]

Interpretation of the Cluster

The purpose of the cluster is to introduce students to complex numbers and have them become proficient performing simple arithmetic and algebraic computations with these numbers, which are precisely the required skills that are necessary to solve quadratic equations with real coefficients and complex roots. Recognizing that the function $f(x) = x^2 + 1$ has no real roots, we invent a root to this function, which we call the imaginary unit, or i, which has the property that $i^2 = -1$. Augmenting the real numbers to include the imaginary unit forms the complex number system, where every number in the complex number system can be represented as $a + bi$, where a and b are real numbers. We call a the real part and b the imaginary part of the complex number.

Students should recognize that the real numbers are a subset of the complex numbers and that real numbers are complex numbers with an imaginary part equaling zero. Also, students should understand that the associative, commutative, and distributive properties govern the complex numbers the same way they do the real number system. Students should recognize the arithmetic closure of the complex numbers and understand how to use complex conjugation to divide complex numbers.

Place in the Course

The topics of the cluster are covered directly in Section 7.8 [7]. The lesson captures the principles of the cluster by defining the complex numbers, demonstrating how to perform addition, subtraction, multiplication, and using complex conjugation for division with complex numbers, and computing powers of i. The topics that use principles learned in the cluster are covered in Sections 8.1 - 8.3 [7]. The principles of the cluster are used in these lessons in covering the principle of square roots, solving quadratic equations, and studying solutions to quadratic equations.

Closest-Related Assessment Problems

HW 7.8 #31 - Perform the indicated operation and simplify. Write each answer in the form $a + bi$. [7]

$(7 - 4i) - (5 - 3i)$

MML 7.8 #5 - Perform the indicated operation and simplify. [8]

$7i \cdot 3i$

Quiz 9 #3 - Perform the indicated operation and simplify. Write each answer in the form $a + bi$. [10]

$\sqrt{-4}\sqrt{-11}$

Exam 3 #8 - Write in the form $a + bi$. [10]

$(4-3i)(6+4i)$

Final Exam H #22 - Rationalize the denominator. Place the solution in the form $a+bi$. [10]

$$\frac{5}{2-4i}$$

Concepts Regularly Discussed

The course does a good job of introducing complex numbers in regards to the cluster. There are plenty of examples that embody the concepts of the cluster, and the assessments regularly challenge the students' ability to perform the basic arithmetic operations on these numbers. The course regularly covers computing powers of i and the concept appears consistently on quizzes and exams. The course has a consistent way of presenting $\sqrt{-1}$ as i and reminding students that $i^2=-1$ whenever i^2 appears.

Concepts Often Overlooked

The course never explains why, for example $\sqrt{-3}\sqrt{-7}\neq\sqrt{21}$. The topic is quickly covered and problems like this appear in the assessments but no explanation is given. The probable reason it is not explained is because a student would need to understand more profound concepts regarding the complex numbers which would be too cumbersome to delve into for the sole purpose of justifying the rule. Also, the closure of the complex numbers is rarely directly addressed. Closure does appear intrinsically as we learn to add, subtract, multiply, and divide complex numbers, but it is rarely the focus of the concept studied.

4.3.2 Use Complex Numbers in Polynomial Identities and Equations

CCSS.Math.Content.HSN-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to recognize the role of complex numbers in solutions to quadratic equations with real coefficients and to be able to find these solutions. To find these solutions, students should be familiar with two methods: completing the square and using the quadratic formula. Students should be able to derive the quadratic formula from completing the square for the general quadratic equation.

In solving quadratic equations, students should recognize that each quadratic equation can have only three possibilities for the types of roots: two real roots, one real root, or two complex roots.

Place in the Course

The course establishes the necessary background information in Section 7.8 [7] by defining the complex numbers and demonstrating how to perform addition, subtraction, multiplication, and using complex conjugation for division with complex numbers, and computing powers of i. The topics of the cluster are covered directly in Sections 8.1 - 8.3 [7]. These lessons reflect the principles of the cluster by understanding the Principle of Square Roots which states that $x^2 = k,\ k \geq 0$ implies $x = \pm\sqrt{k}$, solving quadratic equations, and studying solutions to quadratic equations.

Closest-Related Assessment Problems

HW 8.2 #37 - Solve. (Hint: Factor the difference of cubes. Then use the quadratic formula.) [7]

$x^3 - 8 = 0$

MML 8.2 #4 - Use the quadratic formula to solve the equation. [8]

$x^2 + 6x + 25 = 0$

Quiz 10 #7- Solve by completing the square. [10]

$t^2 - 2t + 4 = 0$

Exam 3 #12 - Solve. [10]

$2w^2 + 22 = 12w$

Final Exam - (none)

Concepts Regularly Discussed

The course fully develops the principles of solving quadratic equations. Early on, the course teaches simple factoring of quadratic equations. Later, the course establishes the Principle of Square Roots, then demonstrates how the principle is applied to solving a quadratic equation by completing the square. After, the quadratic formula is introduced to quickly solve any quadratic equation. The course consistently makes connections between the properties of the quadratic equation and the graphs they produce.

Concepts Often Overlooked

The course thoroughly covers the concepts detailed in the cluster.

CHAPTER 5: ALGEBRA

5.1 Seeing Structure in Expressions

Prior Knowledge

Students progress through the fundamental learning stages of mathematics by slowly engaging with expressions. They first see an individual number as something that expresses a quantity. Then, operators are used to show how these expressions interact with one another to form new expressions. Later on, variables are introduced and used to express something that is unknown. Students learn that individual expressions can be combined or broken apart and recognize the role that each part of the expression plays. They understand that the order of operations dictates how an expression is evaluated. They see exponents as a shorter notation for repeated multiplication and are familiar with basic notational conventions, like writing a coefficient to the left of a variable to indicate multiplication.

Students also have experience recognizing that expressions can often be rewritten in a different forms while still retaining the same value. They have experience of this first when representing rational numbers as fractions of integers and in converting fractions into decimal expansions and vice versa. They recognize the advantage of changing the representation of an expression to aid in bringing out an important feature, for example, completing the square.

5.1.1 Interpret the Structure of Expressions + Write Expressions in Equivalent Forms to Solve Problems

CCSS.Math.Content.HSA-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.★ [1]

CCSS.Math.Content.HSA-SSE.A.1a Interpret parts of an expression, such as terms, factors, and coefficients. [1]

CCSS.Math.Content.HSA-SSE.A.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P*. [1]

CCSS.Math.Content.HSA-SSE.A.2 Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$. [1]

CCSS.Math.Content.HSA-SSE.B.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ [1]

CCSS.Math.Content.HSA-SSE.B.3a Factor a quadratic expression to reveal the zeros of the function it defines. [1]

CCSS.Math.Content.HSA-SSE.B.3b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. [1]

CCSS.Math.Content.HSA-SSE.B.3c Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be* *rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%*. [1]

CCSS.Math.Content.HSA-SSE.B.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★ [1]

Interpretation of the Clusters

The first purpose of these clusters is for students to expand their understanding and use of the elements that compose expressions. Interpreting the structure of expressions is paramount to being able to understand and explain the concepts they represent. Without this understanding of structure, students cannot make a complete connection between a mathematical model and the real-world concept it models. Students should be able to identify the similarities between expressions with similar structure and be able to generalize properties they share. They recognize addition as adding a fixed magnitude to a value and recognize multiplication as proportionally scaling a value linearly by a magnitude. Students should interpret exponents on variables analogously with a sense of dimension.

The second purpose of these clusters is for students to recognize that a single element of an expression can be dissected in different ways for different purposes. Considering an expression like $25x^2$, students can factor the expression in multiple ways but have a sense of the unique prime factorization of both whole numbers and terms with variables. Students should be able to identify like terms and their importance in addition. After identifying elements of the structure of expressions, students should be able to rewrite the expressions differently to reveal properties that can be used to solve problems. Students should be able to factor polynomials to reveal zeros, and to rewrite quadratic functions by completing the square to reveal properties of the function and its graph. Students should be able to rewrite exponential expressions to allow them to multiply with other exponential expressions using the properties of exponents or convert expressions as in annual to monthly interest rates or rates of decay.

The third purpose of these clusters is for students to be able to begin working with series and find the formula for the sum of a finite geometric series.

Place in the Course

The concept of interpreting the structure of expressions is universal and applies to everything learned in the course in some way. Factoring quadratic functions takes place in Chapter 5 [7]; completing the square takes place in Section 8.1 [7]; and transforming exponential expressions takes place in Section 9.6 [7]. Geometric series are not discussed.

Closest-Related Assessment Problems

HW 9.2 #41 - The percentage of smokers P who receive telephone counseling to quit smoking and are still successful t months later can be approximated by $P(t) = 21.4(0.914)^t$ [7]

a) Estimate the percentage of smokers receiving telephone counseling who are successful in quitting for 1 month, 3 months, and 1 year.

b) Graph the function.

MML 5.7 #8 - Factor completely. [8]

$vq + vx + nq + nx$

Quiz 10 #7 - Solve. [10]

$2^{3x} = 64$

Exam 3 #9 - Use completing the square to put the following function by completing the square. State the coordinates of the vertex and state whether it is a maximum or minimum of the function. [10]

$g(x) = x^2 + 6x + 11$

Final Exam G #10 - Factor completely. [10]

$5z^3 - 40$

Concepts Regularly Discussed

Interpreting the structure of expressions is focused on extensively. Regarding direct interpretation, the importance of the structure of expressions is the focus when students work to understand linear equations/functions and their graphs while developing slope-intercept form. Students recognize that a linear equation is one that can be fit into the form $y = mx + b$. From here, students work with different values of m and b and notice how changing the values affects the graph of the function. They recognize that functions of this form are graphed as lines without plotting graphs. Students continue to develop recognition and use of structure when they continue to work with polynomials. Students understand that like terms must be identified if we wish to add or subtract them but multiplication or division does not require this. They can categorize types of polynomials by their structure and recognize linear functions as at-most first-degree polynomials. The course dedicates time to the recognition of polynomials that can be factored by reversing the distributive property and ones that can be factored as a difference of squares or a sum or difference of cubes. They also gain more experience viewing subtraction as addition of the opposites of values and can rewrite expressions featuring subtraction with addition of their opposites instead.

The course also interprets the structure of rational expressions heavily as students rewrite expressions and solve equations involving rational terms. They see that a rational expression might be factored to divide out identical multiplicative terms. Students interpret the structure of expressions when working with roots and recognize like terms when adding or subtracting roots. They must interpret the structure of complex numbers to identify real and imaginary parts and in understanding that every complex number can be put into $a + bi$ form. They also interpret the structure of expressions in quadratic functions when finding roots using the quadratic formula or completing the square to reveal the location of the vertex or axis of symmetry of a parabola, or the maximum or minimum of the function. Properties of exponential and logarithmic functions are developed with a similar approach as other functions studied in the course by changing parameters and observing its effect on aspects of the function and graph.

In every part of the course, students must recognize the structure of expressions to help devise a plan to solve the problem at hand. They recognize that expressions with similar structure will most likely use similar strategies. The problem sets from the assessments focus heavily on interpreting structure and using it to solve the problem.

Concepts Often Overlooked

The assessments problems rarely focus exclusively on interpreting structure for meaning. Students must interpret structure of expressions to complete a task, but no questions are asked that address under what circumstances would it be advantageous to write a particular expression one way or another. Finite series are not discussed.

Illustrative Mathematics Sample Assessments

Alignment: A-SSE.A.1

Mixing Candies [9]

A candy shop sells a box of chocolates for $30. It has $29 worth of chocolates plus $1 for the box. The box includes two kinds of candy: caramels and truffles. Lita knows how much the different types of candies cost per pound and how many pounds are in a box. She said if x is the number of pounds of caramels included in the box and y is the number of pounds of truffles in the box, then I can write the following equations based on what I know about one of these boxes:

- $x + y = 3$
- $8x + 12y + 1 = 30$

Assuming Lita used the information given and her other knowledge of the candies, use her equations to answer the following:

a. How many pounds of candy are in the box?

b. What is the price per pound of the caramels?

c. What does the term $12y$ in the second equation represent?

d. What does $8x + 12y + 1$ in the second equation represent?

Alignment: A-SSE.A.2

An Integer Identity [9]

Let a and b be integers with $a > b > 0$ and $\frac{a^3-b^3}{(a-b)^3} = \frac{73}{3}$. What is $\frac{b}{a}$?

5.2 Arithmetic with Polynomials and Rational Expressions

Prior Knowledge

Students generally begin their journey understanding numbers and quantity by learning how to combine them or break them apart in different ways. First with addition and subtraction and later with multiplication and division, students become more and more comfortable understanding how quantities interact with one another. Students learn that some whole numbers can be divisible by others and can factor whole numbers into prime factorizations. Students can also use a greatest common divisor or least common multiple to aid in computations involving rational numbers. They learn how to perform long division, represent certain quantities as rational numbers, and can represent fractions greater than 1 as mixed numbers, and vice versa. They learn that the basic operations behave the same on all numbers as do the associative, commutative, and distributive properties.

Later, students begin working with unknown quantities. As they do, they find that all of the qualities and properties of the expressions involving these unknown quantities are identical to the familiar properties they studied in basic arithmetic because they stand for numbers.

5.2.1 Perform Arithmetic Operations on Polynomials

CCSS.Math.Content.HSA-APR.A.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. [1]

Interpretation of the Cluster

The first purpose of this cluster is for students to be able to add, subtract, and multiply polynomials together. They should explain why when adding or subtracting polynomials together, they must add or subtract like-terms and that the result will be a polynomial. Students should also be proficient at multiplying polynomials together and using the distributive law in this process. Again, they should explain why when multiplying polynomials together, they are not initially concerned with matching like terms and that the result is again a polynomial. Students should also explain why dividing polynomials does not always result in another polynomial. In this, students should conclude that polynomials are closed under addition, subtraction, and multiplication but not division.

