May 2013

Behavior of Random Dynamical Systems of a Complex Variable

Simon Albert Wagner
University of Wisconsin-Milwaukee

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by

Simon Wagner

A Thesis Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Master of Science
in
Mathematics

at

The University of Wisconsin-Milwaukee
May 2013

ABSTRACT

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Simon Wagner

The University of Wisconsin-Milwaukee, 2013
Under the Supervision of Professor Suzanne Boyd

In this thesis we examine some methods of adding noise to the discrete dynamical system $z \mapsto z^2 + c$, in the complex plane. We compare the "Traditional Random Iteration": choosing a sequence of c-values and applying that sequence of maps to the entire plane, versus what we introduce as "Noisy Random Iteration": for each z and for each iterate calculated, we choose a different c-value. We examine two methods of choices for c: (1) Uniform distribution on a neighborhood of c, versus (2) a Bernoulli choice from two values $\{a, b\}$, with varying probability p in $[0, 1]$ that $c = a$. We show the results of computer investigations, provide definitions and prove some initial results about Noisy Random Iteration. Finally, we leave the audience with some open questions and directions for future research.

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ACKNOWLEDGMENTS

First of all, I would like to express my gratitude to my thesis advisor Suzanne Boyd. You were always willing to answer my questions with pleasure and I really enjoyed working with you a lot. I am deeply grateful for each of our weekly enriching discussions which definitely contributed to the success of this master thesis.

My next thanks goes to my thesis committee, consisting of Prof. Chao Zhu, Prof. Peter Hinow and Prof. Suzanne Boyd. I appreciate that all of you take the time for listening to my defense.

A special thanks goes to Prof. Dr. Hans-Joachim Zwiesler, who made it possible for me to study abroad in the United States. I am very thankful you enabled that I could achieve another masters degree in a different country than Germany.

I want to thank Hiroki Sumi and Mark Comerford for answering questions at the beginning of this investigation.

Finally, I would like to thank you Anja, for generously providing patience and support whenever I needed it, for helping me sustain my joie de vivre and for reminding me what is most important in life.

Chapter 1

Definitions and Examples of Classical and Random Iterations

In order to describe the behavior of Random Iterations of functions, we first need to provide some definitions concerning classical, deterministic chaotic dynamical systems.

1.1 Classical Iteration

The following definitions are provided in [5].

Definition 1.1.1. The **orbit** of a number x_0 under a function f is defined as the sequence of points x_0, $x_1 = f(x_0)$, $x_2 = f^2(x_0)$,$\cdots$, $x_n = f^n(x_0) = f(x_{n-1})$,$\cdots$. The point x_0 is called the **seed** of the orbit. In this thesis, we iterate polynomials of the form $f_c(z) = z^2 + c$, where z and c are complex numbers. $f_c^n(z)$ is meant to be the n-th iteration of the function $f_c(z)$.

Note that instead of writing a complex number in the form $z = a + b\,i$, we will write it in the form $z = (a, b)$.

Now we define a chaotic dynamical system.

Definition 1.1.2. A dynamical system f is called **chaotic** if three conditions are fulfilled:

1. Periodic points of f are dense,
2. the function f is transitive and
3. f depends sensitively on initial conditions.

We say f is transitive if for any pair of points x and y and any $\epsilon > 0$ there is a third point z within ϵ of x whose orbit comes within ϵ of y. The function f depends sensitively on initial conditions if there is a $\beta > 0$ such that for any x and any $\epsilon > 0$ there is a y within ϵ of x and a k such that the distance between $f^k(x)$ and $f^k(y)$ is at least β.

Now we define the filled Julia set K_c, the Julia set J_c, the Basin of Attraction $A_c(\infty)$ and the Mandelbrot set M.

Definition 1.1.3. The **filled Julia set** K_c of a polynomial f_c is the set of points whose orbits are bounded. So $K_c = \{z \mid |f_c^n(z)| \not\to \infty \text{ as } n \to \infty\}$. The **Julia set** J_c, the place where all of the chaotic behavior of a complex function occurs, is the boundary of the filled Julia set. Furthermore, we define the **Basin of Attraction of** ∞ as $A_c(\infty) = \{z \mid |f_c^n(z)| \to \infty \text{ as } n \to \infty\} = \mathbb{C} \setminus K_c$.

Definition 1.1.4. The **Mandelbrot** set M consists of all c-values such that the orbit of 0, which is the critical point of $f_c(z)$, is bounded. That is $M = \{c \in \mathbb{C} \mid |f_c^n(0)| \not\to \infty \text{ as } n \to \infty\} = \{c \in \mathbb{C} \mid c \in K_c\}$. Note that $f_c'(z) = 2z$ and thus the only critical point of $f_c(z) = z^2 + c$ is 0 since $f_c'(0) = 0$.

Remark 1.1.5. Note that M is a subset of the c-plane, whereas the filled Julia set K_c is a subset of the z-plane.

Theorem 1.1.6. *Equivalently to the definition of the Mandelbrot set, it holds that* $M = \{c \mid \ K_c \text{ is connected } \}$. *Further, if* $|c| > 2$, *then the Julia set is equal to the filled Julia set, i.e.* $J_c = K_c$ *and* J_c *is a Cantor set.*

Theorem 1.1.6 is proven in [4] and involves deeper mathematics than we will discuss in this thesis.

Another important theorem that we will use to draw pictures of Julia sets, is the following:

Theorem 1.1.7. *The filled Julia set K_c is contained inside the closed disk of radius $max(|c|, 2)$. That is, $K_c \subseteq \{z \mid |z| \leq max(|c|, 2)\}$.*

To prove this theorem, the following two lemmas are required.

Lemma 1.1.8. ***(The Escape Criterion)*** *If $|z| \geq |c| > 2$, then $|f_c^n(z)| \to \infty$ as $n \to \infty$. This means that $z \notin K_c$ and $z \in A_c(\infty)$.*

Remark 1.1.9. Note that Lemma 1.1.8 implies that if $|c| > 2$, K_c is a subset of the closed disc with center 0 and radius $|c|$, i.e. if $|c| > 2$, then $K_c \subseteq \{z \mid |z| < |c|\}$.

For the case, where $|c| \leq 2$, we can extend the Escape Criterion as follows:

Lemma 1.1.10. *If $|z| > max\{|c|, 2\}$, then $|f_c^n(z)| \to \infty$ as $n \to \infty$. Hence $z \notin K_c$.*

Remark 1.1.11. Notice that Lemma 1.1.10 tells us that if $|c| \leq 2$, then K_c is a subset of the closed disc with center 0 and radius 2, i.e. if $|c| \leq 2$, then $K_c \subseteq \{z \mid |z| < |2|\}$.

Note that the proofs of Lemma 1.1.8 and Lemma 1.1.10 can be found in [5] and therefore we omit those proofs. Thanks to those lemmas, we can prove Theorem 1.1.7.

Proof. (of Theorem 1.1.7) Recall that Lemma 1.1.8 states that if $|c| > 2$, then $K_c \subseteq \{z \mid |z| < |c|\}$ and Lemma 1.1.10 says that if $|c| \leq 2$, then $K_c \subseteq \{z \mid |z| < |2|\}$. Putting both of those lemmas together, we get that for all c-values it holds that $K_c \subseteq \{z \mid z \leq max(|c|, 2)\}$. □

The next result is useful since it helps to state later on an algorithm to draw K_c.

Corollary 1.1.12. *If for some* $k \geq 0$, $|f_c^k(z)| > max\{|c|, 2\}$, *then it follows that* $\exists \lambda > 0$ *s.t.* $|f_c^{k+1}(z)| > (1+\lambda)|f_c^k(z)|$ *and so* $|f_c^n(z)| \to \infty$ *as* $n \to \infty$.

Note that the proof of this corollary is provided in [5].

The following theorem will help us to draw the Mandelbrot set for $f_c(z) = z^2 + c$.

Theorem 1.1.13. *The Mandelbrot set (for* $f_c(z) = z^2 + c$*) is contained in the closed disk with center 0 and radius 2, i.e.* $M = \{c \in \mathbb{C} \mid c \in K_c\} \subseteq \{c \mid |c| \leq 2\}$.

Note this theorem follows from the following result:

Lemma 1.1.14. *If* $|c| > 2$, *then* $|f_c^n(0)| \to \infty$ *as* $n \to \infty$, *so* $0 \notin K_c$ *and* $0 \in A_c(\infty)$.

The proof of this lemma is provided again by [5].

When we want to draw a Julia set or a Mandelbrot set, we do not need to iterate all points in a grid. This is because, due to the stated theorems, it is already clear that the orbit of some points escape to infinity. In the following, we will describe an algorithm to draw either a Julia set or a Mandelbrot set. If the orbit of a point escapes, then it will be colored white, whereas points will be colored black if they do not escape within the first N iterations, where N is the maximum number of iterations we choose. Thus, black points will only yield an approximation of Julia sets and of the Mandelbrot set, since black points could still escape after the first N iterations. Hence, the higher the number of iterations is, the better is the approximation of the respective set. Usually $N = 250$ suffices to draw pictures for $f_c(z) = z^2 + c$.

Algorithm 1.1.15. First, we describe an algorithm, which tells us how to draw a Julia set. Recall that Lemma 1.1.10 says that if $|z| > max\{|c|, 2\}$, then $|f_c^n(z)| \to \infty$ as $n \to \infty$. Thus, for all points outside of the grid $[-R, R]^2$, where $R := max\{|c|, 2\}$, the orbit escapes and so we color those points white. First, we choose a maximum number of iterations N. Now, for each point z within the grid, we compute the first N points of the orbit of z. Recall that Corollary 1.1.12 says, that the orbit of z escapes if for some $k \geq 0$, $|f_c^k(z)| > max\{|c|, 2\}$. Thus, we have to check if $|f_c^k(z)| > max\{|c|, 2\}$ for some $k \leq N$ and if it is the case, then we stop iterating and color the point z white. On the other side, if $|f_c^k(z)| \leq max\{|c|, 2\}$ for all $k \leq N$, then we color the point z black. Hence, the black points yield an approximation of K_c.

Algorithm 1.1.16. Now we describe an algorithm for drawing a Mandelbrot set. Recall that Lemma 1.1.14 says that if $|c| > 2$, then the orbit of 0 under f_c escapes to infinity. So we can color those points white and thus we only have to figure out what happens to c-values with $|c| \leq 2$. So when we draw a Mandelbrot set, we only have to regard c-values within a $[-2, 2] \times [-2, 2]$ - grid. First, we choose a maximum number of iterations N. Then, for each c-value in the grid, we compute the first N points of the orbit of 0 under f_c, i.e. we compute $f_c^k(0)\ \forall\, k = 1, \cdots, N$. Recall that for $|c| \leq 2$, Corollary 1.1.12 says that if for some $k \geq 0$, $|f_c^k(z)| > 2$, then $|f_c^n(z)| \to \infty$ as $n \to \infty$. Thus, if for some $k \leq N$ $|f_c^k(0)| > 2$, then we stop iterating and color the point c white. If $|f_c^k(0)| \leq 2$ for all $k \leq N$, then we color the point c black. So the black points are an appoximation for the Mandelbrot set M.

