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The L^2-Cohomology of Discrete Groups

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by

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ABSTRACT

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Given a space with a proper, cocompact group action, the L^2-cohomology groups are a particularly interesting invariant that incorporates the topology of the space and the geometry of the group action. We are interested in both the algebraic and geometric aspects of these invariants. From the algebraic side, the Strong Atiyah Conjecture claims that the L^2-Betti numbers assume only rational values, with certain prescribed denominators related to the torsion subgroups of the group. We prove this conjecture for the class of virtually cocompact special groups. This implies the Zero Divisor Conjecture holds for such groups. On the geometric side, the Action Dimension Conjecture claims that a group with that acts properly on a contractible n-manifold has vanishing L^2-Betti numbers for $i > n/2$. We will prove this conjecture for many classes of right-angled Artin groups, and all Coxeter groups for $n \leq 4$.

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Chapter 1

Introduction

In the 1970's, Atiyah introduced L^2-methods into topology. The general philosophy behind this theory is to 'lift' classical homological invariants of compact spaces to the universal cover and build a richer theory by incorporating the fundamental group action. Indeed, Atiyah's motivation was to generalize the Atiyah-Singer Index Theorem to the noncompact setting. In this thesis, we are mainly concerned with one particular aspect of this theory–L^2-cohomology.

L^2-cohomology can be defined for any CW-complex with a proper and cocompact action by an infinite discrete group. Heuristically, it comes from restricting to the subspace of cochains which are square summable. Since this set of cochains forms a Hilbert space, we can apply methods from functional analysis, in particular, the theory of *von Neumann algebras*. In particular, the Hilbert space structure combined with the group action allows us to put a dimension on these cohomology groups, the *L^2-Betti numbers*.

Atiyah first defined L^2-Betti numbers as invariants of a Riemannian manifold M. They were defined analytically as traces of the heat kernel on M, and measure the size of the spaces of square-integrable harmonic forms of M. Later, Dodziuk gave a combinatorial definition of L^2-Betti numbers, and showed the equivalence between his and Atiyah's definition. In this thesis, we shall only consider the combinatorial point of view, though some of our results rely implicitly on analytic methods, Indeed, it is the combination of analytic and combinatorial definitions that makes

L^2-cohomology such a useful tool. We will give an introduction to L^2-cohomology in Section 2, mainly following the excellent survey of Eckmann [22].

While L^2-cohomology gives interesting results in its own right, it also importantly provides information about classical homological invariants. For example, L^2-Betti numbers also compute the Euler characteristic, and have stronger vanishing theorems than the compact setting. In fact, the following conjecture is still open:

Singer Conjecture. *If M^n is a closed aspherical manifold then the L^2-Betti numbers of the universal cover $b_i^{(2)}(\widetilde{M^n}, \pi_1(M^n))$ vanish for $i \neq n/2$.*

The Singer Conjecture is strictly stronger than the Hopf Conjecture, which claims that the Euler characteristic of a closed, aspherical $2n$-manifold has sign $(-1)^n$. The Singer Conjecture has been verified in many interesting cases, including Kähler hyperbolic manifolds [28], certain right-angled Coxeter groups [19], and manifolds with amenable fundamental group [14].

In [6], Bestvina, Kapovich and Kleiner introduced the concept of *action dimension* for discrete groups. Here, $\mathrm{actdim}(G)$ is the minimal dimension of a contractible manifold that G acts on properly discontinuously. This can be thought of as a manifold version of the geometric dimension of G.

Our main interest in action dimension is the following conjecture made by Davis and Okun. Here $b_i^{(2)}(G)$ is the L^2-Betti numbers of any contractible G-space.

Action Dimension Conjecture. $b_i^{(2)}(G) = 0$ *for* $i > \mathrm{actdim}(G)/2$.

Applying an L^2-version of Poincaré duality shows that this conjecture implies the Singer conjecture. Some interesting groups for which the conjecture holds is lattices in symmetric spaces [7], [5], mapping class groups of surfaces [20], and $Out(F_n)$ [6].

In this thesis, we will go over joint work with Avramidi, Davis and Okun on the action dimension of right-angled Artin groups in [3]. We were able to prove the Action Dimension Conjecture in most cases. We will also go over joint work with Okun in [45], where we proved the conjecture for Coxeter groups in dimensions ≤ 4.

The algebraic properties of L^2-Betti numbers are also very interesting. By their definition, L^2-Betti numbers are naturally connected to operator theory. One difficulty of L^2-cohomology is that the cohomology groups are generally infinite-dimensional (and hence indistinguishable) if nonzero. To properly assign a finite L^2-Betti number to these spaces requires the theory of group von Neumann algebras. While effective, this makes computing the precise values of L^2-Betti numbers a difficult problem. In fact, though the L^2-Betti numbers are a priori real, it was only shown recently that irrational values occur, and each example of this involves groups with unbounded torsion subgroups [4].

An early conjecture of Atiyah, strengthened by Linnell, Lück, and Schick, restricts the possible values that $b_i^{(2)}(X,G)$ can assume.

Strong Atiyah Conjecture. *Suppose G is a group with bounded torsion, and let* $\mathrm{lcm}(G)$ *be the least common multiple of the orders of the finite subgroups of G. If X is a CW-complex with proper and cocompact G-action, then $b_i^{(2)}(X,G) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.*

In particular, if G is torsion free, then $b_i^{(2)}(X,G)$ are conjecturally integers. A simple argument shows that the Strong Atiyah Conjecture implies the Zero Divisor Conjecture, which claims the integral group ring of a torsion-free groups is a domain. The conjecture was previously known for all elementary amenable groups [36], residually torsion-free elementary amenable groups [47], and right-angled Artin and Coxeter groups [37]. In this thesis, we will show the Strong Atiyah Conjecture holds for all virtually cocompact special groups, a class of groups introduced by Haglund and Wise in [31] which has gained recent prominence in geometric group theory.

This thesis is structured as follows. In Chapter 2, we give a quick introduction to L^2-cohomology. In Chapter 3 we introduce the main classes of groups that we are interested in: Artin groups, Coxeter groups, and special groups. In Chapter 4, we will prove the Strong Atiyah Conjecture for virtually cocompact special groups. We will also prove some new groups where the conjecture holds, including some examples of Coxeter groups and subgroups of limit groups. We will also show that torsion-free virtually cocompact special groups are residually torsion-free elementary amenable, answering a question of Aschenbrenner, Friedl, and Wilton in their survey article. In Chapter 5 we will go over our joint work with Okun and Avramidi-Davis-Okun, on the action dimension of right-angled Artin groups and Coxeter groups.

Chapter 2

Introduction to L^2-cohomology

Throughout this thesis, G will denote a countable, discrete group.

Definition. A CW-complex M is a *G-complex* if G acts on M cellularly, properly discontinuously, and cocompactly. If M is a G-complex and N is a G-stable subcomplex, then (M, N) is a *pair of G-complexes.*

Since G is discrete and M is a CW-complex, properly discontinuous just means that the stabilizer of each point is finite.

The most important example for us is when M is a compact manifold with a CW-complex structure, so that the universal cover $\tilde{M}$ is a $\pi_1(M)$-complex. Of course in this case the action is free.

Definition. Let $L^2(G)$ denote the space of square-summable functions from G to $\mathbb{R}$, i.e.

$$L^2(G) = \{f : G \to \mathbb{R} | \sum_{g \in G} |f(g)|^2 < \infty.\}$$

$L^2(G)$ is a Hilbert space with inner product given by

$$< f_1, f_2 >= \sum_{g \in G} f_1(g) f_2(g)$$

.

Definition. Let M be a G-complex, and let $C_*(M)$ denote the usual real-valued cellular chains of M. The G-action on M induces a natural $\mathbb{Z}G$-module structure on $C_*(M)$. The *square-summable chains of* M are the tensor product

$$C_*^{(2)}(M) = L^2(G) \otimes_{\mathbb{Z}G} C_*(M).$$

The usual boundary and coboundary homomorphisms extend to boundary and coboundary operators $\partial : C_*^{(2)}(M) \to C_{*-1}^{(2)}(M)$ and $\delta : C_*^{(2)}(M) \to C_{*+1}^{(2)}(M)$.

Lemma 2.0.1. *∂ and δ are bounded G-equivariant operators that are adjoint, i.e. $< f, \partial_{i+1} g > = < \delta_i f, g >$ for all $f \in C_i^{(2)}(M)$ and $g \in C_{i+1}^{(2)}(M)$.*

Proof. Both ∂ and δ can be realized as finite matrices with coefficients in $\mathbb{R}G$. An easy computation shows that the norm of such operators is bounded by the largest norm of the group ring elements in this matrix. The fact that these operators follows directly from the finite dimension case, where the matrices are transposes of one another.

$\square$

Definition. The L^2-(co)homology groups can be defined as the kernel of the *combinatorial Laplacian*:

$$L^2H_*(X, G) \cong L^2H^*(X, G) \cong \ker(\partial\delta + \delta\partial) : C_*^{(2)}(X) \to C_*^{(2)}(X).$$

Note that $L^2H_*(X, G)$ can be identified isometrically with a G-stable Hilbert subspace of $L^2(G)^n$. We say that $L^2H_*(X, G)$ is a *Hilbert G-module.* An example of a 1-cycle is shown in Figure 2.

Remark. One would think that the L^2-homology groups of X would be defined as $\ker(\partial_i)/\operatorname{Im}(\partial_{i+1})$ (and similarly for cohomology). The difficulty here is that the image of an operator may not be a closed subspace, so the above is not a Hilbert space. These are called the *unreduced L^2-homology groups of X*, and turn out to be useful objects, but we will not mention much of them, except for a few examples.

The *reduced L^2-homology groups of X* are defined similarly, but now we quotient out by the closure of the image group:

$$L^2H_i^{red}(X, G) = \ker(\partial_i)/\overline{\operatorname{Im}(\partial_{i+1})}.$$

The *reduced L^2-cohomology groups of X*, $L^2H^i_{red}(X, G)$ are defined similarly.

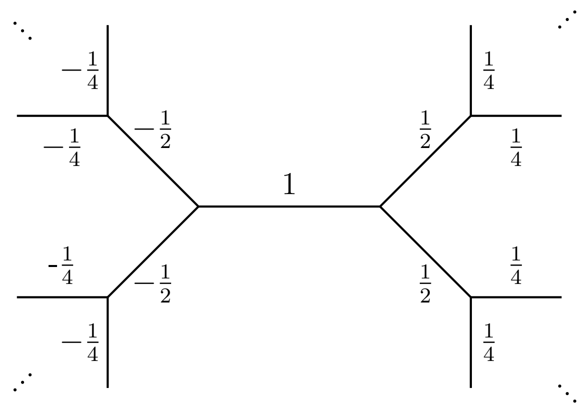


Figure 2.1: An example of an square summable 1-cycle. Note that this cycle is not summable.

Lemma 2.0.2. *$L^2H_*^{red}(X, G) \cong L^2H^*_{red}(X, G) \cong L^2H_*(X, G)$.*

Proof. The proof follows immediately from an L^2-Hodge decomposition. Since ∂ and δ are adjoint, by definition we have

$$< x, \partial_i y > = < \delta_{i-1}x, y >,$$

so that $\ker \delta_i = (\overline{\operatorname{Im} \partial_{i+1}})^\perp$ and $\ker \partial_i = (\overline{\operatorname{Im} \delta_{i+1}})^\perp$. Since $< \delta_{i-1}x, \partial_i y > = 0$, we have $C_i(X) =$

$$\overline{\operatorname{Im}(\delta_{i-1})} \perp \ker(\partial_i) = \overline{\operatorname{Im}(\partial_i)} \perp \ker(\delta_{i-1}) = \overline{\operatorname{Im}(\partial_i)} \perp \overline{\operatorname{Im}(\delta_{i-1})} \perp (\ker(\partial_i) \cap \ker(\delta_{i-1}).$$

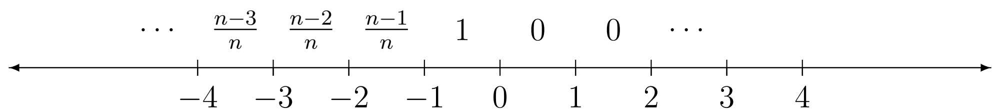
□

Example. Let $X = \mathbb{R}$ with the cellulation by unit intervals and $G = \mathbb{Z}$. It is easy to see that $\partial_1(f) = 0$ if and only if f is constant; therefore $L^2H_1(\mathbb{R}, \mathbb{Z}) = 0$. In fact, $\partial_1(f) = (x - 1)f$, where x is a generator of $\mathbb{Z}$. The image of ∂_1 is obviously contained in (and in fact equal to) the subspace

$$\{f \in C_0^{(2)}(\mathbb{R}) | \sum_n f(n) = 0.\}$$

. This subspace is not closed- for example the sequence $f_n = \chi_0 - \frac{1}{n}\chi_{[1,n]}$ leaves the subspace, where χ_A is the characteristic function. Therefore, the zeroth unreduced

L^2-homology groups are nonzero. Since $\delta_0(f) = 0$ if and only if f is constant, $L^2H^0(\mathbb{R}, \mathbb{Z}) = 0$. In this case it is easy to construct images $\partial_1(g)$ that are arbitrarily close to any $f \in C_0(\mathbb{R})$. For example, χ_0 is approximated by the sequence of functions:



For (M, N) be a pair of G-complexes, the relative L^2-homology $L^2H_*(M, N)$ can be defined analogously.

We record as a lemma some of the basic algebraic properties of L^2-homology that we will need. Since images of maps between Hilbert spaces are rarely closed, we define a *weakly exact sequence* where kernels are equal to the closures of images.

Lemma 2.0.3. *Let (M,N) be a pair of G-complexes.*

- *(Functoriality) If (M_1, N_1) and (M_2, N_2) are pairs of G-spaces and $f : (M_1, N_1) \to (M_2, N_2)$ is a G-equivariant map, then there is an induced map $f_* : L^2H_k(M_1, N_1) \to L^2H_k(M_2, N_2)$. If f is a G-homotopy equivalence, then f_* is an isomorphism.*

- *(Exact sequence of a pair) The sequence*

$$\cdots \to L^2H_i(N) \to L^2H_i(M) \to L^2H_i(M, N) \to \ldots$$

  is weakly exact.

- *(Induction principle) The L^2-homology of M is induced from the L^2-homology of its components:*

$$L^2H_i(M; G) = \bigoplus_{[M^0] \in \pi_0(M)/G} L^2H_i(M^0, \operatorname{St}_G M^0) \nearrow G,$$

  where the sum is over representatives of the orbits of the components of M.

- *(Mayer–Vietoris sequences) Suppose $M = M_1 \cup_{M_0} M_2$ and (M, M_i) is a pair of G-spaces for $i = 1, 2$. Then (M, M_0) is a pair of G-spaces and the sequence*

$$\cdots \to L^2H_i(M_0) \to L^2H_i(M_1) \oplus L^2H_i(M_2) \to L^2H_i(M) \to \ldots$$

  is weakly exact.

- *(Excision) Suppose that M and N are a pair of G-complexes and U is a G-stable subset of Y with $Y - U$ a subcomplex. Then the inclusion $(X - U, Y - U) \to (X, Y)$ induced an isomorphism $L^2H_i(X - U, Y - U) \to L^2H_i(X, Y)$.*

- *(Poincaré Duality) If M is a manifold then $L^2H^i(M) \cong L^2H_{n-i}(M, \partial M)$ and $L^2H_i(M) \cong L^2H^{n-i}(M, \partial M)$*

Example. Suppose X is an infinite G-complex. The same argument as for $(\mathbb{R}, \mathbb{Z})$ shows that $L^2H_0(X, G) = 0$.

Example. By applying Poincaré duality, we see that if M^n is a G-manifold, then $L^2H_n(M^n, G) = 0$. For instance, this implies that the L^2-homology of $\mathbb{H}^2$ with a surface group action is concentrated in dimension one.

Definition. For a discrete group $G, L^2H_i(G)$ is the L^2-homology of a contractible G-space. By the functoriality property, this is well-defined.

2.1 L^2-Betti numbers

The main feature of L^2-cohomology is the notion of an L^2-Betti number. It is remarkable that in this infinite dimensional setting, we can measure the size of these cohomology groups. As we shall see, the cocompact group action allows us to do this: in some sense reducing the problem to finite dimensions.

Definition. A Hilbert space V with isometric G-action is a *Hilbert G-module* if it is isomorphic to a closed, G-stable subspace of $L^2(G)^n$, for some $n \in \mathbb{N}$.

Definition. The *von Neumann algebra* $\mathcal{N}(G)$ is the algebra of all bounded G-equivariant linear endomorphisms from $L^2(G)$ to itself.

Remark. Given a general Hilbert space H, there is the notion of a von Neumann algebra, which is any subspace of bounded operators on H that is closed in the weak topology. It is not hard to check that our group von Neumann algebra is a von Neumann algebra in this sense.

Certainly, $\mathbb{R}G \subset \mathcal{N}(G)$ since $L^2(G)$ is an $\mathbb{R}G$-module. One can also see that $\mathcal{N}(G) \subset L^2(G)$ by mapping $\phi \in \mathcal{N}(G)$ to $\phi(1)$. An equivalent definition of $\mathcal{N}(G)$ is the *weak closure* of $\mathbb{R}G \subset L^2(G)$. Most important for us is that ∂, δ and Δ are in $\mathcal{N}(G)$.