The second purpose of this cluster is for students to see algebra with polynomials as analogous to arithmetic with the integers. They should see that both polynomials and integers can be factored, and that the prime factorizations are unique up to units. Both are closed under addition, subtraction, multiplication, but not division. They see that division is possible for both and each system is not closed under division.

Place in the Course

The course establishes the necessary background in Section 5.1 [7] by establishing the terminology used in working with polynomials and identifying parts of polynomials. The topics of the cluster are covered directly in Sections 5.1 and 5.2 [7]. The lessons reflect the principles of the cluster by defining addition, subtraction, and multiplication of polynomials. The topics that use principles learned in the cluster are covered in Chapters 5, 6, and 8 [7]. The principles of the cluster are used in these lessons to establish methods of factoring polynomials, solving equations involving polynomials, simplifying rational expressions, solving equations involving rational expressions, dividing polynomials, and understanding the roots, graphs, and properties of quadratic functions.

Closest-Related Assessment Problems

HW 5.2 #69 - Multiply and, if possible, simplify. [7]

$(x+7)^2-(x+3)(x-3)$

MML 5.1 #11 - Perform the indicated operations. [8]

$(7x^2+7)-(2x^2+3)+(x^2+3x)$

Quiz 4 #4 - Find $g(x)\cdot g(x)$ [10]

$g(x)=5x+2$

Exam 1 #12 - Subtract: [10]

$(16x^3 - x^2 + 3x) - (-12x^3 + 3x^2 + 2x)$

Final Exam H #9 - Multiply. [10]

$(5A + 2B)^2$

Concepts Regularly Discussed

The course thoroughly establishes how to add, subtract, and multiply polynomials. These skills are such an essential element of working with polynomials that it is impossible to continue studying polynomials without them. Students are questioned extensively on these procedures in the problem set.

Concepts Often Overlooked

The course never directly addresses the concept of closure and it is taken as obvious. The examples and problem sets give students a sense of closure, but the closure of the polynomials is never directly addressed or the main focus. Also, there is not a direct comparison of the properties of polynomials to integers. Many of the properties of polynomials are explored through examples and many of these properties are congruent with those of the integers but the course never stops to make the comparison.

5.2.2 Understand the Relationship Between Zeros and Factors of Polynomials

CCSS.Math.Content.HSA-APR.B.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x – a$ is $p(a)$, so $p(a) = 0$ if and only if $(x – a)$ is a factor of $p(x)$. [1]

CCSS.Math.Content.HSA-APR.B.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to correlate the roots and factors of a given polynomial. As we expand the scope of the division of polynomials from simplification of rational expressions to division featuring a remainder, we observe that the remainders have certain predictable qualities. Using the Remainder Theorem, we see the remainder after division by a binomial of the form $x - a$ can be thought of from the perspective of evaluation of the polynomial function it divides. Students should be able to explain the Remainder Theorem and apply it in the appropriate context. They should understand why the relation holds both ways: $x - a$ is a factor of a polynomial $p(x)$ if and only if $p(a) = 0$. More generally, students should explain why the remainder on division on the polynomial $p(x)$ by $x - a$ will be $p(a)$. Factoring should then be thought of as a way of finding roots and as a way to simplify rational expressions. Students should be able to use the factorization of a polynomial to solve for roots and to construct a rough graph of the function that illustrates some basic qualities.

Place in the Course

The course establishes the necessary background information throughout Chapters 5 and 6 [7] by establishing principles of basic factoring, solving polynomial equations, and division by a binomial of the form $x - a$. The topic of relating factors to roots is covered in Sections 5.8 and 8.3 [7]. The lessons reflect the principles of the cluster by explaining the principle of zero products, demonstrating the advantage of factoring to find roots, and constructing a polynomial function that has given roots by multiplying linear factors. The topics that use principles learned in the cluster are covered in Sections 6.4, 6.5, 7.6, and 8.1 - 8.5 [7]. The principles of the cluster are used in these lessons to solve rational equations, solve equations featuring radicals, and establish a general solution to a quadratic equation. The Remainder Theorem is covered loosely as an afterthought to synthetic division in Section 6.7 [7]. The lesson does not particularly reflect the principles of the cluster because it merely states the Remainder Theorem without proof while providing a few examples. The principle is not used later.

Closest-Related Assessment Problems

HW 5.8 #95 - Fireworks are typically launched from a mortar with an upward velocity (initial speed) of about 64 ft/sec. The height $h(t)$, in feet, of a "weeping willow" display, t seconds after having been launched from an 80-ft high rooftop, is given by $h(t) = -16t^2 + 64t + 80$. How long will it take the cardboard shell from the fireworks to reach the ground? (the graph is given) [7]

MML 6.7 #4 - Use synthetic division to find the indicated function value. [8]

$P(x) = 3x^4 + 5x^2 + 9$, $P(-3)$

Quiz 5 #3 - Solve: [10]

$8p^3 + 4p^2 - 24p = 0$

Exam 2 #6 - Solve: [10]

$2x(3x - 1) + 3(3x - 1) = 0$

Final Exam A #25 - Solve for w. [10]

$w^{-2} - 9w^{-1} + 18 = 0$

Concepts Regularly Discussed

The course thoroughly delves into solving polynomial equations and continually reemphasizes the importance of factoring in finding roots to polynomial functions. The examples and problem set reiterate this importance. Students are regularly assessed by their ability to both find roots of polynomial functions by factoring and produce a polynomial function given roots. This is an essential skill that is used beyond our work with polynomials and continues to be important when we need to find or use roots of other types of functions.

Concepts Often Overlooked

The course only treats the Remainder Theorem as an afterthought to division on a polynomial. The book leaves the proof of the theorem as an exercise, which is not assigned; from this students are not expected to know the proof. The homework set features only one problem regarding the Remainder Theorem and a student can complete the problem without understanding how the Remainder Theorem works. Students only blindly know how to apply the theorem from memorization and are not challenged to further relate the concept to the broader idea of division. The Remainder Theorem is instead used more as a tool to practice synthetic division rather than as a way to reinforce the relation between roots and factors of a polynomial. Few assessments are made to rewrite a polynomial $P(x)$ as $q(x)(x-a)+r$, which better illustrates the purpose of the Remainder Theorem and relates the Remainder Theorem to the similarity of dividing a polynomial by a linear factor to dividing integers.

The course pays very little attention to the fact that constructing a function of a certain type from given roots is related to finding roots from a given function. In this, students see the two processes as different and often do not connect the ideas. Finally, the course does not take the time to have students sketch the graph of a function given its roots, but chooses not to graph non-linear functions until quadratic functions are graphed procedurally.

Illustrative Mathematics Sample Assessment

Alignment: A-APR.B.2

The Missing Coefficient [9]

Consider the polynomial function $P(x) = x^4 - 3x^3 + ax^2 - 6x + 14$, where a is an unknown real number. If $(x - 2)$ is a factor of this polynomial, what is the value of a?

5.2.3 Use Polynomial Identities to Solve Problems

CCSS.Math.Content.HSA-APR.C.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.* [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to write polynomials in equivalent forms for use in simplification or solving equations. Also, we can use polynomial identities to find numerical relationships like Pythagorean triples. Finding equivalent ways of writing a polynomial has many useful applications, especially if the equivalent form factors the polynomial. Once a polynomial function is factored, it becomes easier to identity its roots. If the polynomial is part of a rational expression, factoring by applying a polynomial identity can help simplify the expressions by exposing factors that may divide out in the expression. Students should be able to prove polynomial identities and recognize when to apply them.

Place in the Course

The topics of the cluster are covered directly in Chapter 5 [7]. The lessons reflect the principles of the cluster by formally introducing polynomials, defining arithmetic operations on polynomials, and developing basic factoring techniques on polynomials such as finding common factors, grouping, and applying polynomial identities. The topics that use principles learned in the cluster are covered in Chapter 6 and Chapter 8 [7]. The principles of the cluster are used in these lessons in simplifying rational expressions, solving rational equations, developing a general solution to quadratic equations, and graphing quadratic equations.

Closest-Related Assessment Problems

(none)

Concepts Regularly Discussed

The course dedicates an entire chapter to learning how to recognize appropriate times to apply polynomial identities and working to correctly apply them. The identities that are thoroughly discussed are the use of the distributive property, factoring simple monic trinomials, difference of squares, sum or difference of cubes, and the quadratic formula. The focus of these lessons is to use the polynomial identities in simplification or solving equations. The course does a good job of demonstrating the practicality of applying polynomial identities by dedicating significant class time to having students solve equations that feature polynomials and having students simplify rational expressions.

Concepts Often Overlooked

The course focuses almost entirely on simplification and solving equations when using polynomial identities but neglects to emphasize that they can be used to describe numerical relationships. In Section 5.8 [7] - Applications of Polynomial Equations - the assessments neglect uses of polynomial identities to highlight numerical relationships involving the Pythagorean theorem. Instead, the section only focuses on solving polynomial equations using polynomial identities.

It is often easy to prove a given polynomial identity by expanding the factored side. In the case of the quadratic formula, one method to prove the formula would be to derive it by completing the square in a quadratic function in standard form. Assessment of the students regarding the quadratic formula is almost entirely focused on application rather than derivation.

Illustrative Mathematics Sample Assessment

Alignment: A-APR.C.4

Trina's Triangles [9]

Alice was having a conversation with her friend Trina, who had a discovery to share:

“Pick any two integers. Look at the sum of their squares, the difference of their squares, and twice the product of the two integers you chose. Those three numbers are the sides of a right triangle.”

Trina had tried this several times and found that it worked for every pair of integers she tried. However, she admitted that she wasn't sure whether this "trick" always works, or if there might be cases in which the trick doesn't work.

a. Investigate Trina's conjecture for several pairs of integers. Does her trick appear to work in all cases, or only in some cases?

b. If Trina's conjecture is true, then give a precise statement of the conjecture, using variables to represent the two chosen integers, and prove it. If the conjecture is not true, modify it so that it is a true statement, and prove the new statement.

c. Use Trina's trick to find an example of a right triangle in which all of the sides have integer length, all three sides are longer than 100 units, and the three side lengths do not have any common factors.

5.2.4 Rewrite Rational Expressions

CCSS.Math.Content.HSA-APR.D.6 Rewrite simple rational expressions in different forms; write $^{a(x)}/_{b(x)}$ in the form $q(x) + {}^{r(x)}/_{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to take a rational expression and find equivalent ways of expressing it. Possible ways to write a rational expression would be to factor it and cancel factors from the numerator and denominator, split addition or subtraction in the numerator to have addition or subtraction of two rational expressions and vice versa, simplify rational expressions that feature rational expressions in the numerator or denominator, and performing division with polynomials. When performing division, students should be able to explain why division of a polynomial by another polynomial is continued until the degree of the remainder is less than the degree of the divisor. After using division of polynomials, students should be able to rewrite the original rational expression as a quotient together with a rational expression featuring the remainder and the divisor and check their work.

Place in the Course

The course establishes the necessary background information in Chapter 5 [7] by formally introducing polynomials, defining arithmetic operations on polynomials, and developing basic factoring techniques on polynomials such as finding common factors, grouping, and applying polynomial identities. The topics of the cluster are covered directly in Chapter 6 [7]. The lessons reflect the principles of the cluster by illustrating how to add, subtract, and multiply rational expressions, use factoring to simplify rational expressions, and perform long division.

Closest-Related Assessment Problems

HW 6.6 #13 - Divide and check. [7]

$(16y^3 - 9y^2 - 8y) \div (2y^2)$

MML 6.7 #2 - Use synthetic division to divide. [8]

$(b^2 - 3b + 4) \div (b + 1)$

Quiz 7 #2 - $P(x) = \left(\frac{f}{g}\right)(x)$. Find $P(x)$ using long division. State the quotient and remainder. [10]

$f(x) = x^8 - 2x^6 + 2x^4 - 14x^2$, $g(x) = x^2 - 3$

Exam 2 #13 - Use synthetic division. [10]

$$\frac{6x^3 + 9x^2 - 6x + 60}{x + 3}$$

Final Exam G #17 - Divide using long division or synthetic division. State the quotient and remainder. [10]

$(6x^3 - 19x + 12) \div (x + 2)$

Concepts Regularly Discussed

The course dedicates all of Chapter 6 [7] to rewriting rational expressions. The advantages of rewriting rational expressions becomes apparent in Section 6.4 [7] when the techniques are used to solve rational equations. When working with division, the answers must be in $q(x) + r(x)/b(x)$ form. The tasks are almost entirely computational and the assessments reflect this.

Concepts Often Overlooked

The course often neglects to ask why the degree of the remainder is designed by definition to be less than the degree of the divisor. In this, the course avoids taking extra time to compare division of polynomials to division of integers which happens to be a similar process with similar results. The course passes on the opportunity to revisit the idea that polynomials form a system analogous to the integers and that rational expressions from a system analogous to the rational numbers. From this, the course misses a crucial step to understanding the Remainder Theorem. Also, the course does not require students to use a computer algebra system.

Illustrative Mathematics Sample Assessment

Alignment: A-APR.D.6

Combined Fuel Efficiency [9]

The US Department of Energy keeps track of fuel efficiency for all vehicles sold in the United States. Each car has two fuel economy numbers, one measuring efficient for city driving and one for highway driving. For example, a 2012 Volkswagen Jetta gets 29.0 miles per gallon (mpg) in the city and 39.0 mpg on the highway.

