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On the Generalized Ince Equation

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by

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ABSTRACT

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Under the Supervision of Professor Hans Volkmer

We investigate the Hill differential equation,

$$(1 + A(t))\, y''(t) + B(t)\, y'(t) + (\lambda + D(t))\, y(t) = 0, \tag{0.0.1}$$

where $A(t)$, $B(t)$, and $D(t)$ are trigonometric polynomials. We are interested in solutions that are even or odd, and have period π or semi-period π. Equation (0.0.1) with one of the above conditions constitute a regular Sturm-Liouville eigenvalue problem. Using Fourier series representation each one of the four Sturm-Liouville operators is represented by an infinite banded matrix. In the particular cases of Ince and Lamé equations, the four infinite banded matrices become tridiagonal. We then

investigate the problem of coexistence of periodic solutions and that of existence of polynomial solutions.

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Dedication

Dedicated to the memory of my parents.

Contents

List of Tables

List of Figures

CHAPTER 1

Introduction and Preliminary Concepts

The Ince and Lamé equations reside in the "land beyond Bessel", they are (confluent) Heun equations when brought into algebraic form. The equations have periodic coefficients, so they are Hill equations with spectral parameters λ and h, respectively. Ince's equation has three additional parameters a, b, d, whereas Lamé's equation has two, ν and k. Employing Jacobi's amplitude $t = \operatorname{am} z$, Lamé's equation is transformed to its trigonometric form, and this is a particular Ince equation. Mathieu's equation is an instance of Ince's equation ($a = b = 0$) but not of Lamé's equation. Lamé [**37, 38**] discovered his equation in the 1830's in connection with the problem of determining the steady temperature in an ellipsoidal conductor with three distinct semi-axes when the temperature is prescribed on the surface of the conductor. By introducing ellipsoidal coordinates he found formulas for the temperature in terms of doubly-periodic solutions of Lamé's equation, called Lamé polynomials. Throughout the remainder of the nineteenth century the best analysts of their time worked on the theory of Lamé's equation, among them Heine [**21**], Hermite [**23**], Klein [**36**] and Lindemann [**44**], the latter being famous for his proof that is a transcendental number. In the twentieth century, Ince [**31, 32**] introduced simply-periodic Lamé functions. The well known Handbook of Higher Transcendental Functions, Volume III, by Erdélyi et al. [**1**] contains a very readable overview of the results of Ince and others. Strutt [**65**] gives applications of Lamé functions in engineering and physics. Jansen [**34**] treats simply-periodic Lamé functions and applies them to antenna theory. In the second half of the twentieth century, Arscott was the leading expert on Lamé's equation. He wrote several papers [**4, 5, 6, 7**] on Lamé polynomials and dedicated one chapter of his well known book [**8**] to the Lamé equation.

The first known appearance of the Ince equation is in Whittaker's paper [**83**, Equation (5)] on integral equations. Whittaker emphasized the special case $a = 0$, and this special case was later investigated in more detail by Ince [**27, 30**]. Magnus and Winkler's book [**45**] contains a chapter dealing with the coexistence problem for the Ince equation. Also Arscott [**8**] has a chapter on the Ince equation with $a = 0$. Arscott points out that the Ince equation was never considered by Ince and should be called the generalized Ince equation. We use the name Ince equation following the practice in [**45**]. A large part of the theory of Lamé's equation carries over to the Ince equation, and, in fact, becomes more transparent in this way. For one thing, working with Ince's equation does not require knowledge of the Jacobian elliptic function appearing in Lamé's equation. For instance, Jansen [**34**] preferred to work with the trigonometric form of Lamé's equation. The Ince equation has another advantage over Lamé's equation. The formal adjoint of Ince's equation is again an Ince equation. The adjoint equation is found by the parameter substitution

$$(a, b, \lambda, d) \to (a, -4a - b, \lambda, d - 4a - 2b).$$

If we apply the corresponding substitution to the trigonometric form of Lamé's equation, then, unfortunately, the adjoint equation is not a Lamé equation anymore. This leads to a somewhat awkward theory of Lamé's equation, for example, there are two different Fourier expansions for a simply-periodic Lamé function; see [**1**, Page 65]. Actually, one expansion suffices if we work with Ince's equation. An important feature of the Ince equation is that the corresponding Ince differential operator (whose eigenvalues are λ) when applied to Fourier series can be represented by an infinite tridiagonal matrix. It is this part of the theory that makes the Ince equation particularly interesting. For instance, the coexistence problem which has no simple solution for the general Hill equation has a complete solution for the Ince equation; see Section 3.5.

Lamé functions were originally introduced to solve certain problems in applied mathematics “by hand”. Unfortunately, these computations are involved and never became very popular. Today, however, Lamé and Ince functions excel on modern computers, and so it is not surprising that Lamé and Ince functions enjoy a renaissance in applied mathematics. For example, Lamé functions are used in biomedical engineering [**33**], and Ince functions are used in thermodynamics [**2**]. In addition, symbolic manipulation software makes working with the Ince and Lamé equations so much more enjoyable than it used to be.

This dissertation is an investigation of the theory of Ince and Lamé equations. When studying the Ince equation, it became apparent that many of its properties carry over to a more general class of equations “the generalized Ince equation” . These linear second order differential equations describe important physical phenomena which exhibit a pronounced oscillatory character; behavior of pendulum-like systems, vibrations, resonances and wave propagation are all phenomena of this type in classical mechanics,(see for example [**57**]). while the same is true for the typical behavior of quantum particles (Schrödinger’s equation with periodic potential). Before considering the Ince and Lamé equations and generalization in more detail, we include a summary of some important concepts that will be used throughout this thesis.

1.1. Sturm-Liouville Spectral Theory

Consider the Sturm-Liouville equation

$$-\left(p\left(t\right)y'\left(t\right)\right)'+q\left(t\right)y\left(t\right)=\lambda r\left(t\right)y\left(t\right),\quad a<t<b. \tag{1.1.1}$$

We assume that $p:(a,b)\to(0,\infty)$ is continuously differentiable, $r:(a,b)\to(0,\infty)$ is continuous and $q:(a,b)\to\mathbb{R}$ is continuous. In Coddington and Levinson [**10**] it is assumed that $r\left(t\right)=1$ but this can be always be achieved by the Sturm transformation.

We choose $c \in (a, b)$ and set

$$\xi = \int_c^t r(x)\,dx, \qquad y(t) = Y(\xi).$$

Then we obtain

$$-\left(P(\xi)\,Y'(\xi)\right)' + Q(\xi)\,Y(\xi) = \lambda Y(\xi),$$

where

$$P(\xi) = r(t)\,p(t), \quad Q(\xi) = \frac{q(t)}{r(t)}.$$

In Titchmarsh [**67**] it is assumed that $p(t) = r(t) = 1$. If $p(t)\,r(t)$ is twice continuously differentiable then we can achieve this by Liouville transformation. We choose $c \in (a, b)$ and set

$$\eta = \int_c^t \left(\frac{r(x)}{p(x)}\right)^{1/2} dx, \quad W(\eta) = \left(p(t)\,r(t)\right)^{1/4} \omega(t).$$

Then we obtain

$$-W''(\eta) + Q(\eta)\,W(\eta) = \lambda W(\eta),$$

where

$$Q(\eta) = \frac{f''(\eta)}{f(\eta)} + k(\eta)$$

and

$$f(\eta) = \left(p(t)\,r(t)\right)^{1/4}, \qquad k(\eta) = \frac{q(t)}{r(t)}.$$

Example 1.1.1. Consider

$$-y'' = \lambda \frac{1}{t} y, \tag{1.1.2}$$

so that $p(t) = 1$, $q(t) = 0$, $r(t) = \frac{1}{t}$. We take the interval $(a,\, b) = (0,\, \infty)$. The Sturm transformation is

$$\xi = \int_1^t \frac{dx}{x} = \ln t.$$

Then

$$P(\xi) = \frac{1}{t} = e^{-\xi}.$$

For $Y(\xi) = y(t)$ we obtain the differential equation

$$-\left(e\left(^{-\xi}\right)Y'\right)' = \lambda Y.$$

The Liouville transformation (we can take $c = 0$) is

$$\eta = \int_0^t x^{-1/2}dx = 2t^{1/2}, \quad f(\eta) = t^{-1/4} = \left(\frac{\eta}{2}\right)^{-1/2}, \quad Q(\eta) = \frac{3}{4}\eta^{-2}.$$

We obtain the differential equation

$$-Y'' + \frac{3}{4}\frac{1}{\eta^2}Y = \lambda Y.$$

1.1.1. Regular Sturm-Liouville Problems. The end point a is called regular if $a \in \mathbb{R}$ and the functions p, q, $r : [a, b) \to \mathbb{R}$ satisfy the same assumptions as before but now on $[a, b)$. If a is not regular, it is called singular. Similar definitions apply to b. Suppose that a and b are regular end points. Then we impose boundary conditions

$$\cos\alpha y(a) = \sin\alpha\left(p'\right)(a), \tag{1.1.3}$$

and

$$\cos\beta y(b) = \sin\beta\left(p'\right)(b), \tag{1.1.4}$$

where α, $\beta \in \mathbb{R}$. A complex number λ is called an eigenvalue if there exists a non trivial solution y (eigenfunction corresponding to λ) of (1.1.1), (1.1.3), (1.1.4). In the case of regular Sturm-Liouville problems eigenvalues are real and eigenspaces are one-dimensional.

THEOREM 1.1.2. *The eigenvalues form an increasing sequence λ_n, $n \in \mathbb{N}_0$, converging to infinity. An eigenfunction $\phi_n(t)$ has exactly n zeros in (a, b). If the eigenfunctions are normalized according to*

$$\int_a^b r(t)\left|\phi_n(t)\right|^2 dt = 1,$$

then the sequence $\phi_n(t)$, $n \in \mathbb{N}_0$, *forms an orthonormal basis in the Hilbert space* $L^2_r(a,b)$.

If $\lambda \in \mathbb{C}$ is not an eigenvalue, then the Green's function is

$$G(t,\,x,\,\lambda) = \begin{cases} \psi_1(t,\lambda)\,\psi_2(x,\lambda) & if\; a \le t \le x \le b \\ \psi_2(t,\lambda)\,\psi_1(x,\lambda) & if\; a \le x \le t \le b \end{cases},$$

where $\psi_1(t,\lambda)$ is a solution of (1.1.1), (1.1.3) and $\psi_2(t,\lambda)$ is a solution of (1.1.1), (1.1.4) such that

$$\psi_2(t,\lambda)\,p(t)\,\psi_1'(t,\lambda) - \psi_1(t,\lambda)\,p(t)\,\psi_2'(t,\lambda) = 1.$$

If $f : [a,b] \to \mathbb{C}$ is a continuous function, and λ is not an eigenvalue then

$$y(t) = \int_a^b G(t,x,\lambda)\,f(x)\,dx$$

is the solution of the inhomogeneous differential equation

$$-\left(p(t)\,y'\right)' + q(t)\,y - \lambda r(t)\,y = f(t)$$

which satisfies the boundary conditions (1.1.3), (1.1.4). If we set $y = \phi_n$ and $f(t) = (\lambda_n - \lambda)\,r(t)\,\phi_n(t)$, then

$$\phi_n(t) = (\lambda_n - \lambda)\int_a^b r(x)\,G(t,x,\lambda)\,\phi_n(x)\,dx.$$

1.2. Introduction to the Theory of Hill's Equation

1.2.1. Hill's Equation. The Hill equation is the linear differential equation of the second order

$$c_0(t)\,y''(t) + c_1(t)\,y'(t) + c_2(t)\,y(t) = 0 \tag{1.2.1}$$

with continuous coefficients $c_j : \mathbb{R} \to \mathbb{C}$ with period $\omega > 0$,

$$c_j(t+\omega) = c_j(t) \qquad \forall t \in \mathbb{R},$$

and $c_0(t) \neq 0$ for all $t \in \mathbb{R}$. If $y(t)$ is a solution then also $(Ty)(t) = y(t+\omega)$ is a solution. This defines a linear map T from the two dimentional linear space of solutions into itself. Therefore, there exists a nontrivial solution $y(t)$ (called Floquet solution) and a complex number ρ (called multiplier) such that

$$y(t+\omega) = (Ty)(t) = \rho y(t).$$

When $\rho = \pm 1$ we obtain solutions with period ω or semi-period ω

$$y(t+\omega) = y(t), \qquad y(t+\omega) = -y(t).$$

We choose a fundamental system of solutions y_1, y_2 determined by the initial conditions

$$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

Then

$$\begin{aligned} (Ty_1)(t) &= y_1(t+\omega) = x_{11}y_1(t) + x_{21}y_2(t), \\ (Ty_2)(t) &= y_2(t+\omega) = x_{12}y_1(t) + x_{22}y_2(t). \end{aligned}$$

From the initial conditions for y_1, y_2 we obtain the matrix representation of T with respect to the basis y_1, y_2

$$T = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} y_1(\omega) & y_2(\omega) \\ y_1'(\omega) & y_2'(\omega) \end{bmatrix}. \tag{1.2.2}$$

By Abel's formula for the Wronskian

$$\operatorname{Det} T = \exp\left(-\int_0^\omega \frac{c_1(t)}{c_0(t)}dt\right).$$

Therefore, the multipliers ρ are the roots of the equation

$$\rho^2 - (y_1(\omega) + y_2'(\omega))\rho + \det T = 0. \tag{1.2.3}$$

This quadratic equation has the two non zero solutions ρ_1, ρ_2. The corresponding characteristic exponents m_1, m_2 are defined by $\rho_k = e^{i\omega m_k}$. There are two cases.

(1) $\rho_1 \neq \rho_2$. Let y_1, y_2 be Floquet solutions with multipliers ρ_1, ρ_2 respectively. They form a fundamental system of solutions. If we define

$$p_k(t) = e^{-im_k t} y_k(t),$$

then

$$p_k(t+\omega) = p_k(t),$$

therefore we can write

$$y_k(t) = e^{im_k} p_k(t), \qquad k = 1, 2.$$

(2) $\rho_1 = \rho_2$. There are two possibilities. If ρ_1 is an eigenvalue of T with geometric multiplicity 2 then we can proceed as in the first case. If $\rho = \rho_1 = \rho_2 = e^{i\omega m}$ has geometric multiplicity 1, let y_1 be the corresponding Floquet solution, and let y_2 be a linearly independent solution. we have

$$T\,y_1 = \rho y_1, \quad T\,y_2 = dy_1 + \rho y_2, \quad d \neq 0.$$

If we define

$$p_1(t) = e^{-imt} y_1(t), \quad p_2(t) = e^{-imt} y_2(t) - \frac{d}{\omega\rho} t\, p_1(t)$$

then

$$p_k(t+\omega) = p_k(t), \qquad k = 1, 2,$$

and we can write

$$y_1(t) = e^{imt} p_1(t), \quad y_2(t) = e^{imt}\left(tp_1(t) + p_2(t)\right).$$

Equation (1.2.1) is called *stable* if all solution are bounded in $\mathbb{R}$. Otherwise it is called *unstable*. The equation is stable if and only if $|\rho_k| = 1$ for $k = 1, 2$, and the geometric multiplicity of ρ_1 is 2 when $\rho_1 = \rho_2$.

Next, we add the assumption that the coefficients in (1.2.1) are real-valued and that

$$\int_0^{\omega} \frac{c_1(t)}{c_0(t)} dt = 0. \tag{1.2.4}$$

Assumption (1.2.4) implies $det\, T = 1$ and so $\rho_1\rho_2 = 1$. We define the *discriminant*

$$D = \operatorname{trace} T = y_1(\omega) + y_2'(\omega).$$

If $\rho = e^{i\omega m}$ then equation (1.2.3) becomes

$$D = \rho + \rho^{-1} = e^{i\omega m} + e^{-i\omega m} = 2\cos(\omega m). \tag{1.2.5}$$

Since D is a real number there are these possibilities:

(1) $D > 2$ or $D < -2$. Then ρ_1, ρ_2 are distinct, real and one of them is larger than 1 in absolute value. Thus (1.2.1) is unstable.

(2) $-2 < D < 2$. Then ρ_1,ρ_2 are distinct, conjugates of each other and have absolute value 1. Thus (1.2.1) is stable.

(3) $D = 2$. Then $\rho_1 = \rho_2 = 1$ and there exists a Floquet solution with period ω. The equation is stable if and only if all solutions of (1.2.1) have period ω.

(4) $D = -2$. Then $\rho_1 = \rho_2 = -1$ and there exists a Floquet solution with semi-period ω. The equation is stable if and only if all solutions of (1.2.1) have semi-period ω.

In the case where all solutions have period ω, or all solutions have semi-period ω we speak of *coexistence* of Floquet solutions with period ω or semi-period ω.

Example 1.2.1. We consider the Fourier equation

$$y'' + \lambda y = 0,\ \lambda = c^2,\ c \geq 0. \tag{1.2.6}$$

We can choose $\omega > 0$ arbitrarily. Let us take $\omega = \pi$. We obtain

$$y_1(t) = \cos(ct), \ y_2(t) = \frac{1}{c}\sin(ct)$$

and

$$T = \begin{bmatrix} \cos(c\pi) & c^{-1}\sin(c\pi) \\ -c\sin(c\pi) & \cos(c\pi) \end{bmatrix}.$$

The discriminant is

$$D = 2\cos(c\pi).$$

Therefore, $\rho = e^{\pm ic\pi}$. The equation is stable for all $c > 0$ but unstable for $c = 0$. Floquet solutions with period π exist for $c = 2n$ with $n = 0, 1, 2, \ldots$, and Floquet solutions with semi-period π exist for $c = 2n + 1$ with $n = 0, 1, 2, \ldots$ We have coexistence of Floquet solutions with period ω or semi-period ω for $c = n$, $n = 1, 2, 3, \ldots$, but not for $c = 0$.

1.2.2. Hill's Equation with Spectral Parameter. We consider Hill's equation with parameter λ

$$-(p(t)y')' + q(t)y = \lambda r(t)y, \tag{1.2.7}$$

where p, q, $r : \mathbb{R} \to \mathbb{R}$ are functions with period ω, q continuous, p continuously differentiable and positive, and r continuous and positive. Note that assumption (1.2.4) holds.

The discriminant $D(\lambda)$ becomes real analytic function of the real parameter λ which sometimes is called a *Lypanov* function. In general we can prove the following properties of $D(\lambda)$.

THEOREM 1.2.2. *The discriminant $D(\lambda)$ has the following properties,*

(1) $\lim_{\lambda \to -\infty} D(\lambda) = \infty$;

(2) $D'(\lambda) \neq 0$ *whenever* $-2 < D(\lambda) < 2$;

(3) *if* $D(\lambda) = \pm 2$ *and* $D'(\lambda) = 0$ *then* $\pm D''(\lambda) < 0$;

(4) *there is a sequence* $\delta_k \to \infty$ *such that* $D\left(\right) \leq 2$,

For the proof see Chapter 8, Section 3 in Coddington and Levinson [**10**].

It follows from this theorem that the λ's for which (1.2.7) admits a Floquet solution with period ω form a non decreasing sequence λ_n, converging to infinity. Values of λ for which there is coexistence of Floquet solutions with period ω are counted twice. Similarly the λ's for which (1.2.7) admits a Floquet solution with semi-period ω form a non decreasing sequence μ_n, $n = 0, 1, 2, \ldots$, converging to infinity. We have the inequalities

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_4 < \ldots$$

If $\lambda \in (-\infty, \lambda_0]$ then equation (1.2.7) is unstable. The interval $(-\infty, \lambda_0]$ is the zero-th *instability interval*. The intervals $[\mu_{2m}, \mu_{2m+1}]$, $[\lambda_{2m+1}, \lambda_{2m+2}]$ are the $(2m+1)$ th and $(2m+2)$ th instability intervals, $m = 1, 2, \ldots$, with the exception, if these intervals shrink to one point $[\lambda, \lambda]$ (in the case of coexistence) then (1.2.7) is stable at this value λ. If λ lies in one of the intervals where $D(\lambda) \in (-2, 2)$ then equation (1.2.7) is stable. We call these intervals the *stability* intervals.

1.2.3. The Even Hill Equation. Consider equation (1.2.1) and suppose that c_o, c_2 are even functions and c_1 is an odd function:

$$c_0(-x) = c_0(x), \quad c_2(-x) = c_2(x), \quad c_1(-x) = -c_1(x).$$

This implies that $\det T = 1$. If $y(t)$ is a solution then also $y(-t)$ is a solution. If $y(t)$ is a Floquet solution with multiplier ρ then $y(-t)$ is a Floquet solution with multiplier $1/\rho$. In particular, a Floquet solution with period or semi-period ω must be even or odd unless all solutions have period or semi period ω.

By inverting the matrix T in (1.2.2) we get

$$y_1(t) = y_2'(\omega)\, y_1(t+\omega) - y_1'(\omega)\, y_2(t+\omega).$$

If we substitute $t = -\omega$ we obtain $y_1(-\omega) = y_1(\omega) = y_2'(\omega)$. Therefore the discriminant simplifies to

$$D = 2y_1(\omega).$$

Now consider the Hill equation (1.2.7) with spectral parameter λ under the assumptions of Subsection 1.2.2. In addition we assume that p, q, r are even. Then for every λ we have an even Hill equation. In this case it is customary to introduce a slightly different scheme of notation for the solutions of $D(\lambda) = \pm 2$. Let α_{2n}, $n = 0, 1, 2, \ldots$,denote the increasing sequence of those λ's for which there exists an even Floquet solution with period ω. Let α_{2n+1}, $n = 0, 1, 2, \ldots$, denote the increasing sequence of those λ's for which there exists an even Floquet solution with semi-period ω. Let β_{2n+1}, $n = 0, 1, 2, \ldots$, denote the increasing sequence of those λ's for which there exists an odd Floquet solution with semi-period ω. Let β_{2n+2}, $n = 0, 1, 2, \ldots$,denote the increasing sequence of those λ's for which there exists an odd Floquet solution with period ω. The $n-$th instability interval is then the interval between α_n and β_n. Note that $\alpha_n < \beta_n$, $\alpha_n > \beta_n$, and $\alpha_n = \beta_n$ are possible.

1.3. Outline

In this first chapter we briefly outlined the subject of interest. We discussed the form and basic properties of the Sturm-Liouville problem and the properties of Hill's equation that are needed for the remainder of the thesis. In the remaining chapters, some specific techniques are considered for the analysis of the Ince and Lamé differential equations and their generalization. These chapters are organized as follows.

Chapter two. Introduces a generalization of Ince's equation. The main properties of Ince's equation apply to a more general class of equation that we called the generalized Ince equation.

Chapter three. Discusses in detail the Ince equation, in particular the problem of coexistence of periodic solution, and that of the existence of polynomial solutions in trigonometric form.

Chapter four. investigates Lamé's equation The substitution $t = \frac{\pi}{2} - \operatorname{am} z$ transforms Lamé's to an Ince equation, in this way a large part of the theory of Lamé's equation carries over to the Ince equation.

Chapter five. Generalizes Lamé's equation using the so called generalized elliptic functions. When transformed to its trigonometric form, this equation becomes a generalized Ince equation.

Chapter six. Discusses the technique of separation of variables applied to the wave equation in some special coordinate systems. This precess leads to Ince and Lamé differential equations.

Chapter seven. Presents mathematical and physical applications, and special results, in particular a section on the instability interval of the generalized Ince equation.

Chapter eight. Concludes the thesis.

CHAPTER 2

A Generalization of Ince's Differential Equation

2.1. The Differential Equation

We consider the Hill differential equation

$$(1 + A(t))\, y''(t) + B(t)\, y'(t) + (\lambda + D(t))\, y(t) = 0 \tag{2.1.1}$$

where

$$\begin{aligned} A(t) &= \sum_{j=1}^{\eta} a_j \cos(2jt), \\ B(t) &= \sum_{j=1}^{\eta} b_j \sin(2jt), \\ D(t) &= \sum_{j=1}^{\eta} d_j \cos(2jt). \end{aligned}$$

Here η is a positive integer, the coefficients a_j, b_j, d_j, for $j = 1, 2, ..., \eta$ are specified real numbers. The real number λ is regarded as a spectral parameter. We further assume that $\sum_{j=1}^{\eta} |a_j| < 1$. Unless stated otherwise solutions $y(t)$ are defined for $t \in \mathbb{R}$. We will at times represent the coefficients a_j, b_j, d_j, for $j = 1, 2 \ldots, \eta$ in the vector form: $\mathbf{a} = [a_1, a_2, \ldots, a_\eta]$, $\mathbf{b} = [b_1, b_2, \ldots, b_\eta]$, $\mathbf{d} = [d_1, d_2, \ldots, d_\eta]$.

The polynomials

$$Q_j(\mu) := 2a_j\mu^2 - b_j\mu - \frac{d_j}{2}, \quad j = 1, 2, \ldots, \eta, \tag{2.1.2}$$

will play an important role in the analysis of (2.1.1). For ease of notation we also introduce the polynomials

$$Q_j^{\dagger}(\mu) := Q_j(\mu - 1/2), \quad j = 1, 2, \ldots, \eta. \tag{2.1.3}$$

Equation (2.1.1) is a natural generalization to the original Ince's equation

$$(1 + a\cos(2t))\, y''(t) + (b\sin(2t))\, y'(t) + (\lambda + d\cos(2t))\, y(t) = 0. \tag{2.1.4}$$

Ince's equation by itself includes some important particular cases, if we choose for example $a = b = 0$, $d = -2q$ we obtain the famous Mathieu's Equation

$$y''(t) + (\lambda - 2q\cos(2t))\, y(t) = 0, \tag{2.1.5}$$

with associated polynomial

$$Q(\mu) = q. \tag{2.1.6}$$

If we choose $a = 0$, $b = -4q$, and $d = 4q(\nu - 1)$, where q, ν are real numbers, Ince's equation becomes Whittaker-Hill equation

$$y''(t) - 4q(\sin 2t)\, y'(t) + (\lambda + 4q(\nu - 1)\cos 2t)\, y(t) = 0, \tag{2.1.7}$$

with associated polynomial

$$Q(\mu) = 2q(2\mu - \nu + 1). \tag{2.1.8}$$

Equation (2.1.1) can be brought to algebraic form by applying the transformation $\xi = \cos^2 t$. For example when $\eta = 2$, and $\mathbf{a} = \mathbf{b} = 0$, we obtain

$$\frac{d^2y}{d\xi^2} + \frac{1}{2}\left(\frac{1 - 2\xi}{\xi(1-\xi)}\right)\frac{dy}{d\xi} + \frac{1}{4}\left(\frac{8d_2\xi^2 + (2d_1 - 8d_2)\xi - d_1 + d_2 + \lambda}{\xi(1-\xi)}\right) y = 0. \tag{2.1.9}$$

2.2. Eigenvalues

Equation (2.1.1) is an even Hill equation with period π. We are interested in solutions which are even or odd and have period π or semi period π i.e. $y(t+\pi) = \pm y(t)$. We know that $y(t)$ is a solution to (2.1.1) then $y(t+\pi)$, and $y(-t)$ are also solutions. From the general theory of Hill's equation we obtain the following lemmas:

Lemma 2.2.1. *Let $y(t)$ be a solution of* (2.1.1), *then $y(t)$ is even with period π if and only if*

$$y'(0) = y'(\pi/2) = 0; \tag{2.2.1}$$

$y(t)$ is even with semi period π if and only if

$$y'(0) = y(\pi/2) = 0; \tag{2.2.2}$$

$y(t)$ is odd with semi period π if and only if

$$y(0) = y'(\pi/2) = 0; \tag{2.2.3}$$

$y(t)$ is odd with period π if and only if

$$y(0) = y(\pi/2) = 0. \tag{2.2.4}$$

For the proof see [**14**, Theorem 1.3.4]

Lemma 2.2.2. *Equation* (2.1.1) *can be written in the self adjoint form*

$$-\left(\left(1 + A(t)\right)\omega(t)y'(t)\right)' - D(t)\,\omega(t)\,y(t) = \lambda\omega(t)\,y(t)\,, \tag{2.2.5}$$

where

$$\omega(t) = \exp\left(\int \frac{B(t) - A'(t)}{1 + A(t)}dt\right). \tag{2.2.6}$$

Note that $\omega(t)$ is even and π-periodic since the function $\frac{B(t)-A'(t)}{1+A(t)}$ is continuous, odd, and π-periodic.

Proof. Let $r(t) = (1 + A(t))\,\omega(t)$. (2.2.5) can be written as,

$$(-r(t)y'(t))' - D(t)\omega(t)y(t) = \lambda\omega(t)y(t), \tag{2.2.7}$$

which is equivalent to

$$-r'(t)y'(t)-r(t)y''(t)-D(t)\omega(t)y(t)=\lambda\omega(t)y(t). \tag{2.2.8}$$

Noting that

$$r'(t)=(1+A(t))\omega'(t)+A'(t)\omega(t),$$

and

$$\omega'(t)=\frac{B(t)-A'(t)}{1+A(t)}\omega(t),$$

we see that

$$r'(t)=B(t)\omega(t).$$

Therefore, (2.2.8) can be written as

$$-B(t)\omega(t)y'(t)-(1+A(t))\omega(t)y''(t)-D(t)\omega(t)y(t)=\lambda\omega(t)y(t). \tag{2.2.9}$$

Since $\omega(t)$ is strictly positive, the lemma follows. $\square$

In the case of Ince's equation (2.1.4), we have the following formula for the function ω

$$\omega(t):=\begin{cases}(1+a\cos 2t)^{-1-b/2a} & \text{if } a\neq 0,\\ \exp\left(\frac{-b}{2}\cos 2t\right) & \text{if } a=0.\end{cases} \tag{2.2.10}$$

When $\eta\geq 2$, the function can be computed explicitly using *Maple*. For example, let us consider the case $\eta=2$, with $\mathbf{a}=\left[\frac{1}{4},\frac{1}{8}\right]$, $\mathbf{b}=[1,1]$. Applying (2.2.6), we obtain

$$\omega(t)=\frac{1}{\left(7+2\cos 2t+2(\cos 2t)^2\right)^3}.$$

Equation (2.1.1) with one of the boundary conditions in Lemma 2.2.1 is a regular Sturm-Liouville problem. From the theory of Sturm-Liouville ordinary differential equations it is known that such an eigenvalue problem has a sequence of eigenvalues that converge to infinity. These eigen values are denoted by α_{2m}, α_{2m+1}, β_{2m+1}, and

β_{2m+2}, $m = 0, 1, 2, \ldots$ to correspond to the boundary conditions in lemma 2.2.1 respectively. This notation is consistent with the theory of Mathieu and Ince's equations (see [**45, 79**]). Lemma 2.2.1 implies the following theorem.

THEOREM 2.2.3. *The generalized Ince equation admits a nontrivial even solution with period* π *if and only if* $\lambda = \alpha_{2m}(\mathbf{a}, \mathbf{b}, \mathbf{d})$ *for some* $m \in \mathbb{N}_0$; *it admits a nontrivial even solution with semi-period* π *if and only if* $\lambda = \alpha_{2m+1}(\mathbf{a}, \mathbf{b}, \mathbf{d})$ *for some* $m \in \mathbb{N}_0$; *it admits a nontrivial odd solution with semi-period* π *if and only if* $\lambda = \beta_{2m+1}(\mathbf{a}, \mathbf{b}, \mathbf{d})$ *for some* $m \in \mathbb{N}_0$; *it admits a nontrivial odd solution with period* π *if and only if* $\lambda = \beta_{2m+2}(\mathbf{a}, \mathbf{b}, \mathbf{d})$ *for some* $m \in \mathbb{N}_0$.

EXAMPLE 2.2.4. To gain some understanding about the notation we consider the almost trivial completely solvable example, the so called Cauchy boundary value problem

$$y''(t) + \lambda y(t) = 0, \tag{2.2.11}$$

subject to the boundary conditions of lemma 2.2.1. We have the following for the eigenvalues λ in terms of $m = 0, 1, 2 \ldots$

1. Even with period π we have $\lambda = \alpha_{2m} = (2m)^2$.
2. Even with semi-period π we have $\lambda = \alpha_{2m+1} = (2m+1)^2$.
3. Odd with semi-period π we have $\lambda = \beta_{2m+1} = (2m+1)^2$.
4. Odd with semi-period π we have $\lambda = \beta_{2m+2} = (2m+2)^2$.

The formal adjoint of the generalized Ince equation is

$$((1 + A(t))\, y(t))'' - (B(t)\, y(t))' + (\lambda + D(t))\, y(t) = 0. \tag{2.2.12}$$

By introducing the functions

$$B^*(t) = 2A'(t) - B(t) = \sum_{j=1}^{\eta} -(2ja_j + b_j)\sin(2jt),$$

$$D^*(t) = D(t) + A'(t) - B''(t) = \sum_{j=1}^{\eta} -\left(4j^2 a_j + 2jb_j - d_j\right)\cos(2jt)$$

we note that the adjoint of (2.1.1) has the same form and can be written in the following form:

$$(1 + A(t))y''(t) + B^*(t)y'(t) + (\lambda + D^*(t))y(t) = 0. \tag{2.2.13}$$

Lemma 2.2.5. *If $y(t)$ is twice differentiable defined on $\mathbb{R}$, then, $y(t)$ is a solution to the generalized Ince equation if and only if $\omega(t)y(t)$ is a solution to its adjoint.*

Proof. We Know that

$$B^* = 2A' - B, \quad D^* = D + A'' - B', \quad \omega' = \frac{B - A'}{1 + A}\omega,$$

and

$$\omega'' = \frac{(B' - A'')(1 + A) - A'(B - A') + (B - A')^2}{(1 + A)^2}\omega.$$

For ease of notation, let

$$p = \frac{B - A'}{1 + A}, \quad q = \frac{(B' - A'')(1 + A) - A'(B - A') + (B - A')^2}{(1 + A)^2},$$

then

$$\begin{aligned}
&(1 + A)(\omega y)'' + B^*(\omega y)' + (\lambda + D^*)(\omega y) \\
&= (1 + A)(\omega'' y + 2\omega' y' + \omega y'') + B^*(\omega' y + \omega y') + (\lambda + D^*)(\omega y) \\
&= (1 + A)(q\omega y + 2p\omega y' + \omega y'') + B^*(p\omega y + \omega y') + (\lambda + D^*)(\omega y).
\end{aligned}$$

Substituting for p, q, B^*, and D^* and simplifying we obtain

$$\begin{aligned}
&(1 + A)(\omega y)'' + B^*(\omega y)' + (\lambda + D^*)(\omega y) \\
&= \omega\left((1 + A)y'' + By' + (\lambda + D)y\right).
\end{aligned}$$

□

From lemma 2.2.5 we know that if y is twice differentiable, y is a solution to the generalized Ince's equation with parameters λ, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{d}$ if and only if ωy is a solution to its formal adjoint. Since the function ω is even with period π, the boundary condition for y and ωy are the same. Therefore we have the following theorem.

THEOREM 2.2.6. *We have for* $m \in \mathbb{N}_0$,

$$\alpha_m (a_j, b_j, d_j) = \alpha_m \left(a_j, -4ja_j - b_j, d_j - 4j^2 a_j - 2jb_j\right), \quad j = 1, 2, \ldots \eta, \tag{2.2.14}$$

$$\beta_{m+1}(a_j, b_j, d_j) = \beta_{m+1}(a_j, -4ja_j - b_j, d_j - 4j^2 a_j - 2jb_j), \quad j = 1, 2, \ldots \eta. \tag{2.2.15}$$

From Sturm-Liouville theory we obtain the following statement on the distribution of eigenvalues.

THEOREM 2.2.7. *The eigenvalues of the generalized Ince equation satisfy the inequalities*

$$\alpha_0 < \begin{Bmatrix} \alpha_1 \\ \beta_1 \end{Bmatrix} < \begin{Bmatrix} \alpha_2 \\ \beta_2 \end{Bmatrix} < \begin{Bmatrix} \alpha_3 \\ \beta_3 \end{Bmatrix} < \cdots \tag{2.2.16}$$

The theory of Hill's equation [**45**] gives the following results

THEOREM 2.2.8. *If* $\lambda \le \alpha_0$ *or* λ *belongs to one of the closed intervals with distinct endpoints* α_m, β_m, $m = 0, 1, 2, \ldots$, *then the generalized Ince equation is unstable. For all other real values of* λ *the equation is stable. In the case*

$$\alpha_m (\mathbf{a}, \mathbf{b}, \mathbf{d}) = \beta_m (\mathbf{a}, \mathbf{b}, \mathbf{d}) \tag{2.2.17}$$

for some positive integer m *and the parameters* $\mathbf{a}, \mathbf{b}, \mathbf{d}$ *the degenerate interval* $[\alpha_m, \beta_m]$ *is not an instability interval: The generalized Ince equation is stable if*

$$\lambda = \alpha_m (\mathbf{a}, \mathbf{b}, \mathbf{d}) = \beta_m (\mathbf{a}, \mathbf{b}, \mathbf{d}) .$$

2.3. Eigenfunctions

By theorem 2.2.3, the generalized Ince's equation with $\lambda = \alpha_{2m}(\mathbf{a}, \mathbf{b}, \mathbf{d})$ admits a non trivial even solution with period π. It is uniquely determined up to a constant factor. We denote this Ince function by $Ic_{2m}(t) = Ic_{2m}(t; \mathbf{a}, \mathbf{b}, \mathbf{d})$ when it is normalized by the conditions $Ic_{2m}(0) > 0$ and

$$\int_0^{\pi/2} (Ic_{2m}(t))^2 dt = \frac{\pi}{4}. \tag{2.3.1}$$

The generalized Ince's equation with $\lambda = \alpha_{2m+1}(\mathbf{a}, \mathbf{b}, \mathbf{d})$ admits a non trivial even solution with semi-period π. It is uniquely determined up to a constant factor. We denote this Ince function by $Ic_{2m+1}(t) = Ic_{2m+1}(t; \mathbf{a}, \mathbf{b}, \mathbf{d})$ when it is normalized by the conditions $Ic_{2m+1}(0) > 0$ and

$$\int_0^{\pi/2} (Ic_{2m+1}(t))^2 dt = \frac{\pi}{4}. \tag{2.3.2}$$

The generalized Ince equation with $\lambda = \beta_{2m+1}(\mathbf{a}, \mathbf{b}, \mathbf{d})$ admits a non trivial odd solution with semi-period π. It is uniquely determined up to a constant factor. We denote this Ince function by $Is_{2m+1}(t) = Is_{2m+1}(t; \mathbf{a}, \mathbf{b}, \mathbf{d})$ when it is normalized by the conditions $\frac{d}{dt} Is_{2m+1}(0) > 0$ and

$$\int_0^{\pi/2} (Is_{2m+1}(t))^2 dt = \frac{\pi}{4}. \tag{2.3.3}$$

The generalized Ince equation with $\lambda = \beta_{2m+2}(\mathbf{a}, \mathbf{b}, \mathbf{d})$ admits a non trivial odd solution with period π. It is uniquely determined up to a constant factor. We denote this Ince function by $Is_{2m+2}(t) = Is_{2m+2}(t; \mathbf{a}, \mathbf{b}, \mathbf{d})$ when it is normalized by the conditions $\frac{d}{dt} Is_{2m+2}(0) > 0$ and

$$\int_0^{\pi/2} (Is_{2m+2}(t))^2 dt = \frac{\pi}{4}. \tag{2.3.4}$$

From Sturm-Liouville theory [**10**, Chapter 8, Theorem 2.1] we obtain the following oscillation properties.

THEOREM 2.3.1. *Each of the function systems*

$$\{Ic_{2m}\}_{m=0}^{\infty}, \tag{2.3.5}$$

$$\{Ic_{2m+1}\}_{m=0}^{\infty}, \tag{2.3.6}$$

$$\{Is_{2m+1}\}_{m=0}^{\infty}, \tag{2.3.7}$$

$$\{Is_{2m+2}\}_{m=0}^{\infty} \tag{2.3.8}$$

is orthogonal over $[0, \pi/2]$ *with respect to the weight* $\omega(t)$*, that is, for* $m \neq n$*,*

$$\int_0^{\pi/2} \omega(t)\, Ic_{2m}(t)\, Ic_{2m}(t)\, dt = 0, \tag{2.3.9}$$

$$\int_0^{\pi/2} \omega(t)\, Ic_{2m+1}(t)\, Ic_{2m+1}(t)\, dt = 0, \tag{2.3.10}$$

$$\int_0^{\pi/2} \omega(t)\, Is_{2m+1}(t)\, Is_{2m+1}(t)\, dt = 0, \tag{2.3.11}$$

$$\int_0^{\pi/2} \omega(t)\, Is_{2m+2}(t)\, Is_{2m+2}(t)\, dt = 0. \tag{2.3.12}$$

Moreover, each of the previous system is complete over $[0, \pi/2]$.

Using the transformations that led to Theorem 2.2.6, we obtain the following result.

THEOREM 2.3.2. *We have*

$$Ic_m(t; \mathbf{a}, \mathbf{b}^*, \mathbf{d}^*) = c_m(\mathbf{a}, \mathbf{b}, \mathbf{d})\, \omega(t; \mathbf{a}, \mathbf{b})\, Ic_m(t; \mathbf{a}, \mathbf{b}, \mathbf{d}) \tag{2.3.13}$$

$$Is_m(t; \mathbf{a}, \mathbf{b}^*, \mathbf{d}^*) = s_m(\mathbf{a}, \mathbf{b}^*, \mathbf{d}^*)\, \omega(t; \mathbf{a}, \mathbf{b})\, Is_m(t; \mathbf{a}, \mathbf{b}, \mathbf{d}) \tag{2.3.14}$$

where $c_m(\mathbf{a},\mathbf{b},\mathbf{d})$ *and* $s_m(\mathbf{a},\mathbf{b},\mathbf{d})$ *are positive and independent of* t*, and*

$$\mathbf{b}^* = \left[b_1^*, b_2^*, \ldots, b_\eta^*\right], \qquad \mathbf{d}^* = \left[d_1^*, d_2^*, \ldots, d_\eta^*\right],$$

with

$$b_j^* = -4ja_j - b_j, \qquad d_j^* = d_j - 4j^2 a_j - 2jb_j, \quad j = 1, 2, \ldots, \eta.$$

The adopted normalization of Ince functions is easily expressible in terms of the Fourier coefficients of Ince functions and so is well suited for numerical computations; However, it has the disadvantage that equations (2.3.13) and (2.3.14) require coefficients c_m and s_m which are not explicitly known.

Of course, once the generalized Ince functions Ic_m and Is_m, are known we can express c_m and s_m in the form

$$c_m(\mathbf{a},\mathbf{b},\mathbf{d}) = \frac{1}{\omega(0;\mathbf{a},\mathbf{b})} \frac{Ic_m(0;\mathbf{a},\mathbf{b}^*,\mathbf{d}^*)}{Ic_m(0;\mathbf{a},\mathbf{b},\mathbf{d})}, \tag{2.3.15}$$

$$s_m(\mathbf{a},\mathbf{b},\mathbf{d}) = \frac{1}{\omega(0;\mathbf{a},\mathbf{b})} \frac{Is_m(0;\mathbf{a},\mathbf{b}^*,\mathbf{d}^*)}{Is_m(0;\mathbf{a},\mathbf{b},\mathbf{d})}. \tag{2.3.16}$$

If we square both sides of (2.3.13)and (2.3.14)and integrate, we find that

$$c_m^2 \int_0^{\pi/2} (\omega(t;\mathbf{a},\mathbf{b}) Ic_m(t;\mathbf{a},\mathbf{b},\mathbf{d}))^2 dt = \pi/4, \tag{2.3.17}$$

$$s_m^2 \int_0^{\pi/2} (\omega(t;\mathbf{a},\mathbf{b}) Is_m(t;\mathbf{a},\mathbf{b},\mathbf{d}))^2 dt = \pi/4. \tag{2.3.18}$$

If $\omega(t;\mathbf{a},\mathbf{b})$ is very simple, then it is possible to evaluate the integrals in (2.3.17), (2.3.18) in terms of the Fourier coefficients of the generalized Ince functions. This provides another way to to calculate c_m and s_m.