Definition. The *von Neumann trace* is the linear functional $\mathrm{tr}_G : \mathcal{N}(G) \to \mathbb{R}$ which sends $\phi \in \mathcal{N}(G)$ to $< \phi(1), 1 >$.

Remark. If G is finite, or if we restrict to the subspace $\mathbb{R}G$, the trace is the standard Kaplansky trace.

Suppose now that ϕ is a bounded G-equivariant linear endomorphism from $L^2(G)^n$ to itself. Then ϕ can be represented as an $n \times n$ matrix with coefficients in $\mathcal{N}(G)$. Define $\mathrm{tr}(\phi)$ to be the sum:

$$\mathrm{tr}(\phi) = \sum_{i=1}^{n} \mathrm{tr}(\phi_{i,i}).$$

Let V be a Hilbert G-module, and let $p_V : L^2(G)^n \to L^2(G)^n$ be the orthogonal projection onto V.

Lemma 2.1.1. *For any Hilbert G-module V, p_V is in $\mathcal{N}(G)$.*

Proof. It is obvious that p_V is a bounded operator, in fact p_V is norm decreasing. Let $x \in L^2(G)^n$. Then x decomposes as $p_V(x) + (x - p_V(x))$. Now, for any $g \in G$, we have $gx = p_V(gx) + (gx - p_V(gx))$ and $gx = gp_V(x) + (gx - gp_V(x))$.

Since these are orthogonal decompositions, we have $p_V(gx) = gp_V(x)$, so $p_V \in \mathcal{N}(G)$.

□

Definition. The *von Neumann dimension* of V is defined to be the trace of p_V:

$$\dim_G(V) = \mathrm{tr}_G(p_V).$$

Lemma 2.1.2. $\mathrm{tr}_G(p_A p_B) = \mathrm{tr}_G(p_B p_A)$

Proof. □

Lemma 2.1.3. *The von Neumann dimension of V is well-defined, i.e. does not depend on choice of embedding $V \to L^2(G)^n$.*

Proof. Suppose we have two embeddings $\phi_1 : V \to L^2(G)^n$ and $\phi_2 : V \to L^2(G)^m$. If $n > m$, then we can extend ϕ_2 to an embedding $V \to L^2(G)^m \oplus L^2(G)^{n-m} \cong L^2(G)^n$. Obviously, this does not change the trace, so we can assume $n = m$. Let $h = \phi_1 \circ \phi_2^{-1} : \phi_2(V) \to \phi_1(V)$, and extend h to an operator H in $M_n(\mathcal{N}(G))$ by setting $H(\phi_2(V)^\perp) = 0$. By construction, HH^* is projection onto $\phi_2(V)$ and H^*H is projection onto $\phi_1(V)$. Therefore, we are done since $\mathrm{tr}_G(HH^*) = \mathrm{tr}_G(H^*H)$. □

Remark. In $\mathbb{R}^n$, we can define the dimension of a subspace in a similar way. Namely, if $V \subset \mathbb{R}^n$ is a subspace, let p_V denote the orthogonal projection onto V. Viewing p_V as a matrix, we see that the usual dimension of V is equal to the trace of p_V. To see this, just choose a basis of V and extend to an orthogonal basis of R^n, and use the fact that $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$.

Lemma 2.1.4. *We list some properties of* $\dim_G(V)$.

1. $\dim_G(V) \in [0, \infty)$
2. $\dim_G(V) = 0$ *if and only if* $V = 0$.
3. $\dim_G(L^2(G)) = 1$
4. $\dim_G(V_1 \oplus V_2) = \dim_G(V_1) + \dim_G(V_2)$.
5. *If $f : V \to W$ is a map of Hilbert G-modules, then* $\dim_G(V) = \dim_G(\ker f) + \dim_G(\overline{\mathrm{Im}\, f})$.

6. *If H is a subgroup of finite index m in G, then* $\dim_H(V) = m \dim_G(V)$*. In particular, if G is finite then* $\dim_G(V) = \frac{1}{|G|} \dim V$*.*

7. *If H is a subgroup of G and W is a Hilbert H-module, then* $\dim_G(\mathrm{Ind}_H^G(W)) = \dim_H(W)$*.*

Proof. We shall prove/give ideas for some of the statements, and refer to [22] or [39] for the complete proofs.

1. Since p_V is a projection:

$$\mathrm{tr}_G(p_V) = < p_V(1), 1 > = < p_V \circ p_V(1), 1 > = < p_V(1), p_V(1) > .$$

2. Since p_V is G-equivariant, $\mathrm{tr}_G(p_V) = 0 \iff p_V = 0 \iff V = 0$.

3. $p_V = 1$.

4. Embed $V_1 \oplus V_2$ into $(L^2(G))^n \oplus (L^2(G))^m$, and consider $p_{V_1} \oplus p_{V_2}$.

5. $V/\ker(f) \cong \overline{f(V)}$, which implies $V \cong \ker(f) \oplus \overline{\mathrm{Im}(f)}$.

6. Since $G = \cup_{i=1}^{[G:H]} H.x_i$, we have $L^2(G) \cong L^2(H)^{[G:H]}$. For any $F \in \mathcal{N}(G)$, we have

$$\mathrm{tr}_H(F) = \sum_{i=1}^{[G:H]} < F(x_i), x_i > = < F(1), 1) > = [G : H] \, \mathrm{tr}_G(F).$$

7. See [39]

$\square$

Example ([39])**.** Let $G = \mathbb{Z}$, and let $L^2(\mathcal{T})$ be the Hilbert space of equivalence classes of L^2-integrable complex-valued functions on the circle, where two such functions are equivalent if they differ only on a subset of measure zero. Define the Banach space $L^\infty(S^1)$ by equivalence classes of essentially bounded measurable functions $f : S^1 \to \mathbb{C}$, where essentially bounded means that there is a constant

$C > 0$ such that the set $\{x \in S^1 || f(x)| > C\}$ has measure zero. An element k in $\mathbb{Z}$ acts isometrically on $L^2(S^1)$ by pointwise multiplication with the function $S^1 \to \mathbb{C}$ which maps $z \to z^k$. Fourier transform yields an isometric $\mathbb{Z}$-equivariant isomorphism $L^2(\mathbb{Z}) \to L^2(S^1)$. To be specific, the map is

$$\sum_{n=-\infty}^{n=\infty} a_n \in L^2(\mathbb{Z}) \to \sum_{n=-\infty}^{n=\infty} a_n e^{-2\pi i \omega n} \in L^2(S^1).$$

Hence $\mathcal{N}(\mathbb{Z}) \cong \mathcal{B}(L^2(S^1))$. We obtain an isomorphism

$$L^\infty(S^1) \cong \mathcal{N}(\mathbb{Z}).$$

by sending $f \in L^\infty(S^1)$ to the $\mathbb{Z}$-equivariant operator $M_f : L^2(S^1) \to L^2(S^1), g \to gf$ where $gf(x)$ is defined by $g(x)f(x)$. Under this identification the trace becomes

$$\mathrm{tr}_{\mathcal{N}(\mathbb{Z})} : L^\infty(S^1) \to \mathbb{C}, f \to \int_{S^1} f d\mu.$$

Definition. If X is a G-complex, the $i^{th} L^2$-Betti number $b_i^{(2)}(X, G)$ is defined to be $\dim_G L^2 H_i(Y, G)$.

Lemma 2.1.5. *Most of the properties of L^2-Betti numbers that we need follow from Lemma 2.1.4. We list a couple more that come from L^2-homology.*

- *(Künneth formula) If $G = G_1 \times G_2$ and for $j = 1, 2$, X_j is a geometric G_j-complex, then*

$$b_i^{(2)}(X_1 \times X_2, G) = \sum_{i+j=k} b_i^{(2)}(X_1, G_1) b_i^{(2)}(X_2, G_2).$$

- *(Poincaré Duality) If G acts properly and cocompactly on an n-manifold M, then $b_i^{(2)}(M, G) \cong b_{n-i}^{(2)}(M, G)$.*

Lemma 2.1.6. *(Atiyah's Formula) $\chi^{\mathrm{orb}}(X/G) = \sum_{i=0}^{\dim X} (-1)^i b_i^{(2)}(X, G)$.*

Proof. We will prove the conjecture in the torsion-free case as the torsion case is very similar. Consider the L^2-chain complex $C_*^{(2)}(X) = L^2(G) \otimes_{\mathbb{Z}G} C_*(X)$. Since $C_*^{(2)}(X) \cong L^2(G)_i^\alpha$, where α_i is the number of i-cells of X, we have

$$\sum_i (-1)^i C_i^{(2)}(X) = \sum_i (-1)^i \alpha_i = \chi(X).$$

On the other hand,

$$\sum_i (-1)^i C_i^{(2)}(X) = \sum_i (-1)^i \dim_G \mathcal{H}_i(X),$$

which proves the claim.

□

For those who wish to not think of the previous definition of L^2-Betti numbers, we offer the following fundamental theorem of Lück as an alternative.

Theorem 2.1.7. *Let X be a finite CW-complex with $\pi_1(X) = G$. Suppose $G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \ldots$ be a sequence of finite index normal subgroups terminating at the identiy. The L^2-Betti numbers of $\widetilde{X}$ have the following description:*

$$b_i^2(\tilde{X}, G) = \lim_{i \to \infty} \frac{b_i(\widetilde{X}/G_j)}{[G : G_j]}$$

2.2 The Atiyah and Singer Conjectures

The conjectures in this section are the most well-known in the field.

Strong Atiyah Conjecture. *Let $\operatorname{lcm}(G)$ denote the least common multiple of the orders of finite subgroups of G. Suppose G is a discrete group with $\operatorname{lcm}(G) < \infty$. If $A \in M_n(\mathcal{N}(G))$, then*

$$\dim_G(\ker A) \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}.$$

Recall that the Zero Divisor Conjecture predicts that if G is a torsion-free group then $\mathbb{Z}G$ is an integral domain. In general, this conjecture is still open. The Strong Atiyah Conjecture turns out to be stronger.

Lemma 2.2.1. *The Strong Atiyah Conjecture implies the Zero Divisor Conjecture.*

Proof. Let $a \in \mathbb{Z}G$, and consider the induced operator $a^* \in \mathcal{N}(G)$ which sends f to af. The Strong Atiyah Conjecture predicts that $\dim_G(\ker a^*)$ is 0 or 1. This implies that $\ker a^*$ is either trivial or all of $L^2(G)$. Therefore if a is nonzero then it is not a zero divisor. □

It turns out that the conjectures are equivalent for torsion-free amenable groups. For groups with unbounded torsion, there have been recently constructed counterexamples to weaker versions of the Strong Atiyah Conjecture. First, Grigorchuk, Linnell, Schick, and Zuk showed that a certain lamplighter group with orders of torsion subgroups only powers of 2 could have a L^2-Betti number of $1/3$. Then more recently, Austin produced irrational examples, using a different lamplighter group.

The first major positive result on the Strong Atiyah Conjecture was due to Linnell. Recall that the class of elementary amenable groups is the smallest class which contains all finite and abelian groups, and is closed under taking subgroups, extensions, quotients, and directed unions. For example. solvable groups are elementary amenable, and elementary amenable groups are amenable.

Theorem 2.2.2 ([36]). *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under taking directed unions and extensions with elementary amenable quotients. If $G \in \mathcal{C}$ and $\mathrm{lcm}(G) < \infty$, then the Strong Atiyah Conjecture holds for G.*

For example, the fundamental group of every closed surface is in $\mathcal{C}$, as each of these groups have free commutator. Using some approximation results for L^2-Betti numbers and Linnell's theorem, Schick proved the following. Recall that if $\mathcal{P}$ is a group property, then a group G is residually $\mathcal{P}$ if for every $g \in G$ there is a map $f : G \to M$ such that M satisfies $\mathcal{P}$ and $f(g) \neq 0$.

Theorem 2.2.3 ([47]). *The Strong Atiyah Conjecture holds for all residually torsion-free elementary amenable groups with finite* lcm. *For example, this includes all right-angled Artin groups.*

Another classical conjecture claims that in the manifold setting, that L^2-Betti numbers are essentially trivial invariants.

Singer Conjecture. *If a group G acts properly and cocompactly on a contractible n-manifold $\widetilde{M}^n$, then*

$$L^2H_i(\widetilde{M}^n, G) = 0, \forall i \neq n/2.$$

It seems this is slightly stronger than the classical Singer conjecture, which required G to be torsion-free. If G is virtually torsion-free the conjectures are equivalent by the multiplicativity of L^2-Betti numbers.

The Hopf Conjecture predicts that the Euler characteristic of a closed, aspherical $2n$-manifold has sign $(-1)^n$. Using Atiyah's formula, it is immediate that the Singer Conjecture implies the Hopf Conjecture.

In [5], Borel showed that lattices in Lie groups satisfied the Singer Conjecture. Previously, Dodziuk had shown this for any model space, in particular n-dimensional hyperbolic space $\mathbb{H}^n$.

Theorem 2.2.4. *Let G be a connected semisimple Lie group and Γ a lattice in G. The L^2-Betti numbers of Γ vanish outside the middle dimension.*

One of the strongest vanishing theorems was proven by Cheeger and Gromov in [14].

Theorem 2.2.5. *Suppose that G contains an infinite, normal, amenable subgroup. Then $b_i^{(2)}(G) = 0$ for all i.*

Finally, Gromov proved the Singer conjecture for all Kähler hyperbolic manifolds.

Theorem 2.2.6 ([28])**.** *Fundamental groups of Kähler hyperbolic manifolds satisfy the Singer Conjecture.*

Chapter 3

Coxeter groups and Artin Groups

We will now review the main classes of groups that we will study. We begin with Coxeter groups.

Definition. A *Coxeter group* is given by the following presentation:

$$< s_1, s_2, \ldots, s_n | (s_i s_j)^{m_{ij}} = 1 >$$

where $m_{ij} \in \mathbb{N} \cup \infty$ and $m_{ii} = 1$.

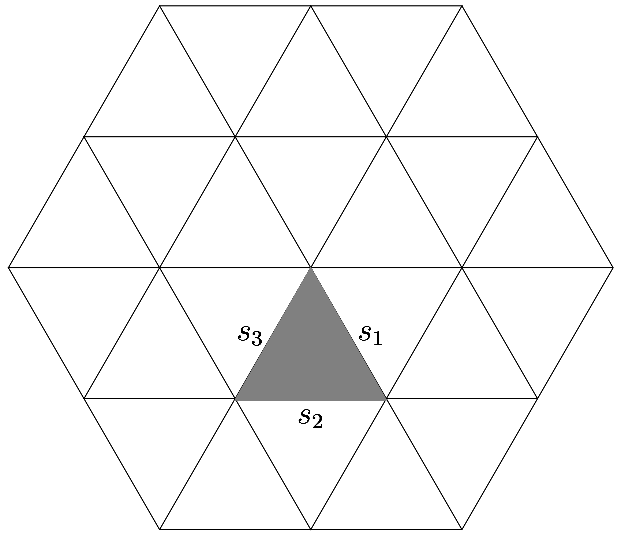
In other words, W is generated by reflections and any two reflections generate a (possible infinite) dihedral group.

Example. Suppose T is an equilateral triangle in $\mathbb{R}^2$, as in the above figure. The group generated by reflections in the hyperplanes corresponding to the sides is the Coxeter group

$$W =< s_1, s_2, s_3 | (s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_1)^3 = 1 > .$$

Definition. The *nerve* of a Coxeter group W is the simplicial complex L which has vertices corresponding to the generators s_i, and $s_{i_1}, s_{i_2}, \ldots, s_{i_n}$ form a simplex of L if and only if $s_{i_1}, s_{i_2}, \ldots, s_{i_n}$ generate a finite subgroup of W.

Definition. A simplicial complex is *flag* if there are no missing simplices, i.e. if the 1-skeleton of a simplex is in L, the simplex is. A subcomplex K of L is *full* if any simplex of L with vertices in K lies in K.



Example. A Coxeter group W is *right-angled* if $m_{ij} \in 2 \cup \infty$, in other words, two generators either commute or have no relation. In this case, the nerve is a flag complex. Conversely, any flag complex L determines a right-angled Coxeter group with nerve L by assigning generators to vertices and having two generators commute if and only if there is an edge between the corresponding vertices.

There are many classical examples of Coxeter groups. The most familiar are the reflection groups in $S^n, \mathbb{R}^n$, and $\mathbb{H}^n$. In these examples, the Coxeter group is the group generated by the set of reflections of faces in a convex polytope. On the other hand, given an arbitrary Coxeter group W, Mike Davis constructed a simplicial complex (the Davis complex) that admits a proper and cocompact W-action by reflections. We will now define the Davis complex, and show that it has many properties that make it a reasonable substitute for the spaces of constant (nonpositive) curvature.

Definition. A *mirror structure* on a space X is an index set S and a collection of subspaces $\{X_s\}_{s \in S}$. For each $x \in X$, let

$$S(x) := \{s \in S \mid x \in X_s\}.$$

An example to keep in mind is a convex polytope in $S^n, \mathbb{R}^n$ or $\mathbb{H}^n$ with mirrors the codimension-one faces. We will assume here that our index set S is finite.