Many banks have "green car loans'' where the interest rate is lowered for loans on cars with high combined fuel economy. This number is not the average of the city and highway economy values. Rather, the combined fuel economy (as defined by the federal

Corporate Average Fuel Economy standard) for x mpg in the city and y mpg on the highway, is computed as

$$combined\ fuel\ economy = \frac{1}{\frac{1}{2}(\frac{1}{x}+\frac{1}{y})} = \frac{2xy}{x+y}$$

a. What is the combined fuel economy for the 2012 Volkswagen Jetta? Give your answer to three significant digits.

b. For most conventional cars, the highway fuel economy is 10 mpg higher than the city fuel economy. If we set the city fuel economy to be x mpg for such a car, what is the combined fuel economy in terms of x? Write your answer as a single rational function, $a(x)/b(x)$.

c. Rewrite your answer from (b) in the form of $q(x) + r(x)b(x)$ where $q(x)$, $r(x)$ and $b(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $b(x)$.

d. Use your answer in (c) to conclude that if the city fuel economy, x, is large, then the combined fuel economy is approximately $x + 5$.

5.3 Creating Equations

Prior Knowledge

Most students are introduced to mathematics as a way of solving real-world problems. If I have four dollars and I spend one, how much do I have left? By translating these concepts into a mathematical language, students learn that simplifying and solving these problems become simpler. Students are expected to enter the course with the ability to translate simple problems with one unknown into a mathematical equation with a single variable.

5.3.1 Create Equations That Describe Numbers or Relationships

CCSS.Math.Content.HSA-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions*. [1]

CCSS.Math.Content.HSA-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. [1]

CCSS.Math.Content.HSA-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods*. [1]

CCSS.Math.Content.HSA-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R*. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to model relationships between quantities using mathematical equations for the purpose of solving problems. Developing this skill is essential for relating real-world concepts to mathematical ideas. Once the concepts have been translated into the mathematical language, we can process them using mathematical principles to narrow down a solution or understand a relationship. Depending on the context of the problem, students should be able to identify an appropriate framework for modeling and accurately apply it.

Students should be able to construct equations and inequalities that feature one variable and solve them to produce a solution. Using two or more variables, students should also be able to construct equations to establish relationships between quantities. For some equations with two variables, students should be able to graph the equation, label axes appropriately, and select a reasonable scale. Also for a given equation featuring multiple variables, students should be able to solve for a particular variable.

Place in the Course

Different aspects of the cluster appear in many parts of the course. Problems where a student must construct equations appear in some form in most sections while problems that require graphing are featured in Chapters 2, 3, 8, and 9 [7]. Constructing equations in one variable and solving them occurs throughout the course in the form of word problems but solving for a particular variable to highlight it in an equation is focused predominantly in Section 6.8 [7]. Students get some practice with this as they rearrange linear equations into slope-intercept form in Chapter 2 [7] as well.

Closest-Related Assessment Problems

HW 6.8 #33 - The formula $P = \frac{A}{1+r}$ is used to determine what principal P should be invested for one year at $(100 \cdot r)\%$ simple interest in order to have A dollars after a year. Solve for r. [7]

MML 2.5 #7 - In 1990, the life expectancy of males in a certain country was 71.3 years. In 1994, it was 74.2 years. Let E represent the life expectancy in year t and let t represent the number of years since 1990. Find a linear function that fits the data. [8]

Quiz 7 #4 - Solve for g. [10]

$$\frac{a}{b} = \frac{c}{d} + \frac{e}{g}$$

Exam 2 #5 - The area of a rectangular rug is 88 sq. ft. Its width is 3 feet less than its length. Find the length and width of the rug. [10]

Final Exam H #16 - Set up an equation to solve the following problem and then solve it. Bill can wash the windows in an apartment in 45 minutes. His partner Sam can wash the same windows in 30 minutes. If they work together, how long will it take them to wash the windows? State the equation and the solution. [10]

Concepts Regularly Discussed

The course uses a few commonly assessed problem types to practice constructing and solving equations. Examples include problems featuring distance = rate x time, constraint problems relating area to perimeter or price and quantity, and rational expressions involving rates. Many examples of solving for a particular variable of interest are given in 6.8. The course intrinsically reiterates that modeling links real-world situations to mathematical concepts by selecting scenarios that are relatable and reasonable.

Concepts Often Overlooked

The course tends to emphasize solving equations over constructing them. From this, students are more estranged from the meaning of each term and the purpose of solving equations. Also, the course never asks students to construct inequalities to solve problems and graphing inequalities is never done when relating two variables. Students are only required to solve inequalities and express their answers in interval notation. Since assessment problems are usually identical to homework problems and in-class examples, students can follow a recipe when attempting to solve problems during assessments, which may mean that the assessments do not accurately reflect their proficiency with the concepts. Also, students are given the units to use and the units never change, making the task of including units as a formality instead of something that provides meaning. When solving for a particular variable of interest, the course tends to use this task as an exercise in practicing algebraic manipulation but doesn't emphasize it as practice building equations.

Illustrative Mathematics Sample Assessment

Alignment: A-REI.C.6, A-CED.A

Cash Box [9]

Nola was selling tickets at the high school dance. At the end of the evening, she picked up the cash box and noticed a dollar lying on the floor next to it. She said,

"I wonder whether the dollar belongs inside the cash box or not."

The price of tickets for the dance was 1 ticket for $5 (for individuals) or 2 tickets for $8 (for couples). She looked inside the cash box and found $200 and ticket stubs for the 47 students in attendance. Does the dollar belong inside the cash box or not?

5.4 Reasoning with Equations and Inequalities

Prior Knowledge

Students enter the course with experience solving simple equations. Students should be able to solve linear equations algebraically in one variable. They understand the solution as the set of all values that make the original equation true. They can graph linear equations in two variables on the coordinate plane. They should understand a graph as the set of all values that make the equation true.

5.4.1 Understand Solving Equations as a Process of Reasoning and Explain the Reasoning

CCSS.Math.Content.HSA-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. [1]

CCSS.Math.Content.HSA-REI.A.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to explain the reasoning of each step used to solve equations. It is just as important that students can justify why we can perform each individual step than producing an answer. If not, students will view solving mathematical equations as an identification of a familiar problem type and a recall of a recipe to solve it. This approach does not explain how or why the process works and aids little in mathematical development. Students must recognize that solving equations starts with an equation that is assumed to be true, and proceeds by applying specific mathematical principles to narrow down a solution, or to show that no solution exists. Students should be able to solve different types of equations in one variable, including rational and radical equations. Also, students should be able to explain why squaring both sides of an equation can cause extraneous solutions and why solving rational equations by some methods sometimes creates solutions that cause division by zero. Students should provide examples in which these happen.

Place in the Course

The concepts of the cluster are universal and apply to everything learned in the course in some way.

Closest-Related Assessment Problems

HW 6.4 #21 - Solve. If no solution exists, state this. [7]

$$\frac{2}{6}+\frac{1}{2x}=\frac{1}{3}$$

MML 7.6 #7 - Solve. [8]

$$\sqrt{9y+8}=\sqrt{8y+12}$$

Quiz 6 #4 - Solve. [10]

$$\frac{8x}{x+1}-4=\frac{2}{x-2}$$

Exam 3 #5 - Solve. [10]

$$\sqrt{6b-35}=7-\sqrt{6b}$$

Final Exam A #15 - Solve. [10]

$$\frac{1}{y+4}-\frac{2}{y-3}=\frac{4}{y^2+y-12}$$

Concepts Regularly Discussed

The course thoroughly explains new methods and individual steps of solving equations. Every time a new mathematical principle is introduced, the book takes time explaining it, giving examples where it is used, and discussing the limitations. The course focuses a lot on solving linear, quadratic, rational, radical, and logarithmic equations as each topic is thoroughly discussed. When solving radical equations and logarithmic equations, students are reminded to test all solutions to confirm they are not extraneous but are not asked to explain why extraneous solutions happen.

Concepts Often Overlooked

The largest oversight the course makes is in the assessment of understanding each step in solving an equation. The vast majority of the course's assessments revolve around simplification and solving but students are not assessed on their justification of the steps of the process. From this, students often think that producing an answer is the most important part of each lesson and may think that mastery of mathematics entails knowing the recipes to solving a large number of known problems. Students then have a much harder time relating similar problems that aren't presented identically. Students understand to test solutions to eliminate extraneous solutions but are not assessed on providing their own examples nor are they asked to check their work.

Illustrative Mathematics Sample Assessment

Alignment: A-REI.A

How Does The Solution Change? [9]

In the equations (a)–(d), the solution x to the equation depends on the constant a. Assuming a is positive, what is the effect of increasing a on the solution? Does it increase, decrease, or remain unchanged? Give a reason for your answer that can be understood without solving the equation.

a. $x - a = 0$

b. $ax = 1$

c. $ax = a$

d. $xa = 1$

5.4.2 Solve Equations and Inequalities in One Variable

CCSS.Math.Content.HSA-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. [1]

CCSS.Math.Content.HSA-REI.B.4 Solve quadratic equations in one variable. [1]

CCSS.Math.Content.HSA-REI.B.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form. [1]

CCSS.Math.Content.HSA-REI.B.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b. [1]

Interpretation of the Cluster

The first purpose of this cluster is for students to be able to solve linear equations and inequalities. Solving linear equations is an essential skill needed in developmental algebra and resurfaces when we address more complicated problems. Solving inequalities is a slightly more challenging task for a few reasons. First, an additional rule must be enforced that states that the sense of the inequalities changes when both sides of the inequality are multiplied or divided by a negative number. Second, the study of inequalities is often coupled with studying absolute values, which introduces additional algebraic complications. Lastly, answers are often written in interval notation, which can be confusing for new learners.

The second purpose of this cluster is for students to be able to solve quadratic equations using a variety of techniques. Solving quadratic equations is another essential skill to master as many problems often later get reduced to solving a quadratic equation. There are a variety of ways we can solve a quadratic equation, and students should understand each method. Among these methods are inspection, factoring, completing the square, and the quadratic formula. Some methods are faster to perform under certain circumstances while others require less sophistication to apply. Understanding how each of these methods is important not only for solving quadratic equations, but because they give us practice using methods we can apply in other situations, for example, we use completing the square on an equation of a circle to highlight the radius and location of the center of the circle. Students should recognize that the quadratic formula can be derived from completing the square on a quadratic function in standard form and perform the steps necessary to derive it. Students should recognize that solving a quadratic function sometimes produces complex answers, so students should be able to identify under what circumstances this happens and to be able to write the complex answers in $a + bi$ form.

Place in the Course

Solving linear equations is something students are expected to be able to do going into the course, but solving inequalities is covered in Chapter 4 [7]. The chapter reflects this principle of the cluster by introducing the addition and multiplication principles for inequalities, introducing intervals and interval notation, and introducing unions, intersections, and absolute values.

Solving quadratic equations is covered in Chapters 5 and 8 [7]. These chapters reflect the principles of the cluster by developing strategies to factor and solve simple quadratic equations, introducing the principle of square roots, use of completing the square and the quadratic formula. Topics that use these principles of the cluster are covered in every chapter after the introduction of polynomials. These chapters use these topics of the cluster by using factoring to solve rational and logarithmic equations and using completing the square to convert quadratic functions into vertex form.

Closest-Related Assessment Problems

HW 8.1 #11 - Solve. [7]

$5y^2 = 30$

MML 4.2 #6 - Solve and graph the solution set. [8]

$x + 2 \leq -5$ or $x + 2 > -6$

Quiz 10 #1 - Solve by completing the square. [10]

$3x^2 + 18x + 9 = 0$

Exam 2 #4 - Let $f(x) = 2x^2 - x - 1$. Find x such that $f(x) = 5$. [10]

Final Exam H #8 - Solve. Place answer in interval notation. [10]

$|3x + 9| \leq 12$

Concepts Regularly Discussed

The course thoroughly investigates all of the procedural and computational aspects of the cluster. Regarding linear equations, the course provides plenty of problems that involve solving linear equations. As for solving inequalities, the course develops a method to solve inequalities involving absolute values, which gives solid practice in conceptualizing intervals. The most thorough coverage is that of quadratic functions. Solving quadratic functions is approached from all the basic angles: factoring using basic methods, by inspection, completing the square, and the quadratic formula. The textbook starts the section on the quadratic formula by completing the square on a quadratic equation in standard form, and so deriving the formula. During the discussion of the use of the quadratic formula, the concept of a discriminant is introduced to give insight into the nature of the roots. Students gain a feel from experience for the appropriate choice of solution method in a given context. When the solutions are non-real, students are instructed to leave their answers in $a \pm bi$ form knowing that their answers will be complex conjugates.

Concepts Often Overlooked

A concept that is not focused on in the course is the derivation of the quadratic formula by completing the square. The book illustrates this but students are not asked to reproduce the derivation. The course mentions the connection between the two concepts but the assessments treat them as separate methods. Students are taught to recognize when the quadratic formula will give complex solutions by analyzing the discriminant but students are not assessed on this task.

Illustrative Mathematics Sample Assessment

Alignment: A-REI.B.4.b

Braking Distance [9]

Suppose that a particular model of car has a braking distance that can be computed as follows: When the car is traveling at *v* miles per hour, its braking distance is given (in feet) by

$$d = 2.2v + \frac{v^2}{20}.$$

a. What is the braking distance, in feet, of a car of this model going 30 mph? 60 mph? 90 mph?

b. Suppose that a specific car of this model took 500 feet to brake. Use your computations in part (a) to make a prediction about how fast it was going when the brakes were applied.

c. Use a graph of the distance equation to determine more precisely how fast it was going when the brakes were applied, and check your answer using the quadratic formula.

5.4.3 Solve Systems of Equations [Algebraically]

CCSS.Math.Content.HSA-REI.C.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions. [1]

CCSS.Math.Content.HSA-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. [1]

CCSS.Math.Content.HSA-REI.C.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to solve systems of equations and explain the reasoning of the process. The idea of a system of equations expands on the idea of establishing a relationship between two variables. If we can establish multiple relationships between the variables, we may narrow down the possibilities of solutions that are true for all stated equations. To simplify a system of equations, we can add or multiply a quantity to both sides of an equation or add a multiple of one equation to another. Students should be able to explain why this does not change the solutions of the system of equations.