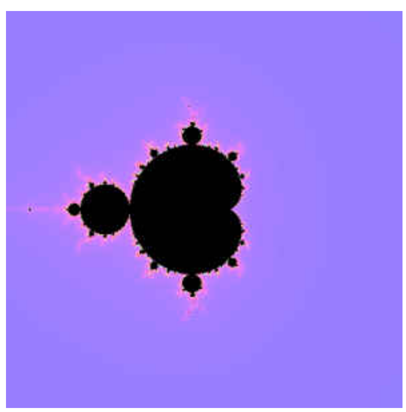
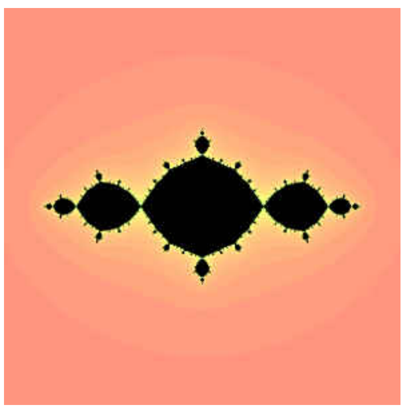


Figure 1.1: The black points in this figure depict an approximation for the Mandelbrot set, calculated with Algorithm 1.1.16.

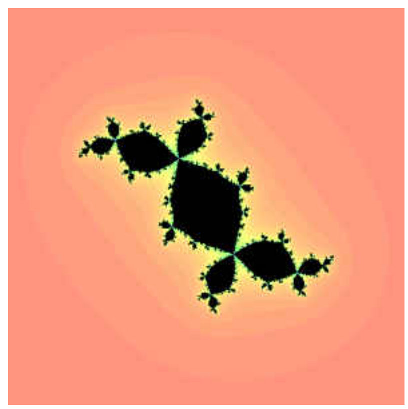
All of the pictures in this thesis are generated with the program Dynamics Explorer, available for download at [2]. In this thesis, the pictures of the Mandelbrot set and the Julia sets will be painted with different colors instead of using only black and white. The color shade of a point is changing accordingly to the number of iterations the point needs before it is clear whether the point escapes to infinity or whether the point belongs to K_c.

Figure 1.1 shows the Mandelbrot set M, drawn in a $[-2,2]^2$ - grid. Recall that the black points are those which do not escape within the first N iterations. Since there might be points, which did not escape within the first N iterations, but which escape after the Nth iteration, the black points are only an approximation for the Mandelbrot set.

Figure 1.2 shows Julia sets obtained by applying Algorithm 1.1.15. In Subfigure 1.2(a) we can see the Julia set for $c = (-1, 0)$. This Julia set is called the “Basilica”. Subfigure 1.2(b) illustrates the Julia set for $c = (-0.12, 0.75)$. This set is called the “Rabbit”.



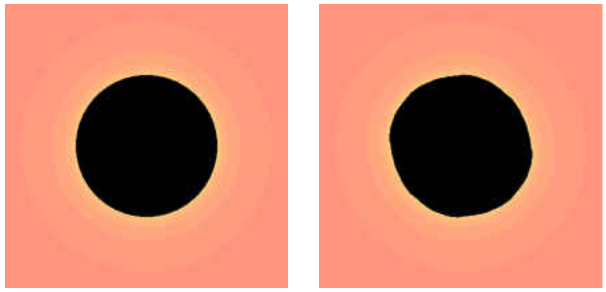
(a) J_c, $c = (-1, 0)$ (Basilica)



(b) J_c, $c = (-0.12, 0.75)$ (Rabbit)

Figure 1.2: Both subfigures show Julia sets, where either $c = (-1, 0)$ (the Basilica) or $c = (-0.12, 0.75)$ (the Rabbit).

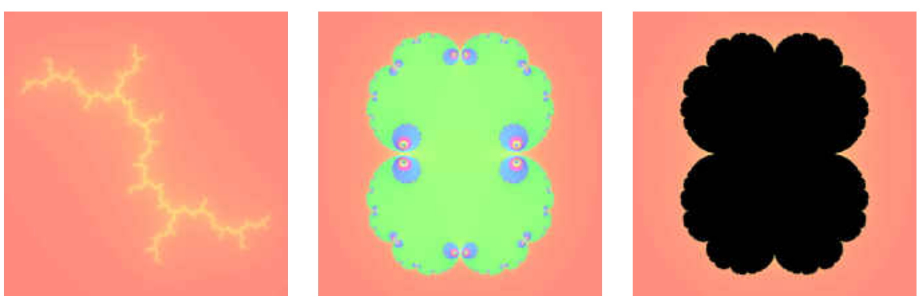
Figure 1.3 shows two Julia sets,whose c-values are in the main cardioid of the Mandelbrot set and close to each other. The c-values of the Julia sets are $c_1 = (0, 0)$ or $c_2 = (0, 0.05)$. We observe that the two Julia sets are very similar, so for $c_1 \approx c_2$ we observe that $J_{c_1} \approx J_{c_2}$ as sets.



(a) J_c, $c = (0, 0)$ (b) J_c, $c = (0, 0.05)$

Figure 1.3: This is a illustration of two Julia sets, whose c-values are close to each other. Subfigure (a) illustrates a Julia set with $c = (0, 0)$ and Subfigure (b) shows a Julia set with $c = (0, 0.05)$.

Subfigure 1.4(a) shows a Julia set for $c = (0, 1)$, which is called a dendrite. In fact, the point $(0, 1)$ is not in the interior of M but instead on its boundary. This Julia set is of importance since $K_c = J_c$. It turns out that this is not a Cantor set and in fact the Julia set is connected. In Subfigure 1.4(b) we can see a Julia set where $K_c = J_c$ but which is a Cantor set, so the Julia set is disconnected. Next, Subfigure 1.4(b) and Subfigure 1.4(c) show that if c varies for example between $c = (0.255, 0)$ and $c = (0.25, 0)$, J_c varies continuously with c but K_c does not vary continously. Instead, K_c only varies continuously inside of a connected component of M. Also, Subfigure 1.4(c) is not a Cantor set.



(a) J_c, $c = (0, 1)$ (b) J_c, $c = (0.255, 0)$ (c) J_c, $c = (0.25, 0)$

Figure 1.4: Subfigures (a) and (b) show Julia sets, where $K_c = J_c$. In Subfigure (a) the Julia set is a dendrite, which is not a Cantor set, whereas in Subfigure (b) the Julia set is a Cantor set. Subfigures (b) and (c) are examples, where J_c varies continuously with c but K_c does not. Subfigure (c) is not a Cantor set.

1.2 Random Iterations of a collection of functions

Overview. In the "classical" case the behavior of iterations of functions $f(z) = z^2 + c$ has been studied already. We will call this the **Classical Iteration**. Now we try to understand the behavior of iterations of a collection of functions $f_e(z) = z^2 + e$, such that the complex number e belongs to a set E. At each step of iteration, a different function is applied, that is another e-value is chosen.

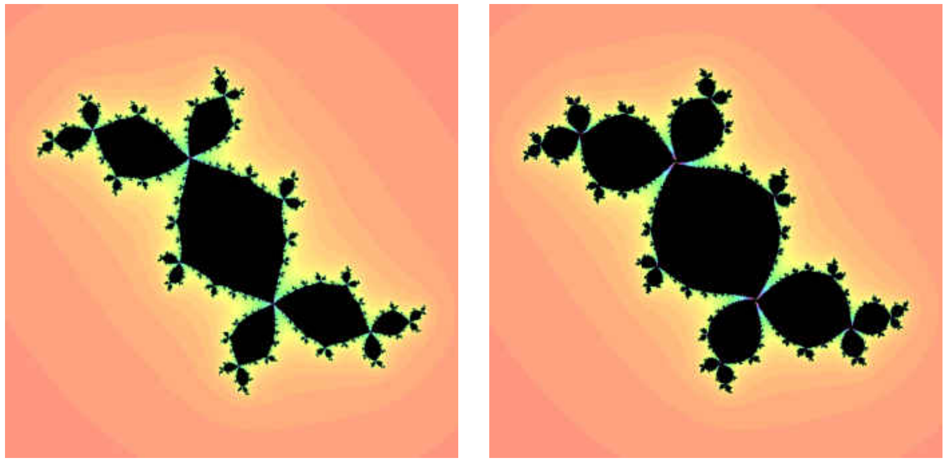
In this thesis we examine two ways of choosing the e-values to iterate the functions. One way is to somehow choose a sequence $\{e_n\}_{n=1}^N$ such that for all n, $e_n \in E$. After that, apply those e_n to the function $f_e(z)$ to compute the N^{th} point in the orbit of the function for each z. Thus, we compute $f_{e_N} \circ f_{e_{N-1}} \circ \cdots \circ f_{e_2} \circ f_{e_1}(z)$, to get the N^{th} point in the orbit. We call this way of choosing the e-values the **Traditional Random Iteration**. This way was for example used by [3], [7] and [1]. Our new approach is to apply for each z at the n^{th} iteration an $e_n(z) \in E$ to the function $f_e(z)$ to compute the next point in the orbit. Thus, for each z, we compute $f_{e_N(z)} \circ f_{e_{N-1}(z)} \circ \cdots \circ f_{e_2(z)} \circ f_{e_1(z)}(z)$, to get the N^{th} point in the orbit. We will call this the **Noisy Random Iteration**. So either, in the Traditional Random Iteration, a sequence $\{e_n\}_{n=1}^N$ is chosen and this sequence is applied to the function $f_e(z)$ for every z in the grid. Or, in the Noisy Random Iteration, for every single z in the grid, there is chosen a different sequence $\{e_n(z)\}_{n=1}^N$ and applied to $f_e(z)$.

Remark 1.2.1. In both, the Traditional Random Iteration and the Noisy Random Iteration, their are many ways of how we could choose the set E and sequences of e-values from E. For example the set E could be $E = \{c + \epsilon \mid \|\epsilon\|_\infty < \delta\}$, for some $\delta > 0$ and some fixed c, where the sequence $\{e_n\}_{n=1}^N$ (or the sequences $\{e_n(z)\}_{n=1}^N$) could be chosen uniformly distributed in $[-\delta, \delta]^2$. The set E could also be $E = \{a, b\}$, where a is chosen if a sample of a Bernoulli distributed random variable takes value 1 and b is chosen if it takes value 0. So choosing $E = \{c + \epsilon \mid \|\epsilon\|_\infty < \delta\}$, where the ϵ-values are uniformly distributed in $[-\delta, \delta]^2$, will be called **Uniform Ball (Noisy or Traditional) Random Iteration**, depending on how the ϵ-values are chosen and how the functions are being iterated. Furthermore, if we choose $E = \{a, b\}$ as stated before, then we will talk about **Bernoulli (Noisy or Traditional) Random Iteration**, also depending on the way of choosing the ϵ-values and iterating the functions.

1.3 Examples of Julia sets close to the Rabbit

In the following, we want to show examples of the Classical Iteration, the Uniform Ball Traditional Random Iteration and the Uniform Ball Noisy Random Iteration where the c-values are close to the Rabbit. Also, we want to compare the two different kinds of iterations with each other.

Examples of Classical Iterations. First, we want to examine some Julia sets close to the Rabbit, that is Julia sets with c-values close to $c = (-1.2, 0.75)$. The Rabbit, which we get by Classical Iteration is illustrated in Subfigure 1.5(a). In Subfigure 1.5(b) we can see a Julia set, also created by Classical Iteration, whose c-value is in some sense “close” to the c-value of the Rabbit. The c-value in this Subfigure is given by $c = (-0.129593, 0.695327)$. Note that we can see a lot of self-similarity in each of those two pictures, that means each component of K_c is shaped alike, so there is almost the same amount of roundness in each component of K_c. Furthermore, Figure 1.5 shows that for both c-values there is a loose “3-fold symmetry” about points where the components of K_c meet. Hence, Julia sets with close c-values created by Classical Iteration seem to be very similar.



