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Contractible n-Manifolds and the Double n-Space Property

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by

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ABSTRACT

CONTRACTIBLE n-MANIFOLDS AND THE DOUBLE n-SPACE PROPERTY

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We are interested in contractible manifolds M^n which decompose or *split* as $M^n = A \cup_C B$ where $A, B, C \approx \mathbb{R}^n$ or $A, B, C \approx \mathbb{B}^n$. We introduce a 4-manifold M containing a spine which can be written as $A \cup_C B$ with A, B, and C all collapsible which in turn implies M splits as $\mathbb{B}^4 \cup_{\mathbb{B}^4} \mathbb{B}^4$. From M we obtain a countably infinite collection of distinct 4-manifolds all of which split as $\mathbb{B}^4 \cup_{\mathbb{B}^4} \mathbb{B}^4$. Connected sums at infinity of interiors of manifolds from sequences contained in this collection constitute an uncountable set of open 4-manifolds each of which splits as $\mathbb{R}^4 \cup_{\mathbb{R}^4} \mathbb{R}^4$.

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Chapter 1

Introduction To Manifold Splitting

1.1 Definitions, Motivation, and Summary of Results

Our results will generally be in the topological category but because of the niceness of the spaces involved we are able to work in both the piecewise linear and smooth categories in our effort to obtain them. We will primarily be working in the PL category. We may choose to construct manifolds (and other objects) to be piecewise-linear or smooth. Unless stated otherwise the reader should view such constructions as PL. By a PL manifold we mean a simplicial complex in which the link of every vertex is a sphere.

Definition 1.1.1. We will write $A \cup_C B$ to indicate a union $A \cup B$ with intersection $C = A \cap B$. We say a manifold M^n *splits* if $M^n = A \cup_C B$ with A, B, and $C = A \cap B \approx \mathbb{B}^n$ or A, B, and $C = A \cap B \approx \mathbb{R}^n$. In the former case we say M "splits into closed balls" or M is a "closed splitter" and write $M^n = \mathbb{B}^n \cup_{\mathbb{B}^n} \mathbb{B}^n$. In the latter case we say M "splits into open balls" or M is an "open splitter" and write $M^n = \mathbb{R}^n \cup_{\mathbb{R}^n} \mathbb{R}^n$.

We are interested in contractible manifolds M^n which are open or closed splitters. We introduce a 4-manifold M containing a spine, which we call a Jester's Hat, that can be written as $A \cup_C B$ with A, B, and C all collapsible. We'll show that this implies M is a closed splitter. From M we obtain a countably infinite collection of distinct 4-manifolds all of which are closed splitters. Connected sums at infinity of interiors of manifolds from sequences contained in this collection constitute an uncountable set of open 4-manifolds each of which splits as $\mathbb{R}^4 \cup_{\mathbb{R}^4} \mathbb{R}^4$. These last two statements constitute our two main theorems.

Our motivation comes from David Gabai's result that the Whitehead 3-manifold, Wh^3, splits into open 3-balls

$$Wh^3 = \mathbb{R}^3 \cup_{\mathbb{R}^3} \mathbb{R}^3 \quad \text{[Gab]}.$$

Other terminology in use which is synonomous with open splitting includes *double n-space property* and *Gabai splitting.*

1.2 Elementary Results

It is clear that the unit ball $\mathbb{B}^n$ splits into two "subballs" overlapping in a n-ball. Likewise, Euclidean space itself splits into two Euclidean spaces meeting in a Euclidean space. More generally, we have the following.

Proposition 1.2.1. *If a manifold M^n splits as $M^n = \mathbb{B}^n \cup_{\mathbb{B}^n} \mathbb{B}^n$ then* $\text{int}M^n$ *splits as int$M^n = \mathbb{R}^n \cup_{\mathbb{R}^n} \mathbb{R}^n$.*

Proof. Suppose $M^n = A \cup_C B$ with $A, B, C \approx \mathbb{B}^n$. We will show that

$$\text{int}M = \text{int}A \cup_{\text{int}C} \text{int}B.$$

In order to do this we show
(1) $\text{int}A \cap \text{int}B = \text{int}C$ and
(2) $\text{int}M = \text{int}A \cup \text{int}B$.

For (1), suppose $x \in \text{int}C$. Then, as $C \approx \mathbb{B}^n$, there exists $N \subset C$ an open (Euclidean) n-ball neighborhood of x. Then N is an open ball neighborhood of x in both A and B and thus $x \in \text{int}A \cap \text{int}B$.

For the reverse inclusion, let $x \in \text{int}A \cap \text{int}B$ and N_A and N_B be neighborhoods of x in A and B each homeomorphic to an open ball of $\mathbb{R}^n$. Then $N_A \cap N_B$ is a neighborhood of x in C and it contains a neighborhood of x homeomorphic to an open $\mathbb{R}^n$ ball as it is a neighborhood of x in M. Thus, x is an interior point of C and we have shown $\mathrm{int}A \cap \mathrm{int}B \subset \mathrm{int}C$.

For (2), it is clear that $\mathrm{int}A \cup \mathrm{int}B \subset \mathrm{int}M$. To see the reverse inclusion suppose for contradiction that there exists $x \in \mathrm{int}M \cap \partial A \cap \partial B$ so we can choose $U_A, V_B \approx \mathbb{R}^n_+$ neighborhoods of x in A and B, respectively. Then $U_A = A \cap U$ and $V_B = B \cap V$ for some open sets $U, V \subset M^n$. Let $W \approx \mathbb{R}^n$ be a neighborhood of x in M^n contained in $U \cap V$, and let $U'_A = A \cap W$ and $V'_B = B \cap W$. Then $U'_A \cup V'_B = W$, with U'_A and V'_B each homeomorphic to an open subset of $\mathbb{R}^n_+$. Notice that $\partial U'_A \subset V'_B$ and $\partial V'_B \subset U'_A$, for if $y \in \partial U'_A$ does not lie in V'_B, then a small half-space neighborhood of y in U'_A is open in W; an impossibility since $W \approx \mathbb{R}^n$. Similarly, we cannot have $y \in \partial V'_B$ that does not lie in U'_A.

Now notice that $U'_A \cap V'_B$ is a neighborhood of x in $A \cap B = C$, which by the previous observation, contains $\partial U'_A \cup \partial V'_B$. Moreover, by (1), every point of $\partial U'_A \cup \partial V'_B$ lies in ∂C. Since $\partial C \approx S^{n-1}$ is a closed $(n-1)$-manifold, small Euclidean $(n-1)$-space neighborhoods must coincide. That is, there exists an $(n-1)$-ball D in ∂C containing x lying in $\partial U'_A \cap \partial V'_B$. We see that D is the intersection of an A neighborhood of x with a B neighborhood of x so that $D \approx \mathbb{B}^{n-1}$ is a neighborhood of x in $A \cap B = C \approx \mathbb{B}^n$. This is our desired contradiction.

□

1.3 History and Current Work

Some classical knowledge about manifold splitting is contained in the following theorem [Gla65], [Gla66].

Theorem 1.3.1. (Glaser) *(a) For each $n \geq 4$ there exists a compact contractible PL n-manifold with boundary W^n not homeomorphic to $\mathbb{B}^n$ such that $W^n \approx \mathbb{B}^n \cup_{\mathbb{B}^n} \mathbb{B}^n$. (b) For each $n \geq 3$ there exist an open contractible n-manifold O^n not homeomorphic to $\mathbb{R}^n$ such that $O^n \approx \mathbb{R}^n \cup_{\mathbb{R}^n} \mathbb{R}^n$.*

For the compact case, Glaser shows the existence of a contractible $(n-2)$-complex piecewise linearly embedded in S^n with non-ball regular neighborhoods which split.

The $n \geq 5$ case was shown in [Gla65] and the $n = 4$ case was shown in [Gla66].

For the noncompact $n \geq 4$ case he takes the interiors of the compact splitters found in (a). For the noncompact $n = 3$ case, Glaser shows that the complement of a certain embedding of a double Fox-Artin arc in S^3 splits and is not a (open) ball [Gla66].

In [Gab], Gabai asks

Question 1.3.2. Is there a reasonable characterization of open contractible 3-manifolds that are the union of two embedded submanifolds each homeomorphic to $\mathbb{R}^3$ and that intersect in a $\mathbb{R}^3$?

Renewed interest in this topic, motivated by Gabai's splitting of the Whitehead manifold and the resulting above question, has led to the following recent results [GRW].

Theorem 1.3.3. (Garity, Repovs, Wright) *There exist uncountably many distinct contractible 3-manifolds that are open splitters.*

Theorem 1.3.4. (Garity, Repovs, Wright) *There are uncountably many distinct contractible 3-manifolds that are not open splitters.*

Note 1.3.5. In dimension 3, the Poincaré conjecture gives that every compact contractible manifold is homeomorphic to $\mathbb{B}^3$ so the question of closed splitters in this case is uninteresting.

Ancel and Guilbault have recently worked out the general compact case for $n \geq 5$ as well as for high dimensional Davis manifolds [AG14+] (see [AG95] for the main ideas).

Theorem 1.3.6. (Ancel and Guilbault) *If C^n $(n \geq 5)$ is a compact, contractible n-manifold then C^n splits as $\mathbb{B}^n \cup_{\mathbb{B}^n} \mathbb{B}^n$.*

Corollary 1.3.7. (Ancel and Guilbault) *For $n \geq 5$:*

1. *the interior of every compact contractible n-manifold is an open splitter, and*

2. *there are uncountably many non-homeomorphic n-manfolds which are open splitters.*

Theorem 1.3.8. (Ancel and Guilbault) *For $n \geq 5$, every Davis n-manifold is an open splitter.*

Note 1.3.9. A result of Ancel and Siebenman states that a Davis manifold generated by C is homeomorphic to the interior of an alternating boundary connected sum $\text{int}(C \overset{\partial}{\sharp} -C \overset{\partial}{\sharp} C \overset{\partial}{\sharp} -C \overset{\partial}{\sharp} \ldots)$. Here $-C$ is a copy of C with the opposite orientation [Gui]. We will show in Section 5.4 that the interior of an infinite boundary connecet sum of closed splitters is an open splitter. Thus there also exists (non-$\mathbb{R}^4$) 4-dimensional Davis manifold splitters.

Chapter 2

The Mazur and Jester's Manifolds

2.1 The Mazur Manifold

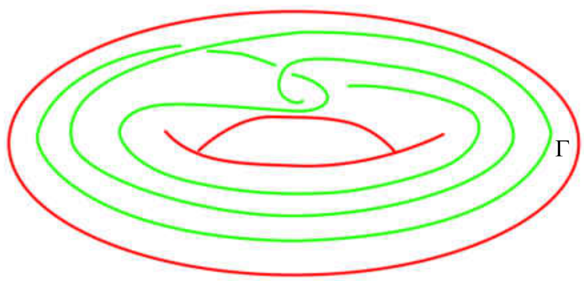

Figure 2.1: $\Gamma \subset \partial(S^1 \times \mathbb{B}^3) \subset$ the Mazur Manifold

In [Maz], Barry Mazur described what are now often called *Mazur manifolds*. Starting with a $S^1 \times \mathbb{B}^3$ one adds a 2-handle $h^{(2)} \approx \mathbb{B}^2 \times \mathbb{B}^2$ along the curve Γ is as in the above figure. That is,

$$Ma^4_\Phi = S^1 \times \mathbb{B}^3 \cup_\Phi \mathbb{B}^2 \times \mathbb{B}^2$$

is a *Mazur manifold*. Here Φ is the framing $\Phi : S^1 \times \mathbb{B}^2 \to T_\Gamma$, T_Γ is a tubular neighborhood of Γ in $\partial(S^1 \times \mathbb{B}^3)$ and the domain $S^1 \times \mathbb{B}^2$ is the first term in the union

$$S^1 \times \mathbb{B}^2 \cup \mathbb{B}^2 \times S^1 = \partial(\mathbb{B}^2 \times \mathbb{B}^2)$$

For each Dehn twist of the $S^1 \times S^1 = \partial(S^1 \times \mathbb{B}^2)$ sending $S^1 \times p$ $(p \in S^1)$ to a closed curve (that is, an integer number of full twists), there exists a framing Φ.

Thus the number of framings is infinite. Mazur chose a specific framing φ yielding a specific manifold, which we'll denote Ma^4, for which he showed $\partial Ma^4 \not\approx S^3$ so $Ma^4 \not\approx \mathbb{B}^4$. The chosen framing corresponds to a parallel copy of Γ say $\Gamma' = \varphi(S^1 \times p)$ which lies at the "top" (the up direction is perpendicular to the page, toward the viewer) of $S^1 \times \mathbb{B}^2$. Thus there are no twists with this framing.

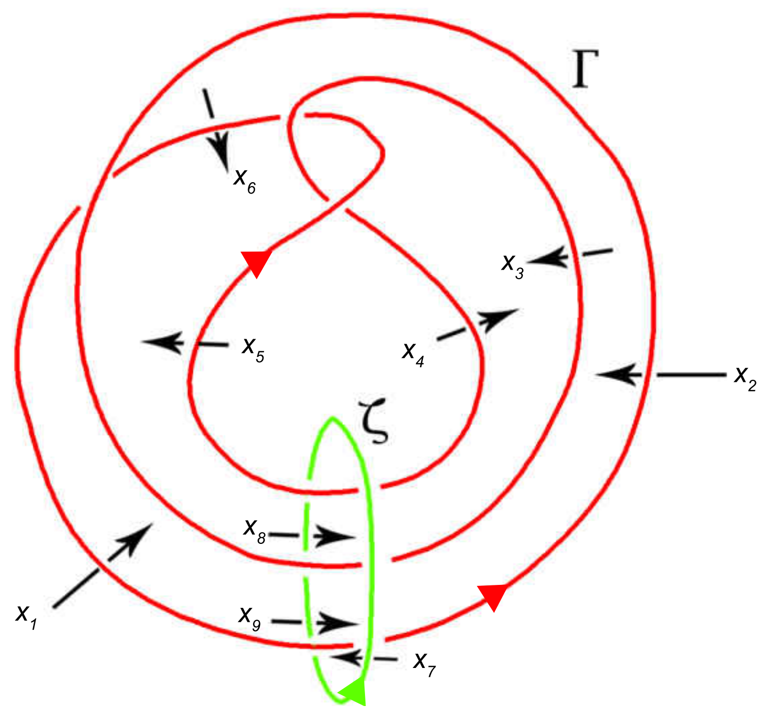

Figure 2.2: Wirtinger diagram of the Mazur link

Here we'll describe our interpretation of his argument for the nontriviality of $\pi_1(\partial Ma^4)$. Starting with the link $\Gamma \cup \zeta$ in S^3 pictured in Figure 2.2, we obtain said figure's Wirtinger presentation (see [Rol, p. 56] for a treatment of Wirtinger presentations). This gives a presentation with exactly one generator for each arc in the link diagram. These generators correspond to the loops in S^3 which start at the viewer's nose (the basepoint), travel under the arc, and then return home (to the nose). Thus in our picture the generators are the x_i as pictured. The relators in the presentation correspond to the undercrossings of pairs of arcs. As there are 9 undercrossings the Wirtinger presentation of this link diagram has 9 generators and 9 relators: $\langle x_1, ..., x_9 | r_1, ..., r_9 \rangle$. We then perform a Dehn drilling on a tubular neighborhood, $N(\zeta) \approx \mathbb{B}^2 \times S^1$, of ζ. That is, we remove $\mathrm{int}N(\zeta)$. Next, we perform a Dehn filling by sewing in $N(\zeta)$ backwards (ie sewing in a $S^1 \times \mathbb{B}^2$) along $\partial N(\zeta)$. This Dehn surgery on $S^3 \approx (S^1 \times \mathbb{B}^2) \cup_{S^1 \times S^1} (\mathbb{B}^2 \times S^1)$ results in an $(S^1 \times \mathbb{B}^2) \cup_{S^1 \times \partial \mathbb{B}^2} (S^1 \times \mathbb{B}^2) \approx S^1 \times S^2$ with Γ embedded as in Figure 2.1. This surgery exchanges $N(\zeta)$'s meridian with its longitude. Thus the group element corresponding to following around ζ is killed and we must add in a relator, say $r_\zeta = x_5 x_2^{-1} x_1^{-1} = 1$, to our presentation to adjust for this.