Students should be able to solve a system of linear equations algebraically that features two variables and two equations. Beyond this, students should also be able to solve a system of equations that features a linear and quadratic equation in two variables.

Place in the Course

The course establishes the necessary background information in Chapter 2 [7] by exploring linear equations algebraically and graphically. The topics of the cluster are covered directly in Sections 3.1 and 3.2 [7]. These lessons reflect the principles of the cluster by defining a system of equations, developing techniques to solve them graphically and algebraically with substitution or elimination. The topics that use principles learned in the cluster are covered in Section 3.3 [7]. The principles of the cluster are used in these lessons in problems that are modeled using systems of equations.

Closest-Related Assessment Problems

HW 8.4 #7 - Solve. [7]

Naoki bikes the 36 mi to Hillsboro averaging a certain speed. The return trip is made at a speed 3 mph slower. Total time for the round trip is 7 hr. Find Naoki's average speed on each part of the trip.

MML 3.2 #5 - Solve by the elimination method. [8]

$3x + 4y = 1$

$6x + 8y = 2$

Quiz 2 #5 - Solve the system graphically. [10]

$4x + 8y = 12$

$-x + 5y = -10$

Exam 1 #5 - Solve the system. [10]

$8x + 9y = 15$

$9x + 6y = 21$

Final Exam G - Solve by elimination or substitution. [10]

$5x - 3y = 7$

$2x = 12y - 8$

Concepts Regularly Discussed

The course does a good job establishing the procedural elements of solving a system of linear equations featuring two equations in two variables. The concepts are reinforced as three methods to solving these equations are explored: graphically, by substitution, and by elimination. The course makes sure to show how and why solutions to these equations come in three forms: a line, a point, or no solution. Both the algebraic and graphical explanations are explored and compared.

Concepts Often Overlooked

The course does not assess students on explaining how the methods of solving a system of equations work. Students then may not understand why adding a multiple of one equation to another results in the same solutions to a system of linear equations. Though Section 8.4 [7] focuses on solving systems of equations that are not linear systems of equations, the assessments focus minimally on these tasks and do not involving graphing solutions. Students may then not make the connection that solving the system graphically or by substitution is a similar process.

Illustrative Mathematics Sample Assessment

Alignment: A-REI.C.7

The Circle and The Line [9]

Sketch the circle with equation $x^2 + y^2 = 1$ and the line with equation $y = 2x - 1$ on the same pair of axes.

a. There is one solution to the pair of equations

$$x^2 + y^2 = 1$$

$$y = 2x - 1$$

that is clearly identifiable from the sketch. What is it? Verify that it is a solution.

b. Find all the solutions to this pair of equations.

5.4.4 Represent and Solve Equations and Inequalities Graphically

CCSS.Math.Content.HSA-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). [1]

CCSS.Math.Content.HSA-REI.D.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★ [1]

CCSS.Math.Content.HSA-REI.D.12 Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to use the graphs of equations to find their common solutions. When students learn about plotting equations, they learn that the graph of an equation represents all ordered pairs of numbers that satisfy the equation—in brief, the solutions of the equation. When multiple equations are plotted, the set of points that are common to all graphs are the solution(s) to each equation and hence, the solution(s) to the system of equations. This is true for any types of equations. After finding the intersections of the graphs, we can interpret the graphs and see that the coordinates of the points of intersection represent simultaneous solutions. We can test the results algebraically to confirm that the values are the same for each equation. In some instances, exact values are not easy to distinguish from a graph, for example, if the units of the graphs are integers and the intersection is not an integer pair. Also, there can be instances in which values can only be approximated, for example, finding common solutions to the functions $f(x) = 2$ and $g(x) = xe^x$. Student should be able to narrow down more accurate solutions by systematic methods, such as using successive approximation.

The second purpose of this cluster is for students to be able to graph the solutions to linear inequalities in two variables. The graphs of these inequalities should be half-planes. Students should also be able to graph systems of linear inequalities as intersections of these half-planes.

Place in the Course

The course establishes the necessary background information in Chapter 2 [7] by exploring linear equations algebraically and graphically. The topics of the cluster are covered directly in Sections 3.1 and 3.2 [7]. These lessons reflect the principles of the cluster by defining a system of equations, developing techniques to solve them graphically and algebraically with substitution or elimination. The topics that use principles learned in the cluster are covered in Section 3.3 [7]. The principles of the cluster are used in these lessons in problems that are modeled using systems of equations.

Closest-Related Assessment Problems

HW 9.5 #65 Graph and state the domain and range of each function. [7]

$f(x) = \ln \mathrm{x} - 2$

MML 9.2 #4 Graph the equation on paper. [8]

$y = (\frac{1}{2})^x$

Quiz 2 #5 - Solve the system graphically. [10]

$4x + 8y = 12$

$-x + 5y = -10$

Exam 1 #4 Place equation into slope-intercept form. Find m, b, and graph. [10]

$9x + 18y = 36$

Final Exam A - Solve by graphing. [10]

$3x + y = 5$

$x - 2y = 4$

Concepts Regularly Discussed

The course makes a solid effort to connect the solutions of equations to their graphs. This is seen heavily in Chapter 2 [7] and is reiterated when the graphical solutions of a system of linear equations is connected with the algebraic solutions. The course uses the graphs of functions when exploring the possibilities of the types of solutions we can find to a system of linear equations.

Concepts Often Overlooked

Solving systems of equations by approximation is not discussed. This is useful when attempting to find solutions graphically. In the assessments, most of the graphical solutions to systems of equations are integers, making the task of identifying them deceptively easy. If students are given a system of equations where the value of a solution were very close to an integer, students can falsely identify the answer as an integer value assuming they don't check their answers. The only systems of equations that are explored graphically are systems of linear equations. If students were presented a system of equations where one or more of the equations are non-linear, they may expect that the types of possible solutions would be the same as the types of possible solutions to a system of linear equations. For example, a third-degree polynomial and a linear equation can have exactly three distinct solutions; this is not possible with a system of linear equations.

A concept that is entirely avoided is graphing the solutions to linear inequalities in two variables. Students solve inequalities in one variable in Chapter 4 [7] but never graph or even discuss linear inequalities in two variables.

Illustrative Mathematics Sample Assessment

Alignment: A-REI.B.4, A-REI.D.11

Two Squares are Equal [9]

Solve the quadratic equation

$$x^2 = (2x - 9)^2$$

using as many different methods as possible.

CHAPTER 6: FUNCTIONS

6.1 Interpreting Functions

Prior Knowledge

Students begin the course with the fundamental idea that we can establish a relationship between elements of different sets. You give me a number n, I can tell you the nth planet from the sun in our solar system. Students understand that models like this have limitations: the number n that is chosen must be an integer between 1 and 8, because a non-positive number makes no sense, as do non-integers. Also, our solar system has just eight planets, so n cannot be more than 8. These integers 1 through 8 are the set of possible inputs, as are the names of planets in our solar system the set of possible outputs. From this, we can see that students also enter the course with an intuition about the domain and range of a function.

From a young age, we continue to formalize how to represent a relationship between sets. Students start by counting objects and assigning each object exactly one number. Later, equations featuring an independent and dependent variable are given. Students learn to substitute values for the independent variable to find the value of the dependent variable. This notation is later substituted for functional notation and students learn to process values in the same way by evaluation of a function.

6.1.1 Understand the Concept of a Function and Use Function Notation

CCSS.Math.Content.HSF-IF.A.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$. [1]

CCSS.Math.Content.HSF-IF.A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. [1]

CCSS.Math.Content.HSF-IF.A.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$*. [1]

Interpretation of the Cluster

The purpose of the cluster is for students to be able to understand the basics of what a function is and how to evaluate it. Functions can be thought of as machines: an element from one set goes in, get processed in some way, and exactly one element comes out. The domain of a function should be thought of as the set of inputs for which the function is defined. The set of outputs are what we call the range of the function. The most elemental concept that defines a function is that it assigns each element of the domain to exactly one element of the range. Understanding functions in very important as we attempt to make sense of relationships between variables. We can graph functions by placing the independent variable x as we have done previously on the horizontal axis and label f(x) on the vertical axis. Students should be able to evaluate functions and use the results to build graphs. Sequences can be thought of as functions and students should be able to make the connection and work with sequences from a functional perspective.

Place in the Course

Functions are covered continually from the beginning of the course. These lessons reflect the principles of the cluster by defining, evaluating, graphing, and exploring the properties of linear, quadratic, exponential and logarithmic functions, and determining the domains and ranges of simpler functions. Sequences are not covered.

Closest-Related Assessment Problems

HW 2.2 #81 - The function F described by $F(C) = \frac{9}{5}C + 32$ gives the Fahrenheit temperature corresponding to the Celsius temperature C. Find the Fahrenheit equivalent to $-5°$ Celsius. [7]

MML 9.1 #10 - Graph the function $f(x)$ and its inverse using the same set of axes. [8]

$f(x) = 3 - x^2, x \geq 0$

Quiz 1 #3 - (A graph of a function is given) [10]

State the domain, range, and find $f(5)$.

Exam 1 #3 - Given $f(x) = x^2 - 2x + 3$. [10]

Find $f(2)$ and $f(3)$.

Final Exam H #12 - Find the domain. Place answer in interval notation. [10]

$$f(x) = \frac{27x}{x^2 + 5x - 14}$$

Concepts Regularly Discussed

The course makes a point to begin discussing functions immediately. Functions are defined along with domain and range and the algebraic understanding of these concepts is united with a graphical understanding. The course is thorough about exploring the properties and graphs of linear, quadratic, exponential and logarithmic functions. Evaluation of these functions is constantly performed as well as solving equations that feature these functions. Performance of the computational components of these concepts is continually assessed.

Concepts Often Overlooked

Assessments tend to favor finding the domain of a function more than finding its range. Notably absent from the course is a discussion of sequences which is reserved for a later course. The course is thorough about examining functions but sequences - are functions - are neglected. The course sometimes treats converting, for example, $y = 3x$ into $f(x) = 3x$, as a formality and students often see it this way. In this case, students may not truly understand the concept of a function. If given the classic diagram of a function that maps one finite set to another represented with arrows, students may not see the connection with that depiction of a function and one in functional notation.

Absent is the task of selecting a type of set for a domain and range. The closest task students are given is to take the predetermined set type, the real numbers, and restrict it so every input is defined. The only types of restrictions that are discussed are those caused by division by zero, by taking the square root of a real number, or by taking the logarithm of a negative number. Students are only graphically presented a situation in which a restricted domain can cause a function to be undefined. This is confusing to students because they cannot link the concept to its algebraic counterpart. A question that asks students to select a domain for a function may be confused with selecting units. Missing is that a function is not defined until a domain is specified.

Though the book introduces functions properly by stating that it assigns each element of the domain to exactly one element of the range, the course inherently continues to assume the domain and codomain to be the real numbers. Students then typically view functions as a strange formality for representing an equation and think that they simply turn numbers into other numbers. The course does not focus on functions as an assignment from one set to another because students are not asked to work with functions that map elements other than numbers to numbers. There is also no distinction made between the codomain and the range; in fact the concept of codomain is not discussed. The course continuously assumes that the codomain is the range, hence asserting that all functions are surjective. The course states that a function is invertible if and only if it is injective. This is true when the codomain and the range are the same, but if not, we must be careful and state that a function is invertible if and only if it is bijective.

Illustrative Mathematics Sample Assessment

Alignment: F-IF.A.2

Yam in the Oven [9]

You put a yam in the oven. After 45 minutes, you take it out. Let $f(t)$ be the temperature of the yam t minutes after you placed it in the oven.

In (a)–(d), explain the meaning of the statement in everyday language.

a. $f(0) = 65$

b. $f(5) < f(10)$

c. $f(40) = f(45)$

d. $f(45) > f(60)$

6.1.2 Interpret Functions That Arise in Applications in Terms of the Context

CCSS.Math.Content.HSF-IF.B.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★ [1]

CCSS.Math.Content.HSF-IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ [1]

CCSS.Math.Content.HSF-IF.B.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★ [1]

Interpretation of the Cluster

The first purpose of this cluster is for students to be able to identify the components and features that make up a function. Such features include x and y-intercepts, intervals where the function is positive, negative, increasing or decreasing, local maximums and minimums, symmetries, and end behavior. Students should be able to relate the key features to graphs and vice versa. Given a graph of a function, students should be able to identify these important features, or given key features be able to draw a graph of a function that has such features. By identifying these components, students can use data to predict future behavior, find roots, reaffirm concepts, and broaden their grasp of how functions relate one set to another. Students should also be able to calculate the average rate of change of a function over a specific interval. This starts the discussion of the relationship between functional values and slope, which has many practical applications and lays a foundation for studying derivatives.

The second purpose of this cluster is for students to be able to relate the domain of a function to its graph and vice versa. By doing this, students can visually affirm how a graph depicts the way a function assigns elements from the domain to the range. Students should also be able to describe appropriate values for data in the domain and range, according to the context of the problem. For example, given a function that gives the number of miles a walker has traveled in t minutes, students should understand that the domain of the function should be non-negative real numbers. Having a grasp of acceptable inputs to a function builds a stronger understanding of what functions actually do and allows students to begin a more formal use of sets.