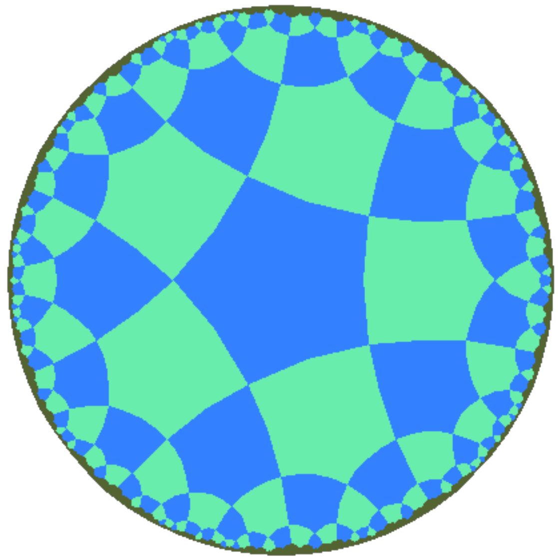


Figure 3.1: The Davis Complex for a right-angled pentagon

Definition. Let X have a mirror structure, and let (W, S) be a Coxeter group with generators $s \in S$. Let W_T denote the subgroup generated by $s \in T \subset S$. Let $\sim$ denote the following equivalence relation on $W \times X$: $(w_1, x) \sim (w_2, y)$ if and only if $x = y$ and $w_1 w_2^{-1} \in W_{S(x)}$. The *basic construction* is the space

$$\mathcal{U}(W, X) := W \times X / \sim .$$

$\mathcal{U}(W, X)$ is therefore constructed by gluing together copies of X along its mirrors. The exact gluing is dictated by the Coxeter group. A standard example is where X is a right-angled pentagon in $\mathbb{H}^2$ with mirrors the edges of X, and W is the right-angled Coxeter group generated by reflections in these edges. Then $\mathcal{U}(W, X) \cong \mathbb{H}^2$, see the above figure.

Definition. Let W be a Coxeter group with nerve L. Let K be the cone on the barycentric subdivision of L. K admits a natural mirror structure with K_s the closed star of the vertex corresponding to s in the barycentric subdivision of L. The *Davis complex* $\Sigma(W, S)$ is defined to be the simplicial complex $\mathcal{U}(W, K)$, see the above figure.

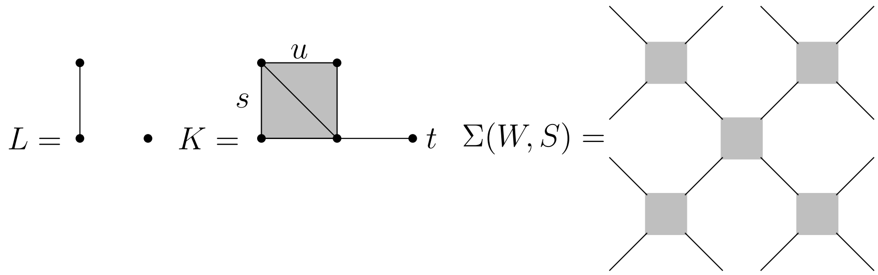


Figure 3.2: The Davis complex for the right-angled Coxeter group $W = D_\infty * \mathbb{Z}_2$.

Lemma 3.0.7 ([16])**.** *$\Sigma(W, S)$ has the following properties.*

- *W acts properly and cocompactly on $\Sigma(W, S)$ with fundamental domain K.*
- *Σ admits a cellulation such that the link of every vertex can be identified with L. Therefore, if L is a triangulation of S^{n-1}, then $\Sigma(W, S)$ is an n-manifold.*
- *$\Sigma(W, S)$ admits a piecewise Euclidean metric that is $CAT(0)$, in particular $\Sigma(W, S)$ is contractible*

We will talk more about CAT(0) metrics in the next section. The existence of a CAT(0) metric is due to Gromov in the right-angled case and to Moussong in general.

Example. Suppose S is a n-dimensional homology sphere that is not simply connected. It is known that S bounds a contractible manifold K. Given a flag triangulation of K such that S is triangulated as a full subcomplex, we can form the basic construction $\mathcal{U}(W, K)$ which is a contractible manifold. It turns out that $\mathcal{U}(W, K)$ is not simply connected at infinity, and therefore not homeomorphic to $\mathbb{R}^n$. Taking a finite index torsion-free subgroup of W, the quotient is one of Davis's famous examples of closed aspherical manifolds with exotic universal covers.

In particular, the Davis complex is contractible, so from this W-action we are able to compute $L^2H_i(W)$. The Singer Conjecture for right-angled Coxeter groups was first studied by Davis and Okun in [19]. Their work inspired much of the work in this thesis. Their main result is a verification of the Singer Conjecture in dimension 4.

Theorem 3.0.8 ([19]). *Let L be a flag triangulation of S^3, and W_L the associated right-angled Coxeter group. Then $b_i^{(2)}(W_L) = 0$ for $i \neq 2$.*

3.1 Cube complexes and right-angled Artin groups

Definition. A *cube complex* is a cell complex obtained by gluing cubes ($\cong [-1,1]^n$) together along their faces by isometries.

A cube complex inherits a natural piecewise Euclidean metric by setting each cube to be isometric to the unit cube in Euclidean space, and then taking the induced path metric.

We will now define a very important geometric concept for us: the notion of a CAT(0) metric space. This generlizes the concept of a non positively curved Riemannian manifolds. Heuristically, a geodesic metric space X is CAT(0) if triangles in X are thinner than triangles in $\mathbb{R}^2$. More precisely, for any $x, y, z \in X$, we can form a *comparison triangle* in $\mathbb{R}^2$ with the same side lengths. The CAT(0) condition is that for any w on the geodesic [yz], the distance from w to x is smaller than the distance between the corresponding points in the comparison triangle, as in the figure below. A geodesic metric space X is locally CAT(0) if the CAT(0) inequality holds in some neighborhood of each point of X. The Gromov version of the Cartan-Hadamard theorem states that the universal cover of a locally CAT(0) space is CAT(0).

Definition. A cube complex is *nonpositively curved* if the link of each vertex is a flag simplicial complex.

The next theorem implies that the combinatorial definition of nonpositive curvature makes sense.

Theorem 3.1.1 (Gromov). *Let K be a cube complex. If the link of each vertex in K is a flag simplicial complex, then the natural piecewise Euclidean metric on K is locally CAT(0).*

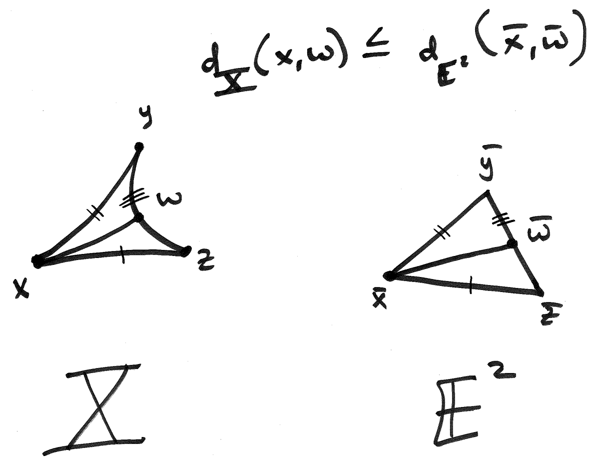


Figure 3.3: The CAT(0) inequality

Remark. In particular, this theorem applies to the Davis complex of a right-angled Coxeter group.

Remark. Given a discrete group Γ, it is in general quite hard to 'cubulate' Γ, i.e. to exhibit Γ as a proper group of isometries on a CAT(0) cube complex. The most important construction was proven by Sageev in his thesis, using the notion of codimension-one subgroups. For example, suppose W is a Coxeter group (cubulated by Niblo and Reeves in [44]). Every fixed set in Σ of a reflection in W separates Σ into two halfspaces. Sageev's idea is to construct a CAT(0) cube complex out of the hyperplane data. Heuristically, vertices are dual to hyperplanes, edges to hyperplane crossings, and higher dimensional cubes get automatically filled to ensure the CAT(0) condition. The construction has enough structure to ensure that W acts on a CAT(0) cube complex.

This next definition will be extremely important in the next section.

Definition. Let X and Y be nonpositively curved cube complexes. A function $f : X \to Y$ is a *local isometry* if for every $x \in X$, $f(\mathrm{Lk}_X(x)) \subset \mathrm{Lk}_Y(f(x))$ as a full subcomplex.

Though this definition is purely combinatorial, it turns out that it is equivalent to the more familiar definition for metric spaces if X and Y are endowed with the usual piecewise Euclidean metric.

Theorem 3.1.2. *Local isometries between nonpositively curved cube complexes are π_1-injective.*

Proof. Let $f : X \to Y$ be a local isometry, and let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be a lift to the universal cover. Since X is locally CAT(0) and complete, every class in $\pi_1(X)$ is represented by a unique local geodesic c, see [9]. Since f is a local isometry, the image $f \circ c$ is a local geodesic in Y of the same length as c, which implies $f \circ c$ is nontrivial in $\pi_1(Y)$ [9].

□

3.2 Right-Angled Artin Groups

Suppose L^1 is a simplicial graph with vertex set V. The *flag complex determined by* L^1 is the simplicial complex L whose simplices are the (vertex sets of) complete subgraphs of L^1. Heuristically, L is obtained by filling in all the "missing" simplifies of L. Associated to L there is a right-angled Artin group (abbreviated RAAG), A_L.

A set of generators for A_L is $\{g_v\}_{v \in V}$, and there are relations $[g_v, g_{v'}] = 1$ whenever $\{v, v'\} \in \operatorname{Edge} L^1$. Therefore, right-angled Artin groups are right-angled Coxeter groups without the idempotent relation.

Let T^V denote the product $(S^1)^V$. Each copy of S^1 is given a (cubical) cell structure with one vertex e_0 and one edge. For each simplex $\sigma \in L$, $T(\sigma)$ denotes the subset of T^V consisting of those points $(x_v)_{v \in V}$ such that $x_v = e_0$ whenever v is not a vertex of σ. So, $T(\sigma)$ is a standard subtorus of T^V; its dimension is $\dim \sigma + 1$. The *Salvetti complex* for A_L is the subcomplex X_L of T^V defined as the union of the subtori $T(\sigma)$ over all simplices σ in L:

$$\mathcal{S}_L := \bigcup_{\sigma \in L} T(\sigma).$$

The 2-skeleton of $\mathcal{S}(L)$ is the presentation complex for A_L; so, $\pi_1(Y_L) = A_L$. There is a natural cubical cell structure on $\mathcal{S}_L$ with a cube of dimension $\dim \sigma + 1$

Figure 3.4: A Salvetti complex

for each $\sigma \in L$. Y_L is a non positively curved cube complex as the link of each vertex is a flag complex. Right-angled Artin groups are one on the most important source of examples in geometric group theory; we refer to Charney's survey article for a wonderful introduction [12].

3.3 Special Groups

If K is a non-positively curved cube complex, Haglund and Wise in [31] made the remarkable observation that $\pi_1(K)$ injects into a right-angled Artin group A_Γ if and only if the hyperplanes of K avoid certain configurations, as shown in the figure below.

Definition. A cube complex with hyperplanes that avoid these configurations is *special*. A group H is then *special* if it is the fundamental group of a special cube complex K, and *compact special* if K is compact.

The only concept which perhaps is unclear from the picture is osculation. To describe this, it is better to identify a hyperplane H with the set of dual edges; since we are assuming H is two sided we can assume our edges have an orientation. Therefore, a hyperplane H self-osculates at v if and only if v is the beginning vertex in two dual edges e_1, e_2 such that e_1 and e_2 do not bound a square in the cube complex.

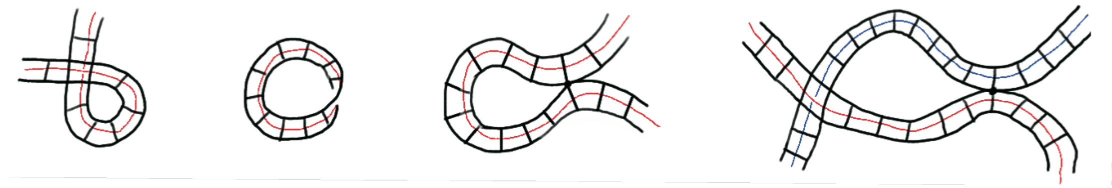


Figure 3.5: Configurations ist verboten. Hyperplanes in special cube complexes are *embedded, one-sided*, do not *self-osculate*, and do not *inter-osculate*

Lemma 3.3.1. *The Salvetti complex associated to any right-angled Artin group is a special cube complex.*

Proof. Let $\mathcal{S}_L$ be a Salvetti complex for A_L. Since $\mathcal{S}_L$ is composed of tori, there is a unique oriented edge dual to each hyperplane in $\mathcal{S}_L$. Therefore, hyperplanes in $\mathcal{S}_L$ are embedded, 2-sided, and do not self-osculate. If two hyperplanes in $\mathcal{S}_L$ intersect, then the corresponding edges a and b bound a square in $\mathcal{S}_L$, which contradicts the hyperplanes osculating. □

Surprisingly, Haglund and Wise's result is not very difficult to prove, so we will reproduce their argument here:

Theorem 3.3.2. *X is a special cube complex with hyperplanes if and only if $\pi_1(X)$ is a subgroup of some right-angled Artin group.*

Proof. If $\pi_1(X)$ is a subgroup of some right-angled Artin group, it corresponds to some cubical cover of a Salvetti complex. Therefore, this direction holds by the observation that the forbidden configurations are preserved by local isometries and that the Salvetti complex is special.

For the other direction, there is a graph Γ_X which has vertices corresponding to the hyperplanes of X, and edges between vertices if the hyperplanes intersect. There is a natural map from X to the Salvetti complex associated to the group A_{Γ_X}. Indeed, each n-cube in X has n-hyperplanes passing through it; since these all intersect there is a corresponding n-torus in Y_{Γ_X}, so we map the cube to the torus. Note that is well-defined because hyperplanes embed and are two sided in X. Avoiding the forbidden configurations precisely means that this map is a local

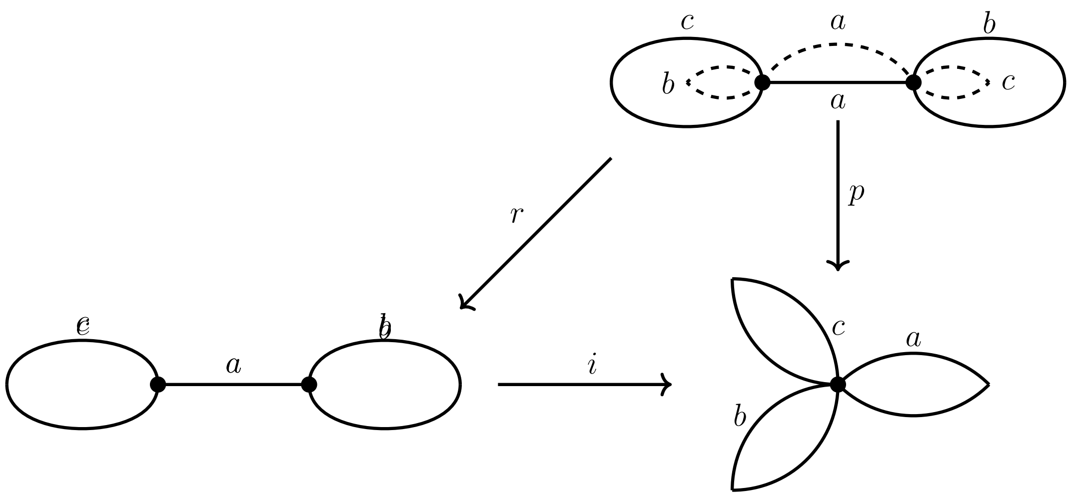


Figure 3.6: For any local isometry from a finite graph to a wedge of circles, there is a finite cover of the wedge that retracts onto the graph. The retraction is induced by collapsing the dotted lines.

isometry, and hence an embedding on π_1. To be more precise, a hyperplane self-osculates at a vertex v if and only if the the induced map on $\mathrm{Lk}_X(v)$ is not injective. Two hyperplanes intersect and osculate at v if and only if the image of $\mathrm{Lk}_X(v)$ in $\mathcal{S}_{\Gamma_X}$ is not a full subcomplex.

□

The next theorem is an immediate consequence of [31, Proposition 6.5]. It is the main property that distinguishes compact special groups from merely special groups.

Theorem 3.3.3. *Let H be a compact special group, and $i : H \to A_\Gamma$ the embedding constructed by Haglund and Wise. Then there is a finite index subgroup K of A_Γ which retracts onto H.*

The above figure indicates why the theorem holds for all finitely generated free groups, and is due to Stallings [51]: the general case is somewhat more complicated.

Chapter 4

The Strong Atiyah Conjecture

After a long setup, we can finally earn our small paycheck and prove something. Suppose we know the Strong Atiyah Conjecture holds for a group G. A natural question is whether the conjecture passes to subgroups of G, or lifts to finite extensions. If G is torsion-free and satisfies the conjecture, then by the induction principle in Lemma 2.0.3 every subgroup satisfies the conjecture as well. If G has torsion, not much is known: the problem is that the subgroup may have less torsion than G, and hence the conjecture predicts something stronger for the subgroup. A similar problem occurs for finite extensions. Suppose the conjecture holds for H and we have a long exact sequence

$$1 \to H \to G \to Q \to 1,$$

where Q is finite. If $\operatorname{lcm}(G) = \operatorname{lcm}(H)|Q|$, then we are done by the multiplicativity of L^2-Betti numbers. In general, G will have less torsion and we are stuck.