Once we know c_m and s_m, we can evaluate the integrals on the left-hand sides of the following equations

$$
\begin{aligned}
& c_m \int_0^{\pi/2} \omega\left(t; \mathbf{a}, \mathbf{b}\right) \left(Ic_m\left(t; \mathbf{a}, \mathbf{b}, \mathbf{d}\right)\right)^2 dt \\
& = \int Ic_m\left(t; \mathbf{a}, \mathbf{b}, \mathbf{d}\right) Ic_m\left(t; \mathbf{a}, \mathbf{b}^*, \mathbf{d}^*\right) dt.
\end{aligned}
\tag{2.3.19}
$$

$$
\begin{aligned}
& s_m \int_0^{\pi/2} \omega\left(t; \mathbf{a}, \mathbf{b}\right) \left(Is_m\left(t; \mathbf{a}, \mathbf{b}, \mathbf{d}\right)\right)^2 dt \\
& = \int Is_m\left(t; \mathbf{a}, \mathbf{b}, \mathbf{d}\right) Is_m\left(t; \mathbf{a}, \mathbf{b}^*, \mathbf{d}^*\right) dt.
\end{aligned}
\tag{2.3.20}
$$

The integrals on the right-hand sides of (2.3.19) and (2.3.20) are easy to calculate once we know the Fourier series of Ince functions.

2.4. Operators and Banded Matrices

In this section we introduce four linear operators associated with equation (2.1.1), and represent them by banded matrices of width $2\eta-1$. It is this simple representation that is fundamental in the theory of the generalized Ince equation. We assume known some basic notions from spectral theory of operators in Hilbert space.

Let H_1 be the Hilbert space consisting of even, locally square-summable functions $f : \mathbb{R} \to \mathbb{C}$ with period π. The inner product is given by

$$
\langle f, g \rangle = \int_0^{\pi/2} f\left(t\right) \overline{g\left(t\right)} dt. \tag{2.4.1}
$$

By restricting functions to $[0, \pi/2]$, H_1 is isometrically isomorphic to the standard $L^2(0, \pi/2)$. We also consider a second inner product

$$
\langle f, g \rangle_\omega = \int_0^{\pi/2} \omega\left(t\right) f\left(t\right) \overline{g\left(t\right)} dt, \tag{2.4.2}
$$

We consider the differential operator

$$
\left(S_1 y\right)\left(t\right) = -\left(1 + A\left(t\right)\right) y''\left(t\right) - B\left(t\right) y'\left(t\right) - D\left(t\right) y\left(t\right). \tag{2.4.3}
$$

The domain $D(S_1)$ of definition of consists of all functions $y \in H_1$ for which y and $y'$are absolutely continuous and $y'' \in H_1$. by restricting functions to $[0, \pi/2]$, this corresponds to the usual domain of a Sturm-Liouville operator associated with the boundary conditions (2.2.1). It is known [**35**, Chapter V, Section 3.6] that S_1 is self-adjoint with compact resolvent when considered as an operator in $(H_1, \langle\, , \,\rangle_\omega)$, and its eigenvalues are $\alpha_{2m}(\mathbf{a}, \mathbf{b}, \mathbf{d})$, $m = 0, 1, 2, \ldots$. All eigenvalues of S_1 are simple. If we consider S_1 as an operator in the Hilbert space$(H_1, \langle\, , \,\rangle)$, then its adjoint S_1^* is given by the operator

$$y \to -((1 + A(t))\, y(t))'' + (B(t)\, y(t))' - D(t)\, y(t),$$

on the same domain $D(S_1)$; see [**35**, Chapter III, Example 5.32]. The adjoint S_1^* is of the same form as S_1 but with $\mathbf{b}$, $\mathbf{d}$ replaced by $\mathbf{b}^*$, $\mathbf{d}^*$, respectively. By Theorem 2.2.6, we see that S_1^* has the same eigenvalues as S_1. Let $\ell^2(\mathbb{N}_0)$ be the space of square-summable sequences $x = \{x_n\}_{n=0}^{\infty}$ with its standard inner product $\langle\, , \,\rangle$. Then

$$(T_1 x) := \frac{x_0}{\sqrt{2}} + \sum_{n=1}^{\infty} x_n \cos(2nt),$$

defines a bijective linear map $T_1 : \ell^2(\mathbb{N}_0) \to H_1$. Consider the operator $M_1 := T_1^{-1} S_1 T_1$ defined on

$$D(M_1) = T_1^{-1}(D(S_1)) = \left\{ x \in \ell^2(\mathbb{N}_0) : \sum_{n=0}^{\infty} n^4 |x_n|^2 < \infty \right\}. \tag{2.4.4}$$

Let e_n denotes the sequence with a 1 in the n^{th} position and 0's in all other positions, we also define $u_n(t) := (T_1 e_n)(t)$, i.e $u_0(t) = \frac{1}{\sqrt{2}}$ and $u_n(t) = \cos(2nt)$ for $n = 1, 2, \ldots$ We find that the operator M_1 can be represented in the following way,

$$M_1 e_n = \begin{cases} \displaystyle\sum_{j=1}^{\eta} \sqrt{2} q_0^j e_j & \text{if } n = 0, \\ \displaystyle r_n e_n + \sum_{j=1}^{\eta} q_{-n}^j \delta_{n-j} e_{|n-j|} + \sum_{j=1}^{\eta} q_n^j e_{n+j} & \text{if } n \geq 1, \end{cases} \tag{2.4.5}$$

where $\delta_0 = \sqrt{2}$ and $\delta_k = 0$ if $k \neq 0$, and $r_n = 4n^2$, $n \in \mathbb{N}$. Note that the factor $\sqrt{2}$ should appear only with e_0.

M_1 is self-adjoint with compact resolvent in $\ell^2(\mathbb{N}_0)$ equipped with the inner product $\langle T_1 x, T_1 y\rangle_\omega$. This inner product generates a norm that is equivalent to the usual $\ell^2(\mathbb{N}_0)$. The operator M_1 has the eigenvalues $\alpha_{2m}(\mathbf{a}, \mathbf{b}, \mathbf{d})$ and the corresponding eigenvectors form sequences of Fourier coefficients for the functions Ic_{2m}.

Now consider the operator S_2 that is defined as S_1 in (2.4.3) but in the Hilbert space H_2 consisting of even functions with semi-period π. This operator has eigenvalues $\alpha_{2m+1}(\mathbf{a}, \mathbf{b}, \mathbf{d})$, with eigenfunctions $Ic_{2m+1}(t)$, $m = 0, 1, 2, \ldots$ Using the basis $\cos(2n+1)t$, $n \in \mathbb{N}_0$, then,

$$(T_2 x)(t) := \sum_{n=0}^{\infty} x_n \cos(2n+1)t$$

defines a bijective linear map $T_2 : \ell^2(\mathbb{N}_0) \to H_2$. Consider the operator $M_2 := T_2^{-1} S_2 T_2$ defined on

$$D(M_2) = T_2^{-1}(D(S_2)) = \left\{ x \in \ell^2(\mathbb{N}_0) : \sum_{n=0}^{\infty} n^4 |x_n|^2 < \infty \right\}.$$

Let $u_n(t) := (T_2 e_n)(t) = \cos(2n+1)t$, for $n = 0, 1, 2, \ldots$, we get the following formula for M_2

$$M_2 e_n = r_n e_n + \sum_{j=1}^{\eta} q_{-n}^{\dagger j} e_{\left|n-j+\frac{1}{2}\right|-\frac{1}{2}} + \sum_{j=1}^{\eta} q_{n+1}^{\dagger j} e_{n+j}, \quad n \geq 0, \tag{2.4.6}$$

where

$$q_n^{\dagger j} = Q_j\left(n - \frac{1}{2}\right), \quad j = 1, 2, \ldots, \eta, \quad r_n = \begin{cases} 1 + q_0^{\dagger j} & \text{if } n = 0, \\ (2n+1)^2 & \text{if } n \geq 1. \end{cases}$$

Now consider the operator S_3 that is defined as S_1 but in the Hilbert space H_3 consisting of odd functions with semi-period π. This operator has the eigenvalues β_{2m+1} with eigenfunctions $Is_{2m+1}(t)$, $m = 0, 1, 2, \ldots$. Using the basis functions

$\sin(2n+1)t$, $n \in \mathbb{N}_0$.

$$(T_3 x)(t) := \sum_{n=0}^{\infty} x_n \sin(2n+1)t$$

defines a bijective linear map $T_3 : \ell^2(\mathbb{N}_0) \to H_3$. Consider the operator $M_3 := T_3^{-1} S_3 T_3$ defined on

$$D(M_3) = T_3^{-1}(D(S_3)) = \left\{ x \in \ell^2(\mathbb{N}_0) : \sum_{n=0}^{\infty} n^4 |x_n|^2 < \infty \right\}.$$

Let $u_n(t) := (T_3 e_n)(t) = \sin(2n+1)t$, for $n = 0, 1, 2, \ldots,$ we have the following formula for M_3,

$$M_3 e_n = r_n^{\dagger} e_n + \sum_{j=1}^{\eta} q_{-n}^{\dagger j} \varepsilon_j e_{\left|n-j+\frac{1}{2}\right|-\frac{1}{2}} + \sum_{j=1}^{\eta} q_{n+1}^{\dagger j} e_{n+j}, \tag{2.4.7}$$

where

$$q_n^{\dagger j} = Q_j\left(n - \frac{1}{2}\right),\ j = 1, 2, \ldots, \eta, \quad r_n^{\dagger} = \begin{cases} 1 - q_0^{\dagger 1} & \text{if } n = 0, \\ (2n+1)^2 & \text{if } n \geq 1, \end{cases}$$

and

$$\varepsilon_j = \begin{cases} 1 & \text{if } n \geq j \\ -1 & \text{if } n < j \end{cases}.$$

Finally, consider the operator S_4 that is defined as S_1 but in the Hilbert space H_4 consisting of odd functions with period π. This operator has the eigenvalues β_{2m+2} with eigenfunctions Is_{2m+2}, $m = 0, 1, 2, \ldots$ Using the basis $\sin(2n+2)t$, $n \in \mathbb{N}_0$,

$$(T_4 x)(t) := \sum_{n=0}^{\infty} x_n \sin(2n+2)t$$

defines a bijective linear map $T_4 : \ell^2(\mathbb{N}_0) \to H_4$. Consider the operator $M_4 := T_4^{-1} S_4 T_4$ defined on

$$D(M_4) = T_4^{-1}(D(S_4)) = \left\{ x \in \ell^2(\mathbb{N}_0) : \sum_{n=0}^{\infty} n^4 |x_n|^2 < \infty \right\}.$$

Let $u_n(t) := (T_4 e_n)(t) = \sin(2n+2)t$, for $n = 0, 1, 2, \ldots$ Then, the formula for M_4 is

$$M_4 e_n = r_n e_n + \sum_{j=1}^{\min(n,\eta)} q_{-n-1}^{j} \varepsilon_j e_{n-j} - \sum_{j=n+2}^{\eta} q_{-n-1}^{j} e_{j-n-2} + \sum_{j=1}^{\eta} q_{n+1}^{j} e_{n+j}, \tag{2.4.8}$$

where

$$r_n = (2n+2)^2, \quad n = 0, 1, 2, \ldots.$$

Example 2.4.1. For the Whittaker-Hill equation (2.1.7) in the following form [**22**]

$$y'' + \left(\lambda + 4\alpha s \cos 2t + 2\alpha^2 \cos 4t\right) y = 0, \quad \alpha \in \mathbb{R},\ s \in \mathbb{N}, \tag{2.4.9}$$

the function $\omega(t)$ from (2.2.6) is equal to 1, therefore the operators S_j, $j = 1, 2, 3, 4$, are self-adjoint on the Hilbert spaces $(H_1, \langle\ ,\ \rangle)$, $j = 1, 2, 3, 4$, respectively. Hence the infinite matrices S_j, $j = 1, 2, 3, 4$, are symmetric. They are represented by

$$M_1 = \begin{pmatrix} 0 & -2\sqrt{2}\alpha s & -\sqrt{2}\alpha^2 & 0 & \cdots \\ -2\sqrt{2}\alpha s & 4-\alpha^2 & -2\alpha s & -\alpha^2 & \cdots \\ -\sqrt{2}\alpha^2 & -2\alpha s & 16 & -2\alpha s & \cdots \\ 0 & -\alpha^2 & -2\alpha s & 36 & \cdots \\ 0 & 0 & -\alpha^2 & -2\alpha s & \cdots \\ 0 & 0 & 0 & -\alpha^2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{2.4.10}$$

$$M_2 = \begin{pmatrix} 1-2\alpha s & -\alpha\,(2s+\alpha) & -\alpha^2 & 0 & \cdots \\ -\alpha\,(2s+\alpha) & 9 & -2\alpha s & -\alpha^2 & \cdots \\ -\alpha^2 & -2\alpha s & 25 & -2\alpha s & \cdots \\ 0 & -\alpha^2 & -2\alpha s & 49 & \cdots \\ 0 & 0 & -\alpha^2 & -2\alpha s & \cdots \\ 0 & 0 & 0 & -\alpha^2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{2.4.11}$$

$$M_3 = \begin{pmatrix} 1+2\alpha s & -\alpha\,(2s-\alpha) & -\alpha^2 & 0 & \cdots \\ -\alpha\,(2s-\alpha) & 9 & -2\alpha s & -\alpha^2 & \cdots \\ -\alpha^2 & -2\alpha s & 25 & -2\alpha s & \cdots \\ 0 & -\alpha^2 & -2\alpha s & 49 & \cdots \\ 0 & 0 & -\alpha^2 & -2\alpha s & \cdots \\ 0 & 0 & 0 & -\alpha^2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{2.4.12}$$

$$M_4 = \begin{pmatrix} 4-\alpha^2 & -2\alpha s & -\alpha^2 & 0 & \cdots \\ -2\alpha s & 16 & -2\alpha s & -\alpha^2 & \cdots \\ -\alpha^2 & -2\alpha s & 36 & -2\alpha s & \cdots \\ 0 & -\alpha^2 & -2\alpha s & 64 & \cdots \\ 0 & 0 & -\alpha^2 & -2\alpha s & \cdots \\ 0 & 0 & 0 & -\alpha^2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{2.4.13}$$

2.5. Fourier Series

The generalized Ince functions admit the following Fourier series expansions

$$Ic_{2m}(t) = \frac{A_0}{\sqrt{2}} + \sum_{n=1}^{\infty} A_{2n} \cos(2nt), \tag{2.5.1}$$

$$Ic_{2m+1}(t) = \sum_{n=0}^{\infty} A_{2n} \cos(2n+1)\,t, \tag{2.5.2}$$

$$Is_{2m+1}(t) = \sum_{n=0}^{\infty} B_{2n+1} \sin(2n+1)\,t, \tag{2.5.3}$$

$$Is_{2m+2}(t) = \sum_{n=0}^{\infty} B_{2n+2} \sin(2n+2)\,t. \tag{2.5.4}$$

We did not indicate the dependence of the Fourier coefficients on m, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$. The normalization of Ince functions implies

$$\sum_{n=1}^{\infty} A_{2n}^2 = 1, \qquad \frac{A_0}{\sqrt{2}} + \sum_{n=1}^{\infty} A_{2n} > 0, \tag{2.5.5}$$

$$\sum_{n=1}^{\infty} A_{2n+1}^2 = 1, \qquad \sum_{n=0}^{\infty} A_{2n+1} > 0, \tag{2.5.6}$$

$$\sum_{n=1}^{\infty} B_{2n+1}^2 = 1, \qquad \sum_{n=0}^{\infty} (2n+1)\, B_{2n+1} > 0, \tag{2.5.7}$$

$$\sum_{n=1}^{\infty} B_{2n+2}^2 = 1, \qquad \sum_{n=0}^{\infty} (2n+1)\, B_{2n+2} > 0. \tag{2.5.8}$$

Using relations (2.3.13) and (2.3.14), we can represent the generalized functions in a different way

$$Ic_m(\mathbf{a}, \mathbf{b}, \mathbf{d}) = \left(\omega(t; \mathbf{a}, \mathbf{b})\, c_m(\mathbf{a}, \mathbf{b}, \mathbf{d})\right)^{-1} Ic_m(\mathbf{a}, \mathbf{b}^*, \mathbf{d}^*), \tag{2.5.9}$$

$$\text{(2.5.10)} \qquad Is_m(\mathbf{a},\mathbf{b},\mathbf{d}) = (\omega(t;\mathbf{a},\mathbf{b})\, s_m(\mathbf{a},\mathbf{b},\mathbf{d}))^{-1}\, Is_m(\mathbf{a},\mathbf{b}^*,\mathbf{d}^*),$$

where

$$b_j^* = -4ja_j - b_j, \qquad d_j^* = d_j - 4j^2a_j - 2jb_j, \qquad j = 1,2,\ldots,\eta.$$

Therefore, we can write

$$\text{(2.5.11)} \qquad Ic_{2m}(\mathbf{a},\mathbf{b},\mathbf{d}) = (\omega(t;\mathbf{a},\mathbf{b}))^{-1}\left(\frac{C_0}{\sqrt{2}} + \sum_{n=1}^{\infty} C_{2n}\cos(2nt)\right),$$

$$\text{(2.5.12)} \qquad Ic_{2m+1}(\mathbf{a},\mathbf{b},\mathbf{d}) = (\omega(t;\mathbf{a},\mathbf{b}))^{-1}\left(\sum_{n=0}^{\infty} C_{2n+1}\cos(2nt)\right),$$

$$\text{(2.5.13)} \qquad Is_{2m+1}(\mathbf{a},\mathbf{b},\mathbf{d}) = (\omega(t;\mathbf{a},\mathbf{b}))^{-1}\left(\sum_{n=0}^{\infty} D_{2n+1}\sin(2nt)\right),$$

$$\text{(2.5.14)} \qquad Is_{2m+2}(\mathbf{a},\mathbf{b},\mathbf{d}) = (\omega(t;\mathbf{a},\mathbf{b}))^{-1}\left(\sum_{n=0}^{\infty} D_{2n+2}\sin(2nt)\right),$$

where

$$C_m = (c_m(\mathbf{a},\mathbf{b},\mathbf{d}))^{-1} A_m, \qquad D_m = (s_m(\mathbf{a},\mathbf{b},\mathbf{d}))^{-1} B_m,$$

and the Fourier coefficients A_n and B_n belong to the parameters $\mathbf{a}$, $\mathbf{b}^*$, $\mathbf{d}^*$. Properties of the coefficients C_n and D_n follow from those of A_n and B_n.

A generalized Ince function is called a generalized Ince polynomial of the first kind if its Fourier series (2.5.1), (2.5.2), (2.5.3), or (2.5.4) terminates. It is called a generalized Ince polynomial of the second kind if its expansion (2.5.11), (2.5.12), (2.5.13), or (2.5.14) terminates. If they exist, These generalized Ince polynomials and their corresponding eigenvalues can be computed from the finite subsections of the matrices M_j, $j = 1,2,3,4$ of Section 2.4.

Example 2.5.1. Consider the equation

$$\text{(2.5.15)} \qquad (1+\cos 2t+\cos 4t)\, y'' + (\sin 2t + \sin 4t)\, y' + \lambda y = 0,$$

one can check that if we set $\lambda = 0$, then any constant function y is an eigenfunction corresponding to the eigenvalue $\alpha_0 = 0$. The adopted normalization of Section 2.3 implies that $Ic_0(t) = \frac{1}{\sqrt{2}}$. It is a generalized Ince polynomial (even with period π).

CHAPTER 3

Ince's Equation

The first known appearance of the Ince equation is in Whittaker's paper [**83**, Equation (5)] on integral equations. Whittaker emphasized the special case $a = 0$, and this special case was later investigated in more detail by Ince [**28, 30**]. Magnus and Winkler's book [**45**] contains a chapter dealing with the coexistence problem for the Ince equation. Also Arscott [**8**] has a chapter on the Ince equation with $a = 0$. Arscott points out that the Ince equation was never considered by Ince and should be called the generalized Ince equation. We use the name Ince equation following the practice in [**45**].

one of the important features of the Ince equation is that the corresponding Ince differential operator when applied to Fourier series can be represented by an infinite tridiagonal matrix. It is this part of the theory that makes the Ince equation particularly interesting. For instance, the coexistence problem which has no simple solution for the general Hill equation has a complete solution for the Ince equation.

In this chapter we further investigate the four infinite matrices of Section 2.4 in the case of Ince's equation . It is their structure that will allow a rigorous discussion of the problems of coexistence of periodic solutions and that of the existence of polynomial solutions.

3.1. Operators and Tridiagonal Matrices

Recall that Ince's equation is:

$$(1 + a\cos(2t))\,y''(t) + (b\sin(2t))\,y'(t) + (\lambda + d\cos(2t))\,y(t) = 0. \tag{3.1.1}$$

In this case relation (2.4.5) reduces to

$$M_1 e_n = \begin{cases} r_0 e_0 + q_0 e_1 & \textbf{if } n = 0 \\ q_{-n} e_{n-1} + r_n e_n + q_n e_{n+1} & \textbf{if } n \geq 1 \end{cases},$$

where

$$r_n := 4n^2, \quad q_n := \begin{cases} \sqrt{2} Q(n) & \text{if } n = -1, 0, \\ Q(n) & \text{otherwise.} \end{cases} \tag{3.1.2}$$

We may represent M_1 by an infinite tridiagonal matrix:

$$M_1 = \begin{pmatrix} r_0 & q_{-1} & 0 & 0 & 0 & 0 & \cdots \\ q_0 & r_1 & q_{-2} & 0 & 0 & 0 & \cdots \\ 0 & q_1 & r_2 & q_{-3} & 0 & 0 & \cdots \\ 0 & 0 & q_2 & r_3 & q_{-4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.1.3}$$

For $x \in D(M_1)$, $M_1 x$ can be computed by the usual multiplication of matrix by column vector. Note that M_1 is self-adjoint with compact resolvent in $\ell^2(\mathbb{N}_0)$ equipped with the inner product $\langle T_1 x, T_1 y \rangle_\omega$. This inner product generates a norm that is equivalent to the usual ℓ^2-norm. The operator M_1 has the eigenvalues $\alpha_{2m}(a, b, d)$ and the corresponding eigenvectors form sequences of Fourier coefficients for the functions Ic_{2m}; see Section 2.5 .

The matrix representation of S_2 from Section 2.4 is

$$M_2 = \begin{pmatrix} r_0^\dagger & q_{-1}^\dagger & 0 & 0 & 0 & 0 & \cdots \\ q_0^\dagger & r_1^\dagger & q_{-2}^\dagger & 0 & 0 & 0 & \cdots \\ 0 & q_1^\dagger & r_2^\dagger & q_{-3}^\dagger & 0 & 0 & \cdots \\ 0 & 0 & q_2^\dagger & r_3^\dagger & q_{-4}^\dagger & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3.1.4}$$

where

$$q_n^{\dagger} := Q\left(n-\frac{1}{2}\right), \quad r_n^{\dagger} := \begin{cases} 1+Q\left(-\frac{1}{2}\right) & if\ n=0, \\ (2n+1)^2 & if\ n\in\mathbb{N}. \end{cases} \tag{3.1.5}$$

The matrix representation M_3 of S_3 is the same as M_2 except for replacing $r_0^{\dagger}$ by $1-Q^{\dagger}(-1/2)$. Finally, the matrix representation M_4 of S_4 is the same as M_1 except that the first row and column in (3.1.3) have to be deleted.

The matrices M_j are instances of the following type of diagonally dominant matrices. Consider the infinite tridiagonal matrix:

$$\begin{pmatrix} \sigma_0 & \tau_1 & 0 & 0 & 0 & 0 & \cdots \\ \rho_1 & \sigma_1 & \tau_2 & 0 & 0 & 0 & \cdots \\ 0 & \rho_2 & \sigma_2 & \tau_3 & 0 & 0 & \cdots \\ 0 & 0 & \rho_3 & \sigma_3 & \tau_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{3.1.6}$$

defined by given real sequences $\{\sigma_n\}_{n=0}^{\infty}$, we assume that the diagonal sequence tends to infinity

$$\lim_{n\to\infty}\sigma_n = +\infty \tag{3.1.7}$$

and that the diagonal is dominant in the following sense: there are $\theta\in(0,1)$ and $n_0\in\mathbb{N}$ such that

$$2\max\left(\tau_{n+1}^2+\rho_n^2, \tau_n^2+\rho_{n+1}^2\right) \le \theta^2\sigma_n^2, \quad n\ge n_0 \tag{3.1.8}$$

The matrix(3.1.6) induces an operator M in the Hilbert space $H=\ell^2(\mathbb{N}_0)$. The domain of definition $D(M)$ of M is

$$D(M) := \left\{\{x_n\}_{n=0}^{\infty}\in H : \sum_{n=0}^{\infty}\sigma_n^2|x_n|^2<\infty\right\},$$

and, for $x \in D(M)$ $(\rho_0 = \tau_0 = 0)$

$$(Mx)_n = \rho_n x_{n-1} + \sigma_n x_n + \tau_{n+1} x_{n+1}.$$

By assumption (3.1.8), $Mx \in H$ is well defined.

LEMMA 3.1.1. *Suppose* (3.1.7) *and* (3.1.8) *hold. The operator M is closed and has compact resolvent. The adjoint M^* of M is is the operator defined in the same manner as M but with ρ_n and τ_n interchanged. The eigenvalues of M and M^* agree.*

PROOF. Let F be the "diagonal part" of M, that is, F is the self-adjoint operator defined by $F(x)_n = \sigma_n x_n$ on $D(F) = D(M)$. Let $G = M - F$ be the "off-diagonal part" of M. By replacing σ_n by $\sigma_n + \omega$ with sufficiently large ω we may assume, without loss of generality, that $\sigma_n > 0$ and that (3.1.8) holds for all $n \geq 0$. Then

$$\|Gx\| \leq \theta \|Fx\|, \qquad \forall x \in D(F). \tag{3.1.9}$$

Since $0 < \sigma_n \to \infty$, F^{-1} exists and is a compact operator. Let $T = GF^{-1} : H \to H$. By (3.1.9), $\|T\| < \theta < 1$. Hence $M^{-1} = F^{-1}(T + I)^{-1}$ is a compact operator; see [**35**, p. 196]. Therefore, M is a closed operator with compact resolvent.

Define N in the same manner as M but with ρ_n, τ_n interchanged. It is easily checked that $\langle Mx, y\rangle = \langle x, Ny\rangle$ for all $x, y \in D(F)$. Since M^{-1} and N^{-1} exist and are bounded operators on H, we conclude that $N = M$. By [**35**, Section III.6.6], the eigenvalues of M are the conjugates of the eigenvalues of M. Since the entries of M are real, this implies that the eigenvalues of M and N agree. □

Eigenvectors of M and M^* are related as follows.

LEMMA 3.1.2. *Suppose* (3.1.7), (3.1.8) *hold, and $\rho_n \neq 0$, $\tau_n \neq 0$ for all $n \in \mathbb{N}$. If $\{x_n\}_{n=0}^{\infty}$ defined by*

$$y_n := \frac{\tau_1 \tau_2 \ldots \tau_n}{\rho_1 \rho_2 \ldots \rho_n} x_n \tag{3.1.10}$$

is an eigenvector of M^* belonging to the same eigenvalue.

Proof. It is easy to verify that formally $M^*y = y$, so it remains to show that y is in the domain of M^*. By Lemma 3.1.2, there is an eigenvector $\{z_n\}$ of M^* belonging to the eigenvalue λ . Since $\tau_n \neq 0$ for all n, $x_0 \neq 0$, and, since $\rho_n \neq 0$ for all n, $z_0 \neq 0$. Therefore, we may assume that $z_0 = x_0$. Then $y_n = z_n$ for all n which shows that y lies in the domain of M^*. □

For a general theory of operators defined by infinite tridiagonal matrices we refer to [**35**, Chapter VII]. The eigenvalue problem for tridiagonal matrices is closely related to the theory of continued fractions and three-term difference equations. We present the definitions and results of these theories that we need in the following section; see also [**56**].

3.2. Three-Term Difference Equations

For given complex sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, consider the three-term recursion

$$x_{n+1} = b_n x_n + a_n x_{n-1}, \quad n \geqslant 1. \tag{3.2.1}$$

A solution $\{x_n\}_{n=0}^{\infty}$ is uniquely determined by its initial values x_0 and x_1. The solutions of (3.2.1) form a two-dimensional linear space. We say that a nontrivial solution $\{u_n\}$ of is recessive if there is a second solution $\{v_n\}$ such that $v_n \neq 0$ for large n and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 0.$$

If $\{u_n\}$, $\{v_n\}$ are like this, then they form a fundamental system of solutions: every solution $\{x_n\}$ has the form

$$x_n = \alpha u_n + \beta v_n.$$

In particular, a recessive solution (if it exists) is uniquely determined up to a constant nonzero factor.

Let $\{C_n\}_{n=0}^{\infty}$, $\{D_n\}_{n=0}^{\infty}$ be the solutions determined by

$$C_0 = 1, \quad C_1 = 0, \quad D_0 = 0, \quad D_1 = 1.$$

Then the finite continued-fraction

$$\frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots + \frac{a_{n-1}}{b_{n-1}}}} = \frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\frac{a_{n-1}}{b_{n-1}} = \frac{C_n}{D_n}$$

is meaningful if $D_n \neq 0$. The infinite continued-fraction

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots \tag{3.2.2}$$

is called convergent ([**56**]) with value $\xi \in \mathbb{C}$ if $D_n \neq 0$ for large n and

$$\lim_{n\to\infty}\frac{C_n}{D_n} = \xi.$$

The continued-fraction diverges unessentially if $C_n \neq 0$, and

$$\lim_{n\to\infty}\frac{D_n}{C_n} = 0.$$

Theorem 3.2.1. *The continued-fraction* (3.2.2) *converges if and only if* (3.2.1) *admits a recessive solution* $\{u_n\}$ *with* $u_0 \neq 0$. *In this case*

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots = -\frac{u_1}{u_0}. \tag{3.2.3}$$

The continued-fraction (3.2.2) *diverges unessentially if and only if* (3.2.1) *admits a recessive solution* $\{u_n\}$ *with* $u_0 = 0$.

Proof. Assume that (3.2.2) converges to ξ. Define $u_n := C_n - \xi D_n$, $v_n := D_n$. Then $\frac{u_n}{v_n} \to 0$ and $u_0 = 1$. If (3.2.2) diverges unessentially then define $u_n := D_n$, and $v_n := C_n$. Then $\frac{u_n}{v_n} \to 0$ and $u_0 = 0$. For the converse statement, let $\{u_n\}$ be a recessive solution of (3.2.1). Let v_n be another solution so that $\frac{u_n}{v_n} \to 0$. Let

$w := v_1u_0 - v_0u_1 \neq 0$, and

$$\alpha = \frac{v_1}{w}, \beta = -\frac{u_1}{w}, \gamma = -\frac{v_0}{w}, \delta = \frac{u_0}{w}.$$

and

$$C_n = \alpha u_n + \beta v_n, \quad D_n = \gamma u_n + \gamma v_n.$$

If $u_0 \neq 0$, then $\delta \neq 0$ and

$$\frac{C_n}{D_n} = \frac{\alpha\frac{u_n}{v_n} + \beta}{\gamma\frac{u_n}{v_n} + \delta} \to \frac{\beta}{\delta} = -\frac{u_1}{u_0}.$$

If $u_0 = 0$, then $\delta = 0, \beta \neq 0$ and $\frac{D_n}{C_n} \to 0$ □

Equation shows that continued-fractions may be used to find recessive solutions.

THEOREM 3.2.2. *Assume that*

$$|b_n| \geq |a_n| + 1 \qquad for\, n \geq 1. \tag{3.2.4}$$

(a) The continued-fraction (3.2.2) *converges to* $\xi \in \mathbb{C}$ *with* $|\xi| \leq 1$. *(b) The difference equation* (3.2.2) *admits a recessive solution* $\{u_n\}$ *with* $|u_1| \leq |u_0| \neq 0$. *(c) For every solution* $\{x_n\}$ *of* (3.2.1) *there is* n_0 *such that* $x_n = 0$ *for* $n \geq n_0$ *or* $x_n \neq 0$ *for* $n \geq n_0$.

PROOF. (*a*) Define

$$E_n := \prod_{j=1}^{n} |a_j|\,, \qquad E_0 := 1.$$

By Induction on n we show that

$$|D_{n+1}| \geq |D_n| + E_n \quad \text{for}\, n \geq 0. \tag{3.2.5}$$

Since $D_1 = 1$, $D_0 = 0$, and $E_0 = 1$, (3.2.5) is true for $n = 0$. Assume (3.2.5) is true for $n = m - 1 \geq 0$. We have

$$\begin{aligned} |D_{m+1}| &\geq |b_m|\,|D_m| - |a_m|\,|D_{m-1}| \\ &\geq (|a_m| + 1)\,|D_m| - |a_m|\,|D_m| + |a_m|\,E_{m-1} \\ &= |D_m| + |a_m|\,E_{m-1} \end{aligned}$$

and so

$$|D_{m+1}| \geq |D_m| + E_m.$$

This proves (3.2.5). Define the Wronskian

$$w_n := C_{n+1}D_n - C_nD_{n+1}.$$

We have

$$\begin{aligned} w_n &= C_{n+1}D_n - C_nD_{n+1} \\ &= (b_nC_n + a_nC_{n-1})\,D_n - C_n\,(b_nD_n + a_nD_{n-1}) \\ &= -a_n\,(C_nD_{n-1} - C_{n-1}D_n) \\ &= -a_nw_{n-1} \end{aligned}$$

and

$$w_n = (-1)^{n-1}\prod_{j=1}^{n} a_j. \tag{3.2.6}$$

We obtain from

$$\frac{C_{n+1}}{D_{n+1}} - \frac{C_n}{Dn} = \frac{w_n}{D_{n+1}D_n}$$

and (3.2.5), (3.2.6) that

$$\left|\frac{C_{n+1}}{D_{n+1}} - \frac{C_n}{Dn}\right| \leq \frac{E_n}{|D_{n+1}D_n|} \leq \frac{|D_{n+1}| - |D_n|}{|D_{n+1}D_n|} \leq \frac{1}{|D_n|} - \frac{1}{|D_{n+1}|}.$$

Hence, for $m > n$,

$$\left|\frac{C_m}{D_m} - \frac{C_n}{Dn}\right| \leq \frac{1}{|D_n|} - \frac{1}{|D_m|}. \tag{3.2.7}$$

Thus

$$\xi = \lim_{n\to\infty} \frac{C_n}{Dn}$$

exists and $|\xi| \leq 1$.(b) Follows from (a) and Theorem 3.2.1.(c) Let $\{x_n\}$ be a solution of (3.2.1). if $x_m = x_{m+1} = 0$, then $x_n = 0$ for all $n \geq m$. If $x_m = 0$, $x_{m+1} \neq 0$, then $x_n \neq 0$ for all $n \geq 0$. □

If (3.2.4) holds for all $n \geq n_0$ and $a_n \neq 0$ for all $n \geq 1$, then we have a recessive solution with $|u_n| \leq |u_{n+1}| \neq 0$ for $n \geq n_0$.

If we assume

$$|b_n| \leq |a_n| + \theta, \qquad n \geq 1 \tag{3.2.8}$$

with a constant $\theta > 1$, then

$$|D_{n+1}| \geq |b_n|\,|D_n| - |a_n|\,|D_{n-1}| \geq (|a_n| + \theta)\,|D_n| - |a_n|\,|D_n| \geq \theta\,|D_{n-1}|\,,$$

gives

$$|D_n| \geq \theta^{n-1} \quad \text{for}\, n \geq 1. \tag{3.2.9}$$

In particular, from (3.2.7) we obtain the error estimate

$$\left|\frac{C_n}{D_n} - \xi\right| \leq \frac{1}{\theta^{n-1}} \quad \text{for}\, n \geq 1. \tag{3.2.10}$$

Theorem 3.2.3. *Assume* (3.2.8) *holds with* $\theta > 1$, *and* $0 \neq a_n \to a \in \mathbb{C}, b_n \to b \in \mathbb{C}$. *Then* $z^2 = bz + a$ *has two solutions* u, v *with* $|u| < 1 < |v|$. *If* $\{x_n\}$ *is a recessive solution of* (3.2.1) *then*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = u.$$

For every nontrivial solution $\{x_n\}$ which is not recessive

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=v.$$

Proof. Let $\{x_n\}$ be a recessive solution, and define $y_n=\frac{x_{n+1}}{x_n}$. Since $|x_n|\le|x_{n+1}|$,we have that $|y_n|\le 1$. Setting $b=u+v$, $a=-uv$, we get

$$\begin{aligned}|y_{n+1}-u| &\ge |y_n|\,|y_{n+1}-u| \\ &= |y_n(b_n-b)+a_n-a+v(y_n-u)| \\ &\ge |v|\,|y_n-u|-|b_n-b|-|a_n-a|\,.\end{aligned}$$

Since $|y_n|\le 1$ this shows that $y_n\to u$.

Set $y_n=\frac{D_{n+1}}{D-n}$, by (3.2.5), $|y_n|\ge 1$. Then

$$\begin{aligned}|y_{n+1}-v| &= \left|b_n-b+\frac{a_n-a}{y_n}+\frac{u}{y_n}(y_n-v)\right| \\ &\le |u|\,|y_n-v|+|b_n-b|+|a_n-a.|\end{aligned}$$

Since $|y_{n+1}|\le|b_n|+|a_n|$, $\{y_n\}$ is bounded. Hence $y_n\to v$. The statement of the theorem follows. □

3.3. Fourier Series

We already know by Section 2.5 that the Ince functions admit the following Fourier series expansions

$$Ic_{2m}(t)=\frac{A_0}{\sqrt{2}}+\sum_{n=1}^{\infty}A_{2n}\cos(2nt)\,, \tag{3.3.1}$$

$$Ic_{2m+1}(t)=\sum_{n=0}^{\infty}A_{2n+1}\cos(2n+1)\,t, \tag{3.3.2}$$

$$Is_{2m+1}(t)=\sum_{n=0}^{\infty}B_{2n+1}\sin(2n+1)\,t, \tag{3.3.3}$$

$$Is_{2m+2}(t) = \sum_{n=0}^{\infty} B_{2n+2} \sin(2n+2)\, t. \tag{3.3.4}$$

The normalization of Ince functions implies

$$\sum_{n=1}^{\infty} A_{2n}^2 = 1, \qquad \frac{A_0}{\sqrt{2}} + \sum_{n=1}^{\infty} A_{2n} > 0, \tag{3.3.5}$$

$$\sum_{n=1}^{\infty} A_{2n+1}^2 = 1, \qquad \sum_{n=0}^{\infty} A_{2n+1} > 0, \tag{3.3.6}$$

$$\sum_{n=1}^{\infty} B_{2n+1}^2 = 1, \qquad \sum_{n=0}^{\infty} (2n+1)\, B_{2n+1} > 0, \tag{3.3.7}$$

$$\sum_{n=1}^{\infty} B_{2n+2}^2 = 1, \qquad \sum_{n=0}^{\infty} (2n+2)\, B_{2n+2} > 0. \tag{3.3.8}$$

We know from Section 2.4 that $\{A_{2n}\}_{n=0}^{\infty}$ is an eigenvector of the infinite matrix M_1 belonging to the eigenvalue α_{2m}. Similarly, $\{A_{2n+1}\}_{n=0}^{\infty}$ is an eigenvector of M_2, $\{B_{2n+1}\}_{n=0}^{\infty}$ is an eigenvector of M_3, and $\{B_{2n+2}\}_{n=0}^{\infty}$ is an eigenvector of M_4. This yields the following difference equations for the Fourier coefficients.

Theorem 3.3.1. *Using the matrix entries* (3.1.3), (3.1.4) *we have*

$$-\alpha_{2m} A_0 + q_{-1} A_2 = 0, \tag{3.3.9}$$

$$q_{n-1} A_{2n-2} + \left(4n^2 - \alpha_{2m}\right) A_{2n} + q_{-n-1} A_{2n+2} = 0,\ n \geq 1, \tag{3.3.10}$$

$$\left(q_0^\dagger + 1 - \alpha_{2m+1}\right) A_1 + q_{-1}^\dagger A_3 = 0, \tag{3.3.11}$$

$$q_n^\dagger A_{2n-1} + \left((2n+1)^2 - \alpha_{2m+1}\right) A_{2n+1} + q_{-n-1}^\dagger A_{2n+3} = 0,\ n \geq 1, \tag{3.3.12}$$

$$\left(q_0^\dagger + 1 - \beta_{2m+1}\right) B_1 + q_{-1}^\dagger B_3 = 0, \tag{3.3.13}$$

$$q_n^\dagger B_{2n-1} + \left((2n+1)^2 - \beta_{2m+1}\right) B_{2n+1} + q_{-n-1}^\dagger B_{2n+3} = 0,\ n \geq 1, \tag{3.3.14}$$

$$\left(4 - \beta_{2m+2}\right) B_2 + q_{-2} B_4 = 0, \tag{3.3.15}$$

$$q_{n-1} B_{2n-2} + \left(4n^2 - \beta_{2m+2}\right) B_{2n} + q_{-n-1} B_{2n+2} = 0,\ n \geq 1 \tag{3.3.16}$$

If Q or $Q^\dagger$ admit integer zeros these difference equations may not allow forward or backward recursion beginning with values for two consecutive Fourier coefficients. Nevertheless, we know from the results in Section 2.4 that the sequences of Fourier coefficients are uniquely determined by the difference equations in Theorem 3.3.1 and the normalizing conditions (3.3.5), (3.3.6), (3.3.7), (3.3.8).

From Section (3.2) we draw the following conclusions.

THEOREM 3.3.2. *Let $\{x_n\}$ be any of the sequences $\{A_{2n}\}$, $\{A_{2n+1}\}$, $\{B_{2n+1}\}$ or $\{B_{2n+2}\}$ of Fourier coefficients of Ince equation. There is n_0 such that either $x_n = 0$ for all $n \geq n_0$ or $x_n \neq 0$ for all $n \geq n_0$. In the latter case, we have*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \begin{cases} \frac{1}{a}\left(\sqrt{1-a^2} - 1\right) & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases} \tag{3.3.17}$$

PROOF. Let $a \neq 0$. By Theorem 3.3.1 we have for $n \geq n_1 > 1$, $\{x_n\}$ satisfies a difference equation of the form (3.2.1) with $a_n \neq 0$ for $n \geq n_1 > 1$. Moreover

$$A_{2n+2} = -\frac{\left(4n^2 - \alpha_{2m}\right)}{q_{-n-1}} A_{2n} - \frac{q_{n-1}}{q_{-n-1}} A_{2n-2},$$

$$B_{2n+2} = -\frac{\left(4n^2 - \beta_{2m+2}\right)}{q_{-n-1}} B_{2n} - \frac{q_{n-1}}{q_{-n-1}} B_{2n+2},$$

$$A_{2n+3} = -\frac{\left((2n+1)^2 - \alpha_{2m+1}\right)}{q_{-n-1}^\dagger} A_{2n+1} - \frac{q_n^\dagger}{q_{-n-1}^\dagger} A_{2n-1},$$

$$B_{2n+3} = -\frac{\left((2n+1)^2 - \beta_{2m+1}\right)}{q^{\dagger}_{-n-1}} B_{2n+1} - \frac{q^{\dagger}_{n}}{q^{\dagger}_{-n-1}} B_{2n-1}.$$

Moreover

$$\lim_{n\to\infty} -\frac{(4n^2 - \alpha_{2m})}{q_{-n-1}} = \lim_{n\to\infty} -\frac{4n^2}{2an^2} = -\frac{2}{a},$$

$$\lim_{n\to\infty} -\frac{\left((2n+1)^2 - \alpha_{2m+1}\right)}{q^{\dagger}_{-n-1}} = \lim_{n\to\infty} -\frac{4n^2}{2an^2} = -\frac{2}{a},$$

$$\lim_{n\to\infty} -\frac{q_{n-1}}{q_{-n-1}} = -1,$$

$$\lim_{n\to\infty} -\frac{q^{\dagger}_{n}}{q^{\dagger}_{-n-1}} = -1.$$

Then $\{x_n\}$ satisfies a difference equation of the (3.2.1) with $a_n \neq 0$ for $n \geq n_1 > 1$, and

$$\lim_{n\to\infty} a_n = -1, \qquad \lim_{n\to\infty} b_n = -\frac{2}{a}.$$

The solutions u, v of $z^2 = -\frac{2}{a}z - 1$ with $|u| < 1 < |v|$ are

$$u = \frac{1}{a}\left(\sqrt{1-a^2} - 1\right), \qquad v = -\frac{1}{a}\left(\sqrt{1-a^2} + 1\right).$$

By Theorem 3.2.2, we must have $x_n = 0$ for all large n or $x_n \neq 0$ for all large n. In the latter case since $\{x_n\} \in \ell^2(\mathbb{N}_0)$, $\{x_n\}$ is a recessive solution of (3.2.1), $n \geq n_1$, and thus $\frac{x_{n+1}}{x_n}$ converges to u by theorem (3.2.3). For $a = 0$, the proof uses the remark after Theorem 3.2.3. □

The case $x_n = 0$ for large n will be considered in more detail in Section 3.4.