Adding a 2-handle along Γ (and throwing out its portion of Ma^4's interior) gives our $\partial Ma^4 = (S^1 \times S^2 - \mathrm{int}N(\Gamma)) \cup_{\partial N(\Gamma)} (\mathbb{B}^2 \times S^1)$. We describe the gluing of $\mathbb{B}^2 \times S^1$ in two steps. We first glue in a thickened meridional disc, D, which kills off Γ' the curve to which it is it is attached (see Figure 2.3). Thus to our Wirtinger presentation we introduce a relator $r_\Gamma = x_7^{-1} x_5^{-1} x_7 x_3^{-1} x_2^{-1} x_7^{-1} = 1$. We next glue on the rest of $\mathbb{B}^2 \times S^1$. The closed complement of D in $\mathbb{B}^2 \times S^1$ is a 3-ball and it is attached along its entire boundary. Adding such does not change the fundamental group and thus $\pi_1(\partial Ma^4) \cong \langle x_1, ..., x_9 | r_1, ..., r_9, r_\zeta, r_\Gamma \rangle$.

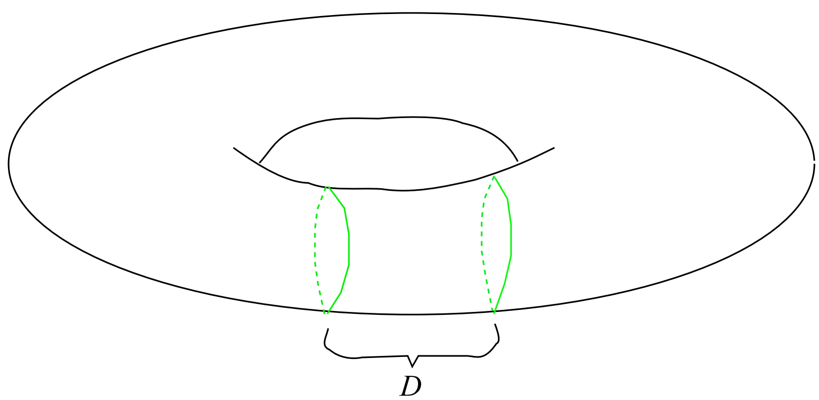

Figure 2.3: Thickened Meridional Disc

Proceeding as in [Maz], let $\beta = x_7$, $\lambda = x_2$, (see fig. 2.2) and $\alpha = \beta\lambda$. Via Tietze transformations (see [Geo, p. 79] for a treatment on Tietze transformations), it was shown in [Maz] that

$$\pi_1(\partial Ma^4) \cong < \alpha, \beta | \beta^5 = \alpha^7, \beta^4 = \alpha^2 \beta \alpha^2 > \text{ and}$$

$$G := \pi_1(\partial Ma^4)/\text{nc}\{\beta^5 = 1\} \cong < \beta, \gamma | \gamma^7 = \beta^5 = (\beta\gamma)^2 = 1 >$$

where $\gamma = \alpha^2$. We claim G maps nontrivally into the subgroup of the isometries of the hyperbolic plane generated by reflections in the geodesics containing the edges of a triangle with angles $\pi/7$, $\pi/5$, and $\pi/2$. That is, there exists a homomorphism

$$h : G \to \text{Isom}(\mathbb{H}^2)$$

so that Imh can be generated by rotations with centers at the vertices of a triangle with angles $\pi/7$, $\pi/5$, and $\pi/2$. See Figure 2.4. Here $h(\beta)$ = rotation with angle $-2\pi/5$ at C and $h(\gamma)$ = rotation with angle $2\pi/7$ at A.

We'll show the relator $h((\beta\gamma)^2) = 1$ is satisfied. Let r_{XY} be reflection in the geodesic containing X and Y. Then $h(\beta) = r_{BC} \circ r_{AC}$ and $h(\gamma) = r_{AC} \circ r_{AB}$, so that $h(\beta)h(\gamma) = r_{BC} \circ r_{AC} \circ r_{AC} \circ r_{AB} = r_{BC} \circ r_{AB}$. This last isometry is a rotation at B with angle $-\pi$ and $h(\beta\gamma)$ is shown to have order 2.

This shows Imh is nontrivial. Hence $\pi_1(\partial Ma^4)$ is nontrivial and thus $\partial Ma^4 \not\approx S^3$.

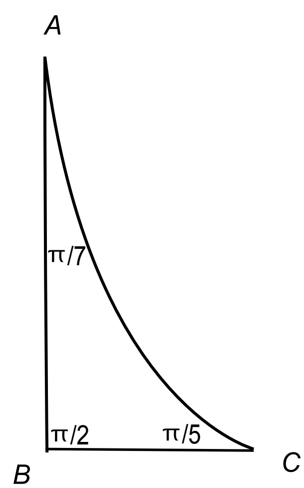

Figure 2.4: Triangle in $\mathbb{H}^2$

We now state and prove the following Proposition which we will employ in Section 4.2.

Proposition 2.1.1. *Let m_Γ be the meridian of the torus ∂T_Γ. Then m_Γ is nontrivial in $S^1 \times S^2 - int(T_\Gamma)$.*

Proof. We choose x_5 as our representative of m_Γ. By the relator

$$r_9 : x_1 = x_7^{-1} x_2 x_7 = \beta^{-1}\lambda\beta = \beta^{-1}(\beta^{-1}\alpha)\beta$$

we get $x_1 = \beta^{-2}\alpha\beta$. By $r_\zeta : x_5 = x_1 x_2$ we obtain

$$x_5 = (\beta^{-2}\alpha\beta)(\beta^{-1}\alpha) = \beta^{-2}\alpha^2 = \beta^{-2}\gamma.$$

Thus

$$\begin{aligned} h(x_5) &= h(\beta^{-2}\gamma) \\ &= h(\beta^{-2})h(\gamma) \\ &= (\text{rotation of } 4\pi/5 \text{ at } C)(\text{rotation of } 2\pi/7 \text{ at } A) \\ &\neq 1_{\mathbb{H}^2} \qquad (\text{since } A \text{ is not fixed}). \end{aligned}$$

Thus x_5 is not trivial in ∂Ma^4. Hence x_5 is nontrivial in $S^1 \times S^2 - \text{int}(T_\Gamma)$. This concludes the proof of Proposition 2.1.1. □

We believe the following question is open.

Question 2.1.2. Does Ma^4 split into closed balls?

Question 2.1.3. Does there exist an infinite number of closed 4-dimensional splitters?

We will give an answer to this question in Section 4.2.

2.2 The Jester's Manifolds

Our definition of the Jester's manifolds is analogous to our definition of the Mazur manifolds. We start with a $S^1 \times \mathbb{B}^3$ and within its $S^1 \times S^2$ boundary we select a curve C as follows. Let T be a tubular neighborhood of C in our $S^1 \times S^2$. We have

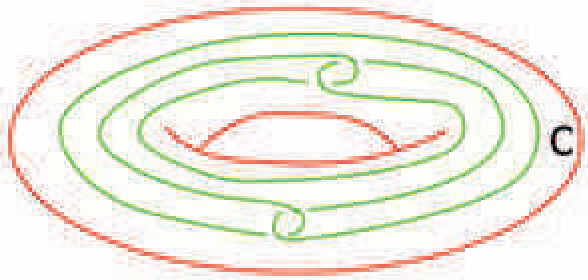

Figure 2.5: $C \subset \partial(S^1 \times \mathbb{B}^3) \subset$ a Jester's Manifold

chosen C so that it is the preimage of the Mazur curve Γ under the standard double covering map $p : S^1 \times \mathbb{B}^3 \to S^1 \times \mathbb{B}^3$ which is a degree 2 map in the first coordinate and the identity in the second.

Then, given a framing $\Psi : S^1 \times \mathbb{B}^2 \to T$, define

$$M_\Psi = S^1 \times \mathbb{B}^3 \cup_\Psi \mathbb{B}^2 \times \mathbb{B}^2$$

where the domain is the $S^1 \times \mathbb{B}^2$ factor in the boundary of our 2-handle $h^{(2)} \approx \mathbb{B}^2 \times \mathbb{B}^2$. We call such an M_Ψ a *Jester's manifold.*

(In Chapter 4, we will expand our definition of Jester's manifold to include analogous handle attachments but using *pseudo*-handles.)

Initially, we had hoped that, by altering the framings, we could prove the existence of an infinite collection of these Jester's manifolds. We proceeded with the aim of showing the fundamental groups of the boundaries were distinct and nontrivial. Unfortunately, due to the significantly more complicated Wirtinger presentations involved, we did not meet this goal. Fortunately, however, we were able to get around this problem by employing a technique of David Wright's (see section 4.2).

The following is still open.

Question 2.2.1. Does there exist a Jester's manifold that is not homeomorphic to a ball? Are there an infinite number of Jester's manifolds (as defined above)?

Chapter 3

Spines

3.1 Collapses

We borrow our definitions (and some figures) of collapse from Marshall Cohen's [Coh, pp. 3,4,14,15]. We will be denoting the join of two simplicial complexes A and B by AB.

Definition 3.1.1. If K and L are finite simplicial complexes we say that there is an *elementary simplicial collapse* from K to L, and write $K \searrow^{e} L$, if L is a subcomplex of K and $K = L \cup aA$ where a is a vertex of K, A and aA are simplexes of K, and $aA \cap L = a(\partial A)$. We call such an A a *free face* of K.

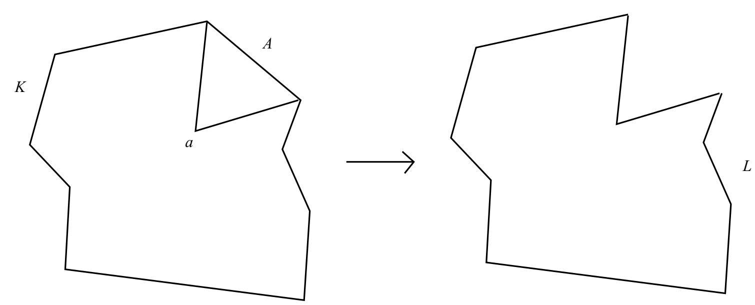

Figure 3.1: Elementary Collapse (simplicial) $K \searrow^{e} L$, A is a free face

Observe that a free face completely specifies an elementary simplicial collapse.

Definition 3.1.2. Suppose that (K, L) is a finite CW pair. Then $K \searrow^{e} L$–i.e. *K collapses to L by an elementary collapse*–iff

1. $K = L \cup e^{n-1} \cup e^n$ where e^n and e^{n-1} are not in L,

2. there exists a ball pair $(Q^n, Q^{n-1}) \approx (\mathbb{B}^n, \mathbb{B}^{n-1})$ and a map $\varphi : Q^n \to K$ such that

   a) φ is a characteristic map for e^n

   b) $\varphi|Q^{n-1}$ is a characteristic map for e^{n-1}

   c) $\varphi(P^{n-1}) \subset L^{n-1}$, where $P^{n-1} \equiv \mathrm{cl}(\partial Q^n - Q^{n-1})$.

In both the simplicial and CW cases we define

Definition 3.1.3. K *collapses* to L, denoted $K \searrow L$, if there is a finite sequence of elementary collapses

$$K = K_0 \searrow^e K_1 \searrow^e K_2 \searrow^e \ldots \searrow^e K_l = L.$$

If K collapses to a point we say K is *collapsible* and write $K \searrow 0$.

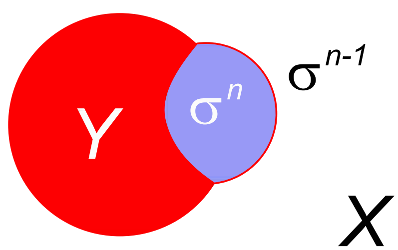

Figure 3.2: Elementary Collapse (CW) $X \searrow Y$

Definition 3.1.4. Suppose M is a compact PL manifold. If K is a PL manifold subcomplex of M contained in int$M$ with $M \searrow K$ we say K is a *spine* of M.

We will make use of the following regular neighborhood theory due to J. H. C. Whitehead. The following two propositions, theorem, and corollary can be found in [RoSa, pp. 40,41].

Proposition 3.1.5. *Suppose $M \supset M_1$ are PL n-manifolds with $M \searrow M_1$. Then there exists a homeomorhism $h : M \to M_1$.*

Theorem 3.1.6. *Suppose $X \subset M$, where M is a PL manifold, X is compact polhedron, and $X \searrow Y$. Then a regular neighborhood of X in M collapses to a regular neighborhood of Y in M.*

Thus if K is a spine of M then for any regular neighborhood $N(K)$ of K in M we have $N(K) \approx M$.

Proposition 3.1.7. *If $X \searrow 0$ then a regular neighborhood of X is a ball.*

Corollary 3.1.8. *Suppose M is a manifold with a spine K and $K \searrow 0$. Then M is a ball.*

Proposition 3.1.9. *Suppose W is a PL manifold and A and B are simplicial complexes $A, B \subset intW$. If $W \searrow A \cup B$ with $A, B, A \cap B \searrow 0$ then W splits into closed balls.*

Proof. Let A, B, and C be such that $W \searrow A \cup_C B$ with $A, B, C \searrow 0$. Regular neighborhoods of collapsible subcomplexes are piecewise linear balls. So given a triangulation of W with A and B as subcomplexes, we construct (with respect to this triangulation) regular neighborhoods N_A of A and N_B of B and we have that N_A and N_B are balls and $N_A \cap N_B$ is a regular neighborhood of C and as such is also a ball. $N_A \cup N_B$ is a regular neighborhood of $A \cup B$, a spine of W, so $N_A \cup N_B$ is homeomorphic to W. □

3.2 The Dunce Hat

The *dunce hat*, D, is defined as the quotient space obtained by identifying the edges of a triangular region as pictured in Figure 3.3. It has a triangulation as shown in Figure 3.4.

D can also be realized by sewing a disc $\mathbb{B}^2$ to a circle S^1 (along the boundary of the disc) with an attaching map as follows. Sew, in the counterclockwise direction,

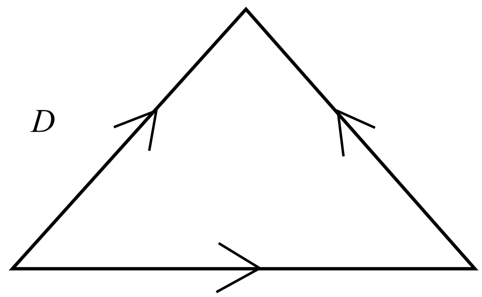

Figure 3.3: The Dunce Hat

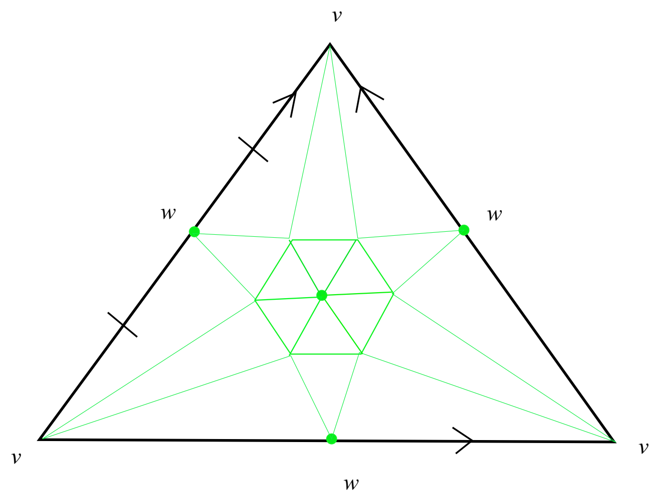

Figure 3.4: Dunce Hat Triangulated

the first third (say $[0, 2\pi/3)$) of the disc's boundary circle bijectively onto S^1. Likewise, continuing in the same direction sew the second third onto S^1. The last third we sew bijectively in the reverse direcction. See Figure 3.5.