Let us give a small example to show the importance of these finite extensions. Suppose we have a right-angled Coxeter group W_L where L is n-dimensional. Let $|S|$ denote the number of generators of W_L. The commutator subgroup C_L is finite index in W_L of index $2^{|S|}$. Since C_L is torsion-free, the conjecture would imply that $b_i(Y, C_L) \in \mathbb{Z}$. On the other hand, if we know the conjecture for W_L, then $b_i(Y, C_L) \in \frac{2^{|S|}}{2^{n+1}}\mathbb{Z}$, a significant upgrade.

Linnell and Schick gave the first conditions on a group that guaranteed the Strong Atiyah Conjecture lifts to finite extensions. They proved that if a group H has a finite $K(H,1)$, has *enough torsion-free quotients*, and is *cohomologically complete*, then the Strong Atiyah Conjecture holds for any group G that fits into the exact sequence

$$1 \to H \to G \to Q \to 1.$$

Loosely speaking, having enough torsion-free quotients says that every map from H to a p-group should factor through a torsion-free elementary amenable group, while being cohomologically complete says that H should have the same cohomology as any pro-p completion.

In the next couple of sections, we will give a "profinite" version of Linnell and Schick's theorem, which most notably will prove the Strong Atiyah Conjecture for finite extensions of cocompact special groups. We strengthen 'enough torsion-free quotients' to the *factorization property*, which requires every map from G to a finite group factor through a torsion-free elementary amenable group. This lets us weaken 'cohomologically complete' to *goodness in the sense of Serre*, which requires a group to have the same cohomology as its profinite completion.

Goodness of groups

In this section we record a few of the facts known about good groups; in particular, a result of Lorensen in [40] that right-angled Artin groups are good.

Definition. For a group G, let $\hat{G} = \varprojlim_{[G:H]<\infty} G/H$, where the inverse limit is taken over the set of finite quotients of G. $\hat{G}$ is the *profinite completion* of G. For every G, there is a canonical homomorphism $i : G \to \hat{G}$ which sends $g \in G$ to the sequence of cosets gH. The homomorphism is injective if and only if G is residually finite.

For example, the profinite completion of $\mathbb{Z}$ is the inverse limit of groups $\varprojlim \mathbb{Z}_n$, sometimes called the Prüfer group. By the Chinese Remainder Theorem, it is isomorphic to the product of all the p-adic integers.

$$\begin{tikzcd}[row sep=small]
 & \mathbb{Z}_2 \arrow[ddl] & & \\
 & & \mathbb{Z}_6 \arrow[ul] \arrow[dl] & \\
1 & \mathbb{Z}_3 \arrow[l] & & \cdots \\
 & & \mathbb{Z}_{15} \arrow[ul] \arrow[dl] & \\
 & \mathbb{Z}_5 \arrow[uul] & & \\
 & \vdots & &
\end{tikzcd}$$

Figure 4.1: A small part of the inverse system of finite index subgroups of $\mathbb{Z}$.

Note that if $\phi : G \to H$ is a group homomorphism, there is an induced map on profinite completions $\hat{\phi} : \hat{G} \to \hat{H}$ which maps

$$(g_0, g_1, g_2, \dots) \to (\phi(g_0), \phi(g_1), \phi(g_2) \dots)$$

Definition. A group G is called *good*, or *good in the sense of Serre* if the homomorphism

$$H^*(\hat{G}, M) = \varinjlim_{[G:H]<\infty} H^*(G/H, M) \xrightarrow{i^*} H^*(G, M)$$

is an isomorphism for every finite G-module M.

The following lemma from Serre gives a useful characterization of goodness.

Lemma 4.0.4. *A group G is good if and only if for every finite index subgroup N of G and cohomology class $\gamma \in H^*(N, M)$ where M is a finite G-module, there is another finite index subgroup $N' < N$ such that γ is in the kernel of the restriction map $H^*(N, M) \to H^*(N, M)$.*

It is now easy to see that $\mathbb{Z}$ is a good group: for any $N = n\mathbb{Z} < \mathbb{Z}$, let $N' = (|M|n)\mathbb{Z}$. Similarly, free groups are good.

Lemma 4.0.5 ([49, Exercise 2(b)])**.** *Let $1 \to H \to G \to K \to 1$ be an extension with H, K good and $H^*(H, M)$ finite for every G-module M. Then G is good.*

Lemma 4.0.6 ([49, Exercise 2(a)])**.** *Suppose we have an exact sequence $1 \to H \to G \to Q \to 1$ with Q finite and H finitely generated. Then the induced sequence of profinite completions $1 \to \hat{H} \to \hat{G} \to Q \to 1$ is exact.*

Corollary 4.0.7. *Suppose G is a good group. If H is commensurable to G then H is good. In particular, goodness passes to finite index subgroups.*

Lemma 4.0.8. *Suppose G is a good group and H is a retract of G. Then H is good.*

Proof. The cohomology of G and $\hat{G}$ are both contravariant functors. Therefore, the canonical map $H^n(\hat{H}, M) \to H^n(H, M)$ is a direct summand of the canonical map $H^n(\hat{G}, M) \to H^n(G, M)$. Since the latter is an isomorphism, so is the former. □

Theorem 4.0.9 ([40, Theorem 3.15])**.** *All right-angled Artin groups are good.*

Proof. Let A_Γ be the right-angled Artin group based on a flag complex Γ, and choose $s \in \Gamma$. Then A_Γ decomposes as the HNN extension $A_{\Gamma-s} *_{A_{\mathrm{Lk}(s)}}$, where $\mathrm{Lk}(s)$ denotes the link of s in Γ. Assume $A_{\Gamma-s}$ and $A_{\mathrm{Lk}(s)}$ are good by induction on the number of generators. On the level of completions $\widehat{A}_{\Gamma-s}$ and $\widehat{A}_{\mathrm{Lk}(s)}$ inject into $\widehat{A}_\Gamma$, as both subgroups are retracts of the latter (this fact is shown in [40].) Therefore, using the Mayer–Vietoris Sequence for HNN extensions and the Five Lemma we conclude that A_Γ is good:

$$\begin{tikzcd}
\cdots \leftarrow H^n(A_{\mathrm{Lk}(s)}, M) \arrow[d, "\hat{i}"'] & H^n(A_{\Gamma-s}, M) \arrow[l, "i"'] \arrow[d, "\hat{i}"'] & H^n(A_\Gamma, M) \leftarrow \dots \arrow[l, "j"'] \arrow[d, "\hat{i}"'] \\
\cdots \leftarrow H^n(\hat{A}_{\mathrm{Lk}(s)}, M) & H^n(\hat{A}_{\Gamma-s}, M) \arrow[l, "i"'] & H^n(\hat{A}_\Gamma, M) \leftarrow \dots \arrow[l, "j"']
\end{tikzcd}$$

In particular, note that if the left and central maps are isomorphisms for all n, then the right map is as well. □

Definition. If Γ is a finite simplicial graph with vertex set S, suppose we are given a family of groups $(G_s)_{s\in S}$. The *graph product* G_Γ is defined as the quotient of the free product of the $(G_s)_{s\in S}$ by the normal subgroup generated by the commutators of the form $[g_s, g_t]$ with $g_s \in G_s, g_t \in G_t$, where s and t span an edge of Γ.

A graph product is a natural generalization of many interesting groups. For example, a right-angled Artin group is a graph product with each vertex group $\mathbb{Z}$, and a right-angled Coxeter group is a graph product with each vertex group $\mathbb{Z}_2$. A similar proof shows that graph products of good groups are good, and graph products of cohomologically complete groups are cohomologically complete.

The factorization property

In this section, we will introduce the factorization property and go over its main properties. Again, our main result is that right-angled Artin group have the factorization property, and that it is passes to finite index subgroups and retracts.

Definition. A group H has *the factorization property* if any map from H to a finite group factors through a torsion-free elementary amenable group.

Remark. Note that a residually finite group that has the factorization property is residually torsion-free elementary amenable.

Lemma 4.0.10. *A finite index subgroup of a group with the factorization property has the factorization property.*

Proof. Let $K < G$ be a finite index subgroup of a group with the factorization property, and let $f : K \to P$ be a map to a finite group. The left action of G on the set of cosets $G/\ker f$ gives a map $g : G \to \mathrm{Sym}(G/\ker f)$ to a finite permutation group, which by assumption factors through a torsion-free elementary amenable group M. Denote by h the map $G \to M$. Since $\ker f$ is the stabilizer of the trivial coset, $\ker g \subset \ker f$, and therefore f factors through $g(K)$. Therefore, f factors through $h(K)$ which is torsion-free elementary amenable as a subgroup of M.

$$\begin{tikzcd}
 & & P \\
 & h(K) \arrow[ur] \arrow[dr] \arrow[dd, hook] & \\
K \arrow[uurr, bend left, "f"] \arrow[ur, "h"] \arrow[rr, "g"] \arrow[dd] & & g(K) \arrow[uu] \arrow[dd, hook] \\
 & M \arrow[dr] & \\
G \arrow[ur, "h"] \arrow[rr, "g"] & & \mathrm{Sym}(G/\ker f)
\end{tikzcd}$$

□

Lemma 4.0.11. *Suppose G has the factorization property. Then any retract of G has the factorization property.*

Proof. Let $p : G \to H$ be a retraction, and let $f : H \to P$ be a map to a finite group. By assumption, $f \circ p$ factors through a torsion-free elementary amenable group M, which induces a factoring of f through a subgroup of M.

$$\begin{tikzcd}
H \arrow[r, "f"] \arrow[d, shift left, "i"] & P \\
G \arrow[u, shift left, "p"] \arrow[r, "g"] & M \arrow[u]
\end{tikzcd}$$

□

Put together, these lemmas imply that the factorization property passes to virtual retracts. In fact, it is easy to slightly generalize this result.

Lemma 4.0.12. *Let H be a subgroup of G such that the completion map $\hat{i} : \hat{H} \to \hat{G}$ is injective. Then H has the factorization property.*

Proof. Let $K \trianglelefteq H$ be a finite index normal subgroup. We must show that there is a normal subgroup $U \trianglelefteq H$ such that H/U is torsion-free elementary amenable.

Since $\hat{H} \to \hat{G}$ is injective, it follows that there is a finite index normal subgroup $K' \trianglelefteq G$ such that $K' \cap H \subset K$. Since G has the factorization property, there is $U' \trianglelefteq K'$ with G/U' torsion-free elementary amenable. This implies the lemma by setting $U = U' \cap H$, since $H/U' \cap H \leq G/U'$. $\square$

The next lemma is well known and follows from the proof of Lemma 5 in [23]. Our argument follows a remark of Ian Agol on mathoverflow.com

Lemma 4.0.13. *If E is a free group and F is a normal subgroup, then $E/[F,F]$ is torsion-free.*

Proof. Let $g \in E - F$, and let G be the subgroup generated by g and F. Note that $[F,F]$ is a normal subgroup of E as $[F,F]$ is characteristic in F. Consider the normal series $G \trianglelefteq [G,G] \trianglelefteq [F,F]$. Since G/F is cyclic, we have $[G,G] \trianglerighteq F$, which makes $[G,G]/[F,F]$ free abelian as a subgroup of $F/[F,F]$. Therefore, G/F is torsion-free by the sequence

$$1 \to [G,G]/[F,F] \to G/[F,F] \to G/[G,G] \to 1.$$

$\square$

Lemma 4.0.14. *Extensions of free groups by torsion-free elementary amenable groups have the factorization property.*

Proof. Let $1 \to E \to H \to M \to 1$ be such an extension with E free and M torsion-free elementary amenable. Let $f : H \to P$ be a map to a finite group. Let $F = E \cap \ker f$. Then E/F is finite, and $E/[F,F]$ is torsion-free by Lemma 4.0.13 and elementary amenable by the exact sequence

$$1 \to F/[F,F] \to E/[F,F] \to E/F \to 1.$$

Now f factors through $H/[F,F]$ which is torsion-free elementary amenable.

$\square$

Theorem 4.0.15. *Graph products of groups with the factorization property have the factorization property.*

Proof. Let $f : G_\Gamma \to P$ be a map to a finite group. By induction on the number of vertices of Γ, assume the restriction of f to $G_{\Gamma-s}$ factors through a torsion-free elementary amenable group N. By hypothesis, the restriction of f to G_s factors through a torsion-free elementary amenable group K. If $G_\Gamma = G_{\Gamma-s} \times G_s$, we can factor f through the product $N \times K$. Otherwise, if $G_{\mathrm{Lk}(s)}$ denotes the graph product based on the link of $s \in \Gamma$, G_Γ splits as an amalgamated product

$$G_\Gamma = G_{\Gamma-s} *_{G_{\mathrm{Lk}(s)}} (G_{\mathrm{Lk}(s)} \times G_s).$$

Let L be the image of $G_{\mathrm{Lk}(s)}$ in N. The above factorizations induce a factorization of f through $N *_L (L \times K)$. By mapping N and $L \times K$ into $N \times K$, we get a map $N *_L (L \times K) \to N \times K$. Since the kernel of this map is free and $N \times K$ is torsion-free elementary amenable, we can apply Lemma 4.0.14 to factor f through a torsion-free elementary amenable group. $\square$

Since $\mathbb{Z}$ trivially has the factorization property, we have:

Corollary 4.0.16. *Right-angled Artin groups have the factorization property.*

4.1 The technical result for the Strong Atiyah Conjecture

We can now state our main result on the Strong Atiyah Conjecture.

Theorem 4.1.1. *Suppose* $1 \to H \to G \to Q \to 1$ *is an exact sequence of groups where* H *satisfies the Strong Atiyah Conjecture and* Q *is elementary amenable. Suppose* H *has a finite classifying space, is a good group, and has the factorization property. Then* G *satisfies the Strong Atiyah Conjecture. Moreover, any group commensurable to* H *satisfies the Strong Atiyah Conjecture.*

Our strategy for Theorem 4.1.1 follows that of Linnell and Schick in [38]. We quickly give an overview of the proof, which will take up the next two sections.

Given a finite extension $1 \to H \to G \xrightarrow{f} Q \to 1$ with H torsion-free, Linnell and Schick give conditions on H so that f factors through an elementary amenable group G/U with $\operatorname{lcm}(G/U) = \operatorname{lcm}(G)$. This is enough to show the conjecture for G, which we record as a key lemma.

Lemma 4.1.2 ([38, Theorem 2.6])**.** *Let* $1 \to H \to G \to M \to 1$ *be an extension where* H *is torsion-free, satisfies the Strong Atiyah Conjecture,* M *is elementary amenable and* $\operatorname{lcm}(M) = \operatorname{lcm}(G)$*. Then* G *satisfies the Strong Atiyah Conjecture.*

The factorization property can be thought of as a condition on H which guarantees a lot of torsion-free elementary amenable quotients. Consider one of these quotients, i.e. a subgroup $U \trianglelefteq H$ with H/U torsion-free elementary amenable. It is natural to make this U into a normal subgroup of G, denoted by U^G, and consider the quotient G/U^G. While G/U^G is always elementary amenable, it is tricky to control its torsion. Using the factorization property, we show that if all the quotients are bad in the sense that they have a lot of torsion, this implies a splitting of Q to the profinite completion $\hat{G}$. With some work, we can use goodness of G to guarantee that at least one of the quotients G/U^G has $\operatorname{lcm}(G/U) = \operatorname{lcm}(G)$.

We now try to make the above ideas precise and prove our main theorem.

Definition. For a CW-complex Y, the *cohomotopy groups of* Y are defined as

$$\pi^n(Y) = [(Y, *), (S^n, *)],$$

the set of pointed homotopy classes of maps from Y to the n-sphere. These can be thought of as dual to the more familiar homotopy groups. The *stable cohomotopy groups of* Y is the direct limit

$$\pi_S^*(Y) := \varinjlim_k [\Sigma^k(X), S^{*+k}],$$

where $\Sigma^k(X)$ is the k-th fold suspension of X. The *reduced stable cohomotopy groups of* Y is the cokernel of the natural map

$$\tilde{\pi}_S^*(Y) = \operatorname{coker} \pi_S^*(pt) \to \pi_S^*(Y).$$

Definition. If Y is a point, then the stable cohomotopy of Y is by definition isomorphic to the stable homotopy groups of spheres — in this case we shorten $\pi_S^*(Y)$

to π_S^*. If G is a discrete group, $\tilde{\pi}_S^*(G)$ is defined to be the reduced stable cohomotopy of the classifying space BG. If $\hat{G}$ is the profinite completion of G, define

$$\tilde{\pi}_S^*(\hat{G}) := \varinjlim_{[G:N]<\infty} \tilde{\pi}_S^*(G/N).$$

More details can be found in Section 4.4 of [38].

The next theorem is a profinite version of Theorem 4.27 in [38]. It is the main tool used to control torsion in these finite extensions.

Theorem 4.1.3. *Suppose we have an exact sequence*

$$1 \to H \to G \to Q \to 1$$

where Q is a finite p-group and H is good and has a finite classifying space. Assume the induced exact sequence of profinite completions

$$1 \to \hat{H} \to \hat{G} \to Q \to 1$$

splits. Then the original sequence also splits.