Place in the Course

The course establishes the necessary background information in Section 2.1 [7] by illustrating how to plot points on a coordinate plane and how an equation can be plotted on a graph. Particular topics of the cluster are covered in different sections throughout the course. In Section 2.2 [7], concepts like slope and relating a domain of a function to its graph are introduced. Intercepts are a focused on in Chapter 2 [7] when discussing linear equations and Chapters 5 and 8 [7] when discussing quadratic functions algebraically and graphically. Symmetries are discussed in Chapter 8 [7] when exploring the symmetries and minima or maxima of quadratic functions. Symmetries are also discussed in Chapter 9 [7] when inverse functions are introduced. The topics that use principles learned in the cluster are covered throughout the course, especially those sections and problems involving modeling.

Closest-Related Assessment Problems

HW 5.8 #95 - Fireworks are typically launched from a mortar with an upward velocity (initial speed) of about 64 ft/sec. The height $h(t)$, in feet, of a "weeping willow" display, t seconds after having been launched from an 80-ft high rooftop, is given by $h(t) = -16t^2 + 64t + 80$. How long will it take the cardboard shell from the fireworks to reach the ground? (the graph is given) [7]

MML 6.1 #2 - Rik usually takes 3 hr more than Pearl does to process a day's order at Liberty Place Photo. If Pearl takes t hr to process a day's orders, the function given by $H(t) = \frac{t^2+3t}{2t+3}$ can be used to determine how long it will take if they worked together. How long will it take them, working together, to complete a day's orders if Pearl can process the orders alone in 4 hours? [8]

Quiz - (none)

Exam - (none)

Final Exam - (none)

Concepts Regularly Discussed

The course is thorough about finding roots of linear and quadratic functions. This idea is explored by continually relating functions to their graphs and using one to extract information about the other. For linear functions, students are presented with the slope and asked to use it to find other information and vice versa. Students learn that slope is an essential part of a linear function as well as understanding its intercepts, particularly its y-intercept when analyzing the function in slope-intercept form. If a linear equation has a positive slope, it will be increasing. If it has a negative slope, it will be decreasing.

Quadratic functions are discussed in great detail. When studying the graphs of these functions, students are asked to identify the axis of symmetry and the maximum and minimum of the function. The book describes how the vertex of the parabola is the point where the graph of a quadratic function changes direction, going from being a decreasing function to being an increasing function, or vice versa.

The course continues the conversation about symmetry when inverse functions are introduced. Students make graphical connections between properties of a function and its inverse by observing that the graphs of a function and its inverse are reflections of one another over the line $y = x$. Also discussed is that a function is an injection if and only if its inverse is a function. Logarithms are introduced as inverses to exponential functions, and this is explored algebraically as well as graphically.

Also, the assessments feature problems where an unknown function is graphed and the student must identify the domain and range. Also, students learn to how a function's properties can be analyzed without graphing to determine qualities of its domain. Specifically, students relate that any input that creates a situation where division by zero happens, that input will not be in the domain because the output value would be undefined.

Concepts Often Overlooked

Most interpretations of the graphs of functions happen outside of the context of a real-world application. This causes a missed opportunity to make conclusions of real-world problems using logical principles.

The only exacting discussion of slope happens when we analyze linear functions. Slope is introduced primarily as a quality of a function rather as the average rate of change over an interval. In linear equations, the slope remains constant so this is true for these types of functions but this sets students up to see slope improperly. Otherwise, students are not asked to compute the slope of a function over a given interval nor are they assessed on estimating the average slope of a non-linear function from its graph. Slope is only referred to as a rate of change in passing.

Many of the features of a function listed earlier are discussed only in passing. Intervals are discussed as solutions to inequalities but not used to represent portions of a function that possess particular qualities. Qualities not discussed in any detail are portions of a function that are increasing or decreasing, positive or negative, or end behavior. This analysis of a function's graph misses an opportunity to use a graph to make decisions based on concepts like optimization. Avoiding these topics delays laying an instinctual foundation for later coursework that uses concepts of limits and derivatives.

As stated before, students are not challenged to select a set with elements of a certain type as the domain. Students are given that the domain is the real numbers and are taught that finding the domain means restricting the real numbers.

Illustrative Mathematics Sample Assessment

Alignment: F-BF.A.1.a, F-IF.B.4, F-IF.B.5

The Canoe Trip, Variation 1 [9]

Mike likes to canoe. He can paddle 150 feet per minute. He is planning a river trip that will take him to a destination about 30,000 feet upstream (that is, against the current). The speed of the current will work against the speed that he can paddle.

a. Let s be the speed of the current in feet per minute. Write an expression for $r(s)$, the speed at which Mike is moving relative to the river bank, in terms of s.

b. Mike wants to know how long it will take him to travel the 30,000 feet upstream. Write an expression for $T(s)$, the time in minutes it will take, in terms of s.

c. What is the vertical intercept of T? What does this point represent in terms of Mike's canoe trip?

d. At what value of s does the graph have a vertical asymptote? Explain why this makes sense in the situation.

e. For what values of s does $T(s)$ make sense in the context of the problem?

6.1.3 Analyze Functions Using Different Representations

CCSS.Math.Content.HSF-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ [1]

CCSS.Math.Content.HSF-IF.C.7a Graph linear and quadratic functions and show intercepts, maxima, and minima. [1]

CCSS.Math.Content.HSF-IF.C.7b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. [1]

CCSS.Math.Content.HSF-IF.C.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. [1]

CCSS.Math.Content.HSF-IF.C.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. [1]

CCSS.Math.Content.HSF-IF.C.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. [1]

CCSS.Math.Content.HSF-IF.C.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. [1]

CCSS.Math.Content.HSF-IF.C.8b Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as y = (1.02)t, y = (0.97)t, y = (1.01)12t, y = (1.2)t/10, and classify them as representing exponential growth or decay. [1]

CCSS.Math.Content.HSF-IF.C.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum*. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to use multiple representations of functions to learn more about them. Specifically, we want to do this by making a conceptual connection between functions and their graphs. Starting from a function presented algebraically, students should be able to graph the important features of the function and relate their meanings algebraically. Particular functions that we want students to graph are: linear, quadratic, square and cube root, piecewise, polynomial, exponential, and logarithmic functions. Each type of graph has important features that should be included when graphing. For linear functions, the intercepts and slope; for quadratic functions, intercepts, axis of symmetry, vertex, and the maximum or minimum; for square root functions, the domain; for polynomial functions, roots when factored; for exponential and logarithmic functions, intercepts and end behavior.

The second purpose of this cluster is for students to be able to represent a function in equivalent forms to reveal properties of the function. For linear equations, writing an equation in slope-intercept form to reveal its slope and y-intercept; for quadratic functions, factoring the function to reveal its roots or rearranging the function into completed square form to reveal the vertex and axis of symmetry. Another skill that helps is representing functions in different forms, for example as a set of ordered pairs, in table form, as an equations, and so on. From this, students should be able to compare different functions in different forms.

Place in the Course

This is a central theme of the course and is featured continuously.

Common Assessment Problems

HW 2.3 #51 - Determine the slope and y-intercept. Then draw a graph. Be sure to check as in Example 5 or Example 8. [7]

$g(x) = 4.5$

MML - Find the domain of the function f. [8]

$f(x) = |x + 1|$

Quiz 10 #5 - Let $f(x) = 4x^2 - 16x + 12$. [10]

Write the function in completed square form.

Find the vertex and the axis of symmetry.

Give a complete graph. Label on the graph the vertex and two other points.

Exam 3 #15 - Let $f(x) = 2x^2 + 12x + 14$. [10]

Write the function in completed square form.

Find the vertex and the axis of symmetry.

Give a complete graph. Label on the graph the vertex and two other points.

Final Exam H - $f(x) = 3^x$. Complete the table and graph. [10]

x	$f(x)$
-2	
-1	
0	
1	
2	

Concepts Regularly Discussed

The course is thorough about connecting algebraic representations of functions to their graphical representations. This is first seen when discussing linear functions as students are given a function in algebraic form or as a table and must graph the function, or vice versa. For quadratic functions, students are given a quadratic function in algebraic form and rearrange it by completing the square to graph it. Students can also deduce features of the graph such as the number of x-intercepts by analyzing the discriminant and the location of the real roots by analyzing factorizations. Given the graphs of quadratic functions, students can find roots and deduce the signs of the coefficient of the x^2 term and the discriminant. Students are taught to graph exponential functions and relate graphing logarithmic functions by reflection over the line $y = x$. Given these functions in their graphical forms, students can deduce the sign of the coefficient and tell if its absolute value is between 0 and 1 or greater than 1.

The algebraic aspect of the cluster is well emphasized. Factoring is a major theme and is used to find roots of polynomial functions. Students learn how to use completing the square to rewrite a quadratic function in completed square form revealing its vertex and making graphing much simpler. From this form, students learn the axis of symmetry is the line $x = -b/2a$ and after can find the axis of symmetry without graphing.

Concepts Often Overlooked

Some functions are not explored very well algebraically or graphically. In particular, the course hardly challenges students to graph square root, cube root, piecewise, or polynomial equations of degree three or higher. Graphing the root functions is explored lightly when studying the relationship between the graphs of functions and their inverses, but certain properties are not explored enough, namely restricting a function's domain so it's inverse will be a function. Students are not challenged to compare functions when they are presented in different forms. Students do change the base of exponential expressions to solve equations but not for the purpose of extracting meaning.

Illustrative Mathematics Sample Assessment

Alignment: F-IF.C.7.c

Graphs of Power Functions [9]

a. Sketch the graphs of the functions described by $f(x) = x^2$ and $g(x) = x^4$ on the same axes, being careful to label any points of intersection. Also, find and label $(\frac{1}{2}, f\left(\frac{1}{2}\right))$ and $(\frac{1}{2}, g\left(\frac{1}{2}\right))$.

b. Sketch the graphs of the functions described by $f(x) = x^3$ and $g(x) = x^5$ on the same axes, being careful to label any points of intersection. Also, find and label $(\frac{1}{2}, f\left(\frac{1}{2}\right))$ and $(\frac{1}{2}, g\left(\frac{1}{2}\right))$.

c. Sketch the graphs of the functions described by $f(x) = x^2$ and $g(x) = x^3$ on the same axes, being careful to label any points of intersection. Also, find and label $(\frac{1}{2}, f\left(\frac{1}{2}\right))$ and $(\frac{1}{2}, g\left(\frac{1}{2}\right))$.

6.2 Building Functions

Prior Knowledge

Students are expected to enter the course understanding that equations describe a relationship between variables. They see that equations with similar structure describe similar relationships. Students also see that changing particular aspects of a function such as its coefficients or constants can produce predictable effects. Students also are expected to know that many aspects of a function can be understood by observing its graph and vice versa.

6.2.1 Build a Function That Models a Relationship Between Two Quantities

CCSS.Math.Content.HSF-BF.A.1 Write a function that describes a relationship between two quantities.★ [1]

CCSS.Math.Content.HSF-BF.A.1a Determine an explicit expression, a recursive process, or steps for calculation from a context. [1]

CCSS.Math.Content.HSF-BF.A.1b Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model*. [1]

CCSS.Math.Content.HSF-BF.A.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★ [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to express relationships between quantities by building functions. Ways to do this would be to state an explicit expression, some type of recursive process, or an algorithm that defines the relationship. This is an important skill to be able to relate real-world ideas to mathematical concepts and vice versa. To expand this, students should be able to combine standard function types using arithmetic operations in modeling.

Place in the Course

Students are asked to write functions that describe relationships between two quantities continuously throughout the course.

Closest-Related Assessment Problems

HW 2.3 #69 - A cross-country skier reaches the 3-km mark of a race in 15 min and the 12-km mark 45 min later. Assuming a constant rate, find the speed of the skier. [7]

MML 2.5 #7 - In 1990, the life expectancy of males in a certain country was 71.3 years. In 1994, it was 74.2 years. Let E represent the life expectancy in year t and let t represent the number of years since 1990. Find a linear function that fits the data. [8]

Quiz 3 #3 - Mary paddled for 12 hours against a 4 km/h current to reach a campsite. The return trip with the same current took 4 hours. How fast can Mary paddle in still water? [10]

Exam 2 #5 - The area of a rectangular rug is 88 sq. ft. Its width is 3 feet less than its length. Find the length and width of the rug. [10]

Final Exam A #2 - Find the equation of the line in slope-intercept form containing $(-3, -1)$ and $(9, -5)$. [10]

Concepts Regularly Discussed

The course illustrates the importance of establishing a formal relationship between quantities to solve problems. Given particular data, students learn to build a linear function from it. Given a point and a rate of change, or two points, students can model a linear function definitively. Given roots of a function, students learn to construct a monic polynomial function that has those roots.

Concepts Often Overlooked

The course tends to favor building equations over building functions. Constructing a function involves defining a domain and codomain and an assignment that relates each element of the domain to exactly one element of the range. When building equations, there is less focus on sets and because of this, students are more estranged from not only the roles that domain and range play but from what a function is or how we can use it. Written word problems too often give the function explicitly and ask students to evaluate it given data from the problem rather than giving students data and having them connect concepts to build a function. With these word problems, students are not given extra data so they have less practice deciphering the usefulness of data according to the context of the problem. Writing equations is done more for the purpose of finding solutions instead of describing relationships. Construction of models that describe relationships between two quantities happens only in a few simpler formats and is not done with sequences or combining function types.

Illustrative Mathematics Sample Assessment

Alignment: F-BF.A.1.a

Susita's Account [9]

At the beginning of January, Susita had some money in her checking account. At the end of each month she deposits enough to double the amount currently in the account. However, she has a loan to pay off, requiring her to withdraw \$10 from the account monthly (immediately after her deposit).

a. Assuming January is the first month, write an equation that describes the amount of money in Susita's account at the end of the nth month, $S(n)$, in terms of the amount of money in Susita's account at the end of the $(n-1)$th month, $S(n-1)$.
b. At the end of May, Susita had \$2 left in the account. How much did she have at the end of January?