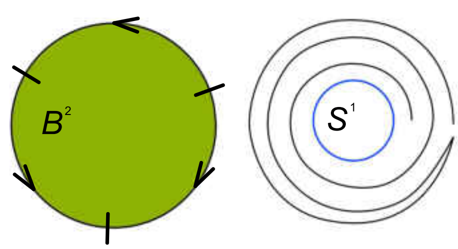

Figure 3.5: The Dunce Hat Attaching Map

The dunce hat was one of the first examples of a contractible but not collapsible simplicial complex. It is contractible since the attaching map described above is homotopic to the identity and thus D is homotopy equivalent to the the disc which is contractible [Hat, p. 16]. It is not collapsible as it has no free face. A well know result by Zeeman is that the Mazur manifold has a dunce hat spine [Zee]. That observation will become clear in the following section, when we identify a spine of a slightly more complicated example.

To the best of our knowledge the following question is open.

Question 3.2.1. Can the dunce hat be expressed as $D = A \cup_C B$ with $A, B, C \searrow 0$? If so, the answer to question 2.1.2 is yes: $Ma^4 \approx \mathbb{B}^4 \cup_{\mathbb{B}^4} \mathbb{B}^4$.

3.3 The Jester's Hat

We define the *Jester's hat*, J, to be the quotient space obtained from gluing the hexagonal region of the plane as in Figure 3.6. Figure 3.7 shows a triagulation of J. We can also realize this space by attaching a disc to a circle with the attaching map in Figure 3.8. We describe said map here. Attach the first third, say $[0, 2\pi/3)$ of the disc boundary to the circle bijectively in the counterclockwise direction. Then map bijectively in the clockwise direction the next sixth of the disc boundary to the bottom half of the circle. Then map the next third all the way around the circle in the clockwise direction. Finally, sew the last sixth to the top half of the circle.

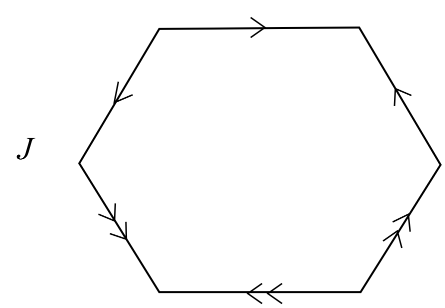

Figure 3.6: The Jester's Hat

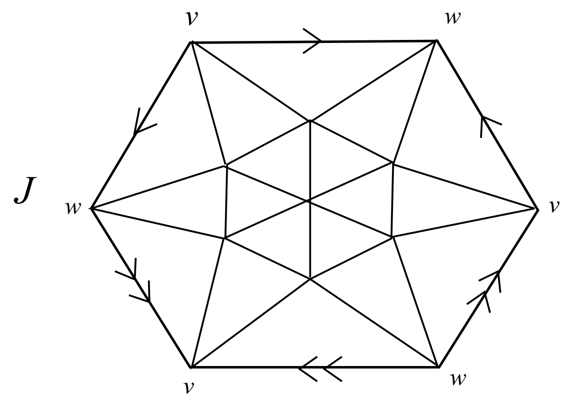

Figure 3.7: J Triangulated

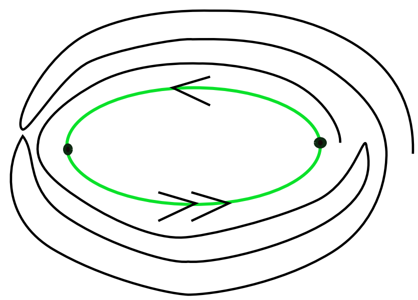

Figure 3.8: Attaching map for J

We observe that since the attaching map is homotopic to the identity, J is contractible. J is not collapsible as it has no free edge. We note that J is the union of two collapsibles which intersect in a collapsible. That is,

$$J = A \cup_C B \text{ with } A, B, C \searrow 0.$$

Figures 3.9 and 3.10 illustrate such a decomposition and associated collapses. Observe $A \cap B$ has no identifications and is thus a PL ball. PL balls are collapsible. We now elaborate on the collapses in 3.10. For $A \cap B$, the first collapse can be obtained from the sequence of elementary collapses specified by the following sequence of free faces: wd, de, ef, fv, fg, dg, cg, ag, d, e, f, g. The second corresponds to the following sequence of free faces: w, c, b, a. For A, we first collapse $A \cap B$ as we did in the first collapse of Figure 3.10. We then perform the collapse with free face sequence cb, ab yielding the "tri-fin" as illustrated.

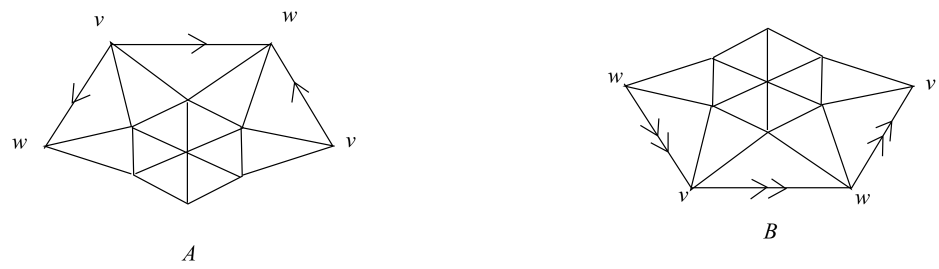

Figure 3.9: J "splits" into Collapsibles

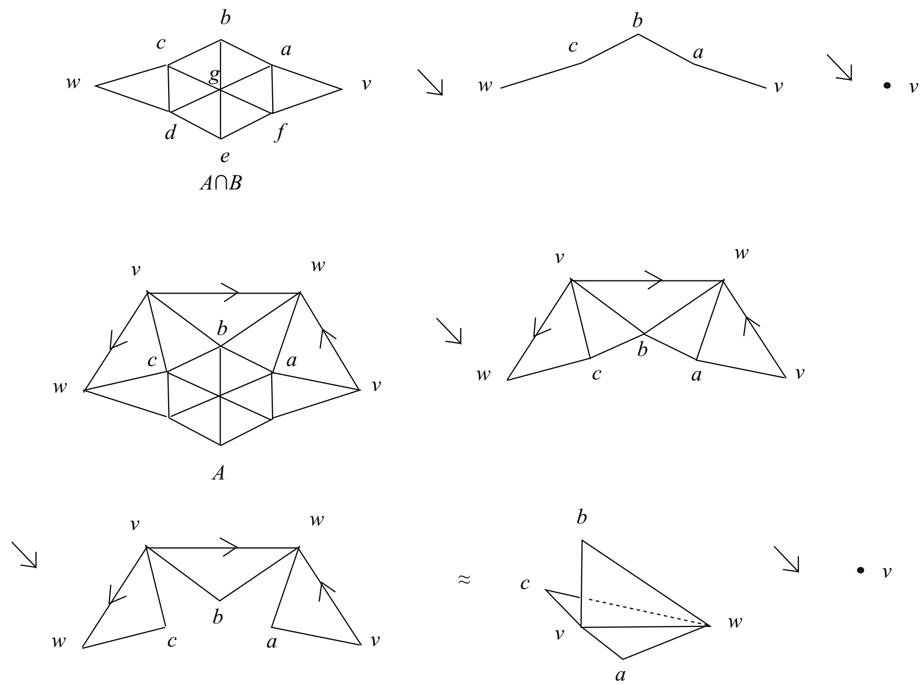

Figure 3.10: Collapses of J's "Splittands"

Proposition 3.3.1. *Every Jester's manifold has a Jester's hat spine.*

Proof. The proof is analogous to Zeeman's proof that the Mazur manifold has a dunce hat spine [Zee]. Let $M = M_\Psi$ be a Jester's manifold for a given framing Ψ. We divide the S^1 of the $S^1 \times S^2$ in which C resides into four arcs I_1, I_2, I_3, and I_4 so that $I_1 \times S^2$ and $I_2 \times S^2$ each contain a "clasp" of C (see Figure 3.11).

For $i = 1, 2$, let $f_i : S^1 \to S^1$ be the map that shrinks I_i to a point, say p_i, and is a homeomorphism on the complement of I_i. Further let $\pi : S^1 \times S^2 \to S^1$ be projection

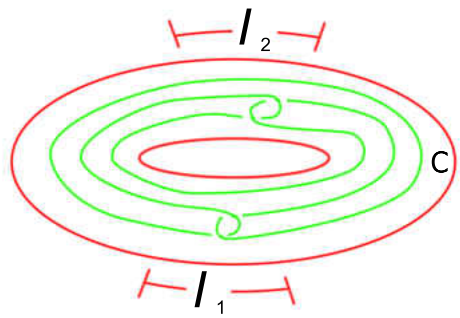

Figure 3.11: Intervals of S^1 and their clasps

onto the first factor, j be the inclusion $C \hookrightarrow S^1 \times S^2$, $g = f_1 \circ f_2 \circ \pi : S^1 \times S^2 \to S^1$ and $h = g \circ j$. Let $M(g)$ and $M(h)$ be the mapping cylinders of g and h, respectively. That is,

$$M(g) = [(S^1 \times S^2 \times [0,1]) \sqcup S^1]/\sim_g \quad \text{and} \quad M(h) = [(C \times [0,1]) \sqcup S^1]/\sim_h$$

where $\sim_g$ and $\sim_h$ are generated by $(x, 0) \sim_g g(x)$ and $(y, 0) \sim_h h(y)$, respectively.

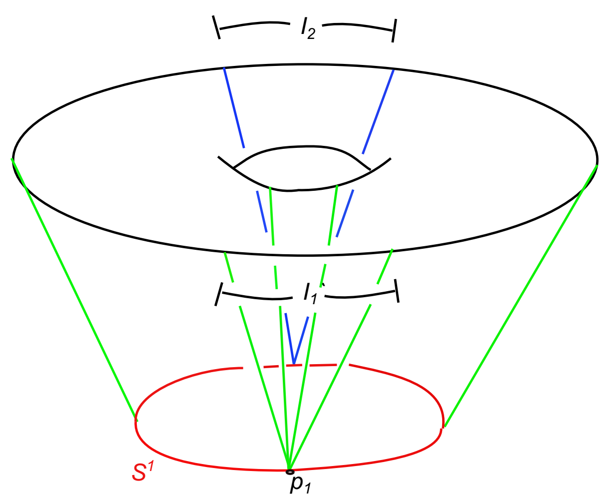

Figure 3.12: $M(g)$ the Mapping Cylinder of g

From the illustrations of $M(g)$ we see that the "cylinder lines fill up" $S^1 \times S^2$ yielding $M(g)$ homeomorphic to $S^1 \times \mathbb{B}^3$. Since $h = g|_C$, $M(h)$ is a subcylinder

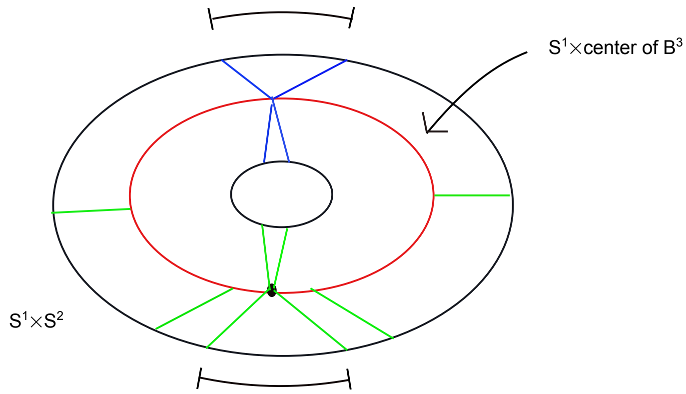

Figure 3.13: $M(g) \approx S^1 \times \mathbb{B}^3$

of $M(g)$ and by a result of J.H.C. Whitehead $M(g) \searrow M(h)$ [Whi]. Further, the 2-handle $h^{(2)}$ viewed as $\mathbb{B}^2 \times \mathbb{B}^2$ in our construction of M collapses onto its core union the attaching tube: $(\mathbb{B}^2 \times \{0\}) \cup (S^1 \times \mathbb{B}^2)$. Follow this with the collapse of $M(g)$ onto $M(h)$ to obtain the collapse:

$$M = S^1 \times \mathbb{B}^3 \cup_\Psi \mathbb{B}^2 \times \mathbb{B}^2 \searrow S^1 \times \mathbb{B}^3 \cup_\Psi [(\mathbb{B}^2 \times \{0\}) \cup (S^1 \times \mathbb{B}^2)] \searrow M(h) \cup_{\Psi|C} B^2.$$

But from the illustration of $M(h)$ (Figure 3.14) we can see that $M(h) \cup_{\Psi|C} B^2$ is our Jester's hat J. $\square$

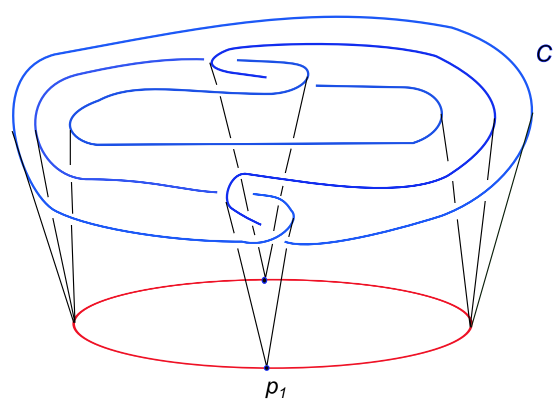

Figure 3.14: The Mapping Cylinder of h

Corollary 3.3.2. *The Jester's manifolds split into closed* 4*-balls.*

Remark 3.3.3. While we now know that the M_Ψ*'s* split into closed balls, we have not demonstrated that any M_Ψ is not just a ball.

Chapter 4

More Jester's Manifolds

For this chapter we let $M = M_\Psi$ be an arbitrary Jester's manifold. Recall Ψ is the framing $\Psi : S^1 \times \mathbb{B}^2 \to T$ and T is a tubular neighborhood of the curve C in $\partial(S^1 \times \mathbb{B}^3)$.

4.1 Pseudo 2-handles

Using M as a model, we apply a construction due to Wright to obtain a collection of manifolds $\{W_i\}$, as follows. To construct W_i, we start with the $S^1 \times \mathbb{B}^3$ of the Jester's manifold construction and attach a "pseudo 2-handle", a $\mathbb{B}^4$, along K_i, the connected sum of i trefoils in the boundary of $\mathbb{B}^4$, to the curve C in $\partial(S^1 \times \mathbb{B}^3)$. (See Figure 4.1.) That is,

$$W_i = S^1 \times \mathbb{B}^3 \cup_{\Psi_i} H.$$

Here Ψ_i is a homeomorphism from a tubular neighborhood T_i of K_i in $\partial\mathbb{B}^4$ to T.

We define the *core of the pseudo handle* to be the cone of K_i with cone point the center of $\mathbb{B}^4$. The core is then a 2-disc whose interior lies in int$\mathbb{B}^4$.

Proposition 4.1.1. *Each* $W_i \searrow J$*.*

Proof. The same proof as for every Jester's manifold collapses to J (Proposition 3.3.1) goes through with the pseudo 2-handle collapsing to its core, a disc B^2. H collapses to its core union its attaching tube defined as $\Psi_i(T_i)$. $M(g)$ again collapses to $M(h)$ with the attaching tube collapsing to the attaching sphere: $\Psi_i(K_i) = C$.

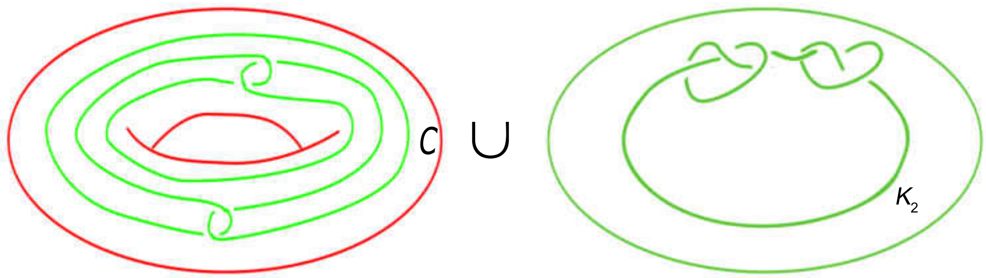

Figure 4.1: $S^1 \times \mathbb{B}^3$ union a degree 2 pseudo 2-handle

□

Corollary 4.1.2. *Each* $W_i = \mathbb{B}^4 \cup_{\mathbb{B}^4} \mathbb{B}^4$.

