December 2012

Pricing of American Lookback Options Using Linear Programming

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by

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A Thesis Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Master of Science
in
Mathematics

at

The University of Wisconsin-Milwaukee
December 2012

ABSTRACT

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The University of Wisconsin-Milwaukee, 2012
Under the Supervision of Professor Stockbridge

We will introduce the American lookback option in the Black-Scholes model. Afterwards we will examine the process it inherits and derive and formulate the linear program needed to price it.

As an approximation, we will apply a time-discretization and a truncation of the infinite space. The requirements for a solution are weakened and the optimization problem is reduced to base functions,being linear functions.

In the end we study the numerical results following from the above computations.

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Chapter 1

Introduction

Options have become extremely popular investment vehicles. Two of the main reasons are the systematic way of pricing them, which gives confidence while buying or selling them, and the corresponding manner of hedging the risks taken on by the sellers of the options. An American lookback option involves the combination of two exotic features: early exercise feature and lookback feature.

In chapter 1 we will introduce the basic model of the thesis. We will explain what an option is and describe a lookback option both of European one and its American counterpart.

The fundamental question of the thesis is how to price the above option. Our way is a linear programming approach. An American option has two uncertainties (running maximum and stock price at execution). In chapter 2 we reduce the dimension of the stochastic process and derive the necessary LP.

Since we cannot model the infinte horizon on time and state space, we have to truncate the space. In chapter 3 we will first discretize the time horizon and then we limit the state space down to finite elements.

The goal in chapter 4 is to show numerical results of our theoretical work. In order to implement the LP, we need to make it more accessible for the computer. So we will rewrite the LP in terms of matrices and vectors. In 4.2 the results of the computations are presented.

1.1 Black-Scholes Model

As a beginning of this chapter we will briefly introduce the standard Black-Scholes-Merton model (see e.g.[8] and [2]) as it forms the basis of our later computations. With the Black-Scholes-Merton model we can try to describe the price of an option and its underlying stock over time.
The model considers two kinds of stocks, a risk-free bank account $B = (B_t)_{t\geq 0}$ earning interest at rate $r > 0$, called bond, and a risky asset $S = (S_t)_{t\geq 0}$, called stock. An asset is a financial object whose value is known at present but is liable to change in the future.[1] We assume that the bank account is a function of time t, satisfying the differential equation

$$dB_t = rB_t dt,\ B_0 > 0\,,$$

which can be solved as

$$B_t = B_0 e^{rt}\,.$$

To generate the paths $S_t(\omega)$ of the price of our underlying asset, we use a geometric Brownian motion $W = (W_t^{\mathbb{P}})_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is the σ-algebra generated by the Brownian motion. That is,

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t^{\mathbb{P}}, \qquad S_0 = s_0 > 0 \tag{1.1}$$

where μ denotes the mean return rate and $\sigma > 0$ the constant volatility.

1.2 Options

Based on the previous section, where we have introduced the model for the financial market, we will cover the idea of options.
Options have become extremely popular for two of the main reasons being their attractiveness for investors with the intention of hedging, and there being a systematic way of pricing them, which gives confidence while buying or selling them.

[1] see [5, chapter 1.1]

This systematic way of pricing for common options such as European Options is the famous Black-Scholes formula. Depending on the character of the option, more exotic options may not have a closed formula, and in those cases we have to rely on computational algorithms.[2]
In this thesis, we will only take European and American options into concern.

Definition 1.2.1. *A European option gives its holder the right, but not the obligation to buy or sell to the writer a prescribed asset for a prescribed price at a prescribed time. A call option gives the holder the right to buy, a put option gives her the right to sell.*[3]

Example 1.2.2. *A simple constructed example for a European call option as in definition 1.2.1 is the following. Two parties agreed on a European call option. The price can either go up from initial* $100 *up to* $110 *in one unit of time (being the expiration date) or down to* $90. *The prescribed price for the asset is the initial stock price. If the stock goes up in case 1, then the holder will make use of the option, since it allows him to buy the price* 10$ *cheaper than at the stock market. On the other hand, if the stock goes down, then he won't use the option, since it is cheaper to buy the asset directly from the market.*

Neither one of the cases from example 1.2.2 results in a loss for the holder, respectively at win for the writer. Therefore the writer will not give out the stock for free and wants compensation for this imbalance. The value of the option is this compensation. So the question arises, how high it has to be, so that it is a fair game for both sides, in order to attract enough market participants.
Example 1.2.2 gives an idea of how the payoff, generated by the option, behaves. The writer has to pay the difference between the prescribed price and the market price of the option, as in the first case, or nothing, as in the second case. Let $\Phi_C(S_T)$ be the payout function of the call option, T be the expiration date and K the prescribed price. Then

$$\Phi_C(S_T) = (S_T - K)_+ = \max\{S_T - K, 0\},$$

[2]see [5, chapter 1.2]
[3]see [5, chapter 1.1]

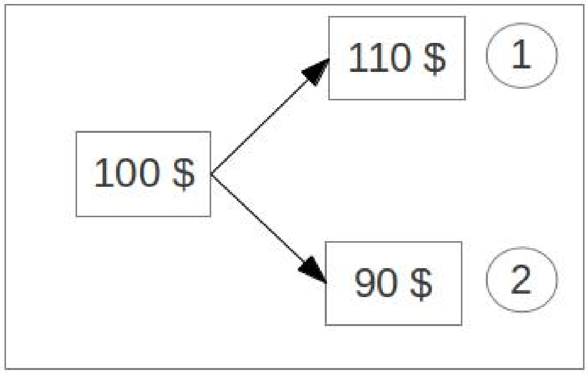

Figure 1.1: Stock movement in example 1.2.2

and a similar approach yields to the payoff function Φ_P for the European put option

$$\Phi_P(S_T) = (K - S_T)_+ = \max\{K - S_T, 0\}.$$

In order to find a value for an option, we have to make the following assumptions (see e.g. [2, Page 640] and [11, Chapter 2.1]):

- the short-term interest rate r is known and constant through time;
- the stock pays no dividend or other distributions;
- there are no transaction costs in buying or selling the stock or the option;
- it is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate;
- there are no penalties to short-selling;
- the underlying asset follows a known stochastic process;
- markets are complete.

Especially noteworthy and of great importance for this thesis is the following type of option.

Definition 1.2.3. *An American option gives its holder the right, but not the obligation to buy or sell to the writer a prescribed asset for a prescribed price at any time until the expiration date.*

When we compare definition 1.2.3 and 1.2.1, we notice the new policy on the exercise time. In contrast to example 1.2.2 with only two time points, one of the previous assumptions was a complete market, which implies that assets and options can be traded any time. So we need to know more about the value process of the option than only its initial worth.
Let $\mathbb{Q}$ be the risk-neutral probability measure introduced via the Radon-Nikodym derivative. Then we get by the Girsanov theorem (see e.g. [6, Chapter 3.5]) that the new process $W^{\mathbb{Q}}$ ia again a $\mathbb{Q}$-Brownian motion with

$$dW_t^{\mathbb{P}} = W_t^{\mathbb{Q}} + \frac{r-\mu}{\sigma}t \quad \text{for } t \geq 0.$$

Under the new probability measure $\mathbb{Q}$ the differential equation (1.1) can be rewritten as

$$\begin{aligned} dS_t &= S_t\left(\mu\, dt + \sigma\, dW_t^{\mathbb{P}}\right) \\ &= S_t\left(\mu\, dt + \sigma\, dW_t^{\mathbb{Q}} + \sigma\frac{r-\mu}{\sigma}\, dt\right) \\ &= S_t\left((\mu + r - \mu)\, dt + \sigma\, dW_t^{\mathbb{Q}}\right) \\ &= S_t\left(r\, dt + \sigma\, dW_t^{\mathbb{Q}}\right). \end{aligned} \tag{1.2}$$

The arbitrage-free, risk-neutral price for an option with payoff $\Phi(S_T)$ at time 0 is

$$e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\Phi(S_T)\right].$$

Let $V(t)$ denote the price of the option at time $t \in [0, T]$. One can prove (see for instance [10, Chapter 5]) that $V(t)$ must be

$$V(t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\Phi(S_T)|\mathcal{F}_t\right].$$

1.3 Lookback Options

As in the previous section, we will start to introduce this class of option by having a look at the European counterpart (compare [4] and [5]).

We can divide European lookback options mainly into fixed strike lookback options and floating strike lookback options.

1.3.1 Fixed strike lookback options

Lookback options are members of the path-dependent options. The payoff of the previous options depended only on the stock price at exercise. But now, the holder "looks back" over the whole path of the stock to determine the payoff. He might be looking at the maximum, minimum, average, etc.

Definition 1.3.1. *A European fixed strike lookback call option with exercise date T, $T > 0$, and strike price K, $K > 0$, is a security whose payoff at time T, is*

$$\Phi_C^{fix}(S_T) = \left(\max\left(L, \max_{0\le t\le T} S_t\right) - K\right)_+ . \tag{1.3}$$

Here, L is a positive constant with $L \ge S_0$. It can be interpreted as the maximum level of the stocks past ($t < 0$) prices. Depending on the size of L and K, (1.3) simplifies into the two following cases:

- $\Phi_C^{fix}(S_T) = \left(\max_{0\le t\le T} S_t - K\right)_+$ for $K \ge L$;
- $\Phi_C^{fix}(S_T) = \max\left(L, \max_{0\le t\le T} S_t\right) - K = L - K + \left(\max_{0\le t\le T} S_t - L\right)_+$ for $K \ge L$.

So the payoff of the option is mainly determined by the difference between the all-time maximum and K, respectively L. A closed form for the price of the option from 1.3.1 can be found in [4, section 4].
We can define a European fixed strike lookback put option equivalently to definition 1.3.1.