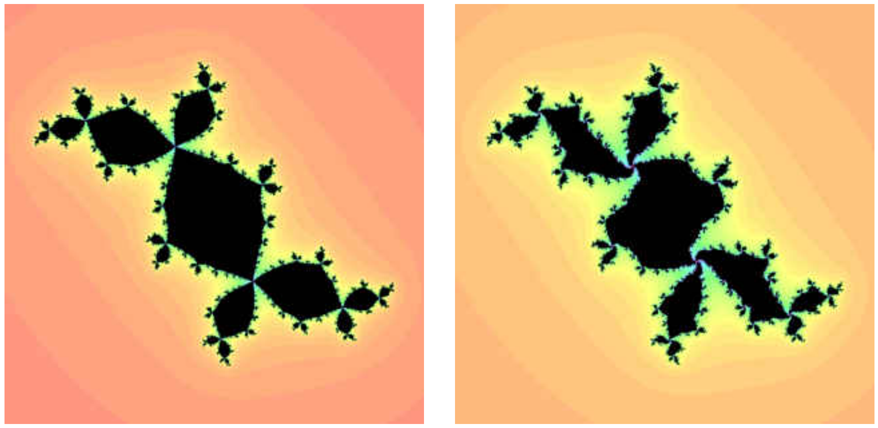
(a) J_c, $c = (-1.2, 0.75)$, Rabbit (b) J_c, $c = (-0.129593, 0.695327)$

Figure 1.5: Both subfigures show Julia sets created by Classical Iteration. Subfigure (a) shows the Rabbit and in Subfigure (b) the c-value is chosen close to the c-value of the Rabbit.

Examples of Uniform Ball Traditional Random Iterations. Next, we choose a c-value close to the Rabbit and a random sequence of e-values, where $E = \{c + \epsilon \mid |\epsilon| \leq \delta\}$. In the following we choose $\delta = 0.09$, so that the ϵ-values influence the behavior of the iteration of $f_e(z)$ not to little, but also not to much. We choose the ϵ-values in the set E to be uniformly distributed in $[-0.09, 0.09]^2$. Recall that in the Uniform Ball Traditional Random Iteration we apply the sequence $\{e_n\}_{n=1}^N$ to $f_e(z) = z^2 + e$, where $e \in E$ for every z.

To see a variety of Julia sets obtained through Traditional Random Iteration, we could either fix a c-value and vary the choice of the sequence $\{e_n\}_{n=1}^N$ or we fix a sequence $\{e_n\}_{n=1}^N$ and vary c. Note that we choose the latter for no particular reason. We observe, that for different chosen c-values in the neighborhood of a certain c-value we get different results.

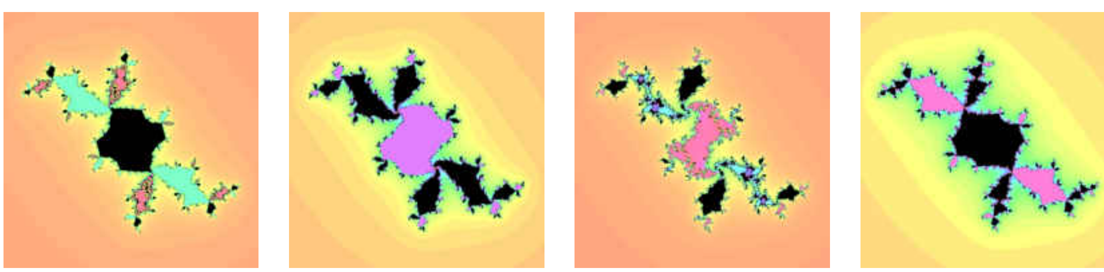
Subfigure 1.6 shows that applying the sequence $\{e_n\}_{n=1}^N$ to $f_e(z)$ and we let $c = (-0.129593, 0.695327)$, then we get a Julia set which still resembles the Rabbit. We can see that this Julia set is still connected, but there is less self-similarity in the shape of each component of K_c.



(a) J_c, $c = (-0.12, 0.75)$, Rabbit (b) J_c, $c = (-0.129593, 0.695327)$

Figure 1.6: Subfigure (a) shows the Rabbit created by Classical Iteration. In Subfigure (b) we can see a Julia set, whose c-value is close to the c-value of the Rabbit and which was created by Uniform Ball Traditional Random Iteration. Subfigure (b) still looks like the Rabbit, but with less self-similarity.

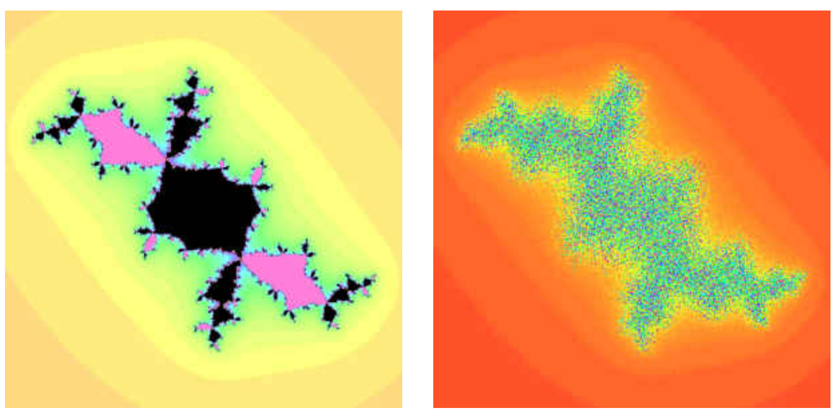
But if we take other c-values in the neighborhood of $c = (-0.12, 0.75)$, then we get Julia sets which in rough outline still look like the Rabbit, but the Julia sets are not connected anymore. Figure 1.7 illustrates some Julia sets, where some portions of K_c are missing. That is, there are many points whose orbits were bounded under J_c with $c = (-0.12, 0.75)$, but which are not bounded for some c-values in the neighborhood $c = (-0.12, 0.75)$.



(a) $J_{(-0.141086,0.724676)}$ (b) $J_{(-0.123511,0.688790)}$ (c) $J_{(-0.097296,0.692368)}$ (d) $J_{(-0.174156,0.730279)}$

Figure 1.7: All subfigures show Julia sets close to the Rabbit, created by Uniform Ball Traditional Random Iteration. For those c-values, the Julia sets are disconnected and large areas of K_c are missing.

Comparison of Uniform Ball Traditional Random Iteration with Uniform Ball Noisy Random Iteration. Now we look at an example, which illustrates the difference between the Uniform Ball Traditional Random Iteration and the Uniform Ball Noisy Random Iteration. Recall that for the Uniform Ball Noisy Random Iteration for every z there is another sequence $\{e_n(z)\}_{n=1}^{N}$ which is applied to $f_e(z)$. For both kinds of iteration, we let $E = \{c + \epsilon \mid \|\epsilon\|_\infty \leq 0.09\}$ and $c = (-0.174156, 0.730279)$. Figure 1.8 illustrates the difference between the picture we get using the Uniform Ball Traditional Random Iteration and the one we get with our Uniform Ball Noisy Random Iteration. The picture, which was created by using the Uniform Ball Noisy Random Iteration looks sandy and messy but in a more uniform way than the picture obtained using the Uniform Ball Traditional Random Iteration. We could say that the picture we get when we are using the Uniform Ball Noisy Random Iteration yields a more accurate approximation of the Julia set for $c = (-0.174156, 0.73279)$ than the picture created by the Uniform Ball Traditional Random Iteration, where the same sequence of ϵ-values was applied for each z in the grid.



(a) J_c, $c = (-0.174156, 0.730279)$, Traditional Random Iteration

(b) J_c, $c = (-0.174156, 0.73279)$, Noisy Random Iteration

Figure 1.8: This figure compares the Uniform Ball Traditional Random Iteration with the Uniform Ball Noisy Random Iteration. Subfigure (a) illustrates a Julia set which was created by using the Uniform Ball Traditional Random Iteration. Subfigure (b) shows a Julia set which was created by using the Uniform Ball Noisy Random Iteration.

Chapter 2

Bernoulli Noisy Random Iteration

In the previous chapter, we iterated $f_e(z) = z^2 + e$ with $E = \{c + \epsilon \mid \|\epsilon\|_\infty < \delta\}$, where $e \in E$ and the ϵ-values were uniformly distributed in $[-\delta, \delta]^2$. In this chapter, we are iterating $f_e(z) = z^2 + e$ for every z under the same sequence $\{e_n\}_{n=1}^N$ chosen from $E = \{a, b\}$. At each step of iteration, a use a sample of a Bernoulli distributed random variable to decide if we iterate $f_a(z)$ or $f_b(z)$. For the Bernoulli distributed random variable it holds that $\mathbb{P}(e = a) = p$ and $\mathbb{P}(e = b) = 1 - p$. Recall that we call this the Bernoulli Noisy Random Iteration.

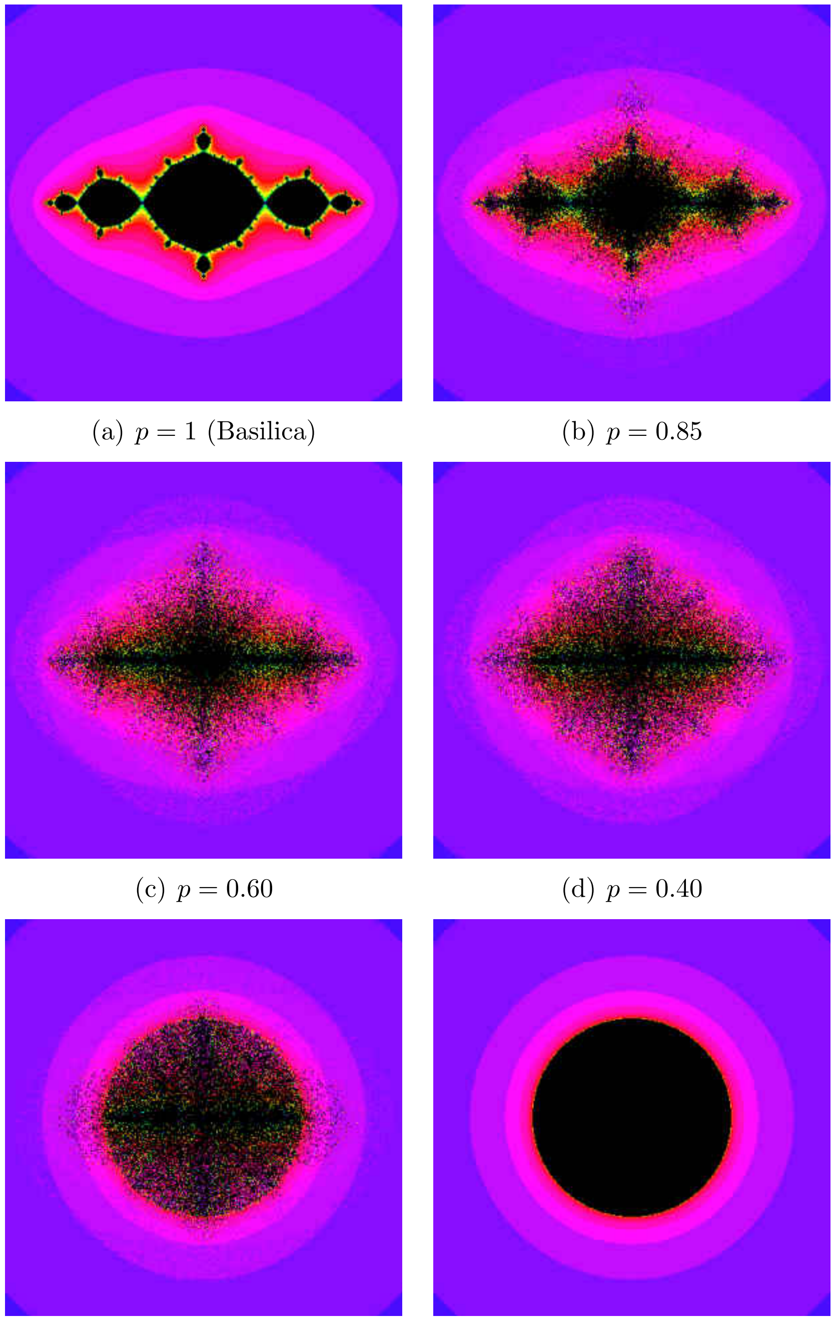
2.1 Bernoulli Noisy Random Iteration including Julia sets

Bernoulli Noisy Random Iteration with Basilica and circle. First, we want to illustrate two examples of a Bernoulli Noisy Random Iteration which involves two different Julia sets. In our first example we choose $a = (-1, 0)$ and $b = (0, 0)$. In our Bernoulli Noisy Random Iteration we draw a sample of a Bernoulli distributed random variable for every z and for every iteration. Suppose we always draw a, then we iterate always $f_e(z) = z^2 + (-1, 0)$. In this case, the Julia set we get is the Basilica. However, if the sample is b, then we iterate $f_e(z) = z^2$ over and over again and thus the Julia set we get a circle with radius 1.