The idea behind Theorem 4.1.3 is that a splitting on the exact sequence of profinite completions should induce an injection of the degree zero reduced stable cohomotopy $i^* : \tilde{\pi}_S^0(\hat{G}) \to \tilde{\pi}_S^0(G)$. We actually prove something slightly weaker than this; but we end up with a description of what an element of the kernel should look like. Since Q splits to $\hat{G}$, $\tilde{\pi}_S^0(Q) \to \tilde{\pi}_S^0(\hat{G})$ is injective, and with some work we conclude an injection $\tilde{\pi}_S^0(Q) \to \tilde{\pi}_S^0(G)$. By the following result of [38], where it is attributed to A. Adem, this guarantees a splitting $Q \to G$.

Theorem 4.1.4 ([38, Theorem 4.28])**.** *Let H be a discrete group with finite cohomological dimension. Suppose we have an extension*

$$1 \to H \to G \xrightarrow{f} Q \to 1$$

where Q is a finite p-group. The extension above splits if and only if the epimorphism $G \to Q$ induces an injection $\tilde{\pi}_S^0(Q) \to \tilde{\pi}_S^0(G)$.

Therefore, Theorem 4.1.1 follows from the following technical lemma, which we prove in the next section.

Lemma 4.1.5. *Suppose that we have an exact sequence*

$$1 \to H \to G \to Q \to 1$$

where Q is a finite p-group and H is good and has a finite classifying space. Assume the induced exact sequence of profinite completions

$$1 \to \hat{H} \to \hat{G} \to Q \to 1$$

splits. Then the induced map $i^* : \tilde{\pi}^0_S(Q) \to \tilde{\pi}^0_S(G)$ is injective.

Before proving our main theorem, we need the following four lemmas — these correspond to Lemmas 4.10, 4.52, 4.54 and 4.59 in [38].

Lemma 4.1.6. *Suppose H is finite index and normal in G, and $\mathcal{U} = \{U\}$ is a collection of normal subgroups of H such that H/U is torsion-free elementary amenable for each $U \in \mathcal{U}$. Let $\mathcal{U}^G = \{U^G\}$ be the corresponding collection of normal subgroups of G, where $U^G = \bigcap_{g\in G} gUg^{-1}$. Then*

1. *This is a finite intersection.*

2. *H/U^G is torsion-free elementary amenable.*

3. *G/U^G is elementary amenable.*

Proof. [(i)]
This follows from H being finite index in G and U being normal in H, so that conjugation by elements of H fixes U.

2. U^G is the kernel of the map

$$H \to H/U \times H/g_1Ug_1^{-1} \times \cdots \times H/g_nUg_n^{-1}$$

where the range is assumed to be torsion-free elementary amenable.

3. We have the exact sequence

$$1 \to H/U^G \to G/U^G \to G/H \to 1,$$

so G/U^G is a finite extension of an elementary amenable group.

$\square$

Lemma 4.1.7. *Let Q be a finite p-group in the exact sequence*

$$1 \to H \to G \xrightarrow{\pi} Q \to 1.$$

Assume that among all normal finite index subgroups of G, there is a cofinal system $U_i \trianglelefteq G$ with $U_i \subset H$, such that for each i, the homomorphism π_i in

$$G \xrightarrow{p_i} G/U_i \xrightarrow{\pi_i} Q$$

has a split $s_i : Q \to G/U_i$. Then the profinite completion map $\hat{\pi} : \hat{G} \to Q$ has a split $Q \to \hat{G}$.

Proof. The proof in [38] works identically in this case. The idea is that for each $q \in Q$, we choose elements $g(q,i) \in G$ with $p_i(g(q,i)) = s_i(q) \in G/U_i$. Since $\hat{G}$ is compact, each sequence $p_i(g(q,i))$ has a convergent subsequence, and since Q is finite, we can assume there is one subsequence with $p_i(g(q,i)) \to g(q)$ for each q. The splitting is then defined as

$$s : Q \to \hat{G}, q \mapsto g(q).$$

$\square$

Lemma 4.1.8. *Suppose H is finitely generated and has the factorization property. Then there exists a collection $\mathcal{U}$ of subgroups $U \trianglelefteq H$ such that every finite index subgroup of H contains a subgroup in $\mathcal{U}$, if $U \in \mathcal{U}$ then H/U is torsion-free elementary amenable, and if $U \in \mathcal{U}, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.*

Proof. Let K_n be the intersection of all subgroups of H of index $\leq n$. Since K_n is finite index in H and H has the factorization property, there is a subgroup V_n

contained in K_n such that H/V_n is torsion-free elementary amenable. Letting $U_n = \cap_{i=1}^{n} V_i$ and $\mathcal{U} = \{U_n\}$ satisfies the above conditions, and by the proof of Lemma 4.1.6(ii), H/U_n is torsion-free elementary amenable for each n.

□

Lemma 4.1.9. *Let Q be a finite p-group and let $1 \to H \to G \xrightarrow{\pi} Q \to 1$ be an exact sequence of groups. Assume H is finitely generated and has the factorization property. Let $\mathcal{U}$ be a collection of normal subgroups of H as in Lemma 4.1.8, and define $\mathcal{U}^G$ as above. If each G/U^G contains a subgroup of order p^k, then there is a subgroup $Q_0 < Q$ of order p^k splitting back to $\hat{G}_0 \leq \hat{G}$, where $G_0 = \pi^{-1}(Q_0)$.*

Proof. By an easy argument in [38], there exists Q_0 of order p^k splitting back to G/U^G for all elements in $\mathcal{U}^G$. If Q_0 splits to G/U^G, then it splits to the quotient G/J for all $U^G \trianglelefteq J \trianglelefteq H$. Amongst the normal finite index subgroups of G (or G_0), those contained in H form a cofinal collection. Since H has the factorization property, each finite index $K \trianglelefteq H$ contains $U^G \in \mathcal{U}^G$, so Q_0 splits to each finite quotient G/K (or G_0/K). By Lemma 4.1.7, this implies that Q_0 splits to $\hat{G}_0$.

□

Proof of Theorem 4.1.1. Using [38, Lemma 2.4, Corollary 2.7], we only need to prove the case of Q being a finite p-group. Let $\mathcal{U}$ be a collection of normal subgroups of H as in Lemma 4.1.8, and let $\mathcal{U}^G$ as above. If $\mathrm{lcm}(G/U^G) = \mathrm{lcm}(G)$ for any $U^G \in \mathcal{U}^G$, we would be done by Lemma 4.1.2, Lemma 4.1.6, and the extension

$$1 \to U^G \to G \to G/U^G \to 1.$$

Since H and H/U^G are torsion-free and Q is a finite p-group, $\mathrm{lcm}(G)$ and $\mathrm{lcm}(G/U^G)$ are powers of p. Now, suppose each of the above groups G/U^G had a torsion subgroup of order p^k. By Lemma 4.1.9, there is a subgroup Q_0 in Q of order p^k and a splitting $Q \to \hat{G}_0$. Theorem 4.1.3 now implies Q_0 splits to G_0, which implies $\mathrm{lcm}(G) \geq p^k$. Therefore, there exists a subgroup U^G such that $\mathrm{lcm}(G/U^G) = \mathrm{lcm}(G)$.

□

Remark. If G is torsion free, there is a quicker proof that does not require Theorem 4.1.3. The idea is the same as above, but a splitting $Q_0 \to \hat{G}_0$ is an easy contradiction as goodness implies $H^*(\hat{G}_0, \mathbb{Z}/p\mathbb{Z}) = H^*(G_0, \mathbb{Z}/p\mathbb{Z})$ is zero above some dimension, while $H^*(Q, \mathbb{Z}/p\mathbb{Z})$ is not. In this case, we can also relax the assumption of H having a finite classifying space to being finitely generated and having finite cohomological dimension.

Suppose H is as in Theorem 4.1.1 and G is commensurable with H with common finite index subgroup K. Since all of our conditions pass to finite index subgroups, we can apply Theorem 1.1 to the core of K in G. We note this as a corollary:

Corollary 4.1.10. *Suppose H is as in Theorem 4.1.1, and G is commensurable with H. Then the Strong Atiyah Conjecture holds for G.*

Remark. Schick has shown that the methods used in [38] apply to the Baum–Connes conjecture with coefficients. Unsurprisingly, our results can be applied in the same way. We will just state our theorem as the proof is identical to his in [48].

Theorem 4.1.11. *Suppose $1 \to H \to G \to Q \to 1$ is an exact sequence of groups, where H and Q satisfy the Baum–Connes conjecture conjecture with coefficients. Suppose H has a finite classifying space, is a good group, and has the factorization property. Suppose G is torsion-free. Then G satisfies the Baum–Connes Conjecture with coefficients.*

4.2 Proof of Lemma 4.1.5

In this section, we complete the proof of Theorem 4.1.1.

Proof of Lemma 4.1.5. We use the same spectral sequence argument as in [38]. Recall that we have an Atiyah–Hirzebruch spectral sequence for $\tilde{\pi}_S^{s+t}(G)$ with

$$E_2^{s,t}(G) = \tilde{H}^s(G, \pi_S^t),$$

and a corresponding spectral sequence for $\tilde{\pi}_S^{s+t}(\hat{G})$ with

$$E_2^{s,t}(\hat{G}) = \tilde{H}^s(\hat{G}, \pi_S^t).$$

This second spectral sequence is defined to be the direct limit of the spectral sequences for the finite quotients of G.

Since G has torsion, our classifying space for G is infinite dimensional, so we need to be careful about the convergence of the Atiyah–Hirzebruch spectral sequence. If X is a connected CW-complex and $X^{(k)}$ its k-skeleton, let $F_k^*(X)$ denote the kernel of the map $j^* : \tilde{\pi}_S^*(X) \to \tilde{\pi}_S^*(X^{(k)})$ induced by the inclusion $j : X^{(k)} \to X$. Assume without loss of generality that $X^{(0)}$ is a point; in this case $F_0^*(X) = \tilde{\pi}_S^*(X)$ as $\tilde{\pi}_S^*$ is trivial. We say the above spectral sequence converges to $\tilde{\pi}_S^{s+t}(X)$ if

$$E_\infty^{s,t}(X) \cong F_s^{s+t}(X)/F_{s+1}^{s+t}(X) \qquad \forall s \geq 0, t \in \mathbb{Z}.$$

Now we compare our two spectral sequences. Recall that π_S^* is trivial if $* > 0$ and finite if $* < 0$. Therefore, these are both fourth quadrant spectral sequences with finite coefficients for $t < 0$. Since G is good by Lemma 4.0.5, we get an isomorphism on $E_2^{s,t}$ terms except for $s > 0, t = 0$ (for $s = t = 0$ both terms are trivial.) This implies isomorphic $E_\infty^{s,t}$ terms for $s + t \leq 0$ as in this range there is no interaction with the possibly non-isomorphic terms, as indicated in the figure below.

In particular, we have an isomorphism on the diagonal $E_\infty^{k,-k}(\hat{G}) \to E_\infty^{k,-k}(G)$ for all k. We now use the convergence results of [38], where it is shown in Propositions 4.43 and 4.47 that $F_k^0(\hat{G})/F_{k+1}^0(\hat{G})$ injects into $E_\infty^{k,-k}(\hat{G})$ and $F_k^0(G)/F_{k+1}^0(G)$ is isomorphic to $E_\infty^{k,-k}(G)$.

(The key idea behind these results is that the E_2-terms of both spectral sequences are finite; for $E_2^{s,t}(G)$ this is trivial, while for $E_2^{s,t}(\hat{G})$ it follows from G being good. Since these terms are finite, they must stabilize after a finite number of sheets of the spectral sequence, and with some care convergence follows from convergence in the finite dimensional case.)

The convergence results imply an injection:

$$F_k^0(\hat{G})/F_{k+1}^0(\hat{G}) \to E_\infty^{k,-k}(\hat{G}) \cong E_\infty^{k,-k}(G) \to F_k^0(G)/F_{k+1}^0(G).$$

We now have a commutative diagram for all k:

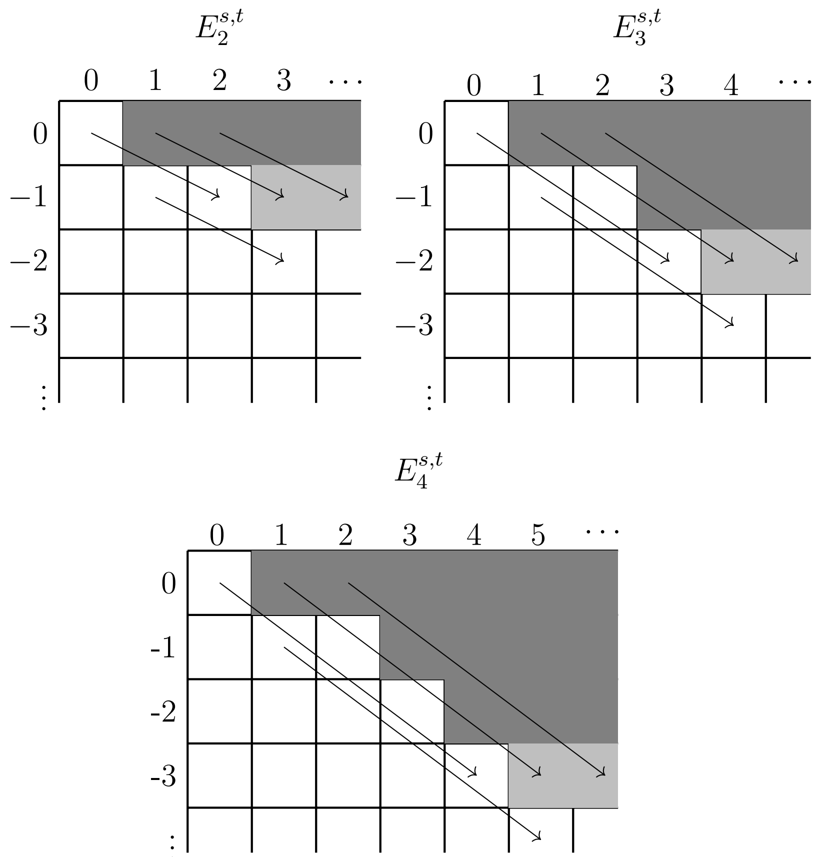


Figure 4.2: The shaded regions indicates terms that interact with the non-isomorphic top row ($s > 0$). These terms have no effect on the lower diagonal of the E_∞ sheet.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_k^0(\hat{G})/F_{k+1}^0(\hat{G}) & \longrightarrow & \tilde{\pi}_S^0(\hat{G})/F_{k+1}^0(\hat{G}) & \longrightarrow & \tilde{\pi}_S^0(\hat{G})/F_k^0(\hat{G}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_k} & & \downarrow{\scriptstyle \Phi_{k+1}} & & \downarrow{\scriptstyle \Phi_k} & & \\
0 & \longrightarrow & F_k^0(G)/F_{k+1}^0(G) & \longrightarrow & \tilde{\pi}_S^0(G)/F_{k+1}^0(G) & \longrightarrow & \tilde{\pi}_S^0(G)/F_k^0(G) & \longrightarrow & 0
\end{array}$$

The injectivity of Φ_{k+1} follows from the injectivity of i_k and Φ_k by a diagram chase. By induction beginning with $k = 0$, we have injections for each $k \geq 0$:

$$\tilde{\pi}_S^0(\hat{G})/F_k^0(\hat{G}) \to \tilde{\pi}_S^0(G)/F_k^0(G).$$

We also have the following commutative diagram, where s is by assumption split-injective:

$$
\begin{array}{ccc}
\tilde{\pi}^0_S(Q) & \longrightarrow & \tilde{\pi}^0_S(Q)/F^0_k(Q) \\
\big\downarrow s & & \big\downarrow s \\
\tilde{\pi}^0_S(\hat{G}) & \longrightarrow & \tilde{\pi}^0_S(\hat{G})/F^0_k(\hat{G}) \\
\big\downarrow & & \big\downarrow \Phi_k \\
\tilde{\pi}^0_S(G) & \longrightarrow & \tilde{\pi}^0_S(G)/F^0_k(G)
\end{array}
$$

We have shown that Φ_k is injective for each $k \geq 0$. This implies any element of the kernel of $\tilde{\pi}^0_S(Q) \to \tilde{\pi}^0_S(G)$ would be contained in $F^0_k(Q)$ for each $k \in \mathbb{N}$. However, it was shown in Proposition 4.40 of [38] that $\cap_k F^0_k(Q)$ is trivial, which implies $\tilde{\pi}^0_S(Q) \to \tilde{\pi}^0_S(G)$ is injective.

□

4.3 Virtually cocompact special groups

As we mentioned before, Linnell, Okun and Schick showed the following:

Theorem 4.3.1 ([37, Theorem 2])**.** *Right-angled Artin groups satisfy the Strong Atiyah Conjecture. Consequently, special groups satisfy the Strong Atiyah Conjecture.*

Obviously, compact special groups have finite classifying spaces. Since goodness and the factorization property pass to finite index subgroups and retracts (Lemmas 4.0.7, 4.0.8, 4.0.10, 4.0.11) and right-angled Artin groups are good and have the factorization property (Theorem 4.0.9 and Corollary 4.0.15), we conclude:

Corollary 4.3.2. *Compact special groups have finite classifying spaces, are good, and have the factorization property.*

Corollary 4.1.10 now immediately implies Theorem 4.1.1, so that the Strong Atiyah Conjecture holds for virtually compact special groups.