Definition 1.3.2. *A European fixed strike lookback put option with exercise date T, $T > 0$, and strike price K, $K > 0$, is a security whose payoff at time T, is*

$$\Phi_P^{fix}(S_T) = \left(K - \min\left(L, \min_{0\le t\le T} S_t\right)\right)_+ . \tag{1.4}$$

$L > 0$ can be interpreted as the minimum level of the stocks past ($t < 0$) prices. Depending on L and K, two cases are possible:

- $\Phi_P^{fix}(S_T) = \left(K - \min_{0\le t\le T} S_t\right)_+$ for $K \le L$;
- $\Phi_P^{fix}(S_T) = K - \min\left(L, \min_{0\le t\le T} S_t\right)$ for $K > L$.

A closed form for the price of the option from 1.3.1 can be found in [4, section 4]. The American counterpart can be exercised at any time until expiration. So from section 1.2 we can derive the American version of (1.3) and (1.4). When we exercise at time τ, the option pays

$$\begin{aligned} \Phi_C^{fix}(S_\tau) &= \left(\max\left(L, \max_{0\le t\le \tau} S_t\right) - K\right)_+ ; \\ \Phi_P^{fix}(S_\tau) &= \left(K - \min\left(L, \min_{0\le t\le \tau} S_t\right)\right)_+ . \end{aligned}$$

1.3.2 Floating strike lookback options

The floating strike lookback option differs from the fixed strike option in terms of the strike. As the strike in the latter option was predetermined, the strike in the floating strike option is the stock price at exercise, and hence it deviates.
The payoff of the European option is

$$\Phi_C^{float}(S_T) = \left(\max\left(L, \max_{0\le t\le T} S_t\right) - S_T\right)_+$$

with $L \ge S_0 > 0$. The payoff of the put is

$$\Phi_P^{float}(S_T) = \left(S_T - \min\left(L, \min_{0\le t\le T} S_t\right)\right)_+ .$$

Analog to section 1.3.1, the American option can be exercised at any time until expiration. When we exercise at time τ, the American floating strike options pay

$$\begin{aligned} \Phi_C^{fix}(S_\tau) &= \left(\max\left(L, \max_{0\le t\le \tau} S_t\right) - S_\tau\right)_+ ; \\ \Phi_P^{fix}(S_\tau) &= \left(S_\tau - \min\left(L, \min_{0\le t\le \tau} S_t\right)\right)_+ . \end{aligned}$$

Chapter 2

Linear Programming

2.1 Dimension reduction

The differential equation in (1.1) can be solved explicitly by applying Ito's formula to $\ln(S_t)$, which gives us

$$S_t = s_0 \exp\left\{(r - \frac{\sigma^2}{2})t + \sigma W_t^{\mathbb{Q}}\right\}, \; t \geq 0\,, \tag{2.1}$$

where $\mathbb{Q}$ is the risk-neutral measure and $W^{\mathbb{Q}}$ is a standard $\mathbb{Q}$-Brownian-motion (see e.g. [10]).

Since lookback options rely on the running extrema of the stock price maximum, we need to introduce the running maximum M_t and the running minimum m_t as

$$\begin{aligned} M_t &= M_0 \vee \max_{0 \leq s \leq t} S_s\,, \\ m_t &= m_0 \wedge \min_{0 \leq s \leq t} S_s. \end{aligned} \tag{2.2}$$

The initial values M_0 and m_0 are included to allow flexibility about the initial extrema.

To price, or equivalently find an optimal strategy, an American lookback option with maturity T at time $t = 0$, we have to solve the following optimization problem for a call

$$\max_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \left(e^{-r\tau}(S_\tau - m_\tau)\right)$$

and for a put

$$\max_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{Q}} \left(e^{-r\tau}(M_\tau - S_\tau)\right)\,.$$

Let $\mathcal{T}$ be the set of all stopping times relative to $\{\mathcal{F}_t\}$, such that $0 \le \tau \le T$. With (2.1) we can transform the optimization problem for the put

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left(e^{-r\tau}(M_\tau - S_\tau)\right) &= \mathbb{E}_{\mathbb{Q}}\left(e^{-r\tau}S_\tau\left(\frac{M_\tau}{S_\tau}-1\right)\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(e^{-r\tau}s_0 e^{(r-\frac{\sigma^2}{2})\tau+\sigma W_\tau^{\mathbb{Q}}}\left(\frac{M_\tau}{S_\tau}-1\right)\right) \\ &= s_0\mathbb{E}_{\mathbb{Q}}\left(e^{-\frac{\sigma^2}{2}\tau+\sigma W_\tau^{\mathbb{Q}}}\left(\frac{M_\tau}{S_\tau}-1\right)\right). \end{aligned} \tag{2.3}$$

Now we define with $e^{-\frac{\sigma^2}{2}\tau+\sigma W_\tau^{\mathbb{Q}}}$ the new measure $\tilde{\mathbb{P}}$ by the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}} = \exp\left(-\frac{\sigma^2}{2}\tau + \sigma W_\tau^{\mathbb{Q}}\right) \text{ on } \mathcal{F}_T\,.$$

Then by a change of measure (2.3) is equivalent to

$$\max_{\tau\in\mathcal{T}} s_0\mathbb{E}_{\tilde{\mathbb{P}}}\left(\frac{M_\tau}{S_\tau}-1\right). \tag{2.4}$$

Let $Y_t := \dfrac{M_t}{S_t}$ with M_t and S_t as before. The following theorems shall give us more insight into the structure of this process. Compare [9, Chapter 3].

Theorem 2.1.1. *The process Y is Markov with respect to the measure $\tilde{\mathbb{P}}$.*

Proof. By the definition, we get

$$\begin{aligned} Y_{t+\Delta} &= \frac{\max\{M_0; \max_{0\le s\le t+\Delta} S_s\}}{S_{t+\Delta}} \\ &= \max\left\{\frac{\max_{0\le s\le t} S_s}{S_t\cdot S_{t+\Delta}/S_t}\,;\ \frac{\max_{t<s\le t+\Delta} S_s/S_t}{S_{t+\Delta}/S_t}\,,\ \frac{M_0}{S_{t+\Delta}}\right\} \\ &= \max\left\{Y_t\cdot\frac{1}{S_{t+\Delta}/S_t}\,;\ \frac{\max_{t<s\le t+\Delta} S_s/S_t}{S_{t+\Delta}/S_t}\right\}. \end{aligned} \tag{2.5}$$

By (2.1) we have for all $t < u \le t+\Delta$ for the $\tilde{\mathbb{P}}$ Brownian motion $W^{\tilde{\mathbb{P}}}$

$$\frac{S_u}{S_t} = \exp\left\{\sigma(W_u^{\tilde{\mathbb{P}}} - W_t^{\tilde{\mathbb{P}}}) + (r + \frac{\sigma^2}{2})(u-t)\right\}.$$

Therefore (2.5) implies by the independent and stationary increments of the Brownian motion that

$$\tilde{\mathbb{P}}\left(Y_{t+\Delta}|\mathcal{F}_t\right) = \tilde{\mathbb{P}}\left(Y_{t+\Delta}|Y_t\right)$$

which proves the Markov property. $\square$

Theorem 2.1.2. *The process Y on $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ is an at $\{1\}$ reflected geometric Brownian motion having drift $-r$ that takes values on $[1, \infty)$*

Proof. From the Girsanov theorem (see for instance [6, Chapter 3.5]), it follows, that $W_t^{\mathbb{Q}} - \sigma t$ is a standard $\tilde{\mathbb{P}}$ Brownian motion and hence $W_t^{\mathbb{Q}} = W_t^{\tilde{\mathbb{P}}} + \sigma t$

$$\begin{aligned}
\mathbb{E}_{\tilde{\mathbb{P}}}\left(\frac{M_t}{S_t}\right) &= \mathbb{E}_{\tilde{\mathbb{P}}}\left(\frac{M_0 \vee \max_{0\le s\le t} s_0 \exp\left\{(r-\frac{\sigma^2}{2})s + \sigma W_s^{\mathbb{Q}}\right\}}{s_0 \exp\left\{(r-\frac{\sigma^2}{2})t + \sigma W_t^{\mathbb{Q}}\right\}}\right) \\
&= \mathbb{E}_{\tilde{\mathbb{P}}}\left(\frac{M_0 \vee \max_{0\le s\le t} s_0 \exp\left\{(r-\frac{\sigma^2}{2})s + \sigma(W_s^{\tilde{\mathbb{P}}} + \sigma s)\right\}}{s_0 \exp\left\{(r-\frac{\sigma^2}{2})t + \sigma(W_t^{\tilde{\mathbb{P}}} + \sigma t)\right\}}\right) \\
&= \mathbb{E}_{\tilde{\mathbb{P}}}\left(\frac{M_0 \vee \max_{0\le s\le t} s_0 \exp\left\{(r+\frac{\sigma^2}{2})s + \sigma W_s^{\tilde{\mathbb{P}}}\right\}}{s_0 \exp\left\{(r+\frac{\sigma^2}{2})t + \sigma W_t^{\tilde{\mathbb{P}}}\right\}}\right).
\end{aligned} \tag{2.6}$$

Now we define

$$\begin{aligned}
\tilde{S}_t &:= s_0 \exp\left\{(r+\frac{\sigma^2}{2})t + \sigma W_t^{\tilde{\mathbb{P}}}\right\}, \\
\tilde{M}_t &:= M_0 \vee \max_{0\le s\le t} \tilde{S}_t
\end{aligned}$$

and

$$\tilde{Y}_t := \frac{\tilde{M}_t}{\tilde{S}_t}.$$

Observe that (2.6) yields

$$\mathbb{E}_{\tilde{\mathbb{P}}}\left(Y_t\right) = \mathbb{E}_{\tilde{\mathbb{P}}}\left(\frac{M_t}{S_t}\right) = \mathbb{E}_{\tilde{\mathbb{P}}}\left(\frac{\tilde{M}_t}{\tilde{S}_t}\right) = \mathbb{E}_{\tilde{\mathbb{P}}}\left(\tilde{Y}_t\right).$$