Now we vary p and take a look at the corresponding Julia sets. Figure 2.1 illustrates this. If p changes from $p = 1$ to $p = 0.85$, we can see only minor changes and so the Julia set for $p = 0.85$ still resembles the Basilica. As p gets smaller and smaller the pictures resembles more and more the circle. As p approaches $p = 0.15$ we can see this very well and finally for $p = 0$ we get the circle.

Bernoulli Noisy Random Iteration with dendrite and Rabbit. Another example of a Bernoulli Noisy Random Iteration which includes two Julia sets would be to choose either $a = (0, 1)$ with probability p and $b = (-0.12, 0.75)$ with probability $1 - p$. If $p = 1$, then the set we get is a dendrite, since we always iterate $f_e = z^2 + (0, 1)$. On the other hand, if $p = 0$, then we iterate $f_e = z^2 + (-0.12, 0.75)$ for every z and at each step of iteration and thus the Julia set we get is the Rabbit.

To get a better imagination of what happens if p varies, Figure 2.2 shows pictures for different values of p. We can see that if p changes from $p = 1$ to $p = 0.85$, the picture we get still looks like the dendrite. As p decreases, the picture looks more and more sandy. As p approaches $p = 0.15$, we can see that the sandy pictures resemble more and more the Rabbit. Finally, for $p = 0$ we get the Rabbit.



(a) $p = 1$ (Basilica)

(b) $p = 0.85$

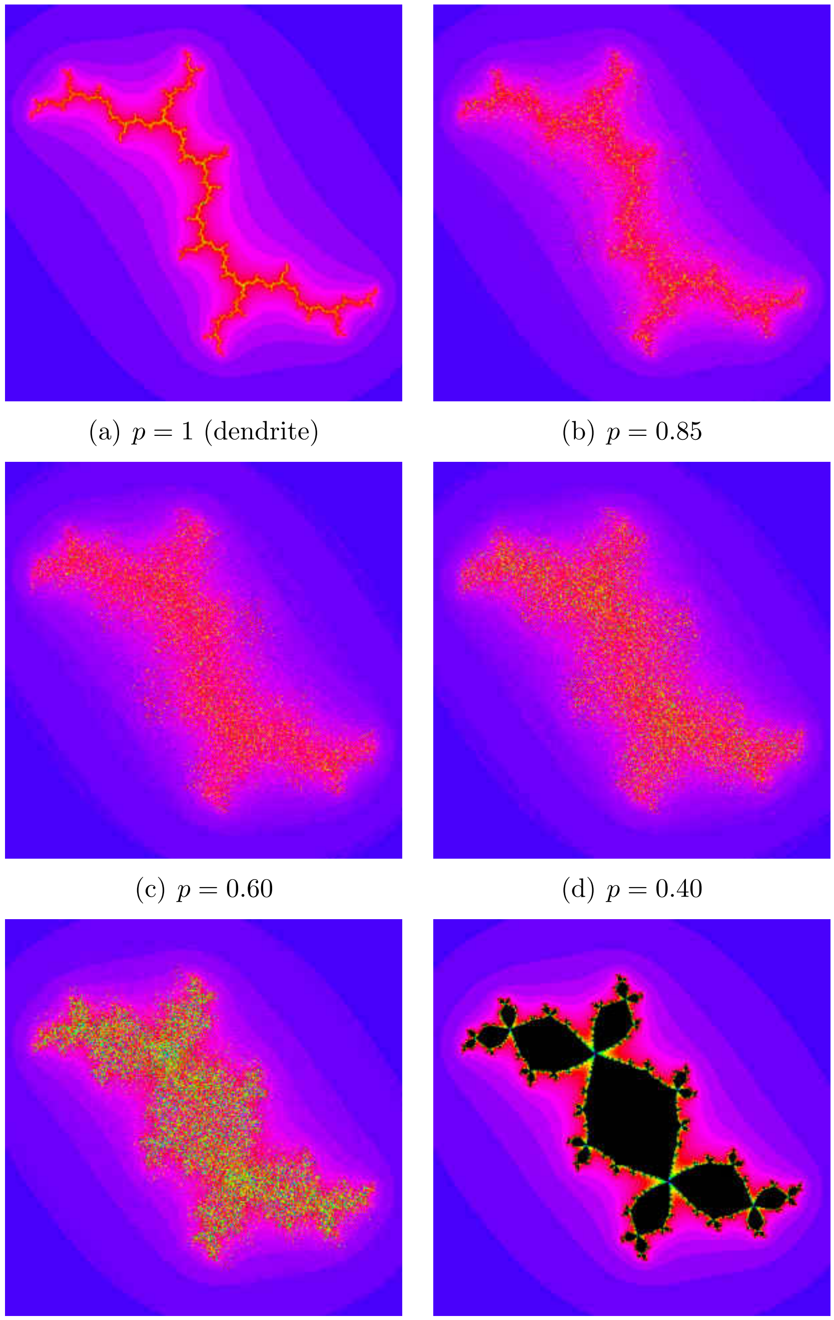
(c) $p = 0.60$

(d) $p = 0.40$

(e) $p = 0.15$

(f) $p = 0$ (circle)

Figure 2.1: This figure illustrates Bernoulli Noisy Random Iterations of $f_e(z) = z^2 + e$ where $e \in \{(-1, 0), (0, 0)\}$ for different probabilities p.



(a) $p = 1$ (dendrite) (b) $p = 0.85$

(c) $p = 0.60$ (d) $p = 0.40$

(e) $p = 0.15$ (f) $p = 0$ (Rabbit)

Figure 2.2: This figure illustrates Bernoulli Noisy Random Iterations of $f_e(z) = z^2 + e$ where $e \in \{(0,1),(0,0)\}$ for different probabilities p.

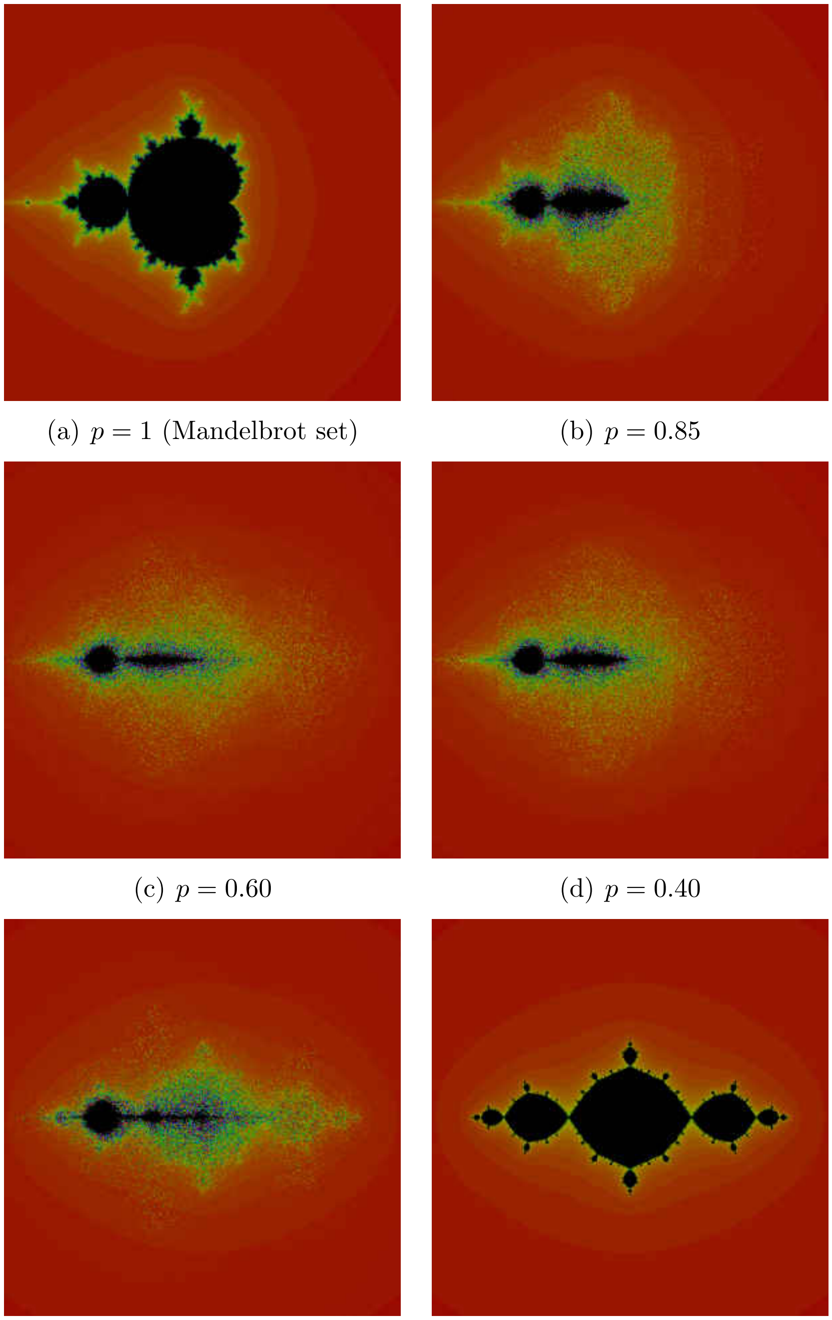
2.2 Bernoulli Noisy Random Iteration including the Mandelbrot set

A different type of iteration that we introduce now is the mixing of a Julia set with the Mandelbrot set under utilization of a Bernoulli distributed random variable. First, we pick a parameter $b \in \mathbb{C}$ and we fix a probability $p \in [0, 1]$. Next, for each point c in the grid and for each step of iteration, we set $E = \{c, b\}$. Again, for every z, we iterate $f_e(z) = z^2 + e$, where $e \in E = \{c, b\}$ and with initial seed $z = 0$, under the same sequence $\{e_n\}_{n=1}^{N}$. Here, $\mathbb{P}(e = c) = p$ and $\mathbb{P}(e = b) = 1 - p$. Finally, we color the point c accordingly to whether the orbit of 0 escapes or not.