Let b^*, d^* be defined as in section 2.3. The Fourier coefficients $\{A_{2n}\}$ of $Ic_{2m}(t; a, b, d)$ form an eigenvector of M_1, whereas the Fourier coefficients $\{A^*_{2n}\}$ of $Ic_{2m}(t; a, b^*, d^*)$ form an eigenvector of M^*_1 belonging to the same eigenvalue $\alpha_{2m}(a, b, d)$. Therefore, up to a constant factor, the sequences $\{A_{2n}\}$ and $\{A^*_{2n}\}$ are related according to Lemma 3.1.2, where ρ_j and σ_j are determined by identifying the matrices (3.1.3) and (3.1.6). We assumed that Q has no integer zero. Similar remarks apply to other Ince functions which yield the following relations

$$A_{2n}^{*} = \frac{q_{-1}q_{-2}\cdots q_{-n}}{q_0 q_1 \cdots q_{n-1}} A_{2n} \tag{3.3.18}$$

$$A_{2n+1}^{*} = \frac{q_{-1}^{\dagger}q_{-2}^{\dagger}\cdots q_{-n}^{\dagger}}{q_1^{\dagger} q_2^{\dagger} \cdots q_{n}^{\dagger}} A_{2n+1}, \tag{3.3.19}$$

$$B_{2n+1}^{*} = \frac{q_{-1}^{\dagger}q_{-2}^{\dagger}\cdots q_{-n}^{\dagger}}{q_1^{\dagger} q_2^{\dagger} \cdots q_{n}^{\dagger}} B_{2n+1}, \tag{3.3.20}$$

$$A_{2n+2}^{*} = \frac{q_{-2}q_{-3}\cdots q_{-n-1}}{q_1 q_2 \cdots q_{n}} A_{2n}. \tag{3.3.21}$$

We now apply results from Section 3.2 to obtain continued-fraction equations for the eigenvalues of Ince's equation.

THEOREM 3.3.3. *(a) Let* $Q(n; a, d, d) \neq 0$ *for all* $n \in \mathbb{Z}$, *and let* $k \in \mathbb{N}_0$. *The eigenvalues* $\lambda = \alpha_{2m}(a, b, d)$, $m \in \mathbb{N}_0$, *are the solutions of the equation*

$$r_k - \lambda - \frac{p_k}{r_{k-1} - \lambda -} \frac{p_{k-1}}{r_{k-2} - \lambda -} \cdots - \frac{p_2}{r_1 - \lambda -} - \frac{p_1}{r_0 - \lambda} \tag{3.3.22}$$

$$= \frac{p_{k+1}}{r_{k+1} - \lambda -} \frac{p_{k+2}}{r_{k+2} - \lambda -} \frac{p_{k+3}}{r_{k+3} - \lambda -} \cdots, \tag{3.3.23}$$

where

$$p_n := q_{-n} q_{n-1},$$

with r_n, q_n *from* (3.1.2). *The left-hand side* (3.3.22) *of the equation is a finite continued-fraction, and the right-hand side* (3.3.23) *is an infinite continued-fraction. If* $k \in \mathbb{N}$, the eigenvalues $\lambda = \beta_{2m+2}(a, b, d)$ $m \in \mathbb{N}_0$, are the solutions of the equation that we obtain by omitting the last fraction $\dfrac{p_1}{r_0 - \lambda}$ in (3.3.22). *(b) Let* $Q^{\dagger}(n; a, d, d) \neq 0$ *for all* $n \in \mathbb{Z}$, *and let* $k \in \mathbb{N}_0$. *The eigenvalues* $\lambda = \alpha_{2m+1}(a, b, d)$,

$m \in \mathbb{N}_0$, *are the solutions of the equation*

$$r_k^\dagger - \lambda - \frac{p_k^\dagger}{r_{k-1}^\dagger - \lambda -} \frac{p_{k-1}^\dagger}{r_{k-2}^\dagger - \lambda -} \cdots - \frac{p_2^\dagger}{r_1^\dagger - \lambda -} - \tag{3.3.24}$$

$$= \frac{p_{k+1}^\dagger}{r_{k+1}^\dagger - \lambda -} \frac{p_{k+2}^\dagger}{r_{k+2}^\dagger - \lambda -} \frac{p_{k+3}^\dagger}{r_{k+3}^\dagger - \lambda -} \cdots, \tag{3.3.25}$$

where

$$p_n^\dagger := q_{-n}^\dagger q_{n-1}^\dagger$$

with $r_n^\dagger$, $q_n^\dagger$ *from* (3.1.5). *The eigenvalues* $\lambda = \beta_{2m+1}(a,b,d)$, $m \in \mathbb{N}_0$, *are the solutions of the equation that we obtain by changing* $r_0^\dagger$ *by* $1 - q_0^\dagger$.

Proof. (a) Consider the sequence $x_n = A_{2n}$ of Fourier coefficients of Ic_{2m}. By Theorem 3.3.1 $\{x_n\}$satisfies

$$(r_0 - \lambda) x_0 + q_{-1} x_1 = 0, \tag{3.3.26}$$

$$q_{n-1} x_{n-1} + (r_n - \lambda) x_n + q_{-n-1} x_{n+1} = 0,\ n \in \mathbb{N}. \tag{3.3.27}$$

Setting

$$z_n := q_{-1} q_{-2} \dots q_{-n} x_n.$$

Substituting in (3.3.26), we have

$$(r_0 - \lambda) z_0 + z_1 = 0,$$

then,

$$z_1 = -(r_0 - \lambda) z_0. \tag{3.3.28}$$

Substituting in (3.3.27), we have

$$q_{n-1}\frac{z_{n-1}}{q_{-1}q_{-2}\dots q_{-n+1}}+(r_n-\lambda)\frac{z_n}{q_{-1}q_{-2}\dots q_{-n+1}q_{-n}}$$
$$+q_{-n-1}\frac{z_{n+1}}{q_{-1}q_{-2}\dots q_{-n+1}q_{-n}q_{n-1}}=0,\ n\in\mathbb{N},$$

which is equivalent to

$$q_{n-1}z_{n-1}+(r_n-\lambda)\frac{z_n}{q_{-n}}+\frac{z_{n+1}}{q_{-n}}=0,\ n\in\mathbb{N},$$

therefore, we have

$$z_{n+1}=-(r_n-\lambda)z_n-p_nz_{n-1},\ n\in\mathbb{N}. \tag{3.3.29}$$

Since $\{z_n\}$ is a recessive solution of (3.3.29), formula (3.2.3) applied to (3.3.29) for $n>k$ gives

$$\frac{p_{k+1}}{r_{k+1}-\lambda-}\frac{p_{k+2}}{r_{k+2}-\lambda-}\frac{p_{k+3}}{r_{k+3}-\lambda-}\dots=-\frac{z_{k+1}}{z_k}. \tag{3.3.30}$$

(3.3.29) also gives

$$-\frac{z_{k+1}}{z_k}=r_k-\lambda+\frac{p_n}{\frac{z_k}{z_{k-1}}}, \tag{3.3.31}$$

From (3.3.28) and (3.3.31) for $n=1,2,...,k$, we obtain the finite continued-fraction

$$-\frac{z_{k+1}}{z_k}=r-\lambda-\frac{p_k}{r_{k-1}-\lambda-}\frac{p_{k-1}}{r_{k-2}-\lambda-}\dots\frac{p_2}{r_1-\lambda-}\frac{p_1}{r_0-\lambda}. \tag{3.3.32}$$

Now (3.3.30), (3.3.32) yield the desired equation. By retracing the steps, we see that this equation has no other solutions than α_{2m}, $m\in\mathbb{N}_0$. □

Using estimate (3.2.10), one can show that the continued fractions appearing in Theorem 3.3.3 are meromorphic functions of λ.

If Q or $Q^{\dagger}$ have integer zeros, then we may use Theorem 2.7.1 to derive continued-fraction equations for the eigenvalues.

3.4. Ince Polynomials

An Ince function is called an Ince polynomial (of the first kind) if its corresponding Fourier series (3.3.1), (3.3.2), (3.3.3) or (3.3.4) terminates. Clearly, such solutions exist if and only if the corresponding infinite matrix M_j has at least one zero in the diagonal below its main diagonal. We now investigate this possibility in more detail.

For $k \in \mathbb{N}$, let $M_{1,k}$ be the principal $k \times k$ submatrix in the north west corner of M_1. Further, let $L_{1,k}$ be the principal submatrix of M_1 complementary to $M_{1,k}$, that is

$$L_{1,k} = \begin{pmatrix} r_k & q_{-k-1} & 0 & 0 & 0 & \dots \\ q_k & r_{k+1} & q_{-k-2} & 0 & 0 & \dots \\ 0 & q_{k+1} & r_{k+2} & q_{-k-3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.4.1}$$

We consider $L_{1,k}$ as an operator in $\ell^2(\mathbb{N}_k)$, $\mathbb{N}_k = \{k, k+1, \dots\}$, defined for sequences $\{x_n\}_{n=k}^{\infty}$ with $\sum_{n=k}^{\infty} n^4 |x_n|^2 < \infty$. We may apply Lemma 3.1.1 to this operator. Similarly, we define matrices $M_{j,k}$, $L_{j,k}$ for $k = 2, 3, 4$.

We will need the following Theorem for the proof of the Theorem 3.4.2

THEOREM 3.4.1. *The eigenvalues α_m and β_m are real-analytic functions of* (a, b, d).

PROOF. We already know by Lemma 2.2.5 that Ince's equation may be written in the formally self-adjoint form

$$-((1 + a\cos 2t)\,\omega(t)\,y')' - d(\cos 2t)\,\omega(t)\,y = \lambda\omega(t)\,y, \tag{3.4.2}$$

where

$$\omega(t) := \begin{cases} (1 + a\cos 2t)^{-1-b/2a} & \text{if } a \neq 0, \\ \exp\left(\frac{-b}{2}\cos 2t\right) & \text{if } a = 0. \end{cases}$$

Let $y(t) = y(t; a, b, d)$ be the solution of Ince's equation with $y(0) = 1$, $y'(0) = 0$. By a theorem on analytic parameter dependence, $f(\lambda, a, b, d) = y'(\pi/2; \lambda, a, b, d)$ is a real-analytic function of its four variables. We have

$$f\left(\alpha_{2m}\left(a,b,d\right),a,b,d\right)=0.$$

Let $z\left(t\right):=\partial\lambda/\partial\lambda$ If we differentiate (3.4.2) with respect to λ, we obtain a differential equation for z. We multiply this equation by y and subtract equation (3.4.2) multiplied by z. Then we integrate between 0 and $\pi/2$ and obtain

$$\left(1-a\right)\omega\left(\pi/2\right)\left(y'\left(\pi/2\right)z\left(\pi/2\right)-y\left(\pi/2\right)z'\left(\pi/2\right)\right)=\int_0^{\pi/2}\omega\left(t\right)y\left(t\right)^2dt.$$

This implies

$$-\left(1-a\right)\omega\left(\pi/2\right)\frac{\partial f}{\partial\lambda}\left(\alpha_{2m}\left(a,b,d\right),a,b,d\right)=\int_0^{\pi/2}\omega\left(t\right)y\left(t\right)^2dt\neq 0,$$

where

$$y\left(t\right)=y\left(t;\alpha_{2m}\left(a,b,d\right),a,b,d\right).$$

Hence, by the implicit function theorem, $\alpha_{2m}\left(a,b,d\right)$ depends analytically on $\left(a,b,d\right)$. In a similar way, we show that the other eigenvalue functions are real-analytic. □

THEOREM 3.4.2. *(a) Let p be an integer zero of Q, and let k be defined by $k:=\frac{1}{2}+\left|\frac{1}{2}+p\right|$. The eigenvalues of $M_{1,k}$ are $\alpha_{2m}\left(a,b,d\right)$, $m=0,1,2,\ldots,k-1$, and The eigenvalues of $L_{1,k}$ are $\alpha_{2m}\left(a,b,d\right)$, $m=k,k+1,\ldots$ Moreover if $k\geq 2$, The eigenvalues of $M_{4,k}$ are $\beta_{2m+2}\left(a,b,d\right)$, $m=0,1,2,\ldots,k-2$, and The eigenvalues of $L_{4,k}$ are $\beta_{2m+2}\left(a,b,d\right)$, $m=k-1,k,\ldots$ (b) Let p be an integer zero of $Q^{\dagger}$, and let k be defined by $k:=\left|p\right|$. The eigenvalues of $M_{2,k}$ are $\alpha_{2m+1}\left(a,b,d\right)$, $m=0,1,2,\ldots,k-1$, and The eigenvalues of $L_{2,k}$ are $\alpha_{2m+1}\left(a,b,d\right)$, $m=k,k+1,\ldots$ The eigenvalues of $M_{3,k}$ are $\beta_{2m+1}\left(a,b,d\right)$, $m=0,1,2,\ldots,k-1$, and The eigenvalues of $L_{3,k}$ are $\beta_{2m+1}\left(a,b,d\right)$, $m=k,k+1,\ldots$*

PROOF. We only prove the statement concerning M_1, the proofs for the other matrices are similar. One of the entries q_{k-1}, q_{-k} in M_1 is zero. We assume first that $q_{k-1}=0$. We abbreviate $M:=M_1$, $K:=M_{1,k}$, $L:=L_{1,k}$.

Observation 1) : K and L do not have a common eigenvalue.

Assume that $Ky = \lambda y$ and $Lz = \lambda z$ with $y = (y_0, y_1, \ldots, y_{k-1}) \neq 0$ and $z = (z_k, z_{k+1}, \ldots) \neq 0$. These vectors should be thought of as column vectors. Since $q_{k-1} \neq 0$, the vector $x = (y_0, y_1, \ldots, y_{k-1,}, 0, 0, \ldots)$ satisfies $Mx = \lambda x$. Assume there is a vector $u = (u_0, \ldots, u_{k-1})$ such that $(k - \lambda)\, u = h$ with $h = (0, \ldots, 0, -q_{-k} z_k)$. Then $v = (u_0, \ldots, u_{k-1}, z_k, z_{k+1}, \ldots)$ satisfies $Mv = \lambda v$. This is impossible since x, v are linearly independent and the eigenvalues of M are simple. On the other hand, if h does not belong to the range of$K - \lambda$, then there are δ and a vector $u = (u_0, \ldots, u_{k-1})$ such that $(K - \lambda)\, u = y + \delta h$. Note that the rank of $K - \lambda$ is $k - 1$ because the eigenspaces of M and thus of K are one-dimensional. It follows that the vector $v = (u_0, \ldots, u_{k-1}, \delta z_{k-1}, \delta z_k, \ldots)$ satisfies $(M - \lambda)\, v = x$. This shows that the root space of M corresponding to the eigenvalue λ has dimension at least 2 which again is impossible because the eigenvalues of M are simple.

Observation 2): The set of eigenvalues of M is the union of the set of eigenvalues of K and the set of eigenvalues of L.

Let $Mx = x$ with $x = (x_0, x_1, \ldots) \neq 0$. Then $z = (z_k, z_{k+1}, \ldots)$ satisfies $Lz = z$. If $x_n \neq 0$ for some $n > k$, then λ is an eigenvalue of L. If $x_n = 0$ for all $n \geq k$, then $(x_0, x_1, \ldots, x_{k-1})$ is an eigenvector of K corresponding to the eigenvalue λ . Let $Ky = \lambda y$ with $y = (y_0, y_1, \ldots, y_{k-1}) \neq 0$. Then $x = (y_0, \ldots, y_{k-1}, 0, 0, \ldots)$ satisfies $Mx = \lambda x$. Hence λ is an eigenvalue of M. Let $Lz = \lambda z$ with $z = (z_k, z_{k+1}, \ldots) \neq 0$. By 1), one can find a vector $(z_0, z_1, \ldots, z_{k-1})$ such that $x = (z_0, z_1, \ldots)$ satisfies $Mx = \lambda x$. Hence is an eigenvalue of M. This completes the proof of 2).

Let $t \in (0, 1]$ Since $Q(\mu; ta, tb, td) = tQ(\mu; a, b, d)$, the zeros of $Q(\cdot; ta, tb, td)$ are independent of t. Hence we may use the results 1), 2) also for ta, tb, td in place of a, b, d. Let $K(t)$ be the matrix K with each element in its subdiagonal and superdiagonal multiplied by t. For every $t \in [0, 1]$, the eigenvalues of $K(t)$ lie in the set $\{\alpha_{2m}(ta, tb, td) : m \in \mathbb{N}_0\}$. Moreover, the eigenvalues of $K(0)$ are $\alpha_{2m}(0, 0, 0) = 4m^2$, $m = 0, 1, ..., k-1$. By Theorem 3.4.1, the functions $\lambda_{2m}(t) = \alpha_{2m}(ta, tb, td)$ are continuous, and we know that $\lambda_0(t) < \lambda_1(t) < \lambda_2(t) < \ldots$ Since the eigenvalues of $K(t)$ depend continuously on t [**35**, Chapter 2], we obtain that $K(t)$ has the eigenvalues $\alpha_{2m}(ta, tb, td)$, $m = 0, 1, ..., k-1$. In particular, the eigenvalues of K are $\alpha_{2m}(a, b, d)$, $m = 0, 1, ..., k-1$. By 1) and 2), it follows that the eigenvalues of L are $\alpha_{2m}(a, b, d)$, $m = k, k+1, ...$ This completes the proof of the theorem if $q_{k-1} = 0$.

If $q_{-k} = 0$ then we consider M^* in place of M. We complete the proof by using Lemma 3.1.1 and the results obtained in the first part of the proof. $\square$

We are now in a position to characterize all Ince polynomials (of the first kind.)

THEOREM 3.4.3. *(a) If $Q(p; a, b, d) \neq 0$ for all $p \in \mathbb{N}_0$, then none of the Ince functions $Ic_{2m}(t; a, b, d)$ and $Is_{2m+2}(t; a, b, d)$ is an Ince polynomial. If $a = b = d = 0$, all of these functions are Ince polynomials. Otherwise set*

$$k := \max\{p \in \mathbb{N}_0 : Q(p) = 0\} + 1.$$

Then $Ic_{2m}(t; a, b, d)$ is an Ince polynomial if and only if $m \in \{0, 1, 2, \ldots, k-1\}$, and $Is_{2m}(t; a, b, d)$ is an Ince polynomial if and only if $m \in \{0, 1, 2, \ldots, k-2\}$. (b) If $Q^\dagger(p; a, b, d) \neq 0$ for all $p \in \mathbb{N}_0$, then none of the Ince functions $Ic_{2m+1}(t; a, b, d)$ and $Is_{2m+1}(t; a, b, d)$ is an Ince polynomial. If $a = b = d = 0$, all of these functions are Ince polynomials. Otherwise set

$$k^\dagger := \max\left\{p \in \mathbb{N} : Q^\dagger(p) = 0\right\}.$$

Then $Ic_{2m}(t; a, b, d)$ is an Ince polynomial if and only if $m \in \left\{0, 1, 2, \ldots, k^\dagger - 1\right\}$, and $Is_{2m}(t; a, b, d)$ is an Ince polynomial if and only if $m \in \left\{0, 1, 2, \ldots, k^\dagger - 1\right\}$.

PROOF. Since the proofs for Ic_{2m}, Ic_{2m+1}, Is_{2m+1}, Is_{2m+2} are similar, we consider only Ic_{2m}. If Ic_{2m} is written in the form with a terminating series, then there is $p \in \mathbb{N}_0$ Such that $A_{2p} \neq 0$, and $A_{2n} = 0$, for $n > p$. By equation (3.3.10) of Theorem (3.3.1), we have

$$q_p A_{2p} + \left(4(p+1)^2 - \alpha_{2m}\right) A_{2p+2} + q_{-p-2} A_{2p+4} = 0,$$

which implies $q_p = Q(p) = 0$, and α_{2m} is an eigenvalue of $M_{1,p+1}$. By Theorem (3.4.2) $m \in \{0, 1, \ldots, p\}$. By definition of of k we have $p < k$. Thus $m \in \{0, 1, \ldots, k-1\}$. Conversely, assume $Q(p) = 0$ with $p \in \mathbb{N}_0$. By Theorem 3.4.2, the eigenvalues of $M_{1,p+1}$are α_{2m}, $m = 0, 1, \ldots, p$, if $(x_0, x1, \ldots, x_p)$ is a corresponding eigenvector, then

$$Ic_{2m}(t) = \frac{x_0}{\sqrt{2}} + \sum_{n=1}^{p} x_n \cos(2nt)$$

up to a constant factor. So Ic_{2m} is an Ince polynomial. □

Ince polynomials and their eigenvalues can be computed from the eigenvalues and eigenvectors of a finite tridiagonal matrix $M_{j,k}$.

EXAMPLE 3.4.4. Let $a < 1$, $b \in \mathbb{R}$, $p \in \mathbb{N}_0$, by setting $Q(p) = 0$, and solving for the parameter d, we can construct an Ince differential equation that have polynomial solutions Ic_{2m} and Is_{2m+2}. Solving $Q(p) = 0$ for d gives $d = 4ap^2 - 2bp$. For example when $a = \frac{1}{2}$, $b = 1$, and $p = 2$ we get $d = 4$. By Theorem 3.4.3 we have that $k = p + 1 = 3$. The corresponding matrix $M_{1,k}$ is

$$M_{1,3} = \begin{pmatrix} 0 & 0 & 0 \\ -2\sqrt{2} & 4 & 4 \\ 0 & -2 & 16 \end{pmatrix},$$

with eigenvalues

$$\alpha_{2m} = \left\{0, 10 - 2\sqrt{7}, 10 + 2\sqrt{7}\right\}.$$

The entries of eigenvectors of $M_{1,3}$ are the coefficient of the associated Ince polynomials Ic_{2m}. That is,

$$\alpha_0 = 0, \quad Ic_0 = 9 + 8\cos 2t + \cos 4t,$$

$$\alpha_2 = 10 - 2\sqrt{7}, \quad Ic_2 = \frac{4}{6 - 2\sqrt{7}}\cos 2t + \cos 4t,$$

$$\alpha_4 = 10 + 2\sqrt{7}, \quad Ic_2 = \frac{4}{6 + 2\sqrt{7}}\cos 2t + \cos 4t.$$

To find Ince polynomials Is_{2m+2}, we consider the submatrix $M_{4,k-1}$,

$$M_{4,2} = \begin{pmatrix} 4 & 4 & 0 \\ -2 & 16 & 10 \\ 0 & 0 & 36 \end{pmatrix},$$

then

$$\beta_{2m+2} = \left\{10 - 2\sqrt{7}, 10 + 2\sqrt{7}\right\},$$

we find two more Ince polynomials

$$\beta_2 = 10 - 2\sqrt{7}, \quad Is_2 = \frac{4}{6 - 2\sqrt{7}} \sin 2t + \sin 4t,$$
$$\beta_4 = 10 + 2\sqrt{7}, \quad Is_4 = \frac{4}{6 + 2\sqrt{7}} \sin 2t + \sin 4t.$$

Let M_1 have a zero in its superdiagonal and let k be defined as in Theorem 3.4.2. Then its adjoint M_1^* has a zero in its subdiagonal. Using Theorem 2.3.2, eigenvectors of of $M_{1,k}^*$ lead to Ince functions which are products of $(\omega(t; a, d, d))^{-1}$and a trigonometric polynomial. Such Ince functions are Ince polynomials of the second kind. For such solutions there holds an obvious analogue of Theorem 3.4.3. Moreover, if we consider an eigenvector $\{x_n\}_{n=k}^{\infty}$ of $L_{1,k}$ and define $x_n = 0$ for $n = 0, 1, 2, \ldots, k-1$, then we obtain an eigenvector of M_1. This leads to Ince functions with a Fourier series whose first k coefficients are zero. Similar remarks apply to the matrices M_j, $j = 2, 3, 4$.

Let us look at some examples.

Example 3.4.5. We choose

$$a = 1/2, \qquad b = 0, \qquad d = 2. \tag{3.4.3}$$

Then $Q(\mu) = \mu^2 - 1$, has zeros ± 1. From Theorem 3.4.3(a) we find three Ince polynomials of the first kind:

$$\begin{aligned} \alpha_0 &= 0, \quad Ic_0(t) = \frac{1}{3}(2 + \cos 2t), \\ \alpha_2 &= 4, \quad Ic_2(t) = \cos 2t, \\ \beta_2 &= 4, \quad Is_2(t) = \sin 2t. \end{aligned}$$

We know that

$$\omega\left(t; \frac{1}{2}, 0, 2\right) = \frac{1}{\left(1 + \frac{1}{2}\cos 2t\right)} = \frac{2}{(2 + \cos 2t)},$$

then Ic_0 is also an Ince polynomial of the second kind:

$$Ic_0(t) = \frac{2}{3}\left(\omega\left(t; \frac{1}{2}, 0, 2\right)^{-1}\right).$$

Example 3.4.6. Now consider

$$a = 1/\sqrt{3}, \qquad b = 6a, \qquad d = -8a. \tag{3.4.4}$$

Then $Q(\mu) = 2a(\mu - 1)(\mu - 2)$ has zeros $\mu = -1, 2$. By Theorem 3.4.3(a), there are five Ince polynomials :

$$\begin{aligned} \alpha_0 &= -4, \quad Ic_0(t) = \frac{1}{\sqrt{7}}\left(\sqrt{3} - \cos 2t\right), \\ \alpha_2 &= 8, \quad Ic_2(t) = \frac{1}{\sqrt{10}}\left(\sqrt{3} + 2\cos 2t\right), \\ \alpha_4 &= 16, \quad Ic_0(t) = \frac{1}{\sqrt{7}\sqrt{13}}\left(3 + 4\sqrt{3}\cos 2t + 5\cos 4t\right), \\ \beta_2 &= 4, \quad Is_2(t) = \sin 2t, \\ \beta_4 &= 16, \quad Is_4(t) = \frac{1}{\sqrt{7}}\left(2\sqrt{3}\sin 2t + \sqrt{3}\sin 4t\right). \end{aligned}$$

There are no Ince polynomials of the second kind for the choice (3.4.3).

3.5. The Coexistence Problem

Ince's equation admits two linearly independent solutions with period π if and only if $\lambda = \alpha_{2m}(a,b,d) = \beta_{2m}(a,b,d)$ for some $m \in \mathbb{N}$. Ince's equation admits two linearly independent solutions with semi-period π if and only if $\lambda = \alpha_{2m+1}(a,b,d) = \beta_{2m+1}(a,b,d)$ for some $m \in \mathbb{N}_0$. In Theorem 3.5.2 we determine all values of m, a, b, d for which $\alpha_m(a,b,d) = \beta_m(a,b,d)$. More generally, we determine the sign of $\alpha_m - \beta_m$ in Theorem 3.5.4.

Theorem 3.5.1. *If $Q(\mu; a,b,d)$ has no integer zero then*

$$\alpha_{2m}(a,b,d) \neq \beta_{2m}(a,b,d) \qquad \text{for all } m \in \mathbb{N}.$$

If $Q^{\dagger}(\mu; a,b,d)$ has no integer zero then

$$\alpha_{2m+1}(a,b,d) \neq \beta_{2m+1}(a,b,d) \qquad \textit{for all } m \in \mathbb{N}_0.$$

Proof. Since the proofs of the two statements are similar, it will be sufficient to prove the first. Assume, if possible, that $\alpha_{2m} = \beta_{2m}$ for some $m \in \mathbb{N}$. Consider the Fourier coefficients $\{A_{2n}\}$, $\{B_{2n}\}$ of Ic_{2m} and Is_{2m}, respectively. By Theorem 3.3.1 the sequences $x_n = A_{2n}$ and $y_n = B_{2n}$ satisfy the same difference equation of the form (3.2.1) for $n \geq 3$. By assumption $a_n \neq 0$ for all n. Since both solutions are recessive, there is a constant c such that $A_{2n} = cB_{2n}$ for all $n \in \mathbb{N}$. from Theorem 3.3.1, equations (3.3.10), (3.3.15) for $n = 1$ yield

$$q_0 A_0 + (4 - \alpha_{2m}) A_2 + q_{-2} A_4 = 0,$$

$$(4 - \beta_{2m+2}) A_2 + q_{-2} A_4 = 0.$$

So $A_0 = 0$, and $A_{2n} = 0$ for all $n \in \mathbb{N}$, which is a contradiction. □

Theorem 3.5.2. *(a) Let $Q(\mu;a,b,d)=0$ have at least one integer root μ, and let ℓ be defined by*

$$\ell := \frac{1}{2} + \min\left\{\left|\frac{1}{2}+\mu\right| : \mu\in\mathbb{Z}\, with\, Q(\mu;a,b,d)=0\right\}. \tag{3.5.1}$$

Then

$$\alpha_{2m}(a,b,d) \neq \beta_{2m}(a,b,d)\ if\ m=1,2,\ldots,\ell-1$$

$$\alpha_{2m}(a,b,d) = \beta_{2m}(a,b,d)\ if\ m=\ell,\ell+1,\ldots$$

(b) Let $Q^{\dagger}(\mu;a,b,d)=0$ have at least one integer root μ, and let $\ell^{\dagger}$ be defined by

$$\ell^{\dagger} := \min\left\{|\mu| : \mu\in\mathbb{Z}\, with\, Q^{\dagger}(\mu;a,b,d)=0\right\}. \tag{3.5.2}$$

Then

$$\alpha_{2m+1}(a,b,d) \neq \beta_{2m+1}(a,b,d)\ if\ m=0,1,2,\ldots,\ell^{\dagger}-1$$

$$\alpha_{2m+1}(a,b,d) = \beta_{2m+1}(a,b,d)\ if\ m=\ell^{\dagger},\ell^{\dagger}+1,\ldots$$

Proof. (a) We apply Theorem 3.4.2. Since $L_{1,\ell}=L_{4,\ell-1}$, $\alpha_{2m}(a,b,d)=\beta_{2m}(a,b,d)$ if $m\geq\ell$. The matrix $M_{4,\ell-1}$ is obtained from $M_{1,\ell}$ by deleting the first row and the first column. Since $M_{1,\ell}$ is a finite tridiagonal matrix with nonzero entries in its subdiagonal and superdiagonal, $M_{1,\ell}$ and $M_{4,\ell-1}$ have no common eigenvalues. Therefore $\alpha_{2m}(a,b,d)\neq\beta_{2m}(a,b,d)$ if $m=1,2,\ldots\ell-1$. (b) Similarly by Theorem 3.4.2, $L_{2,\ell^{\dagger}}=L_{3,\ell^{\dagger}}$, so $\alpha_{2m+1}(a,b,d)=\beta_{2m+1}(a,b,d)$ if $m\geq\ell^{\dagger}$. The matrices $M_{2,\ell^{\dagger}}$ and $M_{3,\ell^{\dagger}}$ are the same except for the first entree. Since both matrices are finite tridiagonal with nonzero entries in their subdiagonals and superdiagonals, $M_{2,\ell^{\dagger}}$and $M_{3,\ell^{\dagger}}$ have no common eigenvalues. Therefore $\alpha_{2m+1}(a,b,d)\neq\beta_{2m+1}(a,b,d)$ if $m=0,1,2,\ldots\ell^{\dagger}-1$. $\square$

Example 3.5.3. Consider Mathieu's equation

$$y"(t)+(\lambda-2q\cos(2t))\,y(t)=0, \tag{3.5.3}$$

where q is a nonzero real number.

We have,

$$Q(\mu) = Q^{\dagger}(\mu) = q.$$

Both polynomials have no zeros. By Theorem 3.4.3, equation (3.5.3) does not have any polynomial solution. Applying Theorem 3.5.2, we see that $\alpha_{2m} \neq \beta_{2m}$ and $\alpha_{2m+1} \neq \beta_{2m+1}$ for all m.

We define sign$x$ for a real number x by $-1, 0, 1$ according to $x < 0$, $x = 0$, or $x > 0$, respectively.

THEOREM 3.5.4. *We have, for* $m \in \mathbb{N}$,

$$\text{sign}\left(\alpha_{2m}(a,b,d) - \beta_{2m}(a,b,d)\right) = \text{sign} \prod_{n=-m}^{m-1} Q(n;a,b,d), \tag{3.5.4}$$

and, for $m \in \mathbb{N}_0$,

$$\text{sign}\left(\alpha_{2m+1}(a,b,d) - \beta_{2m+1}(a,b,d)\right) = \text{sign} \prod_{n=-m}^{m} Q^{\dagger}(n;a,b,d). \tag{3.5.5}$$

PROOF. Since the proofs of (3.5.4) and (3.5.5) are similar, we will only prove (3.5.4). If $Q(n) = 0$ for at least one $n = -m, -m+1, \ldots, m-1$, then Theorem 3.5.2 implies (3.5.4). Hence we assume that

$$\prod_{n=-m}^{m-1} Q(n;a,b,d) \neq 0. \tag{3.5.6}$$

By Theorems 3.5.1 and 3.5.2, (3.5.6) is equivalent to

$$\alpha_{2m}(a,b,d) \neq \beta_{2m}(a,b,d).$$

Since $Q(\mu;ta,tb,td) = tQ(\mu;a,b,d)$, we obtain that

$$F(t) = \alpha_{2m}(ta,tb,td) - \beta_{2m}(ta,tb,td) \tag{3.5.7}$$

is either positive for all $t \in (0,1]$ or negative for all $t \in (0,1]$ By Theorem (3.3.3)(a) with $k=m$, the eigenvalue $\alpha_{2m}(ta,tb,td)$ satisfies the continued-fraction equation

$$f(\lambda,t) = g(\lambda,t), \tag{3.5.8}$$

where

$$f(\lambda,t) = r_m - \lambda - \frac{t^2 p_m}{r_{m-1}-\lambda-}\frac{t^2 p_{m-1}}{r_{m-2}-\lambda-}\cdots - \frac{t^2 p_2}{r_1-\lambda-} - \frac{t^2 p_1}{r_0-\lambda}$$

$$g(\lambda,t) = \frac{t^2 p_{m+1}}{r_{m+1}-\lambda-}\frac{t^2 p_{m+2}}{r_{m+2}-\lambda-}\cdots$$

Similarly, $\beta_{2m}(ta,tb,td)$ satisfies the continued-fraction equation

$$f_1(\lambda,t) = g(\lambda,t), \tag{3.5.9}$$

where

$$f_1(\lambda,t) = r_m - \lambda - \frac{t^2 p_m}{r_{m-1}-\lambda-}\frac{t^2 p_{m-1}}{r_{m-2}-\lambda-}\cdots - \frac{t^2 p_2}{r_1-\lambda}.$$

Claim 1: There is $\delta > 0$ such that, for $t \in (0,\delta)$, $f(\lambda,t) - g(\lambda,t)$ and $f_1(\lambda,t) - g(\lambda,t)$ are decreasing functions of $\lambda \in I$, where

$$I = [r_m - 1, r_m + 1].$$

To prove Claim 1, note that $f(\lambda,t) = r_m - \lambda - t^2 f_0(\lambda,t)$, where $f_0(\lambda,t)$ is analytic for $\lambda \in I$ and $t \in (-\delta,\delta)$ if $\delta > 0$ is sufficiently small. Hence $\frac{df}{d\lambda} \to -1$ as $t \to 0$ uniformly for $\lambda \in I$. The same is true for $f_1(\lambda,t)$. We have $g(\lambda,t) = t^2 g_0(\lambda,t)$, where $g_0(\lambda,t)$ is analytic for $\lambda \in I$ and $t \in (-\delta,\delta)$. Therefore, $\frac{dg}{d\lambda} \to 0$ as $t \to 0$ uniformly for $\lambda \in I$. Hence, $\frac{d(f-g)}{d\lambda} \to -1$, and $\frac{d(f_1-g)}{d\lambda} \to -1$ as $t \to 0$ uniformly for $\lambda \in I$. This establishes Claim 1.

Claim 2: There is $\delta > 0$ such that, for $t \in (0,\delta)$, and $\lambda \in I$,

$$\operatorname{sign}(f(\lambda,t) - f_1(\lambda,t)) = \operatorname{sign}\prod_{n=1}^{m} p_n.$$

To prove Claim 2, note that

$$\operatorname{sign}\frac{t^2 p_1}{r_0-\lambda} = -\operatorname{sign} p_1.$$

If $m=1$, this proves Claim 2. Assume $m \geq 2$. For $0<t<\delta$, with δ sufficiently small we have

$$r_1-\lambda<0, \qquad r_1-\lambda-\frac{t^2 p_1}{r_{0-\lambda}}<0.$$

Hence

$$\operatorname{sign}\left(\frac{t^2 p_2}{r_1-\lambda-\frac{t^2 p_1}{r_0-\lambda}}-\frac{t^2 p_2}{r_1-\lambda}\right) = -\operatorname{sign}\left(p_1 p_2\right).$$

If $m=2$, this proves Claim 2. Continuing in this way we obtain Claim 2 for every m.

We now prove (3.5.4) . Choose $\delta>0$ such that the statements of Claim 1 and Claim 2 hold. Also choose δ so small so that $\alpha_{2m}(ta,tb,td)$ and $\beta_{2m}(ta,tb,td)$ lie in I for $t\in(0,\delta)$. This is possible since $\alpha_{2m}(ta,tb,td)$ and $\beta_{2m}(ta,tb,td)$ are continuous functions of t and their common value at $t=0$ is r_m. By Claim 1, for every $t\in(0,\delta)$, the functions $f(\lambda,t)-g(\lambda,t)$and $f_1(\lambda,t)-g(\lambda,t)$ are decreasing for $\lambda\in I$. They have the zeros $\alpha_{2m}(ta,tb,td)$ and $\beta_{2m}(ta,tb,td)$, respectively. Now Claim 2 yields

$$\operatorname{sign} F(t) = \operatorname{sign}\prod_{n=1}^{m} p_n$$

for $t\in(0,\delta)$. Since $F(t)$ is either positive for all $t\in(0,1]$ or negative for all $t\in(0,1]$, we obtain (3.5.4). □

Note that Theorems 3.5.1 and 3.5.2 are contained in Theorem 3.5.4. Theorem 3.5.4 was proved in [**78**] based on previous results in [**45**].

Example 3.5.5. Choosing $a=1/2$, $b=0$, $d=1$, $m=5$. We obtain,

$$\begin{aligned}
\operatorname{sign}(\alpha_{10}-\beta_{10}) &= -1,\\
\operatorname{sign}(\alpha_{11}-\beta_{11}) &= 1.
\end{aligned}$$

For the Mathieu equation (3.5.3), Theorem 3.5.4 yields

$$\begin{aligned}
\operatorname{sign}(\alpha_{2m}-\beta_{2m}) &= \operatorname{sign}\prod_{n=-m}^{m-1} Q(n) \\
&= \operatorname{sign} q^{2m} \\
&= \operatorname{sign} q.^2
\end{aligned}$$

$$\begin{aligned}
\operatorname{sign}(\alpha_{2m+1}-\beta_{2m+1}) &= \operatorname{sign}\prod_{n=-m}^{m} Q(n) \\
&= \operatorname{sign} q^{2m+1} \\
&= \operatorname{sign} q.
\end{aligned}$$

This is [**48**, Satz 12, page 119].

For the Whittaker-Hill equation (2.1.7), we obtain

$$\begin{aligned}
\operatorname{sign}(\alpha_{2m}-\beta_{2m}) &= \operatorname{sign}\prod_{n=-m}^{m-1} Q(n) \\
&= \operatorname{sign}\prod_{n=-m}^{m-1} 2q(2\mu-\nu+1) \\
&= \operatorname{sign} q^2 \prod_{n=-m}^{m-1} (2\mu-\nu+1)
\end{aligned}$$

$$\begin{aligned}
\operatorname{sign}(\alpha_{2m+1}-\beta_{2m+1}) &= \operatorname{sign}\prod_{n=-m}^{m} Q(n) \\
&= \operatorname{sign}\prod_{n=-m}^{m} (2\mu-\nu+1) \\
&= \operatorname{sign} q \prod_{n=-m}^{m} (2\mu-\nu+1).
\end{aligned}$$

This result improves [**45**, Theorem 7.9].

We may summarize results on (nontrivial) Ince polynomials (of the first or second kind) and coexistence of solutions with period π for the Ince equations as follows.

(1) $Q(\mu)$ has no integer zero. Then there are no Ince polynomials, and no coexistence of solutions with period π occurs.

(2) $Q(\mu)$ has exactly one integer zero. Define ℓ by (3.5.1) Then coexistence of solutions with period π occurs for $\lambda = \alpha_{2m} = \beta_{2m}$ with $m \geq \ell$ and Ince polynomials exist for $\lambda = \alpha_{2m}$, $m = 0, 1, ..., \ell{-}1$ and for $\lambda = \beta_{2m}$, $m = 1, 2, ..., \ell{-}1$ These polynomials are of the first kind if the integer zero of Q is nonnegative and of the second kind if it is negative. In particular, linearly independent Ince polynomials do not coexist, and an Ince polynomial cannot be of the first and second kind simultaneously.

(3) $Q(\mu)$ has precisely two integer zeros. Define ℓ by (3.5.1), and define k in the same way but with min replaced by max. Then coexistence of solutions with period occurs for $\lambda = \alpha_{2m} = \beta_{2m}$, $m \geq \ell$ and Ince polynomials exist for for $\lambda = \alpha_{2m}$, $m = 0, 1, ..., k{-}1$ and for $\lambda = \beta_{2m}$. $m = 1, 2, ..., k{-}1$. In particular, if $\lambda = \alpha_{2m} = \beta_{2m}$ with $m = \ell, \ell+1..., k-1$, then Ince's equations admits a fundamental system of solutions consisting of Ince polynomials. If one of the zeros of Q is nonnegative while the other is negative, there are Ince polynomials which are of the first and second kind simultaneously but such solutions do not coexist with another solution with period .

(4) $a = b = d = 0$. Then Ince's equations is trivial. All solutions with period π are Ince polynomials, and there is coexistence for all eigenvalues except $\lambda{=}\alpha_0$.

We may summarize results on (nontrivial) Ince polynomials (of the first or second kind) and coexistence of solutions with semi-period π for the Ince equations as follows.

(1) $Q^{\dagger}(\mu)$ has no integer zero. Then there are no Ince polynomials, and no coexistence of solutions with semi-period π occurs.

(2) $Q^{\dagger}(\mu)$ has exactly one integer zero. Define $\ell^{\dagger}$ by (3.5.2) Then coexistence of solutions with semi-period π occurs for $\lambda = \alpha_{2m+1} = \beta_{2m+1}$ with $m \geq \ell^{\dagger}$ and Ince polynomials exist for $\lambda = \alpha_{2m+1}$, $m = 0, 1, ..., \ell^{\dagger}{-}1$ and for $\lambda = \beta_{2m+1}$,

$m = 0, 1, ..., \ell^{\dagger}-1$ These polynomials are of the first kind if the integer zero of Q is nonnegative and of the second kind if it is negative. In particular, linearly independent Ince polynomials do not coexist, and an Ince polynomial cannot be of the first and second kind simultaneously.

(3) $Q^{\dagger}(\mu)$ has precisely two integer zeros. Define $\ell^{\dagger}$ by (3.5.2), and define $k^{\dagger}$ in the same way but with min replaced by max. Then coexistence of solutions with semi-period occurs for $\lambda = \alpha_{2m+1} = \beta_{2m+1}$, $m \geq \ell^{\dagger}$ and Ince polynomials exist for for $\lambda = \alpha_{2m+1}$, $m = 0, 1, ..., k^{\dagger}-1$ and for $\lambda = \beta_{2m+1}$, $m = 0, 1, ..., k^{\dagger}-1$. In particular, if $\lambda = \alpha_{2m+1} = \beta_{2m+1}$ with $m = \ell^{\dagger}, \ell^{\dagger}+1..., k^{\dagger}-1$, then Ince's equations admits a fundamental system of solutions consisting of Ince polynomials. If one of the zeros of Q is nonnegative while the other is negative, there are Ince polynomials which are of the first and second kind simultaneously but such solutions do not coexist with another solution with semi-period π.