Remark. The class of virtually compact special groups has been shown to be amazingly large, highlighted by Agol's recent proof of the Virtually Compact Special Theorem.

Theorem 4.3.3 ([1, Theorem 1.1])**.** *Let G be a word-hyperbolic group that acts properly and compactly on a $CAT(0)$ cube complex X. Then G is virtually compact special.*

Recent breakthroughs in 3-manifold theory and Theorem 4.1.1 imply the Strong Atiyah Conjecture for all fundamental groups of finite-volume hyperbolic 3-manifolds.

Remark. We would like to mention here that the proof of the Virtual Fibering Conjecture plus Linnell's theorem also implies the Strong Atiyah Conjecture for such 3-manifold groups. Indeed, these groups are all either elementary amenable extensions of free groups or surface groups.

Coxeter groups were cubulated in [44], and shown to be virtually special in [33]. It is also known that the cubulation is cocompact whenever the group does not contain a Euclidean triangle Coxeter subgroup.

Corollary 4.3.4. *The Strong Atiyah Conjecture holds for all Coxeter groups which do not contain a Euclidean triangle Coxeter subgroup. In particular, it holds for all word-hyperbolic Coxeter groups.*

Knot and Link Complements

In this section we give an example from [10] which illustrates the advantage of our conditions over those of [38]. Recall that a *link group* G is the fundamental group of the complement M of a tamely embedded link in S^3. Similarly, a *knot group* is the fundamental group of the complement of a tame knot in S^3. In [38], it was shown that all knot groups are cohomologically complete. The idea is that the cohomology of knot groups G is well-known by Alexander duality:

$$H^n(G, \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p, & n = 0, 1 \\ 0, & n \geq 2 \end{cases}$$

Therefore, the map to the abelianization $G \to G/[G, G] \cong \mathbb{Z}$ induces an isomorphism on all homology groups. We now use a theorem of Stallings in [50] to conclude

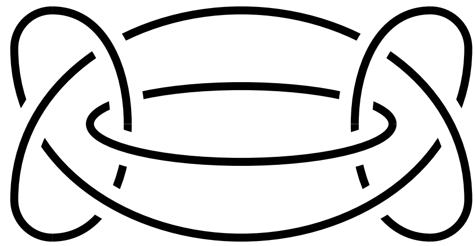


Figure 4.3: The link complement group is good but not cohomologically complete.

that for any prime p, the pro-p completion $\hat{G}^p \cong \hat{\mathbb{Z}}^p$, which implies completeness of G.

Remark. Wilton and Zalesskii prove in [53] that if M is a closed, irreducible, orientable 3-manifold, goodness of $\pi_1(M)$ follows from goodness of all fundamental groups of pieces of the JSJ decomposition of M. We have seen that fundamental groups of closed hyperbolic 3-manifolds, or those with toroidal boundary, are good. Since fundamental groups of Seifert-fibered spaces are good, it follows that all knot groups are good.

Blomer, Linnell and Schick in [8] also show cohomological completeness for certain link complements called *primitive link groups* (this is a combinatorial condition on the linking diagram.) This lets them conclude:

Theorem 4.3.5 ([8, Theorem 1.4])**.** *Let H be a knot group or a primitive link group. If H satisfies the Strong Atiyah Conjecture, then every elementary amenable extension of H satisfies the Strong Atiyah Conjecture.*

On the other hand, Bridson and Reid in [10] reverse the argument in [38] to construct link groups that are not cohomologically complete. These examples are *homology boundary links*, which have the property that the corresponding link group G surjects onto the free group F_2. An example of theirs is shown below.

Again using Alexander duality and Stallings Theorem, Bridson and Reid show that for any prime p, $\hat{G}^p \cong \hat{F}_2^p$, which has homology concentrated in dimension 1 as

F_2 is cohomologically complete. However, link groups with more than 2 components have nontrivial second homology groups, which contradicts completeness in this case.

Clearly, Theorem 4.3.5 does not apply to this homology boundary link example. However, the example shown in the figure (and in general 'most' link complements) are hyperbolic. We have shown above that the link group is good and has the factorization property, so Theorem 4.1.1 applies and the Atiyah conjecture holds for elementary amenable extensions of this group.

Remark. Certainly, it seems the greatest advantage our conditions have is that they pass to finite index subgroups. For instance, though RAAG's are cohomologically complete and have enough torsion-free quotients, the conjecture was unknown (before Theorem 4.1.1) for groups that are virtually RAAG's. The reason being that a finite index subgroup of a RAAG may not be a RAAG.

4.4 Some more examples of groups satisfying the conjecture

A natural question is whether the Strong Atiyah Conjecture holds for all virtually special groups. The methods we used above are not guaranteed to apply here, for instance, it is relatively easy to construct special groups that are not good. However, when the special cube complexes are noncompact in a controlled way, we can use the same arguments as above with a few tricks . The main strategy is to take a group that is not known to be cocompact special, and embed that group as a retract into a larger cocompact special group. Since goodness and the factorization property pass down to retracts, we're done if this occurs, which we record as a lemma. Our main motivation here was to prove the Strong Atiyah Conjecture for all Coxeter groups. Though we haven't been successful in the general case, there are certain special cases where the above strategy is successful.

Lemma 4.4.1. *Suppose that G is a group that contains a finite index subgroup H such that H is torsion-free, good, has the factorization property, and satisfies the*

Strong Atiyah Conjecture. Then any virtual retract of G satisfies the Strong Atiyah Conjecture.

Proof. Let $\rho : G \to K$ be a retraction. Since all our properties pass to finite index subgroups, we can assume H is normal in G. Furthermore, we can assume H retracts onto its image in K by considering the subgroup $H \cap \rho^{-1}(K \cap H)$. The image group $K \cap H$ is finite index in K, is good, has the factorization property and satisfies the Strong Atiyah Conjecture. Therefore, K satisfies the Strong Atiyah Conjecture by Theorem 4.1.1. $\square$

Corollary 4.4.2. *If W_L is a Coxeter group with planar nerve L, then W_L satisfies the Strong Atiyah Conjecture.*

Proof. It is simple to find a triangulation of a 2-sphere that contains L as a full subcomplex, such that the Coxeter 3-manifold group W retracts onto W_L. Since W virtually fibers, W_L satisfies the Strong Atiyah Conjecture by Lemma 4.4.1. $\square$

Definition. A nonpositively curved cube complex is *sparse* if it is quasi-isometric to a wedge of finitely many Euclidean half-spaces.

In other words, a sparse cube complex has a compact core with some number of Euclidean spaces sticking off. A group acts on a CAT(0) cube complex *cosparsely* if the quotient is sparse. A standard example of this is the quotient of the CAT(0) cube complex obtained by applying the Niblo-Reeves cubulation to a Coxeter group whose diagram contains a single Euclidean triangle subgroup.

Theorem 4.4.3. *Let G be the fundamental group of a sparse cube complex X that admits a hierarchy and is relatively hyperbolic to a collection $\mathcal{P}$ of abelian groups. Then G virtually embeds as a retract of a virtually cocompact special group G'. Therefore, any group that contains G as a finite index subgroup satisfies the Strong Atiyah Conjecture.*

Proof. In the above situation, Lemma 16.7 of [54] implies the existence of a group G' that acts properly and cocompactly on a CAT(0) cube complex, is hyperbolic relatively to free abelian groups, and retracts onto G. By Theorem 15.3 of [54], G' is virtually cocompact special, which implies that a finite index subgroup of G is good, has the factorization property, and satisfies the Strong Atiyah Conjecture. Therefore, we are done by Theorem 4.1.1. □

Corollary 4.4.4. *Coxeter groups that are hyperbolic relatively to virtually abelian groups satisfy the Strong Atiyah Conjecture.*

Proof. By the main result of [33], these groups have a finite index subgroup that is virtually the fundamental group of a sparse cube complex. □

Remark. The same argument applies to any group that is virtually the fundamental group of a sparse cube complex and is hyperbolic relative to its peripheral abelian subgroups.

Definition. A group G is *residually free* if for any $g \in G, g \neq 1$ there is a map to a free group $f : G \to F_n$ such that $f(g) \neq 1$. G is a *limit group* if for any finite set X with $1 \notin X$, there is a map $f : G \to F_n$ such that $1 \notin f(X)$.

Theorem 4.4.5 ([54])**.** *Limit groups are virtually cocompact special.*

In [11], Bridson and Wilton prove strong separability properties of products of limit groups. In particular, they show certain subgroups are virtual retracts.

Theorem 4.4.6 (Lemma 7. [11])**.** *Let $G = \Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n$ be a product of limit groups, and let $H \leq G$ be a subgroup of type F_n over $\mathbb{Q}$. Then H is a virtual retract of G.*

In particular, the work of Sela and Baumslag-Myasnikov-Remeslennikov shows that every residually free group is a virtual retract of a product of limit groups. Applying Theorem 4.1.1, we have

Corollary 4.4.7. *Let $G = \Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n$ be a product of limit groups, and let $H \leq G$ be a subgroup of type F_n over $\mathbb{Q}$. Any group that contains H as a subgroup of finite index satisfies the Strong Atiyah Conjecture.*

4.5 TFVCS groups are RTFEA

Theorem 4.5.1. *Suppose that $1 \to H \to G \to Q \to 1$ is an exact sequence of groups with Q finite. Suppose H has a finite classifying space, is a good group, and has the factorization property. Suppose that G is torsion-free and residually finite. Then G is residually torsion-free elementary amenable.*

This theorem immediately follows from the next lemma, where we show that G has the factorization property.

Lemma 4.5.2. *Let π be a finitely generated torsion-free group that has a finite-dimensional classifying space and which is good. If π admits a finite index subgroup G which has the factorization property, then π also has the factorization property.*

Proof. Let $a : \pi \to G$ be a homomorphism to a finite group. We denote by $K \subset \pi$ the intersection of $\ker(a)$ and G^{π}. The subgroup K is of finite index in π and it is clearly contained in G. It is straightforward to see that K also has the factorization property. We write $Q := \pi/K$. Clearly it is enough to show that the projection map $\pi \to Q$ factors through a torsion-free elementary amenable group.

We first consider the case that Q is a p-group. The proof of the Theorem 4.1.1 produces a subgroup $U \trianglelefteq \pi$ such that the map $\pi \to Q$ factors through π/U and such that π/U is torsion-free elementary amenable. The basic idea is as follows: using that K has the factorization property, it is shown that if there is no such U, a non-trivial subgroup Q' of Q splits in the induced sequence of profinite completions $1 \to \hat{K} \to \hat{\pi} \to Q \to 1$. However, putting the following two observations together shows that this is not possible.

1. The cohomology $H^*(Q', \mathbb{F}_p)$ is nonzero in infinitely many dimensions.

2. By [49, Exercise 1, 2.6] any finite-index subgroup L, e.g. K or the preimage of Q' under $\pi \to Q$, of π is also good and it also has a finite-dimensional classifying space. This implies that $H^*(\hat{L}, \mathbb{F}_p) \cong H^*(L, \mathbb{F}_p)$ is nonzero in only finitely many dimensions.

For the general case, we use a trick from [38]. For each Sylow p-subgroup S of Q, consider the exact sequence $1 \to K \to \pi_S \to S \to 1$, where π_S is the preimage of S. By the above, we get a collection of subgroups U_{π_S} such that the quotients π_S/U_{π_S} are torsion-free elementary amenable. Let $U = \cap_S U_{\pi_S}$. Since π/U^π is a finite extension of G/U^π, elementary amenability follows again from [38, Lemma 4.11]. Therefore it suffices to prove the following lemma.

Lemma 4.5.3. *The group π/U^π is torsion-free elementary amenable.*

Suppose that π/U^π has a non-trivial torsion element γ. By raising γ to some power we get an element γ' that is p-torsion for some prime p. We have an exact sequence

$$1 \to U_S^\pi/U^\pi \to \pi_S/U^\pi \to \pi_S/U_S^\pi \to 1$$

where U_S^π/U^π and π_S/U_S^π are torsion-free by Lemma 4.11 in [38]. Since K/U^π is torsion-free, γ' would map to some Sylow p-subgroup, in which case $\gamma' \in \pi_S/U_S^\pi$, which is torsion-free. Therefore, π/U^π is torsion-free. □

In particular, by the work of Agol and Wise this answers Question 9.18 in the book of Aschebrenner, Friedl, and Wilton for all virtually special aspherical 3-manifolds, and for general torsion-free virtually compact special groups.

Remark. There is an application of this result in [25], which is joint work with Stefan Friedl and Stephen Tillmann. The idea here is that the group ring of a torsion-free elementary amenable group $\mathbb{Z}(\Gamma)$ has some very nice properties: in particular it admits an *Ore localization* $\mathbb{K}(\Gamma)$, which is the analogue of a field of fractions in the non-commutative case. If a group is residually torsion-free elementary amenable, then finite sets of group elements can be detected in this localization.

Chapter 5

The Singer Conjecture

We now switch our attention to questions regarding the Singer Conjecture. We begin with a perhaps more familiar concept.

Definition. Let G be a discrete group. The *geometric dimension* of $G, \mathrm{gdim}(G)$ is the smallest dimension of a contractible CW-complex that G acts on properly discontinuously.

If G is torsion-free, then this proper action is actually free and the quotient of the action is a $K(G,1)$. Therefore, the geometric dimension of G is the minimal dimension of a model for $K(G,1)$.

In [6], Bestvina, Kapovich and Kleiner introduced a manifold version of this concept.

Definition. Let G be a group. The *action dimension* of $G, \mathrm{actdim}(G)$ is the smallest dimension of a contractible manifold that G acts on properly discontinuously.

For example, if G acts properly and *cocompactly* on an open contractible n-manifold, then certainly $\mathrm{actdim}(G) \leq n$ and by considering cohomology with compact supports we see that $\mathrm{actdim}(G) = n$. Note that the corresponding statement does not hold for the geometric dimension.

Lemma 5.0.4. *Here are some elementary properties of action dimension.*

1. $\mathrm{actdim}(G_1) \times \mathrm{actdim}(G_2) \leq \mathrm{actdim}(G_1) + \mathrm{actdim}(G_2)$.

2. *If* $H < G$*, then* $\operatorname{actdim}(H) < \operatorname{actdim}(G)$*.*

Proof. Certainly, if G acts on a contractible n-manifold and H acts on a contractible manifold, then $G \times H$ acts on a contractible $(n+m)$-manifold. Similarly, if G acts on a contractible n-manifold then H does as well.

□

We have the following refinement of action dimension.

Definition. Let G be a group. The *cocompact action dimension* of $G, \operatorname{cadim}(G)$ is the smallest dimension of a contractible manifold that G acts on properly discontinuously and cocompactly.

Similar properties as in Lemma 5.0.4 hold for $\operatorname{cadim}(G)$.

Lemma 5.0.5. *If* G *admits a finite* $K(G,1)$*, we have a sequence of inequalities:*

$$\operatorname{actdim}(G) \leq cadim(G) \leq 2\operatorname{gdim}(G).$$

Proof. The first inequality is immediate. The second follows from a theorem of Stallings [**?**]. Note that if a $K(G,1)$ can be embedded into $\mathbb{R}^n$, then any regular neighborhood is an aspherical n-manifold with $\pi_1 = G$, so that $\operatorname{actdim}(G) \leq n$. By general position, any k-dimensional $K(G,1)$ can be embedded into $\mathbb{R}^{2k+1}$, and Stallings' theorem guarantees that we can choose a $K(G,1)$ model that embeds into $\mathbb{R}^{2k}$. □

Shmuel Weinberger pointed out that a recent theorem of Craig Guilbault implies that for groups with finite $K(G,1)$ the difference between cadim and actdim is at most 1, at least in high dimensions.

Theorem 5.0.6 ([30])**.** *For an open manifold* M^n *(*$n \geq 5$*),* $M^n \times \mathbb{R}$ *is homeomorphic to the interior of a compact* $(n+1)$*-manifold with boundary if and only if* M^n *has the homotopy type of a finite complex.*

Corollary 5.0.7. *If* G *is of type* F *and* $\operatorname{actdim}(G) \geq 5$*, then* $\operatorname{cadim}(G) \leq \operatorname{actdim}(G) + 1$*.*

5.1 Singer-type conjectures

One consequence of Bestvina, Kapovich, and Kleiner's work is that an n-fold product of non-abelian free groups does not act properly discontinuously on a contractible $(2n-1)$-manifold. Since non-abelian free groups have $L^2H_1(F_n) \neq 0$, it follows from the Künneth formula for L^2-homology that the n-fold products of free groups have $L^2H_n \neq 0$. Therefore, as noted in [19], their result is implied by the following conjecture of Davis and Okun.

Action dimension (actdim) conjecture. $L^2H_i(G) = 0$ *for* $i > \mathrm{actdim}(G)/2$.

A similar conjecture is open for the cocompact action dimension.

Cocompact action dimension (cadim) conjecture. $L^2H_i(G) = 0$ *for* $i > \mathrm{cadim}(G)/2$.

Since any contractible G-manifold can be used to compute $L^2H_i(G)$, we have an equivalent series of conjectures in terms of manifolds.