Now we want to state two examples of a Bernoulli Noisy Random Iteration which includes a Julia set and the Mandelbrot set. First we want to analyze the behavior of $f_e(z)$ with $b = (-1, 0)$, so $E = \{c, (-1, 0)\}$. If $p = 1$, then we iterate $f_e(z) = z^2 + c$ over and over again, so we simply get the Mandelbrot set. If $p = 0$, then we always iterate $f_e = z^2 + (-1, 0)$ and thus we get the Basilica. If p varies between 0 and 1, then at each step of iteration and for each z, either $f_e(z) = z^2 + c$ or $f_e(z) = z^2 + (-1, 0)$ is being iterated. Figure 2.3 illustrates this for different p-values. This figure shows that if p gets closer to 1, the corresponding picture resembles more the Mandelbrot set whereas if p gets closer to 0, it looks more like the Basilica.

We want to look at another example, so we choose $b = (0, 0)$, so $E = \{c, (0, 0)\}$. Thus, for each c and at each step of iteration, we either iterate $f_e(z) = z^2 + c$ or $f_e(z) = z^2 + (0, 0) = z^2$. As in our last example, for $p = 1$ we get the Mandelbrot set. In the case where $p = 0$, we are always iterating $f_e(z) = z^2$ and thus we get a circle as our Julia set. In Figure 2.4 we can see the according Bernoulli Noisy Random Iterations which include the Mandelbrot set for different values of p. We observe that when p gets closer to 1, the picture resembles the Mandelbrot set and for p-values close to 0 the picture we get looks more like the circle.

We refer the reader to the article [8] in which such an iteration was studied for quadratic maps of one real variable.



(a) $p = 1$ (Mandelbrot set)

(b) $p = 0.85$

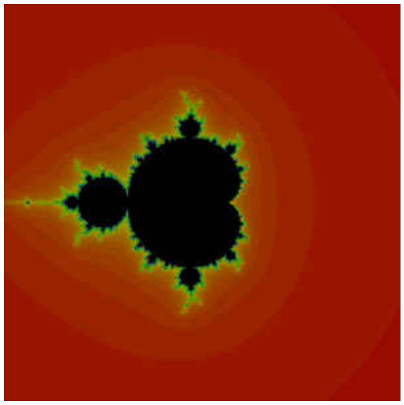
(c) $p = 0.60$

(d) $p = 0.40$

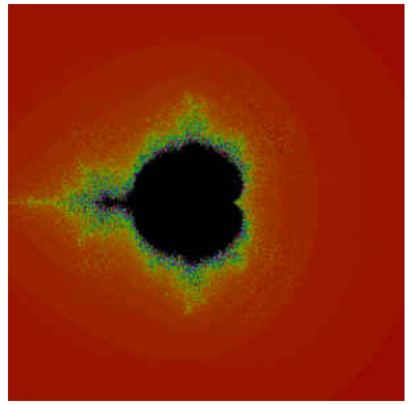
(e) $p = 0.15$

(f) $p = 0$ (Basilica)

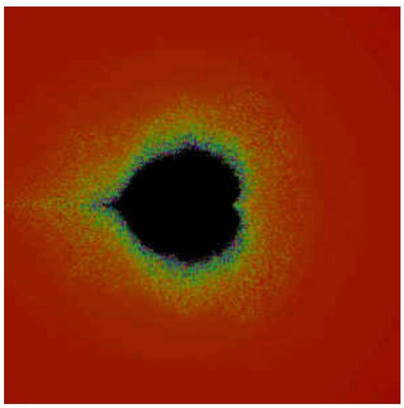
Figure 2.3: Illustration of Bernoulli Noisy Random Iterations of $f_e(z) = z^2 + e$ where $e \in \{(c, (-1, 0)\}$ for different probabilities p.



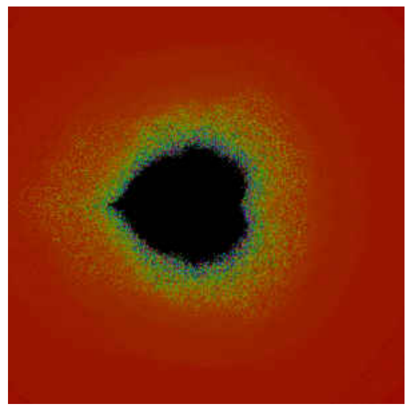
(a) $p = 1$ (Mandelbrot set)



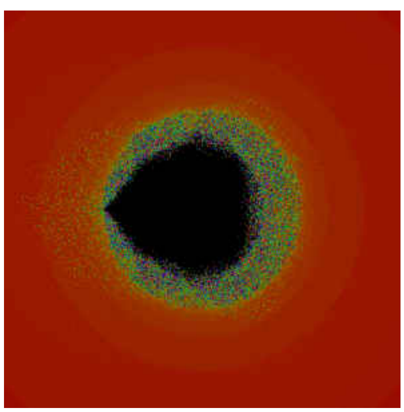
(b) $p = 0.85$



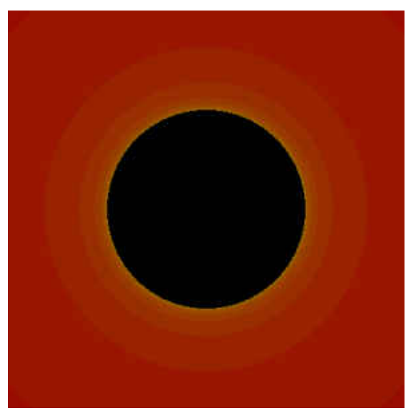
(c) $p = 0.60$



(d) $p = 0.40$



(e) $p = 0.15$



(f) $p = 0$ (circle)

Figure 2.4: This figure illustrates Bernoulli Noisy Random Iterations of $f_e(z) = z^2 + e$ where $e \in \{(c, (0, 0)\}$ for different probabilities p.

Chapter 3

Definitions and Theorems on Noisy Random Iteration

3.1 Definitions

So far, we chose the set E without paying a lot of attention on the probability measure on E. However, we were aware of the fact that elements e in E are drawn with certain probabilities. So for example we said that for $E = \{c + \epsilon \mid \|\epsilon\|_\infty < \delta\}$, the variable ϵ was uniformly distributed and for $E = \{a, b\}$ we said that the probability of drawing a is p and of drawing b is $1 - p$. But now we want to introduce a new notation which we will use from now on.

Notation 3.1.1. Let $\mathbb{B}$ be a σ-algebra of subsets of E and μ be a measure $\mu : \mathbb{B} \to [0, \infty)$ which is σ-additive, $\mu(\emptyset) = 0$ and $\mu(E) = 1$. Then we denote $\mathcal{E} = (E, \mu)$ to be the probability space $(E, \mathbb{B}, \mu)$.

As a shorthand, we will write $\mathcal{E} = \{c + \epsilon \mid \|\epsilon\|_\infty < \delta, \text{uniformly}\}$ for the case where $E = \{c + \epsilon \mid \|\epsilon\|_\infty < \delta\}$ and ϵ is uniformly distributed in $[-\delta, \delta]^2$. Also, we will write $\mathcal{E} = \{\{a, b\}, Bern(p)\}$ for the case where $E = \{a, b\}$ and $\mathbb{P}(e = a) = p$, $\mathbb{P}(e = b) = 1 - p$.

Notation 3.1.2. For $n \in \mathbb{N}$ we define $\{f^n_{\mathcal{E}}(z)\}_{n=1}^\infty$ to be an orbit of z under a choice of $e_1(z), e_2(z), \cdots \in E$. More precise $\{f^n_{\mathcal{E}}(z)\}_{n=1}^\infty := \{\cdots \circ f_{e_n(z)} \circ \cdots \circ f_{e_1(z)}(z)\}$.

Additionally, we define $\{f^n_{\mathcal{E}}(z)\} := \{f^k_{\mathcal{E}}(z)\}^n_{k=1} = \{f_{e_n(z)} \circ \cdots \circ f_{e_1(z)}(z)\}$ to be an orbit until the n-th iteration.

Notation 3.1.3. If the point z escapes to infinity, then we define the random variable $X_{\mathcal{E}-orbit}(z)$ to take value 1. On the other hand, if z does not escape, then $X_{\mathcal{E}-orbit}(z)$ is defined to be 0.

Remark 3.1.4. In general, a probability is assigned to each of the possible outcomes of a random experiment by a probability distribution. In our case, $X_{\mathcal{E}-orbit}(z)$ can take value 0 or 1 and so $X_{\mathcal{E}-orbit}$ is a discrete random variable. A discrete probability distribution assigns to each of the possible outcomes of $X_{\mathcal{E}-orbit}$ a certain probability. Note that the summation of the probabilities for all possible outcomes must be 1. Thus, in our case $P(X_{\mathcal{E}-orbit}(z) = 0) + P(X_{\mathcal{E}-orbit}(z) = 1) = 1$.

Notation 3.1.5. We define $p_{\mathcal{E}-escapes}(z)$ to be the probability that a failure occurs, that is $X_{\mathcal{E}-orbit}(z) = 0$. Thus $p_{\mathcal{E}-escapes}(z)$ is the probability that the orbit of a point z escapes to infinity. Additionally we define $p_{\mathcal{E}-bounded}(z)$ to be the probability that the orbit of z is bounded. Thus we have $p_{\mathcal{E}-escapes}(z) = P(X_{\mathcal{E}-orbit}(z) = 0)$ and $p_{\mathcal{E}-bounded}(z) = P(X_{\mathcal{E}-orbit}(z) = 1)$. Since $P(X_{\mathcal{E}-orbit}(z) = 0) + P(X_{\mathcal{E}-orbit}(z) = 1)$ $= 1$, it follows that $p_{\mathcal{E}-bounded}(z) = 1 - p_{\mathcal{E}-escapes}(z)$.

Notation 3.1.6. Let M be the number of times we calculate $\{f^n_{\mathcal{E}}(z)\}$ for a fixed point z and a fixed number of iterations n.

Law of Large Numbers. By the Law of Large Numbers [6], the sample average $\frac{X_1+X_2+\cdots+X_M}{M}$ converges to the expected value ν for $M \to \infty$, where the random variables $X_1, X_2, \cdots$ is an infinite sequence of independent and identically distributed random variables with $\mathbb{E}(X_1) = \mathbb{E}(X_2) = \cdots = \nu$. Thus, the more times we are calculating $\{f^n_{\mathcal{E}}(z)\}$ in order to calculate $p_{\mathcal{E}-bounded}(z)$, the more precise is the result.

Application of the Law of large numbers. Due to the Law of large numbers, we calculate the probability that the orbit of z escapes as follows:

$$p_{\mathcal{E}-escapes}(z) = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} X_{\mathcal{E}-orbit}(z).$$

Remark 3.1.7. When we regard the set $E = \{a, b\}$, we use in each step of iteration of $\{f^n_{\mathcal{E}}(z)\}$ a sample of a independent and identically distributed Bernoulli(p)-distributed random variable. Also, in the case when $\mathcal{E} = \{c+\epsilon \mid \ |\epsilon| < \delta, \text{uniformly}\}$, the ϵ-values are uniformly distributed random variables which are independent and identically distributed. Thus, in both cases, the law of large numbers holds.

Definition 3.1.8. For a fixed set E and a fixed random process for drawing elements from E to perform iterations on a set of z-values in the plane, we define $K_{\mathcal{E}}(r) := \{z \mid p_{\mathcal{E}-bounded}(z) \geq r\}$, where $r \in (0, 1]$. Furthermore, $\underline{K_{\mathcal{E}}} := K_{\mathcal{E}}(1) = \{z \mid p_{\mathcal{E}-bounded}(z) = 1\} =$ "$K_{\mathcal{E}} - \inf$" and $\overline{K_{\mathcal{E}}} := \bigcup_{r>0} K_{\mathcal{E}}(r) = \{z \mid p_{\mathcal{E}-bounded}(z) > 0\}$ = "$K_{\mathcal{E}} - \sup$". The Basin of Attraction of ∞ can be written in terms of $K_{\mathcal{E}}$ as follows: $A_{\mathcal{E}}(\infty) := \{z \mid p_{\mathcal{E}-bounded}(z) = 0\}$. Note $A_{\mathcal{E}}(\infty) = \mathbb{C} \setminus \underline{K_{\mathcal{E}}} = \{z \mid p_{\mathcal{E}-escapes}(z) = 1\}$.