(4) $a = b = d = 0$. Then Ince's equations is trivial. All solutions with semi-period π are Ince polynomials, and there is coexistence for all eigenvalues.

EXAMPLE 3.5.6. We choose

$$a = \frac{1}{2}, \quad b = -3, \quad d = -4.$$

Then $Q(\mu) = (\mu + 1)(\mu + 2)$ has zeros $\mu = -1$, and $\mu = -2$. We find that $\ell = 1$, $k = 2$. We may draw the following conclusions:

(1) Coexistence of solutions with period occurs for $\lambda = \alpha_{2m} = \beta_{2m}$, $m \geq 1$.

(2) Ince polynomials exist for $\lambda = \alpha_{2m}$, $m = 0, 1$.

$$\alpha_0 = 0, \qquad Ic_0(t) = -1 + \cos 2t,$$

$$\alpha_2 = 4, \qquad Ic_2(t) = \cos 2t.$$

(3) Ince polynomials exist for $\lambda = \beta_{2m}$, $m = 1$.

$$\beta_2 = 4, \qquad Is_2(t) = \cos 2t.$$

(4) The functions $Ic_2(t) = \cos 2t$ and $Is_2(t) = \sin 2t$ coexist for the eigenvalue $\lambda = \alpha_2 = \beta_2 = 4$, and constitute a fundamental system of solutions.

3.6. Separation of Variables

The partial differential equation describing vibrations of an elliptic membrane whose density diminishes radially from the center can by solved by the method of separation of variables in elliptic coordinates; see [**29**]. One is led to ordinary differential equations that can be transformed to Ince's equation with $a = 0$.

We will obtain the Ince equation directly by considering the following partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2b\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) - 2du = 0, \tag{3.6.1}$$

where b and d are real constants. In elliptic coordinates

$$x = \cos\varphi \cosh\xi, \qquad y = \sin\varphi \sinh\xi, \tag{3.6.2}$$

the equation assumes the form

$$\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial \xi^2} + b\left(\sin 2\varphi \frac{\partial u}{\partial \varphi} - \sinh\xi \frac{\partial u}{\partial \xi}\right) + d\left(\cos 2\varphi - \cosh\xi\right) u = 0 \tag{3.6.3}$$

We separate variables $u(\varphi, \xi) = u_1(\varphi)\, u_2(\xi)$, to obtain

$$\begin{aligned} &u_2(\xi)\left(\frac{d^2 u_1}{d\varphi^2} - 2\left(b\sin\varphi\right)\frac{du_1}{d\varphi} + 2\left(d\cos\varphi\right)u_1\right) \\ &+ u_1(\varphi)\left(\frac{d^2 u_2}{d\xi^2} - 2\left(b\sinh\xi\right)\frac{du_2}{d\xi} + 2\left(d\cos\xi\right)u_2\right) = 0, \end{aligned} \tag{3.6.4}$$

which is equivalent to

$$\text{(3.6.5)} \qquad \frac{\frac{d^2u_1}{d\varphi^2} + (b\sin\varphi)\,\frac{du_1}{d\varphi} + (d\cos\varphi)\,u_1}{u_1\,(\varphi)} = -\frac{\frac{d^2u_2}{d\xi^2} - (b\sinh\xi)\,\frac{du_2}{d\xi} - (d\cosh\xi)\,u_2}{u_2\,(\xi)}.$$

Setting both sides equal to the separation constant $-\lambda$, we obtain the following two ordinary differential equations in each of the variables φ and ξ,

$$\text{(3.6.6)} \qquad \frac{d^2u_1}{d\varphi^2} + (b\sin\varphi)\,\frac{du_1}{d\varphi} + (\lambda + d\cos\varphi)\,u_1 = 0,$$

$$\text{(3.6.7)} \qquad \frac{d^2u_2}{d\xi^2} - (b\sinh\xi)\,\frac{du_2}{d\xi} - (\lambda + d\cosh\xi)\,u_2 = 0.$$

Equation (3.6.6) is an Ince equation, and (3.6.7) is a modified Ince equation. Using the substitution $\varphi = i\xi$ in (3.6.6), we obtain

$$\begin{aligned} \frac{d^2v}{d\varphi^2} + (b\sin\varphi)\,\frac{dv}{d\varphi} + (\lambda + d\cos\varphi)\,v &= -\frac{d^2v}{d\xi^2} + (b\sinh\xi)\,\frac{du_1}{d\xi} + (\lambda + d\cosh\xi)\,v \\ &= -\left(\frac{d^2v}{d\xi^2} - (b\sinh\xi)\,\frac{dv}{d\xi} - (\lambda + d\cosh\xi)\,v\right) \\ &= 0, \end{aligned}$$

then, equation (3.6.7) is obtained by substituting $\varphi = i\xi$ in (3.6.6).

Ince polynomials lead to polynomial solutions of (3.6.1).

EXAMPLE 3.6.1. Choose

$$b = \sqrt{3}, \qquad d = -2\sqrt{3}, \qquad \lambda = -2,$$

then $u_1\,(\varphi) = \sqrt{3} - \cos 2\varphi$ solves (3.6.6), so $u_2\,(\xi) = \sqrt{3} - \cosh 2\xi$ solves (3.6.7), and

$$\text{(3.6.8)} \qquad u = \left(\sqrt{3} - \cos 2\varphi\right)\left(\sqrt{3} - \cosh 2\varphi\right) = 2 + \left(2 - \sqrt{3}\right)x^2 - \left(2 + \sqrt{3}\right)y^2$$

satisfies (3.6.1).

To derive the Ince equation with $a \neq 0$ by the method of separation of variables we proceed as follows. Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \left(1 + \frac{b}{a}\right) \frac{1}{z} \frac{\partial u}{\partial z} = 0, \tag{3.6.9}$$

where a and b are real constants with a and real constant with $a \in (-1, 0)$. We introduce sphero-conal coordinates in the half-space $z > 0$ by

$$x = rk \cos\varphi \cosh\xi, \tag{3.6.10}$$

$$y = r\frac{k}{k'} \cos\varphi \cosh\xi, \tag{3.6.11}$$

$$z = r\frac{1}{k'} \left(1 - k^2 \cos^2\varphi\right)^{1/2} \left(1 - k^2 \cos^2\xi\right)^{1/2}, \tag{3.6.12}$$

where

$$r > 0, \qquad 0 \leq \varphi < 2\pi, \qquad 0 < \xi < \operatorname{arcosh}\frac{1}{k}.$$

The numbers k,$k' \in (0, 1)$ are determined by

$$k^2 = \frac{2a}{1-a}, \qquad k'^2 = 1 - k^2.$$

The coordinate surfaces are spheres $x^2+y^2+z^2 = r^2$ and elliptic cones. In spheroconal coordinates equation (3.6.9) becomes

$$\begin{aligned} &(1 + a\cos 2\varphi) \frac{\partial^2 u}{\partial \varphi^2} + (1 + a\cosh 2\xi) \frac{\partial^2 u}{\partial \xi^2} + b\sin 2\varphi \frac{\partial u}{\partial \varphi} \\ &- b\sinh 2\xi \frac{\partial u}{\partial \xi} + a(\cos 2\varphi - \cosh 2\xi) \left(r^2 \frac{\partial^2 u}{\partial r^2} + \left(1 - \frac{b}{a}\right) r \frac{\partial u}{\partial r}\right) = 0. \end{aligned} \tag{3.6.13}$$

We separate variables $u = v(\varphi, \xi)\, u_3(r)$ to obtain

$$\begin{aligned} &\frac{(1 + a\cos 2\varphi) \frac{\partial^2 v}{\partial \varphi^2} + (1 + a\cosh 2\xi) \frac{\partial^2 v}{\partial \xi^2} + b\sin 2\varphi \frac{\partial v}{\partial \varphi} - b\sinh 2\xi \frac{\partial v}{\partial \xi}}{a(\cosh 2\xi - \cos 2\varphi)\, v} \\ &= \frac{r^2 \frac{d^2 u_3}{dr^2} + \left(1 - \frac{b}{a}\right) r \frac{du_3}{dr}}{u_3}. \end{aligned} \tag{3.6.14}$$

Using the separation constant $\frac{d}{a}$, (3.6.14) separates into

$$(1+a\cos 2\varphi)\frac{\partial^2 v}{\partial\varphi^2}+(1+a\cosh 2\xi)\frac{\partial^2 v}{\partial\xi^2} + b\sin 2\varphi\frac{\partial v}{\partial\varphi}-b\sinh 2\xi\frac{\partial v}{\partial\xi}+d\left(\cos 2\varphi-\cosh 2\xi\right)v=0, \tag{3.6.15}$$

$$r^2\frac{d^2u_3}{dr^2}+\left(1-\frac{b}{a}\right)r\frac{du_3}{dr}-\frac{d}{a}u_3=0. \tag{3.6.16}$$

Next, we separate variables $v=u_1(\varphi)\,u_2(\xi)$ in equation (3.6.15) to obtain

$$\frac{(1+a\cos 2\varphi)\frac{d^2u_1}{d\varphi}+b\sin 2\varphi\frac{du_1}{d\varphi}+(d\cos 2\varphi)\,u_1}{u_1} = -\frac{(1+a\cosh 2\xi)\frac{d^2u_2}{d\xi}-b\sinh 2\xi\frac{du_2}{d}-(d\cosh 2\xi)\,u_2}{u_2}. \tag{3.6.17}$$

Setting both sides of (3.6.17) equal to $-\lambda$ we obtain

$$(1+a\cos 2\varphi)\frac{d^2u_1}{d\varphi}+b\sin 2\varphi\frac{du_1}{d\varphi}+(\lambda+d\cos 2\varphi)\,u_1=0, \tag{3.6.18}$$

$$(1+a\cosh 2\xi)\frac{d^2u_1}{d\varphi}-b\sinh 2\xi\frac{du_1}{d\varphi}-(\lambda+d\cos 2\xi)\,u_1=0. \tag{3.6.19}$$

The function $u=u_1(\varphi)\,u_2(\xi)\,u_3(r)$ will satisfy (3.6.9) if there are constants d and λ such that $u_1(\varphi)$ solves the Ince equation (3.6.18), $u_2(\xi)$ solves the modified Ince's equation (3.6.19), and $u_3(r)$ solves the Euler equation (3.6.16). If $a\nu^2-b\nu-d=0$, that is, $Q\left(\frac{\nu}{2}\right)$, then equation (3.6.16) admits the solution r^ν.

Ince polynomials lead to solutions of(3.6.9) which are homogeneous polynomials in x, y, z.

Example 3.6.2. Let,

$$a=-1/\sqrt{3},\qquad b=6a,\qquad d=-8a,\qquad \lambda=-4,$$

then $u_1(\varphi) = \sqrt{3}+\cos 2\varphi$ solves (3.6.18), and so $u_2(\varphi) = \sqrt{3}+\cosh 2\xi$ solves (3.6.19). $Q\left(\frac{\nu}{2}\right) = 0$, has solution $\nu = 2$, therefore $u_3(r) = r^2$solves (3.6.6), and the function

$$u = r^2\left(\sqrt{3}+\cos 2\varphi\right)\left(\sqrt{3}+\cosh 2\xi\right) = \left(6+2\sqrt{3}\right)x^2 + \left(6-2\sqrt{3}\right)y^2 + 2z^2$$

satisfies (3.6.9).

Ince polynomials for $a \neq 0$ are related to special cases of Heun polynomials which again are special cases of Heine-Stieltjes polynomials [**66**, Section 6.8], and, for Heine-Stieltjes polynomials, the corresponding process of separation of variables is treated in [**77**]. The homogeneous polynomials solving (3.6.9) generalize classical spherical harmonics, and a theory parallel to that for spherical harmonics can be created for them.

3.7. Integral Equation for Ince Polynomials

We derive Whittaker's integral equations for Ince polynomials [**84**].

Consider the differential operators

$$Su := -\left(1+a\cos 2s\right)\frac{\partial^2 u}{\partial s^2} - b\sin 2s\frac{\partial u}{\partial s} - d\left(\cos 2s\right)u, \tag{3.7.1}$$

$$Tu := -\left(1+a\cos 2t\right)\frac{\partial^2 u}{\partial t^2} - b\sin 2t\frac{\partial u}{\partial t} - d\left(\cos 2t\right)u. \tag{3.7.2}$$

If $u_1(s)$ and $u_2(t)$ are solutions of the same Ince equation (2.1.4), then $u(s,t) = u_1(s)\,u_2(t)$ satisfies the partial differential equation

$$Su = Tu. \tag{3.7.3}$$

We now apply a well known method to derive integral equations; see [**59**, Section 1.2]. The kernel of the integral equations will be suitable solutions of (3.7.3). Let $n \in \mathbb{N}_0$ be such that $Q(n/2) = 0$. We notice that $(x+iy)^n$ is a solution of the partial differential equations (3.6.1) and (3.6.9). Therefore, by transforming to elliptic and sphero-conal coordinates, respectively, we find that

$$K_n(s,t;a) = \left(\sqrt{1-a}\sin s\sin t\sqrt{1+a}\cos s\cos t\right)^n \tag{3.7.4}$$

satisfies (3.7.3). Now, consider the Fredholm integral operator

$$(Fv)(s) := \int_{-\pi/2}^{\pi/2} \omega(t,a;b)\, K_n(s,t;a)\, v(t)\, dt \tag{3.7.5}$$

which is self-adjoint on the Hilbert space $L^2_\omega(-\pi/2,\pi/2)$ with the weight ω from (2.2.6). The operator has rank $n+1$. Therefore, F has exactly $n+1$ nonzero eigenvalues counted according to multiplicity.

THEOREM 3.7.1. *(a) Let $Q(n/2)) = 0$ with even $n \in \mathbb{N}_0$. The Ince polynomials $Ic_{2m}(t;a,b,d)$, $m = 0,1,...,n/2$, and $Is_{2m}(t;a,b,d)$, $m = 1,2,...,n/2$, are eigenfunctions of the integral operator* (3.7.5) *corresponding to nonzero eigenvalues.*

(b) Let $Q(n/2) = 0$ with odd $n \in \mathbb{N}_0$. The Ince polynomials $Ic_{2m+1}(t;a,b,d)$, $m = 0,1,...,n/2$, and $Is_{2m+1}(t;a,b,d)$, $m = 1,2,...,n/2$, are eigenfunctions of the integral operator (3.7.5) *corresponding to nonzero eigenvalues.*

PROOF. We prove only (a), the proof of (b) being similar. Let $v(t)$ be a solution of Ince's equation with period π . Since K_n satisfies(3.7.3),

$$SFv(s) = \int_{-\pi/2}^{\pi/2} SK_n(s,t)\,\omega(t)\,v(t)\,dt = \int_{-\pi/2}^{\pi/2} TK(s,t)\,\omega(t)\,v(t)\,dt.$$

Taking into account that $K_n(s,\cdot)$, ω and v have period π, integration by parts gives.

$$SFv(s) = \int_{-\pi/2}^{\pi/2} K_n(s,t)\,\tilde{T}(\omega v)(t)\,dt,$$

where $\tilde{T}$ is the formal adjoint of T, that is, the operator we obtain from T by replacing b and d by $b = -4a - b$ and $d = d - 4a - 2b$, respectively. By Theorem 2.3.2, $\tilde{T}(\omega v) = \lambda\omega v$. Hence $SFv = \lambda Fv$ which means that Fv solves the same Ince equation as v. Clearly, Fv has period π and is even or odd when v is even or odd, respectively. This shows that the functions Ic_{2m}, Is_{2m+2}, $m \in \mathbb{N}_0$, are eigenfunctions of F. By

Theorem 2.3.1, these functions form a complete orthogonal set of eigenfunctions for F in $L^2_\omega(-\pi/2, \pi/2)$. Moreover, F maps these functions to trigonometric polynomials of degree at most n. Therefore, the eigenvalues corresponding to Ic_{2m} and Is_{2m} vanish for $m > n/2$, and the eigenvalues are nonzero for the remaining $n+1$ Ince polynomials.

□

3.8. The Lengths of Stability and instability intervals

If $\alpha_n < \beta_n$ or $\beta_n < \alpha_n$ then $[\alpha_n, \beta_n]$ or $[\beta_n, \alpha_n]$ is the $n-$th instability interval of Ince's equation (3.1.1). For fixed a, b, d the signed length

$$\alpha_n(\tau a, \tau b, \tau d) - \beta_n(\tau a, \tau b, \tau d), \qquad n = 1, 2, 3, \ldots \tag{3.8.1}$$

of the nth instability interval is an analytic function of $\tau \in (-1, 1)$ which can be be expanded in a power series about $t = 0$.

Example 3.8.1. Consider the case $a = b = 0$, $d = -2$ (Mathieu's equation) and take $n = 5$. Then

$$\alpha_n(\tau a, \tau b, \tau d) = 25 + \frac{1}{48}\tau^2 + \frac{11}{774144}\tau^4 + \frac{1}{147456}\tau^5 + \frac{37}{891813888}\tau^6 + \ldots$$
$$\beta_n(\tau a, \tau b, \tau d) = 25 + \frac{1}{48}\tau^2 + \frac{11}{774144}\tau^4 - \frac{1}{147456}\tau^5 + \frac{37}{891813888}\tau^6 + \ldots$$

The coefficients of these power series can be computed with software like Maple. We see that power series expansions agree up to the term including t^4. There are no explicit formulas known for those coefficients of these power series.

For example, the signed length of the $5-$th instability interval is given by the difference

$$\alpha_5(0, 0, -2\tau) - \beta_5(0, 0, -2\tau) = \frac{1}{73728}\tau^5 + \frac{7}{169869312}\tau^7 + \ldots \tag{3.8.2}$$

Levy and Keller [**43**] proved the following result for the leading term of the length of instability intervals of example 3.8.1

$$\alpha_m(0,0,-2\tau) - \beta_m(0,0,-2\tau) = \frac{2\tau^m}{(2^{m-1}(m-1)!)^2}(1+O(\tau^2)). \tag{3.8.3}$$

The goal in this section is to generalize the previous result to Ince's equation.

3.8.1. Instability intervals for odd m. Let a, b, d be given real numbers with $|a| < 1$, and Consider as in Chapter 2 the Ince operator

$$Iy(t) = -(1+\tau a\cos 2t)\,y''(t) - \tau b(\sin 2t)\,y'(t) - \tau d(\cos 2t)\,y(t).$$

We represent the operator I by infinite tridiagonal matrices. If I is applied to to Fourier series of the form

$$x(t) = \sum_{n=0}^{\infty} x_k \cos(2n+1)\,t, \tag{3.8.4}$$

we obtain the matrix representation

$$M_2(\tau) = \begin{pmatrix} r_0^\dagger + \tau q_0^\dagger & \tau q_{-1}^\dagger & 0 & 0 & 0 & 0 & \cdots \\ \tau q_0^\dagger & r_1^\dagger & \tau q_{-2}^\dagger & 0 & 0 & 0 & \cdots \\ 0 & \tau q_1^\dagger & r_2^\dagger & \tau q_{-3}^\dagger & 0 & 0 & \cdots \\ 0 & 0 & \tau q_2^\dagger & r_3^\dagger & \tau q_{-4}^\dagger & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3.8.5}$$

where

$$r_n^\dagger = (2n+1)^2, \qquad n = 0,1,2,\ldots$$
$$q_j^\dagger = Q(j-\frac{1}{2}), \qquad Q(\mu) = 2a\mu^2 - b\mu - \frac{d}{2}.$$

The operator $A(\tau)$, $\tau \in (0,1)$ defines an unbounded self-adjoint operator in the sequence Hilbert space $\ell^2(\mathbb{N}_0)$ equipped with the inner product

$$(3.8.6) \quad \begin{aligned} \langle x, y \rangle_{A,\omega} &= \int_0^{\frac{\pi}{2}} \omega(t) \left(\sum_{n=0}^{\infty} x_n \cos(2n+1)t \right) \overline{\left(\sum_{n=0}^{\infty} y_n \cos(2n+1)t \right)} \\ &= \sum_{n=0}^{\infty} \left(\int_0^{\frac{\pi}{2}} \omega(t) \cos(2n+1)t \right) x_n \overline{y_n}. \end{aligned}$$

This inner product and the standard inner product

$$\langle x, y \rangle = \sum_{n=0}^{\infty} x_n \overline{y_n}$$

generate equivalent norms. The operator $A(\tau)$ is bounded below with compact resolvent and its eigenvalues are $\alpha_{2n+1}(\tau) := \alpha_{2n+1}(\tau a, \tau b, \tau d)$, $n = 0, 1, 2, \ldots$

Similarly, if the the Ince operator is applied to Fourier sine series of the form

$$(3.8.7) \quad x(t) = \sum_{n=0}^{\infty} x_k \sin(2n+1)t,$$

we obtain the infinite matrix

$$(3.8.8) \quad M_3(\tau) = \begin{pmatrix} r_0^\dagger - \tau q_0^\dagger & \tau q_{-1}^\dagger & 0 & 0 & 0 & 0 & \cdots \\ \tau q_0^\dagger & r_1^\dagger & \tau q_{-2}^\dagger & 0 & 0 & 0 & \cdots \\ 0 & \tau q_1^\dagger & r_2^\dagger & \tau q_{-3}^\dagger & 0 & 0 & \cdots \\ 0 & 0 & \tau q_2^\dagger & r_3^\dagger & \tau q_{-4}^\dagger & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

note that the matrix $M_3(\tau)$ is the same as $M_2(\tau)$ except $q_0^\dagger$, by $-q_0^\dagger$ in the upper left corner. The operator $M_3(\tau)$, $\tau \in (0,1)$ defines an unbounded self-adjoint operator in the sequence Hilbert space $\ell^2(\mathbb{N}_0)$ equipped with the inner product

$$(3.8.9) \quad \begin{aligned} \langle x, y \rangle &= \int_0^{\frac{\pi}{2}} \omega(t) \left(\sum_{n=0}^{\infty} x_n \sin(2n+1)t \right) \overline{\left(\sum_{n=0}^{\infty} y_n \sin(2n+1)t \right)} \\ &= \sum_{n=0}^{\infty} \left(\int_0^{\frac{\pi}{2}} \omega(t) \sin(2n+1)t \right) x_n \overline{y_n} \end{aligned}$$

This inner product and the standard inner product

$$\langle x, y\rangle = \sum_{n=0}^{\infty} x_n \overline{y_n}$$

generate equivalent norms. The operator $B(\tau)$ is bounded below with compact resolvent and its eigenvalues are $\beta_{2n+1}(\tau) := \beta_{2n+1}(\tau a, \tau b, \tau d)$, $n = 0, 1, 2, \ldots$

Now consider the eigenvalue $\alpha_{2n+1}(\tau)$ for a fixed n, and a corresponding eigenvector $u(\tau) = (u_0(\tau), u_1(\tau), u_2(\tau), \ldots)$. For small $|\tau|$, $u_n(\tau) \neq 0$ and we adopt the normalization $u_n(\tau) = 1$. Then $u_k(\tau)$ is an analytic function of τ in a neighborhood of $\tau = 0$. Similarly, let $v(\tau) = (v_0(\tau), v_1(\tau), v_2(\tau), \ldots)$ be an eigenvalue of $B^*(\tau)$ corresponding to the eigenvalue $\beta_{2n+1}(\tau)$ normalized by $v_n(\tau) = 1$. We obtain the adjoint $B^*(\tau)$ by reflection at the main diagonal as usual. One can verify easily that $B^*(\tau)$ is the same as $B(\tau)$ but with a, b, d replaced by $a^* = a$, $b^* = -4a - b$, $d^* = d - 4a - 2b$, respectively.

Lemma 3.8.2. *For $|\tau|$ sufficiently small, we have*

$$(\alpha_{2n+1}(\tau) - \beta_{2n+1}(\tau)) \langle u(\tau), v(\tau)\rangle = 2\tau q_0^{\dagger} u_0(\tau) v_0(\tau)$$

Proof. We have

$$\begin{aligned}\langle (M_2 - M_3) u, v\rangle &= \langle M_2 u, v\rangle - \langle M_3 u, v\rangle \\ &= \langle M_2 u, v\rangle - \langle M_3^* v, u\rangle \\ &= (\alpha_{2n+1} - \beta_{2n+1}) \langle u, v\rangle .\end{aligned}$$

Since M_2 and M_3 agree except in left upper corner, everything cancels in the left-hand side except $2\tau q_0^{\dagger} u_0(\tau) v_0(\tau)$. □

Lemma 3.8.3. *For $k = 0, 1, \ldots, n-1$, we have*

$$u_k(\tau) = \tau^{n-k} \prod_{j=k}^{n-1} \frac{q\dagger_{-j-1}}{(2n+1)^2 - (2j+1)^2} + O\left(\tau^{n-k+1}\right) \tag{3.8.10}$$

and

$$v_k(\tau) = t^{n-k} \prod_{j=k}^{n-1} \frac{q\dagger_{j+1}}{(2n+1)^2 - (2j+1)^2} + O\left(\tau^{n-k+1}\right). \tag{3.8.11}$$

Proof. Since $A(\tau)\, u(\tau) = \alpha_{2n+1}(\tau)\, u(\tau)$ we obtain that

$$\left(\tau p_0 + r_0^{\dagger} - \alpha_{2n+1}(\tau)\right) u_0(\tau) + \tau p_{-1} u_1(\tau) = 0 \tag{3.8.12}$$

and

$$\tau p_j u_{j-1}(\tau) + \left(r_j^{\dagger} - \alpha_{2n+1}(\tau)\right) u_j(\tau) + \tau p_{-j-1} u_{j+1}(\tau) = 0, \quad j = 1, 2, 3, \ldots \tag{3.8.13}$$

We know that $u_j(0) = 0$ for $j \neq n$. using (3.8.12) and (3.8.13) for $j = 1, 2, \ldots, n-2$ we find that $u_k(\tau) = O(\tau^2)$ when $k \leq n-2$. In a similar way, we see that $u_k(\tau) = O(\tau^3)$ when $k \leq n-3$. In general, we obtain that $u_k(\tau) = O\left(\tau^{n-k}\right)$ for $k < n$. Using $u_n = 1$in (3.8.13) for $j = n-1$ (or (3.8.12) when $n = 1$), we get

$$\left(r_{n-1}^{\dagger} - r_n^{\dagger}\right) u_{n-1}(\tau) + \tau q_{-n}^{\dagger} = O\left(\tau^2\right)$$

which yields claim (3.8.10) when $k = n-1$. In a similar way, we find that

$$\left(r_{n-2}^{\dagger} - r_n^{\dagger}\right) u_{n-2}(\tau) + \tau q_{-n+1}^{\dagger} u_{n-1}(\tau) = O\left(\tau^3\right).$$

Substituting the previous result on $u_{n-1}(\tau)$ this proves (3.8.10) when $n = k-2$. Continuing in this fashion we prove (3.8.10) for all k. The proof of (3.8.11) is almost the same. □

Theorem 3.8.4. *Let a, b, $d \in \mathbb{R}$ with $|a| < 1$. Then, for fixed $n = 0, 1, 2, \ldots$,*

$$\alpha_{2n+1}(\tau a, \tau b, \tau d) - \beta_{2n+1}(\tau a, \tau b, \tau d) = \frac{2\tau^{2n+1}\left(1 + O\left(\tau^2\right)\right)}{\left(2^{2n}(2n!)\right)^2} \prod_{j=-n}^{n} q_j^{\dagger}, \tag{3.8.14}$$

where

$$q_j^{\dagger} := Q\left(j - \frac{1}{2}\right), \qquad Q(\mu) = 2a\mu^2 - b\mu - \frac{1}{2}d.$$

PROOF. Since

$$\langle u(\tau), v(\tau)\rangle = 1 + O(\tau),$$

Lemma 3.8.3 and Lemma 3.8.3 give

$$\begin{aligned}(1+O(\tau))\,\alpha_{2n+1}(\tau)-\beta_{2n+1}(\tau) &= 2\tau q_0^{\dagger}\left(u_0(\tau)\,v_0(\tau)\right)(1+O(\tau))\\ &= 2\tau^{2n+1}q_0^{\dagger}\prod_{j=k}^{n-1}\frac{q_{-j-1}^{\dagger}}{(2n+1)^2-(2j+1)^2}\prod_{j=k}^{n-1}\frac{q_{j+1}^{\dagger}}{(2n+1)^2-(2j+1)^2}(1+O(\tau)).\end{aligned}$$

Since $\alpha_{2n+1}(\tau a, \tau b, \tau d) - \beta_{2n+2}(\tau a, \tau b, \tau d)$ is an odd function of t, this yields (3.8.14).

□

3.8.2. Instability intervals for even m. If the operator I is applied to to Fourier series of the form

$$x(t) = x_0 + \sum_{n=1}^{\infty} x_k \cos 2nt, \tag{3.8.15}$$

we obtain the matrix representation

$$M_1 = \begin{pmatrix} r_0 & \tau q_{-1} & 0 & 0 & 0 & 0 & \cdots \\ \tau q_0 & r_1 & \tau q_{-2} & 0 & 0 & 0 & \cdots \\ 0 & \tau q_1 & r_2 & \tau q_{-3} & 0 & 0 & \cdots \\ 0 & 0 & \tau q_2 & r_3 & \tau q_{-4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.8.16}$$

where

$$\begin{aligned} r_n &= (2n)^2, &\quad n &= 0, 1, 2, \ldots \\ q_j &= Q(j), &\quad Q(\mu) &= 2a\mu^2 - b\mu - \frac{d}{2}. \end{aligned}$$

The operator $M_1(\tau)$, $\tau \in (0,1)$ defines an unbounded self-adjoint operator in the sequence Hilbert space $\ell^2(\mathbb{N}_0)$ equipped with the inner product

(3.8.17)

$$\langle x,y\rangle_{A,\omega} = \int_0^{\frac{\pi}{2}} \omega(t)\left(x_0+\sum_{n=1}^{\infty} x_n \cos(2n+1)t\right)\overline{\left(y_0+\sum_{n=0}^{\infty} y_n \cos(2n+1)t\right)}$$

This inner product and the standard inner product

$$\langle x,y\rangle = \sum_{n=0}^{\infty} x_n\overline{y_n}$$

generate equivalent norms. The operator $A(\tau)$ is bounded below with compact resolvent and its eigenvalues are $\alpha_{2n}(\tau) := \alpha_{2n}(\tau a, \tau b, \tau d)$, $n = 0,1,2,\ldots$

Similarly, if the the Ince operator is applied to Fourier sine series of the form

$$x(t) = \sum_{n=0}^{\infty} x_k \sin(2n+2)t, \tag{3.8.18}$$

we obtain the infinite matrix M_4 which is obtained from M_1 by deleting the first row and the first column. The operator $M_4(\tau)$, $\tau \in (0,1)$ defines an unbounded self-adjoint operator in the sequence Hilbert space $\ell^2(\mathbb{N}_0)$ equipped with the inner product

$$\begin{aligned}\langle x,y\rangle &= \int_0^{\frac{\pi}{2}} \omega(t)\left(\sum_{n=0}^{\infty} x_n \sin(2n+2)t\right)\overline{\left(\sum_{n=0}^{\infty} y_n \sin(2n+2)t\right)}\\ &= \sum_{n=0}^{\infty}\left(\int_0^{\frac{\pi}{2}} \omega(t)\sin(2n+2)t\right) x_n\overline{y_n}\end{aligned} \tag{3.8.19}$$

This inner product and the standard inner product

$$\langle x,y\rangle = \sum_{n=0}^{\infty} x_n\overline{y_n}$$

generate equivalent norms. The operator $M_4(\tau)$ is bounded below with compact resolvent and its eigenvalues are $\beta_{2n+2}(\tau) := \beta_{2n+2}(\tau a, \tau b, \tau d)$, $n = 0,1,2,\ldots$.

The proof of the following theorem for the stability intervals for even m is similar to that of odd m discussed above.

THEOREM 3.8.5. *Let a, b, $d \in \mathbb{R}$ with $|a| < 1$. Then, for fixed $n = 1, 2, 3, \ldots$,*

$$\alpha_{2n}(\tau a, \tau b, \tau d) - \beta_{2n}(\tau a, \tau b, \tau d) = \frac{2\tau^{2n-1}(1 + O(\tau^2))}{(2^{2n-1}(2n-1)!)^2} \prod_{j=-n}^{n} q_j, \tag{3.8.20}$$

where

$$q_j := Q(j), \qquad Q(\mu) = 2a\mu^2 - b\mu - \frac{1}{2}d.$$

3.9. Further Results

Following Eastham [**14**, Section 2.4], one can treat the eigenvalue problem

$$y(t + \pi) = e^{i\nu\pi} y(t)$$

for the Ince equation, where the characteristic exponent is given. If ν is real this leads to self-adjoint operators and infinite tridiagonal matrices as for the eigenvalues problems studied in this chapter. Mennicken [**49**] gives methods for the computation of the characteristic exponent.

Ince [**28, 30**] investigates the asymptotics of Ince functions for $a = 0$. Moreover, Ince's papers also contain bounds for the eigenvalues and other interesting results.

Volkmer [**70**] studies the characteristic polynomials of the matrices $M_{j,k}$ and uses them to approximate eigenvalues of Ince's equation.

When we substitute $\xi = \cos 2t$, Ince's equation becomes

$$\frac{d^2y}{d\xi^2} + \frac{1}{2}\left(\frac{1}{\xi} + \frac{1}{\xi - 1} + \frac{1}{2a\xi + 1 - a}\right)\frac{dy}{d\xi} + \frac{\lambda + d(2\xi - 1)}{4\xi(1 - \xi)(2a\xi + 1 - a)} y = 0. \tag{3.9.1}$$

If $a \neq 0$, this is a Heun equation with regular singular points at 0, 1, $\frac{a-1}{2a}$ and ∞. The indices at 0 and at 1 are 0 and 1/2. The indices at $\frac{a-1}{2a}$ are 0 and $1 + \frac{b}{2a}$ and the indices at infinity are the roots of $Q(-\rho) = 0$. If $a = 0$, then (3.9.1) is a confluent form of Heun's equation. Therefore, results on the Heun equation are applicable to the Ince equation. In particular, Ince polynomials are trigonometric polynomials which can be transformed to ordinary polynomials and then become Heine-Stieltjes polynomials; see [**66**, Section 6.8]. Thus results on Heine-Stieltjes polynomials are applicable to Ince polynomials.

CHAPTER 4

The Lamé Equation

4.1. The Differential Equation

The Lamé differential equation is

$$\frac{d^2w}{dz^2}+\left(h-\nu\left(\nu+1\right)k^2\,\mathrm{sn}^2\left(z,k\right)\right)w=0. \tag{4.1.1}$$

The number k denotes the modulus of the Jacobian elliptic function $\mathrm{sn}\,z=\mathrm{sn}\left(z,k\right).$ For the definitions and properties of the Jacobian elliptic functions sn, cn, dn, the Jacobian amplitude am and the complete elliptic integrals K, and K', we refer to [**86**]. The most important formulas can be found in Appendix C of [**7**].

We will assume that ν is real although we need only that $\nu\left(\nu+1\right)$ is real. Then, without loss of generality, we assume that $\nu\geq-\frac{1}{2}$. The third parameter h is the spectral parameter and will also be always real. The functionsn z is meromorphic on $\mathbb{C}$ with simple poles at each point $z=2pK+(2q+1)K'i$, p, $q\in\mathbb{Z}$.

Since $ku\,\mathrm{sn}\left(u+iK'\right)\to 1$ as $u\to 1$, we find that (4.1.1) has regular singular points $2pK+\left(2q+1\right)K'i$, p, $q\in\mathbb{Z}$, with indices $-\nu$ and $\nu+1$. Unless stated otherwise we will consider solutions of (4.1.1) in $\mathbb{R}$. These solutions can be continued analytically to the horizontal strip $-K'<\Im z<K'$.

The function $\mathrm{sn}^2 z$ has period $2K$. Therefore, if $w\left(z\right)$ is a solution of (4.1.1) then also $w\left(z+2K\right)$ is a solution. Hence Lame's equation is a Hill's equation with period $2K$. Also note that $\mathrm{sn}^2 z$ is an even function. Therefore, if $w\left(z\right)$ is a solution of (4.1.1), then $w\left(-z\right)$ is also a solution. Hence Lamé's equation is an even Hill's equation. The function $\mathrm{sn}^2 z$ has a second period $2iK'$. Therefore, if $w\left(z\right)$ is a solution of (4.1.1) defined on the vertical strip $0<\Re z<2K$ then also $w\left(z+2iK'\right)$ is a solution. Hence Lamé's equation can also be considered as a Hill's equation with period $2iK'$. Since the two lines $\Re z = 0$, $\Im z = K$ intersect at K, instead of asking for even or odd solutions it is more natural to ask for solutions which are even or odd about K, that is, $w(K-z) = \pm w(K+z)$. Note that $\mathrm{sn}^2 z$ is even about K.

If we substitute

$$t = \frac{\pi}{2} - \mathrm{am}\, z, \tag{4.1.2}$$

then

$$\frac{dt}{dz} = -\mathrm{dn}\, z, \mathrm{sn}\, z = \cos t, \mathrm{cn}\, z = \sin t, \mathrm{dn}^2\, z = 1 - k^2 \cos^2 t,$$

and Lamé's equation becomes

$$\left(1 - k^2 \cos^2 t\right) \frac{d^2 w}{dt^2} + k^2 \sin t \cos t \frac{dw}{dt} + \left(h - \nu(\nu+1) k^2 \cos^2 t\right) w = 0. \tag{4.1.3}$$

Equation (4.1.3) is equivalent to

$$\begin{aligned} &\left(1 - \frac{k^2}{2-k^2} \cos t\right) \frac{d^2 w}{dt^2} + \frac{k^2}{2-k^2} \sin 2t \frac{dw}{dt} \\ &\qquad + \left(\frac{2h - \nu(\nu+1) k^2}{2-k^2} - \frac{\nu(\nu+1) k^2}{2-k^2} \cos 2t\right) w = 0. \end{aligned} \tag{4.1.4}$$

Equation (4.1.4) is an Ince equation with parameter

$$-a = b = \frac{k^2}{2-k^2}, \qquad \lambda = \frac{2h - \nu(\nu+1) k^2}{2-k^2}, \qquad d = -\frac{\nu(\nu+1) k^2}{2-k^2}. \tag{4.1.5}$$

If we substitute

$$\xi = \mathrm{sn}^2 z = \cos^2 t, \tag{4.1.6}$$

Lamé equation becomes a particular case of the Heun equation

$$\frac{d^2 w}{d\xi^2} + \frac{1}{2}\left(\frac{1}{\xi} + \frac{1}{\xi - 1} + \frac{1}{\xi - k^{-2}}\right) \frac{dw}{d\xi} + \frac{hk^{-2} - \nu(\nu+1)\xi}{4\xi(\xi-1)(\xi - k^{-2})} w = 0. \tag{4.1.7}$$

We substitute

$$g = (e_1 - e_3)\, h + \nu\,(\nu + 1)\, e_3, \tag{4.1.8}$$

$$\eta = (e_1 - e_3)^{-\frac{1}{2}}\,(z - iK'), \tag{4.1.9}$$

$$\zeta = \wp\,(\eta), \tag{4.1.10}$$

where $\wp$ is the elliptic function of Weierstrass with corresponding constants e_1, e_2, e_3. Then Lamé equation becomes

$$\frac{d^2w}{d\eta^2} + (g - \nu\,(\nu + 1)\,\wp\,(\eta))\, w = 0, \tag{4.1.11}$$

and

$$\frac{d^2w}{d\zeta^2} + \frac{1}{2}\left(\frac{1}{\zeta - e_1} + \frac{1}{\zeta - e_2} + \frac{1}{\zeta - e_3}\right)\frac{dw}{d\zeta} + \frac{g - \nu\,(\nu + 1)\,\zeta}{4\,(\zeta - e_1)\,(\zeta - e_2)\,(\zeta - e_3)} w = 0. \tag{4.1.12}$$

4.2. Eigenvalues

Consider the Lamé equation (4.1.1) with given values for k and ν. A solution $w\,(z)$ is even about K and has period $2K$ if and only if $w\,(z)$ satisfies the boundary conditions

$$w'\,(0) = w'\,(K) = 0. \tag{4.2.1}$$

Lamé's equation (4.1.1) together with the boundary conditions (4.2.1) pose a regular Sturm-Liouville eigenvalue problem with spectral parameter h. Therefore, the corresponding eigenvalues h form a real increasing sequence that tends to infinity. We denote these eigenvalues by $a_\nu^{2m}\,(k^2)$.

Similarly, a solution $y(z)$ of Lamé's equation is even about K and has semi-period $2K$ if and only if

$$w(0) = w'(K) = 0. \tag{4.2.2}$$

The corresponding sequence of eigenvalues is denoted by $a_\nu^{2m+1}(k^2)$.

A solution $y(z)$ of Lamé's equation is odd about K and has semi-period $2K$ if and only if

$$w'(0) = w(K) = 0. \tag{4.2.3}$$

The corresponding sequence of eigenvalues is denoted by $b_\nu^{2m+1}(k^2)$.

A solution $w(z)$ of Lamé's equation is odd about K and has period $2K$ if and only if

$$w(0) = w(K) = 0. \tag{4.2.4}$$

The corresponding sequence of eigenvalues is denoted by $b_\nu^{2m+2}(k^2)$.

All four sequences of eigenvalues are increasing and $m = 0, 1, 2, ...$ The eigenfunctions belonging to these eigenvalues are the Lamé's functions. They will be studied in more detail in Section 4.3 .

The notation of the eigenvalues is chosen in such a way that an even or odd superscript is associated with Lamé's functions with period $2K$ or semi-period $2K$, respectively. The letter a denotes eigenvalues associated with Lamé functions which are even about K, whereas the letter b denotes eigenvalues associated with Lamé functions which are odd about K. Originally, Ince [**32**] had used the letters a and b in connection with even and odd Lamé functions, respectively. We adopted the notation introduced by Erdélyi et al. [**1**] which is of advantage in section about Lamé functions with imaginary period. In order to compare with work of Ince one just has to exchange a_ν^{2m+1} and b_ν^{2m+1}. One should also note that $a_\nu^m(k^2)$ is defined for $m = 0, 1, 2, ...$, whereas $b_\nu^m(k^2)$ is defined only for $m = 1, 2, 3, ...$ If we define a, b, d

by (4.1.5) the eigenvalues of Lamé's equation can be expressed

$$2a_\nu^m\left(k^2\right) = \nu\left(\nu+1\right)k^2 + \left(2-k^2\right)\alpha_m\left(a,b,d\right), \tag{4.2.5}$$

$$2b_\nu^m\left(k^2\right) = \nu\left(\nu+1\right)k^2 + \left(2-k^2\right)\beta_m\left(a,b,d\right). \tag{4.2.6}$$

From Theorems 2.2.7, 3.5.4 we obtain the following results.

THEOREM 4.2.1. *The eigenvalues of Lamé's equation interlace according to*

$$a_\nu^0 < \left\{\begin{array}{c} a_\nu^1 \\ b_\nu^1 \end{array}\right\} < \left\{\begin{array}{c} a_\nu^2 \\ b_\nu^2 \end{array}\right\} < \left\{\begin{array}{c} a_\nu^3 \\ b_\nu^3 \end{array}\right\} < \dots$$

We have, for $m \in \mathbb{N}$,

$$\operatorname{sign}\left(a_\nu^{2m}\left(k^2\right) - b_\nu^{2m}\left(k^2\right)\right) = \operatorname{sign}\prod_{n=-m}^{m-1}\left(2n-\nu\right)\left(2n+\nu+1\right), \tag{4.2.7}$$

and, for $m \in \mathbb{N}_0$,

$$\operatorname{sign}\left(a_\nu^{2m+1}\left(k^2\right) - b_\nu^{2m+1}\left(k^2\right)\right) = \operatorname{sign}\prod_{n=-m}^{m}\left(2n-1-\nu\right)\left(2n+\nu\right). \tag{4.2.8}$$

In particular, we have coexistence

$$a_\nu^m\left(k^2\right) = b_\nu^m\left(k^2\right)$$

if and only if $\nu \in \{0,1,2,\dots,m-1\}$.