Cadim conjecture in dimension n. *If $(M, \partial M)$ is an n-manifold with contractible components which admits a proper and cocompact G-action, then $L^2H_i(M,G) = 0$ for $i > n/2$.*

Remark. These conjectures put restrictions on the embedding dimension of a $K(G,1)$ space. For example, if $L^2H_i(G) \neq 0$, the cadim conjecture implies that any finite $K(G,1)$ space cannot embed in $\mathbb{R}^{2i-1}$.

Lemma 5.1.1. actdim *conjecture* $\Rightarrow$ cadim *conjecture* $\Rightarrow$ *Singer conjecture.*

Proof. The first implication is trivial, and the second follows from applying Poincaré duality, and the fact that a group acting properly and cocompactly on a contractible n-manifold without boundary has adim = cadim = n. □

Although the precise relationship between actdim and cadim is unclear, we can still show the two conjectures are equivalent, at least for type VF groups (groups that virtually admit finite K(G,1)'s).

Theorem 5.1.2. *The cadim and actdim conjectures are equivalent for type VF groups.*

Proof. We need to show that the cadim conjecture implies the actdim conjecture. So, let G be a counterexample to the actdim conjecture, i.e. $\operatorname{actdim}(G) < 2\,\ell^2\dim(G)$. Note that by the Künneth formula, by taking direct products of G with itself, we can assume that G acts on a contractible n-manifold M with $2\,\ell^2\dim(G) - n$ arbitrarily large. If $H < G$ is finite index and torsion-free, then $2\,\ell^2\dim(H) - n$ is still arbitrarily large, and M/H is an aspherical n-manifold. By Theorem 5.0.6, $M/H \times \mathbb{R}$ is the interior of a compact manifold. Therefore, the action of H on the universal cover of this compact manifold is a counterexample to the cadim conjecture. $\square$

The Action Dimension Conjecture has been proven for many classes of interesting groups. Here are some examples:

- If all L^2-Betti numbers of Γ are 0 (e.g., if Γ contains an infinite amenable normal subgroup), then the Action Dimension Conjecture for Γ holds trivially.

- If Γ is a lattice in a semisimple Lie group without compact factors, then the conjecture holds for Γ. In [7], Bestvina and Feighn show that the action dimension of such lattices is the dimension of the corresponding symmetric space, and Borel showed the L^2-Betti numbers of these lattices are concentrated in the middle dimension [5].

- If Γ is the mapping class group of a surface with marked points or punctures, then the conjecture holds for Γ. In [20], it was shown that the action dimension of such a mapping class group is the dimension of the corresponding Teichmüller space, so the conjecture follows by results of Mcmullen [42] and Gromov [28].

- If $\operatorname{actdim}(\Gamma) = 2\operatorname{gdim}(\Gamma)$, then the conjecture holds for Γ. (For example, it is proved in [6] that this is true for $\Gamma = \operatorname{Out}(F_n)$.)

5.2 Manifolds with hierarchies

A somewhat surprising result is the converse to the second implication in Lemma 5.1.1, at least if we somewhat restrict the category. The cadim conjecture with PL boundary restricts the group action to manifolds whose boundaries admit equivariant PL triangulations.

Theorem 5.2.1 ([45])**.** *The Singer conjecture and the* cadim *conjecture with PL boundary are equivalent.*

To prove this theorem, we need to introduce the notion of a hierarchy for a group action. This generalizes the notion of a Haken n-manifold, introduced by Foozwell in his Ph.D. thesis [24].

Definition. A *convex polyhedral cone* C in $\mathbb{R}^n$ is the intersection of a finite collection $\{B_i^+\}$ of linear half-spaces in $\mathbb{R}^n$ (a half-space is linear if its bounding hyperplane B_i is a linear subspace). C is *nondegenerate* if it has nonempty interior. A *hyperplane arrangement in a nondegenerate cone* C is a finite collection $\{A_i\}$ of linear hyperplanes such that each A_i intersects the interior of C.

Definition. Let M be a G-manifold, and $\mathcal{E} = \{E_i\}_{i=0}^r$ a collection of codimension one G-submanifolds. $(M, \mathcal{E})$ is *tidy* if

- The components of M are contractible.
- The components of any intersection of E_i's are contractible.
- $E_i \cap \partial M = \partial E_i$ for all i.
- $(M, \partial M, \mathcal{E})$ locally looks like a hyperplane arrangement in a nondegenerate cone in $\mathbb{R}^n$: every point in M has a chart which maps M into a nondegenerate cone in $\mathbb{R}^n$, ∂M into the boundary of the cone, and the E_i's into a hyperplane arrangement in the cone.

In the case where $\mathcal{E}$ consists of just one submanifold F, this definition is equivalent to requiring that F is locally flat as a submanifold with boundary (sometimes called a neat submanifold), and the components of both M and F are contractible. We will call such a pair (M, F) a *tidy pair.*

In this case, since the components of F are contractible, it admits a collar neighborhood, and since the components of M are contractible, F is two-sided. By *cutting* M along F we mean talking the disjoint union N of the closures in M of components of $M - F$. We say that N is M *cut-open* along F. The action of G on $M - F$ extends by continuity to a proper cocompact action on N. So, N is a G-manifold with boundary ∂M union with two copies of F.

Associated to this cut there is an exact sequence of a triple $(M, F \cup \partial M, \partial M)$

$$\to H_c^{k-1}(F \cup \partial M, \partial M) \to H_c^k(M, F \cup \partial M) \to H_c^k(M, \partial M) \to H_c^k(F \cup \partial M, \partial M) \to$$

By excision, we have $H_c^k(F \cup \partial M, \partial M) \cong H_c^k(F, \partial F)$ and $H_c^k(M, F \cup \partial M) \cong H_c^k(N, \partial N)$. So we have a sequence

$$\to H_c^{k-1}(F, \partial F) \to H_c^k(N, \partial N) \to H_c^k(M, \partial M) \to H_c^k(F, \partial F) \to \tag{5.1}$$

Finally, applying Poincaré duality and reindexing, we obtain a sequence:

$$\to H_k(F) \to H_k(N) \to H_k(M) \to H_{k-1}(F) \to \tag{5.2}$$

Lemma 5.2.2. *If (M, F) is a tidy pair and N is M cut-open along F, then the components of N are contractible manifolds.*

Notice that N may or may not have more G-orbits of components than does M.

Proof. The Van Kampen theorem implies that components of N are simply connected, and the sequence (5.2) shows that N is acyclic. □

Definition. An *n-hierarchy* for an action of a discrete group G on a manifold M is a sequence

$$(M_0, F_0), (M_1, F_1), \ldots, (M_m, F_m), (M_{m+1}, \emptyset),$$

such that

- $M_0 = M$.

- M_{m+1} is a disjoint union of compact, contractible n-manifolds.
- (M_i, F_i) is a tidy pair for each i.
- M_{i+1} is M_i cut-open along F_i.

More generally, if (M, N) is a G-pair of manifolds, we can define a *hierarchy ending in* N in the same way, with the one difference being that $M_{m+1} = N$.

Definition. G *admits an n-hierarchy* if there exists a contractible, n-dimensional G-manifold M and a hierarchy for the action.

For any tidy pair, the same sequence for homology gives a sequence for L^2-homology.

$$\to L^2H^{k-1}(F, \partial F) \to L^2H^k(N, \partial N) \to L^2H^k(M, \partial M) \to L^2H^k(F, \partial F) \to \tag{5.3}$$

Lemma 5.2.3. *Let G act on M with a hierarchy, and let M_1^0 be a component of M_1. Then there is an induced hierarchy for the action of $\mathrm{St}_G(M_1^0)$ on M_1^0, where $\mathrm{St}_G(M_1^0)$ is the stabilizer of M_1^0.*

Proof. We claim the following sequence is a hierarchy for M_1^0:

$$(M_1^0, F_1 \cap M_1^0), (M_2 \cap M_1^0, F_2 \cap M_1^0), \ldots (M_{m+1} \cap M_1^0, \emptyset)$$

We have that M_1^0 is a $\mathrm{St}_G(M_1^0)$-manifold, and by Lemma 5.2.2 is contractible. Since each F_i is G-invariant, $F_i \cap M_1^0$ is $\mathrm{St}_G(M_1^0)$-invariant, and the other conditions of our hierarchy follow immediately. □

Lemma 5.2.4. *Let $(M, \mathcal{E})$ be tidy, and let N be M cut-open along E_0. Then $(N, \{E_i \cap N\}_{i=1}^r)$ is also tidy.*

Proof. Note that (M, E_0) is a tidy pair. We check the conditions of tidiness. Contractibility of the components of N follows immediately from Lemma 5.2.2. By assumption, any intersection $\bigcap E_{i_\alpha}$ has contractible components; let $(\bigcap E_{i_\alpha})'$ denote one orbit of components. Note that since the collection $\mathcal{E}$ is locally modeled on a hyperplane arrangement, either $(\bigcap E_{i_\alpha})'$ is contained in E_0, or $E_0 \cap (\bigcap E_{i_\alpha})'$ is codimension one in $(\bigcap E_{i_\alpha})'$. Since $N \cap \bigcap E_{i_\alpha}$ is precisely $\bigcap E_{i_\alpha}$ cut out by $E_0 \cap \bigcap E_{i_\alpha}$, Lemma 5.2.2 again implies that $N \cap \bigcap E_{i_\alpha}$ has contractible components.

After cutting, the local picture is mostly preserved, we just have to check near E_0. If $x \in E_0$, the new charts come from restricting the old chart to a halfspace bounded by a copy of E_0, the point being that cutting a nondegenerate cone along one of the hyperplanes in a arrangement produces two nondegenerate cones with arrangements.

$\square$

Theorem 5.2.5. *Let M be a G-manifold, and $\mathcal{E} = \{E_i\}_{i=0}^r$ a collection of submanifolds such that $(M, \mathcal{E})$ is tidy. If the components of the complement $M - \cup_i E_i$ have compact closure in M, then the action of G on M admits a hierarchy.*

Proof. The proof is to apply Lemma 5.2.4 repeatedly, as this implies that if we cut along each E_i, we get a hierarchy ending in $M - \cup_i E_i$. To be precise, let $F_j = E_j$ cut-along by $E_0, E_1, \ldots E_{j-1}$ and let $M_0 = M$ and $M_{j+1} = M_j$ cut along by F_j. Since each E_i is G-invariant, (M_j, F_j) is a tidy pair for all j. $\square$

If (M^{2k+1}, F^{2k}) is a tidy G-pair, then Poincaré duality and the cadim conjecture applied to the action of G on M imply $L^2H_k(M, \partial) = L^2H_{k+1}(M) = 0$. In particular, we have the following apparently weaker version of the cadim conjecture.

Weak cadim conjecture. *If (M^{2k+1}, F^{2k}) is a tidy pair, then the map induced by inclusion $i_* : L^2H_k(F, \partial) \to L^2H_k(M, \partial)$ is the zero map.*

Lemma 5.2.6. *Suppose that (M^n, F) is a tidy G-pair, N is M cut-open by F, and the cadim conjecture holds for F. Then the cadim conjecture holds for M if and only if it holds for N and, if $n = 2k+1$ is odd, the weak cadim conjecture holds for (M, F).*

Proof. First, suppose that the cadim conjecture holds for M. We have $L^2H_i(M) = 0$ for $i > n/2$, and $L^2H_i(F) = 0$ for $i > (n-1)/2$, so the claim follows from the sequence (5.3).

Next, suppose the cadim conjecture holds for N, so that we have $L^2H_i(N) = 0$ for $i > n/2$, and $L^2H_i(F) = 0$ for $i > (n-1)/2$. Then sequence (5.3) implies $L^2H_i(M) = 0$ for $i > (n+1)/2$.

Now, we only have to consider the case where $n = 2k+1$ and $i = k+1$. The weak cadim conjecture implies that the map $L^2H_{k+1}(M) \to L^2H_k(F)$ in the sequence (5.3) is the zero map, since it is Poincaré dual to the map $i^* : L^2H_k(M, \partial M) \to L^2H_k(F, \partial F)$ in the sequence (5.3). The result follows. $\square$

Theorem 5.2.7. *The cadim conjecture in dimension $2k-1$ implies the cadim conjecture in dimension $2k$ for manifolds with hierarchies. The cadim conjecture in dimension $2k$ and the weak cadim conjecture in dimension $2k+1$ imply the cadim conjecture in dimension $2k+1$ for manifolds with hierarchies.*

Proof. This is immediate by induction on the length of the hierarchy, using Lemmas 5.2.3 and 5.2.6, and noting that the cadim conjecture holds for manifolds with compact components. $\square$

The proof of Theorem follows immediately from the following key lemma and induction.

Lemma 5.2.8. *The Singer conjecture in dimension n and the cadim conjecture in dimension $(n-1)$ imply the cadim conjecture in dimension n.*

Proof. We use the equivariant Davis reflection group trick as in [18], [16]. The idea is that the trick turns the input of the cadim conjecture (a contractible manifold with boundary and geometric group action) into the input of the Singer conjecture (a contractible manifold without boundary and geometric group action). In addition, the newly constructed manifold action admits a hierarchy ending at a disjoint union of copies of the original. Once this has been established, the proof is more or less the same as that of Theorem 5.2.7.

Suppose that G acts properly and cocompactly on a contractible n-manifold with boundary $(M, \partial M)$. Let L be a flag triangulation of ∂M that is equivariant with respect to the G-action, and suppose that the stabilizer of any simplex fixes the stabilizer pointwise (these triangulations can always be constructed). We can now apply the equivariant reflection group trick. Indeed, L determines a right-angled Coxeter group W, and we can form the basic construction $\mathcal{U} = \mathcal{U}(W, M)$. By the conditions imposed on L, there is an action of G on W which determines a semi-direct product $W \rtimes G$. Since $\mathcal{U}/W \rtimes G \cong M/G$, $W \rtimes G$ acts cocompactly on $\mathcal{U}$. Here are some key properties of the reflection group trick:

- Each wall is a codimension one, contractible submanifold of N.
- There are a finite number of $W \rtimes G$-orbits of walls, and each orbit is a disjoint union of walls.
- Any non-empty intersection of orbits of walls is itself a Davis complex and is therefore contractible.
- The stabilizer of each wall acts properly and cocompactly on the wall.
- The collection of walls looks locally like a right-angled hyperplane arrangement in $\mathbb{R}^n$.

It follows similarly to Theorem 5.2.5 that the $W \rtimes G$ action on $\mathcal{U}$ admits a hierarchy that ends in disjoint copies of M, where the cutting submanifolds are $W \rtimes G$-orbits of walls. Since $\mathcal{U}$ has no boundary, and we are assuming that the Singer conjecture holds for $\mathcal{U}$, the cadim conjecture holds for $\mathcal{U}$. Since we are also assuming the cadim conjecture in dimension $n-1$, it follows by applying Lemma 5.2.6 inductively that the cadim conjecture holds for the original M. $\square$

Coxeter groups admit hierarchies

We assume from now on that W is a Coxeter group with nerve a triangulation of S^{n-1}. If $w \in W$ acts as a reflection on $\Sigma(W, S)$, we call the fixed point set a wall, and denote it Σ^w.

Lemma 5.2.9. *Walls in $\Sigma(W, S)$ have the following properties.*

- *The stabilizer of each wall acts properly and cocompactly on the wall.*
- *Each wall and each half-space is a geodesically convex subset of* $\Sigma(W,S)$*.*
- *The collection of walls separates* $\Sigma(W,S)$ *into disjoint copies of the fundamental domain* K*.*
- *The stabilizer of each point in* $\Sigma(W,S)$ *is a finite Coxeter group, and the walls containing that point can be locally identified with the fixed hyperplanes of the standard action of this Coxeter group on* $\mathbb{R}^n$*.*

Though each wall of Σ is a contractible submanifold, a W-orbit of a wall has in general quite complicated topology. Even in the simple case where W is generated by reflections in a equilateral triangle in $\mathbb{R}^2$ the W-orbit of a wall is not contractible, as W-translates of a wall can intersect nontrivially. However, passing to suitable subgroup fixes this problem.

Theorem 5.2.10. *W has a finite index torsion-free normal subgroup Γ, and the action of Γ on $\Sigma(W,S)$ admits a hierarchy.*

Proof. The existence of such a subgroup Γ is well-known, since Coxeter groups are linear. The cutting submanifolds that we choose will be Γ-orbits of walls in $\Sigma(W,S)$.

A lemma of Millson and Jaffee in [43] shows that any torsion-free normal subgroup of W has the trivial intersection property: for all $\gamma \in \Gamma$, either $\gamma\Sigma^s = \Sigma^s$ or $\gamma\Sigma^s \cap \Sigma^s = \emptyset$. Therefore, each Γ-orbit is a disjoint union of walls and has contractible components.

Once we have removed all the walls, we are left with disjoint copies of the fundamental domain K, and since Γ is finite index in W, there are only finitely many orbits of walls to remove, so by Lemma 5.2.9, this is a tidy collection. Therefore, we are done by Theorem 5.2.5. □

Corollary 5.2.11. *If L is a nerve of a Coxeter group that is a triangulation of S^3, then $b_i^2(W_L) = 0$ for $i > n/2$.*

Remark. If W is a Coxeter group with nerve a triangulation of D^{n-1}, then $\Sigma(W,S)$ is an n-manifold with boundary, and these groups also admit hierarchies.