Note that for Figure 1.8, Figure 2.1 and Figure 2.2, the set E and a random process was fixed, so the definitions above apply to them.

Definition 3.1.9. Now, let $E = E(c)$ be a set of complex numbers which depends continuously on a complex parameter c. So for example fix a parameter b, where b is complex and let $E = \{c, b\}$. Another example would be to fix a $\delta > 0$ and set $\mathcal{E} = \{E, \mu\}$, where $E = \{c + \epsilon \mid \|\epsilon\|_\infty < \delta\}$. We define $M_{\mathcal{E}}(r) := \{c \mid p_{\mathcal{E}-bounded}(0) \geq r\}$, where $r \in (0, 1]$. Additionally, $\underline{M_{\mathcal{E}}} := M_{\mathcal{E}}(1) = \{c \mid p_{\mathcal{E}-bounded}(0) \geq 1\} = \{c \mid p_{\mathcal{E}-bounded}(0) = 1\} =$ "$M_{\mathcal{E}} - \inf$" and $\overline{M_{\mathcal{E}}} := \bigcup_{r>0} M_{\mathcal{E}}(r) = \{c \mid p_{\mathcal{E}-bounded}(0) > 0\} =$ "$M_{\mathcal{E}} - \sup$".

Note that for Figure 2.3 and Figure 2.4 we fixed a $b \in \mathbb{C}$ and we let $E = \{c, b\}$. So the above definitions apply to those figures.

3.2 Statement of theorems and proofs

In this section, we will prove analogs for the Noisy Random Iteration of the theorems in Section 1.1.

Notation 3.2.1. We set $S_E = \sup_{e \in E} |e|$ and $I_E = \inf_{e \in E} |e|$. Here, $|e|$ denotes the modulus of e.

So for example for $E = \{c + \epsilon \mid |\epsilon| < \delta\}, S_E = |c| + \delta$ and $I_E = |c| + \delta$. For $E = \{a, b\}$, $S_E = max\{|a|, |b|\}$ and $I_E = min\{|a|, |b|\}$.

First of all, we want to state the analog of Theorem 1.1.7, which was about the location of the filled Julia set using Classical Iteration.

Theorem 3.2.2. *The set $\overline{K_\mathcal{E}}$ is contained inside the closed disk of radius $max\{2, S_E\}$, i.e. $\overline{K_\mathcal{E}} \subseteq \{z \mid |z| < max\{2, S_E\}\}$.*

In order to be able to prove this theorem, we will prove two lemmas first.

Lemma 3.2.3. ***(Escape Criterion for Noisy Random Iteration.)*** *(analog to Lemma 1.1.8)*
If $|z| \geq S_E > 2$, then $p_{\mathcal{E}-escapes}(z) = 1$, which means that $z \notin K_c$ and so $z \in A_c(\infty)$.

Proof. Suppose $|z| \geq S_E > 2$ and let $\{e_n(z)\}_{n=1}^{\infty} \subseteq E$. We want to prove by induction that $p_{\mathcal{E}-escapes}(z) = 1$.

$$\begin{aligned} |f_{e_1}(z)| &= |z^2 + e_1(z)| \geq |z|^2 - |e_1(z)| \\ &> |z|^2 - S_E \geq |z^2| - |z| \\ &= |z|(|z| - 1) > |z|(1 + \lambda). \end{aligned}$$

Since $|z| > 2$, we can find a $\lambda > 0$ such that $|z| - 1 > 1 + \lambda$ and thus the last inequality holds. So far we have shown that $|f_{e_1(z)}(z)| > |z|(1 + \lambda)$.

Now we suppose that $f_{\mathcal{E}}^{n-1}(z) = f_{e_{n-1}(z)}(f_{\mathcal{E}}^{n-2})(z) > |z|(1+\lambda)^{n-1}$ holds. Then,

$$\begin{aligned}|f_{\mathcal{E}}^{n}(z)| &= f_{e_n(z)}(f_{\mathcal{E}}^{n-1})(z) = |(f_{\mathcal{E}}^{n-1})^2 + e_n(z)| \\ &\geq (f_{\mathcal{E}}^{n-1})^2 - |e_n(z)| > (f_{\mathcal{E}}^{n-1})^2 - (S_E) \\ &> ((1+\lambda))^{n-1}|z|)^2 - |z| > (1+\lambda)^{n-1}|z|^2 - |z| \\ &= (1+\lambda)^{n-1}|z|(|z|-1)) > (1+\lambda)^{n-1}|z|(1+\lambda) \\ &= (1+\lambda)^n|z|.\end{aligned}$$

Thus for this arbitrarily chosen sequence from E, the orbit escapes. Hence, $\{f_{\mathcal{E}}^n(z)\}$ tends surely to infinity for $n \to \infty$, that is $p_{\mathcal{E}-escapes}(z) = 1$. □

As for the Classical Iteration, we can extend the Escape Criterion for the Noisy Random Iteration for the case that $S_E < 2$ as follows.

Lemma 3.2.4. *(analog to Lemma 1.1.10)*
If $|z| \geq max\{S_E, 2\}$, then $\exists\lambda > 0$ s.t. $|f_{\mathcal{E}}^n(z)| > (1+\lambda)^n|z|$ for all n and so $f_{\mathcal{E}}^n(z) \to \infty$ as $n \to \infty$. Thus $p_{\mathcal{E}-escapes}(z) = 1$, which means that $z \notin K_c$ and $z \in A_c(\infty)$.

Proof. Notice that in Lemma 3.2.3 we only used the facts that $|z| \geq S_E$, $|z| > 2$ and $|e_n(z)| < S_E$. Thus it is sufficient to assume that $|z| \geq max\{S_E, 2\}$. Then we can apply the same arguments as in the proof of Lemma 3.2.3 to obtain $p_{\mathcal{E}-escapes}(z) = 1$. □

Proof. (of Theorem 3.2.2) We need to show that $K_{\mathcal{E}} < max\{2, S_E\}\}$. Recall that Lemma 3.2.4 says that if $|z| \geq max\{S_E, 2\}$, then $f_{\mathcal{E}}^n(z) \to \infty$ as $n \to \infty$. Note that if $f_{\mathcal{E}}^n(z) \to \infty$ as $n \to \infty$, then $z \notin K_{\mathcal{E}}$. Thus it follows immediately that $K_{\mathcal{E}} \subseteq \{z \mid |z| < max\{2, S_E\}\}$, which completes the proof. □

Remark 3.2.5. If $S_E > 2$, then $\overline{K_{\mathcal{E}}} = \{z \mid p_{\mathcal{E}-bounded}(z) > 0\} \subseteq \{|z| < S_E\}$. So for example for any $\mathcal{E} = (E, \mu)$ with some probability measure μ we can say that if $E = \{c + \epsilon \mid |\epsilon| < \delta\}$, then $\overline{K_{\mathcal{E}}} \subseteq \{|z| < |c| + \delta\}$. For the case where $E = \{a, b\}$ it follows that $\overline{K_{\mathcal{E}}} \subseteq \{|z| < max\{|a|, |b|\}$.

In the following we will state and prove a corollary, which is helpful to set up and algorithm to draw the sets $M_{\mathcal{E}}(r)$ and $M_{\mathcal{E}}(r)$.

Corollary 3.2.6. *(analog to Corollary 1.1.12)*
If for some $k \geq 0$ and $\{e_1, \cdots, e_k\}$ we get $|f_{\mathcal{E}}^k(z)| > max\{S_E, 2\}$, then for any choices of $\{e_{k+1}, e_{k+2}, \cdots\} \subseteq E$, then $\exists \lambda > 0$ s.t. $|f_{\mathcal{E}}^{j+1}(z)| > (1+\lambda)|f_{\mathcal{E}}^j(z)|$ for every $j \geq k$ and so $|f_{\mathcal{E}}^n(z)|_{n=1}^{\infty}$ escapes.

In order to prove this corollary, we first need to prove the following lemma.

Lemma 3.2.7. *If $|f_{\mathcal{E}}^k(z)| > max\{S_E, 2\}$, then $|f_{\mathcal{E}}^j(z)| > max\{S_E, 2\}$ for $j \geq k$.*

Proof. Suppose $|f_{\mathcal{E}}^k(z)| > max\{S_E, 2\}$, then

$$\begin{aligned} |f_{\mathcal{E}}^{k+1}(z)| &= |f_{e_{k+1}}(f_{\mathcal{E}}^k(z))| = |(f_{\mathcal{E}}^k(z))^2 + e_{k+1}| \\ &\geq |(f_{\mathcal{E}}^k(z))^2| - |e_{k+1}| \geq |(f_{\mathcal{E}}^k(z))^2| - S_E \\ &> max\{S_E, 2\}^2 - S_E. \end{aligned}$$

Now, if $S_E > 2$, then $max\{S_E, 2\} = |S_E|$ and so

$$\begin{aligned} |f_{\mathcal{E}}^{k+1}(z)| &\geq max\{S_E, 2\}^2 - S_E \geq S_E^2 - S_E \\ &= S_E(S_E - 1) > S_E = max\{S_E, 2\}. \end{aligned}$$

Now, if $S_E < 2$, then $max\{S_E, 2\} = 2$ and so

$$\begin{aligned} |f_{\mathcal{E}}^{k+1}(z)| &\geq max\{S_E, 2\}^2 - S_E = 2^2 - S_E \\ &= 4 - S_E > 2 = max\{S_E, 2\}. \end{aligned}$$

So for both cases, $S_E > 2$ and $S_E < 2$ we have shown that $|f_{\mathcal{E}}^{k+1}(z)| \geq max\{S_E, 2\}$. Then it follows via induction that $|f_{\mathcal{E}}^j(z)| > max\{S_E, 2\}$ for every $j \geq k$.

□

Proof. (of Corollary 3.2.6) Suppose $|f_{\mathcal{E}}^k(z)| > max\{S_E, 2\}$ for some $k > 0$ and let $\{e_n\}_{n=1}^{\infty} \subseteq E$, then

$$\begin{aligned}
|f_{\mathcal{E}}^{k+1}(z)| &= f_{e_{k+1}}(f_{\mathcal{E}}^k(z)) = |(f_{\mathcal{E}}^k(z))^2 + e_{k+1}| \\
&\geq |(f_{\mathcal{E}}^k(z))^2| - |e_{k+1}| \\
&\geq |(f_{\mathcal{E}}^k(z))^2| - |(f_{\mathcal{E}}^k(z))| \\
&= |(f_{\mathcal{E}}^k(z))|(|(f_{\mathcal{E}}^k(z))| - 1),
\end{aligned}$$

where the second inequality holds because $|(f_{\mathcal{E}}^k(z))| > max\{S_E, 2\} > |e_{k+1}|$.

Now, since $|(f_{\mathcal{E}}^k(z))| > 2$, we can find a $\lambda > 0$ s.t. $|(f_{\mathcal{E}}^k(z))| - 1 > 1 + \lambda$. Thus we have

$$\begin{aligned}
|f_{\mathcal{E}}^{k+1}| &= |(f_{\mathcal{E}}^k(z))|(|(f_{\mathcal{E}}^k(z))| - 1) \\
&> (1 + \lambda)|(f_{\mathcal{E}}^k(z))|.
\end{aligned}$$

Now recall the statement of Lemma 3.2.7: If $|f_{\mathcal{E}}^k(z)| > max\{S_E, 2\}$, then $|f_{\mathcal{E}}^j(z)| > max\{S_E, 2\}$ for every $j \geq k$. Together with $|f_{\mathcal{E}}^{k+1}(z)| > (1 + \lambda)|(f_{\mathcal{E}}^k(z))|$, we have that
$|f_{\mathcal{E}}^{n+k}(z)| > (1 + \lambda)^n|(f_{\mathcal{E}}^k(z))|$ and finally $|f_{\mathcal{E}}^n(z)| \to \infty$ as n $\to \infty$, which concludes the proof.