Using the relationships (4.2.5), (4.2.6), Theorem 3.3.3 provides continued-fraction equations for the eigenvalues of Lamé's equation.

4.3. Eigenfunctions

The eigenfunctions of Lamé's equation corresponding to the eigenvalues

$$a_\nu^{2m}\left(k^2\right),\ a_\nu^{2m+1}\left(k^2\right),\ b_\nu^{2m+1}\left(k^2\right),\ b_\nu^{2m+2}\left(k^2\right) \tag{4.3.1}$$

are denoted by

$$Ec_\nu^{2m}\left(z,k^2\right),\ Ec_\nu^{2m+1}\left(z,k^2\right),\ Es_\nu^{2m+1}\left(z,k^2\right),\ Es_\nu^{2m+2}\left(z,k^2\right), \tag{4.3.2}$$

respectively. These are the (simply-periodic) Lamé functions. As eigenfunctions these functions are only determined up to a constant factor. We normalize them by the conditions

$$\int_0^K \operatorname{dn} z\left(Ec_\nu^m\left(z,k^2\right)\right)^2 dz = \frac{\pi}{4}, \tag{4.3.3}$$

$$\int_0^K \operatorname{dn} z\left(Es_\nu^m\left(z,k^2\right)\right)^2 dz = \frac{\pi}{4}. \tag{4.3.4}$$

To complete the definition, $Ec_\nu^m\left(K,k^2\right)$ is positive and $\frac{d}{dz}Es_\nu^m\left(K,k^2\right)$ is negative.

Since $\frac{d}{dz}\operatorname{am} z = \operatorname{dn} z$, this agrees with the normalization of Ince functions, and we obtain

$$Ec_\nu^m\left(z,k^2\right) = Ic_m\left(t,a,b,d\right), \tag{4.3.5}$$

$$Es_\nu^m\left(z,k^2\right) = Is_m\left(t,a,b,d\right), \tag{4.3.6}$$

where t, z are related by (4.1.2), and a, b, d are given in (4.1.5).

From Sturm-Liouville theory we derive the following property of Lamé functions.

THEOREM 4.3.1. *Each of the Lamé functions* (4.3.2) *has precisely m simple zeros in the open interval* $(0,K)$. *The superscript* $2m$, $2m+1$, *or* $2m+2$ *equals the number of zeros in the half-open interval* $(0,2K]$.

The analog of Theorem 2.3.1 for Lamé functions is the following

THEOREM 4.3.2. *Each of the function systems*

$$\left\{Ec_\nu^{2m}\left(z,k^2\right)\right\}_{m=0}^{\infty}, \tag{4.3.7}$$

(4.3.8) $$\left\{Ec_{\nu}^{2m+1}\left(z,k^{2}\right)\right\}_{m=0}^{\infty},$$

(4.3.9) $$\left\{Es_{\nu}^{2m+1}\left(z,k^{2}\right)\right\}_{m=0}^{\infty},$$

(4.3.10) $$\left\{Es_{\nu}^{2m+2}\left(z,k^{2}\right)\right\}_{m=0}^{\infty},$$

is orthogonal over$[0,K]$*, that is, for* $m\neq n$,

(4.3.11) $$\int_{0}^{k}Ec_{\nu}^{2m}\left(t\right)Ec_{\nu}^{2n}\left(t\right)dt=0,$$

(4.3.12) $$\int_{0}^{k}Ec_{\nu}^{2m+1}\left(t\right)Ec_{\nu}^{2n+1}\left(t\right)dt=0,$$

(4.3.13) $$\int_{0}^{k}Es_{\nu}^{2m+1}\left(t\right)Es_{\nu}^{2n+1}\left(t\right)dt=0,$$

(4.3.14) $$\int_{0}^{k}Es_{\nu}^{2m+2}\left(t\right)Es_{\nu}^{2n+2}\left(t\right)dt=0,.$$

Moreover, each of the system (4.3.7), (4.3.8), (4.3.9), (4.3.10) *is complete over* $[0,K]$.

4.4. Fourier Series

By (4.3.5), (4.3.6), the Fourier series from Section 2.5 give Fourier series for Lamé functions

(4.4.1) $$Ec_{\nu}^{2m}(z,k^{2})=\frac{A_{0}}{\sqrt{2}}+\sum_{n=1}^{\infty}A_{2n}\cos\left(2nt\right),$$

(4.4.2) $$Ec_{\nu}^{2m+1}(z,k^{2})=\sum_{n=0}^{\infty}A_{2n+1}\cos\left(2n+1\right)t,$$

$$(4.4.3) \qquad Es_\nu^{2m+1}(z,k^2) = \sum_{n=0}^{\infty} B_{2n+1} \sin\left(2n+1\right)t,$$

$$(4.4.4) \qquad Es_\nu^{2m+2}(z,k^2) = \sum_{n=0}^{\infty} B_{2n+2} \sin\left(2n+2\right)t.$$

Where t, z are related by (4.1.2). The coefficients satisfy the normalization relations (2.5.5), (2.5.6), (2.5.7), (2.5.8) and the three-term difference equations given in Theorem 3.3.1 with a, b, d from (4.1.5). If $\{x_n\}$ denotes any of the sequences $\{A_{2n}\}$, $\{A_{2n+1}\}$, $\{B_{2n+1}\}$, $\{B_{2n+2}\}$, Theorem 4.6.3 yields that either $x_n = 0$ for large n, or $x_n \neq 0$ for all large n. In the latter case

$$(4.4.5) \qquad \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{k^2}{\left(1+k'\right)^2},$$

where

$$(4.4.6) \qquad k' := \sqrt{1-k^2}$$

is the complementary modulus.

Using relations (2.3.13), (2.3.14) we can represent Lamé functions in a second way in terms of Ince functions. We first note that with a, b from (4.1.5)

$$(4.4.7) \qquad \begin{aligned} \omega\left(t;a,b\right) &= \left(1+a\cos 2t\right)^{-\frac{1}{2}} = 2^{-\frac{1}{2}}\left(2-k^2\right)^{-\frac{1}{2}}\left(1-k^2\cos 2t\right)^{-\frac{1}{2}} \\ &= 2^{-\frac{1}{2}}\left(2-k^2\right)^{-\frac{1}{2}}\left(\operatorname{dn} z\right)^{-1}. \end{aligned}$$

$$(4.4.8) \qquad Ec_\nu^m(z,k^2) = 2^{-\frac{1}{2}}\left(2-k^2\right)^{-\frac{1}{2}} c_m\left(a,b,d\right)^{-1} \operatorname{dn} z\, Ic_m\left(t;a,b^*,d^*\right),$$

$$(4.4.9) \qquad Es_\nu^m(z,k^2) = 2^{-\frac{1}{2}}\left(2-k^2\right)^{-\frac{1}{2}} s_m\left(a,b,d\right)^{-1} \operatorname{dn} z\, Is_m\left(t;a,b^*,d^*\right),$$

where

$$b^* = -4a - b = \frac{3k^2}{2-k^2}, \quad d^* = d - 4a - 2b = \frac{-k^2}{2-k^2}\left(\nu\left(\nu+1\right)-2\right).$$

Therefore, we can write

$$Ec_\nu^{2m}(z,k^2) = \operatorname{dn} z \left(\frac{C_0}{\sqrt{2}} + \sum_{n=1}^{\infty} C_{2n}\cos\left(2nt\right)\right), \tag{4.4.10}$$

$$Ec_\nu^{2m+1}(z,k^2) = \operatorname{dn} z \left(\sum_{n=0}^{\infty} C_{2n+1}\cos\left(2n+1\right)t\right), \tag{4.4.11}$$

$$Es_\nu^{2m+1}(z,k^2) = \operatorname{dn} z \left(\sum_{n=0}^{\infty} D_{2n+1}\sin\left(2n+1\right)t\right), \tag{4.4.12}$$

$$Es_\nu^{2m+2}(z,k^2) = \operatorname{dn} z \left(\sum_{n=0}^{\infty} D_{2n+2}\sin\left(2n+2\right)t\right), \tag{4.4.13}$$

where

$$C_n = 2^{-\frac{1}{2}}\left(2-k^2\right)^{-\frac{1}{2}} c_m\left(a,b,d\right)^{-1} A_n,$$

$$D_n = 2^{-\frac{1}{2}}\left(2-k^2\right)^{-\frac{1}{2}} c_m\left(a,b,d\right)^{-1} B_n,$$

and the Fourier coefficients A_n and $B_n$belong to the parameters a, b^*, d^*. Properties of the coefficients C_n and D_n follow from those of A_n and B_n; see Section 2.5. For example, we obtain (4.4.5) also for the sequences $\{C_{2n}\}, \{C_{2n+1}\}, \{D_{2n+1}\}, \{D_{2n+2}\}$.

One should note that the sequence $\{A_{2n}\}$ is an eigenvector of the matrix M_1 with a, b, d from (4.1.5), whereas $\{C_{2n}\}$ is an eigenvector of its adjoint. Both eigenvectors belong to the same eigenvalue. In others words, $\{A_{2n}\}$ and $\{C_{2n}\}$ are right and left eigenvectors of the same infinite tridiagonal matrix belonging to the same eigenvalue. Lemma 3.1.2 connects these eigenvectors. Similar remarks apply to the other sequences of Fourier coefficients.

Because of the special form of the weight (4.4.7) we can express the normalization of these sequences more directly as follows.

THEOREM 4.4.1. *We have*

$$\left(1-\frac{k^2}{2}\right)\sum_{n=0}^{\infty} C_{2n}^2 - \frac{k^2}{2}\left(\sqrt{2}C_0C_2+\sum_{n=1}^{\infty} C_{2n}C_{2n+2}\right) = 1, \tag{4.4.14}$$

$$\left(1-\frac{k^2}{2}\right)\sum_{n=0}^{\infty} C_{2n+1}^2 - \frac{k^2}{2}\left(\frac{1}{2}C_1^2+\sum_{n=1}^{\infty} C_{2n+1}C_{2n+3}\right) = 1, \tag{4.4.15}$$

$$\left(1-\frac{k^2}{2}\right)\sum_{n=0}^{\infty} D_{2n+1}^2 + \frac{k^2}{2}\left(\frac{1}{2}D_1^2-\sum_{n=1}^{\infty} D_{2n+1}D_{2n+3}\right) = 1, \tag{4.4.16}$$

$$\left(1-\frac{k^2}{2}\right)\sum_{n=0}^{\infty} D_{2n}^2 - \frac{k^2}{2}\sum_{n=1}^{\infty} D_{2n}D_{2n+2} = 1, \tag{4.4.17}$$

$$\frac{C_0}{\sqrt{2}}+\sum_{n=1}^{\infty} C_{2n} > 0, \quad \sum_{n=0}^{\infty} C_{2n+1} > 0, \tag{4.4.18}$$

$$\sum_{n=0}^{\infty}(2n+1)\,D_{2n+1} > 0, \quad \sum_{n=0}^{\infty}(2n+2)\,D_{2n+2} > 0. \tag{4.4.19}$$

PROOF. Since

$$\mathrm{dn}^2 z = 1-\frac{k^2}{2}-\frac{k^2}{2}\cos 2t,$$

we have

$$\begin{aligned}\frac{\pi}{4} &= \int_0^K \mathrm{dn}z\left(Ec_\nu^{2m}\left(z,k^2\right)\right)^2 dz\\ &= \int_0^{\pi/2}\left(1-\frac{k^2}{2}-\frac{k^2}{2}\cos 2t\right)\left(\frac{C_0}{\sqrt{2}}+\sum_{n=1}^{\infty} C_{2n}\cos(2nt)\right)^2 dt.\end{aligned}$$

Using $2\cos(2t)\cos(2nt) = \cos 2(n+1)t + \cos 2(n-1)t$, we obtain (4.4.14). Formulas (4.4.15), (4.4.16), (4.4.17) are proved similarly, and (4.4.18), (4.4.19) follow from (2.5.5), (2.5.6), (2.5.7), (2.5.8). □

4.5. Lamé Functions with Imaginary Periods

We consider Lamé's equation as a Hill equation with period $2iK'$ along the line $\Re z = K$. In order to transform to real variables, we set

$$z = K + iK' - iu \qquad u \in \mathbb{R}. \tag{4.5.1}$$

By Jacobi's imaginary transformation,

$$\operatorname{sn}(z,k) = \frac{\operatorname{dn}(iu,k)}{k\operatorname{sn}(iu,k)} = \frac{1}{k}\operatorname{dn}(u,k').$$

Hence

$$k^2\operatorname{sn}^2(z,k) = 1 - k'^2\operatorname{sn}^2(z,k'),$$

and Lamé's equation becomes

$$\frac{d^2w}{du^2} + \big(\nu(\nu+1) - h - \nu(\nu+1)k'^2\operatorname{sn}^2(u,k')\big)w = 0. \tag{4.5.2}$$

This is again Lamé's equation with k replaced by k' and the spectral parameter replaced by $\nu(\nu+1) - h$. We have proved the following theorem.

Theorem 4.5.1. *Lamé's equation* (4.1.1) *admits a nontrivial solution which is even about K and has period $2iK'$if and only if $h = \nu(\nu+1) - a_\nu^{2m}(k'^2)$ for some $m \in \mathbb{N}_0$. A corresponding eigenfunction is $Ec_\nu^{2m}(i(z-K-iK'), k'^2)$.*

Lamé's equation admits a nontrivial solution which is even about K and has semi-period $2iK'$if and only if $h = \nu(\nu+1) - a_\nu^{2m+1}(k'^2)$ for some $m \in \mathbb{N}_0$. A corresponding eigenfunction is $Ec_\nu^{2m+1}(i(z-K-iK'), k'^2)$.

Lamé's equation admits a nontrivial solution which is odd about K and has semi-period $2iK'$ if and only if $h = \nu(\nu+1) - b_\nu^{2m+1}(k'^2)$ for some $m \in \mathbb{N}_0$. A corresponding eigenfunction is $Es_\nu^{2m+1}(i(z-K-iK'), k'^2)$.

Lamé's equation (4.1.1) *admits a nontrivial solution which is odd about K and has period $2iK'$ if and only if $h = \nu(\nu+1) - b_\nu^{2m+2}(k'^2)$ for some $m \in \mathbb{N}_0$. A corresponding eigenfunction is $Es_\nu^{2m+2}(i(z-K-iK'), k'^2)$.*

Because of this theorem it is not necessary to introduce new notations for eigenvalues and eigenfunctions of Lamé's equation with period or semi-period $2iK'$.

4.6. Lamé Polynomials

A Lamé function is called a Lamé polynomial of the first kind if its Fourier series (4.4.1), (4.4.2), (4.4.3), or (4.4.4) terminates. It is called a Lamé polynomial of the second kind if its expansion (4.4.10), (4.4.11), (4.4.12), or (4.4.13) terminates.

From Theorem 3.4.3 and its analogue for Ince polynomials of the second kind, and from (4.1.5), we obtain the following result.

THEOREM 4.6.1. *The Lamé function Ec_ν^m is a Lamé polynomial if and only if ν is a nonnegative integer and $m = 0, 1, ..., \nu$. The Lamé function Es_ν^m is a Lamé polynomial if and only if ν is a positive integer and $m = 1, 2, ..., \nu$. The Lamé polynomials are of the first kind or second kind if $\nu - m$ is even or odd, respectively.*

There are the following eight types of Lamé polynomials

(4.6.1)
$$\begin{array}{ll} 1)\ Ec_{2n}^{2m}(z,k^2), & 2)\ Ec_{2n+1}^{2m+1}(z,k^2), \\ 3)\ Es_{2n+1}^{2m+1}(z,k^2), & 4)\ Ec_{2n+1}^{2m}(z,k^2), \\ 5)\ Es_{2n+2}^{2m+2}(z,k^2), & 6)\ Ec_{2n+2}^{2m+1}(z,k^2), \\ 7)\ Es_{2n+2}^{2m+1}(z,k^2), & 8)\ Es_{2n+3}^{2m+2}(z,k^2), \end{array}$$

where $n \in \mathbb{N}_0$ and $m = 0, 1, 2, \ldots, n$ in each case.

These Lamé polynomials and their corresponding eigenvalues can be computed from the finite matrices $M_{j,l}$, $j = 1,2,3,4$, of Section 2.4 as follows. Recall that the polynomial Q defining the entries of the matrices is given by

$$Q(\mu) = -\frac{k^2}{2\left(2-k^2\right)}\left(2\mu-\nu\right)\left(2\mu+\nu+1\right). \tag{4.6.2}$$

1) Let $\nu = 2n$ with $n \in \mathbb{N}_0$. Then

$$a_{2n}^{2m}\left(k^2\right) = \frac{1}{2}\left(2-k^2\right)\alpha_{2m} + \frac{1}{2}\nu\left(\nu+1\right)k^2, \qquad m = 1,2,\ldots,n, \tag{4.6.3}$$

where α_{2m} are the eigenvalues of $M_{1,n+1}$. The corresponding Lamé polynomials are

$$Ec_{2n}^{2m}\left(z,k^2\right) = \frac{A_0}{\sqrt{2}} + \sum_{p=1}^{n} A_{2p}T_{2p}\left(\operatorname{sn} z\right), \tag{4.6.4}$$

where T_p is the Chebyshev polynomial of the first kind defined by $T_p\left(\cos t\right) = \cos\left(pt\right)$. The coefficients $\left(A_0, A_2, \ldots, A_{2n}\right)$ form a right eigenvector of $M_{1,n+1}$ belonging to the eigenvalue α_{2m}.The eigenvector is normalized according to (2.5.5) (with $A_{2p} = 0$ for$p > n$.)

2) Let $\nu = 2n+1$ with $n \in \mathbb{N}_0$. Then

$$a_{2n+1}^{2m+1}\left(k^2\right) = \frac{1}{2}\left(2-k^2\right)\alpha_{2m+1} + \frac{1}{2}\nu\left(\nu+1\right)k^2, \qquad m = 0,1,\ldots,n, \tag{4.6.5}$$

where α_{2m+1} are the eigenvalues of $M_{2,n+1}$. The corresponding Lamé polynomials are

$$Ec_{2n+1}^{2m+1}\left(z,k^2\right) = \sum_{p=0}^{n} A_{2p+1}T_{2p+1}\left(\operatorname{sn} z\right), \tag{4.6.6}$$

The coefficients $\left(A_1, A_3, \ldots, A_{2n+1}\right)$ form a right eigenvector of $M_{2,n+1}$ belonging to the eigenvalue α_{2m+1}. The eigenvector is normalized according to (2.5.6).

3) Let $\nu = 2n+1$ with $n \in \mathbb{N}_0$. Then

$$b_{2n+1}^{2m+1}\left(k^2\right) = \frac{1}{2}\left(2-k^2\right)\beta_{2m+1} + \frac{1}{2}\nu\left(\nu+1\right)k^2, \qquad m = 0,1,\ldots,n, \tag{4.6.7}$$

where β_{2m+1} are the eigenvalues of $M_{3,n+1}$. The corresponding Lamé polynomials are

$$Es_{2n+1}^{2m+1}\left(z,k^2\right) = \operatorname{cn} z \sum_{p=0}^{n} B_{2p+1} U_{2p+1}\left(\operatorname{sn} z\right), \tag{4.6.8}$$

where U_p is the Chebyshev polynomial of the second kind defined by $U_p\left(\cos t\right) = \frac{\sin(p+1)t}{\sin t}$. The coefficients $\left(B_1, B_3, \ldots, B_{2n+1}\right)$ form a right eigenvector of $M_{3,n+1}$ belonging to the eigenvalue B_{2m+1}. The eigenvector is normalized according to (2.5.7).

4) Let $\nu = 2n+1$ with $n \in \mathbb{N}_0$. Then

$$a_{2n+1}^{2m}\left(k^2\right) = \frac{1}{2}\left(2-k^2\right)\alpha_{2m+1} + \frac{1}{2}\nu\left(\nu+1\right)k^2, \qquad m = 0,1,\ldots,n, \tag{4.6.9}$$

where α_{2m+1} are the eigenvalues of $M_{1,n+1}$. The corresponding Lamé polynomials are

$$Ec_{2n+1}^{2m}\left(z,k^2\right) = \operatorname{dn} z \left(\frac{C_0}{\sqrt{2}} + \sum_{p=0}^{n} C_{2p} T_{2p}\left(\operatorname{sn} z\right)\right), \tag{4.6.10}$$

The coefficients $\left(C_0, C_2, \ldots, C_{2n}\right)$ form a right eigenvector of $M_{1,n+1}$ belonging to the eigenvalue α_{2m}.The eigenvector is normalized according to (4.4.14), (4.4.18).

5) Let $\nu = 2n+2$ with $n \in \mathbb{N}_0$. Then

$$b_{2n+2}^{2m+2}\left(k^2\right) = \frac{1}{2}\left(2-k^2\right)\beta_{2m+1} + \frac{1}{2}\nu\left(\nu+1\right)k^2, \qquad m = 0,1,\ldots,n, \tag{4.6.11}$$

where β_{2m+1} are the eigenvalues of $M_{4,n+1}$. The corresponding Lamé polynomials are

$$Ec_{2n+2}^{2m+2}\left(z,k^2\right) = \operatorname{cn} z \sum_{p=0}^{n} B_{2p+2} U_{2p+1}\left(\operatorname{cn} z\right), \tag{4.6.12}$$

The coefficients $\left(B_2, B_4, \ldots, B_{2n+2}\right)$ form a right eigenvector of $M_{4,n+1}$ belonging to the eigenvalue β_{2m+2}. The eigenvector is normalized according to (2.5.8).

6) Let $\nu = 2n+2$ with $n \in \mathbb{N}_0$. Then

$$a_{2n+2}^{2m+1}\left(k^2\right) = \frac{1}{2}\left(2-k^2\right)\alpha_{2m+1} + \frac{1}{2}\nu\left(\nu+1\right)k^2, \qquad m = 0,1,\ldots,n, \tag{4.6.13}$$

where α_{2m+1} are the eigenvalues of $M_{2,n+1}$. The corresponding Lamé polynomials are

$$Ec_{2n+2}^{2m+1}\left(z,k^2\right)=\operatorname{dn}z\sum_{p=0}^{n}C_{2p+1}T_{2p+1}\left(\operatorname{sn}z\right), \tag{4.6.14}$$

The coefficients $(C_1, C_3, \ldots, C_{2n+1})$ form a right eigenvector of $M_{2,n+1}$ belonging to the eigenvalue β_{2m+2}. The eigenvector is normalized according to (4.4.15), (4.4.18).

7) Let $\nu = 2n+2$ with $n \in \mathbb{N}_0$. Then

$$b_{2n+2}^{2m+1}\left(k^2\right)=\frac{1}{2}\left(2-k^2\right)\beta_{2m+1}+\frac{1}{2}\nu\left(\nu+1\right)k^2, \qquad m=0,1,\ldots,n, \tag{4.6.15}$$

where β_{2m+1} are the eigenvalues of $M_{3,n+1}$. The corresponding Lamé polynomials are

$$Es_{2n+2}^{2m+1}\left(z,k^2\right)=\operatorname{dn}z\operatorname{cn}z\sum_{p=0}^{n}D_{2p+1}U_{2p}\left(\operatorname{sn}z\right), \tag{4.6.16}$$

The coefficients $(D_1, D_3, \ldots, D_{2n+1})$ form a right eigenvector of $M_{3,n+1}$ belonging to the eigenvalue β_{2m+1}. The eigenvector is normalized according to (4.4.16), (4.4.19).

8) Let $\nu = 2n+3$ with $n \in \mathbb{N}_0$. Then

$$b_{2n+3}^{2m+2}\left(k^2\right)=\frac{1}{2}\left(2-k^2\right)\beta_{2m+2}+\frac{1}{2}\nu\left(\nu+1\right)k^2, \qquad m=0,1,\ldots,n, \tag{4.6.17}$$

where β_{2m+2} are the eigenvalues of $M_{4,n+1}$. The corresponding Lamé polynomials are

$$Es_{2n+3}^{2m+2}\left(z,k^2\right)=\operatorname{dn}z\operatorname{cn}z\sum_{p=0}^{n}D_{2p+2}U_{2p+1}\left(\operatorname{sn}z\right), \tag{4.6.18}$$

The coefficients $(D_2, D_4, \ldots, D_{2n+2})$ form a right eigenvector of $M_{4,n+1}$ belonging to the eigenvalue β_{2m+2}. The eigenvector is normalized according to (4.4.17), (4.4.19).

Example 4.6.2. Let $n = 2$, $k = 1/2$, and choose $\nu = 2n$, from equation (4.1.5) we have

$$a=-\frac{1}{7}, \quad b=\frac{1}{7}, \quad d=-\frac{6}{7}.$$

The matrix $M_{1,n+1}$ is

$$M_{1,3} = \begin{pmatrix} 0 & \frac{3}{7}\sqrt{2} & 0 \\ \frac{3}{7}\sqrt{2} & 4 & -\frac{3}{7} \\ 0 & 0 & 16 \end{pmatrix},$$

with eigenvalues

$$\alpha_0 = 2 - \frac{4}{7}\sqrt{13}, \quad \alpha_2 = 2 + \frac{4}{7}\sqrt{13}, \quad \alpha_4 = 16,$$

$$\alpha_0 = 2 - \frac{4}{7}\sqrt{13}, \quad \alpha_2 = 2 + \frac{4}{7}\sqrt{13}, \quad \alpha_4 = 16,$$

and corresponding eigenvectors

$$V_0 = \begin{pmatrix} -\frac{\sqrt{2}}{-7+\sqrt{13}} \\ 1 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \frac{\sqrt{2}}{7+\sqrt{13}} \\ 1 \\ 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} -\frac{\sqrt{2}}{1566} \\ -\frac{28}{783} \\ 1 \end{pmatrix}.$$

We also find that the Lamé eigenvalues $a_{2n}^{2m}(k^2)$

$$a_4^0 = \frac{17 - 2\sqrt{13}}{4}, \qquad a_4^2 = \frac{17 + 2\sqrt{13}}{4}, \qquad a_4^4 = \frac{33}{2},$$

with corresponding Lamé polynomials of type 1)

$$\begin{aligned} Ec_4^0\left(z, k^2\right) &= -\frac{1}{-7+\sqrt{13}} + \cos 2t, \\ Ec_4^2\left(z, k^2\right) &= \frac{1}{7+\sqrt{13}} + \cos 2t, \\ Ec_4^4\left(z, k^2\right) &= -\frac{1}{1566} + -\frac{28}{783}\cos 2t + \cos 4t. \end{aligned}$$

We have shown that the eight types of Lamé polynomials can be written as polynomials in sn, cn, dn as follows.

$$
\begin{aligned}
&1)\, w(z) = P\left(\mathrm{sn}^2 z\right) &&= w(2K-z) = w(2K+z) = w(z+2iK')\\
&2)\, w(z) = \mathrm{sn}\, z\, P\left(\mathrm{sn}^2 z\right) &&= w(2K-z) = -w(2K+z) = w(z+2iK')\\
&4)\, w(z) = \mathrm{dn}\, z\, P\left(\mathrm{sn}^2 z\right) &&= w(2K-z) = w(2K+z) = -w(z+2iK')\\
&5)\, w(z) = \mathrm{sn}\, z\ \mathrm{cn}\, z\, P\left(\mathrm{sn}^2 z\right) &&= -w(2K-z) = w(2K+z) = -w(z+2iK')\\
&6)\, w(z) = \mathrm{sn}\, z\ \mathrm{dn}\, z\, P\left(\mathrm{sn}^2 z\right) &&= w(2K-z) = -w(2K+z) = -w(z+2iK')\\
&7)\, w(z) = \mathrm{cn}\, z\ \mathrm{dn}\, z\, P\left(\mathrm{sn}^2 z\right) &&= -w(2K-z) = -w(2K+z) = w(z+2iK')\\
&8)\, w(z) = \mathrm{sn}\, z\ \mathrm{cn}\, z\ \mathrm{dn}\, z\, P\left(\mathrm{sn}^2 z\right) &&= -w(2K-z) = w(2K+z) = w(z+2iK')
\end{aligned}
$$

where P denotes a polynomial. If follows that Lamé polynomials are elliptic functions with periods $4K$ and $4iK'$ that solve Lamé's equation. We now prove the converse statement.

THEOREM 4.6.3. *A nontrivial elliptic function with periods $4K$ and $4iK'$ that solves Lamé's equation is a constant multiple of a Lamé polynomial.*

PROOF. Let $w(z)$ be a nontrivial elliptic function with periods $2K$ and $2iK'$, even about K, which solves Lamé's equation for some given h, ν, k. The function $w(z)$ is also even about iK', so its Laurent expansion at iK' contains only even powers of $z - iK'$. Since sn z is odd about iK' and has a simple pole at iK', there is a polynomial P such that $P(z) := w(z) - P(\mathrm{sn}^2 z)$ is analytic at iK' and $g(iK') = 0$. Since $g(z)$ has periods $2K$ and $2iK'$ and can have poles only at $2pK + (2q+1)K'i$, $p, q \in \mathbb{Z}$, $g(z)$ is an entire elliptic function, and thus $w(z) = P(\mathrm{sn}^2 z)$ by Liouville's theorem. Substituting (4.1.2), we find that $P(\cos^2 t)$ is a solution of the Ince equation that corresponds to Lamé's equation. This solution can be written as a finite linear combination of $\cos(2nt)$, $n \in \mathbb{N}_0$. Therefore, $w(z)$ is a Lamé polynomial of the first kind. Arguing similarly, we see that the statement of the theorem is true if $w(z)$ has period or semi-period $2K$, period or semi-period $2iK'$, and is even or odd about K.

Now let $w(z)$ be a nontrivial elliptic function with periods $4K$ and $4iK'$ which solves Lamé's equation. By Floquet's theorem, $w(z)$ has period or semi-period $2K$ and period or semi-period $2iK'$. If $w(z)$ is neither even nor odd, then all solutions are of this form. It follows from the first part of the proof that the Lamé equation has two linearly independent solutions which are Lamé polynomials. This is impossible. Hence w must be even or odd and the proof is complete. ☐

The following result is due to Erdélyi [**16**].

THEOREM 4.6.4. *Let* $\nu \in \mathbb{N}_0$. *Then*

$$a_\nu^m\left(k^2\right) + a_\nu^{\nu-m}\left(k'^2\right) = \nu\left(\nu+1\right), \qquad m = 0, 1, \dots \nu, \tag{4.6.19}$$

and $Ec_\nu^m\left(z, k^2\right)$ *is a constant multiple of* $Ec_\nu^{\nu-m}\left(i\left(z - K - iK'\right), k'^2\right)$. *Moreover,*

$$b_\nu^m\left(k^2\right) + b_\nu^{\nu-m+1}\left(k'^2\right) = \nu\left(\nu+1\right), \qquad m = 1, 12, \dots \nu, \tag{4.6.20}$$

and $Es_\nu^m\left(z, k^2\right)$ *is a constant multiple of* $Es_\nu^{\nu-m+1}\left(i\left(z - K - iK'\right), k'^2\right)$.

PROOF. Let $\nu = 2n$ with $n \in \mathbb{N}_0$, and consider the Lamé polynomials $Ec_\nu^{2p}\left(z, k^2\right)$ for $p = 0, 1, \dots, n$. Employing the substitution (4.5.1) we find that the function $w_p := Ec_\nu^{2p}\left(i\left(z - K - iK'\right), k'^2\right)$ solves Lamé's equation (4.1.1) with $h_p := \nu\left(\nu+1\right) - a_\nu^{2p}\left(k'^2\right)$. Therefore, $w_p(z)$ is a Lamé polynomial of type 1), and it belongs to the eigenvalue h_p. By Theorem 4.6.1 h_p must equal $a_\nu^{2m}\left(k^2\right)$ for some $m = 0, 1, \dots n$. Taking into account the ordering of the eigenvalues, we obtain

$$\nu\left(\nu+1\right) - a_\nu^{\nu-2m}\left(k'^2\right) = a_\nu^{2m}\left(k^2\right), \qquad m = 0, 1, \dots, n.$$

Moreover, $Ec_\nu^{\nu-2m}\left(i\left(z - K - iK'\right), k'^2\right)$ must be a constant multiple of $Ec_\nu^{2m}\left(z, k^2\right)$. The other seven types of Lamé polynomials are treated similarly. ☐

From Theorems 4.3.1 and 4.6.4 we obtain the following theorem on the distribution of zeros of Lamé polynomials.

THEOREM 4.6.5. *Let $n \in \mathbb{N}_0$ and $m = 0, 1, \ldots, n$. Each of the Lamé polynomials 4.6.1 has m zeros in $(0, K)$ and $n - m$ zeros in $(K, K + iK')$.*

4.7. Lamé Polynomials in Algebraic Form.

Every Lamé polynomial (4.6.1) can be written as

$$w(z) = \left(\mathrm{sn}^2 z\right)^\rho \left(\mathrm{cn}^2 z\right)^\sigma \left(\mathrm{dn}^2 z\right)^\tau P\left(\mathrm{sn}^2 z\right), \tag{4.7.1}$$

where ρ, σ, τ are either 0 or $\frac{1}{2}$. Using the identities

$$\mathrm{dn}^2 z = 1 - k^2 \mathrm{sn}^2 z, \qquad \mathrm{cn}^2 z = 1 - \mathrm{sn}^2 z,$$

and the substitution $\xi = \mathrm{sn}^2 z$, we find that every Lamé polynomial (4.6.1) can be written as a quasi-polynomial

$$\xi^\rho (1 - \xi)^\sigma \left(1 - k^2 \xi\right)^\tau P(\xi). \tag{4.7.2}$$

The polynomial $P(\xi)$ is of degree n, and, by Theorem 4.6.5, it has m simple zeros in $(0, 1)$ and $n - m$ simple zeros in $(1, k^{-2})$. The functions (4.7.2) satisfy the Heun equation (4.1.7). This shows that the functions (4.7.1) are special cases of Heun quasi-polynomials, or, more generally, of Heine-Stieltjes quasipolynomials

The existence of quasi-polynomial solutions of (4.1.7) and various of their properties can be proved directly by the following method due to Stieltjes [**64**].

Let $n \in \mathbb{N}_0$, $m = 0, 1, 2, \ldots n$, $k \in (0, 1)$ and ρ, σ, $\tau \in \{0, 1/2\}$ be given. Let D be the open convex domain in $\mathbb{R}^n$ consisting of $(\xi_1, \xi_2, \ldots, \xi_n)$ satisfying

$$0 < \xi_1 < \xi_2 < \ldots < \xi_m < \xi_{m+1} < \ldots < \xi_n < k^{-2}. \tag{4.7.3}$$

Define the function

$$(4.7.4) \qquad g(\xi_1, \xi_2, \dots, \xi_n) := \left(\prod_{p=1}^{n} \xi_p^{\rho+\frac{1}{4}} \left|1-\xi_p\right|^{\sigma+\frac{1}{4}} \left(1-k^2\xi_p\right)^{\tau+\frac{1}{4}} \right) \prod_{q<r} (\xi_q - \xi_r)$$

for $(\xi_1, \xi_2, \dots, \xi_n)$ in the closure of D, that is , for $(\xi_1, \xi_2, \dots, \xi_n)$ satisfying

$$(4.7.5) \qquad 0 \le \xi_1 \le \xi_2 \le \dots \le \xi_m \le \xi_{m+1} \le \dots \le \xi_n \le k^{-2}.$$

For $(\xi_1, \xi_2, \dots, \xi_n) \in D$, we calculate

$$(4.7.6) \qquad \frac{\partial \ln g}{\partial \xi_p} = \frac{\rho+\frac{1}{4}}{\xi_p} + \frac{\sigma+\frac{1}{4}}{\xi_p - 1} + \frac{\tau+\frac{1}{4}}{\xi_p - k^{-2}} + \sum_{p\neq q=1}^{n} \frac{1}{\xi_p - \xi_q}.$$

If we calculate the Hessian matrix of second partial derivatives of $\ln g$, we see that the entries in its main diagonal are negative, all other entries are positive and the row sums are negative. By looking at Gershgorin circles, we find that the Hessian is negative definite. It follows that $\ln g$ is a strictly concave function on D. Since g is defined and continuous on a compact subset of $\mathbb{R}^n$ it attains its absolute maximum. Since g is nonnegative and vanishes along the boundary of its domain of definition, the maximum is attained at a point in D. Since $\ln g$ is strictly concave, this point is uniquely determined. Therefore, the system of equations

$$(4.7.7) \qquad \frac{\rho+\frac{1}{4}}{\xi_p} + \frac{\sigma+\frac{1}{4}}{\xi_p - 1} + \frac{\tau+\frac{1}{4}}{\xi_p - k^{-2}} + \sum_{p\neq q=1}^{n} \frac{1}{\xi_p - \xi_q} = 0, \qquad p = 1, 2, \dots, n,$$

has a unique solution $(\xi_{1,0}, \xi_{2,0}, \dots, \xi_{n,0}) \in D$. By direct calculation, one can show that the system of equations (4.7.7) holds if and only if the function

$$(4.7.8) \qquad \xi^{\rho} (1-\xi)^{\sigma} \left(1-k^{-2}\xi\right)^{\tau} \prod_{j=1}^{n} (\xi - \xi_{j,0}),$$

satisfies (4.7.6). The method proves that, given n, m, k, ρ, σ, τ there is a unique vector $(\xi_{1,0}, \xi_{2,0}, \dots, \xi_{n,0}) \in D$ such that the function in (4.7.8) satisfies (4.7.6). The system (4.7.7) admits the following electrostatic interpretation: Given three point masses fixed at $\xi = 0$, $\xi = 1$, and $\xi = k^{-2}$ with positive charges $\rho + 1/4$, $\sigma + 1/4$, and

$\tau + 1/4$ respectively, and n movable point masses at $\xi_1, \dots \xi_n$ arranged according to (4.7.3) with unit positive charges, the equilibrium position is attained for $\xi_j = \xi_{j,0}$ for $j = 1, 2, \dots n$.

4.8. Integral Equations

Let $w_1(z)$ and $w_2(z)$ be a pair of solutions of the same Lamé equation for some given parameters h, ν, k. Then the function

$$v(x, y) = w_1(x)\, w_2(y), \qquad x, y \in \mathbb{R}, \tag{4.8.1}$$

satisfies the hyperbolic partial differential equation

$$\partial_1^2 v - \partial_2^2 v - k^2 \nu (v + 1) \left(\operatorname{sn}^2 x - \operatorname{sn}^2 y \right) v = 0. \tag{4.8.2}$$

We apply Riemann's method of integration to this equation. Consider the symmetric function g of four real variables defined by

$$\begin{aligned} g(x, y, x_0, y_0) &:= k^2 \operatorname{sn} x \operatorname{sn} y \operatorname{sn} x_0 \operatorname{sn} y_0 \\ &- \frac{k^2}{k'^2} \operatorname{cn} x \operatorname{cn} y \operatorname{cn} x_0 \operatorname{cn} y_0 + \frac{1}{k'^2} \operatorname{dn} x \operatorname{dn} y \operatorname{dn} x_0 \operatorname{dn} y_0, \end{aligned} \tag{4.8.3}$$

or, equivalently,

$$g(x, y, x_0, y_0) := 1 + 2 \frac{(f(x + y) - f(x_0 + y_0))(f(x - y) - f(x_0 - y_0))}{(f(x + y) + f(x_0 + y_0))(f(x - y) + f(x_0 - y_0))}, \tag{4.8.4}$$

where

$$f(z) := k \operatorname{sn} z + \operatorname{dn} z.$$

Equality of the right-hand sides of (4.8.3) and (4.8.4) follows from the addition formulas for the Jacobian elliptic functions. The calculation is lengthy but nowadays can be carried out easily with mathematical software like *Maple*. The *Maple* command **expand** forces application of the addition theorems, and then the difference of the right-hand sides simplifies to 0. Define the function

$$R(x,y,x_0,y_0) := P_\nu(g(x,y,x_0,y_0)), \tag{4.8.5}$$

where P is the Legendre function. Since $f(z) > 0$ for real z, the representation 4.8.4 shows that $g(x,y,x_0,y_0) \in (-1,\infty)$ On this interval, P_ν is real-analytic, so R is a real-analytic function on $\mathbb{R}^4$. Another calculation shows that R as a function of (x,y) solves (4.8.2) for any (x_0,y_0). This may also be derived from Section 3.10 where equation (4.8.2) is obtained by the method of separation of variables from the Laplace equation. Formula (4.8.4) shows that $g(x,y,x_0,y_0) = 1$ on the lines $y - y_0 = \pm(x - x_0)$. Since $P(1) = 1$, it follows that

$$R(x, y_0 \pm (x - x_0), x_0, y_0) = 1. \tag{4.8.6}$$

Therefore, R is the Riemann function of the partial differential equation (4.8.2). Set

$$a := R\partial_2 v - v\partial_2 R, \qquad b := R\partial_1 v - v\partial_1 R,$$

where v is the function defined in (4.8.1). Since , we obtain $\partial_1 a = \partial_2 b$, we obtain

$$\int_C a(x,y)\,dx + b(x,y)\,dy = 0 \tag{4.8.7}$$

for for every closed rectifiable path C in $\mathbb{R}^2$.

We choose the pentagonal path $C = C_1 + C_2 + C_3 + C_4 + C_5$, see Figure 4.8.1, where x_0, y_0, y_1 are arbitrary real numbers. By (4.8.6), $R = 1$ on C_3 and C_4.

Evaluating the line integrals along C_3 and C_4 using appropriate parameterizations, we find

$$\int_{C_3} a(x,y)\,dx + b(x,y)\,dy = v(x_0,y_0) - v(x_0 + 2K, y_0 + 2K), \tag{4.8.8}$$

$$\int_{C_4} a(x,y)\,dx + b(x,y)\,dy = v(x_0,y_0) - v(x_0-2K, y_0+2K)\,. \tag{4.8.9}$$

We now assume that w_1 has period $4K$. Since $R(.,y,x_0,y_0)$ has period $4K$, $a(.,y)$ and $b(.,y)$ also have period $4K$ for all y. It follows that

$$\int_{C_2} a(x,y)\,dx + b(x,y)\,dy = -\int_{C_5} a(x,y)\,dx + b(x,y)\,dy, \tag{4.8.10}$$

$$\int_{C_1} a(x,y)\,dx + b(x,y)\,dy = \int_{x_0-2K}^{x_0+2K} a(x,y_1)\,dx = \int_{-2K}^{2K} a(x,y_1)\,dx. \tag{4.8.11}$$

Combining (4.8.7), (4.8.8), (4.8.9), (4.8.10), (4.8.11), we obtain

$$\begin{aligned} & w_1(x_0+2K)\,w_2(y_0+2K) - w_1(x_0)\,w_2(y_0) \\ =& \frac{1}{2}\int_{-2K}^{2K} \left(w_2'(y_1)\,R(x,y_1,x_0,y_0) - w_2(y_1)\,\partial_2 R(x,y_1,x_0,y_0)\right) w_1(x)\,dx. \end{aligned} \tag{4.8.12}$$

Now w_1 has period or semi-period $2K$, that is, there is $\sigma \in \{-1,1\}$ such that

$$w_1(x+2K) = \sigma w_1(x)\,. \tag{4.8.13}$$

Hence the left-hand side and so the right-hand side of (4.8.12) vanish when $w_2 = w_1$. Therefore, multiplying (4.8.12) by $w_1(y_1)$, we obtain

$$\begin{aligned} & \left(w_1(x_0+2K)\,w_2(y_0+2K) - w_1(x_0)\,w_2(y_0)\right) w_1(y_1) \\ =& \frac{1}{2} W[w_1,w_2] \int_{-2K}^{2K} R(x,y_1,x_0,y_0)\,w_1(x)\,dx, \end{aligned} \tag{4.8.14}$$

where $W[w_1,w_2]$ denotes the (constant) Wronskian of w_1, w_2. If w_1, w_2 are linearly dependent, then (4.8.14) is trivially true. So let w_1, w_2. be linearly independent. By Floquet's theorem, there is τ such that

$$w_2(x+2K) = \sigma w_1(x) + \tau w_2(x)\,. \tag{4.8.15}$$

Using (4.8.13) and (4.8.15) we rewrite the left-hand side of (4.8.14) as

$$(w_1(x_0+2K)\,w_2(y_0+2K) - w_1(x_0)\,w_2(y_0))\,w_1(y_1) = w_1(x_0)\,w_1(y_0)\,w_1(y_1).$$

We proved the following theorem.