Remark 4.1.3. At this point we don't know if any of the W_i's are not balls. We will address this in the next section.

4.2 A Theorem of Wright

Applying the following theorem will yield an infinite collection of distinct W_i. Before we state the theorem we'll need some definitions.

Definition 4.2.1. A 3-manifold is *irreducible* if every embedded S^2 bounds a $\mathbb{B}^3$.

Definition 4.2.2. A torus S in a 3-manifold X is said to be *incompressible in* X if the homomorphism induced by inclusion $\pi_1(S) \to \pi_1(X)$ is injective.

Definition 4.2.3. A group G is *indecomposible* if for all subgroups A, B such that $G \approx A * B$, either $A = 1$ or $B = 1$. (That is, G contains no nontrivial free factors.)

Theorem 4.2.4. *[Wri] Suppose* X *is a compact* 4*-manifold obtained from the* 4*-manifold* N *by adding a* 2*-handle* H*. If* $\mathrm{cl}(\partial X - H)$ *is an orientable irreducible* 3*-manifold with incompressible boundary, then there exists a countably infinite collection of compact* 4*-manifolds* M_i *such that*
(1) ∂M_i *is not homeomorphic to* ∂M_j *when* $i \neq j$
(2) $\pi_1(\partial M_i) \ncong \mathbb{Z}$ *and is indecomposible*

(3) $\pi_1(\partial M_i) \not\cong \pi_1(\partial M_j)$ *for* $i \neq j$ *and hence,* $\mathrm{int}(M_i)$ *is not homeomorphic to* $\mathrm{int}(M_j)$
(4) $M_i \times I$ *is homeomorphic to* $X \times I$

Note 4.2.5. Conclusion (4) requires a more restrictive choice of attaching map Ψ_i. This conclusion is not necessary for the arguments presented in this thesis thus the omission of these restrictions.

In [Wri], Wright constructs the infinite collection of manifolds $\{M_i\}$ of the theorem as follows. For each $i = 1, 2, ...$ he constructs a manifold by attaching to N a psuedo 2-handle along K_i. From this sequence he exhibits a subsequence $\{M_{i_j}\}$ each term of which has a distict boundary.

For the proof of the following theorem we'll employ the Loop Theorem as stated in [Rol, p. 101].

Theorem 4.2.6. (Loop Theorem) *If* X *is a 3-manifold with boundary and the induced inclusion homomorphism* $\pi_1(\partial X) \to \pi_1(X)$ *has nontrivial kernel, then there exists an embedding of a disc* D *in* X *such that* ∂D *lies in* ∂X*, and represents a nontrivial element of* $\pi_1(\partial X)$*.*

Theorem 4.2.7. *There exists an infinite collection of closed 4-dimensional splitters. The fundamental groups of their boundaries are distinct, indecomposible, and noncyclic.*

Proof. We'll show M meets the hypotheses of Theorem 4.2.4, thus yielding a subsequence of $\{W_i\}$ as our desired collection. Recall T is the tubular neighborhood of the attaching sphere C in the construction of the Jester's manifold so that $\partial T = \partial \mathrm{cl}(\partial M - h^{(2)})$. It suffices to show

Claim 4.2.8. ∂T is incompressible in $\mathrm{cl}(\partial M - h^{(2)}) = S^1 \times S^2 - \mathrm{int}(T)$.

We will show $\ker(\pi_1(\partial T) \to \pi_1(S^1 \times S^2 - \mathrm{int}(T))) = 1$. Recall T_Γ is the tubular neighborhood of the Mazur curve Γ in the $S^1 \times S^2$ in the construction of the Mazur manifold (see Section 2.1). Recall further Proposition 2.1.1: Let m_Γ be the meridian of the torus ∂T_Γ. Then m_Γ is nontrivial in $S^1 \times S^2 - \mathrm{int}(T_\Gamma)$.

By construction $S^1 \times S^2 - \text{int}(T)$ is a double cover of $S^1 \times S^2 - \text{int}(T_\Gamma)$. Call the associated covering map p and let m be a lift of m_Γ so m is a meridian of ∂T. Then $p_*([m]) = [m_\Gamma] \neq 1$ gives $[m] \neq 1$. Suppose by way of contradiction that there exists an embedded disc D in $S^1 \times S^2 - \text{int}(T)$ with ∂D being a nontrivial loop in ∂T. Choose a longitude l on ∂T and let $\mu = [m]$ and $\lambda = [l]$ in $\pi_1(\partial T)$ so that for some $k, j \in \mathbb{Z}$, $[\partial D] = \mu^k \lambda^j$ in $\pi_1(\partial T)$. As C has algebraic index 1 in $S^1 \times S^2$ a nonzero j would imply $[\partial D]$ nontrivial in $\pi_1(S^1 \times S^2 - \text{int}(T))$. Thus $[\partial D] = \mu^k$. But any loop going around meridinally more than once and longitudinally zero will not be embedded. See Figure 4.2. Then it must be that $[\partial D] = [m]^{\pm 1}$. Since m is nontrivial in $S^1 \times S^2 - \text{int}T$ such a D cannot exist and by the Loop Theorem $\ker(\pi_1(\partial T) \to \pi_1(S^1 \times S^2 - \text{int}(T))) = 1$. □

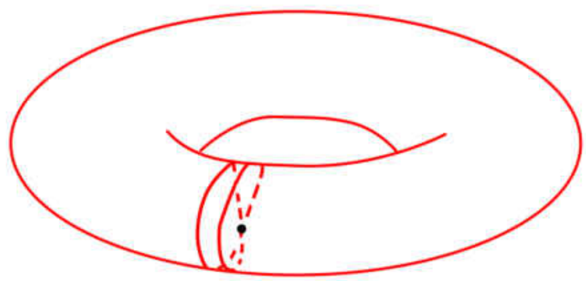

Figure 4.2: $\mu^2\lambda^0 \in \pi_1(\partial T)$

Definition 4.2.9. We call any M_i as yielded by the theorem when applied to any M_Ψ a *Jester's manifold.*

Note that for a given knot K_i, different choices of framing homeomorphism potentially yield different manifolds. So the variety of distinct Jester's manifolds produced by this construction is potentially much greater than we have shown.

We conclude this chapter with a theorem summarizing our accomplishments thus far.

Theorem 4.2.10. *There exists an infinite collection of topologically distinct splittable compact contractible 4-manifolds. The interiors of these are topologiclly distinct contractible splittable open 4-manifolds.*

Chapter 5

Sums of Splitters

In this our concluding chapter, we will exhibit an uncountable collection of contractible open 4-dimensional splitters. We will do so by considering the interiors of infinite boundary connected sums of of our Jester's manifolds. These open manifolds can also be constructed as the connected sum at infinity of the interiors of the same sequence of manifolds. Using the notion of the fundamental group at infinity we will be able to show that any two such sums where one Jester's manifold appears more often as a summand in one than the other are topolgically distinct. We then demostrate a splitting for such manifolds.

5.1 Some Manifold Sums and the Fundamental Group at Infinity

We describe what we mean by the *induced orientation* of the boundary of an oriented manifold X^n. Given a collar neighborhood of ∂X which we identify as $\partial X \times [0,1]$ (∂X identified with $\partial X \times \{0\}$) and a map $h : \mathbb{B}^{n-1} \to \partial X$ we define $\bar{h}$ as

$$\bar{h} : \mathbb{B}^n \to \partial X \times (0,1], \quad \bar{h}(x_1, x_2, ..., x_n) = \left(h(x_1, x_2, ..., x_{n-1}), \frac{3 + x_n}{4} \right).$$

(To be precise the codomain of $\bar{h}$ should be int$X$.) See Fig. 5.1. If $h : \mathbb{B}^{n-1} \to \partial X$ and $\bar{h}$ is a representative of the orientation of X then the ambient isotopy class of h is the *induced orientation* of ∂X. [RoSa, p. 45].

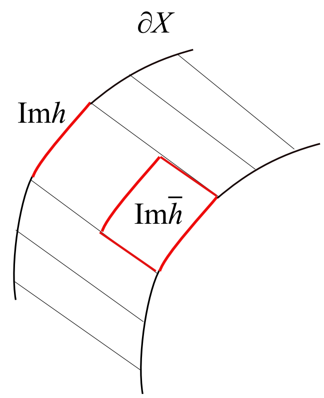

Figure 5.1: h and $\bar{h}$

Definition 5.1.1. Let M^n and N^n be connected oriented manifolds with nonempty boundaries. Orient BdM and BdN with their induced orientations and let B_M and B_N be tame $(n-1)$-balls in ∂M^n and ∂N^n, respectively. Let $\phi : B_M \to B_N$ be an orientaion reversing homeomorphism. Then $M^n \cup_\phi N^n$ is called a *boundary connected sum* (BCS) and is denoted $M^n \overset{\partial}{\sharp} N^n$.

Proposition 5.1.2. *The boundary connected sum is a connected oriented manifold which, provided BdM and BdN are connected, does not depend on the choices of* B_i *or* ϕ_i. *Furthermore the set of connected oriented* n*-dimensional manifolds with connectd boundaries is, under the operation of connected sum, a commutative monoid (that is, associative and contains an identity) the identity being* $\mathbb{B}^n$ *[Kos, p. 97].*

Definition 5.1.3. Let $\{M_i^n\}_{i=1}^m$ (m possibly ∞) be oriented manifolds with nonempty connected boundaries and for each $i = 1, 2, ...$ let $B_{i,L}$ and $B_{i,R}$ be disjoint tame $(n-1)$-balls in ∂M_i^n. For $i > 1$ let $\phi_i : B_{i,L} \to B_{i-1,R}$ be an orientaion reversing homeomorphism. Let $\phi : \sqcup_{i>1} B_{i,L} \to \sqcup_{i\geq 1} B_{i,R}$ with $\phi|_{B_{i,L}} = \phi_i$. Then $(\sqcup M_i)/\phi$ is called a *boundary connected sum* (BCS) and is denoted $M_1 \overset{\partial}{\sharp} M_2 \overset{\partial}{\sharp} \cdots \overset{\partial}{\sharp} M_m$ (or $M_1 \overset{\partial}{\sharp} M_2 \overset{\partial}{\sharp} \cdots$ when $m = \infty$).

We next prepare a description of an analogous sum for open manifolds. But first we need a proposition ensuring the existence of the desired attaching maps.

Definition 5.1.4. By a *proper map* p between spaces Y and X we mean a map $p : Y \to X$ such that for any compact $C \subset X$ we have $p^{-1}(C)$ is compact. A *ray* is a proper embedding $[0, \infty) \to X$.

Note 5.1.5. Unless otherwise stated all rays will piecewise linearly embedded. We will abuse our notation for rays (as well as for some other maps) by using our symbol for the map to also mean its image.

Proposition 5.1.6. *Suppose N is a regular neighborhood of a ray r in an open n-manifold $M (n \geq 4)$. Then $(N, \partial N) \approx (\mathbb{R}^n_+, \mathbb{R}^{n-1})$.*

Proof. The following lemma ensures the existence of a n-space neighborhood $U \approx \mathbb{R}^n$ of r. Let h be such a homeomorphism $h : U \to \mathbb{R}^n$. Since all rays in R^n are ambiently isotopic (see, for example, Lemma 5.2.4), and since $\mathbb{R}^n_+$ is a regular neighborhood of $(0, 0, ..., 0) \times [1, \infty)$ any regular neighborhood N of $h(r)$ will be a half space and then $h^{-1}(N)$ will be a half space regular neighborhood of r. As regular neighborhoods of r are homeomorphic, any regular neighborhood of r will be a half space. □

Note 5.1.7. Our argument for Lemma 5.2.4 is dependent on the dimension n being greater than 4. One must work harder to obtain the above result for $n \leq 3$.

Lemma 5.1.8. *For a ray r in an open n-manifold M there exists U a neighborhood of r such that $U \approx \mathbb{R}^n$.*

Proof. Let N_0 be a regular neighborhood of $r([0, 1.5])$ and note $N_0 \approx \mathbb{B}^n$ as $r([0, 1.5])$ is collapsable. Then we can choose $B^n_0 \approx \mathbb{B}^n$ such that $N_0 \supset B^n_0 \supset r([0, 1])$ and $r([1, \infty]) \cap B_0 = \{r(1)\}$. See Figure 5.2.

By choosing such a B_0 we have that $B_0 \cup r([1, 2.5])$ is collapsible. Next choose N_1 a regular neighborhood of $B_0 \cup r([1, 2.5])$ which is a ball and thus there exists $B_1 \approx \mathbb{B}^n$ such that $B_0 \cup r([1, 2]) \subset \text{int}B_1$, $B_1 \subset N_1$, and $B_1 \cap r([2, \infty)) = \{r(2)\}$. Continuing in this manner we obtain for $i = 0, 1, 2, ...$ n-balls B^n_i, with the properties $B_i \subset \text{int}B_{i+1}$ and B_i is a neighborhood of $r([0, i - 1])$. Then $U = \bigcup_{i \geq 0} \text{int}B_i$ is a neighborhood of r and by the following lemma we see that $U \approx \mathbb{R}^n$. □

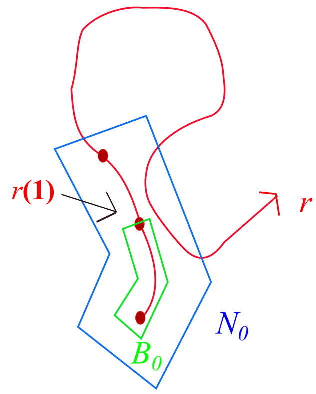

Figure 5.2: Construction of a $\mathbb{R}^n$ neighborhood of r

Lemma 5.1.9. *For $i = 0, 1, ...$ and n-balls B_i in a p.l. manifold with $B_i \subset intB_{i+1}$ the union $\bigcup_{i\geq 0} B_i$ is homeomorphic to $\mathbb{R}^n$.*

We will apply the following theorem in our proof of the previous lemma.

Theorem 5.1.10 (PL Annulus Theorem)**.** *[RoSa, p. 36] Given n-balls A and B with $A \subset intB$ then $cl(B - A) \approx S^{n-1} \times I$.*

Proof of Lemma 5.1.9. Embed B_0 as the unit ball in $\mathbb{R}^n$ via a map $h : B_0 \to \mathbb{R}^n$. We then extend this embedding to an embedding of B_1 onto the radius 2 origin centered ball of $\mathbb{R}^n$. We do this by identifying $(\mathrm{cl}(B_1 - B_0), \partial B_0)$ with $(\partial B_0 \times I, \partial B_0) \approx (S^{n-1} \times I, S^{n-1} \times 0)$ and sending $p = (x, t) \in \partial B_0 \times I$ to $(h(x), t)$. Continue to further extend h to embed $\bigcup B_i$ onto all of $\mathbb{R}^n$. □

Definition 5.1.11. For oriented, piecewise linear, open n-manifolds X and Y, and rays $\alpha_X \subset X$ and $\alpha_Y \subset Y$ we define the *connected sum at infinity (CSI)* of (X, α_X) and (Y, α_Y) as follows. Choose regular neighborhoods N_X and N_Y of α_X and α_Y, respectively. Orient ∂N_X with the induced orientation from the given orientation of $X - \mathrm{int}N_X$ and orient ∂N_Y from the given orientaion of $Y - \mathrm{int}N_Y$. Then the CSI of (X, α_X) and (Y, α_Y) is

$$(X, \alpha_X)\natural(Y, \alpha_Y) = (X - \mathrm{int}N_X) \cup_f (Y - \mathrm{int}N_Y)$$

where f is an orientation reversing p.l. homeomorphism $f : \partial N_X \to \partial N_Y$.