□

Remark 3.2.8. Note that Corollary 3.2.6 is useful for drawing pictures of iterations of functions where any kind of Noisy Random Iteration is involved.

Remark 3.2.9. Naturally, we would like to prove an analog of Theorem 1.1.13, which was about the location of the Mandelbrot set. This proof would be about the location of the set $\overline{M_{\mathcal{E}}}$. Recall that $\overline{M_{\mathcal{E}}} := \bigcup_{r>0} M_{\mathcal{E}}(r) = \{c \mid p_{\mathcal{E}-bounded}(0) > 0\}$. However, we found out that such a theorem must be formulated very carefully to depend on $\mathcal{E}$, not just on S_E and I_E, so we leave this for future research. So the analog for Theorem 1.1.13 does not hold.

Note that the Noisy Random Iteration with a Mandelbrot set and a Julia set depends highly on the set $\mathcal{E}$. This is because for this kind of iteration, the set $\mathcal{E}$ is not fixed for every c-value. The issue of proving an analog of Theorem 1.1.13 is that if S_E and I_E are far apart, even if they are both large, potential cancellation in calculating $f_{\mathcal{E}}(z) = z^2 + e$ means we cannot ensure that the orbit of $z = 0$ will escape. For example suppose $e_n = -9$ if n is odd and $e_n = 3$ if n is even. Then $f_3(0) = 3$, $f_{-9}(3) = 3^2 - 9 = 0$, $f_3(0) = 3$ and so on. Maybe the probability for this event is 0, but there might be other cases where cancellation could occur. Thus, we leave the proof to the reader of this thesis.

Chapter 4

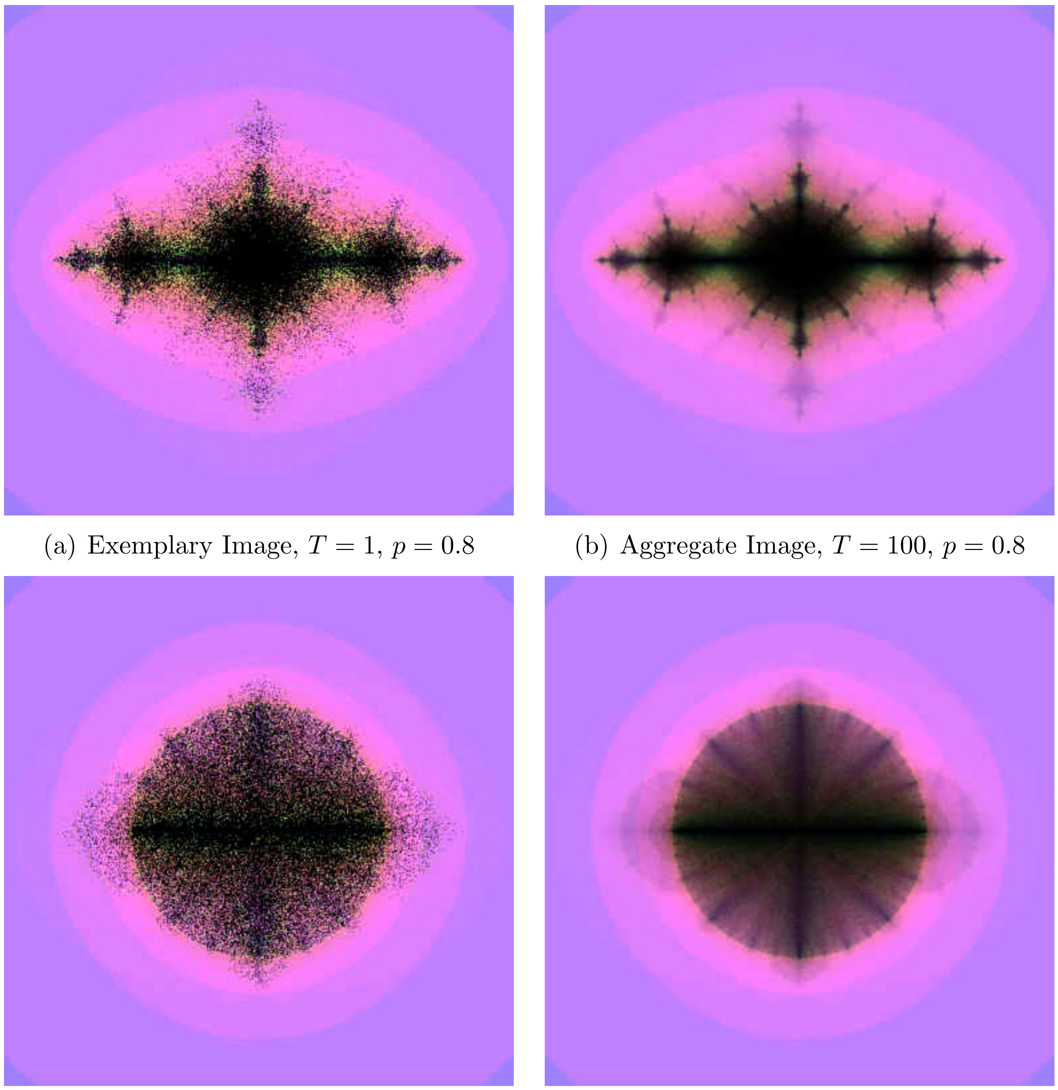
Aggregate Images

When we analysed iterations of a function $f_{\mathcal{E}}(z) = z^2 + e$ with $e \in E$, we paid especially attention to two different cases. First, we choose $E = \{c + \epsilon \mid \|\epsilon\|_\infty \leq \delta\}$, where the ϵ-values were uniformly distributed in $[-\delta, \delta]^2$. Next, for $E = \{a, b\}$, at each step of iteration we either iterated $f_a(z)$ or $f_b(z)$. For both cases, we calculated the orbit of all points z for a sequence $\{e_n\}_{n=1}^{\infty} \subseteq E$ (or sequences $\{e_n(z)\}_{n=1}^{\infty} \subseteq E$) exactly one time. Note that since the orbit of z, i.e. $\{f_{\mathcal{E}}^n(z)\}$ could be different for every single sequence of e-values, the corresponding picture is different for every calculation of the orbits of all z-values in the plane. This was our motivation to introduce $K_{\mathcal{E}}(p) = \{z \mid p_{\mathcal{E}-bounded}(z) \geq p\}$ and to prove where $K_{\mathcal{E}}(p)$ is located. We also defined the set $M_{\mathcal{E}}(r) = \{c \mid p_{\mathcal{E}-bounded}(z) \geq r\}$ and were able to state a corollary, which is helpful to set up an algorithm to draw $K_{\mathcal{E}}(r)$ and $M_{\mathcal{E}}(r)$.

Now we want to illustrate how some sets $K_{\mathcal{E}}(r)$ and $M_{\mathcal{E}}(r)$ look like for different p-values. To get an approximation for $K_{\mathcal{E}}(r)$, we draw T times a picture of some Random Iteration. We will call this picture an **Aggregate Image**. In an Aggregate Image of $K_{\mathcal{E}}(r)$, each pixel color is the average of the colors obtained for that pixel, obtained by calculating the orbit of $f_e(z)$, i.e. $\{f_{\mathcal{E}}^k(z)\}_{k=1}^n$ until the n-th iteration T times.

The higher the number T of pictures we use to create an Aggregate Image is, the more precise it will be. In the case where $T = 1$, the Aggregate Image is the same as the picture we get when we are applying some Random Iteration on $f_{\mathcal{E}}(z)$ exactly one time. We will call an Aggregate Image with $T = 1$ an **Exemplary Image**.

Aggregate Images of a Bernoulli Noisy Random Iterations of two Julia sets. Figure 4.1 compares Exemplary Images of a Bernoulli Noisy Random Iterations, where $f_{\mathcal{E}}(z) = z^2 + e$, $e \in \{(-1, 0), (0, 0)\}$ and $p = 0.8$ or $p = 0.2$ with the according Aggregate Images for $T = 100$. Here, $p = 0.8$ means that $e = (-1, 0)$ with probability 0.8 and $e = (0, 0)$ with probability 0.2, whereas $p = 0.2$ means that $e = (-1, 0)$ with probability 0.2 and $e = (0, 0)$ with probability 0.8. So in Subfigure 4.1(a) we can see a Exemplary Image with $p = 0.8$. Here it is difficult to identify in which areas the probability that the orbit escapes is about the same. However, Subfigure 4.1(b) shows the according Aggregate Image for $T = 100$. We can see that the Aggregate Image looks like a fractal and we can see areas where the probability that the orbit escapes is 1 (for example in the center of the Basilica) and also areas, where the probability that the orbit escapes is smaller but not 0. When we only look at the picture, we cannot tell the exact probabilities that the orbit of points in certain areas escape or not, but one could compute those by calculating the orbit of a point z a large number of times and see how often the point escapes or not. Subfigure 4.1(c) shows a Bernoulli Noisy Random Iteration with $p = 0.8$, whereas Subfigure 4.1(d) illustrates the Aggregate Image for $T = 100$. As before, we can see that there is a certain area, where orbits do not escape and also that there are other areas where the orbit might escape or not.



(a) Exemplary Image, $T = 1$, $p = 0.8$

(b) Aggregate Image, $T = 100$, $p = 0.8$

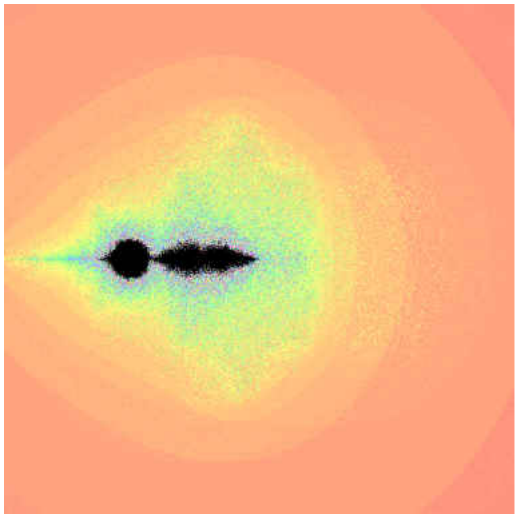
(c) Exemplary Image, $T = 1$, $p = 0.2$

(d) Aggregate Image, $T = 100$, $p = 0.2$

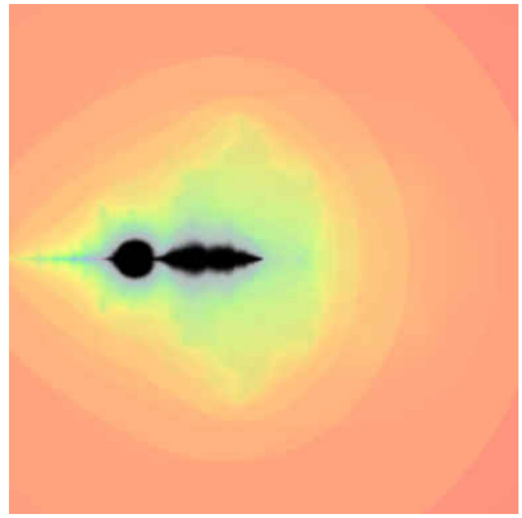
Figure 4.1: This figure compares Exemplary Images of Bernoulli Noisy Random Iterations of $f_{\mathcal{E}}(z) = z^2 + e$ where $e \in \{(-1, 0), (0, 0)\}$ and for certain p-values with Aggregate Images.