6.2.2 Build New Functions from Existing Functions

<u>CCSS.Math.Content.HSF-BF.B.3</u> Identify the effect on the graph of replacing *f*(*x*) by *f*(*x*) + *k*, *k f*(*x*), *f*(*kx*), and *f*(*x* + *k*) for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the

effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. [1]

CCSS.Math.Content.HSF-BF.B.4 Find inverse functions. [1]

CCSS.Math.Content.HSF-BF.B.4a Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. *For example,* *$f(x) = 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$*. [1]

Interpretation of the Cluster

The first purpose of this cluster is for students to be able to identify traits common to all functions. When students begin working with functions, they learn aspects of functions as being specific to a type of function: linear functions have a slope and y-intercept and are graphed as straight lines, quadratic functions have a vertex and axis of symmetry and are graphed as parabolas. They see changing the values of coefficients as something that affects qualities of a particular type of function: for a linear function $f(x) = mx + b$, changing the value of b changes the y-intercept, or for a quadratic function $g(x) = ax^2 + bx + c$, changing the sign of a causes the parabola to change from opening upward to downward or vice versa. This cluster looks to generalize properties and operations common to all functions by having students visually confirm the effects of changing particular aspects of functions. Hence, students can affect a function in a particular predictable way regardless of the function type.

The second purpose of this cluster is for students to be able to find and use inverse functions. In mastering this process, students are given a more concrete experience that illustrates the importance of understanding domain and range, the definition of a function, and identifying a function as injective or not.

Place in the Course

The concepts of the standard are explored somewhat in Chapters 2, 8, and 9 [7]. These chapters reflect the principles of the standard by graphing linear, quadratic, exponential, and logarithmic functions. Inverse functions are introduced in Section 9.1 [7] and are soon used to discuss the relationship between exponential and logarithmic functions.

Closest-Related Assessment Problems

HW 2.3 #57 - Find a linear function whose graph has the given slope and y-intercept.

Slope -7, y-intercept $(0, \frac{1}{3})$ [7]

MML - 9.5 #5 - Graph. [8]

$f(x) = e^{x-5}$

Quiz 10 #5 - Let $f(x) = 4x^2 - 16x + 12$. [10]

Write the function in completed square form.

Find the vertex and the axis of symmetry.

Give a complete graph. Label on the graph the vertex and two other points.

Exam 3 #17 - Find the inverse of $f(x)$. [10]

$f(x) = \sqrt{x-3}$

Final Exam A #30 - Find a formula for the inverse of the following function. [10]

$f(x) = \frac{1}{2}x^3 + 7$

Concepts Regularly Discussed

The course intrinsically covers the effects of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k. Replacing $f(x)$ by $f(x) + k$ is explored when graphing linear, quadratic, exponential, and logarithmic functions; $f(x)$ by $kf(x)$ is covered lightly when graphing exponential and logarithmic functions; replacing $f(x)$ by $f(kx)$ is touched upon when graphing linear, exponential, and logarithmic functions; and replacing $f(x)$ by $f(x + k)$ is seen when graphing quadratic, exponential, and logarithmic functions. Understanding and finding inverses is covered moderately and students are able to find inverses of a variety of functions. The graphical relationship between a function and its inverse is also explored, encouraging students to continue to make connections between the symbolic and graphical representations of functions.

Concepts Often Overlooked

The course doesn't reserve time for discussing replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k for arbitrary functions. These effects are alluded to in the homework set by providing numerous examples that illustrate the principles but the course never stops to discuss this for arbitrary functions. The course does take time to develop a method for finding inverse functions that involves first examining a function to confirm it is injective but students are not required to define the new domain and range of the inverse function. This continues to marginalize the importance of domain and range in a function. Students are not made aware that an inverse function can be used to solve an equation.

Illustrative Mathematics Sample Assessments

Alignment: F-BF.B.3

Building a Quadratic Function from $f(x) = x^2$ [9]

Suppose $f(x) = x^2$ where x can be any real number.

a. Sketch a graph of the function f.

b. Sketch a graph of the function g given by

$$g(x) = f(x) + 2.$$

How do the graphs of f and g compare? Why?

c. Sketch a graph of the function h given by

$$h(x) = -2 \cdot f(x).$$

How do the graphs of f and h compare? Why?

d. Sketch a graph of the function p given by

$$p(x) = f(x+2).$$

How do the graphs of f and p compare? Why?

6.3 Linear, Quadratic, and Exponential Models

Prior Knowledge

Students should enter the course with experience building and interpreting linear models expressed as an equation. From this, they should be familiar with finding the slope and y-intercepts of these equations and determining the value of a dependent variable given the independent variable and vice versa. They should be able to identify the situations in which a linear model would be appropriate and use data to construct a model. Students should be able to interpret graphs of linear equations and make conclusions based on the context of the model.

Students should enter the course with experience evaluating and factoring simple quadratic equations. Students are not expected to have experience working with exponential equations or models.

6.3.1 Construct and Compare Linear and Exponential Models and Solve Problems

CCSS.Math.Content.HSF-LE.A.1 Distinguish between situations that can be modeled with linear functions and with exponential functions. [1]

CCSS.Math.Content.HSF-LE.A.1a Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. [1]

CCSS.Math.Content.HSF-LE.A.1b Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. [1]

CCSS.Math.Content.HSF-LE.A.1c Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. [1]

CCSS.Math.Content.HSF-LE.A.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). [1]

CCSS.Math.Content.HSF-LE.A.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. [1]

CCSS.Math.Content.HSF-LE.A.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to make comparisons between linear and exponential functions by analyzing them and observing their graphs. An early step of fitting an observed real-world situation to an algebraic model is to select a type of function that best suits the properties of what is observed according to the context. If what is modeled expresses a constant rate of change throughout, then a linear model would be a good idea to try. If what is modeled changes by a constant proportion over a unit interval, then an exponential model would be a good idea to try. Students should be able take a linear or exponential function expressed graphically and express it algebraically. This reinforces students' connections of a function with its graph. Students should know that evaluating increasing exponential functions will eventually produce a higher value than evaluating polynomial functions no matter which increasing exponential function or polynomial is selected. This paints a clear picture about the eventual behavior of these functions. Also, students should be able to solve exponential equations using logarithms. By using this approach, we begin to see a more definitive example of how inverse functions can be used to solve problems.

Place in the Course

The course establishes the necessary background information about linear functions in Chapter 2 [7] by defining linear functions, describing their properties, and relating the functions to their graphs. The course establishes the necessary background information about polynomial functions in Chapter 5 and Chapter 8 [7] by defining polynomial and quadratic functions and using the properties of quadratic functions to build their graphs. Recognizing graphs takes place in these chapters respectively, and using logarithms to solve exponential equations takes place in Section 9.6 [7]. Comparing linear functions with exponential functions is not discussed.

Closest-Related Assessment Problems

HW 9.5 #59 - Graph and state the domain and range of each function. [7]

$f(x) = e^{x+2}$

MML 9.6 #4 - Solve for t. [8]

$e^t = 520$

Quiz 2 #3 - Find the linear function $f(x) = mx + b$ whose graph is the line passing through $(3,7)$ and $(-4,9)$. [10]

Exam 3 #16 - Find the equation of the quadratic function with a vertex at $(-2,1)$ and a y-intercept at $(0,9)$. Find the equation in completed square form. [10]

Final Exam G #34 - Solve for x. Round to the nearest hundredth. [10]

$4.7^x = 91.7$

Concepts Regularly Discussed

The course takes time to develop a strong foundation of linear functions by exploring them algebraically and graphically. Students learn how to build a linear function given either a point and a slope or two points. Polynomials in general are introduced and quadratic functions are discussed in great detail algebraically and graphically. Graphing exponential functions is developed, as is solving exponential equations using logarithms. The course provides many examples of solving exponential equations of differing bases, including all positive bases but namely base 2, e, and 10. Students use their calculators to evaluate logarithms with these bases or use a change of base procedure along with calculators to evaluate logarithms of bases different than e and 10.

Concepts Often Overlooked

The course tends to predetermine which types of functions to use in modeling. Instead of students analyzing information and selecting a function of best fit, students understand what function to use according to what is being discussed in the current section. Students are not asked to determine how linear or exponential functions change nor are the functions compared with one another. End behavior is not assessed and a student's awareness of it is often assumed. Sequences are never discussed. Students are able to construct linear functions from limited but adequate information but this skill is not developed for exponential functions.

Illustrative Mathematics Sample Assessment

Alignment: F-LE.A.1.c, F-LE.A.2, F-LE.A.4

Algae Blooms [9]

Algae blooms routinely threaten the health of the Chesapeake Bay. Phosphate compounds supply a rich source of nutrients for the algae, *Prorocentrum minimum*, responsible for particularly harmful spring blooms known as mahogany tides. These compounds are found in fertilizers used by farmers and find their way into the Bay with run-offs resulting from rainstorms. Favorable conditions result in rapid algae growth ranging anywhere from 0.144 to 2.885 cell divisions per day. Algae concentrations are measured and reported in terms of cells per milliliter (cells/ml). Concentrations in excess of 3,000 cells/ml constitute a bloom.

a. Suppose that heavy spring rains followed by sunny days create conditions that support 1 cell division per day and that prior to the rains *Prorocentrum minimum* concentrations measured just 10 cells/ml. Write an equation for a function that models the relationship between the algae concentration and the number of days since the algae began to divide at the rate of 1 cell division per day.
b. Assuming this rate of cell division is sustained for 10 days, present the resulting algae concentrations over that period in a table. Did these conditions result in a bloom?
c. Concentrations in excess of 200,000 cells/ml have been reported in the Bay. If conditions support 2 cell divisions per day, when will these conditions result in a bloom? When will concentrations exceed 200,000 cells/ml?

6.3.2 Interpret Expressions for Functions in Terms of the Situations They Model

CCSS.Math.Content.HSF-LE.B.5 Interpret the parameters in a linear or exponential function in terms of a context. [1]

Interpretation of the Cluster

The purpose of this cluster is for students to be able to take a function that pertains to a model and interpret aspects of it to learn about the model and the mathematics it represents. Among these aspects could be the values of coefficients, constants, and exponents. By understanding how changing these values affects a graph, students learn the role that each value plays. Also, they can relate what they learned about the graphs to other graphs and draw conclusions about similar functions or functions in general.

Place in the Course

The course establishes the necessary background information about linear functions in Chapter 2 [7] by defining linear functions, describing their properties, and relating the functions to their graphs. The course establishes the necessary background information about exponential functions in Chapter 9 [7] by introducing exponential functions and interpreting their properties to build their graphs.

Closest-Related Assessment Problems

HW 2.3 #69 - A cross-country skier reaches the 3-km mark of a race in 15 min and the 12-km mark 45 min later. Assuming a constant rate, find the speed of the skier. [7]

MML - 2.3 #11 - The trade-in value of a snow blower can be determined using the function $v(n) = -150n + 750$. Here $v(n)$ is the trade-in value, in dollars, after n winters of use. What does the number -150 signify? [8]

A. The snow blower was purchased 150 winters ago.

B. The snow blower's value is decreasing by $150 per winter.

C. The snow blower's value is decreasing by $150

D. The snow blower will depreciate complete after 150 winters.

Quiz - (none)

Exam - (none)

Final Exam A #7 - List your variables and two equations for the following problem, and then solve it. [10]

A store mixes Kentucky bluegrass worth $11 per pound with rye grass worth $15 per pound. The mixture is to sell for $12 per pound. How much of each should be used to make a 300-pound mixture?

Concepts Regularly Discussed

The course discusses linear functions thoroughly and exponential functions adequately from a procedural standpoint. Students learn to use the context of a situation to model a function.

Concepts Often Overlooked

The course very rarely takes functions that are modeled after a situation and has students extract meaning from the values of coefficients, constants, or exponents. The class focuses more on building models than interpreting them and this can cause students to have a disconnect between functions and the situations they model. Assessments tend to require students to build functions that fit the functional model that the class is currently studying which will typically be linear or rational functions. Given functional models, students are more often asked to evaluate them or find something computation or procedural such as roots or slope without a context. Some extraction of meaning takes place in the assessments for parameters of linear functions but exponential functions are studied almost completely outside of the context of a situation.

Illustrative Mathematics Sample Assessment

Alignment: F-LE.A.1.c, F-LE.B.5

Illegal Fish [9]

A fisherman illegally introduces some fish into a lake, and they quickly propagate. The growth of the population of this new species (within a period of a few years) is modeled by $P(x) = 5b^x$, where x is the time in weeks following the introduction and b is a positive unknown base.

a. Exactly how many fish did the fisherman release into the lake?

b. Find b if you know the lake contains 33 fish after eight weeks. Show step-by-step work.

c. Instead, now suppose that $P(x) = 5b^x$ and $b = 2$. What is the weekly percent growth rate in this case? What does this mean in every-day language?

CHAPTER 7: SUMMARY OF ANALYSIS

7.1 Generalizations of Course Concepts

What follows is a summary of the analysis of the content standards. I wish to generalize concepts as they appear in the course as a whole.

Concepts Regularly Discussed

The course provides a wide range of topics for study in intermediate algebra. Almost all topics discussed either directly cover what is stated in the content standards or are closely related. The course provides a clear justification for each new procedure and emphasizes relating equations and functions to their graphs. The textbook makes a consistent attempt to relate procedural concepts to applications. Time is taken to relate new concepts to older concepts by using skills learned in an earlier part of the course as a foundational tool for expanding new ideas.

The course also works thoroughly to develop the procedural aspects of intermediate algebra. The course stresses the idea that we use the structure of expressions to identify a method of solving or simplifying. Also emphasized is the importance of proper mathematical notation and attending to the precision of a problem.