Since $\tilde{M}$ is a nondecreasing process of locally bounded variation, Itô's formula is applicable and hence

$$\begin{aligned}
d\tilde{Y}_t &= d(\frac{\tilde{M}_t}{\tilde{S}_t}) = \tilde{M}_t\, d(\frac{1}{\tilde{S}_t}) + \frac{1}{\tilde{S}_t}\, d(\tilde{M}_t) \\
&= \tilde{M}_t \left\{ -\frac{1}{\tilde{S}_t^2}\, d\tilde{S}_t + \frac{1}{2}\frac{1}{\tilde{S}_t^3} 2\sigma^2 \tilde{S}_t^2\, dt \right\} + \frac{d\tilde{M}_t}{\tilde{S}_t} \\
&= \tilde{M}_t \left\{ -\frac{1}{\tilde{S}_t} \left((r+\sigma^2)\, dt + \sigma\, dW_t^{\tilde{\mathbb{P}}} \right) + \frac{\sigma^2}{\tilde{S}_t}\, dt \right\} + \frac{d\tilde{M}_t}{\tilde{S}_t} \\
&= -\frac{\tilde{M}_t}{\tilde{S}_t} \left(r\, dt + \sigma\, dW_t^{\tilde{\mathbb{P}}} \right) + \frac{d\tilde{M}_t}{\tilde{S}_t}.
\end{aligned} \tag{2.7}$$

From (2.7) follows

$$d\tilde{Y}_t = -\tilde{Y}_t(r\, dt + \sigma\, dW_t^{\tilde{\mathbb{P}}}) + \frac{d\tilde{M}_t}{\tilde{S}_t}\,; \tag{2.8}$$

or equivalently in integral notation

$$\tilde{Y}_t = \tilde{Y}_0 - r\int_0^t \tilde{Y}_u\, du - \sigma \int_0^t \tilde{Y}_u\, dW_u^{\tilde{\mathbb{P}}} + \int_0^t \frac{d\tilde{M}_t}{\tilde{S}_t}.$$

Similarly, we get by another application of Itô's formula for any function $f = f(\tilde{Y}) \in \mathcal{C}^2\left([1,\infty)\right)$

$$df(\tilde{Y}_t) = f'(\tilde{Y}_t)\, d\tilde{Y}_t + \frac{1}{2} f''(\tilde{Y}_t)\sigma^2 \tilde{Y}_t^2\, dt\,;$$

or equivalently in integral notation

$$f(\tilde{Y}_t) = f(\tilde{Y}_0) - r\int_0^t Lf(\tilde{Y}_u)\, du - \sigma\int_0^t f(\tilde{Y}_u)\tilde{Y}_u\, dW_u^{\tilde{\mathbb{P}}} + \int_0^t f'(\tilde{Y}_u)\frac{d\tilde{M}_t}{\tilde{S}_t}\,, \tag{2.9}$$

with the differential operator

$$L = -ry\frac{\partial}{\partial y} + \frac{\sigma^2}{2} y^2 \frac{\partial^2}{\partial y^2}.$$

Observe that this is the infinitesimal generator of a geometric Brownian motion. From the definition of $\tilde{M}$ and $\tilde{S}$ we can derive the following properties in a $\tilde{\mathbb{P}}$-a.s. sense. Clearly, $\tilde{M}_t$ is a monotone increasing process and $\tilde{M}_t \geq \tilde{S}_t$. Since $M_t, S_t > 0$ for all $0 \leq t \leq T$, so are $\tilde{M}_t, \tilde{S}_t$.

The process $\tilde{M}_t$ increases if and only if $\tilde{M}_t = \tilde{S}_t$, which is equivalent to $\tilde{Y}_t = 1$. So if $\tilde{Y}_t > 1$ then $d\tilde{M}_t = 0$ and $\int_0^t \mathbb{I}\left(\tilde{Y}_u > 1\right) d\tilde{M}_u = 0$, $t > 0$. So we can write (2.9) as

$$f(\tilde{Y}_t) = f(\tilde{Y}_0) - r\int_0^t Lf(\tilde{Y}_u)\,du - \sigma\int_0^t f(\tilde{Y}_u)\tilde{Y}_u\,dW_u^{\tilde{\mathbb{P}}} + \int_0^t f'(\tilde{Y}_u)\frac{d\tilde{M}_t}{\tilde{S}_t}\mathbb{I}\left(\tilde{Y}_u = 1\right) du\,. \tag{2.10}$$

Still (2.10) does not give a lot of insight into the behavior of the process at the event $\tilde{Y}_t = 1$. So in our next step, we will show that the process is reflected at $\{1\}$ for any $t \in [0, T]$, or equivalently

$$\int_0^t \mathbb{I}\left(\tilde{Y}_u = 1\right) du = 0 \quad (\tilde{\mathbb{P}}\text{-a.s.}). \tag{2.11}$$

With Fubini's theorem we get

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{P}}}\left\{\int_0^T \mathbb{I}\left(\tilde{Y}_u = 1\right) du\right\} &= \int_0^T \mathbb{E}_{\tilde{\mathbb{P}}}\left\{\mathbb{I}\left(\tilde{Y}_u = 1\right)\right\} du \\ &= \int_0^T \tilde{\mathbb{P}}\left(\tilde{Y}_u = 1\right) du\,. \end{aligned} \tag{2.12}$$

Since a Brownian motion has an a.e. absolutely continuous distribution, so has $\tilde{S}$ as a function of $W_{\tilde{\mathbb{P}}}$. Therefore, also the distribution of $\tilde{Y}$ is a.e. absolutely continuous and a.e. $\tilde{\mathbb{P}}\left(\tilde{Y}_u = 1\right) = 0$. Hence, (2.12) is of size 0 and so must (2.11) be.
Property (2.11) tells us, that the process $\tilde{Y}_t$ spends 0 time at the boundary $\{1\}$ and so this point is an instant reflection point or a non-sticky boundary.
The previously introduced differential operator L is the the infinitesimal generator of the process $\tilde{Y}$ on functions $f \in \mathcal{C}^2$ limited by the condition $f'(1^+) = 0$. as already stated L is the infinitesimal generator of a geometric Brownian motion with drift $-r$. So $\tilde{Y}$ is a geometric Brownian motion with instant reflection at $\{1\}$, which proves theorem 2.1.2. $\square$

Let $Y_t^m := \dfrac{m_t}{S_t}$ with m_t and S_t as before. Analog to the previous theorems 2.1.1 and 2.1.2 the following two lemmas can be proved.

Lemma 2.1.3. *The process Y^m is Markov with respect to the measure $\tilde{\mathbb{P}}$.*

Lemma 2.1.4. *The process Y^m on $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ is an at $\{1\}$ reflected geometric Brownian motion having drift $-r$ that takes values on $(0,1]$.*

By applying theorem 2.1.2, and the dimension-reduction it inherits, we can reduce the initial bi-dimensional stopping problem down to a unidimensional. Therefore (2.4) is equivalent to

$$\max_{\tau\in\mathcal{T}} s_0 \mathbb{E}_{\tilde{\mathbb{P}}} \left(Y_\tau - 1\right). \tag{2.13}$$

Denote

$$\varphi_t = \int_0^t \mathbb{I}\left(\tilde{Y}_u = 1\right) \frac{d\tilde{M}_u}{\tilde{S}_u}.$$

Observe that φ_t is a non-negative process, only increasing when $\tilde{Y}_t = 1$. It is the local time process of $\tilde{Y}$ at 1. According to (2.8) we get

$$d\tilde{Y}_t = -\tilde{Y}_t(r\,dt + \sigma\,dW_t^{\tilde{\mathbb{P}}}) + d\varphi_t\,. \tag{2.14}$$

2.2 Deriving the LP

Another formulation of (2.13) of the price for an American lookback put option is

$$\begin{cases} \text{Maximize:} & \max\limits_{\tau\in\mathcal{T}} s_0 \mathbb{E}_{\tilde{\mathbb{P}}} \left(Y_\tau - 1\right) \\ \text{Subject to:} & \tilde{Y} \text{ is a geometric Brownian motion with drift } -r\text{, reflected at 1,} \end{cases} \tag{2.15}$$

and analog for the call option

$$\begin{cases} \text{Maximize:} & \max\limits_{\tau\in\mathcal{T}} s_0 \mathbb{E}_{\tilde{\mathbb{P}}} \left(1 - Y_\tau^m\right) \\ \text{Subject to:} & \tilde{Y}^m \text{ is a geometric Brownian motion with drift } -r\text{, reflected at 1,} \end{cases}$$

The approximation methods in chapter 3 rely on basis functions. Therefore we want to reformulate the optimal stopping problem 2.15 using Itô's formula.
We know from theorem 2.1.2 that $\tilde{Y}$ is a process satisfying the differential equation (2.14). Hence, by Itô's formula, for a function $f = f(\tilde{Y}) \in \mathcal{C}^2([1,\infty))$ we get

$$f(\tilde{Y}_t) - f(\tilde{Y}_0) - \int_0^t A_{\tilde{Y}}[f](\tilde{Y}_u)du - \int_0^t f'(\tilde{Y}_u)\,d\varphi_u \tag{2.16}$$

is again a martingale with $A_{\tilde{Y}}[f](y) = -ryf'(y) + \frac{\sigma^2}{2} y^2 f''(y)$.
The beneficial aspect of constraint (2.16) is the following: If (2.16) is fulfilled by all functions $f \in \mathcal{C}^2([1,\infty))$, then $\tilde{Y}$ complies with the constraint in (2.15). So we can reformulate the optimal stopping problem (2.15) as

$$\begin{cases} \text{Maximize:} & \max_{\tau \in \mathcal{T}} s_0 \mathbb{E}_{\tilde{\mathbb{P}}} \left(Y_\tau - 1 \right) \\ \text{Subject to:} & \text{(2.16) is a martingale for all } f \in \mathcal{C}^2([1,\infty)). \end{cases} \tag{2.17}$$

When we now have again a look at the differential equation of $\tilde{Y}$ in (2.14), then we see, that drift and the instantaneous volatility are state dependent, occurring from the fact that it is a geometric Brownian motion.
In order to simplify later implementations, we try to remodel it by a drifted Brownian motion. By this, we reach constant coefficients for drift and volatility and smaller values.
Apply the ln to $\tilde{Y}$ and denote

$$X_t := \ln \tilde{Y}_t.$$

Lemma 2.2.1. *Let X be as above. X is drifted Brownian motion taking values in $[0,\infty)$ with instant reflection at $\{0\}$ and drift $-r$.*