THEOREM 4.8.1. *Let w_1 be one of the Lamé functions Ec_ν^m or Es_ν^m. Then there exists a constant μ such that, for all $x_0,\ y_0,\ y_1 \in \mathbb{R}$,*

$$\mu w_1(x_0)\,w_1(y_0)\,w_1(y_1) = \int_{-2K}^{2K} R(x, y_1, x_0, y_0)\,w_1(x)\,dx. \tag{4.8.16}$$

The constant μ can be written in the form

$$\mu = \frac{2\sigma\tau}{W[w_1, w_2]}$$

where w_2 is a solution of the same Lamé equation satisfied by w_1, linearly independent of w_1 and σ, τ are determined from (4.8.13), (4.8.15).

As a corollary we obtain the following integral equations for Lamé functions.

THEOREM 4.8.2. *The Lamé function $w_1(z) = Ec_\nu^{2m}(z, k^2)$ satisfies*

$$w_1(z)\,w_2(K) = \int_0^K P_\nu\left(\frac{1}{k'}\operatorname{dn} x \operatorname{dn} z\right) w_1(z)\,dz, \tag{4.8.17}$$

where w_2 is the solution of (4.1.1) *with $h = a_\nu^{2m}(k^2)$ determined by $w_2(0) = 0$,* and *$w_2'(0) = 1$;*

The Lamé function $w_1(z) = Ec_\nu^{2m+1}(z, k^2)$ satisfies

$$w_1(z)\,w_2(K) = -k^2 \operatorname{sn} x \int_0^K \operatorname{sn} z\, P_\nu'\left(\frac{1}{k'}\operatorname{dn} x \operatorname{dn} z\right) w_1(z)\,dz, \tag{4.8.18}$$

where w_2 is the solution of (4.1.1) *with $h = a_\nu^{2m+1}(k^2)$ determined by $w_2(0) = 1$,* and *$w_2'(0) = 0$;*

The Lamé function $w_1(z) = Es_\nu^{2m+1}(z, k^2)$ satisfies

$$w_1(z)\,w_2'(K) = \frac{k^2}{k'} \operatorname{cn} x \int \operatorname{cn} z\, P_\nu'\left(\frac{1}{k'}\operatorname{dn} x \operatorname{dn} z\right) w_1(z)\,dz, \tag{4.8.19}$$

where w_2 *is the solution of* (4.1.1) *with* $h = b_\nu^{2m+1}(k^2)$ *determined by* $w_2(0) = 0$, and $w_2'(0) = 1$;

The Lamé function $w_1(z) = Es_\nu^{2m+1}(z, k^2)$ *satisfies*

$$w_1(z)\, w_2'(K) = \frac{k^4}{k'} \operatorname{sn} x \operatorname{cn} x \int_0^K \operatorname{sn} x \operatorname{cn} z\, P_\nu''\left(\frac{1}{k'} \operatorname{dn} x \operatorname{dn} z\right) w_1(z)\, dz, \tag{4.8.20}$$

where w_2 *is the solution of* (4.1.1) *with* $h = b_\nu^{2m+2}(k^2)$ *determined by* $w_2(0) = 1$, and $w_2'(0) = 0$.

Proof. To prove (4.8.17) we use Theorem 4.8.1 with $x_0 = x$, $y_0 = 0$, $y_1 = K$. We have $W[w_1, w_2] = w_1(0)$, and, by (4.8.15) with $x = -K$, $\tau w_1(K) = 2w_2(K)$ Then (4.8.17) follows by noting that the function under the integral is even with period $2K$. The other equations follow similarly after differentiating equation (4.8.16) with respect to y_0, y_1, and y_1 and y_0, respectively. □

We note that $\nu \in \mathbb{N}_0$ for Lamé polynomials in which case the Legendre function P becomes the Legendre polynomial of degree ν. We also note that the factor of $w_1(x)$ on the left-hand sides of the integral equations vanishes if and only if w_2 also has period $4K$, so if and only if odd and even periodic solutions coexist. By Theorem 4.2.1 this happens if and only if $\nu \in \mathbb{N}_0$ and $m > \nu$. This section is based on [**72, 74**].

4.9. Asymptotic Expansions

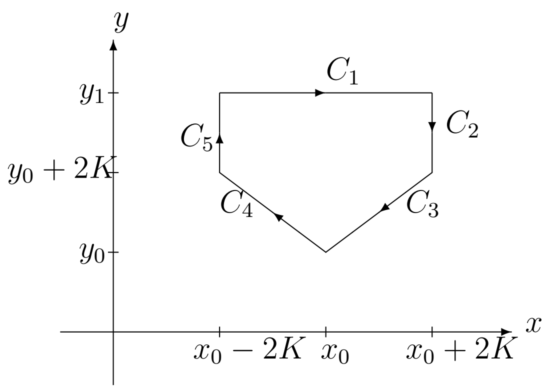
As $\nu \to \infty$ the eigenvalue function $a_\nu^m(k_2)$ has the asymptotic expansion

$$a_\nu^m \sim p\kappa - \tau_0 - \tau_1\kappa^{-1} - \tau_2\kappa^{-2} - ... \tag{4.9.1}$$

Where

$$\kappa = k(\nu(\nu+1))^{1/2}, \qquad p = 2m + 1. \tag{4.9.2}$$

FIGURE 4.8.1. integration path



The coefficients are given by

$$
\begin{aligned}
\tau_0 =& \frac{1}{2^3}\left(1+k^2\right)\left(1+p^2\right), \\
\tau_1 =& \frac{p}{2^6}\left(\left(1+k^2\right)^2\left(p^2+3\right)-4k^2\left(p^2+5\right)\right), \\
\tau_2 =& \frac{1}{2^{10}}\left(1+k^2\right)\left(1-k^2\right)^2\left(5p^4+34p^2+9\right), \\
\tau_3 =& \frac{p}{2^{14}}\left(\left(1+k^2\right)^4\left(33p^4+410p^2+405\right)\right. \\
& \left.-24k^2\left(1+k^2\right)^2\left(7p^4+90p^2+95\right)+16k^2\left(9p^4+130p^2+173\right)\right), \\
\tau_4 =& \frac{1}{2^{16}}\left(\left(1+k^2\right)^5\left(63p^6+1260p^4+2943p^2+486\right)\right. \\
& -8k^2\left(1+k^2\right)^3\left(49p^6+1010p^4+2493p^2+432\right) \\
& \left.16k^4\left(1+k^2\right)\left(35p^6+760p^4+2043p^2+378\right)\right).
\end{aligned}
\tag{4.9.3}
$$

The first three terms in (4.9.1) were given by Ince [**32**] . The expressions for τ_3 and τ_4 are due to Müller [**50**]. Additional terms can be derived using the algorithm from [**50**] with the help of Maple.

The asymptotic expansion (4.9.1) also holds for $b_\nu^m(k^2)$, since the difference $b_\nu^m(k^2) - a_\nu^m(k^2)$ becomes exponentially small as $\nu \to \infty$ according to

$$\text{(4.9.4)}\quad b_\nu^m\left(k^2\right) - a_\nu^m\left(k^2\right) = \frac{\left(1-k^2\right)^{-m-1/2}}{m!\sqrt{2\pi}}(8k\nu)^{m+3/4}\left(\frac{1-k}{1+k}\right)^{\nu+1/2}\left(1+O\left(\nu^{-1/2}\right)\right)$$

as $\nu \to \infty$. This formula was derived in [**79**] based on results from [**82**].

4.10. Further Results

Hargrave and Sleeman [**20**] investigate the asymptotic behavior of Lamé polynomials as $\nu \to \infty$. Mueller [**50, 51, 52**] finds asymptotic expansions for solutions of the Lamé wave equation. Erdélyi [**18**] and Sleeman [**62**] study expansions of Lamé functions in series of Legendre functions. Volkmer [**76**] considers the expansion of analytic functions of two variables in terms of products of Lamé polynomials. Triebel [**68**] applies Lamé functions in the theory of conformal maps. Patera and Winternitz [**53**] find bases for the rotation group. Erdélyi [**17**], Shail [**60, 61**], Whittaker [**85**] and Volkmer [**73, 75**] study integral equations for Lamé functions. Sleeman [**63**] has integral relations for Lamé functions involving double integrals. Lambe [**39, 41**] gives additional results on Lamé polynomials. Volkmer [**76, 79**] gives additional results on the Lamé equation. Arscott and Khabaza [**9**] compute tables of Lam´e polynomials. Jansen [**34**] computes Lamé functions. Ritter [**58**] and Dobner and Ritter [**13**] compute Lam´e polynomials. If $\nu - 1/2$ is an integer, Lamé's equation admits solutions which are non-rational algebraic functions of $\operatorname{sn} z \ \operatorname{cn} z \ \operatorname{dn} z$. Erdélyi [**15**], Ince [**31**] and Lambe [**40**] investigate these algebraic Lamé functions. Maier [**46, 47**] obtains interesting new results on the Lamé equation.

CHAPTER 5

A Generalization of Lamé's Equation

In this Chapter we discuss a generalization of Lamé's equation due to Pawellek [**54**]. This generalization is achieved by the use of so called generalized elliptic functions [**55**].

5.1. The Generalized Jacobi Elliptic Functions

Let $0 < k_2 < k_1 < 1$. We now introduce generalized Jacobian functions s, c, d_1, d_2; [**55**], [**71**]. We set

(5.1.1) $$\kappa := \left(\frac{k_1^2 - k_2^2}{1 - k_2^2}\right)^{1/2} \in (0,1), \quad k_2' = \sqrt{1 - k_2^2}, \quad \kappa' = \sqrt{1 - \kappa^2},$$

and

(5.1.2) $$p(u, k_1, k_2) := \left(k_2'^2 + k_2^2 \operatorname{sn}^2(k_2' u, \kappa)\right)^{-1/2}.$$

We have to specify the choice of root in (5.1.2). The function p is used only on the lines $\Im u = 0$, $k_2' \Re u = K := K(\kappa)$ and $k_2' \Im u = K' := K(\kappa')$. On the first two lines $k_2'^2 + k_2^2 \operatorname{sn}^2(k_2' u, \kappa) > 0$ and we use the positive root to define p. On the third line we define p as follows. We set $u = u' + i\frac{K'}{k_2'}$ with $u' \in \mathbb{R}$. Then

$$k_2'^2 + k_2^2 \operatorname{sn}^2(k_2' u, \kappa) = k_2'^2 + k_2^2 \kappa^{-2} \operatorname{sn}^{-2}(k_2' u', \kappa)$$

and we define

$$p(u, k_1, k_2) = \frac{\operatorname{sn}(k_2' u', \kappa)}{\sqrt{k_2'^2 \operatorname{sn}^2(k_2' u', \kappa) + k_2^2 \kappa^{-2}}} \quad \text{for } u = u' + i\frac{K'}{k_2'}, u' \in \mathbb{R}.$$

Then p is an analytic function of u on each of the three lines.

Now we set

$$
\begin{aligned}
\mathrm{s}(u,k_1,k_2) &= \mathrm{sn}(k_2' u,\kappa)p(u,k_1,k_2),\\
\mathrm{c}(u,k_1,k_2) &= k_2'\,\mathrm{cn}(k_2' u,\kappa)p(u,k_1,k_2),\\
\mathrm{d}_1(u,k_1,k_2) &= k_2'\,\mathrm{dn}(k_2' u,\kappa)p(u,k_1,k_2),\\
\mathrm{d}_2(u,k_1,k_2) &= k_2' p(u,k_1,k_2).
\end{aligned} \tag{5.1.3}
$$

These functions satisfy

$$
\mathrm{s}^2+\mathrm{c}^2 = 1, \qquad \mathrm{d}_i^2 = 1-k_i^2\mathrm{s}^2, \quad i=1,2. \tag{5.1.4}
$$

The derivatives of the elliptic functions together with definitions (5.1.3) gives the first derivatives

$$
\begin{aligned}
\mathrm{s}'(z) &= \mathrm{c}(z)\,\mathrm{d}_1(z)\,\mathrm{d}_2(z),\\
\mathrm{c}'(z) &= -\mathrm{s}(z)\,\mathrm{d}_1(z)\,\mathrm{d}_2(z),\\
\mathrm{d}_1'(z) &= -k_1^2\mathrm{s}(z)\,\mathrm{c}(z)\,\mathrm{d}_2(z),\\
\mathrm{d}_2'(z) &= -k_2^2\mathrm{s}(z)\,\mathrm{c}(z)\,\mathrm{d}_1(z).
\end{aligned} \tag{5.1.5}
$$

From their properties, we can think of the functions (5.1.3) as generalizations of the usual Jacobi elliptic functions $\mathrm{sn}(z,k)$, $\mathrm{cn}(z,k)$, $\mathrm{dn}(z,k)$, and they reduce to them as $k_2 \to 0$. From the doubly-periodic properties of the Jacobian elliptic functions we can deduce that the generalized elliptic functions $\mathrm{s}(z)$, $\mathrm{c}(z)$, $\mathrm{d}_1(z)$, and $\mathrm{d}_2(z)$ are quasi-doubly periodic

$$
\mathrm{s}\left(z+\frac{4K(\kappa)}{k_2'}\right) = \mathrm{s}\left(z+\frac{2iK(\kappa')}{k_2'}\right) = \pm\mathrm{s}(z), \tag{5.1.6}
$$

$$
\mathrm{c}\left(z+\frac{4K(\kappa)}{k_2'}\right) = \mathrm{c}\left(z+\frac{2K(k)+2iK(\kappa')}{k_2'}\right) = \pm\mathrm{c}(z), \tag{5.1.7}
$$

$$
\mathrm{d}_1\left(z+\frac{2K(\kappa)}{k_2'}\right) = \mathrm{d}_1\left(z+\frac{4iK(\kappa')}{k_2'}\right) = \pm\mathrm{d}_1(z), \tag{5.1.8}
$$

$$\text{(5.1.9)} \qquad \mathrm{d}_2\left(z+\frac{2K(\kappa)}{k_2'}\right)=\mathrm{d}_2\left(z+\frac{2iK(\kappa')}{k_2'}\right)=\pm\mathrm{d}_2(z).$$

For more details on the generalized elliptic function see [**55**]. This section is based on [**54, 55**].

5.2. A Generalization of Lamé's Equation.

We consider the equation

$$\text{(5.2.1)} \qquad \frac{d^2w}{dz^2}+(h+V(z))\,w=0,$$

where

(5.2.2)

$$V(z):=\left(\alpha k_1^2k_2^2+\beta k_2^2\right)\mathrm{s}^4(z,k_1,k_2)-\left(\nu(\nu+1)\,k_1^2+\gamma k_2^2+\delta k_1^2k_2^2\right)\mathrm{s}^2(z,k_1,k_2).$$

The number $k_1$and k_2 are such that $0<k_2<k_1<1$, and denote the moduli of the generalized Jacobian elliptic function $\mathrm{s}(z)=\mathrm{s}(z,k_1,k_2)$. We assume that the parameters α, β, γ, δ, ν are real. The parameter h is the spectral parameter and will also be always real. Equation 5.2.2 is a natural generalization of the Lamé equation 4.1.1. Also note that as $k_2\to 0$, 5.2.2 reduces to the original Lamé equation.

If we substitute $\xi=\mathrm{s}^2(z,k_1,k_2)$, we get

$$\text{(5.2.3)} \qquad \begin{aligned}&\frac{d^2w}{d\xi^2}+\frac{1}{2}\left(\frac{1}{\xi}+\frac{1}{\xi-1}+\frac{1}{\xi-k_1^{-2}}+\frac{1}{\xi-k_2^{-2}}\right)\frac{dw}{d\xi}\\ &\qquad\qquad -\frac{hk_1^{-2}k_2^{-2}+Az^2-Bz}{4\xi(\xi-1)\left(\xi-k_1^{-2}\right)\left(\xi-k_2^{-2}\right)}w=0,\end{aligned}$$

with

$$A=\alpha+\beta k_1^{-2}, \qquad B=\nu(\nu+1)\,k_2^{-2}+\gamma k_1^{-2}+\gamma.$$

Equation (5.2.3) is a generalization of the algebraic form of Lamé equation (5.2.3). It is of Fuchsian type with five regular singular points. The exponents are 0 and $\frac{1}{2}$ for $z=0,\,1,\,k_1^{-2},k_2^{-2}$ and $\frac{1}{2}\left[1\pm\left(1+\alpha+\beta k_1^{-2}\right)^{\frac{1}{2}}\right]$ for ∞.

By the substitution of $t = \mathrm{a}(z, k_1, k_2)$, where $\mathrm{a}(z, k_1, k_2)$ is defined by

$$\frac{dt}{dz} = \sqrt{\left(1 - k_1^2 \sin^2 t\right)\left(1 - k_2^2 \sin^2 t\right)}, \tag{5.2.4}$$

(the function $\mathrm{a}(z, k_1, k_2)$ can be understood as the generalization of Jacobi's amplitude function $\mathrm{am}(z, k)$) and using

$$\frac{d^2 t}{dz^2} = \frac{1}{2}\left(k_1^2 k_2^2 - k_1^2 - k_2^2\right) \sin 2t - \frac{1}{4} k_1^2 k_2^2 \sin 4t, \tag{5.2.5}$$

with $w(z) = y(t)$, equation (5.2.1) becomes a generalized Ince equation with $\eta = 2$

(5.2.6)

$$(1 + a_1 \cos 2t + a_2 \cos 4t)\, y'' + (b_1 \sin 2t + b_2 \sin 4t)\, y' + (\lambda + d_1 \cos 2t + d_2 \cos 4t)\, y = 0,$$

where the coefficients a_j, b_j, d_j, $j = 1, 2$, and λ are given by

$$a_1 = \frac{k_1^2 + k_2^2 - k_1^2 k_2^2}{2 + \frac{3}{4} k_1^2 k_2^2 - k_1^2 - k_2^2}, \tag{5.2.7}$$

$$a_2 = \frac{\frac{1}{4} k_1^2 k_2^2}{2 + \frac{3}{4} k_1^2 k_2^2 - k_1^2 - k_2^2}, \tag{5.2.8}$$

$$b_1 = -a_1, \tag{5.2.9}$$

$$b_2 = -2a_1, \tag{5.2.10}$$

$$d_1 = \frac{\nu(\nu + 1) k_1^2 + (\gamma - \beta) k_2^2 - (\delta - \alpha) k_1^2 k_2^2}{2 + \frac{3}{4} k_1^2 k_2^2 - k_1^2 - k_2^2}, \tag{5.2.11}$$

$$d_2 = \frac{\frac{1}{4}\left(\alpha k_1^2 k_2^2 + \beta k_2^2\right)}{2 + \frac{3}{4} k_1^2 k_2^2 - k_1^2 - k_2^2}, \tag{5.2.12}$$

$$\lambda = \frac{2h - \nu(\nu + 1) k_1^2 + \left(\frac{3\beta}{4} - \gamma\right) k_2^2 - \left(\frac{3\alpha}{4} - \delta\right) k_1^2 k_2^2}{2 + \frac{3}{4} k_1^2 k_2^2 - k_1^2 - k_2^2}. \tag{5.2.13}$$

Note that coefficients (5.2.7), (5.2.8), (5.2.9), (5.2.10), (5.2.11), (5.2.12), (5.2.13) reduce to (4.1.5) as $k_2 \to 0$.

From (5.1.6) we know that the function $\mathrm{s}^2(z, k_1, k_2)$ has period $2\mathbf{K}$

$$\mathbf{K} := 2k_2'^{-1} K(\kappa), \tag{5.2.14}$$

where κ is defined by (5.1.1) and $K(\kappa)$ is the complete elliptic integral of the first kind. Therefore, if $w(z)$ is a solution of (5.2.1) then also $w(z + 2\mathbf{K})$ is a solution. Hence the generalized Lamé equation is a Hill's equation with period $2\mathbf{K}$. Also note that $\mathrm{s}^2(z)$ is an even function. Therefore, if $w(z)$ is a solution of (5.2.1), then $w(-z)$ is also a solution. Hence (5.2.1) is an even Hill's equation. The function $\mathrm{s}^2(z)$ has a second period $2i\mathbf{K}'$, where

$$\mathbf{K}' := k_2'^{-1} K(\kappa'). \tag{5.2.15}$$

Hence, (5.2.2) can also be considered as a Hill's equation with period $i\mathbf{K}'$. Since the two lines $\Re z = 0$, $\Im z = \mathbf{K}$ intersect at $\mathbf{K}$, instead of asking for even or odd solutions it is more natural to ask for solutions which are even or odd about $\mathbf{K}$, that is, $w(\mathbf{K} - z) = \pm w(\mathbf{K} + z)$. Note that $\mathrm{s}^2(z)$ is even about $\mathbf{K}$. Solutions to (5.2.1) with period $2\mathbf{K}$ and $4\mathbf{K}$ correspond to solutions to (5.2.6) with period π and 2π respectively.

A solution $w(z)$ is even about $\mathbf{K}$ and has period $2\mathbf{K}$ if and only if $w(z)$ satisfies the boundary conditions

$$w'(0) = w'(\mathbf{K}) = 0. \tag{5.2.16}$$

Equation (5.2.1) together with the boundary conditions (5.2.16) pose a regular Sturm-Liouville eigenvalue problem with spectral parameter h. Therefore, the corresponding eigenvalues h form a real increasing sequence that tends to infinity. We denote these eigenvalues by

$$a_{2m} := a_{2m}\left(k_1^2, k_2^2, \alpha, \beta, \gamma, \delta, \nu\right).$$

Similarly, a solution $w(z)$ of equation (5.2.1) is even about $\mathbf{K}$ and has semi-period $2\mathbf{K}$ if and only if

$$w(0) = w'(\mathbf{K}) = 0. \tag{5.2.17}$$

The corresponding sequence of eigenvalues is denoted by

$$a_{2m+1} := a_{2m+1}\left(k_1^2, k_2^2, \alpha, \beta, \gamma, \delta, \nu\right).$$

A solution $w(z)$ of equation (5.2.1) is odd about $\mathbf{K}$ and has semi-period $2\mathbf{K}$ if and only if

$$w'(0) = w(\mathbf{K}) = 0. \tag{5.2.18}$$

The corresponding sequence of eigenvalues is denoted by

$$b_{2m+1} := b_{2m+1}\left(k_1^2, k_2^2, \alpha, \beta, \gamma, \delta, \nu\right).$$

Finally, solution $w(z)$ of equation (5.2.1) is odd about $\mathbf{K}$ and has semi-period $2\mathbf{K}$ if and only if

$$w(0) = w(\mathbf{K}) = 0. \tag{5.2.19}$$

The corresponding sequence of eigenvalues is denoted by

$$b_{2m+2} := b_{2m+2}\left(k_1^2, k_2^2, \alpha, \beta, \gamma, \delta, \nu\right).$$

All four sequences of eigenvalues are increasing and $m = 0, 1, 2, ...$The eigenfunctions belonging to these eigenvalues are the generalized Lamé functions. The notation of the eigenvalues is chosen in such a way that an even or odd subscript is associated with the generalized Lamé functions with period $2\mathbf{K}$ or semi-period $2\mathbf{K}$, respectively. The letter a denotes eigenvalues associated with the generalized Lamé functions which are even about $\mathbf{K}$, whereas the letter b denotes eigenvalues associated with the generalized Lamé which are odd about $\mathbf{K}$.

One should also note that a_m is defined for $m = 0, 1, 2, ...$, whereas b_m is defined only for $m = 1, 2, 3, ...$ If we define a_j, b_j, d_j, $j = 1, 2$ by (5.2.7), (5.2.8), (5.2.9), (5.2.10), (5.2.11), (5.2.12) the eigenvalues of the generalized Lamé equation can be expressed as

$$\begin{aligned} 2a_m\left(k_1^2, k_2^2\right) &= \nu\left(\nu+1\right)k_1^2 - \left(\frac{3\beta}{4} - \gamma\right)k_2^2 + \left(\frac{3\alpha}{4} - \delta\right)k_1^2 k_2^2 \\ &+ \left(2 + \frac{3}{4}k_1^2 k_2^2 - k_1^2 - k_2^2\right)\alpha_m\left(\mathbf{a}, \mathbf{b}, \mathbf{d}\right), \end{aligned} \tag{5.2.20}$$

$$\begin{aligned} 2b_m\left(k_1^2, k_2^2\right) &= \nu\left(\nu+1\right)k_1^2 - \left(\frac{3\beta}{4} - \gamma\right)k_2^2 + \left(\frac{3\alpha}{4} - \delta\right)k_1^2 k_2^2 \\ &+ \left(2 + \frac{3}{4}k_1^2 k_2^2 - k_1^2 - k_2^2\right)\beta_m\left(\mathbf{a}, \mathbf{b}, \mathbf{d}\right), \end{aligned} \tag{5.2.21}$$

where α_m, β_m are the eigenvalues of equation (5.2.6) corresponding to even and odd eigenfunctions respectevily.

From Sturm-Liouville theory we obtain the following result.

Theorem 5.2.1. *The eigenvalues of Lamé's equation interlace according to*

$$a_0 < \left\{ \begin{array}{c} a_1 \\ b_1 \end{array} \right\} < \left\{ \begin{array}{c} a_2 \\ b_2 \end{array} \right\} < \left\{ \begin{array}{c} a_3 \\ b_3 \end{array} \right\} < \dots$$

The eigenfunctions of Lamé's equation corresponding to the eigenvalues

$$a_{2m},\ a_{2m+1},\ b_{2m+1},\ b_{2m+2} \tag{5.2.22}$$

are denoted by

$$Ec_{2m}\left(z, k_1^2, k_2^2\right),\ Ec_{2m+1}\left(z, k_1^2, k_2^2\right),\ Es_{2m+1}\left(z, k_1^2, k_2^2\right),\ Es_{2m+2}\left(z, k_1^2, k_2^2\right), \tag{5.2.23}$$

respectively. These are the (simply-periodic) generalized Lamé functions. As eigenfunctions these functions are only determined up to a constant factor. We normalize them by the conditions

$$\int_0^{\mathbf{K}} \mathrm{d}_1(z)\,\mathrm{d}_2(z)\left(Ec_m\left(z,k_1^2,k_2^2\right)\right)^2 dz = \frac{\pi}{4}, \tag{5.2.24}$$

$$\int_0^{\mathbf{K}} \mathrm{d}_1(z)\,\mathrm{d}_2(z)\left(Es_m\left(z,k_1^2,k_2^2\right)\right)^2 dz = \frac{\pi}{4}. \tag{5.2.25}$$

To complete the definition, $Ec_m(\mathbf{K},k_1^2,k_2^2)$ is positive and $\frac{d}{dz}Es_m(\mathbf{K},k_1^2,k_2^2)$ is negative.

Since $\frac{d}{dz}\mathrm{a}(z) = \mathrm{d}_1(z)\,\mathrm{d}_2(z)$, this agrees with the normalization of the generalized Ince functions, and we obtain

$$Ec_m\left(z,k_1^2,k_2^2\right) = Ic_m\left(t;a_j,b_j,d_j\right), \tag{5.2.26}$$

$$Es_m\left(z,k_1^2,k_2^2\right) = Is_m\left(t;a_j,b_j,d_j\right), \tag{5.2.27}$$

where t, z are related by (5.2.4), and a_j, b_j, d_j, $j=1,2$ are given by (5.2.7), (5.2.8), (5.2.9), (5.2.10), (5.2.11), (5.2.12)

From [**10**, Chapter 8, Theorem 2.1] we obtain the following oscillation properties.

THEOREM 5.2.2. *Each of the functions* (5.2.23)*has precisely* m *simple zeros in the open interval* $(0,\boldsymbol{K})$*. The superscript* $2m$, $2m+1$, *or* $2m+2$ *equals the number of zeros in the half-open interval* $(0,2\mathbf{K}]$.

The analog of Theorem 2.3.1 for the generalized Lamé functions is the following

THEOREM 5.2.3. *Each of the function systems*

$$\left\{Ec_{2m}\left(z,k_1^2,k_2^2\right)\right\}_{m=0}^{\infty}, \tag{5.2.28}$$

$$\left\{Ec_{2m+1}\left(z,k_1^2,k_2^2\right)\right\}_{m=0}^{\infty}, \tag{5.2.29}$$

$$\left\{Es_{2m+1}\left(z,k_1^2,k_2^2\right)\right\}_{m=0}^{\infty}, \tag{5.2.30}$$

$$\left\{Es_{2m+2}\left(z,k_1^2,k_2^2\right)\right\}_{m=0}^{\infty}, \tag{5.2.31}$$

is orthogonal over$[0, \boldsymbol{K}]$*, that is, for* $m \neq n$,

$$\int_0^{\boldsymbol{k}} Ec_{2m}\left(z,k_1^2,k_2^2\right) Ec_{2n}\left(z,k_1^2,k_2^2\right) dt = 0, \tag{5.2.32}$$

$$\int_0^{\boldsymbol{k}} Ec_{2m+1}\left(z,k_1^2,k_2^2\right) Ec_{2n+1}\left(z,k_1^2,k_2^2\right) dt = 0, \tag{5.2.33}$$

$$\int_0^{\boldsymbol{k}} Es_{2m+1}\left(z,k_1^2,k_2^2\right) Es_{2n+1}\left(z,k_1^2,k_2^2\right) dt = 0, \tag{5.2.34}$$

$$\int_0^{\boldsymbol{k}} Es_{2m+2}\left(z,k_1^2,k_2^2\right) Es_{2n+2}\left(z,k_1^2,k_2^2\right) dt = 0, \tag{5.2.35}$$

Moreover, each of the system (5.2.28), (5.2.29), (5.2.30), (5.2.31) *is complete over*$[0, \boldsymbol{K}]$.

By (5.2.26), (5.2.27), the Fourier series from Section 2.5 give Fourier series for the generalized Lamé functions

(5.2.36)
$$Ec_{2m}(z,k_1^2,k_2^2)=\frac{A_0}{\sqrt{2}}+\sum_{n=1}^{\infty}A_{2n}\cos\left(2nt\right),$$

(5.2.37)
$$Ec_{2m+1}(z,k_1^2,k_2^2)=\sum_{n=0}^{\infty}A_{2n+1}\cos\left(2nt\right),$$

(5.2.38)
$$Es_{2m+1}(z,k_1^2,k_2^2)=\sum_{n=0}^{\infty}B_{2n+1}\cos\left(2nt\right),$$

(5.2.39)
$$Es_{2m+2}(z,k_1^2,k_2^2)=\sum_{n=0}^{\infty}B_{2n+2}\cos\left(2nt\right).$$

With the function $\omega\left(t;a_j,b_j\right),\ j=1,2$ from (2.2.6) and using the relations (2.3.13), (2.3.14), the functions (5.2.23) can be represented in the following way

(5.2.40)
$$Ec_m(z,k_1^2,k_2^2)=\left(\omega\left(t;a_j,b_j\right)c_m\left(a_j,b_j,d_j\right)\right)^{-1}Ic_m\left(a_j^*,b_j^*,d_j^*\right),$$

(5.2.41)
$$Es_m(z,k_1^2,k_2^2)=\left(\omega\left(t;a_j,b_j\right)s_m\left(a_j,b_j,d_j\right)\right)^{-1}Is_m\left(a_j^*,b_j^*,d_j^*\right),$$

where

$$b_j^*=-4ja_j-b_j,\qquad d_j^*=d_j-4j^2a_j-2jb_j,\qquad j=1,2.$$

Therefore, we can write

(5.2.42)
$$Ec_{2m}(z,k_1^2,k_2^2)=\left(\omega\left(t;a_i,b_i\right)\right)^{-1}\left(\frac{C_0}{\sqrt{2}}+\sum_{n=1}^{\infty}C_{2n}\cos\left(2nt\right)\right),$$

(5.2.43)
$$Ec_{2m}(z,k_1^2,k_2^2)=\left(\omega\left(t;a_i,b_i\right)\right)^{-1}\left(\frac{C_0}{\sqrt{2}}+\sum_{n=1}^{\infty}C_{2n}\cos\left(2nt\right)\right),$$

$$\text{(5.2.44)} \qquad Es_{2m+1}(z,k_1^2,k_2^2) = (\omega\,(t;a_i,b_i))^{-1}\left(\sum_{n=0}^{\infty} D_{2n+1}\cos{(2nt)}\right),$$

$$\text{(5.2.45)} \qquad Es_{2m+2}(z,k_1^2,k_2^2) = (\omega\,(t;a_i,b_i))^{-1}\left(\sum_{n=0}^{\infty} D_{2n+2}\cos{(2nt)}\right),$$

where

$$\begin{aligned} C_m &= (c_m\,(a_i,b_i,d_i))^{-1}\,A_m, \\ D_m &= (s_m\,(a_i,b_i,d_i))^{-1}\,B_m, \end{aligned}$$

and the Fourier coefficients A_n and B_n belong to the parameters a_j, b_j^*, d_j^*, $j = 1, 2$. Properties of the coefficients C_n and D_n follow from those of A_n and B_n; see Section 2.5.

A function from (5.2.23) is called a generalized Lamé polynomial of the first kind if its Fourier series (5.2.36), (5.2.37), (5.2.38), or (5.2.39) terminates. It is called a generalized Lamé polynomial of the second kind if its expansion (5.2.42), (5.2.43), (5.2.44), or (5.2.45) terminates. If they exist, These Lamé polynomials and their corresponding eigenvalues can be computed from the finite matrices $M_{j,l}$, $j = 1, 2, 3, 4$, where the matrices M_j are The pentadiogonal infinite matrices

$$\text{(5.2.46)} \qquad M_1 = \begin{pmatrix} r_0 & \sqrt{2}q_{-1}^1 & \sqrt{2}q_{-2}^2 & 0 & \cdots \\ \sqrt{2}q_0^1 & r_1 + q_{-1}^2 & q_{-2}^1 & q_{-3}^2 & \cdots \\ \sqrt{2}q_0^2 & q_1^1 & r_2 & q_{-3}^1 & \cdots \\ 0 & q_1^2 & q_2^1 & r_3 & \cdots \\ 0 & 0 & q_2^2 & q_3^1 & \cdots \\ 0 & 0 & 0 & q_3^2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

$$M_2 = \begin{pmatrix} r_0^\dagger + q_0^{\dagger 1} & q_{-1}^{\dagger 1} + q_{-1}^{\dagger 2} & q_{-2}^{\dagger 2} & 0 & \cdots \\ q_1^{\dagger 1} + q_0^{\dagger 2} & r_1^\dagger & q_{-2}^{\dagger 1} & q_{-3}^{\dagger 2} & \cdots \\ q_1^{\dagger 2} & q_2^{\dagger 1} & r_2^\dagger & q_{-3}^{\dagger 1} & \cdots \\ 0 & q_2^{\dagger 2} & q_3^{\dagger 1} & r_3^\dagger & \cdots \\ 0 & 0 & q_3^{\dagger 2} & q_4^{\dagger 1} & \cdots \\ 0 & 0 & 0 & q_4^{\dagger 2} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{5.2.47}$$

$$M_3 = \begin{pmatrix} r_0^\dagger - q_0^{\dagger 1} & q_{-1}^{\dagger 1} - q_{-1}^{\dagger 2} & q_{-2}^{\dagger 2} & 0 & \cdots \\ q_1^{\dagger 1} - q_0^{\dagger 2} & r_1^\dagger & q_{-2}^{\dagger 1} & q_{-3}^{\dagger 2} & \cdots \\ q_1^{\dagger 2} & q_2^{\dagger 1} & r_2^\dagger & q_{-3}^{\dagger 1} & \cdots \\ 0 & q_2^{\dagger 2} & q_3^{\dagger 1} & r_3^\dagger & \cdots \\ 0 & 0 & q_3^{\dagger 2} & q_4^{\dagger 1} & \cdots \\ 0 & 0 & 0 & q_4^{\dagger 2} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{5.2.48}$$

$$M_4 = \begin{pmatrix} r_1 - q_{-1}^2 & q_{-2}^1 & q_{-3}^2 & 0 & \cdots \\ q_1^1 & r_2 & q_{-3}^1 & q_{-4}^2 & \cdots \\ q_1^2 & q_2^1 & r_3 & q_{-4}^1 & \cdots \\ 0 & q_2^2 & q_3^1 & r_4 & \cdots \\ 0 & 0 & q_3^2 & q_4^1 & \cdots \\ 0 & 0 & 0 & q_4^2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{5.2.49}$$

where

$$q_n^j = Q_j\left(n\right), \qquad r_n = 4n^2.$$

$$q_n^{\dagger j} = Q_j(n-1/2), \qquad r_n^{\dagger} = \left(2n+1\right)^2.$$

M_4 has the same form as M_1, except the first row and the first column are deleted and the first entry in the upper left corner is replaced by $r_1 - q_{-1}^2$. Notice also that M_2 and M_3 have the same form except for the 3 entries in the upper left corner.

From the entries of the matrices M_j, $j = 1, \ldots 4$, we deduce the five term recurrence relations for the coefficients A_n, B_n of equations (5.2.36), (5.2.38), (5.2.38), and (5.2.39)

$$-\alpha_{2m}A_0 + \sqrt{2}q_{-1}^1 A_2 + \sqrt{2}q_{-2}^2 A_4 = 0, \tag{5.2.50}$$

$$\sqrt{2}q_0^1 A_0 + \left(4 + q_{-1}^2 - \alpha_{2m}\right) A_2 + q_{-2}^1 A_4 + q_{-3}^2 A_6 = 0, \tag{5.2.51}$$

$$\sqrt{2}q_0^2 A_0 + q_1^1 A_2 + \left(16 - \alpha_{2m}\right) A_4 + q_{-3}^1 A_6 + q_{-4}^2 A_8 = 0, \tag{5.2.52}$$

$$\begin{aligned} &q_{n-2}^2 A_{2(n-2)} + q_{n-1}^1 A_{2(n-1)} + \left(r_n - \alpha_{2m}\right) A_{2n} \\ &\qquad q_{-n}^1 A_{2(n+1)} + q_{-(n+1)}^2 A_{2(n+2)} = 0 \quad n > 2, \end{aligned} \tag{5.2.53}$$

$$\left(4 - q_0^{\dagger 1} - \alpha_{2m+1}\right) A_1 + \left(q_{-1}^{\dagger 1} + q_{-1}^{\dagger 2}\right) A_3 + q_{-2}^{\dagger 2} A_5 = 0, \tag{5.2.54}$$

$$\left(q_1^{\dagger 1} + q_0^{\dagger 2}\right) A_1 + \left(9 - \alpha_{2m+1}\right) A_3 + q_{-2}^{\dagger 1} A_5 + q_{-3}^{\dagger 2} A_7 = 0, \tag{5.2.55}$$

$$\begin{aligned} &q_{n-1}^{\dagger 2} A_{2n-3} + q_n^{\dagger 1} A_{2n-1} + \left(r_n^{\dagger} - \alpha_{2m+1}\right) A_{2n+1} \\ &\qquad + q_{-(n+1)}^{\dagger 1} A_{2n+3} + q_{-(n+2)}^{\dagger 2} A_{2n+5} = 0, \quad n \geq 2, \end{aligned} \tag{5.2.56}$$

$$\left(1 - q_0^{\dagger 1} - \beta_{2m+1}\right) B_1 + \left(q_{-1}^{\dagger 1} - q_{-1}^{\dagger 2}\right) B_3 + q_{-2}^{\dagger 2} B_5 = 0, \tag{5.2.57}$$

$$\left(q_1^{\dagger 1} - q_0^{\dagger 2}\right) B_1 + (9 - \beta_{2m+1}) B_3 + q_{-2}^{\dagger 1} B_5 + q_{-3}^{\dagger 2} B_7 = 0, \tag{5.2.58}$$

$$\begin{aligned} & q_{n-1}^{\dagger 2} B_{2n-3} + q_n^{\dagger 1} B_{2n-1} + \left(r_n^{\dagger} - \beta_{2m+1}\right) B_{2n+1} \\ & \qquad + q_{-(n+1)}^{\dagger 1} B_{2n+3} + q_{-(n+2)}^{\dagger 2} B_{2n+5} = 0, \quad n \geq 2, \end{aligned} \tag{5.2.59}$$

$$\left(4 - q_1^2 - \beta_{2m+2}\right) B_0 + q_{-2}^1 B_2 + q_{-3}^2 B_4 = 0, \tag{5.2.60}$$

$$q_1^1 B_0 + (16 - \beta_{2m+2}) B_2 + q_{-3}^1 B_4 + q_{-4}^2 B_6 = 0, \tag{5.2.61}$$

where α_m, β_m are the eigenvalues of equation (5.2.6) that correspond to even and odd eigenfunctions respectively. Using (5.2.20), (4.2.6), we can find recurence relations for the eigenvalues a_m, b_m of equation (5.2.2).

CHAPTER 6

The Wave Equation and Separation of Variables

6.1. Elliptic Coordinates

In 1860 Mathieu encountered his equation when he solved the problem of the vibrating elliptic membrane with fixed boundary (elliptical drum) [**8**]. Let the membrane be bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad a>b>0.$$

The problem is to find non trivial solutions $u(x,y)$ of the wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\omega u=0$$

defined inside the ellipse which vanish along ellipse. We consider elliptic coordinates

$$x=c\cosh\xi\cos\eta,\quad y=c\sinh\xi\sin\eta,$$

where $c=\sqrt{a^2-b^2}$, so $(\pm c,0)$ are the foci of the ellipse. The interior of the ellipse is given by $0\le\xi<\xi_0$, $0\le\eta<2\pi$ where ξ_0 is determined from $a=c\cosh\xi_0$, $b=c\sinh\xi_0$. Setting $u(x,y)=v(\xi,\eta)$ we obtain

$$-\frac{\partial^2 v}{\partial \xi^2}-\frac{\partial^2 v}{\partial \eta^2}=\frac{1}{2}c^2\omega\left(\cosh 2\xi-\cos 2\eta\right)v.$$

Separation of variables $v(\xi,\eta)=v_1(\xi)\,v_2(\eta)$ leads to the ordinary differential equations (see section 3.6)

$$\begin{aligned}-v_1''+2\mu\cosh(2\xi)\,v_1 &= \lambda v_1,\\ -v_2''+2\mu\cos(2\eta)\,v_2 &= \lambda v_2,\end{aligned}$$

where λ is the separation constant and $4\mu = c^2\omega$. Since u has to be well-defined function inside the ellipse we need v_2 to have period 2π which forces v_2 to have period π or semi-period π. So we arrive at Mathieu's equation and we have to find its solutions with period π or semi-period π.

6.2. Sphero-Conal Coordinates in $\mathbb{R}^{k+1}$

We introduce sphero-conal coordinates on $\mathbb{R}^{k+1}$; see [**71**]; fix real numbers

$$a_0 < a_1 < a_2 < \cdots < a_k. \tag{6.2.1}$$

Let $(x_0, x_1, \ldots, x_k)$ be in the positive cone of $\mathbb{R}^{k+1}$

$$x_0 > 0, \ldots, x_k > 0. \tag{6.2.2}$$

Its sphero-conal coordinates $r, s_1, \ldots, s_k$ are determined in the intervals

$$r > 0, \quad a_{i-1} < s_i < a_i, \quad i = 1, \ldots, k \tag{6.2.3}$$

by the equations

$$r^2 = \sum_{j=0}^{k} x_j^2 \tag{6.2.4}$$

and

$$\sum_{j=0}^{k} \frac{x_j^2}{s_i - a_j} = 0 \quad \text{for } i = 1, \ldots, k. \tag{6.2.5}$$

The latter equation determines $s_1, s_2, \ldots, s_k$ as the zeros of a polynomial of degree k with coefficients which are polynomials in $x_0^2, \ldots, x_k^2$.