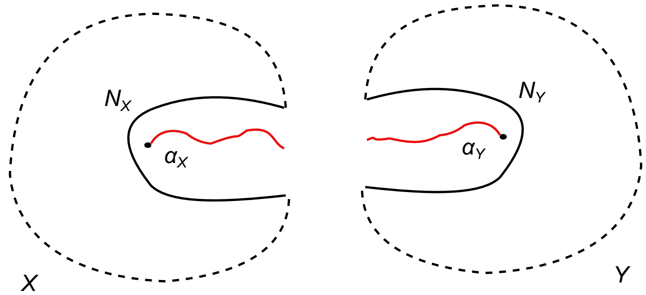

Figure 5.3: $(X, \alpha_X)\natural(Y, \alpha_Y)$

Note that we are considering regular neighborhoods of *noncompact* manifolds and by the uniqueness theorem for regular neighborhoods (see [Coh]) $(X, \alpha_X)\natural(Y, \alpha_Y)$ is independent of the choices of neighborhoods N_X and N_Y.

We note that (for our conditions on the summands) our definiton of $X\natural Y$ is equivalent to both Gompf's definition of *end sum* [Gom] and Calcut, King, and Siebenmann's definition of connected sum at infinity [CKS].

Definition 5.1.12. Let $\{X_i\}_{i=1}^m$ (m possibly ∞), be oriented, piecewise linear, open n-manifolds and for $i = 1, 2, ...$ and $x = L, R$ choose rays $\alpha_{i,x} \subset X_i$. Further choose regular neighborhoods $N_{i,x}$ of $\alpha_{i,x}$ so that $N_{i,L} \cap N_{i,R} = \emptyset$. Orient $\partial N_{i,x}$ with the induced orientation from the given orientation of $X_i - \text{int}(N_{i,L} \cup N_{i,R})$ and choose orientation reversing homeomorphisms $\phi_i : \partial N_{i,R} \to \partial N_{i+1,L}$. Let $\phi : \sqcup_{i\geq 1}\partial N_{i,R} \to \sqcup_{i>1}\partial N_{i,L}$ with $\phi|_{N_{i,R}} = \phi_i$. Let $\check{X}_1 = X_1 - \text{int}N_{1,R}$ and for $i = 2, 3...$ let $\check{X}_i = X_i - \text{int}(N_{i,L} \cup N_{i,R})$. Then $(\sqcup(\check{X}_i))/\phi$ is called the *connected sum at infinity* (*CSI*) of $\{(X_i, \alpha_{i,L}, \alpha_{i,R})\}$. We denote this sum as $(X_1, \alpha_{1,L}, \alpha_{1,R})\natural...\natural(X_m, \alpha_{m,L}, \alpha_{m,R})$ (or $(X_1, \alpha_{1,L}, \alpha_{1,R})\natural(X_2, \alpha_{2,L}, \alpha_{2,R})\natural...$ when m is ∞.) See Fig. 5.4.

Remark 5.1.13. The connected sum at infinity of the interiors of manifolds with connected boundary is homeomorphic to the interior of their boundary connected sum. For a CSI of open manifolds which are not the interiors of compact manifolds (Whitehead's exotic open 3-manifold, for example [Gui, p. 6]) we do not have the luxury of utilizing this result.

We'll now prepare the definition of the fundamental group at infinity of a 1-ended

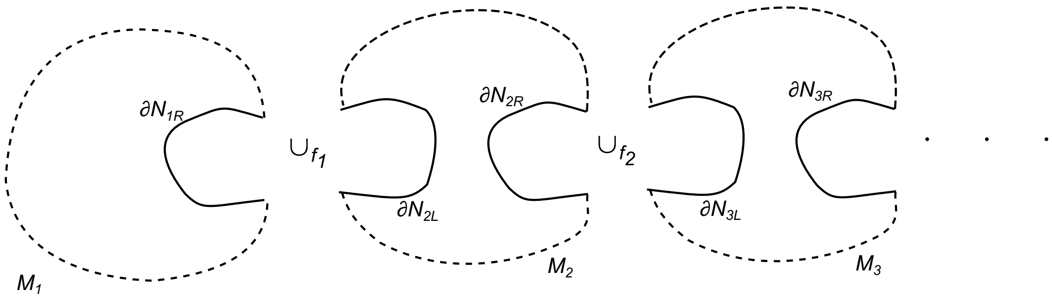

Figure 5.4: $\natural_{i=1}^{\infty} M_i$

topological space. This is an invariant of spaces which are 1-ended and satisfy the condition that any pair of proper rays can be joined by a proper homotopy. (See [Gui] for a much more thorough treatment of this topic.) Let $\{G_j, \varphi_j\}$ be an inverse sequence of groups:

$$G_1 \xleftarrow{\varphi_2} G_2 \xleftarrow{\varphi_3} G_3 \xleftarrow{\varphi_4} \cdots .$$

For an increasing sequence of positive integers $\{j_i\}_{i=1}^{\infty}$, let

$$f_i = \varphi_{j_{i+1}} ... \varphi_{j_i+1}\varphi_{j_i} : G_{j_i} \to G_{j_{i+1}}$$

and call the inverse sequence $\{G_{j_i}, f_i\}$ a *subsequence* of the inverse sequence $\{G_j, \varphi_j\}$.

We say the inverse sequences $\{G_j, \varphi_j\}$ and $\{H_k, \psi_k\}$ are *pro-isomorphic* if there exists subsequences $\{G_{j_i}, f_i\}$ and $\{H_{j_i}, g_i\}$ that may be fit into a commuting ladder diagram of the form

$$\begin{tikzcd}
G_{j_1} & & G_{j_2} \arrow[ll, "f_2"'] \arrow[dl, "d_2"'] & & G_{j_3} \arrow[ll, "f_3"'] \arrow[dl, "d_3"'] & & \cdots \arrow[ll, "f_4"'] & \\
& H_{k_1} \arrow[ul, "u_1"] & & H_{k_2} \arrow[ll, "g_2"'] \arrow[ul, "u_2"] & & H_{k_3} \arrow[ll, "g_3"'] \arrow[ul, "u_3"] & & \cdots \arrow[ll, "g_4"']
\end{tikzcd}$$

Pro-isomorphism is an equivalence relation on the set of inverse sequences of groups.

Definition 5.1.14. We say the inverse sequence of groups $\{G_j, \varphi_j\}$ is *stable* if it is pro-isomorphic to a constant sequence $\{H, \mathrm{id}_H\}$, and we say $\{G_j, \varphi_j\}$ is *semistable* if it is pro-isomorphic to an $\{H_k, \psi_k\}$, where each ψ_k is an epimorphism.

We call $A \subset X$ a *bounded set* (in X) if $\mathrm{cl}(X - A)$ is compact. We define a *neighborhood of infinity* of a topological space X to be the complement of a bounded subset of X. A *closed (open) neighborhood of infinity* in X is one that is closed (open) as a subset of X. A closed neighborhood of infinity N of a manifold M with compact boundary is *clean* if it is a codimension 0 submanifold disjoint from ∂M and $\partial N = \mathrm{Bd}_M N$ has a bicollared neighborhood in M. Here we are using the notation $\mathrm{Bd}_M N$ in the following sense. For A a subset of a topological space Z, $\mathrm{Bd}_Z A$ will denote the (topological) boundary (also known as the frontier) of A in Z (not to be confused with the notion of manifold boundary). We say X is *k-ended* if $k < \infty$ and k is the least upper bound of the set of cardinalities of unbounded components of neighborhoods of infinity of X. That is,

$$k = \sup\{|\{\text{unbounded components of } N\}| : N \text{ a neighborhood of infinity of } X\}.$$

In the case, the above supremum is infinite we say X is *infinite ended*.

By a *cofinal* sequence $\{U_j\}$ of subsets of X we mean $U_j \supset U_{j+1}$ and $\bigcap U_j = \emptyset$. Now let X be a 1-ended space and choose a cofinal sequence $\{U_j\}$ of connected neighborhoods of infinity of X. Choose a ray (called a *base ray*) r in X and base points $x_j \in r \cap U_j$ such that $r([r^{-1}(x_j), \infty)) \subset U_j$. Let $G_j = \pi_1(U_j, x_j)$ and $\tau_j : G_j \to G_{j-1}$ be the homomorphism (called a *bonding homomorphism*) defined as follows. Let $\iota_j : \pi_1(U_j, x_j) \to \pi_1(U_{j-1}, x_j)$ be the homomorphism induced by the inclusion $U_j \hookrightarrow U_{j-1}$ and ρ_j be the cannonical basepoint change isomorphism. This isomorphism is induced by the map that generates a loop α' based at x_{j-1} from a loop α at x_j by starting at x_{j-1} following r to x_j traversing α and returning along r to x_{j-1}. Then τ_j is defined as $\tau_j = \rho_j \circ \iota_j$ and $\{G_j, \tau_j\}$ is a inverse sequence of groups. We then define the *fundamental group of infinity (based at r)* of X (denoted pro-$\pi_1(\epsilon(X), r)$) to be the pro-isomorphism class of $\{G_j, \varphi_j\}$. It can be shown that this class is independent of the choice of $\{U_j\}$.

The following theorem can be found in [Gui, pp. 29-31].

Theorem 5.1.15. *Let X be a 1-ended space. If pro-$\pi_1(\epsilon(X), s)$ is semistable for some ray s then any two rays in X are properly homotopic and conversely. Further in any such space pro-$\pi_1(\epsilon(X), r)$ is independent of base ray r.*

We call any 1-ended manifold X that meets either of the equivalent conditions of Theorem 5.1.15 *semistable*. A *stable* one-ended manifold X is one for which pro-$\pi_1(\epsilon(X), r))$ is stable (hence semistable and thus independent of r).

We now show that if M is a compact manifold with connected boundary (for example any Jester's manifold) then the interior of M is 1-ended and stable. For $j = 1, 2, ...,$ choose compacta C_j in int$M$ such that $M - C_j$ is a product neighborhood of ∂M and the corresponding neighborhoods of infinity $N_j = \text{int}M - C_j$ are cofinal. Note each N_j has one unbounded component. Choose a neighborhood of infinity $N \subset M$. Then there exists k so that $N_k \subset N$. The one unbounded component of N_k must be contained in an unbounded component of N. If N had a second unbounded component then its nonempty intersection with $M - N_j$ would be unbounded. But this would contradict $M - N_j$'s compactness. Thus N must have exactly one unbounded component and we have shown M is 1-ended. As for int$M$ being stable, choose base ray r in int$M$ and base points $x_j \in r \cup N_j$. Then as

$$\begin{aligned} \pi_1(N_j) &\cong \pi_1(\partial M \times (0,1]) \\ &\cong \pi_1(\partial M) \times \pi_1((0,1]) \\ &\cong \pi_1(\partial M) \end{aligned} \tag{5.1}$$

we have $\{\pi_1(N_j), \tau_j\}$ is stable and thus so is int$M$.

5.2 CSI's of Semistable Manifolds

We'll next show that the CSI of a collection of semistable manifolds is independent of the choice of rays.

Definition 5.2.1. We say N is a *half space* of a manifold M^n if N is the image of an embedding $h : \mathbb{R}^n_+ \to M$. We say such an N is a *proper* half space if the embedding h is proper. We say N is a *tame* half space if $h(\partial \mathbb{R}^n_+)$ is bicollared in M^n.

Proposition 5.2.2. *If M is an open, contractible manifold and N is a proper and tame half space of M, then $M - N \approx M$.*

Proof. Let $C \subset N$ be a collar neighborhood of ∂N, $C \approx \partial N \times [0,1]$ and $N' = N - (\partial N \times [0,1))$. Then N and N' are ambient isotopic in M, so that $M - \mathrm{int}N \approx M - \mathrm{int}N'$. See Figure 5.5.

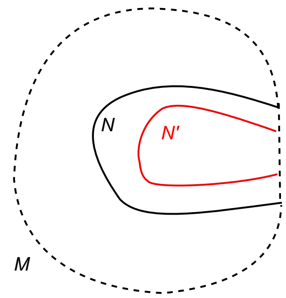

Figure 5.5: $N' \subset N$

Further

$$\begin{aligned}(N - N', \partial N) &\approx (\partial N \times [0,1), \partial N)\\ &= (\mathbb{R}^{n-1} \times [0,1), \mathbb{R}^{n-1})\\ &= (\mathbb{R}^n_+, \mathbb{R}^{n-1})\end{aligned}$$

so $N - N'$ is a half space. Then as $M - N' = (M - \mathrm{int}N) \cup_{\partial N} (N - N')$ we see that $M - N'$ is $M - \mathrm{int}N$ union a half space. Likewise, $M = (M - \mathrm{int}N) \cup_{\partial N} N$ is also a $M - \mathrm{int}N$ union a half space. Thus $M - N \approx M - N' \approx M$. $\square$

Note 5.2.3. Proposition 5.1.6 along with Proposition 5.2.2 imply when

$$(X, \alpha_X) \approx (Y, \alpha_Y) \approx (\mathbb{R}^n, \{0\}^{n-1} \times [0, \infty))$$

we have $(X, \alpha_X) \natural (Y, \alpha_Y) \approx \mathbb{R}^n$.

Lemma 5.2.4. *Suppose M^n $(n \geq 4)$ is a contractible, oriented, piecewise linear, semistable, open manifold. If r and r' are PL rays in M^n and N and N' are regular neighborhoods of r and r', respectively, then there exists an orientation preserving self homeomorphism of M^n taking N to N'.*

Proof. By general position we can assume r and r' are disjoint. By semistability there exists a proper homotopy H between r and r'

$$H : [0, \infty) \times [0, 1] \to M^n, \qquad H_0 = r, \qquad H_1 = r'.$$

We'll approach the cases $n \geq 5$ and $n = 4$ separately. When $n \geq 5$, H can be embedded via the general position theorem for maps [RoSa, p. 61]. Let N, N', and N'' be regular neighborhoods of r, r', and H, respectively (here we are abusing notation by using r, r', and H to denote both the maps and their images). See Figure 5.6. Observe $H \searrow r$, r' and hence N'' is a regular neighborhood of both r and r'. As N and N'' are both regular neighborhoods of r, there exists a self homeomorphism h_1 of M^n sending N to N''

$$h_1 : M^n \to M^n, \qquad h_1(N) = N''.$$

Likewise, there exists h_2 a self homeomorhism of M^n taking N'' to N'. Letting $k = h_2 h_1$ we have k is a self homeomorphism of M^n with $k(N) = N'$.

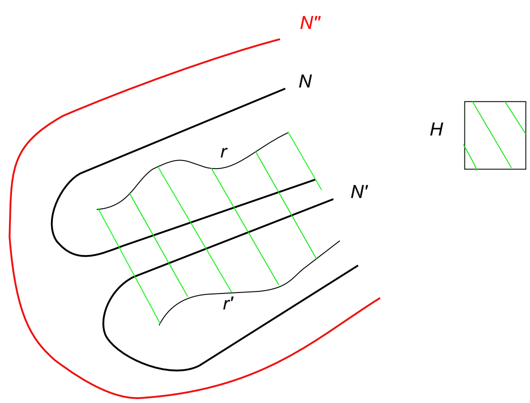

Figure 5.6: Homotopy Between Rays

For $n = 4$ we can cut H into two embeddings and then apply the regular neighborhood theorem as we now show. The singularity set of H is defined as $S(H) = \{x | H^{-1}H(x) \neq x\}$. By general position (and the fact that the dimension of M is twice the dimension of the domain of H) we can arrange so that the singular set is discrete and that the singularities of H are all double points (x such that

$|H^{-1}H(x)| = 2$). We partition $S(H)$ into $\{x_\alpha\}$ and $\{y_\alpha\}$, where $H(x_\alpha) = H(y_\alpha)$. We can then divide the domain of H into two sides, one side containing the x_α's and the other side containing the y_α's. See Figure 5.7 (which is inspired by Fig. 37 from [RoSa, p. 66]).