Proof. Observe that $X_t \in [0,\infty)$, due to $\tilde{Y}_t \in [1,\infty)$ for all $t \geq 0$.

$$\begin{aligned} dX_t &= d \ln \tilde{Y}_t \\ &= \frac{1}{\tilde{Y}_t} d\tilde{Y}_t + \frac{1}{2} \left(\ln \tilde{Y}_t \right)'' d \left[\tilde{Y} \right]_t \\ &= \frac{1}{\tilde{Y}_t} d\tilde{Y}_t - \frac{\sigma^2}{2} dt \\ &= -\left\{ r\, dt + \sigma\, dW_t^{\tilde{\mathbb{P}}} \right\} + \frac{d\varphi_t}{\tilde{Y}_t} - \frac{\sigma^2}{2} dt \\ &= -\left\{ \left(r + \frac{\sigma^2}{2} \right) dt + \sigma\, dW_t^{\tilde{\mathbb{P}}} \right\} + \frac{d\varphi_t}{\tilde{Y}_t} \end{aligned}$$

Observe that

$$\begin{aligned}
\left(\ln \tilde{Y}_t\right)'' d[\tilde{Y}]_t &= \left(\ln \tilde{Y}_t\right)'' (d\tilde{Y}_t)^2 \\
&= \frac{-1}{\tilde{Y}_t^2}\left(-\tilde{Y}_t(r\,dt + \sigma\, dW_t^{\tilde{\mathbb{P}}}) + d\varphi_t\right)^2 \\
&= -\left(r\,dt + \sigma\, dW_t^{\tilde{\mathbb{P}}}\right)^2 + \frac{1}{\tilde{Y}_t}\left(r\,dt + \sigma\, dW_t^{\tilde{\mathbb{P}}}\right)(d\varphi_t) - \frac{1}{\tilde{Y}_t^2}(d\varphi_t)^2 \\
&= -\left(r\,dt + \sigma\, dW_t^{\tilde{\mathbb{P}}}\right)^2 \\
&= -\sigma^2\, dt\,.
\end{aligned}$$

Now compare the dynamics from (2.18) with those we examined in the proof for theorem 2.1.2. By analogy we can find, that the first part in brackets is the diffusion of a ordinary drifted Brownian motion. As before, φ_t increases, when $\tilde{Y}_t = 1$, or equivalently $X_t = \ln \tilde{Y}_t = 0$. So we can precede as earlier and show, that X is a drifted Brownian motion with instant reflection at $\{0\}$ □

The differential equivalent (2.18) can be simplified using the following fact: φ is the local time process of $\tilde{Y}$ at $\{1\}$ and hence also for X and $\{0\}$. Therefore when φ_t increases, then $\tilde{Y}_t = 1$ and the denominator $\tilde{Y}_t$ in (2.18) can be dropped.
So the maximizing function of our optimal stopping problem is

$$\max_{\tau\in\mathcal{T}} s_0 \mathbb{E}_{\tilde{\mathbb{P}}}\left(e^{X_\tau} - 1\right).$$

From lemma 2.2.1 and Itô's formula we can derive for $f \in \mathcal{C}^2([0,\infty))$

$$\begin{aligned}
df(X_t) &= f'(X_t)\, dX_t + \frac{1}{2} f''(X_t) d[X]_t \\
&= f'(X_t)\left\{-\left[\left(r + \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t^{\tilde{\mathbb{P}}}\right] + d\varphi_t\right\} + \frac{\sigma^2}{2} f''(X_t)\, dt \\
&= \left[-f'(X_t)\left(r + \frac{\sigma^2}{2}\right) + \frac{\sigma^2}{2} f''(X_t)\right] dt - \sigma f'(X_t)\, dW_t^{\tilde{\mathbb{P}}} + f'(X_t)\, d\varphi_t\,.
\end{aligned} \tag{2.18}$$

Denote for $f \in \mathcal{C}^2([0,\infty))$

$$A_X[f](x) = -f'(x)\left(r + \frac{\sigma^2}{2}\right) + \frac{\sigma^2}{2} f''(x).$$

Then again by Itô's formula and (2.18),

$$f(X_t) - f(X_0) - \int_0^t A_X[f](X_u)\, du - \int_0^t f'(X_u)\, d\varphi_u \tag{2.19}$$

is a martingale. Therefore the optimal stopping problem (2.17) becomes

$$\begin{cases} \text{Maximize:} & \max\limits_{\tau \in \mathcal{T}} s_0 \mathbb{E}_{\tilde{\mathbb{P}}} \left(e^{X_\tau} - 1 \right) \\ \text{Subject to:} & \text{(2.19) is a martingale for all } f \in \mathcal{C}^2([0,\infty)). \end{cases} \tag{2.20}$$

Let $X_t^m = \ln \tilde{Y}_t^m$. Then we get analog to the previous computations

$$f(X_t^m) - f(X_0^m) - \int_0^t A_X[f](X_u^m)\, du - \int_0^t -f'(X_u^m)\, d\varphi_u\,, \tag{2.21}$$

and the optimal stopping problem for the call is

$$\begin{cases} \text{Maximize:} & \max\limits_{\tau \in \mathcal{T}} s_0 \mathbb{E}_{\tilde{\mathbb{P}}} \left(1 - e^{X_\tau} \right) \\ \text{Subject to:} & \text{(2.21) is a martingale for all } f \in \mathcal{C}^2([0,\infty)). \end{cases}$$

2.3 LP-Formulation

Sections 1.2 and 1.3 yield, that the examined options have an expiration date, denoted by T, which indicates that the optimal exercise strategy τ^* depends on the time horizon. Also the approximation methods in chapter 3 rely on a separation of variables into a time and a space component.

Hence, instead of defining our linear program by the process X in the state space $[0,\infty)$, we consider the process (t, X) in the time-state-space $[0,T] \times [0,\infty)$. Let $\gamma \in \mathcal{C}([0,T])$, then

$$\gamma(t)f(X_t) - \gamma(0)f(X_0) - \int_0^t \left[\gamma(u)A_X[f](X_u) + \gamma'(u)f(X_u)\right] du - \int_0^t \gamma(u)f'(X_u)\, d\varphi_u \tag{2.22}$$

is a martingale with respect to the filtration $\mathcal{F}_t$, if (X, φ) in (2.19) is a martingale for all $f \in \mathcal{C}^2([0,\infty))$.

Clearly, the stopping time τ is bounded by T and by the optional sampling theorem applied to (2.22) we get

$$\begin{aligned} 0 = \mathbb{E}\Big[&\gamma(\tau)f(X_\tau) - \gamma(0)f(X_0) - \int_0^\tau \gamma(u)A_X[f](X_u) + \gamma'(u)f(X_u)\, du \\ &- \int_0^\tau \gamma(u)f'(X_u)\, d\varphi_u\Big]. \end{aligned} \tag{2.23}$$

For a stopping time τ, let υ_τ be the joint distribution of $(\tau, X(\tau))$ and μ_0 the time-space expected occupation measure with

$$\mu_0(\mathcal{G}) = \mathbb{E}\left[\int_0^\tau \mathbb{I}_{\mathcal{G}}(t, X(t))dt\right] \ \forall \mathcal{G} \in \mathbb{B}\left([0,T] \times [0,\infty)\right) .$$

We define the expected occupation measure μ_1 with respect to the local time process as

$$\mu_1(\mathcal{G}) = \mathbb{E}\left[\int_0^\tau \mathbb{I}_{\mathcal{G}}(t, X(t))d\varphi_t\right] \ \forall \mathcal{G} \in \mathbb{B}\left([0,T] \times [0,\infty)\right) ,$$

where ξ denotes the local time process of X at 0.

Since μ_1 is concentrated on $([0,T] \times \{0\})$, we can simplify the notation for μ_1 by $\mu_1(\mathcal{G}) := \mu_1(\mathcal{G} \times \{0\})$ for each Borel set $\mathcal{G}$ on $[0,T]$.

Denote

$$A[\gamma f](t,x) = \gamma(t) A_X[f](x) + \gamma'(t) f(x).$$

When we rewrite the expectation (2.23) in terms of υ_τ, μ_0 and μ_1, we get

$$\begin{aligned} 0 \ = \ & \int_{([0,T]\times[0,\infty))} \gamma(t) f(x)\, d\upsilon_\tau(dt \times dx) - \gamma(0) f(X_0) \\ & - \int_{([0,T]\times[0,\infty))} A[\gamma f](t,x)\mu_0(dt \times dx) \\ & - \int_{([0,T]\times\{0\})} \gamma(t) f'(x)\mu_1(dt \times dx). \end{aligned} \tag{2.24}$$

The original optimal stopping problem for the American lookback put option is equivalent to

$$\left\{ \begin{array}{lll} \text{Maximize} & S_0 \displaystyle\int (e^x - 1)\upsilon_\tau(dt \times dx) & \\ \text{Subject to} & \gamma(0) f(x_0) \ = & \displaystyle\int_{([0,T]\times[0,\infty))} \gamma f d\upsilon_\tau \\ & \quad - & \displaystyle\int_{([0,T]\times[0,\infty))} A[\gamma f](t,x)\mu_0(dt \times dx) \\ & \quad - & \displaystyle\int_{([0,T]\times\{0\})} B[\gamma f](t,0)\mu_1(dt \times dx) \\ & & \forall\, (\gamma, f) \in \mathcal{D}, \\ & \upsilon_\tau \in \mathcal{P}\left([0,T] \times [0,\infty)\right), & \\ & \mu_0 \in \mathcal{M}\left([0,T] \times [0,\infty)\right) \text{ with total mass } <= T, & \\ & \mu_1 \in \mathcal{M}\left([0,T]\right) & . \end{array} \right. \tag{2.25}$$

with $\mathcal{P}$ the set of probability measures, $\mathcal{M}$ the set of finite measures and the domain $\mathcal{D} = \{(\gamma, f) : \gamma \in \mathcal{C}^1[0, T], f \in \mathcal{C}_c^2[0, \infty)\}$. The generators A and B are given by

$$\begin{aligned} A[\gamma f](t, x) &= \gamma(t) \left[-(r + \frac{\sigma^2}{2}) f'(x) + \frac{\sigma^2}{2} f''(x) \right] + \gamma'(t) f(x) \\ B[\gamma f](t, x) &= \gamma(t) f'(x) \,. \end{aligned}$$

Chapter 3

Approximation

3.1 Time discretization

In section 2.3 we have derived the LP-formulation for the American lookback put option (2.25). The first step in our numerical implementation is the finite discretization of the time horizon $[0, T]$.