In this way we obtain a bijective (real-)analytic map from the positive cone in $\mathbb{R}^{k+1}$ to the set of points $(r, s_1, \ldots, s_k)$ satisfying (6.2.3). The inverse map is found by solving a linear system. It is also analytic, and it is given by

$$x_j^2 = r^2 \frac{\prod_{i=1}^k (s_i - a_j)}{\prod_{j\neq i=0}^k (a_i - a_j)}. \tag{6.2.6}$$

Sphero-conal coordinates are orthogonal, and its scale factors (metric coefficients) are given by $H_r = 1$, and

$$H_{s_i}^2 = \frac{1}{4}\sum_{j=0}^k \frac{x_j^2}{(s_i - a_j)^2} = -\frac{1}{4} r^2 \frac{\prod_{i\neq j=1}^k (s_i - s_j)}{\prod_{j=0}^k (s_i - a_j)}, \quad i = 1, 2, \ldots, k. \tag{6.2.7}$$

Consider the Laplace equation

$$\Delta u = \sum_{i=0}^k \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{6.2.8}$$

for a function $u(x_0, x_1, \ldots, x_k)$. Using (6.2.7) we transform this equation to sphero-conal coordinates, and then we apply the method of separation of variables

$$u(x_0, x_1, \ldots, x_k) = u_0(r) u_1(s_1) u_2(s_2) \ldots u_k(s_k). \tag{6.2.9}$$

For the variable r we obtain the Euler equation

$$w_0'' + \frac{k}{r} w_0' + \frac{4\lambda_0}{r^2} w_0 = 0 \tag{6.2.10}$$

while for each of the variables $s_1, s_2, \ldots, s_k$ we obtain the Fuchsian equation

$$\prod_{j=0}^k (s - a_j)\left[u'' + \frac{1}{2}\sum_{j=0}^k \frac{1}{s - a_j} u'\right] + \left[\sum_{i=0}^{k-1} \lambda_i s^{k-1-i}\right] u = 0. \tag{6.2.11}$$

More precisely, if $\lambda_0, \ldots, \lambda_{k-1}$ are any given numbers (separation constants), and if $u_0(r)$, $r > 0$, solves (6.2.10) and $u_i(s_i)$, $a_{i-1} < s_i < a_i$, solve (6.2.11) for each $i = 1, \ldots, k$, then u defined by (6.2.9) solves (6.2.8) in the positive cone of $\mathbb{R}^{k+1}$ (6.2.2).

6.2.1. Special Case $k=2$. Sphero-conal coordinates r, β, γ form an orthogonal coordinate system in $\mathbb{R}^3$. They are connected with Cartesian coordinates x, y, z

$$x = kr \operatorname{sn}\beta \operatorname{sn}\gamma, \tag{6.2.12}$$

$$y = i\frac{k}{k'} r \operatorname{cn}\beta \operatorname{cn}\gamma, \tag{6.2.13}$$

$$z = i\frac{1}{k'} r \operatorname{dn}\beta \operatorname{dn}\gamma. \tag{6.2.14}$$

where

$$r \geq 0,\ \beta = K + i\beta',\ 0 \leq \beta' \leq 2K',\ 0 \leq \gamma \leq 4K. \tag{6.2.15}$$

The coordinate system depends on the modulus $k \in (0,1)$ of the Jacobian elliptic functions. The coordinate surfaces are spheres and confocal cones given by

$$x^2 + y^2 + z^2 = r^2, \tag{6.2.16}$$

$$\frac{x^2}{b^2} + \frac{y^2}{b^2 - 1} - \frac{z^2}{k^{-2} - b^2} = 0, \qquad b = \operatorname{sn}\beta, \tag{6.2.17}$$

$$\frac{x^2}{c^2} - \frac{y^2}{1 - c^2} - \frac{z^2}{k^{-2} - c^2} = 0, \qquad c = \operatorname{sn}\gamma, \tag{6.2.18}$$

where

$$1 \leq b^2 \leq k^{-2}, \qquad 0 \leq c^2 \leq 1.$$

The wave equation

$$\nabla^2 u + \omega^2 u = 0 \tag{6.2.19}$$

transformed to sphero-conal coordinates takes the form

$$k^2 \left(\operatorname{sn}^2\beta - \operatorname{sn}^2\gamma\right) \left(\left(r^2 u_r\right)_r + \omega^2 r^2 u\right) - u_{\beta\beta} + u_{\gamma\gamma} = 0. \tag{6.2.20}$$

This equation admits separated solution

$$u(r,\beta,\gamma) = u_1(r)\, u_2(\beta)\, u_3(\gamma)\,, \tag{6.2.21}$$

where u_1, u_2, u_3 satisfy the differential equations

$$\left(r^2 u_1'\right)' + \left(\omega r^2 - \nu(\nu+1)\right) u_1 = 0, \tag{6.2.22}$$

$$u_2'' + \left(h - \nu(\nu+1)\, k^2 \operatorname{sn}^2 \beta\right) u_2 = 0, \tag{6.2.23}$$

$$u_3'' + \left(h - \nu(\nu+1)\, k^2 \operatorname{sn}^2 \gamma\right) u_3 = 0, \tag{6.2.24}$$

with separation constants h and ν . We obtain the differential equation (6.2.22) of spherical Bessel functions, and twice Lamé's equation (4.1.1).

Assume that $\nu \in \mathbb{N}_0$ and consider the Lamé polynomials Ec_ν^m, $m = 0, 1, ..., \nu$, Ec_ν^m, $m = 0, 1, ..., \nu$; see Section (4.6). Then the functions

$$r^\nu Ec_\nu^m(\beta)\, Ec_\nu^m(\gamma)\,, \qquad r^\nu Es_\nu^m(\beta)\, Es_\nu^m(\gamma) \tag{6.2.25}$$

are solutions of (6.2.19) with $\omega = 0$, so they are a harmonic functions of x, y, z.

Theorem 6.2.1. *The functions* (6.2.25) *are harmonic polynomials in* x, y, z *homogeneous of degree* ν.

Proof. Consider the Lamé polynomial $E = Ec_{2n}^m$, $m = 0, 1, 2, \ldots n$. Then $E(z) = P(\operatorname{sn}^2 z)\,$, where P is the polynomial of degree n with simple real zeros:

$$P(\xi) = d \prod_{j=1}^{n} (\xi - \theta_j)\,. \tag{6.2.26}$$

From the definition of sphero-conal coordinates, we obtain

$$r^2 \left(\operatorname{sn}^2 \beta - \theta\right) \left(\operatorname{sn}^2 \gamma - \theta\right) = \theta(\theta - 1)\left(\theta - k^{-2}\right) \left(\frac{x^2}{\theta} + \frac{y^2}{1-\theta} + \frac{z^2}{\theta - k^{-2}}\right). \tag{6.2.27}$$

By (6.2.26), (6.2.27)

$$r^{2n}E(\beta)\,E(\gamma) = d^2\prod_{j=1}^{n}\theta_j(\theta_j-1)\left(\theta_j-k^{-2}\right)\left(\frac{x^2}{\theta_j}+\frac{y^2}{1-\theta_j}+\frac{z^2}{\theta_j-k^{-2}}\right) \tag{6.2.28}$$

which shows that $r^{2n}E(\beta)\,E(\gamma)$ is a (harmonic) polynomial in$x$, y, z which is homogeneous of degree $2n$. The other types of Lamé polynomials are treated similarly. ☐

We note that the eight types of Lamé polynomials lead to harmonic polynomials $f(x,y,z)$ of the following parities:

$$\begin{aligned}
&1)\ \ f(x,y,z)=\ \ f(-x,y,z)=\ \ f(x,-y,z)=\ \ f(x,y,-z)\\
&2)\ \ f(x,y,z)=-f(-x,y,z)=\ \ f(x,-y,z)=\ \ f(x,y,-z)\\
&3)\ \ f(x,y,z)=\ \ f(-x,y,z)=-f(x,-y,z)=\ \ f(x,y,-z)\\
&4)\ \ f(x,y,z)=\ \ f(-x,y,z)=\ \ f(x,-y,z)=-f(x,y,-z)\\
&5)\ \ f(x,y,z)=-f(-x,y,z)=-f(x,-y,z)=\ \ f(x,y,-z)\\
&6)\ \ f(x,y,z)=-f(-x,y,z)=\ \ f(x,-y,z)=-f(x,y,-z)\\
&7)\ \ f(x,y,z)=\ \ f(-x,y,z)=-f(x,-y,z)=-f(x,y,-z)\\
&8)\ \ f(x,y,z)=-f(-x,y,z)=-f(x,-y,z)=-f(x,y,-z)
\end{aligned} \tag{6.2.29}$$

It follows from Theorem 6.2.1 that, for every $\nu \in \mathbb{N}_0$ the functions

(6.2.30)

$$Ec_\nu^m(\beta)\,Ec_\nu^m(\gamma)\,,\quad m=0,1,2,\ldots\nu,\qquad Es_\nu^m(\beta)\,Es_\nu^m(\gamma)\,,\quad m=1,2,\ldots\nu,$$

are spherical harmonics of degree when considered as functions defined on the unit sphere S in $\mathbb{R}^3$. Let $L^2(S)$ be the space of square-integrable functions defined on S equipped with its natural inner product. The volume element in sphero-conal coordinates is

$$r^2\left(k^2\operatorname{sn}^2\beta-\operatorname{sn}^2\gamma\right)drd\beta' d\gamma.$$

Therefore, if two functions $f,\, g \in L^2(S)$ are expressed in sphero-conal coordinates, their inner product is given by

$$\langle f, g\rangle = \int_0^{4K}\int_0^{2K'} \left(\operatorname{sn}^2\beta - \operatorname{sn}^2\gamma\right) f(\beta,\gamma)\,\overline{g(\beta,\gamma)}\,d\beta' d\gamma. \tag{6.2.31}$$

THEOREM 6.2.2. *The system of* $2\nu+1$ *spherical harmonics forms* (6.2.30) *an orthogonal basis in the linear space of spherical harmonics of degree* ν *.*

PROOF. The linear space of spherical harmonic of degree ν has dimension $2\nu+1$ Thus it is sufficient to show that the spherical harmonics $E(\beta)\,E(\gamma)$ and $\tilde{E}(\beta)\,\tilde{E}(\gamma)$ are orthogonal for different Lamé polynomials E, $\tilde{E}$. If E, $\tilde{E}$ have different types, the spherical harmonics have different parities, thus they are orthogonal. So let E and $\tilde{E}$ be two different Lamé polynomials of the same type. By Theorem 4.3.2

$$\int_0^{K} E(\gamma)\,\tilde{E}(\gamma)\,d\gamma = 0, \tag{6.2.32}$$

and, by the remarks in Section 4.5

$$\int_0^{K'} E(\beta)\,\tilde{E}(\beta)\,d\beta' = 0. \tag{6.2.33}$$

In (6.2.32), (6.2.33) we can replace K by $4K$ and K' by $2K'$, respectively. Then orthogonality of $E(\beta)\,E(\gamma)\,,\ \tilde{E}(\beta)\,\tilde{E}(\gamma)$ with respect to the inner product (6.2.31) follows. □

It is well known that spherical harmonics are complete in $L^2(S)$ that is, if an orthogonal basis is selected in the space of spherical harmonics of degree ν for every $\nu\in\mathbb{N}_0$ then the combined system of all these bases forms an orthogonal basis in $L^2(S)$. Therefore, we obtain the following theorem.

THEOREM 6.2.3. *The system of all functions* (6.2.30) *with* $\nu\in\mathbb{N}_0$ *forms an orthogonal basis of spherical harmonics. Every function* $f\in L^2(S)$ *can be expanded in a Fourier series with respect to this basis which converges to* f *in* $L^2(S)$ *.*

The corresponding expansion of analytic functions in series of the products (6.2.30) is investigated in [**76**].

So far we have considered solutions of (6.2.19) built from Lamé polynomials. Other Lamé functions also lead to solutions which are useful in applications. An important example is the problem of the vibrating elliptic membrane M on the sphere S treated in [**34**]. The set M forms a part of the sphere which in sphero-conal coordinates corresponds to a rectangle in the (β', γ) −plane. The problem is to find eigenfunctions of the Laplace-Beltrami equation (equation (6.2.19) with the radius r separated off) with boundary conditions at the boundary of M. This problem is analogous to the problem of the vibrating elliptical membrane in the plane which can be solved using Mathieu functions. In a similar way, the corresponding problem on the sphere can be solved with the help of simply-periodic Lamé functions.

Now we consider sphero-conal coordinates in the half-space $z > 0$ as in Section 3.6 by

$$x = rk\cos\varphi\cosh\xi, \tag{6.2.34}$$

$$y = r\frac{k}{k'}\cos\varphi\cosh\xi, \tag{6.2.35}$$

$$z = r\frac{1}{k'}\left(1 - k^2\cos^2\varphi\right)^{1/2}\left(1 - k^2\cos^2\xi\right)^{1/2}, \tag{6.2.36}$$

where

$$r > 0, \qquad 0 \le \varphi < 2\pi, \qquad 0 < \xi < \operatorname{arcosh}\frac{1}{k}, \qquad k, k' \in (0,1), \qquad k'^2 = 1 - k^2.$$

In this case equation (6.2.19) with $\omega = 0$ becomes

$$\left(1 + \frac{k^2}{2+k^2}\cos 2\varphi\right)\frac{\partial^2 u}{\partial\varphi^2} + \left(1 + \frac{k^2}{2+k^2}\cosh 2\xi\right)\frac{\partial^2 u}{\partial\xi^2} + \frac{k^2}{2+k^2}\sin 2\varphi\frac{\partial u}{\partial\varphi} \tag{6.2.37}$$

$$-\frac{k^2}{2+k^2}\sinh 2\xi\frac{\partial u}{\partial \xi}+\frac{k^2}{2+k^2}\left(\cos 2\varphi-\cosh 2\xi\right)r^2\frac{\partial^2 u}{\partial r^2}=0.$$

We separate variables $u=u_1\left(\varphi\right)u_2\left(\xi\right)u_3\left(r\right)$ using separation constants d and λ to obtain

$$\left(1+\frac{k^2}{2+k^2}\cos 2\varphi\right)\frac{d^2u_1}{d\varphi}+\frac{k^2}{2+k^2}\sin 2\varphi\frac{du_1}{d\varphi}+\left(\lambda+d\cos 2\varphi\right)u_1=0, \tag{6.2.38}$$

$$\left(1+\frac{k^2}{2+k^2}\cosh 2\xi\right)\frac{d^2u_2}{d\varphi}-\frac{k^2}{2+k^2}\sinh 2\xi\frac{du_2}{d\varphi}-\left(\lambda+d\cos 2\xi\right)u_2=0. \tag{6.2.39}$$

$$r^2\frac{d^2u_3}{dr^2}+-\frac{d\left(2+k^2\right)}{k^2}u_3=0. \tag{6.2.40}$$

6.2.2. Special Case $K=3$. Taking $k=3$, sphero-conal coordinates in $\mathbb{R}^4$ can be written in terms of Jacobian elliptic functions (but this will not be possible in dimension higher than 4.) Using a linear substitution $s=c\tilde{s}+d$, we assume without loss of generality that $a_0=0$ $a_1=1$, $a_2=k_1^{-2}$ and $a_3=k_2^{-2}$ with $0<k_2<k_1<1$. Then (6.2.6) becomes

$$\begin{aligned}
x_0^2 &= r^2k_1^2k_2^2s_1s_2s_3,\\
x_1^2 &= r^2\frac{k_1^2k_2^2}{k_1'^2k_2'^2}(1-s_1)(s_2-1)(s_3-1),\\
x_2^2 &= r^2\frac{k_2^2}{k_1'^2(k_1^2-k_2^2)}(1-k_1^2s_1)(1-k_1^2s_2)(k_1^2s_3-1),\\
x_3^2 &= r^2\frac{k_1^2}{k_2'^2(k_1^2-k_2^2)}(1-k_2^2s_1)(1-k_2^2s_2)(1-k_2^2s_3).
\end{aligned}$$

We now substitut

$$s_1 = \mathrm{s}^2(\alpha_1, k_1, k_2), \quad 0 < k_2'\alpha_1 < K,$$

$$s_2 = \mathrm{s}^2(\alpha_2, k_1, k_2), \quad \alpha_2 = \frac{K}{k_2'} + i\alpha_2',\ 0 < k_2'\alpha_2' < K',$$

$$s_3 = \mathrm{s}^2(\alpha_3, k_1, k_2), \quad \alpha_3 = \alpha_3' + \frac{K'}{k_2'},\ 0 < k_2'\alpha_3' < K,$$

where $K := K(\kappa)$ and $K' := K(\kappa')$ are the elliptic integral of the first kind, with κ and κ' defined the same way as in Section 5.1.

The maps $\alpha_i \to s_i$ are bijections. Using (5.1.4) , we obtain

$$\begin{aligned}
x_0 &= rk_1k_2\mathrm{s}(\alpha_1, k_1, k_2)\,\mathrm{s}(\alpha_2, k_1, k_2)\,\mathrm{s}(\alpha_3, k_1, k_2),\\
x_1 &= -r\frac{k_1k_2}{k_1'k_2'}\mathrm{c}(\alpha_1, k_1, k_2)\,\mathrm{c}(\alpha_2, k_1, k_2)\,\mathrm{c}(\alpha_3, k_1, k_2),\\
x_2 &= ir\frac{k_2}{k_1'k_2'\kappa}\mathrm{d}_1(\alpha_1, k_1, k_2)\,\mathrm{d}_1(\alpha_2, k_1, k_2)\,\mathrm{d}_1(\alpha_3, k_1, k_2),\\
x_3 &= r\frac{k_1}{k_2'^2\kappa}\mathrm{d}_2(\alpha_1, k_1, k_2)\,\mathrm{d}_2(\alpha_2, k_1, k_2)\,\mathrm{d}_2(\alpha_3, k_1, k_2).
\end{aligned}$$

This representation is only valid for the positive cone $\mathbb{R}^4$. But now we can allow $0 < k_2'\alpha_1 < 4K$, $0 < k_2'\alpha_2' < K'$, $-K < k_2'\alpha_3' < K$ to cover (almost) the whole space $\mathbb{R}^4$.

6.3. Ellipsoidal Coordinates

Ellipsoidal coordinates α, β, γ form an orthogonal coordinate system in $\mathbb{R}^3$. They are connected with Cartesian coordinates x, y, z by

$$x = k\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma, \tag{6.3.1}$$

$$y = -\frac{k}{k'}\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma, \tag{6.3.2}$$

$$z = \frac{i}{kk'}\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma. \tag{6.3.3}$$

where

$$(6.3.4)\quad \alpha = K + iK' - \alpha',\ 0 \le \alpha' \le K,\ \beta = K + i\beta',\ 0 \le \beta' \le 2K',\ 0 \le \gamma \le 4K.$$

The coordinate surfaces are confocal ellipsoids and hyperboloids with one or two sheets given by

$$(6.3.5)\qquad \frac{x^2}{a^2} + \frac{y^2}{a^2-1} - \frac{z^2}{a^2-k^{-2}} = 1, \qquad a = \operatorname{sn}\alpha,$$

$$(6.3.6)\qquad \frac{x^2}{b^2} + \frac{y^2}{b^2-1} - \frac{z^2}{k^{-2}-b^2} = 1, \qquad b = \operatorname{sn}\beta,$$

$$(6.3.7)\qquad \frac{x^2}{c^2} - \frac{y^2}{1-c^2} - \frac{z^2}{k^{-2}-c^2} = 1, \qquad c = \operatorname{sn}\gamma,$$

where

$$k^{-2} \le a^2 < \infty, \quad 1 \le b^2 \le k^{-2}, \quad 0 \le c^2 \le 1.$$

The wave equation (**??**) transformed to ellipsoidal coordinates α, β, γ is

$$(6.3.8)\qquad \begin{aligned} &\left(\operatorname{sn}^2\beta - \operatorname{sn}^2\gamma\right) u_{\alpha\alpha} + \left(\operatorname{sn}^2\gamma - \operatorname{sn}^2\alpha\right) u_{\beta\beta} + \left(\operatorname{sn}^2\alpha - \operatorname{sn}^2\beta\right) u_{\gamma\gamma} \\ &+ \omega^2 k^2 \left(\operatorname{sn}^2\alpha - \operatorname{sn}^2\beta\right)\left(\operatorname{sn}^2\beta - \operatorname{sn}^2\gamma\right)\left(\operatorname{sn}^2\beta - \operatorname{sn}^2\gamma\right) = 0. \end{aligned}$$

It admits solutions of the form

$$(6.3.9)\qquad u(\alpha,\beta,\gamma) = u_1(\alpha)\, u_2(\beta)\, u_3(\gamma),$$

where u_1, u_2, u_3 each satisfy the Lamé wave equation

$$(6.3.10)\qquad u'' + \left(h - \nu(\nu+1) k^2 \operatorname{sn}^2(z,k) + k^2\omega^2 \operatorname{sn}^4(z,k)\right) u = 0.$$

When $\omega = 0$, this is the Lamé equation. For $\omega \neq 0$, (6.3.10) can be considered as a generalization of Lamé's equation.

If we substitute

(6.3.11) $$t = \frac{\pi}{2} - \operatorname{am} z,$$

then

$$\frac{dt}{dz} = -\operatorname{dn} z, \quad \operatorname{sn} z = \cos t, \quad \operatorname{cn} z = \sin t, \quad \operatorname{dn}^2 z = 1 - k^2 \cos^2 t,$$

and the Lamé wave equation (6.3.10) becomes

(6.3.12)

$$\left(1 - k^2 \cos^2 t\right) u'' + k \left(\sin t \cos t\right) u' + \left(h - \nu \left(\nu + 1\right) k^2 \cos^2 t + k^2 \omega^2 \cos^4 t\right) u = 0.$$

Equation (6.3.12) is equivalent to the generalized Ince equation

(6.3.13) $$\left(1 + a_1 \cos 2t\right) u'' + b_1 \left(\sin 2t\right) u' + \left(\lambda + d_1 \cos 2t + d_2 \cos 4t\right) u = 0,$$

where,

(6.3.14)
$$\begin{aligned} -a_1 = b_1 &= \frac{k^2}{2 - k^2}, \\ d_1 &= -\frac{k}{2 - k^2} \left(\nu \left(\nu + 1\right) - \omega^2\right). \\ d_2 &= \frac{k^2 \omega^2}{4 \left(2 - k^2\right)} \\ \lambda &= \frac{2h - \left(\nu \left(\nu + 1\right) - \frac{3\omega^2}{4}\right) k^2}{2 - k^2} \end{aligned}$$

Consider a Lamé polynomial E and form the function

(6.3.15) $$E\left(\alpha\right) E\left(\beta\right) E\left(\gamma\right)$$

which is a harmonic functions of x, y, z.

THEOREM 6.3.1. *Let E be a Lamé polynomial. Then* (6.3.15) *is a harmonic polynomial in x, y, z.*

Proof. The definition of ellipsoidal coordinates leads to the identity

(6.3.16)

$$\left(\operatorname{sn}^2\alpha-\theta\right)\left(\operatorname{sn}^2\beta-\theta\right)\left(\operatorname{sn}^2\gamma-\theta\right)=\theta\left(\theta-1\right)\left(\theta-k^{-2}\right)\left(\frac{x^2}{\theta}+\frac{y^2}{\theta-1}+\frac{z^2}{\theta-k^{-2}}-1\right).$$

Let E be a Lamé polynomial of the first type written as $E\left(z\right)=P\left(\operatorname{sn}^2 z\right)$ with P as in (6.2.26). Then (6.3.16) shows that the function (6.3.15) is a (harmonic) polynomial. The proof for the other types of Lamé polynomials is similar. □

The harmonic polynomials derived from (6.3.15) are called ellipsoidal harmonics; see [**8**, §9.8.1], [**24**, Chapter XI] and [**86**, Chapter 23]. The parities of these polynomials for the eight types of Lamé polynomials are again given by (6.2.29). Ellipsoidal harmonics can be used to solve boundary value problems for harmonic functions involving ellipsoids.

CHAPTER 7

Mathematical Applications

7.1. Instability Intervals

We consider the generalized Ince equation in the following form

$$(1 + \epsilon A(t))\, y''(t) + \epsilon B(t)\, y'(t) + (\lambda + \epsilon D(t))\, y(t) = 0. \tag{7.1.1}$$

where $A(t)$, $B(t)$, $D(t)$ are trigonometric polynomials defined the same way as in Chapter 2 and

$$|\epsilon| \sum_{j=1}^{\eta} |a_j| < 1.$$

Equation (7.1.1) contains the spectral parameter λ and the perturbation parameter ϵ.

In this section we investigate the length L_m of the m-th instability intervals of equation (7.1.1). Volkmer [**80**] finds The leading term in the expansion of L_m in terms of ϵ. These results are extension of earlier work of Levy and Keller [**43**].

From Chapter 2, we know that The eigenvalue problem of (7.1.1) splits into four problems with eigenfunctions that are even or odd, and have period or semi period π. The eigenvalues λ form two increasing sequences $\{\alpha_m(\epsilon)\}_{m=0}^{\infty}$ and $\{\beta_m(\epsilon)\}_{m=0}^{\infty}$ converging to infinity, where the eigenvalues α_m, β_m correspond to even and odd eigenfunctions, respectively, and an even or odd subscripts m indicates that the corresponding eigenfunction have period π, respectively. in the unperturbed case $\epsilon = 0$, we have

$$\alpha_m(0) = \beta_m(0) = m^2. \tag{7.1.2}$$

The m-th instability interval of (7.1.1) is the interval with end points α_m, β_m for $m \geq 1$. The eigenvalue α_m may be the left or right end point of this interval. We may call $\alpha_m - \beta_m$ the signed length of the m-th instability interval. The analytic functions $\alpha_m(\epsilon)$ and $\beta_m(\epsilon)$ can be expanded of powers of ϵ. In this expansion of the signed length $\alpha_m - \beta_m$ many terms cancel, the goal is to find the term with lowest power of ϵ which does not vanish.

In the case $A(t) = B(t) = 0$, Levy and Keller [**43**] (see also Arnold [**3**]) proved that

$$\alpha_m(\epsilon) - \beta_m(\epsilon) = \omega_m \epsilon^{p+1} + O\left(\epsilon^{p+2}\right) \qquad \epsilon \to 0, \tag{7.1.3}$$

where

$$m = p\eta + q, \quad q = 1, 2, \ldots \eta, \quad p = 0, 1, 2, \ldots \tag{7.1.4}$$

Moreover, Levy and Keller [**43**] gave an explicit formula for ω_m when $q = \eta$. One should note that that ω_m may be zero for particular values of m and the coefficients a_j,b_j, d_j. Several of the improvements made by Volkmer [**80**] to the results found in [**43**], are presented in this section. Our contribution to this discussion consists of developing a Maple code capable of computing these instability intervals symbolically and numerically (see Appendix A.) For additional results on the lengths of instability intervals see also [**11, 12, 26**]. For a general account of perturbation theory see Kato [**35**]. All proofs of results presented in this section can be found in [**80**].

7.1.1. The m-th instability intervals for odd m. Let m be a given positive odd integer. Let y be an eigenfunction of (7.1.1) belonging to the eigenvalue $\lambda = \alpha_m$. it admits the Fourier expansion

$$y = \sum_{k=0}^{\infty} A_{2k+1} \cos(2k+1)t \tag{7.1.5}$$

where

$$\sum_{k=0}^{\infty} A_{2k+1}^2 (2k+1)^2 < \infty.$$

If $\epsilon = 0$ then $A_m \neq 0$ and $A_{2k+1} = 0$ for $2k+1 \neq m$. If $|\epsilon|$ is small enough, A_m is nonzero, and we are permitted to normalize by

$$A_m = 1. \tag{7.1.6}$$

The Fourier coefficients $A_{k+1}(\epsilon)$ are defined uniquely, and they are analytic functions of ϵ in a neighborhood of $\epsilon = 0$. Substituting (7.1.5) in (7.1.1) and comparing coefficients, we obtain the formula

$$\left(\alpha_m - (2k+1)^2\right) A_{2k+1} = \epsilon \sum_{j=1}^{k\wedge\eta} C_j (2j-2k-1) A_{2k-2j+1} \tag{7.1.7}$$

$$+\epsilon \sum_{j=1}^{\eta} C_j (2j+2k+1) A_{2k+2j+1} + \epsilon \sum_{i=1}^{\eta-k-1} C_{i+k+1} (2i+1) A_{2i+1}$$

where $k = 0, 1, 2, \ldots$, and C_j is the quadratic polynomial

$$C_j(\mu) := \frac{1}{2}\left(a_j\mu^2 + b_j\mu - d_j\right). \tag{7.1.8}$$

If $k \geq \eta$ then (7.1.7) contains the empty sum $\sum_{i=1}^{\eta-k-1}$ which is defined as 0.

The adjoint equation to (7.1.1) (see Section 2.2) is

$$(1+\epsilon A(t))z'' + \epsilon B^*(t)z' + (\lambda + \epsilon D^*(t))z = 0, \tag{7.1.9}$$

where

$$B^* = 2A' - B, \quad D^* = D + A'' - B'.$$

We obtain (7.1.9) from (7.1.1) by replacing b_j, d_j by

$$b_j^* = -4ja_j - b_j, \quad d_j^* = -4j^2a_j - 2jb_j + d_j. \tag{7.1.10}$$

The polynomial $C_j^*(\mu) := \frac{1}{2}\left(a_j\mu^2 + b_j^*\mu - d_j^*\right)$ is related to $C_j(\mu)$ by

$$C_j^*(\mu) = C_j(2j - \mu). \tag{7.1.11}$$

Since (7.1.1) and its adjoint (7.1.9) have the same eigenvalues, there is an eigenfunction z of (7.1.9) belonging to the eigenvalue $\lambda = \beta_m$. It admits the Fourier expansion

$$z = \sum_{k=0}^{\infty} B_{k+1} \sin(2k+1)t. \tag{7.1.12}$$

We again use the normalization

$$B_m = 1.$$

Substituting (7.1.12) in (7.1.1) and comparing coefficients, we obtain the formula

$$\left(\beta_m - (2k+1)^2\right) B_{k+1} = \epsilon \sum_{j=1}^{k\wedge\eta} C_j(2k+1) B_{2k-2j+1} \tag{7.1.13}$$

$$+\epsilon \sum_{j=1}^{\eta} C_j(-2k-1) B_{2k+2j+1} - \epsilon \sum_{i=1}^{\eta-k-1} C_{i+k+1}(2k+1) B_{2i+1},$$

where $k = 0, 1, 2, \ldots$

From Standard methods from perturbation theory, we have the following estimate on the domains of analyticity of the eigenvalues $\alpha_m(\epsilon)$ and $\beta_m(\epsilon)$.

Theorem 7.1.1. *If m is positive and odd then the eigenvalues functions $\alpha_m(\epsilon)$ and $\beta_m(\epsilon)$ are analytic in the disk $|\epsilon| < r_m$, where*

$$r_m := \min\left\{\frac{2m-2}{C(m) + C(m-2)}, \frac{2m+2}{C(m) + C(m+2)}\right\}, \tag{7.1.14}$$

and

$$C(\mu) := \frac{1}{2}\left(a\mu^2 + b\mu + d\right), \quad a := \sum_{j=1}^{\eta} |a_j|, \ b := \sum_{j=1}^{\eta} |b_j|, \ d := \sum_{j=1}^{\eta} |d_j|.$$

If $m = 1$ the first term on the right-hand side of (7.1.14) is to be omitted, and if $a = b = c = 0$ we set $r_m := \infty$.

Example 7.1.2. Taking $\mathbf{a}=\left[\frac{1}{2},0\right]$, $\mathbf{b}=[2,2]$, $\mathbf{d}=[2,1]$ in Theorem 7.1.1, with the help of *Maple* we obtain the following formula for r_m

$$r_m=\min\left\{\frac{2m-2}{\frac{1}{4}\left(m^2+(m-2)^2\right)+4m-1},\frac{2m+2}{\frac{1}{4}\left(m^2+(m+2)^2\right)+4m+7}\right\}.$$

The following formula will be essential to calculate ω_m in (7.1.3).

Theorem 7.1.3. *For all positive add* m, *we have*

$$(\alpha_m-\beta_m)\sum_{k=0}^{\infty}A_{2k+1}B_{2k+1}=2\epsilon\sum_{k=0}^{\eta-1}\sum_{i=0}^{\eta-k-1}C_{i+k+1}\left(2i+1\right)A_{2i+1}B_{2k+1}. \tag{7.1.15}$$

Next, define the sequence $\{u_{2k+1}\}_{k=0}$ by $u_m:=1$, $u_{2k+1}:=0$ for $2k+1>m$, and the recursively, for $\ell=1,2,\ldots,$ $j=0,1,2,\ldots,\eta-1$,

$$u_{m_\ell+2j}:=e_{m_\ell+2j}\sum_{i=0}^{j}C_{\eta-j+i}\left(m_{\ell-1}+2i\right)u_{m_{\ell-1}+2i}, \tag{7.1.16}$$

where

$$m_\ell:=m-2\ell\eta, \tag{7.1.17}$$

and

$$e_k:=\frac{1}{m^2-k^2}. \tag{7.1.18}$$

Also, define u^*_{2k+1} the same way as u_{2k+1}but with C_j replaced with C^*_j.

Theorem 7.1.4. *Let*

$$m=2n+1,\ n=r\eta+q\qquad \textit{where } q=0,1,\ldots,\eta-1,\ r=0,1,2,\ldots \tag{7.1.19}$$

(a) If $m_\ell\le 2k+1\le m_{\ell-1}$ *with* $\ell=0,1,2,\ldots,r$, *then the Taylor expansions of* A_{2k+1} *and* B_{2k+1} *in powers of* ϵ *have the form*

$$A_{2k+1}=u_{2k+1}\epsilon^{\ell}+O\left(\epsilon^{\ell+1}\right), \tag{7.1.20}$$

$$B_{2k+1} = u^*_{2k+1}\epsilon^{\ell} + O\left(\epsilon^{\ell+1}\right). \tag{7.1.21}$$

(*b*) *If* $k = 0, 1, \ldots, q-1$ *then*

$$A_{2k+1} = \left(u_{2k+1} + e_{2k+1}\sum_{j=q}^{\eta-k-1} C_{j+k+1}\left(2j+1\right)u_{2j+1}\right)\epsilon^{r+1} + O\left(\epsilon^{r+2}\right), \tag{7.1.22}$$

$$B_{2k+1} = \left(u^*_{2k+1} - e_{2k+1}\sum_{j=q}^{\eta-k-1} C_{j+k+1}\left(2j+1\right)u^*_{2j+1}\right)\epsilon^{r+1} + O\left(\epsilon^{r+2}\right). \tag{7.1.23}$$

ω_m can be determined in the following result by combining theorems 7.1.3 and 7.1.4.

THEOREM 7.1.5. *Let* m *be of the form* (7.1.19).(*a*) *If* $2q < \eta$ *then* (7.1.3)*holds with* $p = 2r$ *and*

$$\omega_m = 2\sum_{k=q}^{\eta-q-1}\sum_{i=q}^{\eta-k-1} C_{i+k+1}\left(2i+1\right)u_{2i+1}u^*_{2k+1}. \tag{7.1.24}$$

(*b*) *If* $2q \geq \eta$ *then* (7.1.3)*holds with* $p = 2r+1$ *and*

$$\omega_m = 2\sum_{k=q}^{\eta-1}\sum_{i=0}^{\eta-k-1} C_{i+k+1}\left(2i+1\right)u_{2i+1}u^*_{2k+1} + C_{i+k+1}\left(2k+1\right)u_{2k+1}u^*_{2i+1}. \tag{7.1.25}$$

7.1.2. The m-th instability intervals for even m. We now consider eigenvalues α_m, β_m of (7.1.1) for a given positive even integer m. Let y be an eigenfunction (7.1.1) belonging to eigenvalue $\lambda = \alpha_m$. It admits the Fourier expansion

$$y = \sum_{k=0}^{\infty} A_{2k}\cos\left(2kt\right). \tag{7.1.26}$$

We again normalize using equation (7.1.6). Substituting (7.1.26) in (7.1.1) and comparing coefficients, we get

$$\left(\alpha_m - (2k)^2\right)A_{2k} = \epsilon\sum_{j=1}^{k\wedge\eta} C_j\left(2j-2k\right)A_{2k-2j} \tag{7.1.27}$$

$$+\epsilon\sum_{j=1}^{\eta}C_j\left(2j+2k\right)A_{2k+2j}+\epsilon\sum_{i=1}^{\eta-k}C_{i+k+1}\left(2i\right)A_{2i}$$

for $k=1,2,\ldots,$ and

$$\alpha_m y_0=\epsilon\sum_{j=1}^{\eta}C_j\left(2j\right)y_{2j}. \tag{7.1.28}$$

Consider an eigenfunction z of (7.1.9) belonging to the eigenvalue $\lambda=\beta_m$. It admits the fourier expansion

$$z=\sum_{k=0}^{\infty}B_{k+1}\sin\left(2kt\right). \tag{7.1.29}$$

Using the normalization

$$B_m=1,$$

substituting (7.1.29) in (7.1.1) and comparing coefficients, we obtain the formula

(7.1.30)

$$\left(\beta_m-(2k)^2\right)B_{2k}=\epsilon\sum_{j=1}^{(k-1)\wedge\eta}C_j\left(2k\right)B_{2k-2j}$$
$$+\,\epsilon\sum_{j=1}^{\eta}C_j\left(-2k\right)B_{2k+2j}-\epsilon\sum_{i=1}^{\eta-k}C_{i+k}\left(2k\right)B_{2i},\qquad k=1,2,\ldots$$

THEOREM 7.1.6. *For all positive even m, we have*

$$\left(\alpha_m-\beta_m\right)\sum_{k=1}^{\infty}A_{2k}B_{2k}=2\epsilon\sum_{k=i}^{\eta}\sum_{i=0}^{\eta-k}C_{i+k}\left(2i\right)A_{2i}B_{2i}. \tag{7.1.31}$$

The sequences $\{u_{2k}\}$ and $\{u_{2k}^*\}$ are defined as in Subsection 7.1.1.

THEOREM 7.1.7. *Let*

$$m=2n,\ n=r\eta+q\qquad \textit{where } q=1,2,\ldots,\eta,\ r=0,1,2,\ldots \tag{7.1.32}$$

(a)If $m_\ell \leq 2k \leq m_{\ell-1}$ *where* $\ell = 0, 1, 2, \ldots, r$, *then the taylor expansions of* A_{2k} *and* B_{2k} *in powers of* ϵ *have the following form*

$$A_{2k} = u_{2k}\epsilon^{\ell} + O\left(\epsilon^{\ell+1}\right), \tag{7.1.33}$$

$$B_{2k} = u^*_{2k}\epsilon^{\ell} + O\left(\epsilon^{\ell+1}\right). \tag{7.1.34}$$

(b) If $k = 1, \ldots, q-1$ *then*

$$A_{2k} = \left(u_{2k} + e_{2k}\sum_{j=q}^{\eta-k} C_{j+k}\left(2j\right)u_{2j}\right)\epsilon^{r+1} + O\left(\epsilon^{r+2}\right), \tag{7.1.35}$$

$$B_{2k} = \left(u^*_{2k} - e_{2k}\sum_{j=q}^{\eta-k} C_{j+k}\left(2j\right)u^*_{2j}\right)\epsilon^{r+1} + O\left(\epsilon^{r+2}\right). \tag{7.1.36}$$

(c) If $k = 0$ *then*

$$A_0 = u_0\epsilon r^{r+1} + O\left(\epsilon^{r+2}\right). \tag{7.1.37}$$

The next theorem is the equivalent to theorem 7.1.5 for even m.

THEOREM 7.1.8. *Let* m *be of the form* (7.1.32).

(a) If $2q < \eta$ *then* (7.1.3)*holds with* $p = 2r$ *and*

$$\omega_m = 2\sum_{k=q}^{\eta-q}\sum_{i=q}^{\eta-k} C_{i+k}\left(2i\right)u_{2i}u^*_{2k}. \tag{7.1.38}$$

(b) If $2q \geq \eta$ *then* (7.1.3)*holds with* $p = 2r + 1$ *and*

$$\omega_m = 2\sum_{k=q}^{\eta} C_k\left(0\right) u_0 u_{2k}^* + 2\sum_{k=q}^{\eta}\sum_{i=1}^{\eta-k} C_{i+k}\left(2i\right) u_{2i}u_{2k}^* + C_{i+k}\left(2k\right) u_{2k}u_{2i}^*. \tag{7.1.39}$$

7.1.3. A matrix formula for ω_m. Formulas (7.1.24), (7.1.25), (7.1.38), (7.1.39) are sufficient to compute ω_m. However there is another formula for ω_m in terms of matrix algebra. This formula is the basis for our maple code to compute ω_m (see Appendix A)

Define the $\eta \times \eta$ lower triangular matrices

$$F_k := \begin{pmatrix} C_\eta\left(m_k\right) & 0 & 0 & 0 & \dots & 0 \\ C_{\eta-1}\left(m_k\right) & C_\eta\left(m_k+2\right) & 0 & 0 & \dots & 0 \\ C_{\eta-2}\left(m_k\right) & C_{\eta-1}\left(m_k+2\right) & C_\eta\left(m_k+4\right) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_1\left(m_k\right) & C_2\left(m_k+2\right) & C_3\left(m_k+4\right) & \dots & \dots & C_\eta\left(m_k+2\eta-2\right) \end{pmatrix} \tag{7.1.40}$$

and the$\eta \times \eta$ diagonal matrices

$$E_k := diag\left(e_{m_k}, e_{m_k+2}, \dots, e_{m_k+2s-2}\right).$$

Then (7.1.16) leads to

$$\begin{pmatrix} u_{m_\ell} \\ u_{m_\ell+2} \\ \vdots \\ u_{m_\ell+2s-2} \end{pmatrix} = E_\ell F_{\ell-1} E_{\ell-1} F_{\ell-2} \dots E_2 F_1 E_1 F_0 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{7.1.41}$$

TABLE 1. instability intervals example

m	1	2	3	4	5
ω_m	2.50000000	0.62500000	0.04150391	0.00352648	0.00020829

where $\ell = 1, 2, 3, \ldots$ Let G_k be the same matrix as F_k but with D_j replaced by D_j^*. Then

$$\begin{pmatrix} u^*_{m_\ell} \\ u^*_{m_\ell+2} \\ \vdots \\ u^*_{m_\ell+2s-2} \end{pmatrix} = E_\ell G_{\ell-1} E_{\ell-1} G_{\ell-2} \ldots E_2 G_1 E_1 G_0 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{7.1.42}$$

THEOREM 7.1.9. *Let m be of the form* (7.1.4)*, then* (7.1.3) *holds, where ω_m is the entry in the first column and $(\eta - q + 1)$st row of the matrix*

$$W := 2F_p E_p F_{p-1} E_{p-1} \ldots E_2 F_1 E_1 F_0. \tag{7.1.43}$$

EXAMPLE 7.1.10. In the case of the Ince equation, we have $\eta = 1$, which forces q to be equal to 1 in (7.1.4). Since we can write

$$m = p + 1, \qquad p = 0, 1, 2, \ldots$$

we obtain the following for the Ince equation with coefficients $a = 1/2$, $b = 1$, $d = -1$; see Table 1

By writing out a product of lower triangular matrices explicitly, (7.1.41) can be written in the alternate form

$$u_{2m-2i} = \sum_{i_1+i_2+\cdots+i_\ell=i} \prod_{k=1}^{\ell} e_{m-I_{k+1}} D_{i_k}(m - I_k) \quad \text{for } (\ell-1)\eta < i < \ell\eta \tag{7.1.44}$$

where the sum is taken over all $(i_1, i_2, \ldots, i_\ell) \in \{1, 2, \ldots, \eta\}^\ell$ with sum i, and $I_k := 2\sum_{\nu=1}^{k-1} i_\nu$. Similarly we can formulate Theorem 7.1.9 in the following way.

Theorem 7.1.11. *Let m be of the form* (7.1.4)*, then* (7.1.3) *holds with*

$$\omega_m = \sum_{j_1+j_2+\cdots+j_p=m} \prod_{k=1}^{p} e_{m-J_k} \prod_{\sigma=0}^{p} D_{J_\sigma}\left(m-J_\sigma\right), \tag{7.1.45}$$

where the sum is extended over all $(j_0, j_1, \ldots, j_p) \in \{1, 2, \ldots, \eta\}^{p+1}$ with sum i, and $I_k := 2\sum_{\nu=1}^{k-1} j_\nu$.