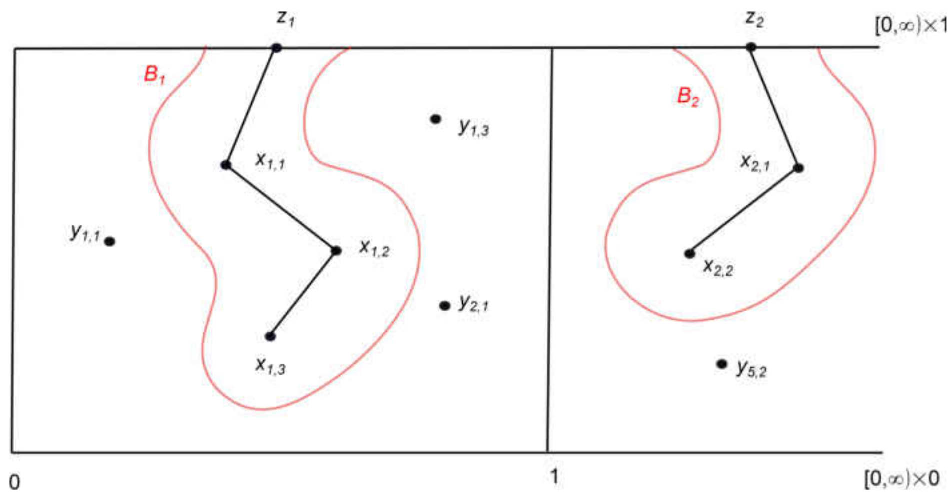

Figure 5.7: Dividing H's Domain

For $j = 1, 2, \ldots$, choose a point $z_j \in [j-1, j) \times 1$ and arc $\beta_j \subset [j-1, j) \times [0, 1)$ joining z_j and every x_α in $[j-1, j) \times (0, 1)$ but missing all the y_α's with the property that β_j meets $[0, \infty) \times 1$ only at z_j. Then choose B_j regular neighborhood of β_j missing each of the sets: the y_α's, $[0, \infty) \times 0$, and the union of the B_i's for $i < j$. Then B_j is a ball of which $B_j \cap [0, \infty) \times 0$ is a face. As the singular set is discrete there exists a product neighborhood of $[0, \infty) \times 1$ say $([0, \infty) \times 1) \times [a, b]$ which misses all the y_α's. Then the strip S_1 defined as

$$S_1 = (([0, \infty) \times 1) \times [a, b]) \cup (\cup_j B_j)$$

is embedded by H as is its closed complement $S_2 = \mathrm{cl}(([0, \infty) \times [0, 1]) - S_1))$. See Figure 5.8.

Let r'' be the image of the "lower border of S_1", $r'' = H(\mathrm{cl}(\partial S_1 \cap ((0, \infty) \times (0, 1))))$. Note S_1 collapses to $[0, \infty) \times 1$ and to the lower border of S_1. Likewise, S_2 collapses to the lower border of S_1 and to $[0, \infty) \times 0$. Thus we have, $H(S_1) \searrow r, r''$ and

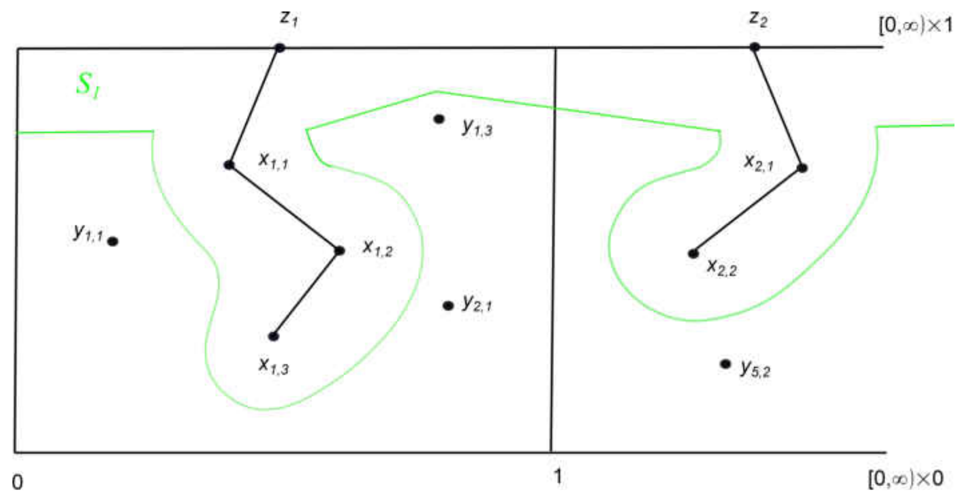

Figure 5.8: Strip S_1

$H(S_2) \searrow r'', r'$. Choose regular neighborhoods N_1, N_2, N, N', and N'' of $H(S_1)$, $H(S_2)$, r, r', and r'', respectively. Then N_1 is a regular neighborhood of both r and r'' and N_2 is a regular neighborhood of each of r'' and r. By the regular neighborhood theorem there exists homeomorphisms $(M, N) \to (M, N_1) \to (M, N'') \to (M, N_2) \to (M, N')$.

□

Corollary 5.2.5. *For $n \geq 4$ and 1-ended semistable manifolds $X^n, Y^n, X_1^n, X_2^n, \ldots$ $(X, \alpha_X)\natural(Y, \alpha_Y)$, $(X_1, \alpha_{1,L}, \alpha_{1,R})\natural\ldots\natural(X_m, \alpha_{m,L}, \alpha_{m,R})$ and $(X_1, \alpha_{1,L}, \alpha_{1,R})\natural(X_2, \alpha_{2,L}, \alpha_{2,R})\natural\ldots$ are independent of choices of rays $\alpha_X,\ \alpha_Y, \alpha_{1,L}, \alpha_{1,R}, \alpha_{2,L}, \alpha_{2,R}, \ldots$.*

As a result of the corollary, when considering 1-ended semistable n-manifolds $(n \geq 4)$ X and Y we will use the notations $X\natural Y$, $X_1\natural\ldots\natural X_m$, and $X_1\natural X_2\natural\ldots$ for the unique CSI's of X and Y, $X_1, X_2, \ldots, X_m$, and $X_1, X_2, \ldots$.

The following propostion can be justified by an application of Van Kampen's Theorem.

Proposition 5.2.6. *Let X and Y be 1-ended semistable open n-manifolds $(n \geq 4)$. Then $\pi_1(X\natural Y) \cong \pi_1(X) * \pi_1(Y)$.*

5.3 Some Combinatorial Group Theory and Uncountable Jester's Manifold Sums

The primary goal of this section is the following.

Theorem 5.3.1. *The set of homeomorphism classes of all possible CSI's of interiors of Jester's manifolds is uncountable.*

This Theorem can be obtained from Theorem 4.2.7 by an application of theorem (4.1) of Curtis and Kwun [CuKw]. Since the approach used there is a bit outdated, we will supply an alternate version of their theorem. The essence of our proof is the same as theirs, but ours will take advantage of the rigorous development of the fundamental group at infinity that has taken place in the intervening years. The new approach is also more direct in that it compares open manifolds directly, without reference to some discarded boundaries. We will demostrate shortly the following more general result, for which Theorem 5.3.1 will be a corollary.

Theorem 5.3.2. *Let $\mathcal{G}$ be a collection of distinct indecomposible groups, none of which are infinite cyclic and let $\{X_i^n\}$and $\left\{Y_j^n\right\}$ be countably infinite collections of simply connected, 1-ended open n-manifolds with each* pro-$\pi_1(\varepsilon(X_i))$ *and* pro-$\pi_1(\varepsilon(Y_j))$ *being stable and pro-isomorphic to an element of $\mathcal{G}$. Then $X_1\natural X_2\natural ...$ and $Y_1\natural Y_2\natural ...$ are 1-ended and semistable and if any element of $\mathcal{G}$ appears more times in one of the sequences, $\{\text{pro-}\pi_1(\varepsilon(X_i))\}$ and $\{\text{pro-}\pi_1(\varepsilon(Y_j))\}$, than it does in the other, then* pro-$\pi_1\left(\varepsilon\left(X_1\natural X_2\natural X_3\natural\cdots\right)\right)$ *is not pro-isomorphic to*
pro-$\pi_1\left(\varepsilon\left(Y_1\natural Y_2\natural Y_3\natural\cdots\right)\right)$.

First we'll state and prove a theorem about certain types of inverse sequences of groups that will help us determine when two infinite CSI's of our Jester's manifolds are distinct. This theorem (or its discovery) and its proof are motivated by Theorem (4.1) (and its proof) in [CuKw].

Theorem 5.3.3. *Let $A_1, A_2, ...$ and $B_1, B_2, ...$ be indecomposible groups none of which are infinite cyclic, and for all positive integers j and k let G_j and H_k be the* *free products*

$$G_j = A_1 * A_2 * \ldots * A_j$$

$$H_k = B_1 * B_2 * \ldots * B_k.$$

Further let $\varphi_j : G_j \to G_{j-1}$ *and* $\psi_k : H_k \to H_{k-1}$ *be the obvious projections such that*

$$\varphi_j|_{G_{j-1}} = id_{G_{j-1}},\ \varphi_j(A_j) = 1,$$

$$\psi_k|_{H_{k-1}} = id_{H_{k-1}} \text{ and, } \psi_k(B_k) = 1.$$

Suppose the inverse sequences $\{G_j, \varphi_j\}$ *and* $\{H_k, \psi_k\}$ *are pro-isomorphic. That is, there exists a commutative ladder diagram as below.*

$$\begin{tikzcd}
G_{j_1} & & G_{j_2} \arrow[ll, "f_2"'] \arrow[dl, "d_2"'] & & G_{j_3} \arrow[ll, "f_3"'] \arrow[dl, "d_3"'] & & \cdots \arrow[ll, "f_4"'] & \\
& H_{k_1} \arrow[ul, "u_1"] & & H_{k_2} \arrow[ll, "g_2"'] \arrow[ul, "u_2"] & & H_{k_3} \arrow[ll, "g_3"'] \arrow[ul, "u_3"] & & \cdots \arrow[ll, "g_4"']
\end{tikzcd}$$

Here the bonding homomorphisms are the compositions

$$f_i = \varphi_{j_{i+1}} \ldots \varphi_{j_i+1} \varphi_{j_i} \text{ and } g_i = \psi_{k_{i+1}} \ldots \psi_{k_i+1} \psi_{k_i}.$$

Then there exists a self bijection Φ *of* $\mathbb{Z}_+$ *such that* $A_j \cong B_{\Phi(j)}$.

Proof. It suffices to show the following two claims.

Claim 1: For each positive integer pair (l, s) with $l \leq j_s$ and $s > 1$ there exists at least as many isomorphic copies of A_l among $B_1, \ldots, B_{k_s}$ as there are among $A_1, \ldots, A_{j_s}$.

Claim 2: For each positive integer pair (r, m) with $r \leq k_m$ there exists at least as many isomorphic copies of B_r among $A_1, \ldots, A_{l_m}$ as there are among $B_1, \ldots, B_{k_r}$.

We prove claim 1 and by a similar argument one can prove claim 2. We will use the following facts: in a group $C = C_1 * C_2 * \ldots * C_n$ (1) no nontrivial free factor C_i is a subgroup of a conjugate of some other free factor C_j and (2) every conjugate of C_i meets every other factor C_j, $j \neq i$ trivially. These facts can be verified using normal forms [LySc, p. 175]. Consider the following commutative ladder diagram.

$$\begin{tikzcd}
G_{j_s} & & G_{j_{s+1}} \arrow[ll, "f_{s+1}"'] \arrow[dl, "d_{s+1}"'] & & G_{j_{s+2}} \arrow[ll, "f_{s+2}"'] \arrow[dl, "d_{s+2}"'] & & \cdots \arrow[ll, "f_{s+3}"'] & \\
& H_{k_s} \arrow[ul, "u_s"] & & H_{k_{s+1}} \arrow[ll, "g_{s+1}"'] \arrow[ul, "u_{s+1}"'] & & H_{k_{s+2}} \arrow[ll, "g_{s+2}"'] \arrow[ul, "u_{s+2}"'] & & \cdots \arrow[ll, "g_{s+3}"']
\end{tikzcd}$$

We observe that for $i > 1$, $d_i|_{G_{j_{i-1}}} := d_i \circ (G_{j_{i-1}} \hookrightarrow G_{j_i})$ and $u_i|_{H_{k_{i-1}}}$ are monomorphisms since $f_i|_{G_{j_{i-1}}}$ and $g_i|_{H_{k_{i-1}}}$ are. Thus $A_i \cong d_{s+1}(A_i)$ and $B_k \cong u_{s+1}(B_k)$ for $i \leq j_s$ and $k \leq k_s$.

Choose $l \leq j_s$. We'll show there exists $t \leq k_s$ such that $u_{s+1}(B_t)$ is a conjugate of A_l thus exhibiting B_t as an isomorphic copy of A_l. Since $d_{s+2}(A_l) \cong A_l$ is indecomposible and not infinite cyclic the Kurosh Subgroup Theorem [Mas, p. 219] gives $d_{s+2}(A_l) \leq \beta B_t \beta^{-1}$ for some $t \leq k_{s+1}$ and $\beta \in H_{k_{s+1}}$. Moreover, since A_l survives into G_{j_s} we have $t \leq k_s$. Then the restriction $u_{s+1}|_{B_t}$ is injective and thus so is $u_{s+1}|_{\beta B_t \beta^{-1}}$ and we know $u_{s+1}(\beta B_t \beta^{-1})$ is indecomposible and not infinite cyclic. We again apply Kurosh yielding $u_{s+1}(\beta B_t \beta^{-1})$ is a subgroup of a conjugate of some A_r.Thus in $G_{j_{s+1}}$ we have $A_l = f_{s+2}(A_l) = u_{s+1}d_{s+2}(A_l) \leq u_{s+1}(\beta B_t \beta^{-1}) \leq$ congugate of A_r. By our facts $l = r$ and we have $A_l = \beta B_t \beta^{-1} \cong B_t$. More specifically, t is the unique integer less than or equal to for which $u_{s+1}(B_t)$ is conjugate to A_l.

Thus we have shown the map

$$\Psi : \{1, 2, ..., j_s\} \to \{1, 2, ..., k_s\};\ l \mapsto t$$

is injective and $B_{\Psi(i)} \cong A_i$. This completes the proof of claim 1 and the proof of the proposition.

$\square$

We now apply Theorem 5.3.3 to prove Theorem 5.3.2.

Proof of Theorem 5.3.2. Let A_i and B_j be groups such that pro-$\pi_1(\varepsilon(X_i))$ and pro-$\pi_1(\varepsilon(Y_j))$ are pro-isomorphic to the constant sequences $\{A_i, id_{A_i}\}$ and $\{B_j, id_{B_j}\}$. Then the hypothesis "an element of $\mathcal{G}$ appears more times in one of the sequences, $\{\text{pro-}\pi_1(\varepsilon(X_i))\}$ and $\{\text{pro-}\pi_1(\varepsilon(Y_j))\}$, than it does in the other," translates as there does not exist the bijection Φ as in the conclusion of Theorem 5.3.3. Thus if we can show that $X_1 \natural X_2 \natural ...$

and $Y_1\natural Y_2\natural\ldots$ are 1-ended and semistable and also that pro-$\pi_1(\varepsilon(X_1\natural X_2\natural\ldots))$ and pro-$\pi_1(\varepsilon(Y_1\natural Y_2\natural\ldots))$ are of the forms $\{G_j,\varphi_j\}$ and $\{H_k,\psi_k\}$ in the statement of Theorem 5.3.3 we will have the desired result.

For $i=1,2,\ldots$ let $U_{i,1}\supset U_{i,2}\supset\ldots$ be a cofinal sequence of clean neighborhoods of infinity in X_i so that $\{\pi_1(U_{i,j}),\tau_{i,j}\}\in$ pro-$\pi_1(\varepsilon(X_i))$ can be fit into a commuting ladder diagram with $\{A_i, id_{A_i}\}$

$$\begin{tikzcd}
\pi_1(U_{i,1}) & & \pi_1(U_{i,2}) \arrow[ll, "\tau_{i,2}"'] \arrow[dl, "d_{i,2}"'] & & \pi_1(U_{i,3}) \arrow[ll, "\tau_{i,3}"'] \arrow[dl, "d_{i,3}"'] & & \cdots \arrow[ll, "\tau_{i,4}"'] \\
& A_i \arrow[ul, "u_{i,1}"] & & A_i \arrow[ll, "id"'] \arrow[ul, "u_{i,2}"] & & A_i \arrow[ll, "id"'] \arrow[ul, "u_{i,3}"] & & \cdots \arrow[ll, "id"']
\end{tikzcd} \tag{5.2}$$

Here $\tau_{i,j}$ is the bonding homomorphism discussed in the definition of the fundamental group at infinity.