5.3 The van Kampen obstruction

The main idea in the paper of Bestvina, Kapovich, and Kleiner relates action dimension to a classical construction in embedding theory, due to van Kampen.

Definition. If K be a simplicial complex, let K^* denote the *simplicial configuration space of* K, which is the space of unordered pairs of disjoint simplices of K:

$$K^* = \{\{\sigma, \tau\} : \sigma, \tau \in K, \sigma \cap \tau = \emptyset\}$$

Definition. Let K be an k-dimensional simplicial complex, and let $f : K \to \mathbb{R}^{2k}$ be a general position map. The *van Kampen obstruction* $\mathrm{vk}_{\mathbb{Z}/2}(K) \in H^{2k}(K^*, \mathbb{Z}_2)$ is defined by $\mathrm{vk}_{\mathbb{Z}/2}(\{\sigma, \tau\}) = |f(\sigma) \cap f(\tau)| \mod 2$.

The class of this cocycle turns out not to depend on the choice of f, which implies that if $\mathrm{vk}_{\mathbb{Z}/2}(K) \neq 0$ then K does not embed into $\mathbb{R}^{2k}$. In this case, we say K is an *n-obstructor.* An easy argument shows that such K cannot embed into any contractible n-manifold.

There is a stronger van Kampen obstruction vk^{2k} in $H^{2k}(K^*, \mathbb{Z})$, which is defined similarly but takes orientations into account. This is known to be complete in high dimensions, which we record as a theorem.

Theorem 5.3.1. *Let K be a k-dimensional simplicial complex.*

- *If $\mathrm{vk}^{2k}_{\mathbb{Z}/2}(K) \neq 0$, then K does not embed into $\mathbb{R}^{2k}$.*
- *If $\mathrm{vk}^{2k}(K) = 0$ and $k \neq 2$, then K embeds into $\mathbb{R}^{2k}$.*

On the other hand, Freedman, Krushkal, and Teichner constructed 2-dimensional complexes that have trivial van Kampen obstruction but still do not embed into $\mathbb{R}^4$.

In [41], Matousek, Tancer, and Wagner gave an explicit formula for the van Kampen obstruction

Theorem 5.3.2. *Let K be a k-dimensional simplicial complex, and fix a total order $<$ on the vertices of K. Let $\sigma = [v_0, v_1, \dots v_k]$ and let $\tau = [w_0, w_1, \dots, w_k]$. Define* *the van Kampen cocycle by*

$$\mu(\sigma,\tau)=\begin{cases}1, & \textit{if } v_0<w_0<v_1<\cdots<v_k<w_k,\\ 0, & \textit{otherwise.}\end{cases}$$

We would now like to relate the van Kampen obstruction to action dimension. This requires the following concept of Bestvina.

Definition. A Z-structure on a group Γ is a pair $(\widetilde{X},Z)$ of spaces satisfying the following four axioms:

- $\widetilde{X}$ is a Euclidean retract.
- Z is a Z-set in $\widetilde{X}$.
- $X=\widetilde{X}/Z$ admits a covering space action of γ with compact quotient.
- The collection of translates of a compact set in X forms a null-sequence in $\widetilde{X}$, i.e. for every open cover $\mathcal{U}$ of $\widetilde{X}$ all but finitely many translates are $\mathcal{U}$-small.

A space Z is a *boundary* of Γ if there is a Z-structure $(\widetilde{X},Z)$ on Γ. For example, CAT(0) and torsion-free hyperbolic groups admit Z-structures. The next theorem is a special case of the main result of [6].

Theorem 5.3.3. *Suppose Z is a boundary of a group Γ, and that K is an embedded n-obstructor in Z. Then* $\operatorname{obdim}(G)\geq n+2$, *and hence* $\operatorname{actdim}(G)\geq n+2$.

Let me briefly give an idea of why their theorem holds. Suppose for simplicity that Γ acts properly and cocompactly on a contractible complex X and on a contractible manifold M. Furthermore, suppose that K is an embedded k-obstructor in X. Since X and M are quasi-isometric, we would like to transfer K over to M and get a contradiction. Since M is contractible we can assume our quasi-isometry is continuous, but unfortunately, there is no reason that it should restrict to an embedding of K. However, if we assume that our map $K\to X$ mapped disjoint simplifies very far apart, then this property will be preserved by this quasi-isometry. In other words, we get an induced *almost embedding* $K \to X$ in that disjoint simplices get mapped disjointly, which is enough for the van Kampen obstruction to provide a contradiction. If we can find an embedded obstructor in the boundary of our group, then we can map a coned obstructor to our group with simplices mapped far apart.

5.4 The action dimension of right-angled Artin groups

We shall briefly go over the main theorem of [3]. We first need to introduce the octahedralization of a simplicial complex.

The octahedralization of a simplicial complex Given a finite set V, let $\Delta(V)$ denote the full simplex on V and let $O(V)$ denote the boundary complex of the octahedron on V. In other words, $O(V)$ is the simplicial complex with vertex set $V \times \{\pm 1\}$ such that a subset $\{(v_0, \varepsilon_0), \dots, (v_k, \varepsilon_k)\}$ of $V \times \{\pm 1\}$ spans a k-simplex if and only if its first coordinates $v_0, \dots v_k$ are distinct. Projection onto the first factor $V \times \{\pm 1\} \to V$ induces a simplicial projection $p : O(V) \to \Delta(V)$. We denote the vertex $(v, +1)$ or $(v, -1)$ by v^+ or v^-, respectively.

Any finite simplicial complex L with vertex set V is a subcomplex of $\Delta(V)$. The *octahedralization* OL of L is the inverse image of L in $O(V)$:

$$OL := p^{-1}(L) \subset O(V).$$

We also say that OL is the result of "doubling the vertices of L."

Heuristically, OL is constructed by gluing together n-octahedra, one for each n-simplex of L. If two simplifies of L intersect, then the corresponding octahedra intersect in a sub-octahedron, and the whole construction is canonical.

Now, if L is a flag complex and A_L the associated right-angled Artin group, recall that one model for $K(A_L, 1)$ is the Salvetti complex Y_L. Since Y_L has an $n+1$-torus for every n-simplex in L, in the universal cover $\widetilde{Y}_L$ there is an associated $n+1$-plane. The boundary of this plane is an n-octahedron. The different planes corresponding to different simplifies exactly intersect in such a way so as ensure that

OL is contained in a boundary of A_L. Therefore, if we determine the embedding dimension of OL, we are well on our way to determining the action dimension of A_L.

Main Theorem. *Suppose L is a k-dimensional flag complex.*

1. *If $H_k(L;\mathbb{Z}/2) \neq 0$, then $\mathrm{vk}_{\mathbb{Z}/2}(OL) \neq 0$. Consequently,*
$$\operatorname{actdim} A_L = 2k+2 = 2\operatorname{gd} A_L.$$
2. *If $H_k(L;\mathbb{Z}/2) = 0$, then $\mathrm{vk}_{\mathbb{Z}/2}(OL) = 0$. So, for $k \neq 2$, $\operatorname{embdim} OL \leq 2k$. Consequently, $\operatorname{actdim} A_L \leq 2k+1$.*

Corollary 5.4.1. *Suppose $\dim L = k$ with $k \neq 2$. Then A_L is the fundamental group of an aspherical $(2k+1)$-manifold if and only if $H_k(L;\mathbb{Z}/2) = 0$.*

Remark. For $k = 1$, the corollary was proved previously by Droms [21]. In [26] Gordon extended this to all Artin groups as follows: Suppose L is the nerve of a Coxeter group where the edges of L are labeled by integers ≥ 2 and that A_L is the corresponding Artin group. Then A_L is a 3-manifold group if and only if each component of L is either a tree or a 2-simplex with edges labeled 2. (In the case where all edge labels of L are required to be even, this had been proved earlier by Hermiller and Meier [32].)

Remark. Interestingly, the octahedralization of a simplicial complex also contains the obstructor considered by Bestvina and Feighn in their computation of the action dimension of lattices in Lie groups.

Remark. The L^2-Betti numbers of right-angled Artin groups were explicitly computed by Davis and Leary; the formula is
$$b_i^{(2)}(A_L) = \overline{b_{i-1}}(L),$$
where $\overline{b_{i-1}}(L)$ is the reduced Betti number of L. Therefore, Theorem 5.4 is a verification of the Action Dimension conjecture in many cases.

In the next two sections, we will sketch the proof of the the Main Theorem.

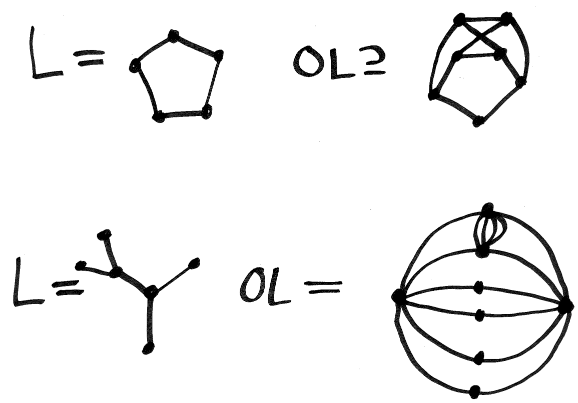


Figure 5.1: Embedding octahedalizations in dimension 1

5.5 The case where $H_k(L;\mathbb{Z}_2) \neq 0$

We shall construct an explicit van Kampen obstruction as follows.

Suppose M is a $\mathbb{Z}/2$-valued k-cycle on L. Identify M with its support (i.e., M is identified with the subcomplex which is the union of those k-simplices σ which have nonzero coefficient in M.) Choose a k-simplex $\Delta \in M$ with vertices $v_0, \ldots, v_k$.

Let $v_i^{\pm}$ denote the two vertices in OM lying above v_i. Let D be the full subcomplex of OL containing M^- and the doubled vertices $v_0^{\pm}, \ldots, v_k^{\pm}$ of Δ. We say that D is M *doubled over the simplex* Δ. Define a chain $\Omega \in C_{2k}(\mathcal{C}(D);\mathbb{Z}/2)$ by declaring the $2k$-cell $[\sigma,\tau]$ of $\mathcal{C}(D)$ to be in Ω if and only if

- $\sigma \cap \tau = \emptyset$, and
- $\Delta^0 \subset p(\sigma) \cup p(\tau)$.

  (Here Δ^0 denotes the 0-skeleton of Δ.)

Lemma 5.5.1. *Ω is a cycle*

Proof. We need to check that for any disjoint k-cell σ and $(k-1)$-cell α, there are an even number of τ such that $\{\sigma,\tau\} \in \Omega$ and $\alpha \subset \tau$.

The proof breaks down into two cases.

- Δ^0 is not contained in $p(\sigma)^0 \cup p(\alpha)^0$.

  In this case, there are either two or zero options for τ.

- Δ^0 is contained in $p(\sigma)^0 \cup p(\alpha)^0$.

  In this case, note that p is injective on the set of vertices v such that $\alpha * v \in D$, and since M is a cycle there are an even number of such vertices.

□

We shall not go through the proof that Ω is nontrivial; it turns out to be an explicit computation using the description of the van Kampen cocycle in [41].

5.6 The case where $H_k(L, \mathbb{Z}_2) = 0$.

To show that $\mathrm{vk}^{2k}(OL) = 0$, we will show that the van Kampen cocycle evaluates to 0 on every cycle. The key idea is that $H_k(L) = 0$ restricts the possible cycles that can occur. The following lemma is key to this approach.

Lemma 5.6.1. *Let Σ be a cycle in $H_{2k}(OL^*, \mathbb{Z}_2)$. For any k-dimensional $\sigma \in OL$, the collection*

$$\{\tau | \{\sigma, \tau\} \in \Sigma$$

is a cycle in $H_k(OL, \mathbb{Z}_2)$.

Proof. This is immediate from the definition. Since Σ is a cycle, for every k-cell σ and $(k-1)$-cell α, there is an even number of τ containing α such that $\{\sigma, \tau\} \in \Sigma$. Therefore, the collection of such τ forms a cycle. □

This lemma of course holds for every simplicial complex, not just OL. Note that this lemma only holds in the top dimension. It is essentially this fact that makes the van Kampen obstructions in other dimensions so difficult to work with.

To study the van Kampen obstruction we can examine the cycles in OL. If $H_k(L, \mathbb{Z}_2) = 0$, then each of the cycles vanishes under the projection map $p : OL \to L$. Therefore, we have the following:

Lemma 5.6.2. *If $H_k(L, \mathbb{Z}_2) = 0$, then for any $\sigma \in L$ and any cycle Γ in $H_k(OL, \mathbb{Z}_2)$, the collection*

$$\{\tau \in \Gamma | \tau \in p^{-1}(\sigma)\}$$

is even.

Theorem 5.6.3. *If L is a k-dimensional flag complex with $H_k(L, \mathbb{Z}_2) = 0$, then the $\mathbb{Z}_2$-valued van Kampen obstruction $\mathrm{vk}^2_{\mathbb{Z}/2}\, k \in H_{2k}(OL, \mathbb{Z}_2)$ is trivial.*

Proof. Fix a total order $<$ on the vertices of L, and extend to the vertices of OL by

$$v_0 < v_0^+ < v_1 < v_1^+ < \cdots < v_n < v_n^+.$$

Let Σ be any cycle in $\mathcal{C}(OL)$. We will show that the sum

$$\sum_{\{\sigma,\tau\}\in\Sigma} \mathrm{vk}^2_{\mathbb{Z}/2}\, k(\{\sigma,\tau\}) = 0 \mod 2.$$

Note that

$$\sum_{\{\sigma,\tau\}\in\Sigma} \mathrm{vk}^2_{\mathbb{Z}/2}\, k(\{\sigma,\tau\}) = \sum_{c\in L\cup\emptyset} \sum_{\substack{\{\sigma,\tau\}\in\Sigma \\ p(\sigma)\cap\hat{p}(\tau)=c}} \mathrm{vk}^2_{\mathbb{Z}/2}\, k(\{\sigma,\tau)$$

which further decomposes as

$$\sum_{c\in L\cup\emptyset} \sum_{\substack{a,b\in L^k \\ a\cap b=c}} \sum_{\substack{\{\sigma,\tau\}\in\Sigma \\ p(\sigma)\cap\hat{p}(\tau)=c}} \mathrm{vk}^2_{\mathbb{Z}/2}\, k(\{\sigma,\tau).$$

In the second sum, we are summing over all *unordered* pairs of k-cells of L which intersect at c. Finally, the last sum decomposes as

$$\sum_{\sigma\in p^{-1}(a)} \sum_{\substack{\{\sigma,\tau\}\in\Sigma \\ \tau\in p^{-1}(b)}} \mathrm{vk}^2_{\mathbb{Z}/2}\, k(\{\sigma,\tau)$$

and we will show that this innermost sum is even.

Fix $\sigma \in p^{-1}(a)$ and suppose that $\mathrm{vk}^{2k}_{\mathbb{Z}/2}(\{\sigma,\tau\} = 1$ for $p(\tau) = b$. By our choice of ordering, $\mathrm{vk}^{2k}_{\mathbb{Z}/2}(\{\sigma,\tau\} = 1$ for all τ' such that $p(\tau') = b$ and $\tau = \tau'$ on $\hat{p}^{-1}(c)$. Note that such τ' are precisely the simplices in $p^{-1}(b)$ that are disjoint from σ. By Lemma 5.6.2, $\{\sigma,\tau'\} \in \Sigma$ for an even number of such τ', which implies the innermost sum, and hence the total sum, is even. $\square$

Remark. Technically, we have only shown that $\mathrm{vk}^{2k}_{\mathbb{Z}/2}(OL) = 0$ when we need to show that $\mathrm{vk}^{2k}(OL) = 0$. This requires a little more effort, the key idea is that $H_k(L, \mathbb{Z}_2) = 0$ implies that $H_k(L, \mathbb{Z}_{2^k}) = 0$ for all k, which is enough to show the $\mathbb{Z}$-valued obstruction vanishes.

Questions

We conclude with a number of questions.

1. Suppose $1 \to H \to G \to Q \to 1$ is an exact sequence with Q finite, and G torsion-free. Suppose that the profinite completion $\hat{H}$ of H is torsion-free. What conditions on H ensure that $\hat{G}$ is torsion-free? Note that is H is good, then G is good and this follows.

2. Are Coxeter groups virtually cocompact special?

3. Does the Strong Atiyah Conjecture hold for virtually sparse special groups?

4. Suppose L is a flag 2-complex with $H_2(L, \mathbb{Z}_2) = 0$. What is $\operatorname{actdim}(A_L)$?

5. The Action Dimension Conjecture for RAAG's claims that if $b_{i-1}(L) \neq 0$, then $\operatorname{actdim}(A_L) \geq 2i$. Note that Theorem 5.4 implies this if each homology class could be realized by a flag cycle. In fact, we only need a weaker sort of "local flagness", given here:

$$\text{For all } \sigma, \tau \in L \text{ with } \Delta^0 \subset \sigma \cup \tau \text{ we have } \sigma \cap \tau \subset \Delta. \tag{$*$}$$

   Can a cycle always be chosen to satisfy the $*$ condition?

6. Are there classes of general Artin groups where the Action Dimension Conjectture can be investigated in the same way as in [3]?

7. Can the methods of [45] calculate the L^2-cohomology of virtually special groups?

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