Example 7.1.12. If $m = 10$, and $\eta = 3$, then according to (7.1.4) we have $p = 3$, $q = 1$, and (7.1.45) reads

$$\begin{aligned}\omega_{10} =& \frac{1}{352800} D_3(-4) D_2(0) D_2(4) D_3(10) + \frac{1}{268800} D_3(-4) D_2(0) D_3(6) D_2(10)\\ &+ \frac{1}{268800} D_2(-6) D_3(0) D_2(4) D_3(10) + \frac{1}{204800} D_2(-6) D_3(0) D_3(6) D_2(10)\\ &+ \frac{1}{338688} D_3(-4) D_3(2) D_1(4) D_3(10) + \frac{1}{258048} D_3(-4) D_3(2) D_2(6) D_2(10)\\ &\frac{1}{145152} D_3(-4) D_3(2) D_3(8) D_1(10) + \frac{1}{338688} D_3(-4) D_1(-2) D_3(4) D_3(10)\\ &+ \frac{1}{258048} D_2(-6) D_2(-2) D_3(4) D_3(10) + \frac{1}{145152} D_1(-8) D_3(-2) D_3(4) D_3(10).\end{aligned}$$

7.2. A Hochstadt Type Estimate

Consider the generalized Ince Equation

$$\left(1 + A(t)\right) y''(t) + \lambda y(t) = 0, \tag{7.2.1}$$

where

$$A(t) = \sum_{j=1}^{\eta} a_j \cos 2jt, \quad \eta \in \mathbb{N}.$$

Dividing both sides of (7.2.1) by $\left(1 + A(t)\right)$, and letting $\omega(t) = \left(1 + A(t)\right)^{-1}$we obtain

$$y''(t) + \lambda\omega(t) y(t) = 0. \tag{7.2.2}$$

We now follow Hochstadt [**25**] who gave estimate for the stability intervals using the Prüfer angle. Let $y(t)$ be a non trivial real solution of (7.2.1) with $\lambda > 0$. There are continuously differentable functions $\theta, r : \mathbb{R} \to \mathbb{R}$ such that

$$\sqrt{\lambda\omega(t)}y(t) = r(t)\sin\theta(t), \quad y'(t) = r(t)\cos\theta(t).$$

$\theta(t)$ is called the modified Prüfer angle. Then r and θ satisfy the differential equations

$$r' = \frac{1}{2}\frac{\omega'(t)}{\omega(t)} r\sin^2\theta, \tag{7.2.3}$$

$$\theta' = \sqrt{\lambda\omega(t)} + \frac{1}{4}\frac{\omega'(t)}{\omega(t)} r\sin(2\theta). \tag{7.2.4}$$

Let

$$0 \le \lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \ldots$$

denote the values of λ for which (7.2.1) admits Floquet solutions with period π, and let

$$\mu_0 \le \mu_1 < \mu_2 \le \mu_3 < \ldots$$

denote the values of λ for which (7.2.1) admits Floquet solutions with semi-period π (this is the notation used in Section 1.2.) The stability intervals for (7.2.1) are

$$S_0 = (\lambda_0, \mu_0), \; S_1 = (\mu_1, \lambda_1), \; S_2 = (\lambda_2, \mu_2), \ldots$$

and the instability intervals are

$$I_0 = (-\infty, 0], \; I_1 = [\mu_0, \mu_1], \; I_2 = [\lambda_1, \lambda_2], \ldots$$

Lemma 7.2.1. *let $\lambda > 0$. (a) $\lambda \in I_k$ if and only if there is a real solution θ of* (7.2.4) *such that*

$$\theta(\pi) - \theta(0) = k\pi. \tag{7.2.5}$$

(b) $\lambda \in I_k$ if and only if

$$\theta(\pi) - \theta(0) \in (k\pi, (k+1)\pi) \tag{7.2.6}$$

for every solution of (7.2.4).

Proof. (a) let $\lambda \in I_k$. There is a real Floquet solution y of (7.2.1) such that

$$y(t+\pi) = \rho y(t),$$

with $\rho \in \mathbb{R}$. Then the corresponding Prüfer angle θ satisfies

$$\theta(\pi) - \theta(0) = m\pi,$$

where m is a non negative integer. Since y has m zeros in $[0,\pi)$ and using Theorem 3.1.3 in [**8**] , we obtain that $m = k$.

Conversely let θ be a real solution of (7.2.4) with (7.2.5). Then there is a real solution y of (7.2.1) that generates the Prüfer angle θ and this y satisfies $y(t+\pi) = \rho y(t)$ with real ρ. Than λ must be in one of the instability intervals. Part (b) follows from (a). □

Theorem 7.2.2. *Set*

$$u := \int_0^\pi \sqrt{\omega(t)}dt, \qquad v := \frac{1}{4}\int_0^\pi \frac{|\omega'(t)|}{\omega(t)}dt.$$

Suppose that $v < \pi/2$. *Then for every* $k = 0, 1, 2, \ldots$

$$\left[\left(\frac{k\pi + v}{u}\right)^2, \left(\frac{(k+1)\pi - v}{u}\right)^2\right] \in S_k. \tag{7.2.7}$$

Proof. Integrating (7.2.4) between $t = 0$ and $t = \pi$, we obtain

$$\begin{aligned} \int_0^\pi \theta' dt &= \int_0^\pi \left(\sqrt{\lambda\omega(t)} + \frac{1}{4}\frac{\omega'(t)}{\omega(t)} r\sin(2\theta)\right) dt \\ &\leq \sqrt{\lambda}u + \int_0^\pi \frac{1}{4}\frac{\omega'(t)}{\omega(t)} r\sin(2\theta)\, dt. \end{aligned}$$

And we see that every real solution (7.2.1) satisfies

$$\left|\theta(\pi) - \theta(0) - \sqrt{\lambda}u\right| \leq \left|\int \frac{1}{4}\frac{\omega'(t)}{\omega(t)} r\sin(2\theta)\, dt\right| \leq v. \tag{7.2.8}$$

Equation (7.2.8) gives

$$-v+\sqrt{\lambda}u\leq\theta\left(\pi\right)-\theta\left(0\right)\leq v+\sqrt{\lambda}u, \tag{7.2.9}$$

if λ is in the interval on the left hand side of (7.2.7) it follows that

$$k\pi<\theta\left(\pi\right)-\theta\left(0\right)<\left(k+1\right)\pi. \tag{7.2.10}$$

By Lemma 7.2.1(b) , $\lambda\in S_k$. □

As an application, consider the the frequency modulation equation[**45**]

$$\left(1+a\cos 2t\right)y''\left(t\right)+\lambda y\left(t\right)=0,\qquad 0<a<1. \tag{7.2.11}$$

From Theorem 7.2.2 we obtain the following result.

THEOREM 7.2.3. *The stability intervals S_k, $k=0,1,2,\ldots$ of equation 7.2.11 satisfy*

(7.2.12)

$$\left[4\left(1+a\right)\left(\frac{k\pi+\frac{1}{2}\ln\left(\frac{1+a}{1-a}\right)}{K\left(\sqrt{\frac{2a}{1+a}}\right)}\right)^2,4\left(1+a\right)\left(\frac{\left(k+1\right)\pi-\frac{1}{2}\ln\left(\frac{1+a}{1-a}\right)}{K\left(\sqrt{\frac{2a}{1+a}}\right)}\right)^2\right]\in S_k.$$

Where K is the complete elliptic integral of the first kind.

7.3. A Special Case

We consider the equation

$$y''\left(t\right)+b_1\left(\sin 2t\right)y'\left(t\right)+\left(\lambda+d_1\cos 2t+d_2\cos 4t\right)y\left(t\right)=0, \tag{7.3.1}$$

where b_1, d_1, d_2 are given real numbers and λ is the spectral parameter. Equation (7.3.1) is a special case of the generalized Ince equation with $\eta=2$. Adding the assumption $b_1^2+8d_2\geq 0$, equation (7.3.1) can be transformed to the Ince equation

$$z''\left(t\right)+b\left(\sin 2t\right)z'\left(t\right)+\left(\widetilde{\lambda}+d\cos 2t\right)z\left(t\right)=0, \tag{7.3.2}$$

where

$$b = -\sqrt{b_1^2 + 8d_2}, \quad d = d_1 - b_1 - \sqrt{b_1^2 + 8d_2}, \quad \widetilde{\lambda} = \lambda + d_2, \tag{7.3.3}$$

by mean of the transformation

$$y(t) = z(t) \exp\left(\frac{b_1 + \sqrt{b_1^2 + 8d_2}}{4} \cos 2t\right). \tag{7.3.4}$$

The associated polynomials to equation (7.3.2) are

$$Q(\mu) = \sqrt{b_1^2 + 8d_2}\mu + \frac{1}{2}\left(b_1 - d_1 + \sqrt{b_1^2 + 8d_2}\right), \tag{7.3.5}$$

$$Q^\dagger(\mu) = \sqrt{b_1^2 + 8d_2}\mu + \frac{1}{2}(b_1 - d_1), \tag{7.3.6}$$

whereas the associated polynomials to equation (7.3.1) are

$$Q_1(\mu) = -b_1\mu - \frac{d_1}{2}, \tag{7.3.7}$$

$$Q_2(\mu) = -\frac{d_2}{2}, \tag{7.3.8}$$

$$Q_1^\dagger(\mu) = -b_1\mu + \frac{b_1 - d_1}{2}, \tag{7.3.9}$$

$$Q_1^\dagger(\mu) = -\frac{d_2}{2}. \tag{7.3.10}$$

If $b_1 + 8d_2 > 0$, define the real numbers p and $p^\dagger$ by

$$p := -\frac{1}{2}\frac{b_1 - d_1 + \sqrt{b_1^2 + 8d_2}}{\sqrt{b_1^2 + 8d_2}}, \qquad p^\dagger := \frac{1}{2}\frac{d_1 - b_1}{\sqrt{b_1^2 + 8d_2}}. \tag{7.3.11}$$

By Section 3.5 we have the following result on coexistence of solutions with period π and semi period π for equation (7.3.1)

THEOREM 7.3.1.

(a) If $p \notin \mathbb{Z}$, then no coexistence of solutions with period π occurs.

(b) If $p^\dagger \notin \mathbb{Z}$, then no coexistence of solutions with semi-period π occurs.

(*c*) *If* $p \in \mathbb{Z}$, *define* ℓ *by*

$$\ell := \frac{1}{2} + \left| p + \frac{1}{2} \right|, \tag{7.3.12}$$

then coexistence of solutions with period π *occurs for* $\lambda = \alpha_{2m}^2 = \beta_{2m}^2$ *with* $m \geq \ell$.

(*d*) *If* $p^\dagger \in \mathbb{Z}$, *define* $\ell^\dagger$ *by*

$$\ell^\dagger := \left| p^\dagger \right|, \tag{7.3.13}$$

then coexistence of solutions with semi-period π *occurs for* $\lambda = \alpha_{2m+1}^2 = \beta_{2m+1}^2$ *with* $m \geq \ell^\dagger$.

Example 7.3.2. We consider equation (7.3.1) with $b_1 = d_1$. We have $p = -1/2$ and $p^\dagger = 0$, therefore $\alpha_{2m}^2 \neq \beta_{2m}^2$ and $\alpha_{2m+1}^2 = \beta_{2m+1}^2$ for $m = 0, 1, 2, \ldots$

In the case $b_1 = 0$, $d_1 = 2\theta_1$, $d_2 = 2\theta_2$. Equation (7.3.1) becomes

$$y''(t) + (\lambda + 2\theta_1 \cos 2t + 2\theta_2 \cos 4t)\, y(t) = 0. \tag{7.3.14}$$

Note that the substitution (7.3.2) transforms (7.3.14) to an Ince equation with

$$a = 0, \quad b = -4\sqrt{\theta_2}, \quad d = 2\theta_1 - 4\sqrt{\theta_2}, \quad \widetilde{\lambda} = \lambda + 2\theta_2. \tag{7.3.15}$$

Urwin and Arscott [**69**] investigate equation (7.3.14). Lebedev and Pergamenceva [**42**] find integral equations for periodic solutions of the same differential equation.

7.4. Nonlinear Evolution Equation

The combined KdV-mKdV equation is the is the nonlinear, dispersive partial differential equation for a function u of two real variables, space x and time t

$$\frac{\partial u}{\partial t} + (\alpha + \gamma u)\, u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0, \tag{7.4.1}$$

where α, β, γ are fixed real numbers.

Equation (7.4.1) is widely used in various branches of physics, such as plasma physics, fluid physics, quantum field theory. It has been investigated thoroughly in the literature as it is used to model a variety of nonlinear phenomena (see [**81**]).

Following [**19**], we seek a travelling wave solutions to (7.4.1) of the form

$$u := u(\xi), \qquad \xi := k(x - ct), \tag{7.4.2}$$

where k and c are wave number and wave speed, respectively.

Substituting (7.4.2) into (7.4.1), we have

$$((\alpha + \gamma u)\, u - c)\frac{du}{d\xi} + \beta k^2 \frac{d^3 u}{d\xi^3} = 0. \tag{7.4.3}$$

Integrating equation (7.4.3) with respect to ξ and setting the integration constant to zero, we obtain

$$\beta k^2 \frac{d^2 u}{d\xi^2} + \frac{\gamma}{3}u^3 + \frac{\alpha}{2}u^2 - cu = 0. \tag{7.4.4}$$

Next, we consider the perturbation method (see [**35**]) by setting

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots, \tag{7.4.5}$$

where ϵ ($0 < \epsilon \ll 1$) is a small perturbation parameter. Substituting (7.4.5) into (7.4.4), we obtain the following zeroth-order and first-order equations

$$\epsilon^0 : \qquad \beta k^2 \frac{d^2 u_0}{d\xi^2} + \frac{\gamma}{3}u_0^3 + \frac{\alpha}{2}u_0^2 - cu_0 = 0, \tag{7.4.6}$$

$$\epsilon^1 : \qquad \beta k^2 \frac{d^2 u_1}{d\xi^2} + \left(\gamma u_0^2 + \alpha u_0 - c\right) u_1 = 0. \tag{7.4.7}$$

To solve the zeroth-order equation (7.4.6), assume a solution of the form

$$u_0 = a_0 + a_1 \operatorname{sn} \xi, \tag{7.4.8}$$

where the function $\operatorname{sn} \xi$ has modulus m ($0 < m < 1$). Substituting equation (7.4.8) into equation (7.4.6), the expansion coefficients a_0 and a_1 can easily be determined as

$$a_0 = -\frac{\alpha}{2\gamma}, \qquad a_1 = \pm\sqrt{-\frac{6\beta}{\gamma}}mk, \tag{7.4.9}$$
$$c = -\frac{\alpha^2}{6\gamma}, \qquad k^2 = -\frac{\alpha^2}{12\beta\gamma\left(1+m^2\right)},$$

and the zeroth-order solution is

$$u_0 = -\frac{\alpha}{2\gamma} \pm \sqrt{-\frac{6\beta}{\gamma}}mk\,\mathrm{sn}\,\xi. \tag{7.4.10}$$

Substituting the zeroth-order exact solution (7.4.10) into the first-order equation (7.4.7) yields

$$\frac{d^2u_1}{d\xi^2} + \left(\left(1+m^2\right) - 6m^2\,\mathrm{sn}^2\,\xi\right)u_1 = 0. \tag{7.4.11}$$

Equation (7.4.11) is a Lamé equation, one can check that its solution is

$$u_1 = A\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \tag{7.4.12}$$

where A is an arbitrary constant. Hence, equation (7.4.12) is the first-order exact solution of combined mKdV-KdV equation (7.4.1).

7.5. Two Degree of Freedom Systems and Vibration Theory

Some dynamic systems that require two independent coordinates, or degrees of freedom, to describe their motion, are called "two degree of freedom systems". For a two degree of freedom system there are two equations of motion, each one describing the motion of one of the degrees of freedom. In general, the two equations are in the form of coupled differential equations. Assuming a harmonic solution for each coordinate, the equations of motion can be used to determine two natural frequencies, or modes, for the system.

In many of such dynamical systems, one is faced with the concept of free vibrations; this means that although an outside agent may have participated in causing an initial displacement or velocity, or both, of the system, the outside agent plays no further role, and the subsequent motion depends only up on the inherent properties of the system. This is in contrast to "forced" motion in which the system is continually driven by an external force.

As an example, Recktenwald and Rand [**57**] study autoparametric excitation in a class of systems having the following very general expressions for kinetic energy T and potential energy V :

$$T = \frac{1}{2}\dot{x}^2 + \beta(x,y)\,\dot{y}^2, \tag{7.5.1}$$

$$V = \frac{1}{2}x^2 + \frac{1}{2}\omega^2 y^2 + \alpha_{22}x^2y^2 + \alpha_{13}xy^3 + \alpha_{04}y^4, \tag{7.5.2}$$

with the assumption that the function $\beta(x,y)$ has the form

$$\beta(x,y) = \beta_{00} + \beta_{01}x + \beta_{10}y + \beta_{02}x^2 + \beta_{11}xy + \beta_{20}y^2. \tag{7.5.3}$$

To investigate the linear stability of the $x-$mode

$$x = \cos t, \qquad y = 0, \tag{7.5.4}$$

set

$$x = \cos t + u, \quad y = v,$$

in Lagrange's equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{x}}\right) - \frac{\partial(T-V)}{\partial x} &= 0, \\ \frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{y}}\right) - \frac{\partial(T-V)}{\partial y} &= 0, \end{aligned}$$

which gives

$$\ddot{u} + u = 0, \tag{7.5.5}$$

and

$$\begin{aligned}
&\left(2\beta_{00} + A^2\beta_{02} + 2A\beta_{01}\cos t + A^2\beta_{02}\cos 2t\right)\ddot{v}\\
&\quad + \left(-2A\beta_{01}\sin t - 2A^2\beta_{02}\sin 2t\right)\dot{v}\\
&\quad + \left(\lambda + A^2\alpha_{22} + A^2\alpha_{22}\cos 2t\right)v = 0.
\end{aligned} \tag{7.5.6}$$

Equation (7.5.6) can be put in the form of a generalized Ince equation

$$(1 + a_1\cos t + a_2\cos 2t)\,y'' + (b_1\sin t + b_2\sin 2t)\,y' + (\lambda + d_1\cos t + d_2\cos 2t)\,y = 0 \tag{7.5.7}$$

where the coefficients a_1, a_2, b_1, b_2, d_1, and d_2 are given by

$$\begin{aligned}
a_1 =& \frac{2A\beta_{01}}{2\beta_{00} + A^2\beta_{02}},\\
a_2 =& \frac{A^2\beta_{01}}{2\beta_{00} + A^2\beta_{02}},\\
b_1 =& - a_{1,}\\
b_2 =& - 2a_2,\\
d_1 =& 0,\\
d_2 =& \frac{A^2\alpha_{22}}{2\beta_{00} + A^2\beta_{02}},\\
\lambda =& \frac{\omega^2 + A^2\alpha_{22}}{2\beta_{00} + A^2\beta_{02}}.
\end{aligned}$$

In [**57**], the authors find sufficient conditions for the coexistence of solutions with period and semi-period 2π.

CHAPTER 8

Conclusion

This dissertation is an attempt of a thorough investigation of Ince and Lamé equations, and their generalizations. All of these equation are linear second order ordinary differential equations with periodic coefficients. In particular, they are even Hill equations with period ω (with period π for Ince's equation and $2K$ for Lamé's equation). We are interested in solutions which are even or odd and have period ω or semi-period ω. From the general theory of Hill's equation the problem splits into four regular Sturm-Liouville problems:

- Even with period ω.
- Even with semi-period ω.
- Odd with semi-period ω.
- Odd with period ω.

Using Fourier series representation of the solution, each one of these Sturm-Liouville operators is represented by a banded infinite matrix.

When studying the Ince equation, it became apparent that many of the techniques can be useful in treating a more general class of equation "the generalized Ince equation". In chapter two we introduced the general frame work and gave formulas to calculate the entries of the banded infinite matrices associated with the generalized Ince equation. For example in the case of Ince's equation this process gives rise to four tridiagonal infinite matrices, which were discussed in detail in chapter three, the tridiagonal structure also allowed the investigation of the problem of coexistence of periodic solution (Section 3.5) and that of the existence of polynomial solutions in trigonometric form.

Chapter four was dedicated to Lamé's equation. Employing Jacobi's amplitude $t = \operatorname{am} z$, Lamé's equation is transformed to its trigonometric form, and this is a particular Ince equation. In a similar fashion we discussed a generalization of Lamé's equation found in [**54**], which is then transformed to a particular case ($\eta = 2$) of the generalized Ince equation.

As in the case of Mathieu's equation, Lamé and Ince equations appears in the process of separation of variables of some partial differential equation problems in certain special coordinate systems, such as sphero-conal coordinates. These ideas were discussed in chapter six (in the case of the wave equation).

chapter seven was a collection of mathematical and physical applications that we felt were relevant to the discussion. The analysis was supplemented by Maple codes that can be found in the Appendix.

In section 6.3, we considered the problem of separation of variables for the wave equation

$$\nabla^2 u + \omega^2 u = 0.$$

In ellipsoidal coordinate system, the wave equation admits separable solutions of the form

$$u(\alpha, \beta, \gamma) = u_1(\alpha)\, u_2(\beta)\, u_3(\gamma), \tag{8.0.8}$$

where u_1, u_2, u_3 each satisfy the Lamé wave equation

$$u'' + \big(h - \nu(\nu+1)\, k^2 \operatorname{sn}^2(z,k) + k^2\omega^2 \operatorname{sn}^4(z,k)\big)\, u = 0. \tag{8.0.9}$$

For future work, we would like to investigate equation (8.0.9) as another generalization of Lamé's equation. The transformation $t = \frac{\pi}{2} - \operatorname{am} z$, transforms (8.0.9) to a generalized Ince equation of the form

$$(1 + a_1 \cos 2t)\, u'' + b_1 (\sin 2t)\, u' + (\lambda + d_1 \cos 2t + d_2 \cos 4t)\, u = 0, \tag{8.0.10}$$

in particular, we would like to find condition for coexistence of periodic solutions.

APPENDIX A

Maple Code

Infinite matrix M_1 (even with period π boundary conditions) of Section 2.4

```
Loading LinearAlgebra
# Functions A(t), B(t), and D(t)
A := proc (a, t)
local s; s := Dimension(a);
sum('a[i]'*cos(2*i*t), i = 1 .. s)
end proc
B := proc (b::Vector, t)
local s; s := Dimension(b);
sum('b[i]'*sin(2*i*t), i =1 .. s)
end proc
# The generalized Ince operator
T := proc (a, b, d, y, t)
-(1+A(a, t))*(diff(diff(y, t), t))-B(b, t)*(diff(y,t))-A(d, t)*y
end proc
u := proc (n, t)
options operator, arrow;
cos(2*n*t)
end proc
# Coefficients a_j, b_j, d_j, j = 1, ..., η
a := Vector([a1, a2]);
b := Vector([b1, b2]);
```

```
d := Vector([d1, d2]);
# Computation the entries Infinite Matrix M1
r := proc (n, j)
options operator, arrow;
simplify(4*(int(T(a, b, d, u(n, t),t)*u(j, t), t = 0 .. (1/2)*Pi))/Pi)
end proc
# Display of a finite section (upper left corner of M1)
N := 10
Aevp := proc (a, b, d, n)
local A, k, l, v, w;
A := Matrix(n, n, 0);
for k to n do
for l to n do w := k-1; v := l-1;
if w = 0 or v = 0 then A[k, l] := r(v,w)/sqrt(2)
else A[k, l] := r(v, w)
end if
end do
end do;
A
end proc
B := Aevp(a, b, d, 7)
```

Infinite matrix M_2 (even with semi-period π boundary conditions) of Section 2.4

```
Loading LinearAlgebra
# Functions A(t), B(t), and D(t)
A := proc (a, t)
local s; s := Dimension(a);
```

```
sum('a[i]'*cos(2*i*t), i = 1 .. s)
end proc
B := proc (b::Vector, t)
local s; s := Dimension(b);
sum('b[i]'*sin(2*i*t), i =1 .. s)
end proc
# The generalized Ince operator
T := proc (a, b, d, y, t)
-(1+A(a, t))*(diff(diff(y, t), t))-B(b, t)*(diff(y,t))-A(d, t)*y
end proc
u := proc (n, t)
options operator, arrow;
cos((2*n+1)*t)
end proc
# Coefficients a_j, b_j, d_j, j = 1, ..., η
a := Vector([a1, a2]);
b := Vector([b1, b2]);
d := Vector([d1, d2]);
# Computation the entries Infinite Matrix M_1
r := proc (n, j)
options operator, arrow;
simplify(4*(int(T(a, b, d, u(n, t),t)*u(j, t), t = 0 .. (1/2)*Pi))/Pi)
end proc
# Display of a finite section (upper left corner of M_1)
Aevp := proc (a, b, d, n)
local A, k, l, v, w;
A := Matrix(n, n, 0);
for k to n do
```

```
for l to n do
w := k-1; v := l-1;
A[k, l] := r(v, w)
end do
end do;
A
end proc
B := Aevp(a, b, d, 7)
```

Infinite matrix M_3 (odd with semi-period π boundary conditions) of Section 2.4

```
Loading LinearAlgebra
# Functions A(t), B(t), and D(t)
A := proc (a, t)
local s; s := Dimension(a);
sum('a[i]'*cos(2*i*t), i = 1 .. s)
end proc
B := proc (b::Vector, t)
local s; s := Dimension(b);
sum('b[i]'*sin(2*i*t), i =1 .. s)
end proc
# The generalized Ince operator
T := proc (a, b, d, y, t)
-(1+A(a, t))*(diff(diff(y, t), t))-B(b, t)*(diff(y,t))-A(d, t)*y
end proc
u := proc (n, t)
options operator, arrow;
sin((2*n+1)*t)
```

```
end proc
```

Coefficients a_j, b_j, d_j, $j = 1, ..., \eta$

```
a := Vector([a1, a2]);
b := Vector([b1, b2]);
d := Vector([d1, d2]);
```

Computation the entries Infinite Matrix M_1

```
r := proc (n, j)
options operator, arrow;
simplify(4*(int(T(a, b, d, u(n, t),t)*u(j, t), t = 0 .. (1/2)*Pi))/Pi)
end proc
```

Display of a finite section (upper left corner of M_1)

```
Aevp := proc (a, b, d, n)
local A, k, l, v, w;
A := Matrix(n, n, 0);
for k to n do
for l to n do
w := k-1; v := l-1;
A[k, l] := r(v, w)
end do
end do;
A
end proc
B := Aevp(a, b, d, 7)
```

Infinite matrix M_4 (odd with period π boundary conditions) of Section 2.4

```
Loading LinearAlgebra
```

Functions $A(t)$, $B(t)$, and $D(t)$

```
A := proc (a, t)
local s; s := Dimension(a);
sum('a[i]'*cos(2*i*t), i = 1 .. s)
end proc
B := proc (b::Vector, t)
local s; s := Dimension(b);
sum('b[i]'*sin(2*i*t), i =1 .. s)
end proc
# The generalized Ince operator
T := proc (a, b, d, y, t)
-(1+A(a, t))*(diff(diff(y, t), t))-B(b, t)*(diff(y,t))-A(d, t)*y
end proc
u := proc (n, t)
options operator, arrow;
sin((2*n+2)*t)
end proc
# Coefficients a_j, b_j, d_j, j = 1, ..., η
a := Vector([a1, a2]);
b := Vector([b1, b2]);
d := Vector([d1, d2]);
# Computation the entries Infinite Matrix M_1
r := proc (n, j)
options operator, arrow;
simplify(4*(int(T(a, b, d, u(n, t),t)*u(j, t), t = 0 .. (1/2)*Pi))/Pi)
end proc
# Display of a finite section (upper left corner of M_1)
Aevp := proc (a, b, d, n)
local A, k, l, v, w;
```

```
A := Matrix(n, n, 0);
for k to n do
for l to n do
w := k-1; v := l-1;
A[k, l] := r(v, w)
end do
end do;
A
end proc
B := Aevp(a, b, d, 7)
```

Weight ω in the self adjoint form of the generalized Ince equation

---

Loading LinearAlgebra

Functions $A(t)$, $B(t)$, and $D(t)$

```
A := proc (a, t)
local s; s := Dimension(a);
sum('a[i]'*cos(2*i*t), i = 1 .. s)
end proc
B := proc (b::Vector, t)
local s; s := Dimension(b);
sum('b[i]'*sin(2*i*t), i =1 .. s)
end proc
```

Loading PDEtools

Coefficients a_j, b_j, d_j, $j = 1, ..., \eta$

```
a := Vector([a1, a2]);
b := Vector([b1, b2]);
d := Vector([d1, d2]);
```

```
# ω (t) formula
"r(t):=(B(b,t)-diff(A(a,t),t))/(1+A(a,t));"
"s(t):=&int;r(t) &DifferentialD;t;"
"omega(t):=(e)^(s(t) );"
"simplify( omega(t), 'size' )"
```

Test for Ince's polynomials and coexistence of solutions with period π

```
Loading LinearAlgebra
Q := proc (a, b, d)
local T; T := Vector(1, 2, 0);
T :=[(1/4)*(b+sqrt(b^2+4*a*d))/a, -(1/4)*(-b+sqrt(b^2+4*a*d))/a]
end proc
Incepoly := proc (T)
local p, q, l, k, ma, mi;
p := type(T[1], integer);
q :=type(T[2], integer);
if p = false and q = false then
print('There is no Ince polynomials,
no coexistence of solutions with period Pi occurs')
elif p = true and q = false
then l := 1/2+abs(1/2+T[1]);
if l = 1 then
print('Ince polynomials with eigenvalues α_2m when m is equal to);
print(0)
elif
l = 2 then
print('Ince polynomials with eigenvalues α_2m when m is equal to');
```

print(0, 1);

print('Ince polynomials with eigenvalues β_{2m} when m is equal to');

print(1)

elif l = 3 then

print('Ince polynomials with eigenvalues α_{2m} when m is equal to');

print(0,1,2);

print('Ince polynomials with eigenvalues β_{2m} when m is equal to');

print(1,2);

else

print('Ince polynomials with eigenvalues α_{2m} when m is equal to');

print(0, 1, () .. (), l-1);

print('Ince polynomials with eigenvalues β_{2m} when m is equal to');

print(1, () ..(), l-1)

end if;

print('solutions with period π coexist, $\alpha_{2m} = \beta_{2m}$, for m equals');

print(l, l+1, () .. (), infinity);

if T[1] < 0 then

print('All Ince polynomials are of the second kind polynomials')

else

print('All Ince polynomials are of the first kind polynomials')

end if

elif p = false and q = true then

l := 1/2+abs(1/2+T[2]);

if l = 1 then

print('Ince polynomials with eigenvalues α_{2m} when m is equal to');

print(0)

elif

l = 2 then

print('Ince polynomials with eigenvaluesα_{2m} when m is equal to');

print(0, 1);

print('Ince polynomials with eigenvaluesβ_{2m} when m is equal to');

print(1);

elif l = 3 then

print('Ince polynomials with eigenvaluesα_{2m} when m is equal to');

print(0, 1,2);

print('Ince polynomials with eigenvaluesβ_{2m} when m is equal to');

print(1,2)

else

print('Ince polynomials with eigenvaluesα_{2m} when m is equal to');

print(0, 1, () .. (), l-1);

print('Ince polynomials with eigenvaluesβ_{2m} when m is equal to');

print(1, () ..(), l-1)

end if;

print('Solutions of period π coexist.');

print('$\alpha_{2m} = \beta_{2m}$ when m is equal to');

print(l, l+1, () .. (), infinity);

if T[1] $<$ 0 then

print('All Ince polynomials are second kind polynomials')

else

print('All Ince polynomials are first kind polynomials')

end if

else l :=1/2+min(abs(1/2+T[1]), abs(1/2+T[2]));

k := 1/2+max(abs(1/2+T[1]),abs(1/2+T[2]));

print('Solutions of period π coexist.');

print('$\alpha_{2m} = \beta_{2m}$ when m is equal to');

print(l, l+1, () .. (), infinity);

```
if k = 1 then
print('Solutions of period π coexist.');
print('α_{2m} = β_{2m} when m is equal to');
print(0)
elif k = 2 then
print('Ince polynomials with eigenvalues α_{2m} when m equals');
print(0, 1);
print('Ince polynomials with eigenvalues β_{2m} when m equals');
print(1)
elif k = 3 then
print('Ince polynomials with eigenvalues α_{2m} when m equals');
print(0, 1,2);
print('Ince polynomials with eigenvaluesβ_{2m} when m equals');
print(1,2)
else
print('Ince polynomials with eigenvaluesα_{2m} when m equals');
print(0, 1, () .. (), k-1);
print('Ince polynomials with eigenvaluesβ_{2m} when m equals');
print(1, () ..(), k-1)
end if
end if
end proc
```

Test for Ince's polynomials and coexistence of solutions with semi-period π

```
Loading LinearAlgebra
> Q := proc (a, b, d)
local T; T := Vector(1, 2, 0);
```

T := [(1/4)*(b+sqrt(bˆ2+4*a*d))/a+1/2, 1/2-(1/4)*(-b+sqrt(bˆ2+4*a*d))/a]

end proc;

> Incepoly := proc (T)

local p, q, l, k, ma, mi;

p := type(T[1], integer);

q := type(T[2], integer);

if p = false and q = false then

print('There is no Ince polynomials, and no coexistence of solutions with semi-period π occurs')

elif p = true and q = false then

l := abs(T[1]);

if l = 1 then

print('Ince polynomials with eigenvalues α_{2m+1} when m equals');

print(0);

print('Ince polynomials with eigenvalues β_{2m+1} when m equals');

print(0)

elif l = 2 then

print('Ince polynomials with eigenvalues α_{2m+1} when m equals');

print(0, 1);

print('Ince polynomials with eigenvalues β_{2m+1} when m equals');

print(0, 1)

elif l = 3 then

print(Ince polynomials with eigenvalues α_{2m+1} when m equals');

print(0, 1, 2);

print('Ince polynomials with eigenvalues β_{2m+1} when m equals');

print(0, 1, 2)

else

print('Ince polynomials with eigenvalues α_{2m+1} when m equals');

print(0, 1, () .. (), l-1);

print('Ince polynomials with eigenvalues β_{2m+1} when m equals');

print(0, 1, () .. (), l-1)

end if;

print('Solutions with semi-period π coexist, $\alpha_{2m+1} = \beta_{2m+1}$, when m equals');

print(l, l+1, () .. (), infinity);

if T[1] $<$ 0 then

print('All Ince polynomials are second kind polynomials')

else

print('All Ince polynomials are first kind polynomials')

end if

elif

p = false and q = true then

l := abs(T[2]); if l = 1 then

print('Ince polynomials with eigenvalues α_{2m+1} when m equals');

print(0);

print('Ince polynomials with eigenvalues β_{2m+1} when m equals');

print(0)

elif l = 2 then

print('Ince polynomials with eigenvalues α_{2m+1} when m equals');

print(0, 1);

print('Ince polynomials with eigenvalues β_{2m+1} when m equals');

print(0, 1)

elif l = 3 then

print('Ince polynomials with eigenvalues α_{2m+1} when m equals');

print(0, 1, 2);

print('Ince polynomials with eigenvalues β_{2m+1} when m equals');

print(0, 1, 2)

else

print('Ince polynomials with eigenvalues α_{2m+1} when m equals');

print(0, 1, () .. (), l-1);

print('Ince polynomials with eigenvalues β_{2m+1} when m equals');

print(0, 1, () .. (), l-1)

end if;

print('Solutions of period coexist with $\alpha_{2m+1} = \beta_{2m+1}$ when n equals');

print(l, l+1, () .. (), infinity);

if T[1] < 0 then

print('All Ince polynomials are second kind polynomials')

else print('All Ince polynomials are first kind polynomials')

end if

else

l := min(abs(T[1]), abs(T[2]));

k := max(abs(T[1]), abs(T[2]));

print('Solutions of period coexist with $\alpha_{2m+1} = \beta_{2m+1}$ when n equals');

print(l, l+1, () .. (), infinity);

if k = 1 then

print('Ince polynomials with eigenvalues α_{2m+1} when m equals');

print(0);

print('Ince polynomials with eigenvalues β_{2m+1} when m equals');

print(0)

elif k = 2 then

print('Ince polynomials with eigenvalues α_{2m+1} when m equals');

print(0, 1);

print('Ince polynomials with eigenvalues β_{2m+1} when m equals');

print(0, 1)

elif k = 3 then

```
print('Ince polynomials with eigenvalues α_{2m+1} when m equals');
print(0, 1, 2);
print('Ince polynomials with eigenvalues β_{2m+1} when m equals');
print(0, 1, 2)
else
print('Ince polynomials with eigenvalues α_{2m+1} when m equals');
print(0, 1, () .. (), k-1);
print('Ince polynomials with eigenvalues β_{2m+1} when m equals');
print(0, 1, () .. (), k-1)
end if
end if
end proc;
```

Transformation of Ince's equation to algebraic form

```
Loading PDEtools
Ince := (1+a*cos(2*t))*(diff(diff(y, t), t))+b*sin(2*t)*(diff(y, t))
        +(lambda+d*cos(2*t))*y = 0;
f := y(t);
y := g(t);
tr := {t = arccos(sqrt(z))};
R := dchange(tr, Ince)
```

Transformation of the generalized Ince equation to algebraic form

---

```
Loading LinearAlgebra
A := proc (a, t)
local s;
```

```
s := Dimension(a);
sum('a[i]'*cos(2*i*t), i = 1 .. s);
end proc
B := proc (b::Vector, t)
local s;
s := Dimension(b);
sum('b[i]'*sin(2*i*t), i =1 .. s)
end proc;
f := y(t):
Loading PDEtools
a := Vector([a1, a2, a3]);
b := Vector([b1, b2, b3]);
d := Vector([d1, d2, d3]);
GInce := (1+A(a, t))*(diff(diff(f, t), t))+B(b, t)*(diff(f, t))
            +(lambda+A(d,t))*f = 0;
tr := {t = arccos(sqrt(z))};
dchange(tr, GInce);
```

Instability intervals of Section 7.1 A matrix formula for ω_m

---

Loading LinearAlgebra

Polynomials D_j, $j = 1, 2, \ldots, \eta$

```
C := proc (j, t) local s;
s := Dimension(a);
(1/2)*a[j]*t^2+(1/2)*b[j]*t-(1/2)*d[j]
end proc
```

Coefficients **a**, **b**, **d**

```
a := Vector([a1, a2, a3])
```

```
b := Vector([b1, b2, b3])
d := Vector([d1, d2, d3])
s := Dimension(a)
```

Definition of m_l: Equation (7.1.17)

```
ml := proc (m, l)
m-2*l*s
end proc
ek := proc (m, k)
1/(m^2-k^2)
```

Definition of e_k: Equation (7.1.18)

```
end proc
Ek := proc (k, m)
local V, i;
V := Matrix(s, s, 0);
for i to s do
V[i, i] :=ek(m, ml(m, k)+2*i-2)
end do;
V
end proc
```

Definition of matrix F_k: Equation (7.1.40)

```
Fk := proc (k, m)
local V, mk, i, j;
V := Matrix(s, s, 0);
for i to s do
for j to i do
V[i, j] := C(s-i+j, ml(m, k)+2*j-2)
end do
end do;
```

```
V
end proc
# Definition of m: Equation (7.1.4)
m := proc (p, q)
p*s+q
end proc
# Matrix W : Theorem 7.1.9
W := proc (p, q) local V, i;
V := 2.*Fk(0, m(p, q));
for i to p do
V := Fk(i,m(p, q)).Ek(i, m(p, q)).V
end do;
V
end proc
# For example if m = 10, η = 3, then according to (7.1.4) p = 3, and q = 1
T := W(3, 1);
# ω_m is the entry in the first column and (η − q + 1) = 3 row of the matrix T
ω_m:= T[3, 1]
```

Separation of variables in Section 3.6 : Ince equation when $a = 0$

```
Loading PDEtools
pde := diff(u(x, y), x, x)+diff(u(x, y), y, y)-2*b*(x*(diff(u(x, y),x))
    +y*(diff(u(x, y), y)))-2*d*u(x, y) = 0
# Change of variables to elliptic coordinates
tr := {x = cos(eta)*cosh(xi), y = sin(eta)*sinh(xi)}
pde1 := dchange(tr, pde)
pde1 := subs(u(eta, xi) = v(eta)*w(xi), pde1)
```

```
pde1 := simplify(pde1)

pde1 := simplify(-pde1*(-cosh(xi)^2+cos(eta)^2))

pde1 := combine(pde1, 'trig')
```

Separation of variables in Section 3.6 : Ince equation when $a = 0$

```
Loading PDEtools

Loading Student:-MultivariateCalculus

pde := diff(u(x, y, z), x, x)+diff(u(x, y, z), y, y)
    +diff(u(x, y, z), z,z)-(1+b/a)*(diff(u(x, y, z), z))/z = 0

# choose a and b in (-1,0)

a := 'a';

b := 'b'

# Define k and k' ( both are in (0,1))

k := sqrt(2*a/(a-1));

k1 := sqrt(1-k^2);

# change of variables to sphero-conal coordinates

tr := {x = r*k*cos(eta)*cosh(xi), y = r*k*sin(eta)*sinh(xi)/k1,
    z =r*(1-k^2*cos(eta)^2)^(1/2)*(1-k^2*cosh(xi)^2)^(1/2)/k1}

pde1 := dchange(tr, pde, {eta, r, xi}, simplify)
```

APPENDIX B

Matlab Code

Infinite matrix M_1 (Even with period π boundary conditions) of Section 2.4

```
function y=M1(a,b,d,n)
%Matrix M1 for even with period pi (nu=2)
M=zeros(n,n);
for i=1:n
    M(i,i)=r(i);
end
for i=2:n
    M(i,i-1)=Q1(a,b,d,i-2);
end
for i=3:n
    M(i,i-2)=Q2(a,b,d,i-3);
end
for i=1:n-1
    M(i,i+1)=Q1(a,b,d,-i);
end
for i=1:n-2
    M(i,i+2)=Q2(a,b,d,-i-1);
end
M(2,2)=r(2)+Q2(a,b,d,-1);
M(1,2)=(sqrt(2))*Q1(a,b,d,-1);
```

```
M(1,3)=(sqrt(2))*Q2(a,b,d,-2);
M(2,1)=(sqrt(2))*Q1(a,b,d,0);
M(3,1)=(sqrt(2))*Q2(a,b,d,0);
y=M;
end
```

Infinite matrix M_2 (Even with semi-period π boundary conditions) of Section 2.4

```
function y=M2(a,b,d,n)
%matrix M2 for Ince's equation (s=2) even with semi period pi
M=zeros(n,n);
for i=2:n
    M(i,i)=rd(i);
end
M(1,1)=1+Qd1(a,b,d,0);
for i=3:n
    M(i,i-1)=Qd1(a,b,d,i-1);
end
M(2,1)=Qd1(a,b,d,1)+Qd2(a,b,d,0);
for i=3:n
    M(i,i-2)=Qd2(a,b,d,i-2);
end
for i=2:n-1
    M(i,i+1)=Qd1(a,b,d,-i);
end
M(1,2)=Qd1(a,b,d,-1)+Qd2(a,b,d,-1);
```

```
for i=1:n-2
    M(i,i+2)=Qd2(a,b,d,-i-1);
end
y=M;
end
```

Infinite matrix M_3 (Odd with semi- period π boundary conditions) of Section 2.4

```
function y=M3(a,b,d,n)
% Matrix M2 for Ince's equation (s=2) even with semi period pi
M=zeros(n,n);
for i=2:n
    M(i,i)=rd(i);
end
M(1,1)=1-Qd1(a,b,d,0);
for i=3:n
    M(i,i-1)=Qd1(a,b,d,i-1);
end
M(2,1)=Qd1(a,b,d,1)-Qd2(a,b,d,0);
for i=3:n
    M(i,i-2)=Qd2(a,b,d,i-2);
end
for i=2:n-1
    M(i,i+1)=Qd1(a,b,d,-i);
end
```

```
M(1,2)=Qd1(a,b,d,-1)-Qd2(a,b,d,-1);


 for i=1:n-2
      M(i,i+2)=Qd2(a,b,d,-i-1);
 end
 y=M;
 end
```

Infinite matrix M_4 (Odd with period π boundary conditions) of Section 2.4

```
 function y = M4(a,b,d,n)
% Matrix M4 odd with period pi (s=2) Ince equation
M=M1(a,b,d,n+1);
N=zeros(n,n);
 for i=1:n
 for j=1:n
N(i,j)=M(i+1,j+1);
 end
 end
 y=N;
 end
```

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