For this purpose, let n_t be a sufficiently large number of time steps $\{t_1, t_2, \ldots, t_{n_t}\}$, which are to be chosen equidistant in $[0, T]$.

For the time derivatives we choose a one-sided finite difference approach (see e.g. [7, chapter 5]) for the derivative over time. We can choose between either a forward or a backward difference approach. If the velocity at a point is non-negative, then we need to use a forward difference. If it is negative on the other hand, we have to use backward difference. This scheme is called "upwind" approximation method. As we move forward in time, we have to choose the forward-difference approach.

So our lp-program (2.25) becomes

$$
\left\{
\begin{array}{lll}
\text{Maximize} & S_0 \sum_{j=1}^{n_t} \int (e^x - 1)\upsilon_\tau(t_j, dx) & \\
\text{Subject to} & \gamma(0) f(x_0) = & \sum_{j=1}^{n_t} \int_{[0,\infty)} \gamma(t_j) f \upsilon_\tau(t_j, dx) \\
& & - \sum_{j=1}^{n_t} \int_{[0,\infty)} \tilde{A}[\gamma f](t_j, x)\mu_0(t_j, dx) \\
& & - \sum_{j=1}^{n_t} \tilde{B}[\gamma f](t_j, 0)\mu_1(t_j, 0) \\
& & \forall\, (\gamma, f) \in \mathcal{D}, \\
& \upsilon_\tau \in \mathcal{P}\left([0,T] \times [0,\infty)\right), & \\
& \mu_0 \in \mathcal{M}\left([0,T] \times [0,\infty)\right) \text{ with total mass } \leq T, & \\
& \mu_1 \in \mathcal{M}\left([0,T]\right), &
\end{array}
\right.
\tag{3.1}
$$

in which the generators $\tilde{A}$ and $\tilde{B}$ are given by

$$
\begin{aligned}
\tilde{A}[\gamma f](t_j, x) &= \gamma(t_j)\left[-(r + \frac{\sigma^2}{2}) f'(x) + \frac{\sigma^2}{2} f''(x)\right] + \frac{\gamma(t_{j+1}) - \gamma(t_j)}{T/n_t} f(x) \\
\tilde{B}[\gamma f](t_j, x) &= \gamma(t_j) f'(x)\,.
\end{aligned}
$$

Since there are only finitely-many states in $[0,T]$, it is sufficient to use finitely-many functions γ_i. A simple choice is the indicator function on the time steps $\gamma_i(t) = \mathbb{I}_{t_i}(t)$, for all $i = 1, \ldots, n_t$. So the measures υ_τ, μ_0 and mu_1 have on the time axis only mass on the time points $t_0 \ldots t_{n_t}$.

With this choice for the γ_i, the lp-program (3.1) turns into

$$\left\{\begin{array}{lll} \text{Maximize} & S_0\displaystyle\sum_{j=1}^{n_t}\int (e^x-1)\upsilon_\tau(t_j,dx) & \\ \text{Subject to} & \mathbb{I}_{t_i}(0)f(x_0) = & \displaystyle\sum_{j=1}^{n_t}\int_{[0,\infty)} \mathbb{I}_{t_i}(t_j)f(x)\upsilon_\tau(t_j,dx) \\ & - & \displaystyle\sum_{j=1}^{n_t}\int_{[0,\infty)} \tilde{A}[\mathbb{I}_{t_i}f](t_j,x)\mu_0(t_j,dx) \\ & - & \displaystyle\sum_{j=1}^{n_t} \tilde{B}[\mathbb{I}_{t_i}f](t_j,0)\mu_1(t_j,0) \\ & & \forall i\in\{1,\ldots,n_t\}\forall\,(\gamma_i,f)\in\mathcal{D}, \\ & \upsilon_\tau(t_j,\cdot),\ \mu_0(t_j,\cdot)\in\mathcal{M}\left([0,\infty)\right), & \\ & \mu_1(t_j)\geq 0\,,\ j=1,\ldots,n_t, & \\ & \displaystyle\sum_{j=1}^{n_t}\upsilon_\tau(t_j,[0,\infty))=1, & \\ & \displaystyle\sum_{j=1}^{n_t}\mu_0(t_j,[0,\infty))\leq T, & \end{array}\right. \tag{3.2}$$

The generators $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are

$$\begin{aligned} \tilde{A}[\mathbb{I}_{t_i}f](t_j,x) &= \mathbb{I}_{t_i}(t_j)\left[-(r+\frac{\sigma^2}{2})f'(x)+\frac{\sigma^2}{2}f''(x)\right]+\frac{\mathbb{I}_{t_i}(t_{j+1})-\mathbb{I}_{t_i}(t_j)}{T/n_t}f(x), \\ \tilde{B}[\mathbb{I}_{t_i}f](t_j,x) &= \mathbb{I}_{t_i}(t_j)f'(x). \end{aligned}$$

After simplifications we get as the martingale constraint in (3.2)

$$\begin{aligned} \mathbb{I}_{t_i}(0)f(x_0) = \ & \int_{[0,\infty)} f(x)\upsilon_\tau(t_i,dx) - \int_{[0,\infty)}\frac{n_t}{T}f(x)\mu_0(t_{i-1},dx) \\ & -\int_{[0,\infty)}\left(\left[-(r+\frac{\sigma^2}{2})f'(x)+\frac{\sigma^2}{2}f''(x)\right]-\frac{n_t}{T}f(x)\right)\mu_0(t_i,dx) \\ & -f'(0)\mu_1(t_i,0) \\ & \forall i\in\{1,\ldots,n_t\}\forall\,(\gamma_i,f)\in\mathcal{D}. \end{aligned}$$

In order to reduce the degree of the generators, we use partial integration for the term $\int_{([0,T]\times[0,\infty))} A[\gamma f](t,x)\mu_0(dt\times dx)$ from the Martingal-constraint

$$\begin{aligned}\int_0^\infty \frac{\sigma^2}{2} f''(x)\mu_0(t_i,x)dx &= \left[\frac{\sigma^2}{2} f'(x)\mu_0(t_i,x)\right|_{x=0}^{\infty} - \int_0^\infty \frac{\sigma^2}{2} f'(x)\frac{d}{dx}\mu_0(t_i,x)dx \\ &= \frac{\sigma^2}{2} f'(\infty)\mu_0(t_i,\infty) - \frac{\sigma^2}{2} f'(0)\mu_0(t_i,0) \\ &- \int_0^\infty \frac{\sigma^2}{2} f'(x)\frac{d}{dx}\mu_0(t_i,x)dx.\end{aligned} \tag{3.3}$$

So with (3.3) the martingale-constraint in the lp-program (3.2) becomes

$$\begin{aligned}\mathbb{I}_{t_i}(0)f(x_0) = \quad & \int_{[0,\infty)} f(x)\upsilon_\tau(t_i,dx) - \int_{[0,\infty)} \frac{n_t}{T} f(x)\mu_0(t_{i-1},dx) \\ & - \int_{[0,\infty)} \left(-(r+\frac{\sigma^2}{2})f'(x) - \frac{n_t}{T}f(x)\right)\mu_0(t_i,dx) \\ & -\frac{\sigma^2}{2} f'(\infty)\mu_0(t_i,\infty) + \frac{\sigma^2}{2} f'(0)\mu_0(t_i,0) + \int_0^\infty \frac{\sigma^2}{2} f'(x)\frac{d}{dx}\mu_0(t_i,x)dx \\ & -f'(0)\mu_1(t_i,0) \\ & \forall i \in \{1,\ldots,n_t\} \forall\, (\gamma_i, f) \in \mathcal{D}.\end{aligned} \tag{3.4}$$

We presumed f has compact support, hence $f'(\infty)$.

3.2 Finite elements in space

In the previous section 3.1 we discretized the time horizon $[0,T]$ into $\{t_1,t_2,\ldots,t_{n_t}\}$. Our next step is the truncation of the infinite-space $[0,\infty)$ into $[0,M]$ for a sufficiently big M in order to create the approximation space for the numerical method. With this truncation our measures and densities have mass only on $[0,M]$. But due to the normal distribution of the increments of the Brownian motion, the process can exceed the border M with positive probability. Therefore we have to make sure, that also our process stays in $[0,M]$.