Concepts Often Overlooked

The course focuses almost entirely on a few procedures: simplifying, solving, evaluating, and graphing. This can have a negative impact on a student's perception of mathematics by presenting mathematics as an endeavor that involves taking some mysterious mathematical object and turning into another mysterious mathematical object. Also, students may picture proficiency in mathematics as having a large memorized bank of equations and matching them to very specific situations. Frequently, students pass the course with a C or better, hence satisfying their general education requirement, solely by unknowingly applying predetermined procedures. Students can memorize formulas without needing to understand their derivation or purpose according to a context.

The course rarely requires a student to explain their reasoning or the reasoning of others. The assessments ask students to produce an answer that is typically a number, quantity, equation, or graph. There are no assessment items that require students to answer questions in their own words nor are there assessment items where a student must examine the work of another to determine what is being done.

Modeling in the course is done on a very algorithmic basis. For problems that require modeling, students usually try to appropriate a framework for the model from examples used in the textbook or lecture. Students typically only need to take the framework and change the numbers. Units are given to students in the problems so students do not need to decide a suitable unit to use according to the context. As stated earlier, units are treated more as a formality than as a way to understand a problem. This is also true for word or story problems.

The word problems are meant to be a way for students to apply concepts of a section to real-world situations. Most word problems describe a situation and then give students a function or equation that decontextualizes the situation. Students then usually evaluate or solve accordingly. It is possible for students to produce correct answers simply by evaluating these functions without having an understanding of the context. There are few problems that have students fully decontextualize a situation on their own, process it mathematically, and contextualize it according to a context.

CHAPTER 8: SUGGESTIONS FOR BETTER ALIGNMENT TO THE STANDARDS

8.1 Motivation

To better align Math 105 to the goals of the Common Core State Standards, there are many adjustments that could be made. The most important deciding factor that dictates course coverage is lecture time. There is a tight window when all topics are covered so time is very valuable. To better align Math 105 with the goals of the Common Core State Standards, some course content must be eliminated while different content must take its place. The content that replaces it will either be new content or current content will be developed more thoroughly. There are many methods to decide which content to eliminate and the most practical solution will most likely consist of a combination of the different methods.

8.2 Possible Content to Eliminate

Eliminate Topics Covered That Are Not Detailed In the Standards

There are a few topics the course covers that are not directly included in the standards. Topics covered that are not directly included are solving equations and inequalities with absolute values, synthetic division, and equations reducible to quadratic. This is not to say that these topics are entirely unrelated to the content of the standards but these represent the topics that are farthest from the goals of the standards and are perhaps the most expendable. Also, each topic listed is not used as a foundational aspect of a later topic discussed.

Eliminate Topics Listed as Eighth Grade Standards

Eliminating topics that are meant to be covered before high school can free up valuable time. The course makes assumptions about what students should know coming into the course. Most of the eight grade standards are assumed to be mastered while others are reintroduced to refresh students' memories as they begin expanding the topics.

Eliminate Overlap with Math 095

Earning a C or better in Math 095 is one possible prerequisite for the course; the other prerequisites demand students have a familiarity with the material that is comparable. This method of eliminating material would remove similar content as the method of eliminating topics listed as eighth grade standards. By enforcing prerequisite knowledge more strictly, this will free up additional course time.

8.3 Possible Content to Add or Emphasize

Adding Content Listed in the Standards

Listed in the Appendix are standards that are not included in the course. Adding content from this list will diversify students' knowledge of mathematics.

Developing Inadequately Covered Standards

The standards that are inadequately covered in the course (listed in the Appendix) could be delved into more thoroughly. With the time freed up from eliminated topics, instructors can take time in lectures for more hands-on learning and develop a more comprehensive understanding of the concepts.

8.4 Adjustments to the Assessments

The assessments of the course could be changed by creating a new list of written homework and/or changing the final exam. By changing these assessment items to better align to the goals of the Common Core State Standards, the instructors would also be motivated to change the types of assessment problems included on their quizzes and exams to better reflect the homework and better prepare students for the final exam.

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APPENDIX

A.1 Content Standards Indexed by Coverage

Standards Covered Thoroughly

CCSS.Math.Content.HSN-RN.A.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents. [1]

CCSS.Math.Content.HSN-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real. [1]

CCSS.Math.Content.HSN-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers. [1]

CCSS.Math.Content.HSN-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions. [1]

CCSS.Math.Content.HSA-SSE.A.1a Interpret parts of an expression, such as terms, factors, and coefficients. [1]

CCSS.Math.Content.HSA-SSE.A.2 Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$. [1]

CCSS.Math.Content.HSA-SSE.B.3a Factor a quadratic expression to reveal the zeros of the function it defines. [1]

CCSS.Math.Content.HSA-APR.D.6 Rewrite simple rational expressions in different forms; write ${}^{a(x)}/_{b(x)}$ in the form $q(x) + {}^{r(x)}/_{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system. [1]

CCSS.Math.Content.HSA-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions*. [1]

CCSS.Math.Content.HSA-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R*. [1]

CCSS.Math.Content.HSA-REI.A.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. [1]

CCSS.Math.Content.HSA-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. [1]

CCSS.Math.Content.HSA-REI.B.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form. [1]

CCSS.Math.Content.HSA-REI.B.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b. [1]

CCSS.Math.Content.HSA-REI.C.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions. [1]

CCSS.Math.Content.HSA-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. [1]

CCSS.Math.Content.HSA-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). [1]

CCSS.Math.Content.HSF-IF.A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. [1]

CCSS.Math.Content.HSF-IF.C.7a Graph linear and quadratic functions and show intercepts, maxima, and minima. [1]

CCSS.Math.Content.HSF-IF.C.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. [1]

CCSS.Math.Content.HSF-IF.C.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. [1]

CCSS.Math.Content.HSF-BF.B.4a Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x+1)/(x-1)$ *for* $x \neq 1$. [1]

CCSS.Math.Content.HSF-LE.A.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where *a*, *c*, and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology. [1]

Standards Covered Inadequately

CCSS.Math.Content.HSN-RN.A.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we* *define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5*. [1]

CCSS.Math.Content.HSN-Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. [1]

CCSS.Math.Content.HSN-Q.A.2 Define appropriate quantities for the purpose of descriptive modeling. [1]

CCSS.Math.Content.HSA-SSE.A.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P*. [1]

CCSS.Math.Content.HSA-SSE.B.3b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. [1]

CCSS.Math.Content.HSA-APR.A.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. [1]

CCSS.Math.Content.HSA-APR.B.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$. [1]

CCSS.Math.Content.HSA-APR.B.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial. [1]

CCSS.Math.Content.HSA-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. [1]

CCSS.Math.Content.HSA-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods*. [1]

CCSS.Math.Content.HSA-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. [1]

CCSS.Math.Content.HSA-REI.D.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★ [1]

CCSS.Math.Content.HSF-IF.A.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$. [1]

CCSS.Math.Content.HSF-IF.B.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity*.★ [1]

CCSS.Math.Content.HSF-IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n)* *gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ [1]

CCSS.Math.Content.HSF-IF.C.7b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. [1]

CCSS.Math.Content.HSF-BF.A.1a Determine an explicit expression, a recursive process, or steps for calculation from a context. [1]

CCSS.Math.Content.HSF-BF.B.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\ f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. [1]

CCSS.Math.Content.HSF-LE.A.1b Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. [1]

CCSS.Math.Content.HSF-LE.A.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). [1]

CCSS.Math.Content.HSF-LE.B.5 Interpret the parameters in a linear or exponential function in terms of a context. [1]

Standards Not Covered

CCSS.Math.Content.HSN-RN.B.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. [1]

CCSS.Math.Content.HSN-Q.A.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. [1]

CCSS.Math.Content.HSA-SSE.B.3c Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%*. [1]

CCSS.Math.Content.HSA-SSE.B.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★ [1]

CCSS.Math.Content.HSA-APR.C.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.* [1]

CCSS.Math.Content.HSA-REI.C.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$. [1]

CCSS.Math.Content.HSA-REI.D.12 Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes. [1]

CCSS.Math.Content.HSF-IF.A.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$. [1]

CCSS.Math.Content.HSF-IF.B.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★ [1]

<u>CCSS.Math.Content.HSF-IF.C.7c</u> Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. [1]

<u>CCSS.Math.Content.HSF-IF.C.8b</u> Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as y = (1.02)t, y = (0.97)t, y = (1.01)12t, y = (1.2)t/10, and classify them as representing exponential growth or decay. [1]

<u>CCSS.Math.Content.HSF-IF.C.9</u> Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum*. [1]

<u>CCSS.Math.Content.HSF-BF.A.1b</u> Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model*. [1]

<u>CCSS.Math.Content.HSF-BF.A.2</u> Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★ [1]

CCSS.Math.Content.HSF-LE.A.1a Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. [1]

CCSS.Math.Content.HSF-LE.A.1c Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. [1]

CCSS.Math.Content.HSF-LE.A.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. [1]

Plussed Standards Covered

CCSS.Math.Content.HSN-CN.A.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers. [1]

CCSS.Math.Content.HSN-CN.C.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x – 2i)$. [1]

CCSS.Math.Content.HSF-BF.B.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents. [1]

Eighth Grade Standards Covered

CCSS.Math.Content.8.EE.A.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$. [1]

CCSS.Math.Content.8.EE.A.2 Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational. [1]

CCSS.Math.Content.8.EE.C.7a Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers). [1]

CCSS.Math.Content.8.EE.C.8a Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously. [1]

CCSS.Math.Content.8.EE.C.8b Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example,* $3x + 2y = 5$ *and* $3x + 2y = 6$ *have no solution because* $3x + 2y$ *cannot simultaneously be 5 and 6*. [1]

CCSS.Math.Content.8.EE.C.8c Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair*. [1]

CCSS.Math.Content.8.F.A.1 Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.[1] [1]

CCSS.Math.Content.8.F.A.3 Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function* $A = s^2$ *giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line*. [1]

CCSS.Math.Content.8.F.B.4 Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (*x, y*) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values. [1]

A.2 Math 095 and Math 105 Content Coverage Table

Please note that all of the standards are quoted directly from [CCSSM]. A standard with an asterisk (*) has been identified by the CCSSM authors as having particularly strong connections to the Modeling conceptual category.

Note that standards starting with “N” are from the Number and Quantity high school conceptual category; those starting with “A” refer to items from the Algebra high school conceptual category; those starting with “F” refer to the Function high school conceptual category.

The brief summary comments in the Math 095 column were resulting from the Hayley Nathan's thesis exploring the connections between Math 095 and the CCSSM standards [12]. The Math 105 column information was determined from my analysis of the content in Math 105 and the CCSSM standards.

A blank cell translates to content that is absent from a given course.

Bolded items represent some degree of overlap between content from the two courses.

Standard Number	Standard	Math 095	Math 105
N-RN 1	Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5*.		Students are given a formal definition of the nth root
N-RN 2	Rewrite expressions involving radicals and rational exponents using the properties of exponents.	Rewrite radical expressions (square root only) and those involving rational exponents (integers only)	Rewrite radical expressions (nth roots) and those involving rational exponents
N-RN 3	Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational		The classification of rational and irrational numbers is addressed in the course; there are no assessment items for this standard and the said properties of sums and products are not addressed

Standard Number	Standard	Math 095	Math 105
	number is irrational.		
N-Q 1	Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.		Addressed in the course but there are no specific assessment items
N-Q 2	Define appropriate quantities for the purpose of descriptive modeling.		Addressed in the course through application problems related to solving equations and systems of equations
N-Q 3	Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.		Addressed in the course but no specific assessment items
N-CN 1	Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.		Students know the existence of i and write complex numbers in the form $a + bi$
N-CN 2	Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex		Students apply the quadratic equation and find complex roots

Standard Number	Standard	Math 095	Math 105
	numbers.		
N-CN 7	Solve quadratic equations with real coefficients that have complex solutions.		
A-SSE 1	Interpret expressions that represent a quantity in terms of its context. *		
A-SSE 1a	Interpret parts of an expression, such as terms, factors, and coefficients.		Students are not directly assessed on this but they identify terms, factors, and coefficients; it is unclear if they *interpret* these items; students also identify the degree of polynomials
A-SSE 1b	Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P*.		
A-SSE 2	Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*		Students see structure in linear equations of the form $y = mx + b$ and identify the real and imaginary components of a complex number $a + bi$; interpret the structure of quadratic equations in vertex form (identify vertex, maximum/minimum, opening up or down); rewrite polynomials in different forms in order

Standard Number	Standard	Math 095	Math 105
			to factor them
A-SSE 3	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. *	Limited to choosing and producing an equivalent form	See above
A-SSE 3a	Factor a quadratic expression to reveal the zeros of the function it defines.	**Limited to factoring quadratic expressions**	Students solve quadratic equations by **factoring** (and other methods).
A-SSE 3b	Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.		Students complete the square to convert quadratic functions into vertex form so that they can identify the maximum or the minimum
A-SSE 3c	Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%*.		
A-SSE 4	Derive the formula for the sum of a finite geometric series (when the		[Listed as being addressed in the course but no assessment items or discussion presented]

Standard Number	Standard	Math 095	Math 105
	common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* *		
A-APR 1	Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.	**Students add, subtract, and multiply polynomials**	**Students add, subtract, and multiply polynomials**; they also multiply *functions* represented by polynomials
A-APR 2	Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.		Addressed in one homework problem, but students can solve the homework problem without applying the Remainder Theorem; the Remainder Theorem is used more as a practice tool for synthetic division
A-APR 3	Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.	**Students solve polynomial equations by factoring**	**Students solve polynomials equations by factoring**
A-APR 4	Prove polynomial identities and use them to describe		These ideas are developed in the lessons in class, but no