As in the definition of $X_1\natural X_2\natural\ldots$, for $i=1,2,\ldots$ choose disjoint rays $r_{i,L},r_{i,R}\subset X_i$ and disjoint regular neighborhoods $N_{i,L},N_{i,R}\subset X_i$ of said rays with the additional property that for each j, $r_{i,x}$ meets $\mathrm{Bd}_{X_i}U_{i,j}$ transversely in a single point.

For $i=2,3,\ldots$ and for $j=1,2,\ldots$ let

$$\hat{U}_{1,j}=U_{1,j}-\mathrm{int}N_{1,R}\ \text{ and }\ \hat{U}_{i,j}=U_{i,j}-\mathrm{int}(N_{i,L}\cup N_{i,R}).$$

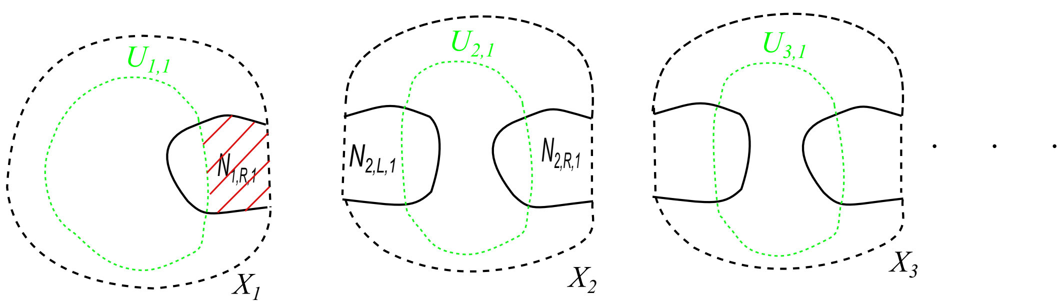

Figure 5.9: Neighborhoods of ∞

We claim $\pi_1(\hat{U}_{i,j})\cong\pi_1(U_{i,j})$. For $i,j=1,2,\ldots$ and $x=L,R$ let $N_{i,x,j}=U_{i,j}\cap N_{i,x}$ which is homeomorphic to $r_{i,x}((a,\infty))\times\mathbb{B}^{n-1}$ for some $a>0$ since $r_{i,x}$ meets $\mathrm{Bd}_{X_i}U_{i,j}$

transversely in a single point. We see that
$\hat{U}_{i,j} \cap N_{i,x,j} \approx r_{i,x}((a,\infty)) \times S^{n-2}$ which is simply connected as $n \geq 4$. Thus

$$\pi_1(U_{i,j}) = \pi_1(\hat{U}_{i,j} \cup N_{i,L,j} \cup N_{i,R,j}) \approx \pi_1(\hat{U}_{i,j}).$$

For $i = 1, 2, ..,$ let $\hat{X}_i = X_i - N_{i,L} \approx X_i$ and

$$W_i \quad = \quad \hat{U}_{1,i} \cup_\phi \hat{U}_{2,i} \cup_\phi ... \cup_\phi \hat{U}_{i,i} \cup_\phi \hat{X}_{i+1} \natural X_{i+2} \natural X_{i+3} ...$$

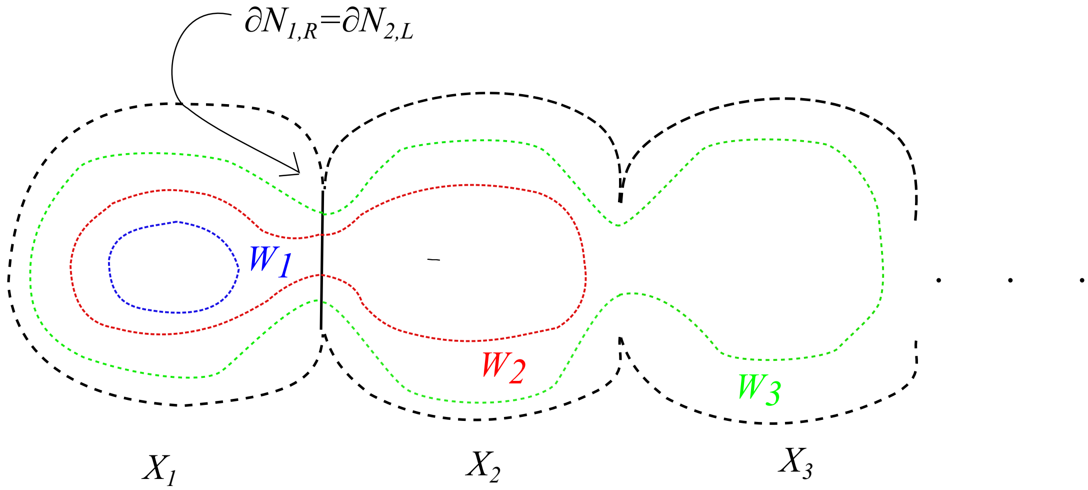

Figure 5.10: $W_1 \supset W_2 \supset W_3$ in $X_1 \natural X_2 \natural ...$

Observe that $W_1, W_2, ...$ form a cofinal sequence of connected neighborhoods of infinity in $X_1 \natural X_2 \natural ...$ and thus if U is a neighborhood of infinity in $X_1 \natural X_2 \natural ...$ then $U \supset W_i$ for some i. This shows $X_1 \natural X_2 \natural ...$ is 1-ended. Then as

$$\hat{U}_{i,j} \cap \hat{U}_{i+1,j} = \partial N_{i,L,j} = \partial N_{i+1,R,j} \approx (a, \infty) \times S^{n-2},$$

$$\hat{U}_{i,i} \cap_\varphi \hat{X}_{i+1} = \partial N_{i,R,j},$$

and the X_i are all simply connected we have

$$\pi_1(W_j) \cong \pi_1(U_{1,j}) * \pi_1(U_{2,j}) * ... * \pi_1(U_{j,j}).$$

We will show $\{\pi_1(W_j), \tau_{1,j}\}$ is pro-isomorphic to $\{G_j, \varphi_j\}$. For our base ray we choose r_1 the choosen base ray for X_1. Let

$$1_{i,j} : \pi_1(U_{i,j}) \to 1,$$

$$d_j' = d_{1,j} * d_{2,j} * ... * d_{j,j-1} * 1_{j,j},$$

$$d_j' : \pi_1(U_{1,j}) * \pi_1(U_{2,j}) * ... * \pi_1(U_{j,j}) \to A_1 * A_2 * ... * A_{j-1},$$

$$u_j' = u_{1,j} * u_{2,j} * ... * u_{j,j-1} * u_{j,j},$$

$$u_j' : \pi_1(U_{1,j}) * \pi_1(U_{2,j}) * ... * \pi_1(U_{j,j}) \to A_1 * A_2 * ... * A_j, \text{ and}$$

$$\tau_j' = \tau_{1,j} * \tau_{2,j} * ... * \tau_{j-1,j} * 1_{j,j}$$

$$\tau_j' : \pi_1(U_{1,j}) * \pi_1(U_{2,j}) * ... * \pi_1(U_{j,j}) \to \pi_1(U_{1,j}) * \pi_1(U_{2,j}) * ... * \pi_1(U_{j-1,j})$$

where $d_{i,j}$, $u_{i,j}$, and $\tau_{i,j}$ are the "up","down", and bonding homomorphisms of the previous ladder diagram (5.2). We then have the following commutative diagram:

$$\begin{tikzcd}
\pi_1(U_{1,1}) & & \pi_1(U_{1,2}) * \pi_1(U_{2,2}) \arrow[ll, "\tau_2'"'] \arrow[dl, "d_2'"'] & & \pi_1(U_{1,2}) * \pi_1(U_{2,2}) * \pi_1(U_{3,3}) \arrow[ll, "\tau_3'"'] \arrow[dl, "d_3'"'] & & \cdots \arrow[ll, "\tau_{i,4}"'] & \\
& A_1 \arrow[ul, "u_1'"] & & A_1 * A_2 \arrow[ll, "\varphi_2"'] \arrow[ul, "u_2'"'] & & A_1 * A_2 * A_3 \arrow[ll, "\varphi_3"'] \arrow[ul, "u_3'"'] & & \cdots \arrow[ll, "\varphi_4"']
\end{tikzcd}$$

Thus $X_1 \natural X_2 \natural ...$ is semistable and $\{\pi_1(W_j), \tau_j'\}$ is pro-isomorphic to $\{G_j, \varphi_j\}$. Similarly, one can show pro-$\pi_1(\varepsilon(Y))$ is of the form $\{H_k, \psi_k\}$. □

Theorem 5.3.4. *Let $\mathcal{G}$ be a collection of distinct indecomposible groups none of which are infinite cyclic and let $\{C_i^n\}$and $\left\{D_j^n\right\}$ be countably infinite collections of compact simply connected n-manifolds with connected boundaries that have fundamental groups lying in $\mathcal{G}$. If any element of $\mathcal{G}$ appears more times in one of the sequences, $\{\pi_1(\partial C_i^n)\}$ and $\left\{\pi_1\left(\partial D_j^n\right)\right\}$, than it does in the other, then*

$$\operatorname{int}\left(C_1 \overset{\partial}{\#} C_2 \overset{\partial}{\#} C_3 \overset{\partial}{\#} \cdots\right) \not\approx \operatorname{int}\left(D_1 \overset{\partial}{\#} D_2 \overset{\partial}{\#} D_3 \overset{\partial}{\#} \cdots\right).$$

Proof. Since C_i and D_i are compact with connected boundaries $X_i = \operatorname{int} C_i$ and $Y_i = \operatorname{int} D_i$ are 1-ended and stable and thus meet the hypotheses of Theorem 5.3.2. Since the CSI's of the interiors are homeomorphic to the interiors of the BCS's we have the desired result. □

Theorem 5.3.1, which we repeat below, can now be seen to be a corollary to Theorem 5.3.4.

Theorem 5.3.1. *The set of homeomorphism classes of all possible infinite CSI's of interiors of Jester's manifolds is uncountable.*

In the next section we will show that these manifolds split.

5.4 Sums of Splitters Split

In this section we demonstrate our main result:

Theorem 5.4.1. *There exists an uncountable collection of contractible open 4-manifolds which split as* $\mathbb{R}^4 \cup_{\mathbb{R}^4} \mathbb{R}^4$.

We'll demonstrate the above result by showing that the infinite CSI $X_1 \natural X_2 \natural ...$ of certain types of splitters $X_i \approx \mathbb{R}^n \cup_{\mathbb{R}^n} \mathbb{R}^n$ $(n \geq 4)$ also splits. Our argument consists of choosing our ray, regular neighborhood pairs in the definition of the CSI to lie in the intersections (the C_i's) of the splittings $A_i \cup_{C_i} B_i \approx \mathbb{R}^n \cup_{\mathbb{R}^n} \mathbb{R}^n$. This will yield the CSI to be of the form

$$(A_1 \natural A_2 \natural ...) \cup_{C_1 \natural C_2 \natural ...} (B_1 \natural B_2 \natural ...) \tag{5.3}$$

which is itself an open splitting. We apply this result to our infinite sums of Jester's manifolds, an uncounatble collection. The work comes in showing the existence of the desired ray, regular neighborhood pair mentioned above. We desire, for all i, that our ray not only lies in C_i but also that it is proper in both A_i and B_i thus ensuring we obtain a splitting of the form (5.3).

Proposition 5.4.2. *If* Σ *is a smooth properly embedded line in* $\mathbb{R}^n$ *and* M^{n-1} *is a closed smooth submanifold of* $\mathbb{R}^n$ *intersecting* Σ *transversely then* $|\Sigma \cap M^{n-1}|$ *is even.*

Proof. As M^{n-1} is a codimension 1, closed submanifold of Euclidean space, the Jordan-Brouwer separation theorem gives it has an inside and an outside [Ale].

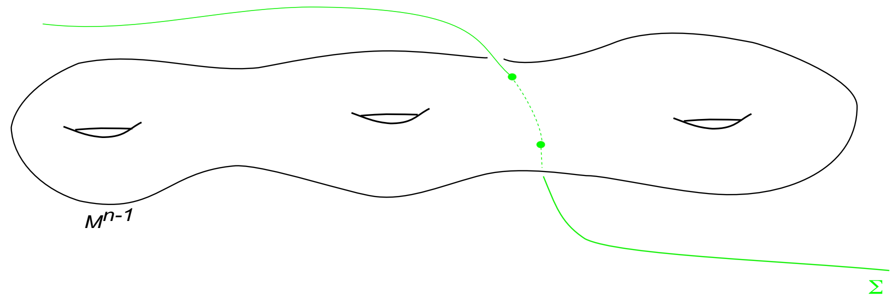

Figure 5.11: $|\Sigma \cap M^{n-1}|$ is even

Since at each intersection point of Σ with M, Σ meets M transversely, Σ passes from M's inside to M's outside or vice versa. $\square$

Lemma 5.4.3. *Suppose M is a contractible n-manifold which splits as $M = A \cup_C B$, $A, B, C \approx \mathbb{R}^n$. Then there exists a ray r in C which is also proper in both A and B.*

Proof. We will describe a proof that uses differential topology. Analogous proofs are possible in the PL or topological categories. Let $S = A \cap \mathrm{Bd}_{M^n}(C)$ and $T = B \cap \mathrm{Bd}_{M^n}(C)$ so that $\mathrm{Bd}_{M^n}(C) = S \sqcup T$. Let $\overline{C} = \mathrm{cl}_{M^n}(C)$ so $\overline{C} = C \cup S \cup T$. Note S and T are closed in $\overline{C}$. Let $\alpha = [-1, 1]$ be an arc in $\overline{C}$ so that $\alpha \cap S = \{-1\}$ and $\alpha \cap T = \{1\}$. Choose $N \approx \mathrm{int}\alpha \times \mathbb{B}^{n-1}$ a tapered product neighborhood of $\mathrm{int}\alpha$ in C. That is, $\mathrm{Bd}_{M^n}(N) - N = \partial\alpha$.

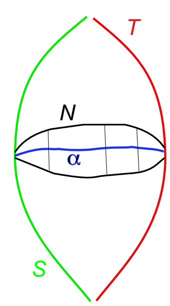

Figure 5.12: Tapered Product Neighborhood of α

Let $f : S \cup N \cup T \to \alpha$ be a retraction so that $f^{-1}(-1) = S$, $f^{-1}(1) = T$ and for $x \in \mathrm{int}\alpha$, $f(x \times \mathbb{B}^{n-1}) = \{x\}$. That is, f collapses N along product lines. Note that for $x \in \mathrm{int}\alpha$, $f^{-1}(x)$ intersects α transversly precisely at x. We then apply the Tietze extension theorem to get a retraction $f : \overline{C} \to \alpha$. We choose such an f that is smooth. We will now adjust f with the aim that C maps to intα. Let $W = f^{-1}([-1,0]) \cup N \cup T$ and $b \in C - W$. Via Urysohn's Lemma choose $\eta : \overline{C} \to [0,1]$ such that $\eta^{-1}(0) = W$ and $\eta^{-1}(1) = \{b\}$. Let $g = f - \eta$ so $g|_W = f_W$. If $x \notin W$ then $\eta(x) > 0$ and $g(x) = f(x) - \eta(x) < f(x) - 0 \leq 1$. Thus $g^{-1}(1) = T$. Similarly we can adjust g to get, say h, so $h^{-1}(1) = T$, $h^{-1}(-1) = S$, and $h|_{S \cup N \cup T} = f|_{S \cup N \cup T}$.

Via Sard's Theorem we can choose a regular value v of h in intα and let V be the component of $h^{-1}(v)$ containing v [Kos, p. 227]. We observe that V is a smooth $(n-1)$-submanifold of C without boundary which is closed in C and intersects α (transversely) precisely at v. If V were compact, the previous propositon would yield that the number of intersections of the C properly embedded line intα with V would be even. Thus V is noncompact and hence is C unbounded. We claim V is embedded properly in C. For suppose K is a compactum in C and let $\iota : V \hookrightarrow C$ be the inclusion map. Then $V \cap K = \iota^{-1}(K)$ is a closed subset of K and is hence compact thus showing ι is proper. There then exists a ray r in V which is proper in C.