We want to reflect the process at M so as to keep it bounded. We introduce the new generator B_M

$$B_M[\gamma f](t,x) = -\gamma(t)f'(x)$$

with the reflection measure only having mass when $x = M$.
So the optimal stopping problem is

$$\begin{cases} \text{Maximize} & S_0 \displaystyle\int (e^x - 1)\upsilon_\tau(dt \times dx) \\ \text{Subject to} & \gamma(0)f(x_0) = \displaystyle\int_{([0,T]\times[0,M))} \gamma(t)f(x)\upsilon_\tau(dt \times dx) \\ & \quad - \displaystyle\int_{([0,T]\times[0,M))} A[\gamma f](t,x)\mu_0(dt \times dx) \\ & \quad - \displaystyle\int_{([0,T]\times\{0\})} B[\gamma f](t,0)\mu_1(dt \times dx) \\ & \quad + \displaystyle\int_{([0,T]\times\{M\})} B_M[\gamma f](t,M)\mu_1(dt \times dx) \\ & \quad \forall\,(\gamma, f) \in \mathcal{D}, \\ & \upsilon_\tau \in \mathcal{P}\left([0,T] \times [0,\infty)\right), \\ & \mu_0 \in \mathcal{M}\left([0,T] \times [0,\infty)\right) \text{ with total mass } \le T, \\ & \mu_1 \in \mathcal{M}\left([0,T]\right). \end{cases} \tag{3.5}$$

So as in section 3.1 the martingale constraint in (3.5) becomes

$$\begin{aligned} \mathbb{I}_{t_i}(0)f(x_0) = & \int_{[0,\infty)} f\upsilon_\tau(t_i, dx) - \int_{[0,\infty)} \frac{n_t}{T} f(x)\mu_0(t_{i-1}, dx) \\ & - \int_{[0,\infty)} \left(-(r + \frac{\sigma^2}{2})f'(x) - \frac{n_t}{T} f(x) \right) \mu_0(t_i, dx) \\ & + \frac{\sigma^2}{2} f'(0)\mu_0(t_i, 0) + \int_0^\infty \frac{\sigma^2}{2} f'(x) \frac{d}{dx}\mu_0(t_i, x)dx \\ & - f'(0)\mu_1(t_i, 0) + f'(M)\mu_1(t_i, M) \\ & \forall i \in \{1, \ldots, n_t\} \forall\,(\gamma_i, f) \in \mathcal{D}. \end{aligned} \tag{3.6}$$

With the results from section 3.1 we could limit our search for the densities and measures υ_τ, μ_0 and μ_1 on $([0,T] \times [0,\infty))$ down to the discrete cases $\upsilon_\tau^i(x) := \upsilon_\tau(t_i, x)$, $\mu_0^i(x) := \mu_0(t_i, x)$, $\mu_1^i := \mu_1(t_i, 0)$ and $\mu_M^i := \mu_1(t_i, M)$ on $(\{t_i\} \times [0, M])$ for all $i \in \{1, \ldots n_t\}$.
Now we seek the approximated densities $\upsilon_\tau^1 \ldots \upsilon_\tau^{n_t}$ and $\mu_0^1 \ldots \mu_0^{n_t}$ and point masses $\mu_1^1 \ldots \mu_1^{n_t}$ and $\mu_M^1 \ldots \mu_M^{n_t}$ using only finitely-many elements.
For each t_k we write

$$\begin{aligned} \upsilon_\tau^k(x) &= a_{k,0}g_0(x) + \ldots a_{k,n_s}g_{n_s}(x), \\ \mu_0^k(x) &= b_{k,0}g_0(x) + \ldots b_{k,n_s}g_{n_s}(x), \\ \mu_1^k &= c_k, \\ \mu_M^k &= d_k, \end{aligned}$$

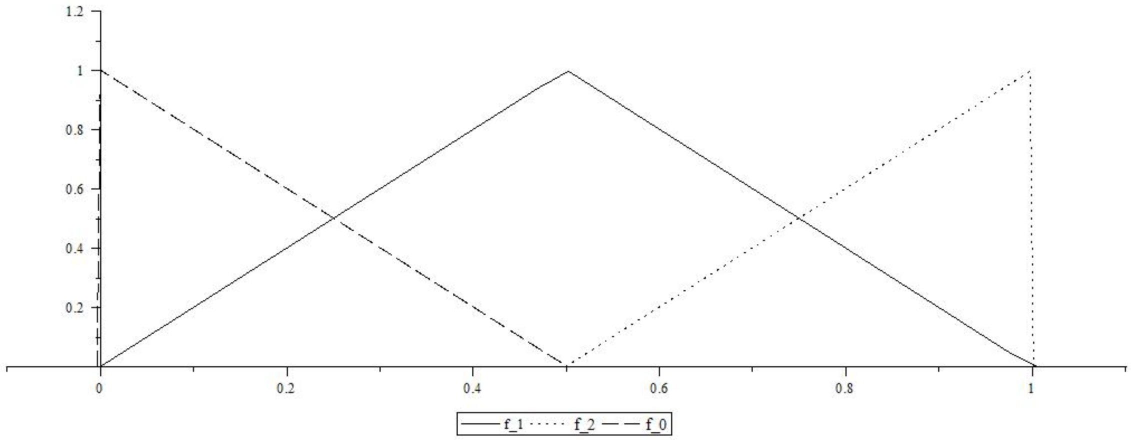

Figure 3.1: Linear functions for $n_s = 2$

where $g_0(x) \dots g_{n_s}(x)$ are base functions.
Continuous, piecewise linear functions will be the class for those base functions. So $g_0(x) \dots g_{n_s}(x)$ will be of the form $f_0(x)$, $f_j(x)$ or $f_{n_s}(x)$, with

$$f_0(x) = \begin{cases} -\frac{n}{M}\left(x - \frac{M}{n}\right), & x \in \left[0, \frac{M}{n}\right], \\ 0, & otherwise, \end{cases}$$

$$f_j(x) = \begin{cases} \frac{n}{M}\left(x - \frac{M}{n}(j-1)\right), & x \in \left[(j-1)\frac{M}{n}, j\frac{M}{n}\right], \\ -\frac{n}{M}\left(x - \frac{M}{n}(j+1)\right), & x \in \left[j\frac{M}{n}, (j+1)\frac{M}{n}\right], \\ 0, & otherwise, \end{cases}$$

$$f_{n_s}(x) = \begin{cases} \frac{n}{M}\left(x - (n-1)\frac{M}{n}\right), & x \in \left[(n-1)\frac{M}{n}, M\right], \\ 0, & otherwise. \end{cases}$$

Figure 3.1 is an example for $M = 1$ and $n_s = 2$. With those base functions all piecewise linear functions on the interval $[0, M]$ can be formed. With those new base polynomials our maximizing function in the lp-program (3.2) becomes

$$\text{Maximize } S_0 \sum_{j=1}^{n_t} \sum_{i=0}^{n_s} \int_0^M (e^x - 1) a_{ji} g_i(x) dx,$$

and the martingale-constraint from (3.4) is

$$\begin{aligned}
\mathbb{I}_{t_i}(0)f(0) = & \sum_{j=0}^{n_s}\int_0^M f(x)a_{i,j}g_j(x)dx - \sum_{j=0}^{n_s}\int_0^M \frac{n_t}{T}f(x)b_{i-1,j}g_j(x)dx \\
& -\sum_{j=0}^{n_s}\int_0^M \left(-(r+\frac{\sigma^2}{2})f'(x) - \frac{n_t}{T}f(x)\right) b_{i,j}g_j(x)dx \\
& -\sum_{j=0}^{n_s}\frac{\sigma^2}{2}f'(M)b_{i,j}g_j(M) + \sum_{j=0}^{n_s}\frac{\sigma^2}{2}f'(0)b_{i,j}g_j(0) \\
& +\sum_{j=0}^{n_s}\int_0^M \frac{\sigma^2}{2}f'(x)b_{i,j}g_j'(x)dx - f'(0)c_i + f'(M)d_i \\
& \forall i \in \{1,\ldots,n_t\}\forall(\gamma_i, f) \in \mathcal{D}.
\end{aligned} \tag{3.7}$$

In the next step, we will replace $f(x)$ in turns by the g_j's. So (3.7) becomes

$$\begin{aligned}
\mathbb{I}_{t_i}(0)g_k(0) = & \sum_{j=0}^{n_s}\int_0^M g_k(x)a_{i,j}g_j(x)dx - \sum_{j=0}^{n_s}\int_0^M \frac{n_t}{T}g_k(x)b_{i-1,j}g_j(x)dx \\
& -\sum_{j=0}^{n_s}\int_0^M \left(-(r+\frac{\sigma^2}{2})g_k'(x) - \frac{n_t}{T}g_k(x)\right) b_{i,j}g_j(x)dx \\
& -\sum_{j=0}^{n_s}\frac{\sigma^2}{2}g_k'(M)b_{i,j}g_j(M) + \sum_{j=0}^{n_s}\frac{\sigma^2}{2}g_k'(0)b_{i,j}g_j(0) \\
& +\sum_{j=0}^{n_s}\int_0^M \frac{\sigma^2}{2}g_k'(x)b_{i,j}g_j'(x)dx - g_k'(0)c_i + g_k'(M)d_i \\
& \forall i \in \{1,\ldots,n_t\}\forall k \in \{0,\ldots,n_s\}.
\end{aligned}$$

Instead of searching for densities and measures, our problem reduces to finding the maximizing factors $a_{1,1}\ldots a_{n_t,n_s}$, $b_{1,1}\ldots b_{n_t,n_s}$, $c_1\ldots c_{n_t}$ and $d_1\ldots d_{n_t}$.
Since υ_τ is a probability density, we get the additional constraint

$$\sum_{i=0}^{n_t}\sum_{j=0}^{n_s}\int_0^M a_{ij}g_j(x)dx = 1, \tag{3.8}$$

and the time-space occupation measure μ_0 is a finite measure with total mass $\leq T$, therefore

$$\sum_{i=0}^{n_t}\sum_{j=0}^{n_s}\int_0^M b_{ij}g_j(x)dx \leq T. \tag{3.9}$$

Chapter 4

Numerical Results

Based on the work derived until section 3.2 we will first implement a matrix version of the previous lp problems and then show the numerical results. Since there exists no closed form for the price of American Lookback options, the only waz to justify the results is to compare it to previous works. For instance, the examples [1] and [3] give a reference to justify the numbers.