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	numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*		assessment items are given
A-APR 6	Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.	**Students write rational expressions in the form $\frac{a(\square)}{b(x)}$ in simplified form by factoring**; the degree of the polynomial is not addressed and, as such, the division algorithm for polynomials is not utilized; long division is not performed	**Students simplify rational expressions by dividing out common factors**; students perform long division and synthetic division; students are asked to state the quotient and remainder
A-CED 1	Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions*.*	Create linear equations and linear inequalities in one variable and use them to solve problems; also create rational equations	
A-CED 2	Create equations in two or more variables to represent	Create linear equations in two variables; graph linear equations on coordinate axes with labels and scales; more emphasis on	Students create simple linear equations and systems of linear equations; uses

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	relationships between quantities; graph equations on coordinate axes with labels and scales.*	the solving than the creating	distance/rate/time problems, constraint problems relating area and perimeter, and rational equations involving rates; more emphasis on the solving rather than creating
A-CED 3	Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.**	Represent constraints by linear inequalities in one variable	See above
A-CED 4	Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.**	**Solve linear and rational equations for a quantity of interest**	**Solve linear and rational equations for a quantity of interest**
A-REI 1	Explain each step in solving a simple equation as following from the		Students use steps but there is no assessment of understanding each step in solving an equation;

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	equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.		students solve many different types of equations
A-REI 2	Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.	**Solve simple rational and radical (square root) equations in one variable; students test their answers but are not asked to explain or describe why the extraneous solutions arise in the first place**	**Students solve rational and radical (square root) equations in one variable; students understand they should test solutions but assessment items to do not individually assess the extraneous solution component**
A-REI 3	Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.	**Solve linear equations and inequalities in one variable**	**Solve linear inequalities in one variable**; solve inequalities involving absolute value in one variable
A-REI 4	Solve quadratic equations in one variable.		Solve quadratic equations and find inputs resulting in a given output of a quadratic function
A-REI 4a	Use the method of completing the square to transform any quadratic equation in *x* into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive		Solve by completing the square; the quadratic formula is used in the course but not derived

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	the quadratic formula from this form.		
A-REI 4b	Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers *a* and *b*.	**Quadratic equations are solved only by factoring**.	**Quadratic equations are solved** using all methods noted here; students write complex solutions in the prescribed form
A-REI 5	Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.		Addressed but students are not assessed
A-REI 6	Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.	**Solve systems of linear equations in two variables exactly using substitution and the method of adding a multiple of one equation to another; graphically show solutions**	**Solve systems graphically or algebraically (elimination and substitution methods)**
A-REI 7	Solve a simple system consisting		Addressed in the chapter but the assessment items

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	of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.		deemphasize the nonlinear case; graphing is not completed
A-REI 10	Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).		Students graph linear, quadratic, exponential, and logarithmic functions; the understanding is not officially assessed
A-REI 11	Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial,		Systems of linear (only) equations are solved graphically. Function notation is not used in the assessment items.

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	rational, absolute value, exponential, and logarithmic functions. *		
A-REI 12	Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.		
F-IF 1	Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.		Students identify the domain and range (rather than the domain being given in advance)
F-IF 2	Use function notation, evaluate functions for inputs in their		Students use function notation for a variety of types of common families of functions;

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	domains, and interpret statements that use function notation in terms of a context.		students evaluate functions at given inputs
F-IF 3	Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by f(0) = f(1) = 1, f(n+1) = f(n) + f(n-1) for n ≥ 1.*		
F-IF 4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and*		Students distinguish between positive and negative slope of a linear equation; students describe the symmetry and maximum/minimum of a parabola; symmetries of inverse functions are addressed [Note there are no assessment items for any of these topics]

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	minimums; symmetries; end behavior; and periodicity. *		
F-IF 5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* *		
F-IF 6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. *	**Calculate** and (on occasion) interpret **the slope of a linear equation**, including estimating the rate of change from a graph	**Students determine the slope of a linear equation**
F-IF 7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. *		

Standard Number	Standard	Math 095	Math 105
F-IF 7a	Graph linear and quadratic functions and show intercepts, maxima, and minima.	**Graph linear equations and find intercepts.**	**Students find intercepts of linear equations**; graph quadratic functions and show intercepts, maxima, and minima
F-IF 7b	Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.		
F-IF 7c	Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.		Students identify zeros for suitable factorizations
F-IF 7e	Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.		Graph exponential and logarithmic functions (only one assessment item for graphing an exponential function)
F-IF 8	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.	Write linear equations in multiple forms (for example, slope-intercept and standard form)	
F-IF 8a	Use the process of factoring and completing the		Students convert quadratic functions to vertex form

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	square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.		
F-IF 8b	Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as y = (1.02)t, y = (0.97)t, y = (1.01)12t, y = (1.2)t/10, and classify them as representing exponential growth or decay.		
F-IF 9	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum*.		

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F-BF 1	Write a function that describes a relationship between two quantities. *		
F-BF 1a	Determine an explicit expression, a recursive process, or steps for calculation from a context.		
F-BF 1b	Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model*.		
F-BF 2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. *		
F-BF 3	Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\ f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive		Intrinsically studied when working with quadratic functions in vertex form

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	and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.		
F-BF 4	Find inverse functions.		Inverse functions are found
F-BF 4a	Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x+1)/(x-1)$ *for* $x \neq 1$.		
F-LE 1	Distinguish between situations that can be modeled with linear functions and with exponential functions.		
F-LE 1a	Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by		

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	equal factors over equal intervals.		
F-LE 1b	Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.		
F-LE 1c	Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.		
F-LE 2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).		Students construct linear functions given a point and the slope or given two points
F-LE 3	Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.		
F-LE 4	For exponential		Students solve

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	models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.		exponential equations using logarithms
F-LE 5	Interpret the parameters in a linear or exponential function in terms of a context.		Students interpret parameters in linear equations.

A.3 Math 095 Final Exam

Give a complete solution of each problem, including the problem statement, on a separate piece of paper. All graphs are to be on graph paper with axes suitably scaled and labeled. In word problems all variables must be identified. Work must be neat, organized and easy to follow. Answers must be identified in clear complete English sentences. You must demonstrate in some way that your answers are plausible or, in fact, correct. [11]

1. Solve for x if

$$\frac{x}{x-4} + 3 = \frac{4}{x-4}.$$

2. Solve for x if

$$-2(x-1) + 8x = 9x - 1.$$

3. Perform the indicated subtraction and combine like terms and factor if possible:

$$\frac{6}{y-4} - \frac{1}{y+1}.$$

4. If $p = -4$ and $q = -1$, what is the value of $2p^2 + 3q$?

5. Tickets for a play at the community theatre cost \$12.00 for an adult and \$2.00 for a child. If 160 tickets were sold and the total receipts were \$1420, how many of each type of ticket were sold?

6. Solve for x if

$$x^2 + 6x - 27 = 0.$$

7. A triangle has side length $2x - 5$, $3x + 1$, and $4x + 2$ respectively. Express the perimeter of this triangle as a polynomial in x. Be sure to combine like terms.

8. Write an equation for a line whose slope is $-1/2$ and whose y-intercept is $(0,3)$. Graph this line.

9. Perform the indicated division:

$$\frac{9x^2-48x+56}{3x-7}.$$

10. The sum of two numbers is 76. The second is 1 more than 2 times the first. What are these two numbers?

11. Rewrite the following using only positive exponents. Reduce any fractions if possible.

$$\frac{(a^6)^{-4}}{a(a^6)^{-3}}.$$

12. Find the slope and y-intercept of the line represented by the following equation:

$$-8x - 10y = 60.$$

13. Perform the following division:

$$\left(\frac{x^2+2x-15}{4x^2}\right) \div \left(\frac{x^2-25}{2x-10}\right).$$

Be careful not to divide by 0.

14. Factor the following completely into linear factors:

$$15x^2 - 37x + 20.$$

15. Multiply out and collect like terms:

$$(-3x + 3)^2.$$

16. Factor completely into linear factors:

$$20x^3 + 24x^2 - 5x - 6.$$

17. Solve for y if

$$\sqrt{y+9} = 4.$$

18. Solve this system of equations for (x, y) by addition:

$$3x - 4y = 17,$$

$$x + 2y = 9.$$

19. Subtract $4d^2 - 8d - 3$ from $10d^2 + 10d - 6$ and combine like terms.

20. Factor completely into linear terms:

$$6b^4 - 18b^3 - 60b^2.$$

21. Solve this system of equations by graphing:

$$3x + y = 6,$$

$$3x - y = 0.$$

22. Find two consecutive integers such that the sum of 6 times the first integer and 4 times the second integer is 34.

23. A car uses 8 gallons of gasoline on a trip of 168 miles. At that rate, how much gasoline will a trip of 273 miles require?
24. Find an integer with the property that when it is added to its square we get 72.
25. Solve the following inequality and graph the solution on a number line:

$$3(x-4) > 7x - 9.$$

26. Write $\sqrt{45} - \sqrt{5}$ in the form $a\sqrt{b}$ where b has no perfect square factors (other than ± 1).
27. Write an equation of the line passing through $(-2,-3)$ and $(3,17)$. If you have not already done so, and it is possible, write this equation in slope-intercept form.
28. Graph the inequality

$$2x + 3y > 6.$$

29. The legs of a right triangle have lengths 20 cm and 15 cm. What is the length of a hypotenuse of this right triangle? Express your answer in the form $a\sqrt{b}$ where b has no perfect square factors (other than ± 1).
30. Graph $2x - 5y = 10$.

A.4 Math 105 Final Exam

Give a complete solution of each problem on a separate piece of paper. All graphs are to be on graph paper with axes suitably scaled and labeled. In word problems all variables must be identified. Work must be neat, organized and easy to follow. Answers must be identified in clear, complete English sentences. You must demonstrate in some way that your answers are plausible or, in fact, correct. [10]

1. Find the slope of the line containing the points with coordinates $(3,4)$ and $(-9,-4)$.
2. Find the equation of the line in slope-intercept form containing $(3,4)$ and $(-9,-4)$.
3. Determine the slope and y-intercept of the line with equation $5x - 3y = -6$ and graph this line.
4. Find an equation of the line in point-slope form that passes through $(7,-2)$ and is perpendicular to the line $3x + 5y = 10$.
5. Solve by graphing:

$$2x + y = 8$$

$$3x - 2y = -2$$

6. Solve by substitution or elimination:

$$3x - y = 1$$

$$4x - 2y = -4$$

7. Just Nuts sells pistachios for $6.50 per pound and almonds for $8.00 per pound. How many pounds of each should be used to make a 50 pound bag that sells for $7.40 per pound?
8. Solve for x and write your answer using interval notation:

$$|3x - 2| \leq 16.$$

9. Multiply $(6a + 2b)^2$.

10. Factor $2w^3 - 54$ completely.

11. Solve by factoring:

$$x^3 + 12x^2 = -20x.$$

12. Find the domain of

$$f(x) = \frac{5x}{x^2 - 49}$$

with the understanding that it is a subset of the real numbers. Use interval notation to write your answer.

13. Divide and simplify:

$$\left(\frac{a^2-4a+4}{2a+6}\right) \div \left(\frac{a^3-4a}{3a}\right).$$

14. Perform the following subtraction and write the result as a ratio of polynomials of lowest degree:

$$\frac{w}{w^2+5w+6} - \frac{2}{w^2+3w+2}.$$

15. Find the value of x if

$$\frac{4}{x+3} = \frac{1}{x-4} + \frac{x+11}{x^2-x-12}.$$

16. It takes Elle 90 minutes to clean her apartment and it takes Gretchen 60 minutes to clean the same apartment. If they work together (and don't get in each other's way!) how long would it take them to clean the apartment working together?

17. Perform the following division using either synthetic division or long division:

$$(x^3 - 5x^2 + 15) \div (x - 4).$$

18. Find the variation constant k and an equation of variation if y varies directly as x and $y = 62$ when $x = 40$.

19. Simplify the following product and write the answer using radicals (not fractional exponents). Assume that the variables represent non-negative real numbers.

$$\sqrt[3]{2xy^3} \times \sqrt[3]{12x^4y}.$$

20. Perform the indicated multiplication and simplify the radical:

$$\sqrt{-8} \times \sqrt{-7}.$$

21. Write

$$\frac{\sqrt[4]{3G}}{\sqrt[4]{2E^2F^3}}.$$

22. Write

$$\frac{7}{4-3i}$$

as $a + bi$ for two real numbers a and b.

23. (095) Solve the following equation of y:

$\sqrt{y-2} - 9 = 1.$

24. Solve the equation $x^2 + 6x = 5$ for x by using the quadratic formula or by completing the square.

25. Solve for y if $y^{1/2} - 6y^{1/4} + 8 = 0.$

26. The local train travels 15 miles per hour slower than the express train. The local trains travels 240 miles in the same time that the express train travels 315 miles. Find the speed of each train.

27. Find the vertex and the axis of symmetry of the quadratic function $f(x) = -3(x-2)^2 + 6$ and then graph the function, plotting at least 5 points.

28. Suppose $f(x) = x^2 - 5$ and $g(x) = 4x + 6$. Find $(f \circ g)(x)$.

29. Suppose $f(x) = x^2 - 5$ and $g(x) = 4x + 6$. Find $(f \circ g)(-3)$.

30. Find a formula for the inverse of $f(x) = x^3 + 8$.

31. Find x if $\log_x(64) = 3$.

32. Solve the following equation for x: $\log_2(15 - 8x) = 5$.

33. Express

$$\log_a\left(\frac{x\sqrt{y}}{z^2}\right)$$

in terms of $\log_a(x)$, $\log_a(y)$ and $\log_a(z)$. You should assume $x > 0$, $y > 0$, and $z > 0$.

34. Solve the following equation for x and round your exact answer to the nearest hundredth:

$$3.7^x = 72.$$

35. Suppose that $f(x) = (1/2)^x$. Graph this function, and include the points where $x = 0, \pm 1, \pm 2$.