Aggregate Images of a Bernoulli Noisy Random Iterations including the Mandelbrot set. We want to take a look at another example. Figure 4.2 compares the Bernoulli Noisy Random Iterations of $f_{\mathcal{E}}(z) = z^2 + e$ where $e \in \{z, (0,0)\}$ and for certain p-values with the Aggregate Images of the Bernoulli Noisy Random Iterations. We can see that the Aggregate Images (see Subfigures 4.2(b) and 4.2(d)) do not look as sandy as the Exemplary Images (see Subfigures 4.2(a) and 4.2(c)). The Aggregate Images show very precisely, in which area the probability is 1 that the orbit does not escape. Furthermore we can see the Mandelbrot set in Subfigure 4.2(b) more clearly than in Subfigure 4.2(a). Also, the Basilica in Subfigure 4.2(d) is more clear than in Subfigure 4.2(c).

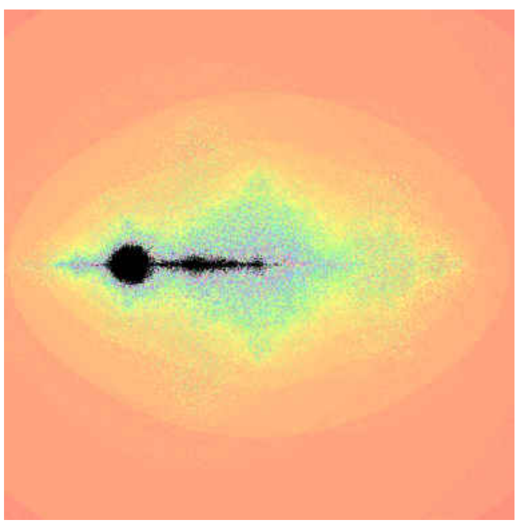
Aggregate Images of a Uniform Ball Noisy Random Iteration. Now, we want to take a look at Aggregate Images of Uniform Ball Noisy Random Iterations. Figure 4.3 compares the Uniform Ball Noisy Random Iterations of $f_{\mathcal{E}}(z) = z^2 + e$ with an Exemplary Image of the Uniform Ball Noisy Random Iterations. We choose $E = \{(-0.1225, 0.7325) + \epsilon \mid |\epsilon| \leq \delta\}$ and vary the variable δ. Comparing the Exemplary Image in Subfigure 4.3(a) with the Aggregate Image in Subfigure 4.3(b), we can see that there are only two areas, where the Aggregate Image looks more fuzzy than the Exemplary Image. In both of those pictures we chose $\delta = 0.01$. But if we increase δ, we can see that larger areas in the Aggreate Images look fuzzy (regard Subfigure 4.3(d) and 4.3(f)). Regarding the Subfigures 4.3(f) and 4.3(f), we can observe that the more sandy the Exemplary Image looks like, the more fuzzy gets the Aggreate Image.



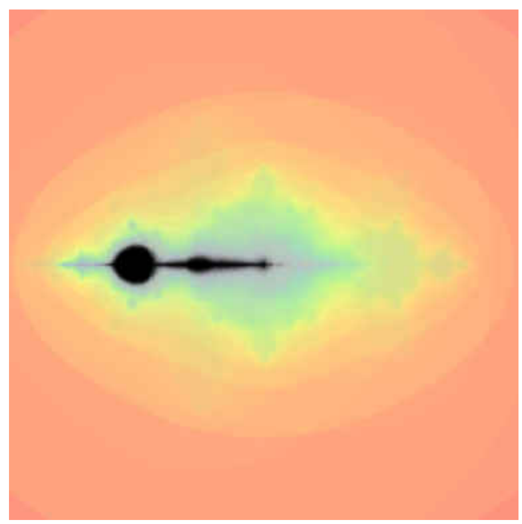
(a) Exemplary Image, $T = 1$, $p = 0.75$



(b) Aggregate Image, $T = 100$, $p = 0.75$

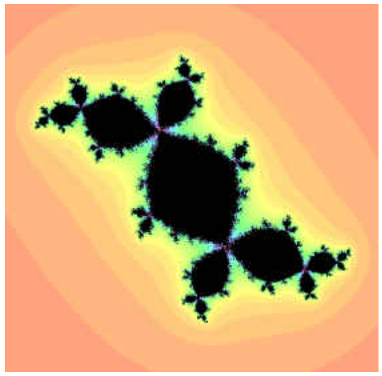


(c) Exemplary Image, $T = 1$, $p = 0.25$

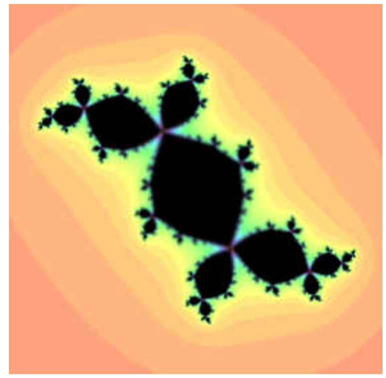


(d) Aggregate Image, $T = 100$, $p = 0.25$

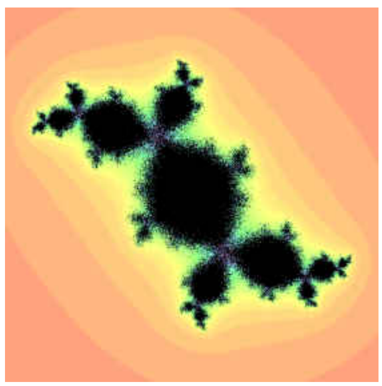
Figure 4.2: This figure compares Exemplary Images of Bernoulli Noisy Random Iterations of $f_{\mathcal{E}}(z) = z^2 + e$ where $e \in \{c, (0,0)\}$ and for certain p-values with Aggregate Images.



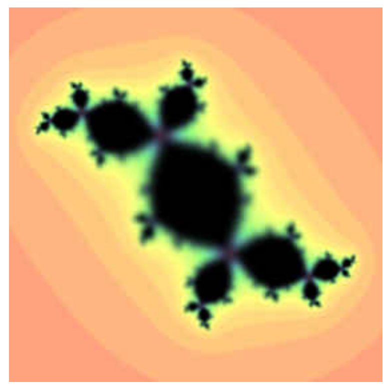
(a) Exemplary Image, $T = 1$, $\delta = 0.01$



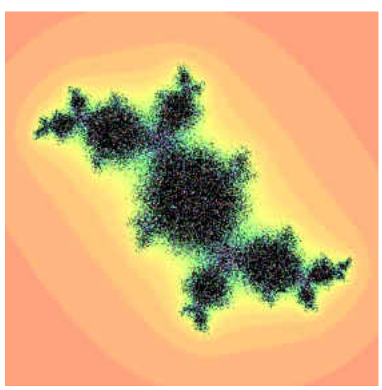
(b) Aggregate Image, $T = 450$, $\delta = 0.01$



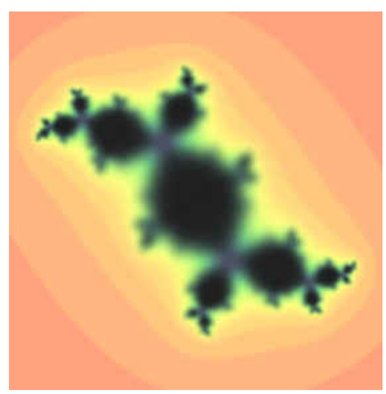
(c) Exemplary Image, $T = 1$, $\delta = 0.03$



(d) Aggregate Image, $T = 350$, $\delta = 0.03$



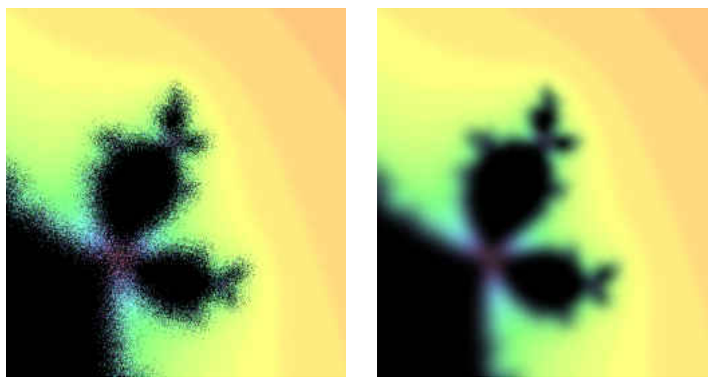
(e) Exemplary Image, $T = 1$, $\delta = 0.05$



(f) Aggregate Image, $T = 500$, $\delta = 0.05$

Figure 4.3: This figure compares Examplary Images of Uniform Ball Noisy Random Iterations of $f_{\mathcal{E}}(z) = z^2 + e$ where $E = \{(-0.1225, 0.7325) + \epsilon \mid \|\epsilon\|_\infty \leq \delta\}$ with Aggregate Images.

Aggregate Images of an Enlagement of a Uniform Ball Noisy Random Iteration. Finally, we want to regard an enlargement of an area in the Rabbit, using Uniform Ball Noisy Random Iterations. Figure 4.4 compares an enlargement of an Exemplary Image of Uniform Ball Noisy Random Iterations of $f_{\mathcal{E}}(z) = z^2 + e$ where $E = \{(-0.1225, 0.7325) + \epsilon \mid |\epsilon| \leq \delta\}$ with an enlargement of an Aggregate Image. We can see that in the Aggregate Image it is more clear, which points to not escape to infinity with probability 1. Again, we can observe, that the Exemplary Image is sandy whereas the Aggregate Image is fuzzy.



(a) Exemplary Image, $T = 1$, $\delta = 0.01$ (b) Aggregate Image, $T = 750$, $\delta = 0.01$

Figure 4.4: This figure compares an enlargement of an Exemplary Image of a Uniform Ball Noisy Random Iteration of $f_{\mathcal{E}}(z) = z^2 + e$ where $E = \{(-0.1225, 0.7325) + \epsilon \mid \|\epsilon\|_\infty \leq \delta\}$ with an enlargement of an Aggregate Image.

Chapter 5

Conjectures

In this thesis, we probably could find a huge amount of conjectures which we could state. Any reader can look at the pictures in this thesis and formulate a lot more conjectures. Here is a start:

Conjecture 5.0.1. Suppose we iterate $f_{\mathcal{E}}(z)$ where $\mathcal{E} = \{\{c, b\}, p\}$. For any $p \in [0, 1]$, if $b \in M$, then $\exists$ neighborhood $B \ni b$, s.t. $B \subseteq M_{\mathcal{E}}(1)$.

Conjecture 5.0.2. Suppose we iterate $f_{\mathcal{E}}(z)$ where $E = \{\{a, b\}, p\}$. Then, $p_{\mathcal{E}-bounded}(z) \to$ P(orbit of z is bounded under $z^2 + a$) as $p \to 1$. On the other hand, $p_{\mathcal{E}-bounded}(z) \to$ P(orbit of z is bounded under $z^2 + b$) as $p \to 0$.

Question 5.0.3. If $\mathcal{E} = \{E, \mu\}$ depends continuously on a complex parameter c, then for each r does the set $K_{\mathcal{E}}(r)$ depend continuously on c?

Question 5.0.4. How does Aggregate Images of Traditional Random Iterations compare to Aggregate Images of Noisy Random Iterations?

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