4.1 Matrix Formulation

In this section we will implement the lp-program derived in chapter 2.
In order to tweak computations, we will try to formulate our lp-program in terms of matrices. We start by rearranging terms

$$\begin{aligned}
\mathbb{I}_{t_i}(0)g_k(0) &= \sum_{j=0}^{n_s} a_{i,j} \int_0^M g_k(x)g_j(x)dx \\
&\quad - \sum_{j=0}^{n_s} b_{i-1,j} \frac{n_t}{T} \int_0^M g_k(x)g_j(x)dx \\
&\quad - \sum_{j=0}^{n_s} b_{i,j} \left(\int_0^M -(r+\frac{\sigma^2}{2})g_k'(x)g_j(x)dx - \frac{n_t}{T}\int_0^M g_k(x)g_j(x)dx \right) \\
&\quad + \sum_{j=0}^{n_s} b_{i,j} \left(\frac{\sigma^2}{2} g_k'(0)g_j(0) \right) \\
&\quad + \sum_{j=0}^{n_s} b_{i,j} \int_0^M \frac{\sigma^2}{2} g_k'(x)g_j'(x)dx - c_i g_k'(0) + d_i g_k'(M)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n_s} \left(a_{i,j} - \frac{n_t}{T} b_{i-1,j} + \frac{n_t}{T} b_{i,j} \right) \int_0^M g_k(x) g_j(x) dx \\
&\quad - \sum_{j=0}^{n_s} b_{i,j} \Bigg(\int_0^M -(r + \frac{\sigma^2}{2}) g_k'(x) g_j(x) dx + \frac{\sigma^2}{2} g_k'(M) g_j(M) \\
&\quad - \frac{\sigma^2}{2} g_k'(0) g_j(0) - \int_0^M \frac{\sigma^2}{2} g_k'(x) g_j'(x) dx \Bigg) \\
&\quad - c_i g_k'(0) + d_i g_k'(M).
\end{aligned} \tag{4.1}
$$

We define the matrices E and F as follows

$$
\begin{aligned}
E_{kj} &= \int_0^M g_k(x) g_j(x) dx \\
F_{jk} &= \int_0^M -(r + \frac{\sigma^2}{2}) g_k'(x) g_j(x) dx - \frac{\sigma^2}{2} g_k'(0) g_j(0) \\
&\quad - \int_0^M \frac{\sigma^2}{2} g_k'(x) g_j'(x) dx.
\end{aligned}
$$

Then we can rewrite (4.1) as vector products

$$
\mathbb{I}_{t_i}(0) g_k(0) = \left(A_{i,\dots} - \frac{n_t}{T} B_{i-1,\dots} + \frac{n_t}{T} B_{i,\dots} \right) E_{\dots,k} - B_{i,\dots} F_{\dots,k} - C_i g_k'(0) + D_i g_k'(M),
$$

with $A = (a_{ij})_{i,j=1}^{n_t,n_s}$, $B = (b_{ij})_{i,j=1}^{n_t,n_s}$, $C = (c_i)_{i=1}^{n_t}$ and $D = (d_i)_{i=1}^{n_t}$.
The constraints (3.8) and (3.9) become

$$
\begin{aligned}
1 &= \sum_{i=0}^{n_t} \sum_{j=0}^{n_s} A_{ij} \int_0^M g_j(x) dx, \\
T &\geq \sum_{i=0}^{n_t} \sum_{j=0}^{n_s} B_{ij} \int_0^M g_j(x) dx \quad \geq \quad 0.
\end{aligned} \tag{4.2}
$$

4.2 American Lookback Put

We will examine a newly issued six-month American put option. As in [1] and [3] we will assume a volatility $\sigma = 0.2$, an initial price $S_0 = 100$ and an interest rate $r = 0.1$. The problem files of the lp's were written with MATLAB and they were solved by CPLEX on a multyprocessor-system with 4 x Dual-Core UltraSPARC-IV+, 1.8 GHz, 32 MB L2 Cache and 96 GB RAM working on Solaris 10.

n_s	Time steps n_t 50	60	70	80	90	100
50	10.565765	10.591681	10.609761	10.623192	10.633930	10.642717
60	10.474508	10.499860	10.518057	10.531501	10.542473	10.551187
70	10.409084	10.434215	10.452588	10.466191	10.477007	10.485665
80	10.359970	10.385325	10.403425	10.417170	10.427864	10.436495
90	10.321679	10.347019	10.365201	10.379028	10.389539	10.398287
100	10.291223	10.316563	10.334742	10.348354	10.359051	10.367724

Table 4.1: Price

n_s	Time steps n_t 50	60	70	80	90	100
50	4500	8215	7321	5706	7575	9086
60	6219	6357	7427	9050	5605	7100
70	5946	8531	6022	6782	11225	20199
80	6638	9150	11646	6599	9423	10655
90	9262	6007	12621	14573	63871	19701
100	17835	14578	14037	17482	12813	21214

Table 4.2: Iterations

n_s	Time steps n_t 50	60	70	80	90	100
50	7.84	19.89	20.23	18.13	23.91	39.03
60	13.03	15.85	23.17	37.83	22.91	35.96
70	14.9	27.64	22.74	34.21	87.25	230.28
80	18.77	34.25	49.93	33.42	65.52	101.2
90	29.51	25.65	62.1	87.3	908.12	160.97
100	101.24	97.07	83.55	114.93	150.54	198.26

Table 4.3: Time in seconds

n_s	Time steps n_t 50	100	200
50	10.236346	10.312639	10.682042
100	10.126262	10.202569	
200	10.071213	10.146185	
300	10.032130		

Table 4.4: Price for $M = 0.2$

We will start by a brief sensitivity analysis. Tables 4.1, 4.2 and 4.3 show the results of the computations with above assumptions and the limit for the state space $M = 0.35$. Without surprise, with an increasing number of time steps and base functions, more iterations are needed and hence the computations need more time.
With the number of time points fixed and the number of base functions increasing, the price drops. So the limitation to linear base functions, leads to an overpricing of the option. As we dense the time steps, the price increases. Hence, the error due to time discretization underestimates the price of the option.
In the next step, we will decrease the limit M for the state space. With $M = 0.2$ and increased number of base functions and time steps, we observe the results as in figures 4.4, 4.5 and 4.6.
In [1, Table 1], Babbs computes the price of an American lookback put option using a binomial model. Her computations point out, that the price of the option should be around \$10.17. So with $M = 0.35$ we will overestimate the price as figure 4.1 shows for a small amount of time points and base functions. With $M = 0.2$ the price is rather underestimated. Hence, the truncation of the state horizon leads to an underestimation of the option.
In [3, Table I], Conze and Viswanathan derive an upper bound for the American lookback option. For a put option with expiration six month, they found the upper bound to be \$14, 89 for. So for neither $M = 0.2$ nor $M = 0.35$ this upper bound is violated.

In figure In figure 4.1 we observe the stopping probabilities υ_τ^k. In figure 4.2 we

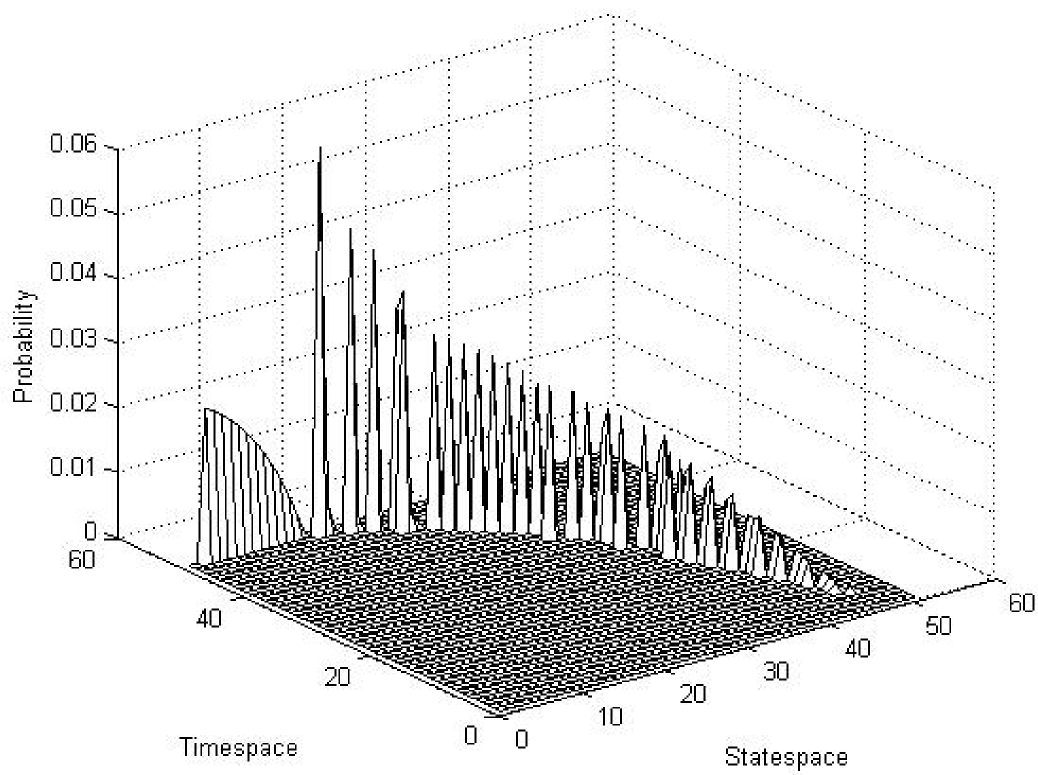

Figure 4.1: Stopping probabilities for $n_t = 50$ and $n_s = 50$.

n_s	Time steps n_t 50	100	200
50	2559	6046	10814
100	6171	7701	
200	12563	12762	
300	9204		

Table 4.5: Iterations for $M = 0.2$

n_s	Time steps n_t 50	100	200
50	3.86	28.71	114.83
100	21.14	48.54	
200	91.61	166.27	
300	94.28		

Table 4.6: Time in seconds for $M = 0.2$

observe the maximizing function

$$\text{Maximize } S_0 \sum_{j=1}^{n_t} \sum_{i=0}^{n_s} \int_0^M (e^x - 1) a_{ji} g_i(x) dx.$$

The timespace spreads from 0 to n_t, where n_t represents the end of the time horizon, and the spacespace from 0 to n_s, where n_s represents the limit M. As time progresses the areas with positive probabilities are closer to 0. At the beginning of the lifetime of the option the possible payoff must be very high in order to make the execution of the option attractive enough, because the holder of the option gives up on the chance of achieving an even higher payoff. At maturity the option is exercised in any case. Even if the stock price is equal to the running maximum and the quotient $X_t = \ln(\frac{M_t}{S_t}) = 0$.