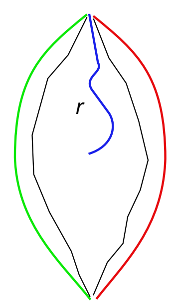

Figure 5.13: $N' \supset S \sqcup T$

We now show r is proper in both A and B. Let K be a compact subset of A. We claim the end of r lies outside of K. Again by Sard, there exists ϵ_1 and ϵ_2 sufficiently small so that $-1 + \epsilon_1 < v < 1 - \epsilon_2$ are regular values of h. Let $T' = h^{-1}([-1 + \epsilon_1, 1 - \epsilon_2])$, a closed subset of C. Then $K' = K \cap T'$ is a compact subset of C. Therefore, r eventually stays outside of K'. But since r lives in T', when it leaves K' it also leaves K. Thus r is proper in A and a similar argument can be made to show r is proper in B. □

Recall Proposition 1.2.1 which says that the interior of a closed splitter is an open splitter.

Corollary 5.4.4. *Suppose M^n is a compact contractible n-manifold such that $M = A \cup_C B$, with $A, B, C \approx \mathbb{B}^n$. Then there exists a ray r in* int$C$ *which is also proper in both* int$A$ *and* int$B$.

Proposition 5.4.5. *Let M_1 and M_2 be contractible, piecewise linear, open n-manifolds ($n \geq 4$) which split as $M_i = A_i \cup_{C_i} B_i$, $A_i, B_i, C_i \approx \mathbb{R}^n$. Further let $r_i \subset C_i$ be a ray in C_i which is also proper in both A_i and B_i. Then the connected sum at infinity of (M_1, r_1) and (M_2, r_2) also splits: $(M_1, r_1)\natural(M_2, r_2) = A \cup_C B$ with $A, B, C \approx \mathbb{R}^n$.*

An immediate corollary is:

Corollary 5.4.6. *Let M_1 and M_2 be contractible, piecewise linear, semistable, open n-manifolds ($n \geq 4$) which split as $M_i = A_i \cup_{C_i} B_i$, $A_i, B_i, C_i \approx \mathbb{R}^n$. Then the connected sum at infinity of M_1 and M_2 also splits: $M_1 \natural M_2 = A \cup_C B$ with $A, B, C \approx \mathbb{R}^n$.*

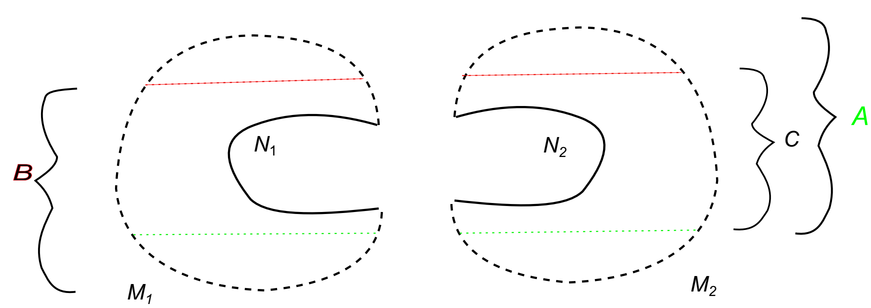

Figure 5.14: $M_1 \natural M_2$ Splits

Proof of Proposition 5.4.5. For $i = 1, 2$, let N_i be a (A_i, B_i, and C_i) regular neighborhood of r_i. For $X_i = M_i, A_i, B_i, C_i$, let $\hat{X}_i = X_i - \text{int}(N_i)$. Given an orientation reversing homeomorphism $f : \partial N_1 \to \partial N_2$ we have $(M_1, r_1)\natural(M_2, r_2) = \hat{M}_1 \cup_f \hat{M}_2$. Let $A = \hat{A}_1 \cup_f \hat{A}_2$ and observe that $A = (A_1, r_1)\natural(A_2, r_2)$. Likewise we let $B = \hat{B}_1 \cup_f \hat{B}_2 = (B_1, r_1)\natural(B_2, r_2)$ and $C = \hat{C}_1 \cup_f \hat{C}_2 = (C_1, r_1)\natural(C_2, r_2)$ and we see that $(M_1, r_1)\natural(M_2, r_2) = A \cup_C B$. From Note 5.2.3 we know each of A, B, and C are $\mathbb{R}^n$'s as they are each the connected sum at infinity of two $\mathbb{R}^n$'s. See Figure 5.14. $\square$

Proposition 5.4.7. *For $i = 1, 2, ...,$ let M_i be a contractible, open n-manifold ($n \geq 4$) such that $M_i = A_i \cup_{C_i} B_i$ with $A_i, B_i, C_i \approx \mathbb{R}^n$ for all i. Further let $r_{i,L}$ and $r_{i,R}$ be disjoint rays in C_i that are also proper in both A_i and B_i. Then*

$$M := \natural_{i=1}^{\infty}(M_i, r_{i,L}, r_{i,R}) \approx A \cup_C B$$

with $A, B, C \approx \mathbb{R}^n$.

Corollary 5.4.8. *For $i = 1, 2, ...,$ let M_i be a contractible, semistable, open n-manifold ($n \geq 4$). If $M_i = A_i \cup_{C_i} B_i$ with $A_i, B_i, C_i \approx \mathbb{R}^n$ for all i then*

$$M := \natural_{i=1}^{\infty} M_i \approx A \cup_C B$$

with $A, B, C \approx \mathbb{R}^n$.

Proof of Proposition 5.4.7. For $i = 1, 2, ...,$ choose disjoint A_i, B_i, and C_i regular neighborhoods $N_{i,L}$, $N_{i,R}$ of $r_{i,L}$ and $r_{1,R}$, respectively. For $j = 1, 2, ...,$ let

$$\check{C}_j = (C_1 - N_{1R}) \cup (C_2 - [\text{int} N_{2L} \cup N_{2R}]) \cup (C_3 - [\text{int} N_{3L} \cup N_{3R}]) \cup ... \cup (C_j - [\text{int} N_{j,L} \cup N_{j,R}])$$

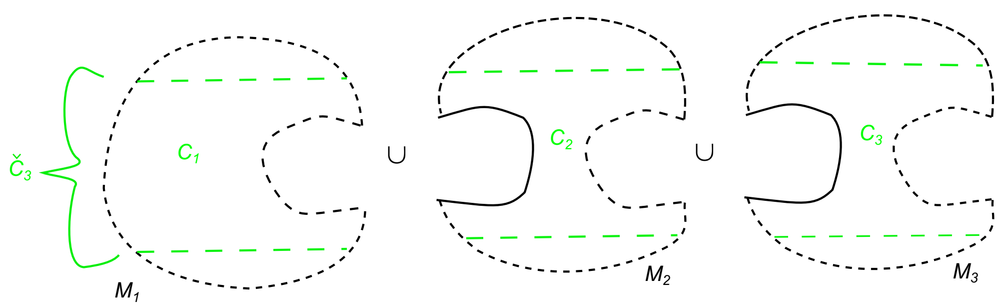

Figure 5.15: $\check{C}_3$

Then $\check{C}_j = (\natural_{i=1}^{j}(C_i, r_i)) - N_{j,R} \approx \mathbb{R}^n - \mathbb{R}^n_+ \approx \mathbb{R}^n$ and $\check{C}_j \subset \check{C}_{j+1}$. Let $C = \cup \check{C}_j$, so that C is an ascending union of $\mathbb{R}^n$'s and thus is itself an $\mathbb{R}^n$ [Bro]. Let

$$\check{A}_j = (A_1 - \text{int} N_{1R}) \cup (A_2 - [\text{int} N_{2L} \cup N_{2R}]) \cup (A_3 - [\text{int} N_{3L} \cup N_{3R}]) \cup ... \cup (A_j - [\text{int} N_{j,L} \cup N_{j,R}]),$$

$\check{B}_j = (B_1 - \text{int}N_{1R}) \cup (B_2 - [\text{int}N_{2L} \cup N_{2R}]) \cup (B_3 - [\text{int}N_{3L} \cup N_{3R}]) \cup ... \cup (B_j - [\text{int}N_{j,L} \cup N_{j,R}])$,

$A = \cup \check{A}_j$, and $B = \cup \check{B}_j$ so that $A, B \approx \mathbb{R}^n$ and $M = A \cup_C B$. $\square$

We have demonstrated that any CSI of interiors of Jester's manifolds splits and thus have demonstrated

Theorem 5.4.1. *There exists an uncountable collection of contractible open 4-manifolds which split as* $\mathbb{R}^4 \cup_{\mathbb{R}^4} \mathbb{R}^4$.

Recall Note 1.3.9 in which we reported the result of Ancel and Siebenman which states that a Davis manifold generated by C is homeomorphic to the interior of an alternating boundary connected sum $\text{int}(C \overset{\partial}{\sharp} -C \overset{\partial}{\sharp} C \overset{\partial}{\sharp} -C \overset{\partial}{\sharp} ...)$ where $-C$ is a copy of C with the opposite orientation. We have now proved

Corollary 5.4.9. *There exists (non-*$\mathbb{R}^4$*) 4-dimensional Davis manifold splitters.*

Bibliography

[Ale] J.W. Alexander, *A proof and extension of the Jordan-Brouwer separation theorem*, Trans. Amer. Math Soc. **23** (1922), 333-349

[AG95] Fredric D. Ancel and Craig R. Guilbault, *Compact Contractible n-Manifolds Have Arc Spines* ($n \geq 5$), Pacific Jounal of Mathematics, Vol. 168, No. 1, 1995

[AG14+] Fredric D. Ancel and Craig R. Guilbault, *Infinite Connected Sums of Manifolds,* In progress

[Bro] M. Brown, *A monotone union of open n-cells is an open n-cell*, Proc. Am. Math. Soc. 12, 812-814 (1961)

[CKS] J.S. Calcut, H.C. King, and L.C. Siebenmann, *Connected sum at infinity and Cantrell-Stallings hyperplane unknotting.* Rocky Mountain J. Math. 42 (2012), 1803-1862

[Coh] M. M. Cohen, *A general theory of relative regular neighborhoods*, Trans. Amer. Math. Soc. 136, 189229 (1969)

[CuKw] M.L. Curtis and Kyung Whan Kwun *Infinite Sums of Manifolds.* Topology 3 (1965), 31 - 42

[Gab] David Gabai *The Whitehead manifold is a union of two Euclidean spaces.* Journal of Topology 4 (2011), 529 - 534

[GRW] Garity, Repovs, Wright *Contractible 3-manifolds and the double 3-space property.* Preprint

[Geo] Ross Geoghegan *Topological Methods in Group Theory.* Springer 2008.

[Gla65] Leslie Glaser *Contractible complexes in S_n.* Proc. Amer. Math. Soc. 16, 1357-1364 (1965)

[Gla66] Leslie Glaser *Intersections of combinatorial balls of Euclidean spaces.* Bull. Am. Math. Soc. 72, 68-71 (1966)

[Gla72] Leslie Glaser *Geometric Combinatorial Topology.* Van Nostrand Reinhold Company 1970.

[Gom] R.E. Gompf, *An infinite set of exotic $\mathbb{R}^4$'s.* J. Differential Geom. 21 (1985), 283-300

[Gui] C. R. Guilbault *Ends, Shapes, and Boundaries in Manifold Topology and Geometric Group Theory* arXiv:1210.6741v3 [math.GT] 22 Jul 2013

[Hat] Hatcher, Allen, *Algebraic Topology*, Cambridge University Press, New York, 2001

[Kos] Kosinski, Antoni A., *Differential Manifolds*, Academic Press, Inc., San Diego, 1992

[LySc] Lyndon, Roger C. and Schupp, Paul E., *Combinatorial Group Theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[Mas] William S. Massey, *Algebraic Topology: An Introduction*, Harbrace college mathematics series, Harcourt, Brace and World, New York, 1967.

[Maz] Mazur, Barry *A note on some contractible* 4*-manifolds.* Ann. of Math. (2) 73 1961 221–228.

[Mun] Munkres, James R. *Elements of Algebaic Topology*, The Benjamin/Cummings Publishing Company, Inc. (1984)

[Rol] Dale Rolfsen *Knots and links*, Publish or Perish, (1976)

[RoSa] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise-Linear Topology.* Springer-Verlag, Berlin Heidelberg New York 1982

[Whi] J.H.C. Whitehead *Simplicial Spaces, nuclei and m-groups*, Proc. Lond. Mat. Soc. **45** (1939), 243-327

[Wri] David G. Wright *On 4-Manifolds Cross I*. Proceedings of the American Mathematical Society Vol. 58 (1976), 315 - 318

[Zee] E. C. Zeeman, *On the Dunce Hat* Topology 2 (1963), 341 – 358

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- Finite Mathematics (probability and linear programming)
- Liberal Arts Mathematics (graph theory and applications)

- Comtemporary Applications of Mathematics (social choice, apportionment, graph theory, probability and statistics, transformational geometry)
- Lower level algebra courses

TEACHING SKILLS

Experienced with D2L and Blackboard. I have experience with Illuminate. I have given group work lessons as well as graphing calculator, Excel, and R presentations.

PUBLICATIONS AND PAPERS

- Primary author of the solutions manual for “Survival Analysis Using S: Analysis of Time-to-Event Data”, Tableman, M., and Kim, J. S., (2003) Chapman and Hall/CRC, Boca Raton.
- “Analysis of Termination Time in the Presence of Competing Risks: Cumulative Incidence Applied to Employment Decisions”, Tableman, M., and Sparks, P., (2006)
- “Statistical Results for Research Hypothesis One in the GHS Nursing Research Proposal: First Stage Labor Duration in Women with Pregnancy Induced Hypertension”, Sparks, P., (2006)

PRESENTATIONS

- “Analysis of Failure Time Data in the Presence of Competing Risks: An Overview and Examples”, given at Portland State University, Portland, Oregon, December 2003.
- “Hat Spines and Splittings”, given at the Geometric Topology Workshop, Calvin College, Grand Rapids, Michigan, June, 2013.

- "Sums of Splittable Manifolds", given at the Topology Seminar, University of Wisconsin-Milwaukee, November, 2013.

COURSES TAKEN

- **Topology Courses**: Piecewise Linear Morse Theory, Ends Shapes and Boundaries in Manifold Topology and Geometric Group Theory, Fiber Bundles, Geometric Group Theory, Coxeter Groups (two semester sequence), Relatively Hyperbolic Groups

- **Algebra Courses**: Representation Theory of the Symmetric Group, The Classical Groups: Their Representations, The Classical Groups: Their Invariants, Homology

- **Statistics Courses**: Resampling Methods, Survival Analysis, Sampling Methods, Advanced Mathematical Statistics (three quarter sequence), Theory of Linear Models (three quarter sequence)

- **Other Electives**: Simulation, Applied Probability (two quarter sequence), Graph Theory, Numerical Analysis, Number Theory, Set Theory, Ordinary Differential Equations, Transformational Geometry, Various Math Education Courses

COMPUTER SKILLS

I have experience with Minitab, Matlab, S-Plus, R, SAS, Excel, Maple, Derive, and GAP.

STATISTICAL PROCEDURES USED

Regression Models, ANOVA, Orthogonal Polynomials, Factorial Designs, Blocking in Factorial Designs, 2k and 3k Factorial Designs, Multiple Comparisons, Random Effects Models, Nested and Split-Plot Designs, Randomized Blocking, Latin Square Designs, Proportional Hazards Models, Log-linear Regression, Competing

Risks Models, Bootstrapping, and Resampling.

MEMBERSHIPS

- American Mathematical Society
- Society for Industrial and Applied Mathematics
- Tosa East Towne Neighborhood Association (Wauwatosa, WI)
- Milwaukee Graduate Assistant Association
- Urban Ecology Center (Milwaukee, WI)
- Milwaukee Art Museum

HONORS

- Full tuition teaching assistantship, University of Wisconsin-Milwaukee
- Full tuition research assistantship, University of Wisconsin-Milwaukee
- Summer Research Excellence Award, University of Wisconsin-Milwaukee
- University of Wisconsin-Milwaukee Chancellor's Fellowship Award
- Full tuition teaching assistantship, Portland State University

INTERESTS

I enjoy hiking, biking, camping, reading, movies, spending time with my wife and son, and cooking as well as following spectator sports especially football.