As we introduced the running maximumin chapter 2, we included a preexisting maximum M_0 to allow more flexibility. So with $M_0 > S_0$ and hence $X_0 > 0$, the option can be interpreted as an option issued earlier. We set $M_0 = 110$ and choose $n_t = 50$ and $n_s = 50$. Figure 4.7 shows the price. The probabilities curve looks in shape exactly the same as before. The probabilities are shifted towards option-start and the upper part of the space horizon.

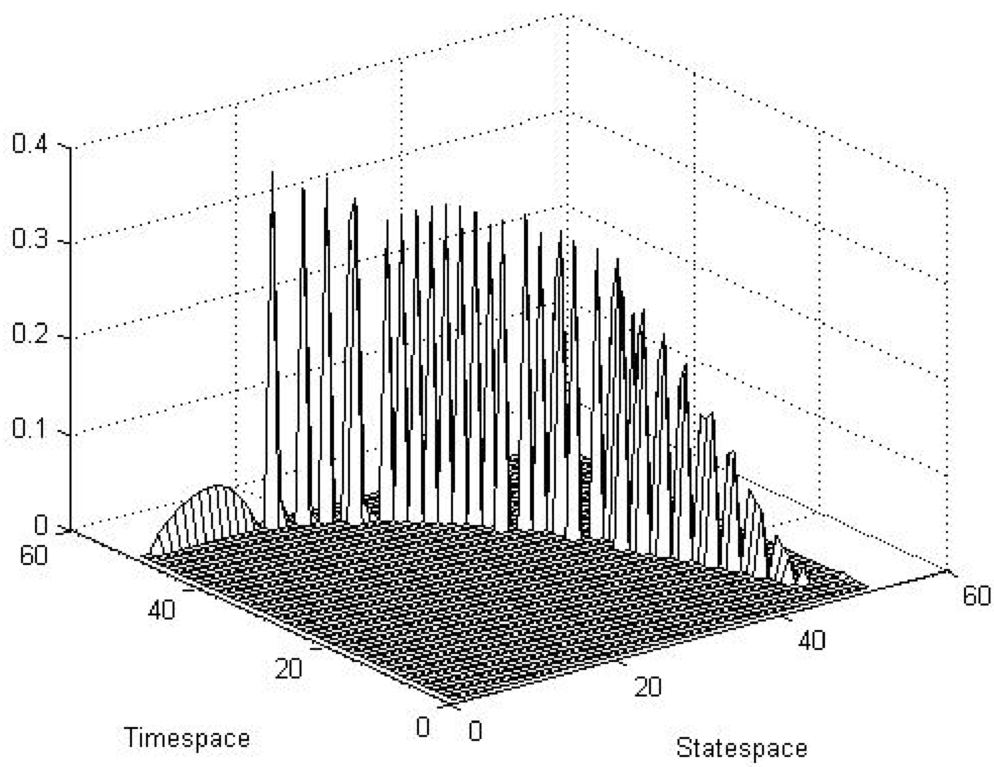

Figure 4.2: Maximising function

Price	Iteration	Time
12.644335	2659	4,03 sec

Table 4.7: Option with $M_0 > S_0$

We can interpret this as that since the maximum starts up higher, exercising early becomes more attractive.

As we compare the results of this chapter to, for instance, [1], we see that when we increase only the amount of timepoints, or base functions, the price will wander off. We need to increase both of them to get better results. Also, the linear programming approach does not seem to be very efficient when we just want to get the price of the option. Babbs got in her work results in way less time and Iterations. But her binomial approach can only show the price of the option. Our approach not only computes the price, but also the stopping probabilities.

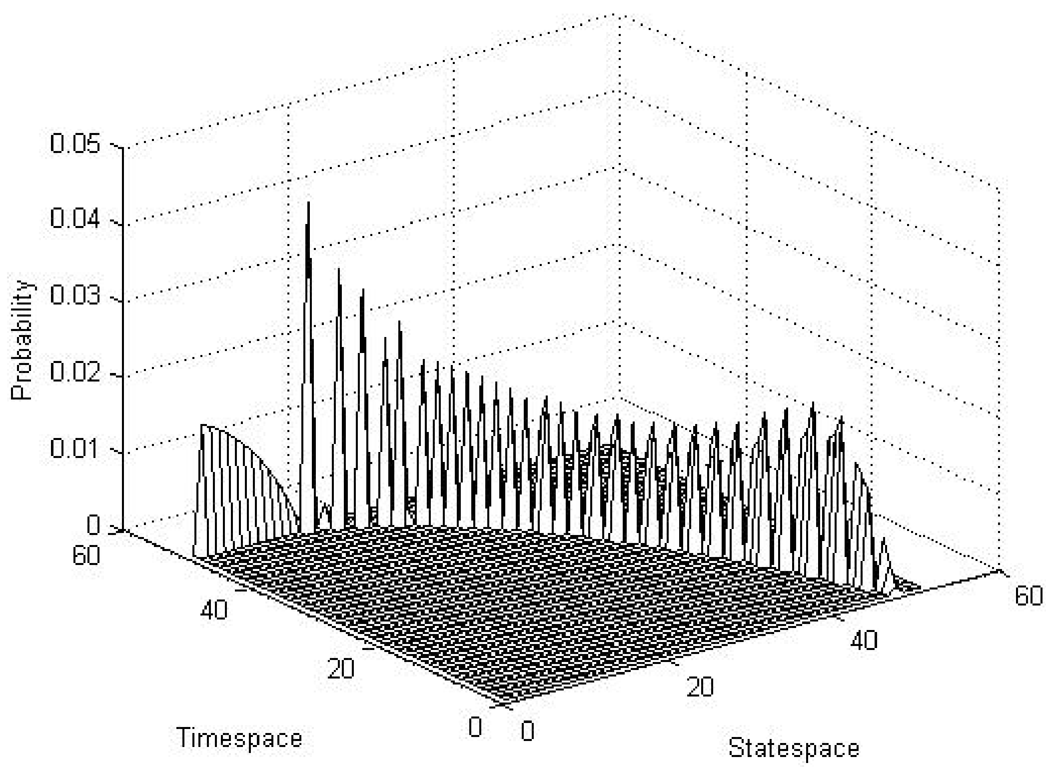

Figure 4.3: Stopping probabilities for $M_0 = 110$

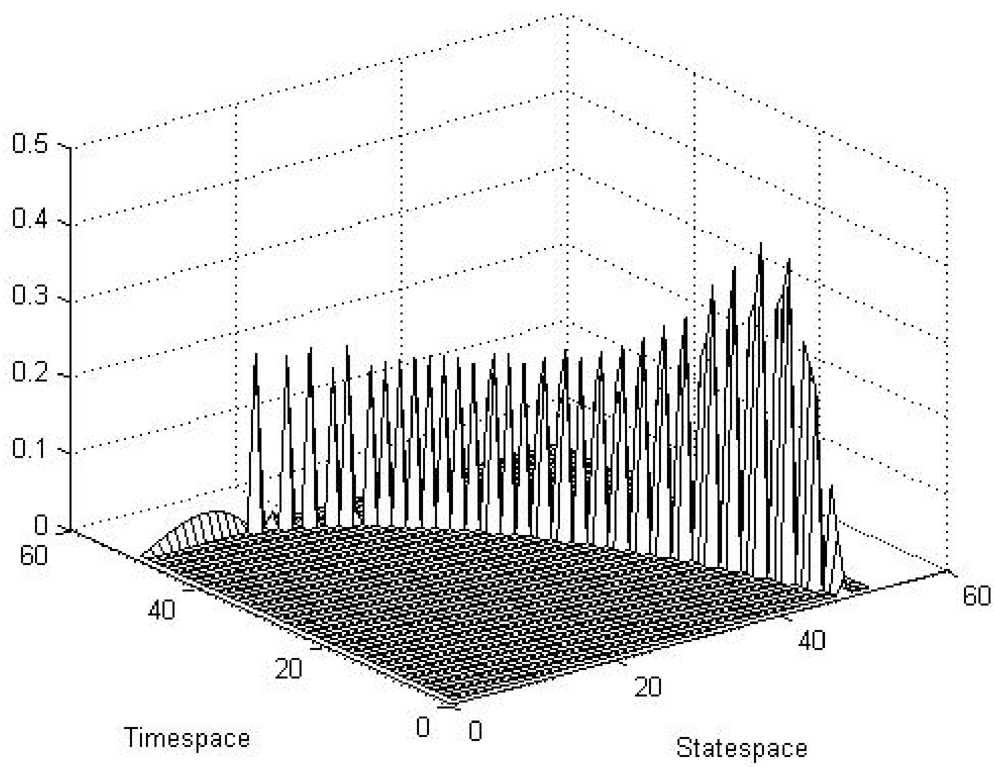

Figure 4.4: Maximising function

Chapter 5

Conclusion

After we introduced the model and described the character of lookback options, the next step was to reduce the dimension odf the problem. We showed that the logarithm of the ratio of maximum/ minimum and stock price is more feasible. Next, we translated the problem into an LP. Since we are not able to consider the infinite time-state space,we have to apply approximations.

Our first step was to disretize the time space and in a second step, we limited the problem to only piecewise linear functions to approximate the test functions.

We reformulated the LP with above approximations and rewrote it in terms of matrices and vectors. This was followed by the numerical results of the computations. We were able to present the price of newly issued American options, together with the resulting probabilities.

We examined the impact of increasing numbers of base functions and time points. With the number of time points fixed and the number of base functions increasing, the price drops. So the limitation to linear base functions, leads to an overpricing of the option. As we dense the time steps, the price increases. Hence, the error due to time discretization underestimates the price of the option.

Also, the effect of different truncations of the state space was presented. The chapter about numerical results ended with an example of an earlier issued option. So we tried to price an option during its lifetime, instead of prior to the life of the option. The linear programming approach does not seem to be very efficient when we just want to get the price of the option. Babbs got in her work results in much less time and iterations. But her binomial approach can only show the price of the option. Our approach not only computes the price, but also the stopping probabilities.

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